

## Week 2: Unit roots tests

Handling strongly dependent time series,  
Spurious regression, Cointegration and error  
correction model (ECM)

Advanced Econometrics 4EK608

Vysoká škola ekonomická v Praze

# Outline

- 1 Stochastic processes
- 2 Unit roots
- 3 Cointegration
- 4 Error correction models

# Czech terminology

Testy jednotkových kořenů,  
silně závislé časové řady,  
zdánlivá regrese, kointegrace,  
model korekce chyby,  
MA proces, AR proces,  
náhodná procházka s driftem,  
integrovaná časová řada, řád integrace,  
korelogram, ADF test,  
stochastický trend, deterministický trend,  
trendově stacionární časová řada,

# Examples of stochastic processes

## Weakly dependent time series

- Moving average process of order one MA(1)  
 $x_t = e_t + \alpha_1 e_{t-1}$ , where  $e_t$  is *i.i.d.* time series.  
Observations with higher time distance than 1 are uncorrelated. This process is stationary.
- Autoregressive process of order 1: AR(1)  
 $y_t = \rho_1 y_{t-1} + e_t \Rightarrow \text{Corr}(y_t, y_{t+h}) = \rho_1^h$
- If stability condition  $|\rho| < 1$  holds, the process is weakly dependent because correlation converges to zero with growing  $h$ . Also, this process is stationary for  $y_0 = 0$ .

# Examples of stochastic processes

Random walk:

$$y_t = y_{t-1} + e_t$$

$$y_t = y_{t-2} + e_{t-1} + e_t$$

...

$$y_t = e_t + e_{t-1} + \cdots + e_1 + y_0$$

Shocks have permanent effects, the series is not covariance stationary and is strongly dependent.

$$E(y_t) = E(y_0)$$

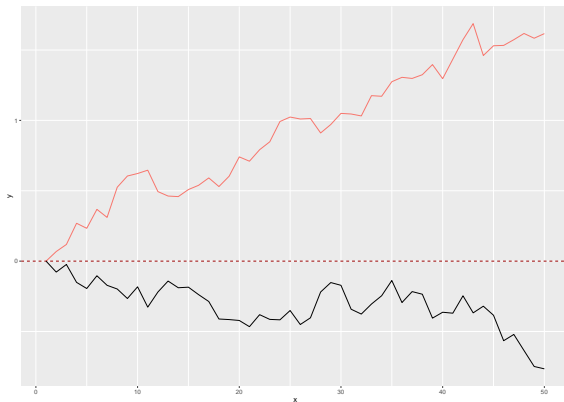
$$Var(y_t) = \sigma_e^2 t$$

$$Corr(y_t, y_{t+h}) = \sqrt{t/(t+h)}$$

Correlation decreases very slowly and speed depends on  $t$ .

# Examples of stochastic processes

- Two realizations of a random walk



# Examples of stochastic processes

- Random walk with a drift

$$y_t = \alpha_0 + y_{t-1} + e_t \Rightarrow y_t = \alpha_0 t + e_t + e_{t-1} + \cdots + e_1 + y_0$$

A linear trend with random walk around the trend.

It is neither covariance stationary nor weakly dependent.

$$E(y_t) = \alpha_0 t + E(y_0)$$

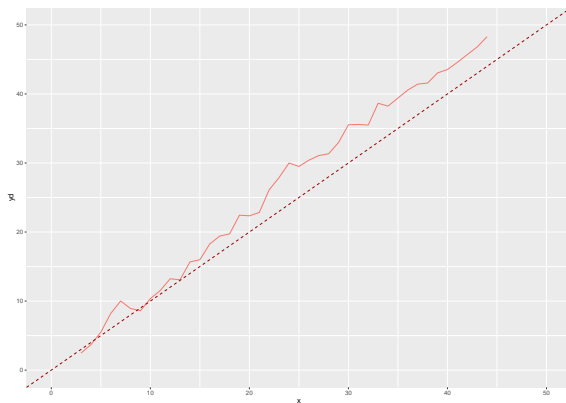
$$Var(y_t) = \sigma_e^2 t$$

$$Corr(y_t, y_{t+h}) = \sqrt{t/(t+h)}$$

Correlation decreases very slowly and decline speed depends on  $t$ .

# Examples of stochastic processes

- Realization of random walk with a drift



- Different realizations of trending TS may produce similar time series.



# Examples of stochastic processes

$$y_t = [1 \cdot y_{t-1}] + [u_t] = y_{t-1} + u_t$$

- Unit root process:  $y_t = y_{t-1} + u_t$ ;  $u_t$  is a weakly dependent series.
- Random walk is a special case of the unit root process where:  $u_t \sim \text{Distr}(0, \sigma_u^2), iid$

We need to distinguish strongly and weakly dependent TS:

- Economic reasons:  
In strongly dependent series, shocks or policy changes have long or permanent effects; in weakly dependent series, their effect is only temporary.
- Statistical reasons:  
Analysis with strongly dependent series must be handled in specific ways.

# Integrated series

## Terminology - Order of integration

- Weakly dependent TS are integrated of order zero:  $I(0)$ .
- If we have to difference a TS once to get a weakly dependent TS, then it is integrated of order 1:  $I(1)$ .
- Example of a  $I(1)$  process:

$$y_t = y_{t-1} + e_t \quad \Rightarrow \quad \Delta y_t = y_t - y_{t-1} = e_t$$

$$\log y_t = \log y_{t-1} + e_t \Rightarrow \Delta \log y_t = e_t$$

- A time series is integrated of order  $d$ :  $I(d)$ , if it becomes a weakly dependent TS after being differenced  $d$  times.

# Unit roots tests

Unit root tests help to decide if a time series is  $I(0)$  or not

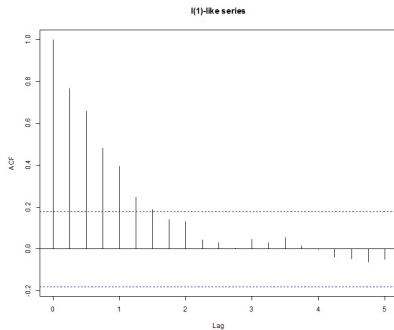
- Use either some informal procedure or a unit root test
- Informal procedures
  - Analyze autocorrelation of the first order

$$\hat{\rho}_1 = \hat{Corr}(y_t, y_{t-1})$$

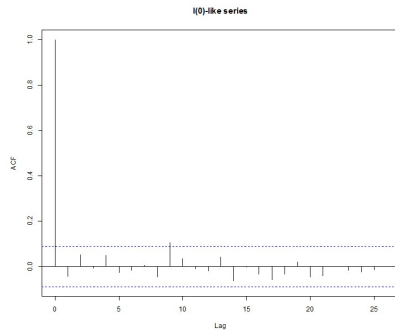
- If  $\hat{\rho}_1$  approaches 1, it indicates that the series can have unit root. Alternatively, it could have a deterministic trend.
- We can analyze sample autocorrelations using a correlogram

# Unit root tests

Correlogram:  $\rho_h = \frac{cov(y_t, y_{t-h})}{\sigma_{y_t} \cdot \sigma_{y_{t-h}}}$



### $I(1)$ -like series



### $I(0)$ -like series

# Unit root tests

## Dickey-Fuller (DF) test – motivation

Unit root test in an  $ar(1)$  process:

$$y_t = \alpha + \rho y_{t-1} + e_t$$

$$H_0 : \rho = 1, \quad H_1 : \rho < 1$$

- Under  $H_0$ ,  $y_t$  has a unit root.
  - For  $\rho = 1 \wedge \alpha = 0 \rightarrow y_t$  is a random walk.
  - For  $\rho = 1 \wedge \alpha \neq 0 \rightarrow y_t$  is a random walk with a drift and  $E(y_t)$  is a linear function of  $t$ .
- Under  $H_1$ ,  $y_t$  is a weakly dependent  $ar(1)$  process.

# Unit root tests

## Dickey-Fuller (DF) test – motivation

Unit root test in an  $ar(1)$  process:

$$y_t = \alpha + \rho y_{t-1} + e_t$$

$$H_0 : \rho = 1, \quad H_1 : \rho < 1$$

For DF tests,  $H_1 : \rho < 1$  is a common simplification to the full space of alternatives to  $H_0 : \rho = 1$ .

- For  $|\rho| < 1$ ,  $y_t$  is weakly dependent (as  $\text{plim } \rho^h = 0$ )  
However, if unit root is likely to be present, the probability of  $\rho < 0$  is negligible.
- We usually ignore the possibility of  $\rho > 1$ , as it would lead to explosive behavior in  $y_t$ .  
...  $|\rho| > 1$  would allow for explosive oscillations in  $y_t$ .

# Dickey Fuller (DF) test

- Basic equation for unit root test in an  $ar(1)$  process:

$$y_t = \alpha + \rho y_{t-1} + e_t$$

- For DF test, we apply a suitable transformation to  $y_t$ :  
we subtract  $y_{t-1}$  from both sides of the equation:

$$\Delta y_t = \alpha + (\rho - 1)y_{t-1} + e_t; \text{ apply substitution: } \theta = (\rho - 1)$$

i.e.

$$H_0 : \rho = 1 \Leftrightarrow H_0 : \theta = 0$$

$$\Delta y_t = \alpha + \theta y_{t-1} + e_t; \text{ now:}$$

$$H_1 : \rho < 1 \Leftrightarrow H_1 : \theta < 0$$

- We use a  $t$ -ratio for testing  $H_0 : \theta = 0$ . However:  
Under  $H_0$ ,  $t$ -ratios don't have a  $t$ -distribution, but follow a  $DF$ -distribution. (-negative- critical values of the  $DF$  distribution are much farther from zero)
- Critical values for the  $DF$  distribution are available from statistical tables and implemented in most relevant SW packages.

# DF test & ADF test

Unit root time series can manifest various levels of complexity. Hence, DF test is usually performed using the following three specifications:

$$\begin{aligned}
 \Delta y_t &= \theta y_{t-1} + e_t && \text{random walk} \\
 \Delta y_t &= \alpha + \theta y_{t-1} + e_t && \text{random walk with a drift} \\
 \Delta y_t &= \alpha + \theta y_{t-1} + \delta t + e_t && \text{random walk with a drift and trend}
 \end{aligned}$$

DF test is the same ( $H_0 : \theta = 0$ ) for all specifications /critical values differ/

Augmented Dickey-Fuller (ADF) test is a common generalization of DF test  
(example: Augmentation of the DF test for the 2<sup>nd</sup> specification)

$$\Delta y_t = \alpha + \theta y_{t-1} + \gamma_1 \Delta y_{t-1} + \cdots + \gamma_p \Delta y_{t-p} + e_t$$

- When estimating  $\theta$ , we control for possible  $ar(p)$  behavior in  $\Delta y_t$ .
- ADF test has the same null hypothesis as a DF test  $\rightarrow H_0 : \theta = 0$ .



## Unit root tests in R: package {urca}

Description of the options for the `ur.df()` function:

❶ type "none"

$$\Delta y_t = \gamma y_{t-1} + e_t$$

tau1: we test for  $H_0 : \gamma = 0$  (unit root)

❷ type "drift"

$$\Delta y_t = a_0 + \gamma y_{t-1} + e_t$$

tau2:  $H_0 : \gamma = 0$  (unit root)

phi1:  $H_0 : \gamma = a_0 = 0$  (unit root and no drift)

❸ type "trend"

$$\Delta y_t = a_0 + \gamma y_{t-1} + a_2 t + e_t$$

tau3:  $H_0 : \gamma = 0$  (unit root)

phi2:  $H_0 : \gamma = a_0 = a_2 = 0$  (unit root, no drift, no trend)

phi3:  $H_0 : \gamma = a_2 = 0$  (unit root and no trend)

## Unit root tests

- ADF test for TS with trend

$$\Delta y_t = \alpha + \delta t + \theta y_{t-1} + \gamma_1 \Delta y_{t-1} + \cdots + \gamma_p \Delta y_{t-p} + e_t$$

Under the alternative hypothesis of no unit root, the process is trend-stationary.

- The critical values in the ADF distribution with time trend are even more negative as compared to random walk and random walk with a drift.
- When using DF/ADF specification 1 or 2 (R-W, R-W with drift) to test for unit root in a clearly trending TS, the test would not have sufficient power (we would not reject  $H_0$  for trending weakly dependent TS).

# Unit roots and trend-stationary series

- $\Delta y_t = \alpha + \delta t + \theta y_{t-1} + \gamma_1 \Delta y_{t-1} + \cdots + \gamma_p \Delta y_{t-p} + e_t$
- Terminology:
  - Stochastic trend:  $\theta = 0$   
Also called **difference-stationary process**:  $y_t$  can be turned into  $I(0)$  series by differencing. Terminology emphasizes stationarity after differencing  $y_t$  instead of weak dependence in differenced TS.
  - Deterministic trend:  $\delta \neq 0, \quad \theta < 0$   
Also called **trend-stationary process**: has a linear trend, not a unit root.  $y_t$  is weakly dependent -  $I(0)$  - around its trend. We can use such series in LRMs, if trend is also used as regressor.
- DF/ADF tests are not precise tools. Distinguishing between stochastic and deterministic trend is not easy (sample size!).

# Handling trend-stationary time series

- Trend-stationary TS fulfill TS.1' assumption (look at Week1 presentation).

We can use them in regressions if we have time trend among regressors.

# Handling strongly dependent time series

- Strongly dependent time series do not fulfill TS.1' assumption (look at Week1 presentation). We cannot use them in regressions directly.
- Sometimes, taking logarithms helps.
- Sometimes, we can transform such series into weakly dependent time series.
- Differencing is popular, but it has drawbacks.

# Handling strongly dependent time series

## Example

$$Y_t = \beta_1 + \beta_2 X_t + \varepsilon_t \qquad Y_t, X_t \sim I(1) \qquad (1)$$

$$Y_{t-1} = \beta_1 + \beta_2 X_{t-1} + \varepsilon_{t-1} \qquad \varepsilon_t \sim i.i.d. \qquad (2)$$

$$\Delta Y_t = \beta_2 \Delta X_t + v_t \qquad v_t = \varepsilon_t - \varepsilon_{t-1} \qquad (3)$$

- ❶ If we work with logarithms, it has an additional advantage:  
 $\log Y_t - \log Y_{t-1} = \log \left( \frac{Y_t}{Y_{t-1}} \right) \doteq \frac{Y_t - Y_{t-1}}{Y_{t-1}}$   
i.e.: nice interpretation as rate of growth of  $Y_t$
- ❷ Three problems
  - ❶  $v_t$  is no more i.i.d.
  - ❷ We loose information linked with the levels of variables, short term relation are stressed
  - ❸ Estimates often generate bad long-term predictions:  
 $\Delta \hat{Y}_t = \hat{\beta}_2 \Delta X_t; \dots$  what if  $\beta_1 \neq 0$ ?

# Handling strongly dependent time series

## Some properties of integrated processes

- ❶ The sum of stationary and non-stationary series must be non-stationary.
- ❷ Consider a process  $y_t = \alpha + \beta x_t$ :
  - If  $x_t$  is stationary then  $y_t$  will be stationary.
  - If  $x_t$  is non-stationary then  $y_t$  will be non-stationary.
- ❸ If two time series are integrated of different orders, then any linear combination of the series will be integrated at the higher of the two orders of integration.
- ❹ Sometimes it turns out a linear combination of two  $I(d)$  series is integrated of order less than  $d$ .

# Spurious regression or cointegration

- **Spurious regression** Regressing one  $I(1)$ -series on another  $I(1)$ -series may lead to extremely high  $t$ -statistics even if the series are completely independent. Similarly, the  $R^2$  of such regressions tend to be very high. Regression analysis involving time series that have a unit root may generate completely misleading inferences.
- **Cointegration** Fortunately, regressions with  $I(1)$ -variables are not always spurious: If there is a stable relationship between time series that, individually, display unit root behavior, these time series are called “cointegrated”.



# Spurious regression or cointegration

## General definition of cointegration

Two  $I(1)$ -time series  $y_t, x_t$  are said to be cointegrated if there exists a stable relationship between them, where:

$$y_t = \alpha + \beta x_t + e_t, \quad e_t \sim I(0)$$

## Cointegration (CI) test if CI parameters are known

For residuals of the known CI relationship:

$$e_t := y_t - \alpha - \beta x_t,$$

test whether the residuals have a unit root. If the unit root  $H_0$  is rejected,  $y_t, x_t$  are cointegrated.

## Spurious regression or cointegration

- **Testing for CI if the parameters are unknown**

If the potential relationship is unknown, it can be estimated by OLS. After that, we test whether the regression residuals have a unit root. If the unit root is rejected, this means that  $y_t, x_t$  are cointegrated. Due to the pre-estimation of parameters, critical values are different than in the case of known parameters.

(Software handles this automatically.)

- **The CI relationship may include a time trend**

If the two series have differential time trends (drifts in this case), the deviation between them may still be  $I(0)$  but with a linear time trend. In this case one should include a time trend in the CI-regression. Also, we have to use different critical values when testing residuals.

(Software handles this automatically.)

## Error correction model (ECM)

- It can be shown that when variables are cointegrated, i.e. when there exists a long-term relationship among them, their short-term dynamics are related as in a so-called error correction model (ECM).

# Error Correction Models - motivation

## Autoregressive distributed lag models

- Autoregressive distributed lag model with one regressor

$$\text{ADL}(p, q): y_t = \beta_0 + \sum_{i=1}^p \beta_i y_{t-i} + \sum_{j=0}^q \gamma_j x_{t-j} + u_t, \quad u_t \sim iid(0, \sigma^2)$$

- There are many useful modifications/simplifications to the  $\text{ADL}(p, q)$  process. For example:

$$\text{ADL}(1, 1): y_t = \beta_0 + \beta_1 y_{t-1} + \gamma_0 x_t + \gamma_1 x_{t-1} + u_t. \quad (4)$$

Additional  $\text{ADL}(1, 1)$  restriction:  $\beta_1 = 1$  and  $\gamma_1 = -\gamma_0$

gives a model in 1<sup>st</sup> diffs.:  $\Delta y_t = \beta_0 + \gamma_0 \Delta x_t + u_t$ .

## Error Correction Models - motivation

For ADL(1,1) model (4), suppose there is an equilibrium value  $x^\circ$  and in the absence of shocks,  $x_t \rightarrow x^\circ$  as  $t \rightarrow \infty$ . Then, assuming absence of  $u_t$  errors,  $y_t$  converges to steady state:  $y^\circ$ .

Hence, the ADL(1,1) model (4) can be re-written as:

$$y^\circ = \beta_0 + \beta_1 y^\circ + (\gamma_0 + \gamma_1) x^\circ$$

Solving this for  $y^\circ$  as a function of  $x^\circ$ , we get

$$y^\circ = \frac{\beta_0}{1 - \beta_1} + \frac{\gamma_0 + \gamma_1}{1 - \beta_1} x^\circ = \frac{\beta_0}{1 - \beta_1} + \lambda x^\circ$$

where  $\lambda \equiv \frac{\gamma_0 + \gamma_1}{1 - \beta_1}$  and  $|\beta_1| < 1$  is assumed.

## Error Correction Models - motivation

$$y^{\circ} = \frac{\beta_0}{1 - \beta_1} + \lambda x^{\circ}$$

$$\lambda \equiv \frac{\gamma_0 + \gamma_1}{1 - \beta_1}$$

- $\lambda$  is the long-run derivative of  $y^{\circ}$  with respect to  $x^{\circ}$ .
- $\lambda$  is an elasticity if both  $y^{\circ}$  and  $x^{\circ}$  are in logs.
- $\hat{\lambda}$  can be computed directly from the estimated parameters of the ADL(1,1) model (4).

## Error Correction Models - motivation

The ADL(1,1) equation (4) - repeated here for convenience:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \gamma_0 x_t + \gamma_1 x_{t-1} + u_t,$$

can be equivalently rewritten as follows:

$$\Delta y_t = \beta_0 + (\beta_1 - 1)(y_{t-1} - \lambda x_{t-1}) + \gamma_0 \Delta x_t + u_t. \quad (5)$$

Again,  $\lambda \equiv \frac{\gamma_0 + \gamma_1}{1 - \beta_1}$  and  $|\beta_1| < 1$  is assumed.

Equation (5) is an error-correction model (ECM).

# Error Correction Models

$$\text{ECM: } \Delta y_t = \beta_0 + (\beta_1 - 1)(y_{t-1} - \lambda x_{t-1}) + \gamma_0 \Delta x_t + u_t.$$

- $(y_{t-1} - \lambda x_{t-1})$  measures the extent to which the long run equilibrium between  $y_t$  and  $x_t$  is not satisfied (at  $t - 1$ ).
- Consequently,  $(\beta_1 - 1)$  can be interpreted as the proportion of the disequilibrium  $(y_{t-1} - \lambda x_{t-1})$  that is reflected in the movement of  $y_t$ , i.e. in  $\Delta y_t$ .
- $(\beta_1 - 1)(y_{t-1} - \lambda x_{t-1})$  is the **error-correction term**.
- Many ADL( $p, q$ ) specifications can be re-written as ECMs.
- **ECMs can be used with non stationary TS** (Week 3).
- ECMs  $(\beta_1 - 1)$  is essentially the same as  $\theta$  from Partial adjustment model (see Week 4).



## Box 1: Partial Adjustment Models

(Will be put into context in the 4th week)

$$Y_t^* = \alpha + \beta X_t + u_t \quad Y_t^*: \text{optimal value or target value or long-run equilibrium value} \quad (6)$$

$$Y_t - Y_{t-1} = \theta(Y_t^* - Y_{t-1}) \quad 0 < \theta < 1 \quad (7)$$

$\theta \sim$  coefficient of adjustment  
 $Y_t = \theta Y_t^* + (1 - \theta)Y_{t-1}$

As  $\theta \rightarrow 1$ , the speed of contemporaneous adjustment of  $Y_t$  towards  $Y_t^*$  grows.

Substituting  $Y_t^*$  from (6) to (7) yields

$$Y_t = \alpha\theta + \beta\theta X_t + (1 - \theta)Y_{t-1} + \theta u_t \quad (8)$$

We estimate (8) and then calculate parameters in (6) and (7).

# Error Correction Models

Some more complicated ECMs:

- 1) We can use higher order lags, e.g. ADL(2,2):

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \gamma_0 x_t + \gamma_1 x_{t-1} + \gamma_2 x_{t-2} + u_t,$$

to establish ECMs. It is again possible to rearrange and re-parametrize ADL(2,2) to get an ECM. More than one re-parameterization is possible.

- 2) More than two variables can enter into an equilibrium relationship.