第一次作业

第一题

判断方程的类型。

$$u_{xx} + xyu_{yy} = 0 \tag{1}$$

解：特征方程为

$$y_x^2 + xy = 0 \tag{2}$$

判别式为

$$\Delta = -xy \tag{3}$$

1. $xy > 0$：椭圆型方程。
2. $xy = 0$：抛物型方程。
3. $xy < 0$：双曲型方程。

第二题

化下列方程为标准形式。

第一问

$$u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0 \tag{4}$$

解：特征方程为

$$y_x^2 - 4y_x + 5 = 0 \tag{5}$$

特征解为

$$(2x - y) + ix = C_1, \qquad (2x - y) - ix = C_2 \tag{6}$$

作变量代换

$$\xi = 2x - y, \qquad \eta = x \tag{7}$$

那么

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = 2u_\xi + u_\eta \tag{8}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = -u_\xi \tag{9}$$

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta} \tag{10}$$

$$u_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\xi\xi} \tag{11}$$

$$u_{xy} = \frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = -2u_{\xi\xi} - u_{\xi\eta} \tag{12}$$

代入原方程，化为椭圆型方程的标准型

$$u_{\xi\xi} + u_{\eta\eta} + u_\eta = 0 \tag{13}$$

第二问

$$u_{xx} + yu_{yy} = 0 \tag{14}$$

解：特征方程为

$$y_x^2 + y = 0 \tag{15}$$

当$y > 0$时，特征解为

$$2\sqrt{y} + ix = C_1, \qquad 2\sqrt{y} - ix = C_2 \tag{16}$$

作变量代换

$$\xi = x, \qquad \eta = 2\sqrt{y} \tag{17}$$

那么

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi \tag{18}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{2}{\eta}u_\eta \tag{19}$$

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} = u_{\xi\xi} \tag{20}$$

$$u_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{4}{\eta^2}u_{\eta\eta} - \frac{4}{\eta^3}u_\eta \tag{21}$$

代入原方程，化为椭圆型方程的标准型

$$u_{\xi\xi} + u_{\eta\eta} = \frac{u_\eta}{\eta} \tag{22}$$

当$y < 0$时，特征解为

$$x + 2\sqrt{-y} = C_1, \qquad x - 2\sqrt{-y} = C_2 \tag{23}$$

作变量代换

$$\xi = x + 2\sqrt{-y}, \qquad \eta = x - 2\sqrt{-y} \tag{24}$$

那么

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + u_\eta \tag{25}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{4}{\xi - \eta}(u_\eta - u_\xi) \tag{26}$$

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \tag{27}$$

$$u_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{y}(2u_{\xi\eta} - u_{\xi\xi} - u_{\eta\eta}) \tag{28}$$

代入原方程，化为双曲型方程的第一标准型

$$u_{\xi\eta} = \frac{u_\eta - u_\xi}{2(\xi - \eta)} \tag{29}$$

第三题

确定下列方程的通解。

$$u_{xx} - 3u_{xy} + 2u_{yy} = 0 \tag{30}$$

解：特征方程为

$$y_x^2 + 3y_x + 2 = 0 \tag{31}$$

特征解为

$$x + y = C_1, \qquad 2x + y = C_2 \tag{32}$$

作变量代换

$$\xi = x + y, \qquad \eta = 2x + y \tag{33}$$

那么

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + 2u_\eta \tag{34}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\xi + u_\eta \tag{35}$$

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} = u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta} \tag{36}$$

$$u_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \tag{37}$$

$$u_{xy} = \frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\xi\xi} + 3u_{\xi\eta} + 2u_{\eta\eta} \tag{38}$$

代入原方程，化为双曲型方程的第一标准型

$$u_{\xi\eta} = 0 \tag{39}$$

从而通解为

$$u(\xi, \eta) = f(\xi) + g(\eta) \tag{40}$$

即

$$u(x, y) = f(x + y) + g(2x + y) \tag{41}$$

其中f, g为任意二阶连续可微函数。

第二次作业

第一题

求解下列特征值问题：

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=X'(l)=0 \end{cases} \tag{42}$$

解：如果$\lambda<0$，通解为

$$X(x)=C\mathrm{e}^{\sqrt{-\lambda}x}+D\mathrm{e}^{-\sqrt{-\lambda}x} \tag{43}$$
$$X'(x)=C\sqrt{-\lambda}\mathrm{e}^{\sqrt{-\lambda}x}-D\sqrt{-\lambda}\mathrm{e}^{-\sqrt{-\lambda}x} \tag{44}$$

代入初值条件

$$C\sqrt{-\lambda}-D\sqrt{-\lambda}=0,\qquad C\sqrt{-\lambda}\mathrm{e}^{\sqrt{-\lambda}l}-D\sqrt{-\lambda}\mathrm{e}^{-\sqrt{-\lambda}l}=0 \tag{45}$$

解得

$$C=D=0 \tag{46}$$

于是原方程仅存在零解。

如果$\lambda=0$，通解为

$$X(x)=Cx+D \tag{47}$$
$$X'(x)=C \tag{48}$$

代入初值条件

$$C=0 \tag{49}$$

于是原方程的通解为

$$X(x)=D \tag{50}$$

如果$\lambda>0$，通解为

$$X(x)=C\cos\sqrt{\lambda}x+D\sin\sqrt{\lambda}x \tag{51}$$
$$X'(x)=-C\sqrt{\lambda}\sin\sqrt{\lambda}x+D\sqrt{\lambda}\cos\sqrt{\lambda}x \tag{52}$$

代入初值条件

$$D\sqrt{\lambda}=0,\qquad -C\sqrt{\lambda}\sin\sqrt{\lambda}l+D\sqrt{\lambda}\cos\sqrt{\lambda}l=0 \tag{53}$$

解得

$$C=\begin{cases} \text{任意}, & \lambda=\frac{n^2\pi^2}{l^2},n\in\mathbb{N}^* \\ 0, & \lambda\neq\frac{n^2\pi^2}{l^2},n\in\mathbb{N}^* \end{cases},\qquad D=0 \tag{54}$$

于是若$\lambda\neq n^2\pi^2/l^2$，则原方程仅存在零解；若$\lambda=n^2\pi^2/l^2$，则原方程存在非零解

$$X(x)=C_n\cos\frac{n\pi}{l}x,\qquad n\in\mathbb{N} \tag{55}$$

第二题

求解下列特征值问题：

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=X(l)=0 \end{cases} \tag{56}$$

解：如果$\lambda<0$，通解为

$$X(x)=C\mathrm{e}^{\sqrt{-\lambda}x}+D\mathrm{e}^{-\sqrt{-\lambda}x} \tag{57}$$

$$X'(x)=C\sqrt{-\lambda}\mathrm{e}^{\sqrt{-\lambda}x}-D\sqrt{-\lambda}\mathrm{e}^{-\sqrt{-\lambda}x} \tag{58}$$

代入初值条件

$$C\sqrt{-\lambda}-D\sqrt{-\lambda}=0,\qquad C\mathrm{e}^{\sqrt{-\lambda}l}+D\mathrm{e}^{-\sqrt{-\lambda}l}=0 \tag{59}$$

解得

$$C=D=0 \tag{60}$$

于是原方程仅存在零解。

如果$\lambda=0$，通解为

$$X(x)=Cx+D \tag{61}$$

$$X'(x)=C \tag{62}$$

代入初值条件

$$C=0,\qquad Cl+D=0 \tag{63}$$

解得

$$C=D=0 \tag{64}$$

于是原方程仅存在零解。

如果$\lambda>0$，通解为

$$X(x)=C\cos\sqrt{\lambda}x+D\sin\sqrt{\lambda}x \tag{65}$$

$$X'(x)=-C\sqrt{\lambda}\sin\sqrt{\lambda}x+D\sqrt{\lambda}\cos\sqrt{\lambda}x \tag{66}$$

代入初值条件

$$D\sqrt{\lambda}=0,\qquad C\cos\sqrt{\lambda}l+D\sin\sqrt{\lambda}l=0 \tag{67}$$

解得

$$C=\begin{cases} \text{任意}, & \lambda=\frac{(n-1/2)^2\pi^2}{l^2}, n\in\mathbb{N}^* \\ 0, & \lambda\neq\frac{(n-1/2)^2\pi^2}{l^2}, n\in\mathbb{N}^* \end{cases},\qquad D=0 \tag{68}$$

于是若$\lambda\neq(n-1/2)^2\pi^2/l^2$，则原方程仅存在零解；若$\lambda=(n-1/2)^2\pi^2/l^2$，则原方程存在非零解

$$X(x)=C_n\cos\frac{(n-1/2)\pi}{l}x,\qquad n\in\mathbb{N}^* \tag{69}$$

第三题

求解下列特征值问题：

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=X'(l)=0 \end{cases} \tag{70}$$

解：如果$\lambda<0$，通解为

$$X(x) = C\mathrm{e}^{\sqrt{-\lambda}x} + D\mathrm{e}^{-\sqrt{-\lambda}x} \tag{71}$$

$$X'(x) = C\sqrt{-\lambda}\mathrm{e}^{\sqrt{-\lambda}x} - D\sqrt{-\lambda}\mathrm{e}^{-\sqrt{-\lambda}x} \tag{72}$$

代入初值条件

$$C + D = 0, \qquad C\sqrt{-\lambda}\mathrm{e}^{\sqrt{-\lambda}l} - D\sqrt{-\lambda}\mathrm{e}^{-\sqrt{-\lambda}l} = 0 \tag{73}$$

解得

$$C = D = 0 \tag{74}$$

于是原方程仅存在零解。

如果$\lambda = 0$，通解为

$$X(x) = Cx + D \tag{75}$$

$$X'(x) = C \tag{76}$$

代入初值条件

$$C = D = 0 \tag{77}$$

于是原方程仅存在零解。

如果$\lambda > 0$，通解为

$$X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x \tag{78}$$

$$X'(x) = -C\sqrt{\lambda}\sin\sqrt{\lambda}x + D\sqrt{\lambda}\cos\sqrt{\lambda}x \tag{79}$$

代入初值条件

$$C = 0, \qquad -C\sqrt{\lambda}\sin\sqrt{\lambda}l + D\sqrt{\lambda}\cos\sqrt{\lambda}l = 0 \tag{80}$$

解得

$$C = 0, \qquad D = \begin{cases} \text{任意}, & \lambda = \frac{(n-1/2)^2\pi^2}{l^2}, n \in \mathbb{N}^* \\ 0, & \lambda \neq \frac{(n-1/2)^2\pi^2}{l^2}, n \in \mathbb{N}^* \end{cases} \tag{81}$$

于是若$\lambda \neq (n-1/2)^2\pi^2/l^2$，则原方程仅存在零解；若$\lambda = (n-1/2)^2\pi^2/l^2$，则原方程存在非零解

$$X(x) = D_n \sin\frac{(n-1/2)\pi}{l}x, \qquad n \in \mathbb{N}^* \tag{82}$$

第三次作业

使用分离变量法求解定解问题：

$$\begin{cases}u_t=a^2u_{xx}, & 0<x<l,t>0\\ u(x,0)=x, & 0\le x\le l\\ u_x(0,t)=u_x(l,t)=0, & t\ge 0\end{cases}\tag{83}$$

解：令$u(x,t)=T(t)X(x)$，代入方程

$$T'(t)X(x)=a^2T(t)X''(x)\tag{84}$$

于是存在$\lambda\in\mathbb{R}$，使得成立

$$\frac{T'(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda\tag{85}$$

即

$$\begin{cases}T'(t)+a^2\lambda T(t)=0\\ X''(x)+\lambda X(x)=0\end{cases}\tag{86}$$

考虑到边界条件

$$\begin{cases}X''(x)+\lambda X(x)=0\\ X'(0)=X'(l)=0\end{cases}\tag{87}$$

求解特征值问题

$$\lambda_n=\frac{n^2\pi^2}{l^2},\qquad X_n(x)=\cos\frac{n\pi}{l}x,\qquad n\in\mathbb{N}\tag{88}$$

代入原方程

$$T_n'(t)+\left(\frac{an\pi}{l}\right)^2T_n(t)=0,\qquad n\in\mathbb{N}\tag{89}$$

通解为

$$T_n(t)=C_n\mathrm{e}^{-\left(\frac{an\pi}{l}\right)^2t},\qquad n\in\mathbb{N}\tag{90}$$

从而原微分方程的解为

$$u_n(x,t)=C_n\mathrm{e}^{-\left(\frac{an\pi}{l}\right)^2t}\cos\frac{n\pi}{l}x,\qquad n\in\mathbb{N}\tag{91}$$

由迭加原理

$$u(x,t)=\frac{C_0}{2}+\sum_{n=1}^{\infty}C_n\mathrm{e}^{-\left(\frac{an\pi}{l}\right)^2t}\cos\frac{n\pi}{l}x\tag{92}$$

考虑到初始条件

$$x=\frac{C_0}{2}+\sum_{n=1}^{\infty}C_n\cos\frac{n\pi}{l}x\tag{93}$$

由Fourier级数

$$C_n = \frac{2}{l}\int_0^l \xi \cos\frac{n\pi}{l}\xi \mathrm{d}\xi = \begin{cases} l, & n = 0 \\ 0, & n \geq 1 且 2 \mid n \\ \frac{-4l}{n^2\pi^2}, & n \geq 1 且 2 \nmid n \end{cases} \tag{94}$$

从而原微分方程的形式解为

$$u(x,t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\left(\frac{a(2n-1)\pi}{l}\right)^2 t}}{(2n-1)^2}\cos\frac{(2n-1)\pi}{l}x \tag{95}$$

第四次作业

第一题

求解Laplace方程

$$\begin{cases} u_{xx}+u_{yy}=0, & x^2+y^2<a^2 \\ u|_{x^2+y^2=a^2}=x+y \end{cases} \tag{96}$$

解：引入极坐标变换

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \tag{97}$$

那么原方程化为

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=0, & 0<\rho<a \\ u|_{\rho=a}=a\cos\theta+a\sin\theta \end{cases} \tag{98}$$

求解函数$f(\tau)=a\cos\tau+a\sin\tau$的Fourier系数

$$A_n=\frac{1}{\pi}\int_0^{2\pi}f(\tau)\cos n\tau\mathrm{d}\tau=\frac{a}{\pi}\int_0^{2\pi}(\cos\theta+\sin\theta)\cos n\tau\mathrm{d}\tau=\begin{cases} a, & n=1 \\ 0, & n\neq 1 \end{cases} \tag{99}$$

$$B_n=\frac{1}{\pi}\int_0^{2\pi}f(\tau)\sin n\tau\mathrm{d}\tau=\frac{a}{\pi}\int_0^{2\pi}(\cos\theta+\sin\theta)\sin n\tau\mathrm{d}\tau=\begin{cases} a, & n=1 \\ 0, & n\neq 1 \end{cases} \tag{100}$$

从而微分方程的形式解为

$$u(\rho,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left(\frac{\rho}{a}\right)^n(A_n\cos n\theta+B_n\sin n\theta) \tag{101}$$

$$=\rho\cos\theta+\rho\sin\theta \tag{102}$$

即

$$u(x,y)=x+y \tag{103}$$

第二题

求解Laplace方程

$$\begin{cases} u_{xx}+u_{yy}=0, & x^2+y^2<a^2 \\ u|_{x^2+y^2=a^2}=\sin\theta\cos 2\theta \end{cases} \tag{104}$$

其中$\theta=\arctan(y/x)$。

解：引入极坐标变换

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \tag{105}$$

那么原方程化为

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=0, & 0<\rho<a \\ u|_{\rho=a}=\sin\theta\cos 2\theta \end{cases} \tag{106}$$

求解函数$f(\tau)=\sin\theta\cos 2\theta$的Fourier系数

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(\tau)\cos n\tau \mathrm{d}\tau = \frac{1}{\pi}\int_0^{2\pi} \sin\tau\cos 2\tau\cos n\tau \mathrm{d}\tau = 0 \tag{107}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\tau)\sin n\tau \mathrm{d}\tau = \frac{1}{\pi}\int_0^{2\pi} \sin\tau\cos 2\tau\sin n\tau \mathrm{d}\tau = \begin{cases} -1/2, & n=1 \\ 1/2, & n=3 \\ 0, & n\neq 1 且 n\neq 3 \end{cases} \tag{108}$$

从而微分方程的形式解为

$$u(\rho,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{a}\right)^n (A_n\cos n\theta + B_n\sin n\theta) \tag{109}$$

$$= \frac{\rho^3}{2a^3}\sin 3\theta - \frac{\rho}{2a}\sin\theta \tag{110}$$

即

$$u(x,y) = \frac{\rho^3}{2a^3}\sin 3\theta - \frac{\rho}{2a}\sin\theta \tag{111}$$

$$= \frac{\rho^3}{2a^3}(3\sin\theta - 4\sin^3\theta) - \frac{\rho}{2a}\sin\theta \tag{112}$$

$$= \frac{1}{2a^3}(3(x^2+y^2)y - 4y^3) - \frac{y}{2a} \tag{113}$$

$$= \frac{1}{2a^3}(3x^2y - y^3) - \frac{y}{2a} \tag{114}$$

$$= \frac{3x^2y - y^3}{2a^3} - \frac{y}{2a} \tag{115}$$

第五次作业

第一题

对于$\eta > 0$，求函数的Fourier变换：

$$f(x) = \mathrm{e}^{-\eta x^2}, \qquad x \in \mathbb{R} \tag{116}$$

解：

$$\begin{aligned}
\mathscr{F}[f] &= \int_{-\infty}^{+\infty} f(\xi)\mathrm{e}^{-i\lambda\xi}\mathrm{d}\xi && (117)\\
&= \int_{-\infty}^{+\infty} \mathrm{e}^{-\eta\xi^2-i\lambda\xi}\mathrm{d}\xi && (118)\\
&= \frac{1}{\sqrt{\eta}}\mathrm{e}^{-\frac{\lambda^2}{4\eta}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\zeta^2}\mathrm{d}\zeta \qquad \left(\zeta = \sqrt{\eta}\left(\xi + \frac{i\lambda}{2\eta}\right)\right) && (119)\\
&= \frac{1}{\sqrt{\eta}}\mathrm{e}^{-\frac{\lambda^2}{4\eta}}\Gamma(1/2) && (120)\\
&= \sqrt{\frac{\pi}{\eta}}\mathrm{e}^{-\frac{\lambda^2}{4\eta}} && (121)
\end{aligned}$$

第二题

直接用解的公式求解定解问题：

$$\begin{cases} u_t = a^2 u_{xx}, & x \in \mathbb{R}, t > 0 \\ u(x,0) = x^2 + 1, & x \in \mathbb{R} \end{cases} \tag{122}$$

解：

$$\begin{aligned}
u(x,t) &= \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}(\xi^2+1)\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2t}}\mathrm{d}\xi && (123)\\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}((2a\sqrt{t}\zeta + x)^2 + 1)\mathrm{e}^{-\zeta^2}\mathrm{d}\zeta \qquad \left(\zeta = \frac{\xi - x}{2a\sqrt{t}}\right) && (124)\\
&= \frac{4a^2t}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\zeta^2\mathrm{e}^{-\zeta^2}\mathrm{d}\zeta + \frac{4ax\sqrt{t}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\zeta\mathrm{e}^{-\zeta^2}\mathrm{d}\zeta + \frac{x^2+1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\zeta^2}\mathrm{d}\zeta && (125)\\
&= \frac{8a^2t}{\sqrt{\pi}}\int_{0}^{+\infty}\zeta^2\mathrm{e}^{-\zeta^2}\mathrm{d}\zeta + \frac{2(x^2+1)}{\sqrt{\pi}}\int_{0}^{+\infty}\mathrm{e}^{-\zeta^2}\mathrm{d}\zeta && (126)\\
&= \frac{4a^2t}{\sqrt{\pi}}\Gamma(3/2) + \frac{x^2+1}{\sqrt{\pi}}\Gamma(1/2) && (127)\\
&= x^2 + 2a^2t + 1 && (128)
\end{aligned}$$

第三题

求函数的Laptops变换：

$$f(t) = \sinh\omega t \tag{129}$$

解：

$$
\begin{aligned}
\mathscr{L}[f(t)] &= \int_0^{+\infty} \mathrm{e}^{-pt} \sinh \omega t \mathrm{d}t && (130)\\
&= \int_0^{+\infty} \mathrm{e}^{-pt} \frac{\mathrm{e}^{\omega t} - \mathrm{e}^{-\omega t}}{2} \mathrm{d}t && (131)\\
&= \frac{1}{2} \int_0^{+\infty} \mathrm{e}^{-(p-\omega)t} \mathrm{d}t - \frac{1}{2} \int_0^{+\infty} \mathrm{e}^{-(p+\omega)t} \mathrm{d}t && (132)\\
&= \frac{\omega}{p^2 - \omega^2}, \qquad \mathrm{Re}(p) > |\mathrm{Re}(\omega)| && (133)
\end{aligned}
$$

第六次作业

第一题

用D'Alembert公式求解定解问题：

$$\begin{cases} u_{tt} = a^2 u_{xx}, & x \in \mathbb{R}, t > 0 \\ u(x,0) = x^2, & x \in \mathbb{R} \\ u_t(x,0) = x, & x \in \mathbb{R} \end{cases} \tag{134}$$

解：由D'Alembert公式

$$u(x,t) = \frac{1}{2}((x+at)^2 + (x-at)^2) + \frac{1}{2a}\int_{x-at}^{x+at} \xi \mathrm{d}\xi \tag{135}$$

$$= x^2 + a^2t^2 + xt \tag{136}$$

第二题

在上半平面$\{(x,t): x \in \mathbb{R}, t > 0\}$上给出一点$M(2,5)$，对于弦振动$u_{tt} = u_{xx}$方程来说，点$M$的依赖区间是什么？它是否落在点$(1,0)$的影响区间内？

解：由于点(x,t)的依赖区间为$[x-t, x+t]$，因此点M的依赖区间为$[-3,7]$。

由于点x_0的影响区域为

$$D_3 = \{(x,t): x_0 - at \le x \le x_0 + at, t \ge 0\} \tag{137}$$

因此点$(1,0)$的影响区域为

$$D_3 = \{(x,t): 1 - t \le x \le 1 + t, t \ge 0\} \tag{138}$$

显然$(2,5) \in D_3$，因此M落在点$(1,0)$的影响区域内。

第七次作业

第一题

第一问

说明如下函数是否为调和函数。

$$u(x,y)=x^3-3xy^2 \tag{139}$$

解:

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=6x-6x=0 \tag{140}$$

因此u为调和函数。

第二问

说明如下函数是否为调和函数。

$$u(x,y)=3x^2y-y^3 \tag{141}$$

解:

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=6y-6y=0 \tag{142}$$

因此u为调和函数。

第二题

证明：Green函数在区域Ω内成立不等式

$$0<G(P,P_0)<\frac{1}{4\pi|PP_0|} \tag{143}$$

证明：由Green函数的定义

$$G(P,P_0)=\frac{1}{4\pi|PP_0|}-g(P,P_0),\qquad \begin{cases}\Delta g=0, & P\in\Omega\\ g\mid_{\partial\Omega}=\frac{1}{4\pi|PP_0|}\end{cases} \tag{144}$$

从而g在Ω内调和，且

$$g\mid_{\partial\Omega}=\frac{1}{4\pi|PP_0|}>0 \tag{145}$$

由调和函数极值原理，在Ω内成立$g>0$，因此

$$G(P,P_0)=\frac{1}{4\pi|PP_0|}-g(P,P_0)<\frac{1}{4\pi|PP_0|} \tag{146}$$

另一方面，由于$\lim\limits_{P\to P_0}G(P,P_0)=+\infty$，从而存在充分小的$\varepsilon>0$，使得成立$B_\varepsilon(P_0)\subset\Omega$，且在$\overline{B}_\varepsilon(P_0)$上成立$G>0$。在$\Omega\setminus B_\varepsilon(P_0)$上，$\Delta G=0$，且$G\mid_{\partial\Omega}=0$，同时$G\mid_{\partial B_\varepsilon(P_0)}>0$。由调和函数极值原理，在$\Omega\setminus B_\varepsilon(P_0)$上成立$G>0$，进而在$\Omega$内成立$G>0$。

综上所述

$$0 < G(P, P_0) < \frac{1}{4\pi|PP_0|} \tag{147}$$