ISE 426
Optimization models and applications

Lecture 4 — September 8, 2015

- Linear programming: more examples
- Basic properties of LP problems

Reading: WV p. 56.

LP example #3: Short term financial planning[1]

- Your company makes tape recorders (TR) and radios (RD).
- Returns are:

  TR: 100$ (price) − 50$ (labor) − 30$ (raw m.)= 20$

  RD: 90$ (price)− 35$ (labor) − 40$ (raw m.) = 15$
- Raw material is sufficient to produce 100 TRs and 100 RDs

[1]See Winston & Venkataramanan, page 82.

LP example #3: Short term financial planning

Balance sheet (today):	Assets	Liabilities
Cash	\$10,000	
Accts. recv.	\$3,000	
Inv. outst.	\$7,000[2]	
Bank loan		\$10,000

Before the end of the month,

- we will collect soon \$2,000 of accts.
- we will receive new inventory worth \$2000
- we must pay \$1,000 of loan and another \$1,000 for rental
- management: "on 09/30 cash has to be at least \$4,000"
- bank requires that assets / liability ratio be at least 2

⇒ How many TRs and RDs do we produce this month to maximize return?

[2] $\$7000 = \$30 \times 100 + \$40 \times 100.$

LP example #3: Short term financial planning

- return on each TR is \$20, RD is \$15
- suppose t is #TR and r is #RD
- Balance sheet in a month:

BS (09/30):	Assets	Liabilities
Cash	\$10,000	
	$+\$2{,}000 - \$1{,}000 - \$1{,}000$	
	$-\$50t - \$35r$	
Accts. recv.	\$3,000	
	$-\$2{,}000 + \$100t + \$90r$	
Inv. outst.	\$7,000	
	$+\$2{,}000 - \$30t - \$40r$	
Bank loan		\$10,000
		$+\$2{,}000 - \$1{,}000$

LP example #3: Short term financial planning

- return on each TR is \$20, RD is \$15
- suppose t is #TR and r is #RD
- Balance sheet in a month:

BS (09/30):	Assets	Liabilities
Cash	$\$10{,}000 - \$50t - \$35r$	
Accts. recv.	$\$1{,}000 + \$100t + \$90r$	
Inv. outst.	$\$9{,}000 - \$30t - \$40r$	
Bank loan		\$11,000

- Cash $\geq \$4,000$ means $\$10,000 - \$50t - \$35r \geq \$4,000$

$\Rightarrow$ $\$50t + \$35r \leq \$6,000$

- Ratio ≥ 2 means $\dfrac{\text{Cash + Accts. recv. + Inv. outst.}}{\text{Bank loan}} \geq 2$

$$\Rightarrow \quad \frac{\$20,000 + \$20t + \$15r}{\$11,000} \geq 2$$

$\Rightarrow$ $\$20,000 + \$20t + \$15r \geq \$22,000$

$\Rightarrow$ $\$20t + \$15r \geq \$2,000$

LP example #3: Short term financial planning

$$
\begin{array}{llll}
\max & \$20t & +15r & \\
 & \$50t & +\$35r & \leq \$6,000 \\
 & \$20t & +\$15r & \geq \$2,000 \\
 & t & & \leq 100 \\
 & & r & \leq 100 \\
 & t, & r & \geq 0
\end{array}
$$

LP example #4: Project selection[4]

We have 5 investment opportunities over a 2-year term.

- i.e., we'll invest in the same funds this and next year
- each has two cash outflows, for 2009 and for 2010, and
- a Net Present Value (NPV)[3]
- available cash: 40 M$ this year, estimate 20 M$ next year

Investment		1	2	3	4	5
(a_i) Cash outflow,	2009	11	53	5	5	29
(b_i)	2010	3	6	5	1	34
(v_i) NPV		13	16	16	14	39

What investment(s) get the **maximum** total NPV? What percentage of each?

[3] The amount by which the investment will increase the company's value.
[4] Winston&Venkataramanan, example 10, page 80.

LP example #4: Project selection

- **Variables:** for each opportunity 1, 2,. . . , 5, the percentage of investment: $x_i \in [0, 1] \forall i = 1, 2, \ldots, 5$
- **Constraints:** limited cash to expend in 2009 and in 2010:

$$\sum_{i=1}^{5} a_i x_i \leq 40 \qquad \sum_{i=1}^{5} b_i x_i \leq 20$$

- **Objective function:** the total NPV (to be maximized)

$$\sum_{i=1}^{5} v_i x_i$$

LP example #4: Project selection

$$
\begin{array}{lrrrrrr}
\max & 13x_1 & +16x_2 & +16x_3 & +14x_4 & +39x_5 & \\
 & 11x_1 & +53x_2 & +5x_3 & +5x_4 & +29x_5 & \leq 40 \\
 & 3x_1 & +6x_2 & +5x_3 & +1x_4 & +34x_5 & \leq 20 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \in [0,1]
\end{array}
$$

Graphical solution of LP problems

Consider an LP problem with m constraints and **two variables**.

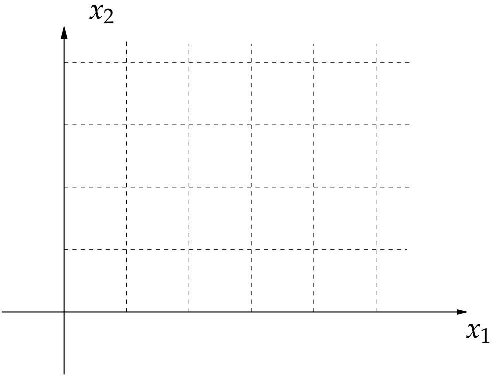
$$
\begin{array}{llll}
\min & c_1 x_1 & + c_2 x_2 & \\
 & a_{11} x_1 & + a_{12} x_2 & \leq b_1 \\
 & a_{21} x_1 & + a_{22} x_2 & \leq b_2 \\
 & \vdots & & \\
 & a_{m1} x_1 & + a_{m2} x_2 & \leq b_m
\end{array}
$$

- the objective function is associated with vector (c_1, c_2) in $\mathbb{R}^2$
- lines defined by $c_1 x_1 + c_2 x_2 = c_0$ correspond to solutions with the same objective function, c_0
- "$\leq$" and "$\geq$" constraints (i.e., *inequality* constraints) are associated with a half-plane of $\mathbb{R}^2$
- "$=$" constraints (or *equality* constraints) are associated with a line on the $\mathbb{R}^2$ plane.

Graphical solution of LP problems

Intersect all constraints $\Rightarrow$ the feasible set is a polyhedron.
$\Rightarrow$ Solve $\equiv$ find the point(s) with minimum $c_1x_1 + c_2x_2$.

$$\begin{array}{rl} \min & x_1 + x_2 \\ & 2x_1 + 3x_2 \geq 6 \\ & -x_1 + 2x_2 \leq 2 \\ & 5x_1 + x_2 \geq 10 \end{array}$$



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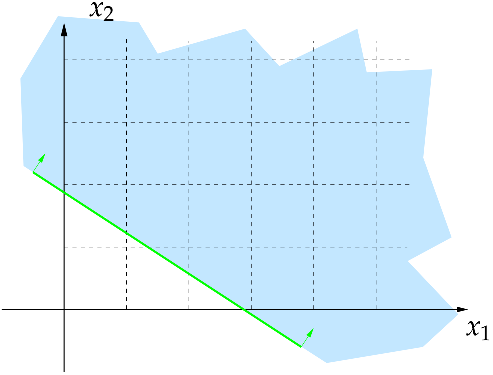

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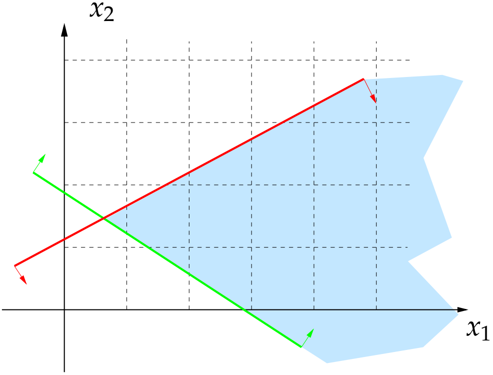

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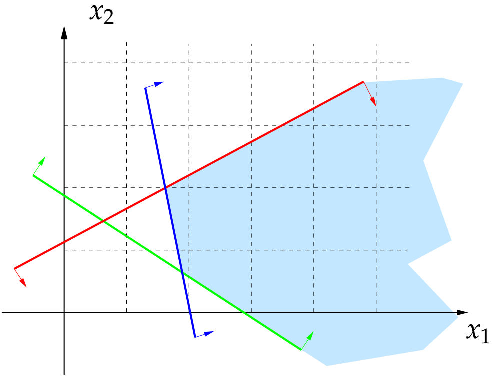

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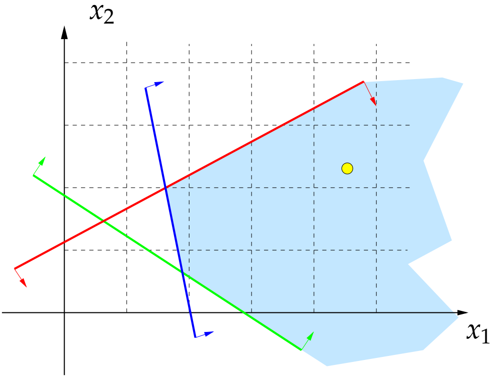

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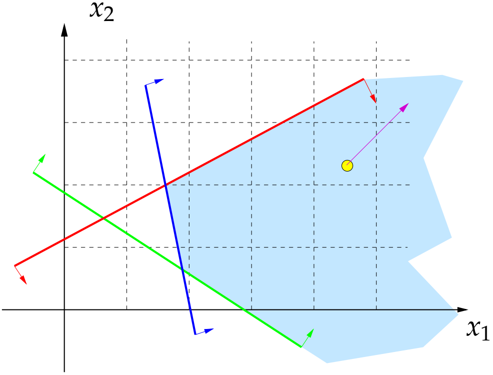

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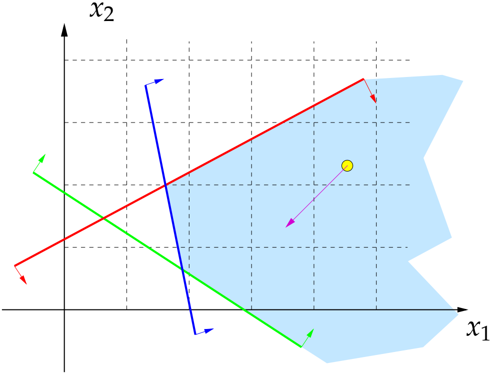

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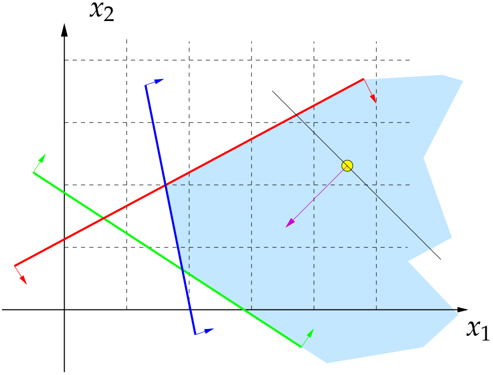

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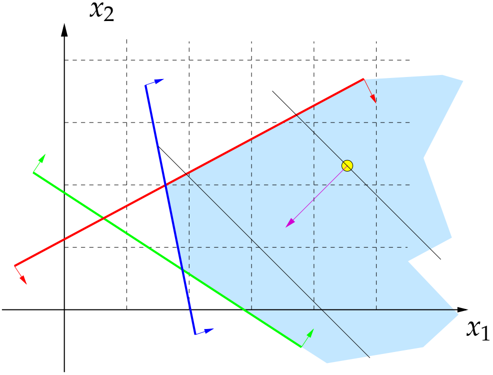

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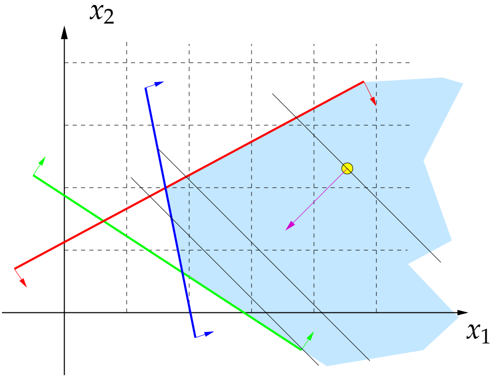

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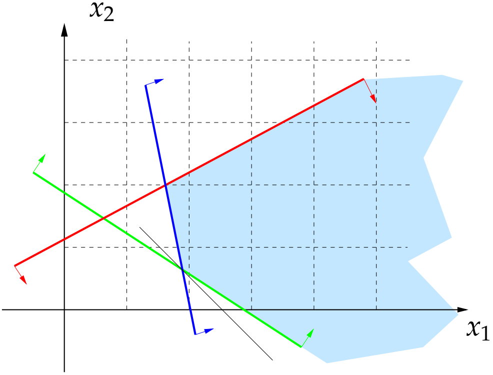

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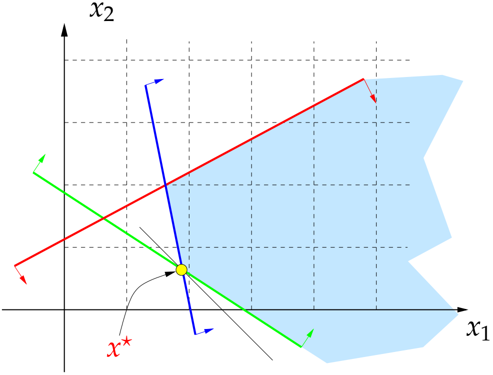

Remember the financial planning problem?

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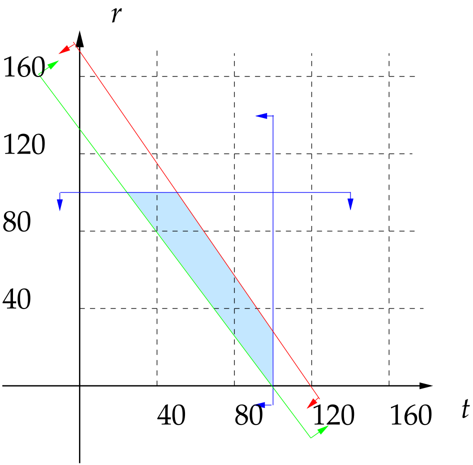

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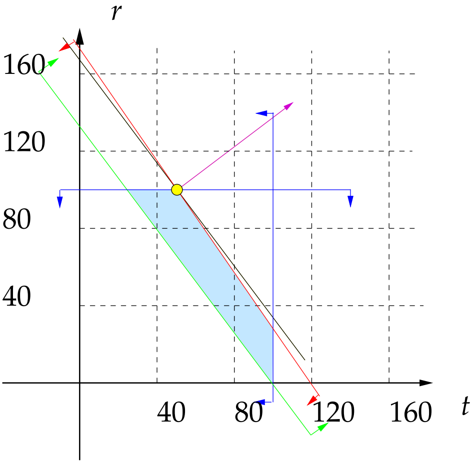
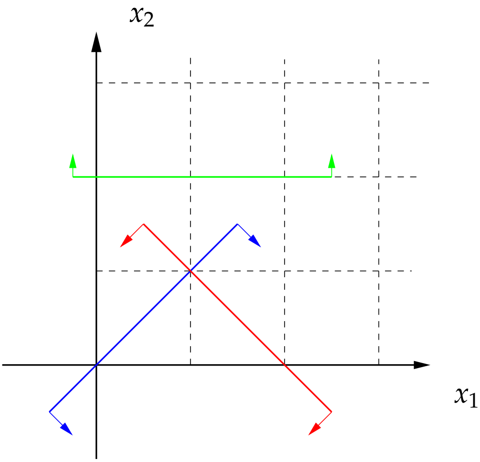


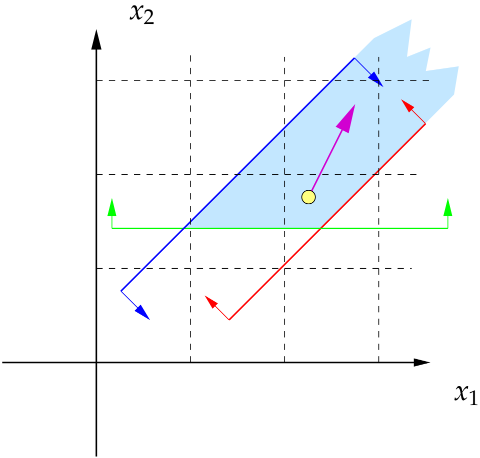
Example: Infeasible problem

$$\begin{aligned} \min \quad & x_1 + x_2 \\ & x_1 - x_2 \geq 0 \\ & x_1 + x_2 \leq 2 \\ & \quad\; x_2 \geq 2 \end{aligned}$$



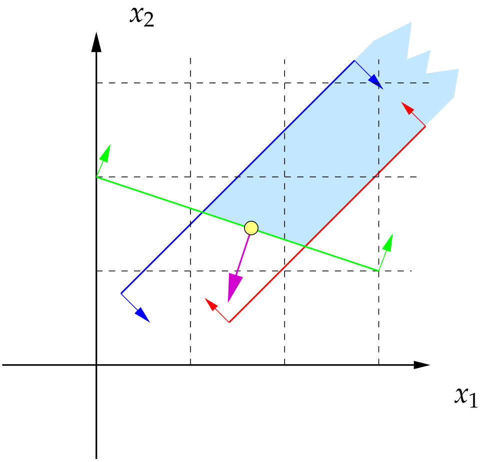
Example: Unbounded problem

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ & 2x_1 - 2x_2 \geq -1 \\ & x_1 - \ \ x_2 \leq 1 \\ & \qquad\ \ x_2 \geq \tfrac{3}{2} \end{aligned}$$



Example: Multiple optima

$$
\begin{array}{ll}
\min & x_1 + 3x_2 \\
& 2x_1 - 2x_2 \geq -1 \\
& x_1 - x_2 \leq 1 \\
& x_1 + 3x_2 \geq 6
\end{array}
$$



An LP problem can be...

Problems with two variables are easily classified as

- feasible and bounded (more than one optimum)
- unbounded
- infeasible

Degenerate problem

$$\begin{array}{rl} \min & x_1 + 3x_2 \\ & x_1 - 2x_2 \leq 0 \\ & 7x_1 + 2x_2 \geq 18 \\ & x_1 + \ \ x_2 \geq 3 \end{array}$$