ISE 426
Optimization models and applications

Lecture 9 — September 29, 2015

Duality, continued

Reading:

- W.&V. Sections 6.5–6.7, pages 295-308
- H.&L. Section 6.1–6.4, pages 151-169

Reminders:

- Quiz on 10/08, practice on 10/06.

Primal problem, dual problem

Primal

$$\begin{array}{lll} \min & 3x_1 + 4x_2 & \\ & 5x_1 + 6x_2 & \geq 7 \\ & 8x_1 + 9x_2 & \geq 10 \\ & 11x_1 + 12x_2 & \geq 13 \\ & x_1, x_2 \geq 0 & \end{array}$$

Dual

$$\begin{array}{lll} \max & 7u_1 + 10u_2 + 13u_3 & \\ & 5u_1 + 8u_2 + 11u_3 & \leq 3 \\ & 6u_1 + 9u_2 + 12u_3 & \leq 4 \\ & u_1, u_2, u_3 \geq 0 & \end{array}$$

In general:

$$\begin{array}{ll} \min & c^\top x \\ & Ax \geq b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \max & b^\top u \\ & A^\top u \leq c \\ & u \geq 0 \end{array}$$

The **primal** has n variables and m constraints

$\Rightarrow$ The **dual** has m variables and n constraints

Properties of duality in LP

Weak duality: Given a primal $\min\{c^\top x : Ax \geq b, x \geq 0\}$ and its dual $\max\{b^\top u : A^\top u \leq c, u \geq 0\}$,

$$b^\top \bar{u} \leq c^\top \bar{x}$$

for any $\bar{x}$ and $\bar{u}$ feasible for their respective problems.

Strong duality: If a problem $\min\{c^\top x : Ax \geq b, x \geq 0\}$ is bounded and its dual $\max\{b^\top u : A^\top u \leq c, u \geq 0\}$ is bounded, their optimal solutions $\bar{x}$ and $\bar{u}$ coincide in value:

$$c^\top \bar{x} = b^\top \bar{u}$$

Properties of duality in LP (cont.)

Consequence: solving the dual or the primal doesn't matter: we get the same objective function value.

What if the primal (or the dual) is infeasible or unbounded?

Four cases:

- Primal bounded, dual bounded;
- Primal infeasible, dual infeasible;
- Primal unbounded ($c^\top x = -\infty$), dual infeasible;
- Primal infeasible, dual unbounded ($b^\top u = +\infty$).

		Dual		
		bounded	unbounded	infeasible
Primal	bounded	Possible	–	–
	unbounded	–	–	Possible
	infeasible	–	Possible	Possible

Unbounded LP problem

Consider the following **minimization** problem:

$$
\begin{array}{lll}
\min & -5x_1 - 4x_2 & \\
 & 2x_1 - x_2 & \geq 1 \\
 & -x_1 + 2x_2 & \geq 1 \\
 & x_1, x_2 \geq 0 &
\end{array}
$$

- How to show that the solution is unbounded?
- Consider direction (d_1, d_2) in which one can move infinitely while decreasing the objective function.

Unbounded LP problem $\Rightarrow$ Infeasible dual

$$\begin{array}{ll} -5d_1 - 4d_2 < 0 & \\ 2d_1 - d_2 & \geq 0 \\ -d_1 + 2d_2 & \geq 0 \\ d_1, d_2 \geq 0 & \end{array}$$

For example $(d_1, d_2) = (1, 1)$.

The dual

$$\begin{array}{lll} \max & u_1 + u_2 & \\ & 2u_1 - u_2 & \leq -5 \\ & -u_1 + 2u_2 & \leq -4 \\ & u_1, u_2 \geq 0 & \end{array}$$

$$\begin{array}{rll} 0 \leq & u_1(2d_1 - d_2) + u_2(-d_1 + 2d_2) & = \\ & d_1(2u_1 - u_2) + d_2(-u_1 + 2u_2) \leq & -5d_1 - 4d_2 < 0 \end{array}$$

primal unboundedness - dual infeasibility

Primal - unbounded

$$\begin{aligned} \min \quad & c^\top x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} & c^\top d < 0 \\ & Ad \geq 0 \\ & d \geq 0 \end{aligned}$$

Dual - infeasible

$$\begin{aligned} \max \quad & b^\top u \\ & A^\top u \leq c \\ & u \geq 0 \end{aligned}$$

for any feasible u

$0 \leq d^\top A^\top u \leq d^\top c < 0$

$u \geq 0$

Primal problem and dual problem with equality constraints

Primal	Dual
$$\begin{array}{ll} \min & 3x_1 + 4x_2 \\ & 5x_1 + 6x_2 \geq 7 \\ & 8x_1 + 9x_2 \geq 10 \\ & -8x_1 - 9x_2 \geq -10 \\ & 11x_1 + 12x_2 \geq 13 \\ & x_1, x_2 \geq 0 \end{array}$$	$$\begin{array}{ll} \max & 7u_1 + 10u_2' - 10u_2'' + 13u_3 \\ & 5u_1 + 8u_2' - 8u_2'' + 11u_3 \leq 3 \\ & 6u_1 + 9u_2' - 9u_2'' + 12u_3 \leq 4 \\ & u_1, u_2', u_2'', u_3 \geq 0 \end{array}$$
$$\begin{array}{ll} \min & 3x_1 + 4x_2 \\ & 5x_1 + 6x_2 \geq 7 \\ & 8x_1 + 9x_2 = 10 \\ & 11x_1 + 12x_2 \geq 13 \\ & x_1, x_2 \geq 0 \end{array}$$	$$\begin{array}{ll} \max & 7u_1 + 10u_2 + 13u_3 \\ & 5u_1 + 8u_2 + 11u_3 \leq 3 \\ & 6u_1 + 9u_2 + 12u_3 \leq 4 \\ & u_1, u_3 \geq 0, \ u_2 \ - \ \text{unrstrtd} \end{array}$$

Primal problem and dual problem with unrestricted variables

Primal	Dual
$\begin{array}{ll} \min & 3x_1 + 4(x_2' - x_2'') \\ & 5x_1 + 6(x_2' - x_2'') \geq 7 \\ & 8x_1 + 9(x_2' - x_2'') \geq 10 \\ & 11x_1 + 12(x_2' - x_2'') \geq 13 \\ & x_1, x_2', x_2'' \geq 0 \end{array}$	$\begin{array}{ll} \max & 7u_1 + 10u_2 + 13u_3 \\ & 5u_1 + 8u_2 + 11u_3 \leq 3 \\ & 6u_1 + 9u_2 + 12u_3 \leq 4 \\ & -6u_1 - 9u_2 - 12u_3 \leq -4 \\ & u_1, u_2, u_3 \geq 0 \end{array}$
$\begin{array}{ll} \min & 3x_1 + 4x_2 \\ & 5x_1 + 6x_2 \geq 7 \\ & 8x_1 + 9x_2 \geq 10 \\ & 11x_1 + 12x_2 \geq 13 \\ & x_1 \geq 0,\ x_2 \ - \ \text{unrstrtd} \end{array}$	$\begin{array}{ll} \max & 7u_1 + 10u_2 + 13u_3 \\ & 5u_1 + 8u_2 + 11u_3 \leq 3 \\ & 6u_1 + 9u_2 + 12u_3 = 4 \\ & u_1, u_2, u_3 \geq 0 \end{array}$

How to construct the dual of an LP

Variable	Constraint
Constraint	Variable
Minimize	Maximize
Variable ≥ 0	Constraint $\leq$
Variable ≤ 0	Constraint $\geq$
Var. Unrestricted	Constraint $=$
Constraint $\leq$	Variable ≤ 0
Constraint $\geq$	Variable ≥ 0
Constraint $=$	Var. Unrestricted

LP primal and dual problem, standard form

Primal	Dual
$\begin{aligned} \min \quad & c^\top x \\ & Ax = b \\ & x \geq 0 \end{aligned}$	$\begin{aligned} \max \quad & b^\top u \\ & A^\top u \leq c \end{aligned}$

LP primal and dual problem, standard form

Primal	Dual
$\min \ c^\top x$	$\max \ b^\top u$
$Ax = b$	$A^\top u + s = c$
$x \geq 0$	$s \geq 0$

What is the dual of the dual in standard form?

$$
\begin{array}{ll} \min & c^\top x \\ & Ax = b \\ & x \geq 0 \end{array}
\quad \rightarrow \quad
\begin{array}{ll} \max & b^\top u \\ & A^\top u + s = c \\ & s \geq 0 \\ & u - \text{ unrstrtd} \end{array}
\quad \rightarrow \quad
\begin{array}{ll} \min & c^\top x \\ & Ax = b \\ & Ix \geq 0 \\ & x - \text{ unrstrtd} \end{array}
$$

Complementary slackness

- Given a primal-dual pair, now we know how to solve one and get the optimal objective function of the other.

e.g. Solve primal $\Rightarrow$ get optimal obj.f. $c^\top \bar{x}$, an optimal solution $\bar{x}$, and the optimal dual obj.f. $b^\top \bar{u}$. **How do we get $\bar{u}$?**

Complementary Slackness: If the primal problem

$$\min\{c^\top x : \sum_{i=1}^n a_{ji} x_i \geq b_j \ \forall j = 1, 2 \ldots, m, x \geq 0\}$$

is bounded and admits optimum $\bar{x}$, and its dual

$$\max\{b^\top u : \sum_{j=1}^m a_{ji} u_j \leq c_i \ \forall i = 1, 2 \ldots, n, u \geq 0\}$$

is bounded and admits optimal solution $\bar{u}$, then

$$\begin{array}{c} \bar{u}_j (\sum_{i=1}^n a_{ji} \bar{x}_i - b_j) = 0 \quad \forall j = 1, 2 \ldots, m; \\ \bar{x}_i (\sum_{j=1}^m a_{ji} \bar{u}_j - c_i) = 0 \quad \forall i = 1, 2 \ldots, n \end{array}$$

So if we solve the primal and get $\bar{x}$, we can get $\bar{u}$ by solving a system of equations.

LP primal and dual problem, standard form

Primal		Dual	
min	$c^\top x$	max	$b^\top u$
	$Ax = b$		$A^\top u + s = c$
	$x \geq 0$		$s \geq 0$

$$s_i x_i = 0 \;\Rightarrow\; (c_i - \sum_{j=1}^{m} a_{ji} u_j) x_i = 0$$

Either the primal variable is zero of the dual constraint is tight.

Example

$$\begin{array}{ll} \min & 3x_1 + 4x_2 \\ & 5x_1 + 6x_2 \geq 7 \\ & 8x_1 + 9x_2 \geq 10 \\ & 11x_1 + 12x_2 \geq 13 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{ll} \max & 7u_1 + 10u_2 + 13u_3 \\ & 5u_1 + 8u_2 + 11u_3 \leq 3 \\ & 6u_1 + 9u_2 + 12u_3 \leq 4 \\ & u_1, u_2, u_3 \geq 0 \end{array}$$

Solve the dual (with AMPL+CPLEX): get $(u_1, u_2, u_3) = (0.6, 0, 0)$.
Find (x_1, x_2) with complementary slackness:

$$\left.\begin{array}{l} u_1(5x_1 + 6x_2 - 7) = 0 \\ u_2(8x_1 + 9x_2 - 10) = 0 \\ u_3(11x_1 + 12x_2 - 13) = 0 \\ x_1(5u_1 + 8u_2 + 11u_3 - 3) = 0 \\ x_2(6u_1 + 9u_2 + 12u_3 - 4) = 0 \end{array}\right. \Rightarrow \begin{array}{l} 0.6(5x_1 + 6x_2 - 7) = 0 \\ 0(8x_1 + 9x_2 - 10) = 0 \\ 0(11x_1 + 12x_2 - 13) = 0 \\ x_1(5 \cdot 0.6 + 8 \cdot 0 + 11 \cdot 0 - 3) = 0 \\ x_2(6 \cdot 0.6 + 9 \cdot 0 + 12 \cdot 0 - 4) = 0 \end{array}$$

$$\begin{array}{l} 5x_1 + 6x_2 = 7 \\ x_1 \cdot 0 = 0 \\ x_2 \cdot (-0.4) = 0 \end{array} \Rightarrow \begin{array}{l} 5x_1 + 6x_2 = 7 \\ x_1 \cdot 0 = 0 \\ x_2 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = \frac{7}{5} \\ x_2 = 0 \end{array}$$

Another example

Consider the following LP problem:

$$
\begin{array}{rllllll}
\min & x_1 & +2x_2 & +3x_3 & -4x_4 & -3x_5 & \\
\text{s.t.} & -2x_1 & & -x_3 & -x_4 & & \geq 1 \\
& -x_1 & +x_2 & -x_3 & & +x_5 & \leq 2 \\
\multicolumn{7}{l}{x_1, x_2, x_3 \geq 0, x_4, x_5 \leq 0}
\end{array}
$$

1. Write its dual.
2. Solve the dual through the graphical method.
3. After finding the optimal value of the dual variables, use complementary slackness to find the optimal value of the primal variables.

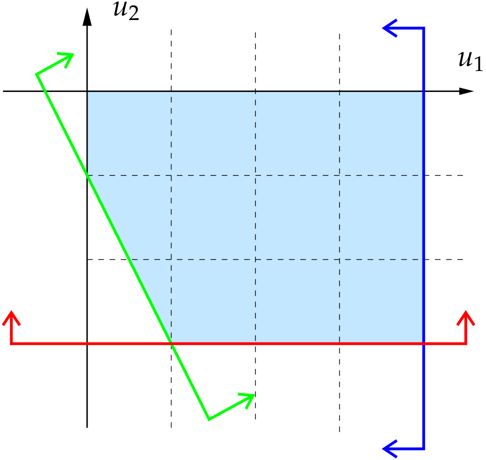
Another example: solution

The dual is

$$
\begin{array}{rrrl}
\max & u_1 & +2u_2 & \\
\text{s.t.} & -2u_1 & -u_2 & \leq 1 \\
 & & u_2 & \leq 2 \\
 & -u_1 & -u_2 & \leq 3 \\
 & -u_1 & & \geq -4 \\
 & & u_2 & \geq -3 \\
 & \multicolumn{3}{l}{u_1 \geq 0, u_2 \leq 0}
\end{array}
$$

Another example: solution

The second and third dual constraints are ignored as they are redundant to solve the problem.



The solution is clearly $(u_1, u_2) = (4, 0)$, corresponding to a value of 4 of the objective function.

Another example: solution

Complementarity slackness implies that:

$$
\begin{aligned}
&x_1(-2u_1 - u_2 - 1) = 0\\
&x_2(u_2 - 2) = 0\\
&x_3(-u_1 - u_2 - 3) = 0\\
&x_4(-u_1 + 4) = 0\\
&x_5(u_2 + 3) = 0\\
&u_1(-2x_1 - x_3 - x_4 - 1) = 0\\
&u_2(-x_1 + x_2 - x_3 + x_5 - 2) = 0
\end{aligned}
$$

which reduces, once we know the values of u_1 and u_2, to:

$$
\begin{aligned}
&x_1(-8 - 0 - 1) = 0\\
&x_2(0 - 2) = 0\\
&x_3(-4 - 0 - 3) = 0\\
&x_4(-4 + 4) = 0\\
&x_5(0 + 3) = 0\\
&-2x_1 - x_3 - x_4 = 1
\end{aligned}
$$

which implies $(x_1, x_2, x_3, x_5) = (0, 0, 0, 0)$, while
$x_4 = -2 \cdot 0 - 0 - 1 = -1$.

Maximum Flow

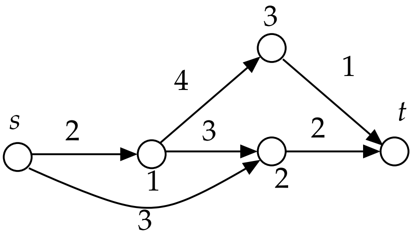
$$
\begin{array}{rll}
\max & \sum_{j \in V:(j,t) \in A} x_{jt} & \\
\text{s.t.} & \sum_{j \in V:(i,j) \in A} x_{ij} = \sum_{j \in V:(j,i) \in A} x_{ji} & \forall i \in V : s \neq i \neq t \\
& 0 \leq x_{ij} \leq c_{ij} & \forall (i,j) \in A
\end{array}
$$

- Variables for each node u_i for flow conservation constraints
- Variables for each arc z_{ij} for capacity constraints

$$
\begin{array}{rll}
\min & \sum_{(i,j) \in A} c_{ij} z_{ij} & \\
\text{s.t.} & z_{ji} \geq u_j - u_i & \forall (i,j) \in A,\ i \neq s\ j \neq t \\
& z_{si} \geq u_i & \forall (s,i) \in A \\
& z_{it} \geq 1 - u_{it} & \forall (i,t) \in A \\
& 0 \leq z_{ij} & \forall (i,j) \in A
\end{array}
$$

From complementarity slackness: $z_{ij}(x_{ij} - c_{ij}) = 0$.
What does it mean?

Recall the Max-Flow example



$$
\begin{aligned}
\max \quad & x_{2t} + x_{3t} \\
\\
x_{s1} &= x_{12} + x_{13} \\
x_{s2} + x_{12} &= x_{2t} \\
x_{13} &= x_{3t} \\
\\
0 \le x_{s1} \le 2 \quad & 0 \le x_{s2} \le 3 \\
0 \le x_{12} \le 3 \quad & 0 \le x_{13} \le 4 \\
0 \le x_{2t} \le 2 \quad & 0 \le x_{3t} \le 1
\end{aligned}
$$

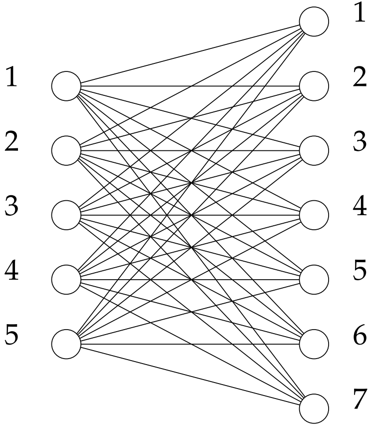
Transportation problem

Variables: qty of product from producer $i \in P$ to distributor $j \in D$: x_{ij} (non-negative)

Constraints:

1. capacity:
   $\sum_{j \in D} x_{ij} \leq p_i \quad \forall i \in P$
2. demand:
   $\sum_{i \in P} x_{ij} \geq d_j \quad \forall j \in D$

Objective function: total transportation cost, $\min \sum_{i \in P} \sum_{j \in D} c_{ij} x_{ij}$



Dual of the transportation problem

- Variables for each supplier u_i for each supplier capacity constraints
- Variables for each distributer v_j for each distributer demand constraints

$$\begin{array}{rll} \max & \sum_{j\in D} d_j v_j - \sum_{i\in P} p_i u_i & \\ \text{s.t.} & v_j - u_i \leq c_{ij} \ \forall i \in P,\ j \in D & \\ & 0 \leq u_i, v_j & \forall i \in P,\ j \in D \end{array}$$

From complementarity slackness: $u_i(\sum_{j\in D} x_{ij} - p_i) = 0$ and $v_j(\sum_{i\in P} x_{ij} - d_j) = 0$.
From complementarity slackness: $x_{ij}(c_{ij} - v_j + u_i) = 0$.
What does it mean? Only send product from i to j if the difference between the “fair market” buy price for i and cell price for j equals the transportation cost.

Shadow prices

Consider an LP problem $\min\{c^T x : Ax \leq b\}$. Suppose we solved it to the optimum and an optimal solution is $x^\star$.

- associated with constraints
- if nonzero, the constraint is active : for inequality $a^T x \leq b$, we have $a^T x^\star = b$ (equality constraints are always active)
- it can be interpreted as the "marginal value" of the constraint (or of the resource/budget/limit/…the constraint is associated with)

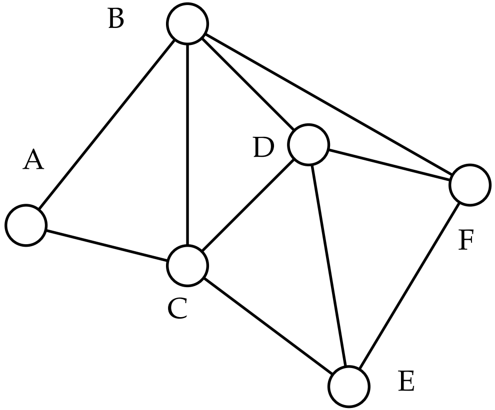
Reduced costs

- associated with variables
- if nonzero, the variable x_i is at its *lower* or *upper* bound
- gives an estimate of the "marginal value" of x_i
- i.e., if the coefficient of x_i in the objective function were lowered by that amount, the optimal solution would have $x_i \neq 0$.

Another example: the shortest path problem

For simplicity, the graph below is undirected, but we can assume for each edge there are two oppositely oriented arcs.

Suppose the problem is to compute the shortest path A $\rightarrow$ F.



The shortest path problem: primal

$$
\begin{array}{llll}
\min & c_{AB}x_{AB} + c_{BA}x_{BA} + \ldots + c_{EF}x_{EF} & +c_{FE}x_{FE} & \\
& x_{AB} + x_{AC} & -x_{BA} - x_{CA} & = 1 \\
& x_{BA} + x_{BC} + x_{BD} + x_{BF} & -x_{AB} - x_{CB} - x_{DB} - x_{FB} & = 0 \\
& x_{CA} + x_{CB} + x_{CD} + x_{CE} & -x_{AC} - x_{BC} - x_{DC} - x_{EC} & = 0 \\
& x_{DB} + x_{DC} + x_{DE} + x_{DF} & -x_{BD} - x_{CD} - x_{ED} - x_{FD} & = 0 \\
& x_{EC} + x_{ED} + x_{EF} & -x_{CE} - x_{DE} - x_{FE} & = 0 \\
& x_{FB} + x_{FD} + x_{FE} & -x_{BF} - x_{DF} - x_{DF} & = -1 \\
& x_{AB}, x_{BA} \ldots, x_{EF}, x_{FE} \geq 0 & &
\end{array}
$$

- We can express this as $\min\{c^\top x : Ax = b, x \geq 0\}$
- A is the adjacency matrix of G
- $|V|$ constraints, $|A|$ variables
- All constraints are equalities

The shortest path problem: dual

$$
\begin{aligned}
\max \quad & u_A - u_F \\
& u_A - u_B \leq c_{AB} \\
& u_B - u_A \leq c_{BA} \\
& u_A - u_C \leq c_{AC} \\
& u_C - u_A \leq c_{CA} \\
& \vdots \\
& u_E - u_F \leq c_{EF} \\
& u_F - u_E \leq c_{FE}
\end{aligned}
$$

- This is $\max\{b^\top u : A^\top u \leq c\}$
- $A^\top$ is the **transposed** adjacency matrix of G
- $|V|$ variables, $|A|$ constraints
- All variables are unrestricted in sign