第一次作业

第一题

已知曲线$\boldsymbol{r}=\left\{\cos^3(t),\sin^3(t),\cos(2t)\right\}$，求

第一问

基本向量$\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$。

解：由于

$$\begin{gathered}\boldsymbol{r}'=\left\{-3\sin(t)\cos^2(t),3\sin^2(t)\cos(t),-2\sin(2t)\right\}\\ \frac{\mathrm{d}s}{\mathrm{d}t}=|\boldsymbol{r}'|=5|\sin(t)\cos(t)|\end{gathered}\tag{1}$$

那么

$$\boldsymbol{\alpha}=\dot{\boldsymbol{r}}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}=\frac{\boldsymbol{r}'}{|\boldsymbol{r}'|}=\begin{cases}\left\{-\dfrac{3}{5}\cos(t),\dfrac{3}{5}\sin(t),-\dfrac{4}{5}\right\}, & \sin(2t)\geq 0\\ \left\{\dfrac{3}{5}\cos(t),-\dfrac{3}{5}\sin(t),\dfrac{4}{5}\right\}, & \sin(2t)\leq 0\end{cases}\tag{2}$$

从而

$$\dot{\boldsymbol{\alpha}}=\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}s}=\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}=\left\{\frac{3}{25\cos(t)},\frac{3}{25\sin(t)},0\right\},\qquad|\dot{\boldsymbol{\alpha}}|=\frac{3}{25|\sin(t)\cos(t)|}\tag{3}$$

因此

$$\boldsymbol{\beta}=\frac{\dot{\boldsymbol{\alpha}}}{|\dot{\boldsymbol{\alpha}}|}=\begin{cases}\{\sin(t),\cos(t),0\}, & \sin(2t)\geq 0\\ \{-\sin(t),-\cos(t),0\}, & \sin(2t)\leq 0\end{cases}\tag{4}$$

进而

$$\boldsymbol{\gamma}=\boldsymbol{\alpha}\times\boldsymbol{\beta}=\left\{\frac{4}{5}\cos(t),-\frac{4}{5}\sin(t),-\frac{3}{5}\right\}\tag{5}$$

第二问

曲率和挠率。

解：曲率为

$$\kappa=|\dot{\boldsymbol{\alpha}}|=\frac{3}{25|\sin(t)\cos(t)|}\tag{6}$$

由于

$$\dot{\boldsymbol{\gamma}}=\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}s}=\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}=\begin{cases}\left\{-\dfrac{4}{25\cos(t)},-\dfrac{4}{25\sin(t)},0\right\}, & \sin(2t)\geq 0\\ \left\{\dfrac{4}{25\cos(t)},\dfrac{4}{25\sin(t)},0\right\}, & \sin(2t)\leq 0\end{cases}\tag{7}$$

那么当$\sin(2t)\geq 0$时，$\dot{\boldsymbol{\gamma}}$与$\boldsymbol{\beta}$异向，于是挠率为

$$\tau=|\dot{\boldsymbol{\gamma}}|=\frac{4}{25|\sin(t)\cos(t)|}=\frac{4}{25\sin(t)\cos(t)}\tag{8}$$

当$\sin(2t) \leq 0$时，$\dot{\boldsymbol{\gamma}}$与$\boldsymbol{\beta}$同向，于是挠率为

$$\tau = -|\dot{\boldsymbol{\gamma}}| = -\frac{4}{25|\sin(t)\cos(t)|} = \frac{4}{25\sin(t)\cos(t)} \tag{9}$$

因此挠率为

$$\tau = \frac{4}{25\sin(t)\cos(t)} \tag{10}$$

第二次作业

确定螺旋面

$$\boldsymbol{r}=\{u\cos v,u\sin v,cv\} \tag{11}$$

上的曲率线。

解：一阶偏导为

$$\boldsymbol{r}_u=\{\cos v,\sin v,0\},\qquad \boldsymbol{r}_v=\{-u\sin v,u\cos v,c\} \tag{12}$$

二阶偏导为

$$\boldsymbol{r}_{uu}=\{0,0,0\},\qquad \boldsymbol{r}_{uv}=\{-\sin v,\cos v,0\},\qquad \boldsymbol{r}_{vv}=\{-u\cos v,-u\sin v,0\} \tag{13}$$

单位法向量为

$$\boldsymbol{n}=\frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{|\boldsymbol{r}_u\times\boldsymbol{r}_v|}=\frac{\{c\sin v,-c\cos v,u\}}{\sqrt{u^2+c^2}} \tag{14}$$

第一基本形式为

$$E=\boldsymbol{r}_u\cdot\boldsymbol{r}_u=1,\qquad F=\boldsymbol{r}_u\cdot\boldsymbol{r}_v=0,\qquad G=\boldsymbol{r}_v\cdot\boldsymbol{r}_v=u^2+c^2 \tag{15}$$

第二类基本量为

$$L=\boldsymbol{r}_{uu}\cdot\boldsymbol{n}=0,\qquad M=\boldsymbol{r}_{uv}\cdot\boldsymbol{n}=-\frac{c}{\sqrt{u^2+c^2}},\qquad N=\boldsymbol{r}_{vv}\cdot\boldsymbol{n}=0 \tag{16}$$

曲面上曲率线方程为

$$\begin{vmatrix}\mathrm{d}v^2 & -\mathrm{d}u\mathrm{d}v & \mathrm{d}u^2\\ E & F & G\\ L & M & N\end{vmatrix}=\begin{vmatrix}\mathrm{d}v^2 & -\mathrm{d}u\mathrm{d}v & \mathrm{d}u^2\\ 1 & 0 & u^2+c^2\\ 0 & -\frac{c}{\sqrt{u^2+c^2}} & 0\end{vmatrix}=\frac{c}{\sqrt{u^2+c^2}}((u^2+c^2)\mathrm{d}v^2-\mathrm{d}u^2)=0 \tag{17}$$

因此

$$\frac{\mathrm{d}u}{\mathrm{d}v}=\pm\sqrt{u^2+c^2} \tag{18}$$

解得

$$\ln\left(\sqrt{u^2+c^2}-u\right)=C\pm v,\qquad C\in\mathbb{R} \tag{19}$$

因此螺旋面上的曲率线族为

$$\ln\left(\sqrt{u^2+c^2}-u\right)=C\pm v,\qquad C\in\mathbb{R} \tag{20}$$