L07: Particle Filter and Monte-Carlo localization

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1 Particle filter (PF)

$$\begin{cases} \overline{bel}(x_t) = & \int p(x_t|x_{t-1}, u_t) bel(x_{t-1}) dx_{t-1} \\ bel(x_t) = & \eta p(z|x_t) \overline{bel}(x_t) \end{cases} \tag{1}$$

Gaussian filters (Unimodal distributions):

- Kalman Filter — Linear system
- Extended KF — Non-Linear system
- Unscented KF — Non-Linear system (more at extra notes on L06)

Non-parametric : Particle Filter (PF)

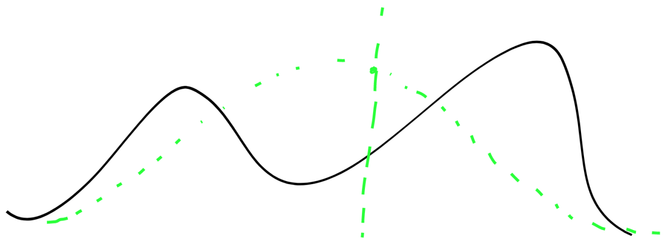

Figure 1: Non-parametric filters do not assume a unimodal distribution, such as KF which always approximates the solution to a Gaussian dsitribution (green line).

Particle set: $X_t = \{\langle x_t^{[1]}, \omega_t^{[1]} \rangle, \dots \langle x_t^{[M]}, \omega_t^{[M]} \rangle\}$.
The particle set consists of M particles, each of them is a pair of a state $x^{[m]}$ and a weight $\omega^{[m]}$.

Figure 2: Example of a particle set. On top, samples from a 1D PDF, on the bottom, a 2D Gaussian PDF with few samples drawn and the 1-σ iso-contour plotted.

$$\begin{aligned} x^{[m]} &\sim p(x) && \text{Weighted samples. Particles become a good representation of PDFs} \\ \omega^{[m]} &= p_z(x^{[m]}) && \text{(if } M is \text{ large enough)} \end{aligned}$$

Q: What are the weights on sample mean and sample covariance?

1. Particle filter (X_{t-1}, u_t, z_t):
2. for $m = 1 : M$
3. $\quad x_t^{[m]} \sim p(x_t | u_t, x_{t-1}^{[m]})$
4. $\quad \omega_t^{[m]} = p(z_t | x_t^{[m]})$ propagation $\overline{bel}(x_t)$
5. $\quad \overline{X}_t = \overline{X}_t \cup \langle x_t^{[m]}, \omega_t^{[m]} \rangle)$ correction
6. $X_t = \text{resampe}^*(\overline{X}_t)$ (better correction)X are "down" from $bel(x_t)$and not $\overline{bel}$

[Gordon] reading introduces resampling as a requirement for the PF to work property

2 Bayes filter for full states

$$bel(x_{0:t}) = p(x_{0:t} | u_{1:t}, z_{1:t})$$

particles $x_{0:t}^{[m]} = x_0^{[m]}, x_1^{[m]}, \ldots x_t^{[m]}$ Sequence of samples of states over time.

$$\begin{aligned} bel(x_{0:t}) &= \eta p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) p(x_{0:t} | z_{1:t-1}, u_{1:t}) \\ \text{(Markov + Product rule)} &= \eta p(z_t | x_t) p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t}) p(x_{0:t-1} | z_{1:t-1}, u_{1:t}) = \\ &= \eta p(z_t | x_t) p(x_t | x_{t-1}, u_t) \underbrace{p(x_{0:t-1} | z_{1:t-1}, u_{1:t-1})}_{bel(x_{0:t-1})} \end{aligned}$$

We have obtained a new recursive form of the Bayes filter, but now considering the state variable to be a sequnece of all estimates at all instants of time, i.e., time $0 : t$.

$$\begin{aligned} \overline{bel}(x_{0:t}) &= p(x_t | x_{t-1}, u_t) bel(x_{0:t-1}) \\ bel(x_{0:t}) &= \eta p(z_t | x_t) \overline{bel}(x_{0:t}) \end{aligned}$$

2.1 Prediction step

From this full state Bayes (no marginalization) given a particle $x_{t-1}^{[m]} \sim bel(x_{0:t-1})$

$$\overline{bel}(x_{0:t}) \begin{cases} \overline{x}_t^{[m]} & \sim \quad p(x_t | x_{t-1}^{[m]}, u_t) \cdot \omega_{t-1}^{[m]} \quad \text{Sample drawn from the previous sample.} \\ \overline{\omega}_t^{[m]} & = \quad 1 \cdot \omega_{t-1}^{[m]} \quad \text{importance factor from } bel. \end{cases} \tag{2}$$

In Fig. 3 is depicted an example of particles propagated (predicted)

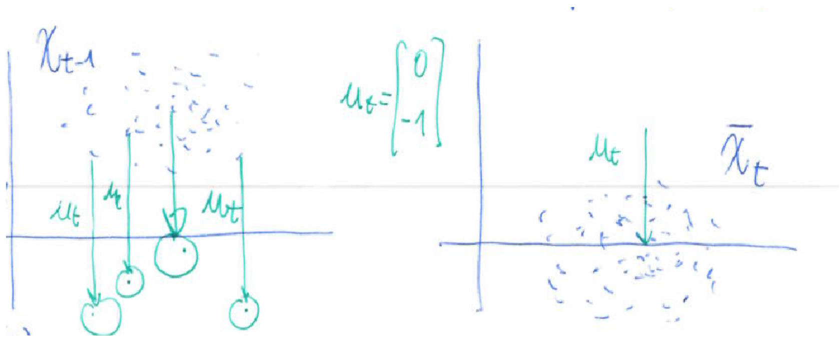

Figure 3: Example of the prediction step. For each particle, a propagation is sampled and this conforms the new particle set $\bar{X}_t$.

Importance sampling

We will briefly introduce the concept of *importance sampling* before we continue with the PF derivation, since it plays an essential role on it.

$$\begin{aligned} \mathbb{E}_{x\sim p(x)}\{I(x\in A)\} &= \int I(x\in A)p(x)dx = \int I(x\in A)\frac{p(x)}{q(x)}\cdot q(x)dx \\ &= \mathbb{E}_{x\sim q}\left\{I(x\in A)\cdot\omega(x)\right\}, \end{aligned} \tag{3}$$

where $\omega(x) = \frac{p(x)}{q(x)}$ is the Importance factor, $p(x)$ is the target distribution, which we usually can't use directly and $q(x)$ is the proposal distribution, more accessible and ready to use. The function $I()$ in this context is the indicator function.

Example: Probability of sample a 1d r.v X in the interval [15,17] if

$$p(x) = \mathcal{N}(0,1) \qquad A = \{x\ :\ 15 \le x \le 17\}$$

$$1)\ \ p(x\in A) = \sum I(x^m\in A)p(x^m), \quad x^m \sim p(x^m)$$

$$2)\ \text{Importance Sampling:}\ p(x\in A) = \Sigma I(x^m\in A)\underbrace{\frac{p(x^m)}{q(x^m)}}_{\omega^m}\cdot q(x^m), \quad x^m\sim q(x)$$

for instance $q(x) = N(16,1)$ (proposal distribution)

With this proposal distribution we don't need trillions of samples but only hundreds.

$$\omega^m = \frac{\mathcal{N}(x^m;0,1)}{\mathcal{N}(x^m;16,1)} \tag{4}$$

2.2 Correction step

$$\underbrace{bel(x_{0:t})}_{\text{target distribution}} = \eta p(z_t|x_t)\cdot \underbrace{\overline{bel}(x_{0:t})}_{\text{proposal distribution}}, \tag{5}$$

where $\overline{X}_t$ is the particle set representing the belief PDF $\overline{bel}(x_{0:t})$, our proposal distribution.

In order to correctly characterize the posterior $bel(x_{0:t})$ we are going to weight the particles already drawn $\overline{X}_t$ with proper weights ω (importance factors).

$$x_t^{[m]} = \overline{x}_t^{[m]} \qquad \text{(previously drawn) from proposal distribution.}$$

$$\omega_t^{[m]} = \frac{\text{target distribution}}{\text{proposal distribution}} = \frac{\eta p(z_t|x_t^{[m]}) \cdot \overline{bel}(x_t^{[m]})}{\overline{\text{bel}}(x_t^{[m]})} = \eta p(z_t|x_t^{[m]}) \tag{6}$$

Example: Correction step applied to the particle set $\bar{X}_t$

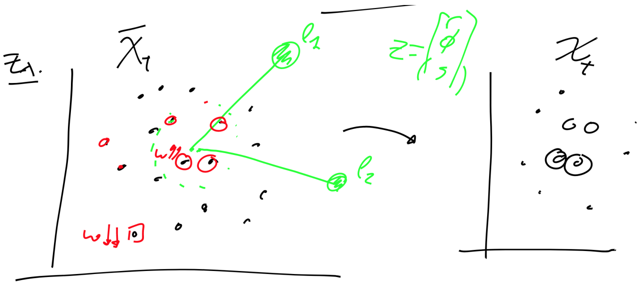

Figure 4: Example of correction step for a particle set.

Problem: creates an almost empty set of particles with weights non-zero and many particles with low weights $\Rightarrow$ Degenerating over time.

3 PF Resampling

Resampling is the solution to the degeneracy occurring when propagating and correcting multiple times a particle set.
Idea: "survival of the fittest". Only the most likely particles ($\omega^m \uparrow$) 'might' survive

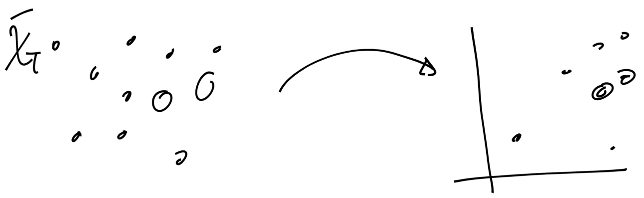

Figure 5: Resampling (the solution). Now, resampling guarantees that the highest values of ω^m are more likely to *survive* but it also give chances to the particles with small values of importance factors to represent the next particle set. From M samples on $\overline{X}_t$ we get M samples on X_t (closest to the $bel(x_0)$)

3.1 Independent Resampling. First solution

We create a cumulative distribution function:

$$c_m = c_{m-1} + \omega^{[m]} \qquad \text{(normalization should be considered)} \tag{7}$$

$$\begin{aligned}
&\text{for } m = 1 : M \\
&\quad a \sim U[0;1] && \text{(uniform distribution)} \\
&\quad j = \text{find}(c_m, a) \\
&\quad X_t = X_t \cup \langle x_t^{[j]}, \omega_t^{[j]} \rangle
\end{aligned}$$

Problem: over time, independent resampling induces a loss of diversity in the particle population X_t.

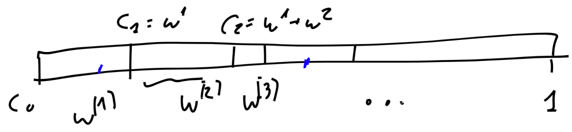

Figure 6: Independent Resampling.

3.2 Low-variance sampling

We create a similar distribution as in the independent sampling algorithm:

$$c_m = c_{m-1} + \omega^{[m]}.$$

The difference is that we no longer sample from this discrete distribution. Only an initial random configuration r is sampled, and then we add particles at internals $1/M$ over the full set.

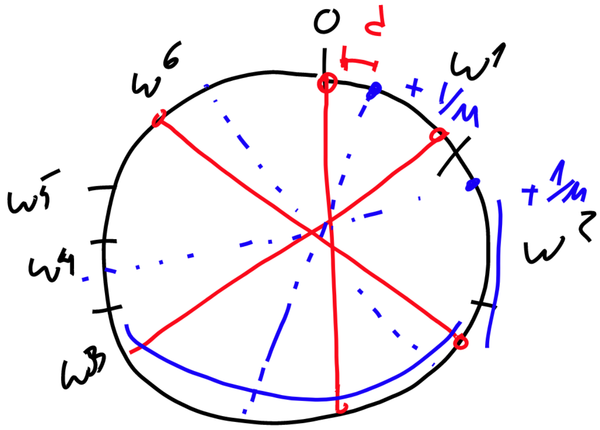

Figure 7: Low-variance sampling scheme. We select particles as equally spaced intervals.

Algorithm: low-variance sampling $(\overline{X}_t)$: (ProbRob 110)
$X_t = \phi,\ c = \omega_t^{[1]},\ i = 1;$
$r \sim U(0, 1/M);$
for m =*1:M* **do**
  $a = r + (m-1) \cdot \frac{1}{M}$
  while a¿c **do**
    $i++;$
    $c = c + \overline{\omega}^{[i]};$
  end
  $X_t = X_t \cup \langle \overline{x}_t^{[i]}, \frac{1}{M} \rangle;$
end
return X_t

4 Monte-Carlo localization (MCL)

Reading: Dellaert'99.
We want to solve the Markov localization (L06) using PF.

$$bel(x_t) = p(x_t | U, Z, m) \rightarrow X_t, \tag{8}$$

where X_t is the particle set representing the posterior belief and m is the map of landmarks.

Algorithm: MCL (X_{t-1}, u_t, z_t, m) :

$\overline{X}_t = X_t = \phi$;
for *m =1:M* **do**
  $x_t^{[m]}$ = sample_motion_model$(u_t, x_{t-1}^{[m]})$ (Section 2.1 and L05)
  $\omega_t^{[m]}$ = measurement_model $(z_t, x_t^{[m]}, m)$ (Section 2.2 and L06)
  $\overline{X}_t = \overline{X}_t \cup \langle x_t^{[i]}, \omega_t^{[m]} \rangle$
end
X_t = low_variance_sampling $(\overline{X}_t)$

5 Summary

- Particle filter is a version of the Bayes filter for full sequences $x_{0:t}$.
- Prediction step $\overline{x}_t^{[m]} \sim p(x_t | x_{t-1}^{[m]}, u_t) \cdot \omega_{t-1}^{[m]}$.
- Correction step $\omega_t^{[m]} = \eta p(z_t | x_t^{[m]})$.
- Resampling regenerates the particle set.