#4 HW ISE429 Name: Bolun Xu.

(1) Let A be the ~~ra~~ random varable of second roll.

Let B be the r.v. of third roll.

$$Y = X + A + B.$$

$\because$ X, A, B are indepent.

$$\therefore E(Y|X) = E(X|X) + E(A|X) + E(B|X)$$

$$E(A|X) = E(A) = \frac{1+2+3+4+5+6}{6} = 3.5 = E(B)$$

$$\therefore E(Y|X) = X + 7$$

(2) $E(X|Y) = E(X|X+A+B) = E(Y|X+A+B) - E(A|X+A+B) - E(B|X+A+B)$

$$= \frac{1}{3}(X+A+B) = \frac{Y}{3}$$

$$\therefore E(X|Y) = \frac{Y}{3}$$

To show this, we need to prove the following three:

(1) M_n is $\mathcal{F}_n$ - measurable.

(2) $E|M_n| < \infty$

(3) $E(M_{n+1} | \mathcal{F}_n) = M_n \quad \forall n$.

For part (1): $M_n = \dfrac{e^{t(X_1 + \cdots + X_n)}}{(E e^{tX_1})^n}$ is obviously $\mathcal{F}_n$ - measurable.

For part (2):

$$E|M_n| = \left| E\left(\frac{e^{t(X_1 + \cdots + X_n)}}{E(e^{tX_1})^n} \right) \right| = \left| \frac{E(e^{t(X_1 + \cdots + X_n)})}{E(e^{tX_1})^n} \right|$$

$$= \frac{|E(e^{tX_1})| \cdot |E(e^{tX_2})| \cdots |E(e^{tX_n})|}{|E(e^{tX_1})|^n} < \infty$$

For part (3):

$$E(M_{n+1} | \mathcal{F}_n) = E\left(\frac{e^{t(X_1 + X_2 + \cdots + X_{n+1})}}{[E(e^{tX_1})]^{n+1}} \middle| F_n \right)$$

$$= \frac{e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}}{m^n(t)} \cdot E\left(\frac{e^{tX_{n+1}}}{E(e^{tX_1})} \middle| F_n \right)$$

$$= M_n \cdot \frac{E(e^{tX_{n+1}})}{E(e^{tX_n})} = M_n.$$

$\therefore \{M_n\}$ is a martingale w.r.t. $\mathcal{F}_n$.

3. we first prove that

$\{M_n = (\frac{q}{p})^{X_n}\}$ is a martingale w.r.t. $\mathcal{F}_n$.

we have to prove:

(1) M_n is $\mathcal{F}_n$-measurable.

(2) $E|M_n| < \infty$

(3) $E(M_{n+1}|\mathcal{F}_n) = M_n$.

For (1) M_n only concerns about X_n, so it is $\mathcal{F}_n$-measurable

For (2) $E(|M_n|) = \sum_{k=-\infty}^{\infty} (\frac{q}{p})^k p(X_n = k) =$

$$= \sum_{k=-\infty}^{a} (\frac{q}{p})^k p(X_n=k) + \sum_{k=a+1}^{\infty} (\frac{q}{p})^k \cdot p(X_n=k) \leq$$

$$\leq \sum_{k=-\infty}^{a} (\frac{q}{p})^k q^{a-k} + \sum_{k=a+1}^{\infty} p^{k-a} \cdot (\frac{q}{p})^k \leq$$

$$\leq \frac{q^a \cdot p^{-a}}{1-p} + \frac{p^{-a} \cdot q^{a+1}}{1-q} < \infty.$$

For (3). $E(M_{n+1}|\mathcal{F}_n) = E((\frac{q}{p})^{X_{n+1}} | \mathcal{F}_n) = E[(\frac{q}{p})^{X_n} \cdot (\frac{q}{p})^{X_{n+1}-X_n} | \mathcal{F}_n] =$

$$= M_n \cdot E((\frac{q}{p})^{X_{n+1}-X_n} | \mathcal{F}_n).$$

$\because$ $X_{n+1} - X_n$ is independent of $\mathcal{F}_n$, $X_{n+1} - X_n = \begin{cases} 1 & \text{w.p. } (p) \\ -1 & \text{w.p. } (1-p) \end{cases}$

$\therefore E(M_{n+1}|\mathcal{F}_n) = M_n \cdot [(\frac{q}{p}) \cdot p + (\frac{p}{q}) \cdot q] = M_n.$

$\therefore \{M_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$.

3. Let G_n be a Filtration of natural number.

$\{T=n\}$ Is obviously G_n-measurable since when reach state 0 or N, it will stop.

T is a stopping time.

Then we can assume that.

$P(T>n) \le C \cdot \rho^n$, for some C and $0<\rho<1$

$\therefore E[|M_n| \cdot I(T>n)] \to 0$ is $n \to \infty$.

~~Using the~~ As is proved in class.

r. v. $Y \geq 0$ $E(Y)<\infty$, $A_1, A_2, \cdots A_n$ s.t.

$P\{A_1\} \to 0$ as $n \to \infty$

Then $E[Y \cdot I\{A_n\}] \to 0$ as $n \to \infty$.

now apply optimal samply theorem 2.

$E(M_T) = E(M_0)$

let $P_1 = P(X_T = 0)$.

then $P(X_T = N) = 1 - P_1$

$\therefore P_1 \left(\frac{q}{p}\right)^0 + (1-P_1)\cdot\left(\frac{q}{p}\right)^N = \left(\frac{q}{p}\right)^a \Rightarrow P_1 = \dfrac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N}$

$\therefore$ if $\frac{q}{p} > 1 \Leftrightarrow p < \frac{1}{2}$ $\lim_{N\to\infty} P\{X_T = 0\} = 1$

if $\frac{q}{p} < 1 \Leftrightarrow p > \frac{1}{2}$ $\lim_{N\to\infty} P\{X_T = 0\} = 0$.

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4. (1) If $T=0$.

$M_n = u(X_{T_n}) - 0 = u(X_0)$ is obviously a martingale

Now we discuss the case that $T \neq 0$.

$$M_n = u(X_{T_n}) - \sum_{j=0}^{T_n - 1} g(X_j), \quad n=0,1,2 \ldots$$

We want to prove that:

(a) M_n is $\mathcal{F}_n$-measurable.

(b) $E(|M_n|) < \infty$.

(c) $E(M_{n+1} | \mathcal{F}_n) = M_n$.

For (a) part since $T_n = \min(T, n)$.

$u(X_{T_n})$ is $\mathcal{F}_n$-measurable

$\sum_{j=1}^{T_n - 1} g(X_j)$ is $\mathcal{F}_n$-measurable.

$\therefore M_n$ is $\mathcal{F}_n$-measurable.

For (b). assume $g(x), f(x)$ are finite.

then $u(x)$ is finite, since it is finite-space.

$$E(|M_n|) = E\left(\left|u(X_{T_n}) - \sum_{j=0}^{T_n - 1} g(X_j)\right|\right) \leq$$

$$\leq E|u(X_{T_n})| + E\left(\left|\sum g(X_j)\right|\right) < \infty.$$

4. For (c)

$$M_{n+1}-M_n = I(T>n)(M_{n+1}-M_n).$$

$$= I(T>n)\left[u(X_{n+1})-u(X_n)-g(X_n)\right]$$

Therefore.

$$E(M_{n+1}|F_n) = E(M_n+M_{n+1}-M_n|F_n).$$

$$= M_n+E(M_{n+1}-M_n|F_n).$$

$$= M_n+E\Big(I(T>n)\big(u(X_{n+1})-u(X_n)-g(X_n)\big)\Big|F_n\Big).$$

$$= M_n+I(T>n)\,E(u(X_{n+1})-u(X_n)-g(X_n)|F_n).$$

$$= M_n+I(T>n)\left[E(u(X_{n+1}|F_n)-u(X_n)-g(X_n)\right]$$

$$= M_n+I(T>n)\left[u(X_n)+g(X_n)-u(X_n)-g(X_n)\right]$$

$$= M_n.$$

$\therefore \{M_n\}$ is a matingale w.r.t F_n.

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5. step1: if $\mu \neq 0$ suppose $X_1' = X_1 - \mu$ then $\mu' = EX_1' = 0$.

$EX_1 = \mu = 0$.

Step2-step3 ...

Step4: proof: we need to prove

(1) M_n is $\mathcal{F}_n$-measurable.

(2) $E(|M_n|) < \infty$.

(3) $E(M_{n+1} | \mathcal{F}_n) = M_n$.

part(1) obvious because $M_n = X_1 + \cdots + X_{T_n}$ is $\mathcal{F}_n$.

part(2) $E(|M_n|) \leq \sum_{i=1}^{T_n} E(|X_i|) < \infty$ by letting $T = T_n$.

part(3) $E(M_{n+1} | \mathcal{F}_n) = E(M_n + M_{n+1} - M_n | \mathcal{F}_n)$.

$$= M_n + E(M_{n+1} - M_n | \mathcal{F}_n)$$

$$= M_n + E(I(T > n) X_{n+1} | \mathcal{F}_n).$$

$$= M_n + I(T > n) E(X_{n+1} | \mathcal{F}_n) = M_n$$

$\therefore \{M_n\}$ is martingale w.r.t. $\mathcal{F}_n$.

Step5: From step 1-4, we satisfied the requirement of optimal sampling Theorem 3.

$\therefore EM_T = EM_0 = 0$.

$\therefore EM_T = 0$.