Week 4: Estimating Dynamic Models

Advanced Econometrics 4EK608

Vysoká škola ekonomická v Praze

Outline

1. Finite and infinite distributed lag models
2. Polynomial distributed lag
3. Geometric distributed lag (Koyck)
4. PAM, AEH, Rational expectations
5. FWL theorem - repetition from BSc courses

Czech terminology

Modely konečných a nekonečných rozložených zpoždění,
řád modelu, okamžitý a dlouhodobý multiplikátor,
rozdělení zpoždění, dynamicky úplné modely,
polynomicky (geometricky, racionálně) rozdělené zpoždění,
Koyckova transformace, model částečného přizpůsobení,
cílová (optimální) úroveň vysvětlované proměnné,
koeficient přizpůsobení, rychlost přizpůsobení,
adaptivní a racionální očekávání,
hypotéza efektivních trhů.

Finite and infinite distributed lag models

- Static models

$$y_t = \beta_0 + \beta_1 x_t + u_t, \qquad t = 1, 2, \ldots, T$$

- Finite distributed lag (FDL) Models

$$y_t = \alpha_0 + \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + u_t$$

- Order of the FDL model,
  impact multiplier vs. long-run multiplier,
  temporary vs. permanent change in x,
  lag distribution

Dynamically complete models

- Model is dynamically complete if we have a “sufficient” number of lags of regressors, so that no more additional lags would help with explanation of variance in the dependent variable.
- In dynamically incomplete models, we usually detect autocorrelation in the error term of the LRM.

Finite and infinite distributed lag models

Infinite distributed lag (IDL) models

- Lagged regressors extend back to infinity
- We cannot estimate IDL models without the use of simplifying restrictions on parameters,
  i.e. restrictions on lag distribution
- IDL models are useful under the assumption of lagged coefficients converging to zero as lag increases
- Order of the IDL model (∞),
  impact multiplier vs. long-run multiplier,
  temporary vs. permanent change in x,
  … all analogical to FDL models

FDL: Polynomial distributed lag (Almon)

Used in Finite distributed lag models
. . . example below also extends to higher order polynomials

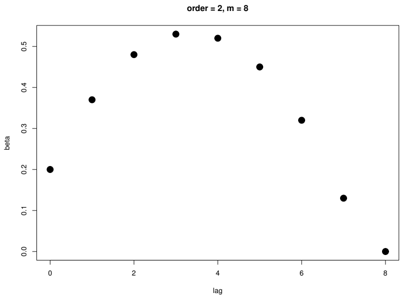
$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \cdots + \beta_m x_{t-m} + u_t \tag{1}$$

$$y_t = \alpha + (\sum_{i=0}^{m} \beta_i x_{t-i}) + u_t \ , \quad (i \text{ is the lag operator here})$$

Simplifying assumption:

$$\beta_i = k_0 + k_1 i + k_2 i^2 \tag{2}$$

$$\begin{aligned} \beta_0 &= k_0 \\ \beta_1 &= k_0 + k_1 + k_2 \\ \beta_2 &= k_0 + k_1 \cdot 2 + k_2 \cdot 4 \\ &\dots \\ \beta_m &= k_0 + k_1 m + k_2 m^2 \end{aligned}$$



Polynomial distributed lag

- Almon-type transformation of (1) for $m = 8$ and $k = 2$:

$$y_t = \alpha + k_0 x_t + (k_0 + k_1 + k_2) x_{t-1} + (k_0 + 2k_1 + 4k_2) x_{t-2} + \\ + \cdots + (k_0 + 8k_1 + 64k_2) x_{t-8} + u_t \tag{3}$$

$$y_t = \alpha + k_0 (x_t + x_{t-1} + \cdots + x_{t-8}) + \\ + k_1 (x_{t-1} + 2x_{t-2} + \cdots + 8x_{t-8}) + \\ + k_2 (x_{t-1} + 4x_{t-2} + \cdots + 64x_{t-8}) =$$

$$y_t = \alpha + k_0 \underbrace{\sum_{i=0}^{8} x_{t-i}}_{W_{0t}} + k_1 \underbrace{\sum_{i=1}^{8} i\, x_{t-i}}_{W_{1t}} + k_2 \underbrace{\sum_{i=1}^{8} i^2 x_{t-i}}_{W_{2t}} + u_t \tag{4}$$

$$y_t = \alpha + k_0 W_{0t} + k_1 W_{1t} + k_2 W_{2t} + u_t \tag{5}$$

We estimate (5), then calculate β_i (lags 0 to 8) as in (2) … note the reduction in estimated parameters (10 vs 4).

Polynomial distributed lag

- Method developed by Shirley Almon in the 60ies.
- Equation (5) can be generalized: for m lags, sums go to m, for higher order polynomials, we add more W-terms.
- Advantages of this approach:
  - Saves degrees of freedom
  - Removes the problem of multicollinearity
  - Does not affect the assumptions for u, because errors do not change during transformation
- In EViews, transformation is slightly modified.
- In R, routines are available.

Geometric distributed lag (Koyck)

IDL linear regression model: $y_t = f(x_t, x_{t-1}, x_{t-2}, \dots)$:

$y_t = \alpha + \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + \delta_3 x_{t-3} + \cdots + u_t$

Assumption for the geometric δ_j weights:

- $\delta_j = \gamma \rho^j, \quad 0 < \rho < 1, \quad j = 0, 1, 2, \dots$
  $|\rho| < 1$ is a more general (valid) stability assumption.
  (j is the lag operator here)
  Sometimes, the assumption is written as
  $\delta_j = \delta_{j-1}\,\rho, \quad 0 < \rho < 1.$

Instantaneous propensity (multiplier): $\delta_0 = \gamma \rho^0 = \gamma$

Long-run propensity (multiplier):

$\delta_0 + \delta_1 + \cdots = \gamma(1 + \rho + \rho^2 + \rho^3 + \dots) = \frac{\gamma}{1-\rho}$

Koyck transformation of the IDL model:

$$y_t = \alpha + \delta_0 x_t \quad + \delta_1 x_{t-1} \; + \delta_2 x_{t-2} + \cdots + u_t$$

$$y_t = \alpha + \gamma(\rho^0) x_t + \gamma\rho x_{t-1} + \gamma\rho^2 x_{t-2} + \cdots + u_t \tag{6}$$

$$y_{t-1} = \alpha + \gamma x_{t-1} + \gamma\rho x_{t-2} + \gamma\rho^2 x_{t-3} + \cdots + u_{t-1} \quad | \times \rho \tag{7}$$

$$\rho y_{t-1} = \alpha\rho + \gamma\rho x_{t-1} + \gamma\rho^2 x_{t-2} + \cdots + \rho u_{t-1} \tag{8}$$

Now, we subtract (8) from (6):

$$y_t - \rho y_{t-1} = \underbrace{\alpha(1-\rho)}_{\alpha_0} + \gamma x_t + \underbrace{u_t - \rho u_{t-1}}_{v_t} \tag{9}$$

$$y_t = \alpha_0 + \gamma x_t + \rho y_{t-1} + v_t \tag{10}$$

Geometric distributed lag (Koyck)

IDL model: $y_t = \alpha + \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + \cdots + u_t$

Koyck transf.: $y_t = \alpha_0 + \gamma x_t + \rho y_{t-1} + v_t$

Using the Koyck transformation, we can calculate parameters of the IDL model from the estimated model after Koyck transformation:

$$\hat{\delta}_0 = \hat{\gamma}$$

$$\hat{\delta}_j = \hat{\gamma}\,\hat{\rho}^j \ ; \ j = 1, 2, 3, \dots$$

$$\hat{\alpha} = \frac{\hat{\alpha}_0}{1-\hat{\rho}}$$

Problems of the Koyck transformation:

- in (10), regressor y_{t-1} is not exogenous ($v_t = u_t - \rho u_{t-1}$) …IVR will be discussed in Week 8
- $v_t = u_t - \rho u_{t-1}$ is not i.i.d.

Rational distributed lag

The geometric distributed lag is a special case of rational distributed lag (RDL) model:

$$y_t = \alpha_0 + \gamma x_t + \rho y_{t-1} + v_t \quad \text{(geometric distributed lag)}$$
$$y_t = \alpha_0 + \gamma_0 x_t + \rho y_{t-1} + \gamma_1 x_{t-1} + v_t \quad \text{(RDL)} \tag{11}$$

This can be shown by successive substitution ($|\rho| < 1$):

$$\begin{aligned} y_t &= \alpha + \gamma_0(x_t + \rho x_{t-1} + \rho^2 x_{t-2} + \dots) \\ &\quad + \gamma_1(x_{t-1} + \rho x_{t-2} + \rho^2 x_{t-3} + \dots) + u_t \end{aligned} \tag{12}$$

$$\begin{aligned} y_t &= \alpha + \gamma_0 x_t + (\rho\gamma_0 + \gamma_1)x_{t-1} + \rho(\rho\gamma_0 + \gamma_1)x_{t-2} + \\ &\quad + \rho^2(\rho\gamma_0 + \gamma_1)x_{t-3} + \dots + u_t \end{aligned} \tag{13}$$

After estimating (11), we can calculate lag distributions for (13)

Rational distributed lag

RDL specification:

$$y_t = \alpha_0 + \gamma_0 x_t + \rho y_{t-1} + \gamma_1 x_{t-1} + v_t$$

can be used to calculate δ_h in the IDL model:

$$y_t = \alpha + \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + \cdots + u_t$$

With RDLs, impact propensity $\gamma_0 \equiv \delta_0$ can differ in sign from lagged coefficients.

$\delta_h = \rho^{h-1}(\rho\gamma_0 + \gamma_1)$ corresponds to the x_{t-h} variable for $h \geq 1$.
...δ_0 may differ in sign from "lags", even if $\rho > 0$.
...for $\rho > 0$, δ_h doesn't change sign with growing $h \geq 1$.

Long-run propensity: $LRP = \frac{\gamma_0+\gamma_1}{1-\rho}$,
where $|\rho| < 1 \Rightarrow$ the sign of LRP follows the sign of $(\gamma_0 + \gamma_1)$.

Also, $u_t = v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots \quad MA(\infty)$

IDL/Koyck and RDL – summary

$$y_t = \alpha + \delta_0 x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + \delta_3 x_{t-3} + \cdots + u_t$$

IDL/Koyck: $y_t = \alpha_0 + \gamma x_t + \rho y_{t-1} + v_t$

RDL: $y_t = \alpha_0 + \gamma_0 x_t + \rho y_{t-1} + \gamma_1 x_{t-1} + v_t$

Table 1: Koyck vs. RDL coefficients

	IDL/Koyck	RDL
Impact multiplier	$\delta_0 = \gamma$	$\delta_0 = \gamma_0$
Lagged multiplier (lag h)	$\delta_h = \gamma\rho^h$	$\delta_h = \rho^{h-1}(\rho\gamma_0 + \gamma_1)$
LRP	$\frac{\gamma}{1-\rho}$	$\frac{\gamma_0+\gamma_1}{1-\rho}$

Partial adjustment model

Partial adjustment model (PAM)
is based on two main assumptions:

1. LRM describes behavior of y_t^*, which is the unobserved, expected/equilibrium/target/optimum value of y_t:

$$y_t^* = \alpha + \beta x_t + u_t \tag{14}$$

2. Between two time periods, y_t follows the process:

$$y_t - y_{t-1} = \theta(y_t^* - y_{t-1}), \qquad 0 < \theta < 1 \tag{15}$$

Hence, the actual Δy_t is only a fraction of the "desirable" change from y_{t-1} to the optimum value of y_t^*.
...in the special case of $\theta = 1$, Δy_t leads to optimum.

Note: (15) can be re-written as: $y_t = \theta y_t^* + (1 - \theta) y_{t-1}$

Partial adjustment model

Parameter estimation of PAM:

$$y_t^* = \alpha + \beta x_t + u_t \tag{14}$$

$$y_t = \theta y_t^* + (1-\theta) y_{t-1}, \qquad 0 < \theta < 1 \tag{15}$$

1. Substitute for y_t^* in (15) from (14):

$$\begin{aligned} y_t &= \alpha\theta \quad + \beta\theta x_t + \ (1-\theta) y_{t-1} \quad + \theta u_t \\ y_t &= \beta_0' \quad + \beta_1' x_t + \ \beta_2' y_{t-1} \qquad\quad + \theta u_t \end{aligned} \tag{16}$$

2. Estimate (16) and then calculate sought parameters of the PAM in (14) and (15):
   $\hat{\theta} = 1 - \hat{\beta}_2'$
   $\hat{\alpha} = \hat{\beta}_0' / \hat{\theta}$
   $\hat{\beta} = \hat{\beta}_1' / \hat{\theta}$

Note that θu_t can be independent of y_{t-1} and i.i.d.

Partial adjustment model

Parameter interpretation of PAM:

$$
\begin{aligned}
y_t^* &= \alpha + \beta x_t + u_t \\
\underline{y_t} &\underline{= \theta y_t^* + (1-\theta) y_{t-1}, \qquad 0 < \theta < 1}
\end{aligned}
$$

$$
\begin{aligned}
y_t &= \alpha\theta + \ \beta\theta x_t + \ (1-\theta) y_{t-1} + \ \theta u_t \\
y_t &= \beta_0' \ + \ \beta_1' x_t + \ \beta_2' y_{t-1} + \ \theta u_t
\end{aligned}
$$

Parameters:

θ : $\theta = (1 - \beta_2')$ is the adjustment coefficient; higher $\hat{\theta}$ indicates higher speed of adjustment towards equilibrium.

β_1' : impact multiplier (short-run marginal propensity)

β : $\beta = \beta_1'/\theta$ is the long run multiplier.

Adaptive expectations hypothesis

Adaptive expectations hypothesis (AEH) model is based on two main assumptions:

1. LRM describes behavior of y_t, as a function of x_t^*: the unobserved, expected/equilibrium/target/optimum value of x_t (permanent income, potential output, etc.):

$$y_t = \alpha + \beta x_t^* + u_t. \tag{17}$$

2. The unobserved x_t^* process is defined as:

$$\begin{gathered} x_t^* - x_{t-1}^* = \phi(x_t - x_{t-1}^*), \qquad 0 < \phi < 1 \\ \Downarrow \\ x_t^* = \phi x_t + (1-\phi) x_{t-1}^*. \end{gathered} \tag{18}$$

with $\phi = 0$ for static expectations
and $\phi = 1$ for immediate adjustment.

Note: alternative 2^{nd} hypothesis: $x_t^* = \phi x_{t-1} + (1-\phi) x_{t-1}^*$.

Adaptive expectations hypothesis

Parameter estimation of AEH model:

$$y_t = \alpha + \beta x_t^* + u_t, \tag{17}$$

$$x_t^* = \phi x_t + (1-\phi) x_{t-1}^* \tag{18}$$

Successive substitution for x_t^* from (18) to (17): IDL process

$$y_t = \alpha + \beta\phi x_t + \beta\phi(1-\phi)x_{t-1} + \beta\phi(1-\phi)^2 x_{t-2} + \cdots + u_t \tag{19}$$

After applying Koyck transformation, we get

$$\begin{aligned} y_t &= \alpha\phi \quad + \beta\phi x_t + \ (1-\phi)y_{t-1} \quad + v_t \\ y_t &= \beta_0' \quad + \beta_1' x_t + \ \beta_2' y_{t-1} \quad + v_t \end{aligned} \tag{20}$$

Estimate (20), then calculate parameters in (17) and (18).
$\hat{\phi} = 1 - \hat{\beta}_2'$ (ϕ is the “adaptive expectations coefficient”)

$\hat{\alpha} = \hat{\beta}_0'/\hat{\phi}$

$\hat{\beta} = \hat{\beta}_1'/\hat{\phi}$ (β_1' and β are the SR and LR propensities)

Note: Both problems of Koyck-transformed model estimation apply.

Koyck, PAM, AEH: regression of y_t on x_t and y_{t-1}

The same underlying regression model (statistical form) is used:

- The Koyck transformation:
  $y_t = \alpha_0 + \gamma x_t + \rho y_{t-1} + v_t$
- The Partial adjustment model (PAM):
  $y_t = \alpha\theta + \beta\theta x_t + (1-\theta)y_{t-1} + \theta u_t$
- The Model with adaptive expectations (AEM):
  $y_t = \alpha\phi + \beta\phi x_t + (1-\phi)y_{t-1} + v_t$

We can make three different interpretations from one estimated equation.

Of course, not all interpretations are always relevant, we must choose according to application and test assumptions.

Koyck, PAM, AEH: regression of y_t on x_t and y_{t-1}

Example: Model for $c_t \leftarrow f(x_t, x_{t-1}, x_{t-2}, \dots)$:
private consumption (c_t) as a function of disposable income (x_t).
Same estimated model for Koyck, PAM, AEH:

$$
\begin{aligned}
\hat{c}_t &= \; 1,038 + \; 0.404 x_t + \; 0.501 c_{t-1} \\
\textbf{Koyck:}\quad \hat{c}_t &= \; \hat{\alpha}_0 + \; \hat{\gamma} x_t + \; \hat{\rho} c_{t-1}
\end{aligned}
$$

Koyck: IDL, geometric decay in δ parameters assumed:

- $\hat{\rho} = 0.501$
- $\hat{\alpha}_0 = 1,038 = \hat{\alpha}(1-\hat{\rho}) = \hat{\alpha}(1-0.501) \Rightarrow \hat{\alpha} = \frac{1,038}{0.499} = 2,080$
- $\hat{\delta}_j = \hat{\gamma}\hat{\rho}^j = 0.404 \times 0.501^j$
- $LRP = \frac{\hat{\gamma}}{1-\hat{\rho}} = \frac{0.404}{0.499} \doteq 0.81$
- IDL: $\hat{c}_t = 2,080 + \underbrace{0.404}_{\hat{\gamma}\hat{\rho}^0} x_t + \underbrace{0.202}_{\hat{\gamma}\hat{\rho}} x_{t-1} + \underbrace{0.101}_{\hat{\gamma}\hat{\rho}^2} x_{t-2} + \dots$

Koyck, PAM, AEH: regression of y_t on x_t and y_{t-1}

Example continued:

$$\begin{aligned} \hat{c}_t &= \quad 1,038 + \quad 0.404 x_t + \quad 0.501 c_{t-1} \\ \textbf{PAM:}\ \hat{c}_t &= \quad \hat{\alpha}\hat{\theta} + \quad \hat{\beta}\hat{\theta} x_t + \quad (1-\hat{\theta}) c_{t-1} \end{aligned}$$

- $(1-\hat{\theta}) = 0.501 \Rightarrow \hat{\theta} = 0.499$
- $\hat{\alpha}\hat{\theta} = 1,038 \Rightarrow \hat{\alpha} = \frac{1{,}038}{0.499} = 2,080$
- $\hat{\beta}\hat{\theta} = \ 0.404 \Rightarrow \hat{\beta} = \frac{0.404}{0.499} \doteq 0.81$

- PAM: $\hat{c}_t^* = 2,080 + 0.81 x_t$

  $c_t - c_{t-1} = 0.499 \cdot (c_t^* - c_{t-1})$

- If c_t has a prominent inertia and Δc_t significantly follows changes in habits, we might use the PAM approach.

Koyck, PAM, AEH: regression of y_t on x_t and y_{t-1}

Example continued:

$$\begin{array}{rrrrr} & \hat{c}_t = & 1,038 + & 0.404x_t + & 0.501c_{t-1} \\ \textbf{AEH:} & \hat{c}_t = & \hat{\alpha}\hat{\phi} + & \hat{\beta}\hat{\phi}x_t + & (1-\hat{\phi})c_{t-1} \end{array}$$

- $(1-\hat{\phi}) = 0.501 \Rightarrow \hat{\phi} = 0.499$
- $\hat{\alpha}\hat{\phi} = 1,038 \Rightarrow \hat{\alpha} = \frac{1{,}038}{0.499} = 2,080$
- $\hat{\beta}\hat{\phi} = 0.404 \Rightarrow \hat{\beta} = \frac{0.404}{0.499} \doteq 0.81$

- AEH: $\hat{c}_t = 2,080 + 0.81x_t^*$

  $x_t^* = 0.499x_t + 0.501x_{t-1}^*$

- If c_t if formed as a function of expected (e.g. permanent) income, we might prefer AEH.

Rational expectations

- Rational expectations

$$\mathbf{E}_{t-1}(x_t) = a_0 + a_1 x_{t-1} + b_1 z_{1,t-1} + b_2 z_{2,t-2} + \dots$$

$\mathbf{E}_{t-1}(x_t)$: expected value of x_t at time $t-1$
$z_{k,t-j}$: exogenous variables with impact on $\mathbf{E}_{t-1}(x_t)$

We put $x_t^* = \mathbf{E}_{t-1}(x_t)$ into (17): $y_t = \alpha + \beta x_t^* + u_t$

We assume that agents:

  - know all relevant information
  - know how to use this information

Agents can make prediction errors (v_t), so:

$$x_t = x_t^* + v_t$$

Rational vs. adaptive expectations

Under rational expectations:

- Expected value of prediction errors $[v_t = x_t - x_t^*]$ must be zero. If they were systematically different from zero, rational agents would immediately adjust their forecasting methods accordingly.
- $[v_t = x_t - x_t^*]$ prediction error must be uncorrelated with any information available when the prediction is made $(t-1)$. If not, this would imply that the forecaster has not made use of all available information.
- These properties can be used for testing the rational expectations hypothesis in different applications (e.g. through ex-post simulated dynamic forecasts).

Some economic application that use expectations

- Philips curve (Expectations-augmented Phillips curve)
- Efficient market hypothesis (EMH)
- Consumption function - Permanent income hypothesis

FWL theorem - repetition from BSc courses

Frisch and Waugh in the 30-ies: detrending
Lovell in the 60-ies: deseasonalizing

- Example: Spurious regression with trend-stationary series: e.g. Regression of the US GDP on salmon production in Norway

Solution (identical $\hat{\beta}_2$ estimates)

M1: $gdp_t = \beta_0 + \beta_1 year_t + \beta_2 salmon_t + u_t$
M2: $gdp.detrended_t = \beta_2 salmon.detrended_t + u_t$
M2: $gdp.detrended_t = \beta_0 + \beta_1 year_t + \beta_2 salmon_t + u_t$
M4: $gdp_t = \beta_2 salmon.detrended_t + u_t$

- Note that β_0, β_1 and u_t may differ among equations.

FWL theorem

- Partitioned regression

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + u = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u} \tag{21}$$

- Projection matrices and residual makers

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}', & \boldsymbol{M} &= \boldsymbol{I} - \boldsymbol{P} \\ \boldsymbol{P}_1 &= \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1', & \boldsymbol{M}_1 &= \boldsymbol{I} - \boldsymbol{P}_1 \\ \boldsymbol{P}_2 &= \boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2', & \boldsymbol{M}_2 &= \boldsymbol{I} - \boldsymbol{P}_2 \end{aligned}$$

- Some properties

$$\begin{aligned} \boldsymbol{P}\boldsymbol{P}_1 &= \boldsymbol{P}_1, \;\; \boldsymbol{P}\boldsymbol{X}_1 = \boldsymbol{X}_1 \;\; (\boldsymbol{X}_1 \text{ is in the span } (\boldsymbol{X})) \\ \boldsymbol{P}\boldsymbol{P}_1 &= (\boldsymbol{P}_1\boldsymbol{P})' = \boldsymbol{P}_1\boldsymbol{P} = \boldsymbol{P}_1 \\ \boldsymbol{M}\boldsymbol{M}_1 &= (\boldsymbol{M}_1\boldsymbol{M})' = \boldsymbol{M}_1\boldsymbol{M} = \boldsymbol{M} \\ \boldsymbol{M}_1\boldsymbol{X}_1 &= \boldsymbol{0}, \quad \boldsymbol{X}_2'\boldsymbol{M} = \boldsymbol{0} \end{aligned}$$

FWL theorem

M1: $gdp_t = \beta_0 + \beta_1 year_t + \beta_2 salmon_t + u_t$
$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u} = \boldsymbol{X}\boldsymbol{\beta} + u$

M2: $gdp.detrended_t = \beta_2 salmon.detrended_t + u_t$
$\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$

M3: $gdp.detrended_t = \beta_0 + \beta_1 year_t + \beta_2 salmon_t + u_t$
$\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$

M4: $gdp_t = \beta_2 salmon.detrended_t + u_t$
$\boldsymbol{y} = \boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}$

FWL theorem & applications

1. Estimates of β_2 for a partioned model (21) in all M1 - M4 models are the same
2. Residuals from M1, M2, M3 are the same

Lemma: Based on properties of projection matrices and residual makers, we can show that $\hat{\boldsymbol{\beta}}_2$ estimators in all four (M1 to M4) specifications can be expressed as:

$$\hat{\boldsymbol{\beta}}_2 = (\boldsymbol{X}_2' \boldsymbol{M}_1 \boldsymbol{X}_2)^{-1} \boldsymbol{X}_2' \boldsymbol{M}_1 \boldsymbol{y}$$

Application: deseasonalizing, centered coefficient of determination, interchangeability of reference categories (e.g. quarterly dummies), simplification of many proofs in econometrics

FWL theorem (detrending)

$$\boldsymbol{y} \;=\; \boldsymbol{X}\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{u}} \;=\; \boldsymbol{P}\boldsymbol{y} + \boldsymbol{M}\boldsymbol{y} \;=\; \boldsymbol{X}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{X}_2\hat{\boldsymbol{\beta}}_2 + \boldsymbol{M}\boldsymbol{y}$$

Skeleton of the proof (for M1 equation):

$$\boldsymbol{y} = \boldsymbol{X}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{X}_2\hat{\boldsymbol{\beta}}_2 + \boldsymbol{M}\boldsymbol{y} \quad | \quad \boldsymbol{X}_2'\boldsymbol{M}_1 \times$$

$$\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y} = \underbrace{\boldsymbol{X}_2'\underbrace{\boldsymbol{M}_1\boldsymbol{X}_1}_{\boldsymbol{0}}\hat{\boldsymbol{\beta}}_1}_{\boldsymbol{0}} + \boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2\hat{\boldsymbol{\beta}}_2 + \underbrace{\boldsymbol{X}_2'\underbrace{\underbrace{\boldsymbol{M}_1\boldsymbol{M}}_{\boldsymbol{M}}}_{\boldsymbol{0}}\boldsymbol{y}}_{\boldsymbol{0}}$$

$$\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2\hat{\boldsymbol{\beta}}_2 \Rightarrow \hat{\boldsymbol{\beta}}_2 = (\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{M}_1\boldsymbol{y} \Rightarrow \text{QED}$$