ISE 426
Optimization models and applications

Lecture 11 — October 2, 2014

- Basic feasible solutions
- simplex method
- sensitivity analysis

Reminders:

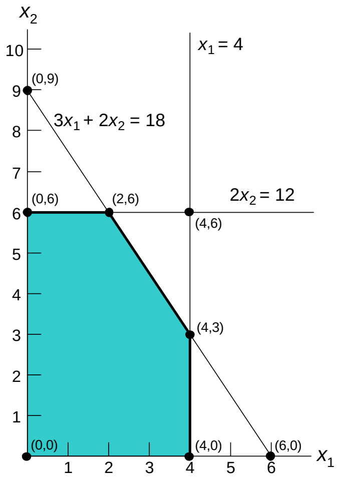
- Quiz on 10/14, practice on 10/09.

Simplex Method

Example

$$
\begin{array}{lrcrcr}
\text{maximize} & 3x_1 & + & 5x_2 & & \\
\text{subject to} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

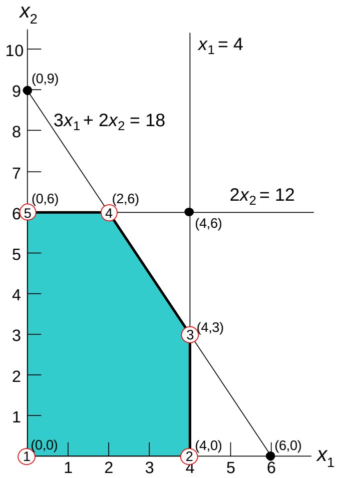
The lines are the **constraint boundaries**.



Corner-Point Solutions

5 **corner-point feasible (CPF) solutions**:

1. (0,0)
2. (4,0)
3. (4,3)
4. (2,6)
5. (0,6)



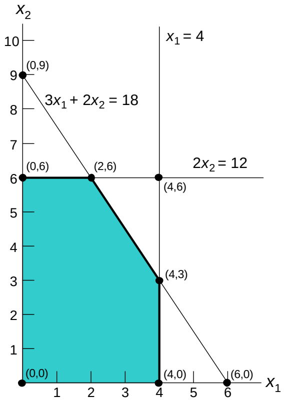
The Key Idea

Taken together, the two properties mean we can find an optimal solution by:

1. Starting at any CPF solution
2. Moving to a better adjacent CPF solution, if one exists
3. Continuing until the current CPF solution has no adjacent CPF solutions that are better

This is the essence of the simplex method.

The Dual



$$
\begin{array}{lrcrcrcr}
\text{minimize} & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{subject to} & u_1 & & & + & 3u_3 & \geq & 3 \\
 & & & 2u_2 & + & 2u_3 & \geq & 5 \\
 & u_1 & & & & & \geq & 0 \\
 & & & u_2 & & & \geq & 0 \\
 & & & & & u_3 & \geq & 0
\end{array}
$$

Example

$$\begin{array}{lrcrcl}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}$$

$$\begin{array}{lrcrcrcl}
\min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\
 & & & 2u_2 & + & 2u_3 & \geq & 5 \\
 & u_1 & & & & & \geq & 0 \\
 & & & u_2 & & & \geq & 0 \\
 & & & & & u_3 & \geq & 0
\end{array}$$

Consider optimal $(x_1, x_2) = (2, 6)$, compute dual from complementary slackness:

$$\begin{array}{l}
u_1(x_1 - 4) = 0 \\
u_2(+2x_2 - 12) = 0 \\
u_3(3x_1 + 2x_2 - 18) = 0 \\
x_1(u_1 + 3u_3 - 3) = 0 \\
x_2(2u_2 + 2u_3 - 5) = 0
\end{array}
\quad \Rightarrow \quad
\begin{array}{l}
u_1(-2) = 0 \\
u_2(0) = 0 \\
u_3(0) = 0 \\
2(u_1 + 3u_3 - 3) = 0 \\
6(2u_2 + 2u_3 - 5) = 0
\end{array}$$

Example

$$
\begin{array}{llllll}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{llllllll}
\min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\
 & & & 2u_2 & + & 2u_3 & \geq & 5 \\
 & u_1 & & & & & \geq & 0 \\
 & & & u_2 & & & \geq & 0 \\
 & & & & & u_3 & \geq & 0
\end{array}
$$

Consider optimal $(x_1, x_2) = (2, 6)$, compute dual from complementary slackness:

$$
\begin{aligned}
u_1 &= 0 \\
3u_3 &= 3 \\
2u_2 + 2u_3 &= 5
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
u_1 &= 0 \\
u_2 &= \tfrac{3}{2} \\
u_3 &= 1
\end{aligned}
$$

Example

$$\begin{array}{lrcrcl}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}$$

$$\begin{array}{lrcrcrcl}
\min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\
 & & & 2u_2 & + & 2u_3 & \geq & 5 \\
 & u_1 & & & & & \geq & 0 \\
 & & & u_2 & & & \geq & 0 \\
 & & & & & u_3 & \geq & 0
\end{array}$$

Consider a feasible CPF $(x_1, x_2) = (4, 3)$, compute dual complementary solution:

$$\begin{aligned}
u_1(x_1 - 4) &= 0 \\
u_2(+2x_2 - 12) &= 0 \\
u_3(3x_1 + 2x_2 - 18) &= 0 \\
x_1(u_1 + 3u_3 - 3) &= 0 \\
x_2(2u_2 + 2u_3 - 5) &= 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
u_1(0) &= 0 \\
u_2(-4) &= 0 \\
u_3(0) &= 0 \\
2(u_1 + 3u_3 - 3) &= 0 \\
6(2u_2 + 2u_3 - 5) &= 0
\end{aligned}$$

Example

$$\begin{array}{llllll} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{llllllll} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (4, 3)$, compute dual complementary solution:

$$\begin{aligned} u_2 &= 0 \\ u_1 + 3u_3 &= 3 \\ 2u_3 &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_1 &= -\tfrac{9}{2} \\ u_2 &= 0 \\ u_3 &= \tfrac{5}{2} \end{aligned}$$

Example

$$\begin{array}{lrcrcr} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcrcr} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (4, 3)$, compute dual complementary solution:

- $u_1 = -\frac{9}{2}$ is the shadow price for the constraint $x_1 \leq 4$ at the solution $x_1 = 4$.
- $u_1 < 0$ means that the objective function will improve if we allow $x < 4$ while we keep $3x_1 + 2x_2 = 18$.
- Consider $x_1 = 3$, then $x_2 = \frac{9}{2}$ and $3x_1 + 5x_2 = 3 * 4 + 5 * 3 + \frac{9}{2}$.

Example

$$\begin{array}{lccccc} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lccccccc} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (0, 6)$, compute dual complementary solution:

$$\begin{aligned} u_1(x_1 - 4) &= 0 \\ u_2(+2x_2 - 12) &= 0 \\ u_3(3x_1 + 2x_2 - 18) &= 0 \\ x_1(u_1 + 3u_3 - 3) &= 0 \\ x_2(2u_2 + 2u_3 - 5) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_1(-4) &= 0 \\ u_2(0) &= 0 \\ u_3(-6) &= 0 \\ 0(u_1 + 3u_3 - 3) &= 0 \\ 6(2u_2 + 2u_3 - 5) &= 0 \end{aligned}$$

Example

$$\begin{array}{llllll} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \le & 4 \\ & & & 2x_2 & \le & 12 \\ & 3x_1 & + & 2x_2 & \le & 18 \\ & x_1 & & & \ge & 0 \\ & & & x_2 & \ge & 0 \end{array}$$

$$\begin{array}{llllllll} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \ge & 3 \\ & & & 2u_2 & + & 2u_3 & \ge & 5 \\ & u_1 & & & & & \ge & 0 \\ & & & u_2 & & & \ge & 0 \\ & & & & & u_3 & \ge & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (0, 6)$, compute dual complementary solution:

$$\begin{aligned} u_1 &= 0 \\ 2u_2 &= 2 \\ u_3 &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} u_1 &= 0 \\ u_2 &= \tfrac{5}{2} \\ u_3 &= 0 \\ u_1 + 3u_3 - 3 &= -3 \end{aligned}$$

Example

$$\begin{array}{lrcrcr} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcrcr} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (0, 6)$, compute dual complementary solution:

- $s_1 = u_1 + 3u_3 - 3 = -3$ is the reduced cost for the variable $x_1 \geq 0$ at the solution $x_1 = 0$.
- $s_1 < 0$ means that the objective function will improve if we allow $x > 0$ while we keep $2x_2 = 12$.
- Consider $x_1 = 1$, then $x_2 = 6$ then $3x_1 + 5x_2 = 3 * 0 + 5 * 6 + 3$.

Changes in Objective Function Coefficients

- Suppose x^* is the optimal solution for an LP.
- Z^* is its optimal objective value.
- Suppose that some objective function coefficient c_j changes.

$$
\begin{array}{llcl}
\max Z = & 3x_1 + 5x_2 & & \\
\text{s.t.} & x_1 & \leq & 4 \\
 & 2x_2 & \leq & 12 \\
 & 3x_1 + 2x_2 & \leq & 18 \\
 & x_1 \;,\; x_2 & \geq & 0
\end{array}
$$

- What if the 3 changed?

Changes in c, cont'd

$$
\begin{array}{llcl}
\max Z = 3x_1 + 5x_2 & & & \\
\text{s.t.} & x_1 & & \leq 4 \\
 & & 2x_2 & \leq 12 \\
 & 3x_1 + & 2x_2 & \leq 18 \\
 & x_1 \quad , & x_2 & \geq 0
\end{array}
$$

- If 3 changed to 0, what would the new solution be?
- If 3 changed to 30, what would the new solution be?
- If the 3 changed to $3 \pm \delta$, where δ is tiny, what would the new solution be?

Example: Maximization

$$
\boxed{\begin{array}{lrcrl}
\max Z = & 3x_1 & + & 5x_2 & \\
\text{s.t.} & x_1 & & & \leq 4 \\
 & & & 2x_2 & \leq 12 \\
 & 3x_1 & + & 2x_2 & \leq 18 \\
 & x_1 & , & x_2 & \geq 0
\end{array}}
$$

- Optimal solution is $(x_1^*, x_2^*) = (2, 6)$, $Z^* = 36$.
- Suppose the 3 increased to 7.
- Which of the following is true?
  1. Z^* will increase.
  2. Z^* will decrease.
  3. Z^* will stay the same.
  4. We can't say.

Example: Maximization, cont'd

$$
\boxed{\begin{array}{llcl}
\max Z = 3x_1 + 5x_2 & & & \\
\text{s.t.} & x_1 & & \leq 4 \\
& & 2x_2 & \leq 12 \\
& 3x_1 + & 2x_2 & \leq 18 \\
& x_1 \quad , & x_2 & \geq 0
\end{array}}
$$

- $(x_1^*, x_2^*) = (2, 6)$, $Z^* = 36$.
- 3 increases to 7.
- The local rate of increase is derived from $x_1^* \delta = 2 \times 4 = 8$.
- Which of the following is true?
    1. Z^* will increase by *exactly* 8.
    2. Z^* will increase by *at most* 8.
    3. Z^* will increase by *at least* 8.
    4. Z^* will increase, but we don't know by how much.

Changes in Constraint Right-Hand Sides

$$
\boxed{\begin{array}{llcl}
\max Z = & 3x_1 + 5x_2 & & \\
\text{s.t.} & x_1 & \leq & 4 \\
& 2x_2 & \leq & 12 \\
& 3x_1 + 2x_2 & \leq & 18 \\
& x_1 \; , \; x_2 & \geq & 0
\end{array}}
$$

- Suppose 12 increased.
- Would optimal solution change?
- Would optimal objective value change?
- Would optimal basis change?

Example

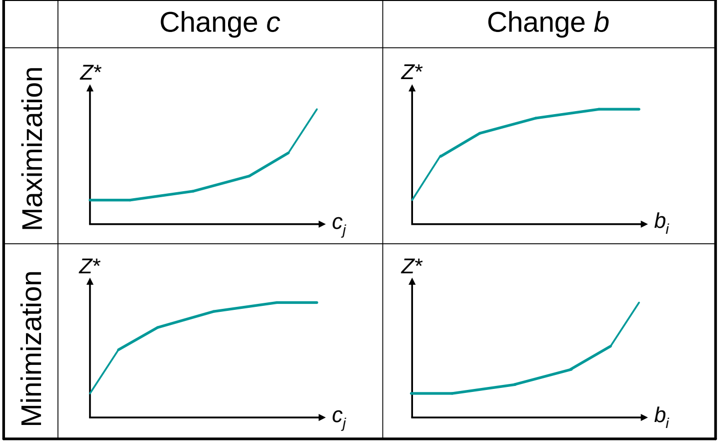
$$\begin{array}{llcl} \max\ Z = 3x_1 + 5x_2 & & & \\ \text{s.t.} & x_1 & & \leq 4 \\ & & 2x_2 & \leq 12 \\ & 3x_1 + & 2x_2 & \leq 18 \\ & x_1\ , & x_2 & \geq 0 \end{array}$$

- Optimal solutions are $(x_1^*, x_2^*) = (2, 6)$, $(u_1^*, u_2^*, u_3^*) = (0, \frac{3}{2}, 1)$, $Z^* = 36$.
- Suppose the 12 increased to 16.
- Which of the following is true?
  1. Z^* will increase.
  2. Z^* will decrease.
  3. Z^* will stay the same.
  4. We can't say.

Example, cont'd

$$\begin{array}{llcl}
\max Z = 3x_1 + 5x_2 & & & \\
\text{s.t.} & x_1 & & \leq 4 \\
& & 2x_2 & \leq 12 \\
& 3x_1 + & 2x_2 & \leq 18 \\
& x_1 \quad , & x_2 & \geq 0
\end{array}$$

- $(x_1^*, x_2^*) = (2, 6)$, $(u_1^*, u_2^*, u_3^*) = (0, \frac{3}{2}, 1)$, $Z^* = 36$.
- 12 increases to 16.
- The local rate of increase come from the shadow price: $u_2^* \delta = \frac{3}{2} \times 4 = 6$.
- Which of the following is true?
  1. Z^* will increase by *exactly* 6.
  2. Z^* will increase by *at most* 6.
  3. Z^* will increase by *at least* 6.
  4. Z^* will increase, but we don't know by how much.



Change c
Change b
Maximization
Minimization
Z^*
c_j
Z^*
b_i
Z^*
c_j
Z^*
b_i

Relationship to Complementary Slackness

- Suppose there is slack in the ith primal constraint.
  - Increasing the RHS would not change the optimal solution.
  - By complementary slackness, u_i^* must equal 0 (in the dual).
  - Using the statement on the previous slides, the optimal objective function changes by u_i^*, or 0.
- Suppose there is no slack in the ith primal constraint.
  - Increasing the RHS *would* change the optimal solution.
  - u_i^* *probably* (!) is greater than 0.
  - Using the statement above, the optimal objective function changes by u_i^*.
- This agrees with our interpretation of the dual values as *shadow prices*.

Relationship to Complementary Slackness, cont'd

- Now suppose there is slack in the jth dual constraint.
  - By complementary slackness, $x_j^* = 0$ (in the primal).
  - If we increase c_j slightly, we'll still want to set $x_j^* = 0$.
  - We argued that for each unit increase in c_j, Z^* changes by x_j^* (if optimal basis stays the same).
  - So Z^* increases by 0 when c_j increases.
- Suppose there is no slack in the jth dual constraint (reduced cost is 0).
  - $x_j^* > 0$ (probably).
  - If we increase c_j by 1, the objective value will go up by x_j^*.