IE426 – Optimization models and applications

Fall 2015 – Homework #4

This homework accounts for 5% of the final grade. It is due on Tuesday, November 17, in class. There are 20 points available for Problems 1-4. Problems 5-8 ARE NOT REQUIRED. No credit will be given for them, they are simply additional practice problems.

1 Integer Programming/Scheduling Formulation (5 pts.)

A computing cluster has n processors and $m > n$ tasks to be run, each on a single processor. Each task $j \in M = \{1, 2 \ldots, m\}$ requires t_j hours to complete (t_j is an integer parameter), and when a task is spawned on a processor $i \in N = \{1, 2 \ldots, n\}$, that processor only runs task j until its completion.

Formulate a model for loading the tasks onto the processors in such a way that the whole cluster becomes totally free again as soon as possible, that is, we want to minimize the time of completion of all tasks given that all of them are executed.

For each variable and constraint, briefly describe what they mean.

2 Integer Programming/Matching Formulation (5 pts.) This problem is from final of 2011.

After finishing his laundry, Bob realizes that he has 20 different socks, none of which really match any other (probably a familiar situation). Many of them are sort of similar though, and some pairs would be less embarrassing to wear together than others. Say a white sock from GAP and a white sock from Eddie Bauer are quite similar; but a black and a white one cannot be worn together without considerable embarrassment.

A measure of how well two socks would fit together is given by the number

$$e_{ij},$$

which is the embarrassment caused by wearing sock i and sock j together.

1. Set up an MIP whose solution will match the socks to minimize Bob's embarrassment throughout the next 10 days. We assume that he changes socks every day.

2. Suppose that Bob plans to wear shorts on 5 days, and long pants on 5 other days. Some pairs of socks are more embarrassing to wear with shorts than with long pants. Precisely,

   $$s_{ij}$$

   is the embarrassment caused if socks i and j are worn together while wearing shorts. Also,

   $$\ell_{ij}$$

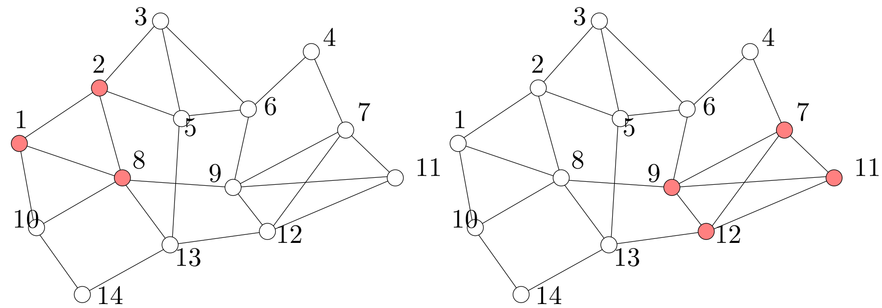


Figure 1: Examples of groups of friends: $\{1, 2, 8\}$ and $\{7, 9, 11, 12\}$. I believe the latter is the largest one for this example.

is the embarrassment caused if socks i and j are worn together while wearing long pants. Modify your MIP formulation to accommodate this change. We still assume that he changes socks every day.

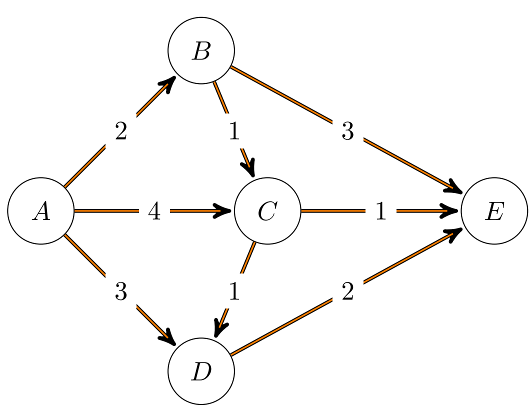
3 Graph optimization (5 pts.)

A social network has n people registered and connected to their friends just like any other social network. Among one of the projects running in the company is one that will definitely need some Optimization knowledge. They are looking for the largest group S of people that are completely connected to one another, i.e., the largest subset S of users such that any two people i and j in S are mutual friends, with a *direct* connection (aka a "clique"). Examples of subsets S are given in Figure 1.

1. Formulate the problem of finding the largest group S of mutual friends as an Integer Programming problem. Clearly express the meaning of each variable and constraints. (Hint: this formulation is not trivial to figure out, but it has a very simple form)

2. Given the optimal solutions to the maximum clique problem, formulate the problem of finding the next largest clique, which is not part of the original largest clique (at least one person should be new.

4 Flow problem and goal programming (5 pts.)



1. Formulate the problem of sending 8 units of flow from A to E as a linear programming problem, using the formulations studied in this course. The numbers on the arcs are the capacities. Do not use an objective function - the problem is infeasible. Show this by finding the min cut whose value is smaller than 8.

2. Assume that the condition that 8 units of flow have to leave A is not relaxable (it has to be satisfied). You have two groups of other constraints - the flow conservation constraints and the capacity constraints.

   Consider preemptive goal programming formulation where you first want to satisfy the capacity constraints and only after that you care to satisfy the flow conservation constraints. Show the formulation that you need to solve for this goal programming.

3. Now, consider preemptive goal programming where you switch the priorities of the constraints (flow conservation is more important than capacities). Show the formulation.

4. Can you figure out the solutions for each of the preemptive approaches and show how they are different from each other?

5. Now, formulate a problem that keeps all flow constraints feasible (and sends 8 units of flow from A to E), but relaxes the capacity constraints in the following way: each arc can have its capacity doubled or not increased at all. Minimize the **number** of arcs for which the capacity is doubled.

5 Reformulate using constraints of a Linear Programming problem or, if necessary, Mixed Integer Linear Programming Problem (12pts.)

1. $\max\{|x|, |y|\} \leq 1$.(6pts)
2. $\max\{|x|, |y|\} \geq 1$.(6pts)

6 Formulation, Mixed Integer/Goal Programming

Kyra is organizing a large dinner party. There are k tables, each sitting n people. There are m men attending and w women. She needs to assign seats at the tables in such a way that the number of men and women at each table does not differ by more than two. Formulate this as a feasible set of an integer linear programming problem.

Is the above problem *always* feasible? Explain.

Write an integer linear optimization problem that minimizes the number of tables that violate the condition on the maximum difference between the number of men and women.

7 Reformulation using binary variables (8 pts.)

Consider a set of vectors $x \in R^n$ described by the following conditions.

$$\max_i \{x_1, x_2, x_3, \ldots, x_n\} \geq 1$$

Describe this set by using a set of linear constraints and binary variables, as we did in homework and class. In other words, you only should use linear constraints that can involve continuous and/or binary variables, and all feasible solutions for this set of constraints should give x that is feasible for the the above set and vice versa.

8 Integer Programming

Consider a graph $G = (V, E)$, in Figure 2, and a cost C_{ij} for each edge $\{i, j\} \in E$. Suppose you want to find the subset S of V with at least k nodes, such that the cost of all edges, that link a node in S with a node outside of S is minimized. For example, if S is the set of four dark nodes is the graph in Figure 2, then the total cost of all edges connecting S to other nodes is $C_{12}+C_{14}+C_{15}+C_{23}+C_{34}+C_{36}$

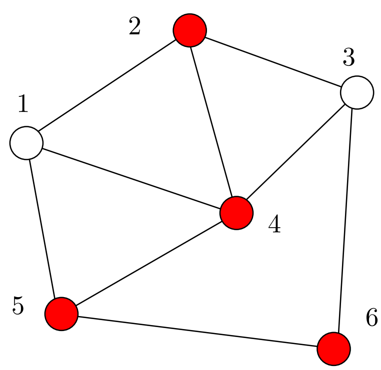


Figure 2:

1. Consider binary variables that indicate if a node is in S or not. Consider also binary variables that indicate if an edge is connecting a node in S with a node outside of S. Now write down conditions between these types of variables, which ensure logical implications: for all $\{i, j\} \in E$, if $i \in S$ and $j \in V/S$ then edge $\{i, j\}$ connects node in S with a node outside S. (5pts)

2. Write the full formulation of the problem of selecting at least k nodes so that the edge cost is minimized. (5pts).