Discrete Mathematics for Computer Science

Lecture 1b: Propositional Logic

Dr. Ming Tang

Department of Computer Science and Engineering
Southern University of Science and Technology (SUSTech)
Email: tangm3@sustech.edu.cn

This Lecture

Logic and Mathematical Proofs

Sets and Functions

Relations

Complexity of Algorithms

Graphs I

Number Theory — Cryptography

Graphs II

Mathematical Induction and Recursion

Trees

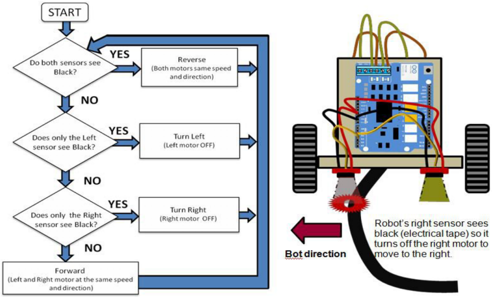
Counting

Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers

What is Logic?

Logic is the basis of all mathematical reasoning:

- Syntax of statements
- The meaning of statements
- The rules of logical inference



What is Propositional Logic?

Proposition: a declarative sentence that is either true or false (not both).

- Declarative sentence: a sentence that makes a statement, while it does not ask a question or give an order
- Either true or false: fixed; no variable involved

Truth value of a proposition: true, denoted by T; false, denoted by F.

Propositional variables: variables that represent propositions

- Conventional letters used for propositional variables are p, q, r, s, ...

Examples

Examples of propositions:

- SUSTech is located in Shenzhen. (T)
- $2 + 2 = 3$ (F)
- It is raining today. (either T or F)(The date is specified)

Examples which are not propositions:

- No parking. $\rightarrow$ Not a declarative sentence
- How old are you? $\rightarrow$ Not a declarative sentence
- $x + 2 = 5$ $\rightarrow$ Neither true nor false (Related to predicate logic!)
- Computer x is functioning properly.
  (Computer "x" is not specified) $\rightarrow$ Neither true nor false (Related to predicate logic!)

How about the following?

- Do not pass go. Not a proposition
- What time is it? Not a proposition
- There is no pollution in New Jersey. A proposition; either T or F
- $2^n \geq 100$ Not a proposition
- 13 is a prime number. A proposition; T

Questions from Students: Proposition

Proposition: a declarative sentence that is either true or false (not both).

Are paradox propositions?

- A paradox is a declarative sentence that is true and false at the same time — thus, a paradox is not a proposition.[1]

	Determine the type of Sentence	If a proposition determine its truth value
5 is a prime number.	Declarative and Proposition	T
8 is an odd number.	Declarative and proposition	F
Did you lock the door?	Interrogative	
Happy Birthday!	Exclamatory	
Jane Austen is the author of Pride and Prejudice.	Declarative and Proposition	T
Please pass the salt.	Imperative	
She walks to school.	Declarative	
$\lvert x+y\rvert \leq \lvert x\rvert + \lvert y\rvert$	Declarative	

[1] https://calcworkshop.com/logic/propositional-logic/

Questions from Students: Proposition

Proposition: a declarative sentence that is either true or false (not both).

Is $x^2 \geq 0$ a proposition? Note that $x^2 \geq 0$ is true whenever x is a real number.

- No, because x is variable and could be anything, e.g., a car, a person.

Predicate $P(x)$: $x^2 \geq 0$

- $P(2)$ is a proposition
- "$\forall x P(x)$ whenever x is a real number" is a proposition

Compound Propositions

Many mathematical statements are constructed by combining one or more propositions $\rightarrow$ compound propositions.

- p: It rains outside.
- q: We will watch a movie.
- A new proposition r: If it rains outside, then we will watch a movie.

(Recall that p, q, r are propositional variables that represent propositions.)

Compound propositions are build using logical connectives:

- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Exclusive or $\oplus$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Negation

Let p be a proposition. The negation of p, denoted by $\neg p$, is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p".

Example:

- p: SUSTech is located in Shenzhen. (T)
- $\neg p$: It is not the case that SUSTech is located in Shenzhen. That is, SUSTech is not located in Shenzhen. (F)

Negation

Negation of the following propositions?

- $5 + 2 \neq 8$ (T)
- 10 is not a prime number. (T)
- Class does not begin at 8:30am. (F)

Negation:

- It is not the case that $5 + 2 \neq 8$. That is, $5 + 2 = 8$. (F)
- It is not the case that 10 is not a prime number. That is, 10 is a prime number. (F)
- It is not the case that class does not begin at 8:30am. That is, class begins at 8:30am. (T)

Negation: Truth Table

A **truth table** displays the relationships between truth values (T or F) of different propositions.

The truth table for the negation of a proposition:

p	$\neg p$
T	F
F	T

- Each row corresponds to a possible truth value of p.
- Given the truth value of p, obtain the truth value of $\neg p$.

Conjunction (And)

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q".

The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Example:

- p: SUSTech is located in Shenzhen. (T)
- q: $5 + 2 = 8$ (F)
- $p \wedge q$: SUSTech is located in Shenzhen, and $5 + 2 = 8$ (F)

Conjunction (And)

Conjunction of the following?

- p: Rebecca's PC has more than 16 GB free hard disk space.
- q: The processor in Rebecca's PC runs faster than 1 GHz.

Conjunction:

- $p \wedge q$: Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz.

Disjunction (Or)

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q" (inclusive or).

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Example:

- p: SUSTech is located in Shenzhen. (T)
- q: $5 + 2 = 8$ (F)
- $p \vee q$: SUSTech is located in Shenzhen, or $5 + 2 = 8$. (T)

Disjunction (Or)

Disjunction of the following proposition?

- p: Students who have taken calculus can take this class.
- q: Students who have taken computer science can take this class.

Disjunction:

- $p \vee q$: Students who have taken calculus or computer science can take this class.

  Note: This is an inclusive or. We mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects.

Conjunction and Disjunction: Truth Table

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

- Each row corresponds to a possible pair truth values of p and q.
- Given the truth value of p and q, obtain the truth values of $p \wedge q$ and $p \vee q$.

Extend to $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ or $p_1 \vee p_2 \vee \ldots \vee p_n$

- If there are n propositional variables, there are 2^n rows.
- Given $p_1, p_2, \ldots, p_n$, obtain the truth values of the above compound propositions.

Exclusive Or

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exclusive Or

Exclusive or of the following proposition?

- p: Students who have taken calculus can take this class.
- q: Students who have taken computer science can take this class.

Exclusive or:

- $p \oplus q$: Students who have taken calculus or computer science, but not both, can enroll in this class.

Conditional Statement (Implication)

Let p and q be propositions. The conditional statement (a.k.a. implication) $p \to q$, is the proposition "if p, then q".

Proposition $p \to q$ is false when p is true and q is false, and true otherwise.

In $p \to q$, p is called the hypothesis and q is called the conclusion.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional Statement (Implication)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- p: It doesn't rain today (F)
- q: I will go to the store today
- p$\rightarrow$ q: If it doesn't rain today, then I will go to the store today (T)

Suppose it rains today. Then,

- No matter whether I go to the store today or not, my statement is true, i.e., I am not lying.

Questions from Students: Implication

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Essentially, $\to$ **is a logical operator**: given two logical values, produces a third logical value, using a common defined rule

Using "if ..., then ..." to express this operator:

- "If it is sunny tomorrow, then we will go hiking."

However, "if ..., then ..." may not be the most accurate expression:

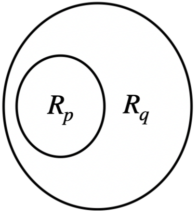
- "Not A; or, A implies B" (useful law)
- BUT this expression is NOT commonly accepted!

Please use "if ..., then ..." as the English interpretation.

Conditional Statement (Implication)

$p \to q$ is read in a variety of equivalent ways:

- if p then q
- p implies q
- p is sufficient for q
- q is necessary for p
- q follows from p
- p only if q



Example:

- p: Point A is in R_p.
- q: Point A is in R_q.
- If point A is in R_p, then point A is in R_q.

Note: It is about English Expression but NOT inference.

Conditional Statement (Implication)

- The converse of $p \rightarrow q$ is $q \rightarrow p$.
- The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.
- The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

Examples:

- If you get 100 on the final, then you will get an A. $(p \rightarrow q)$
- If you get an A, then you get 100 on the final. $(q \rightarrow p)$
- If you don't get an A, then you don't get 100 on the final. $(\neg q \rightarrow \neg p)$
- If you don't get 100 on the final, then you don't get an A. $(\neg p \rightarrow \neg q)$

Which is equivalent to $p \rightarrow q$?

$\neg q \rightarrow \neg p$ is equivalent to $p \rightarrow q$

- **Equivalent**: given any possible truth values of the propositions, two compound propositions always have the same truth value.
- Try to write the truth table of $p \rightarrow q$ and $\neg q \rightarrow \neg p$?

Equivalent

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	**T**	F	F	**T**
T	F	**F**	T	F	**F**
F	T	**T**	F	T	**T**
F	F	**T**	T	T	**T**

Equivalent: given any possible truth values of p and q, two compound propositions $p \to q$ and $\neg q \to \neg p$ always have the same truth value

How about

- $p \to q$ and its converse $q \to p$?
- $p \to q$ and its inverse $\neg p \to \neg q$?
- the converse $q \to p$ and the inverse $\neg p \to \neg q$?
  [Prove equivalence (next lecture): truth table and logical equivalences]

Biconditional

Let p and q be propositions. The biconditional statement (a.k.a. bi-implications), denoted by $p \leftrightarrow q$, is the proposition "p if and only if q", is true when p and q have the same truth values, and false otherwise.

- p is necessary and sufficient for q
- if p then q, and conversely
- p iff q

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

A Quick Summary of Compound Proposition

A proposition is a declarative statement that is either true or false.

Compound propositions are build using logical connectives:

- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Exclusive or $\oplus$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Given the truth value of one or more propositions, the truth value for compound proposition?

Determining the Truth Value

- p: 2 is a prime (T)
- q: 6 is a prime (F)

Determine the truth value of the following:

- $\neg p$ F
- $p \wedge q$ F
- $p \wedge \neg q$ T
- $p \vee q$ T
- $p \oplus q$ T
- $p \to q$ F
- $q \to p$ T

Constructing the Truth Table

Construct a truth table for $p \vee q \to \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \to \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Computer Representation of True and False

- A **bit** is a symbol with two possible values: 0 (false) or 1 (true)
- A variable that takes on values 0 and 1 is called a Boolean variable.
- A bit string is a sequence of zero or more bits. The length of this string is the number of bits in the string.

Computer Representation of True and False

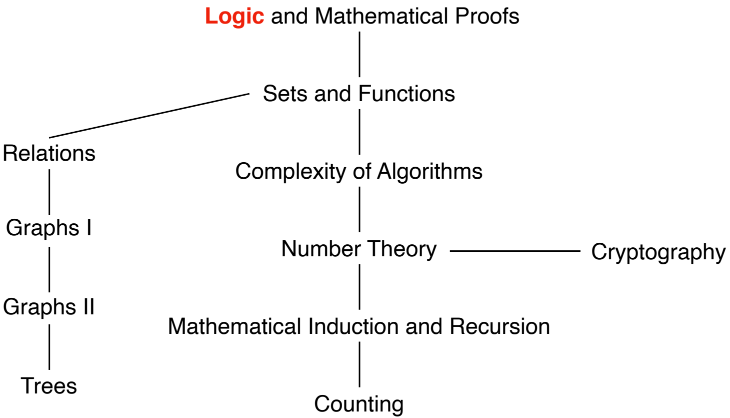
Bitwise operations:

- Each bit is represented by 1 (True) and 0 (False)
- Compute OR ($\vee$), AND ($\wedge$), XOR ($\oplus$) in a bitwise fashion

01 1011 0110	
11 0001 1101	
	bitwise *OR*
	bitwise *AND*
	bitwise *XOR*

01 1011 0110	
11 0001 1101	
11 1011 1111	bitwise *OR*
01 0001 0100	bitwise *AND*
10 1010 1011	bitwise *XOR*

Next Lecture



Logic: Propositional logic, applications of propositional logic, propositional equivalence, predicates and quantifiers, nested quantifiers