ISE 426
Optimization models and applications

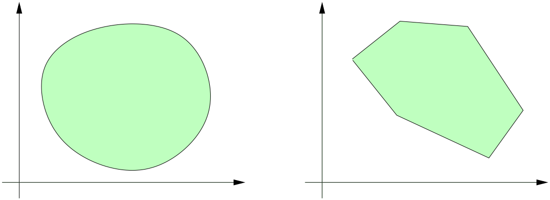
Lecture 2 — September 2, 2014

Convexity; Relaxations; Lower and upper bounds.

- Winston, chapter 1, **or**
- Winston & Venkataramanan, chapter 1, **or**
- Hillier & Lieberman, chapter 2.

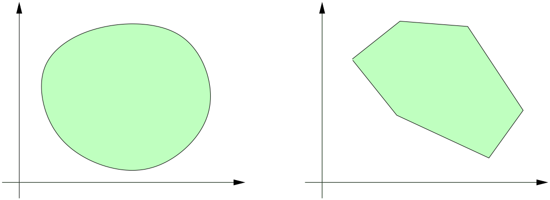
Convexity

Convex sets



Def.: A set $S \subseteq \mathbb{R}^n$ is convex if any two points x' and x'' of S are joined by a segment **entirely** contained in S:

Convex sets



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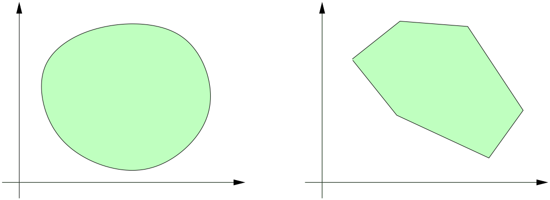

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The intersection of two convex sets is convex.

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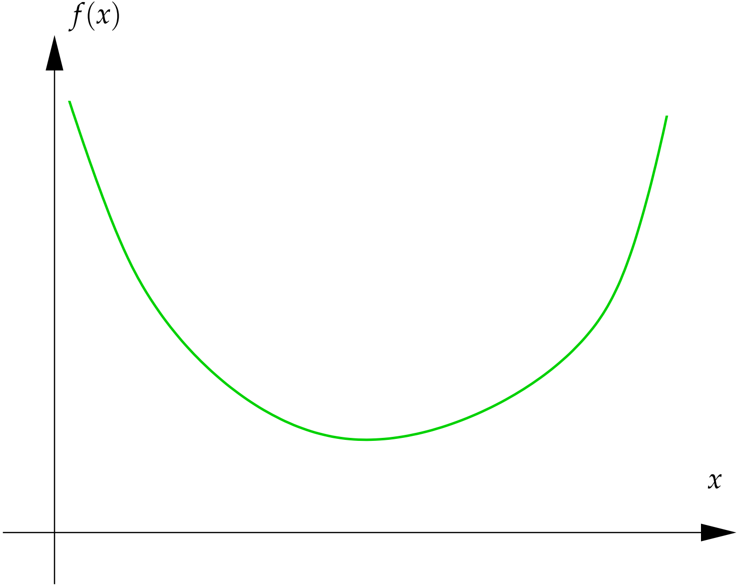

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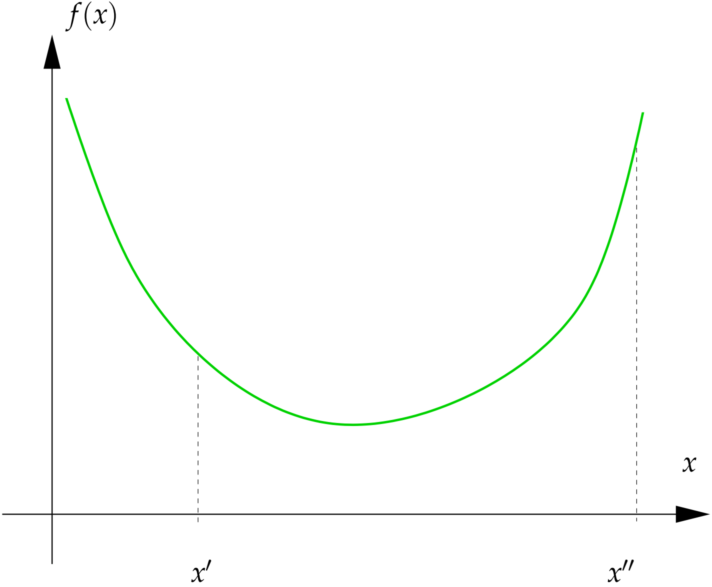

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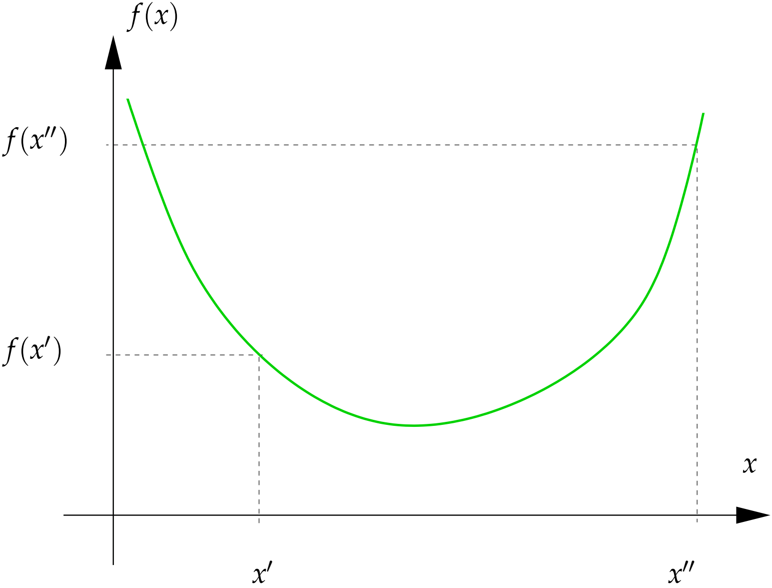

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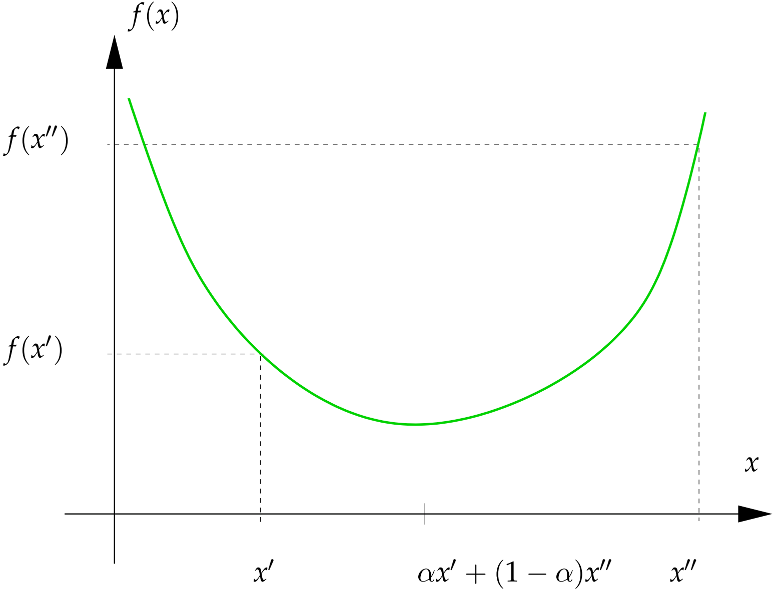

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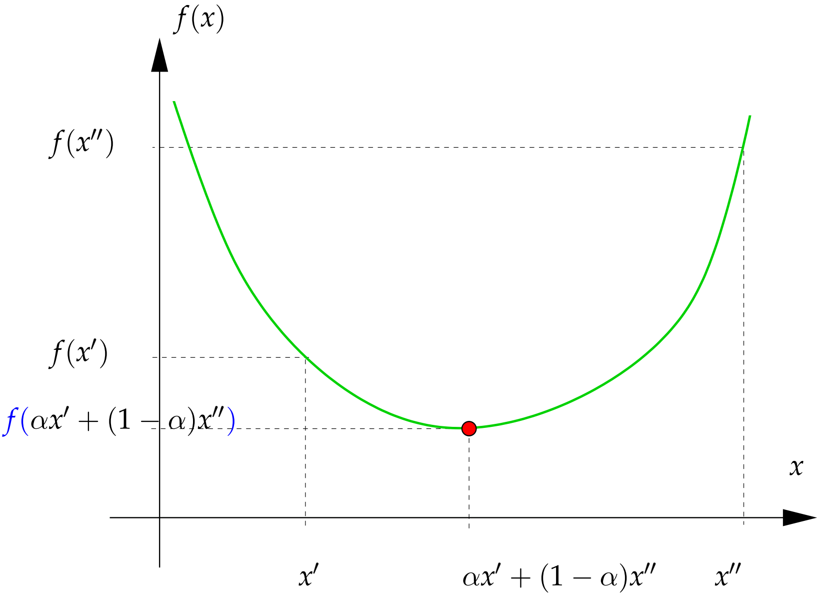

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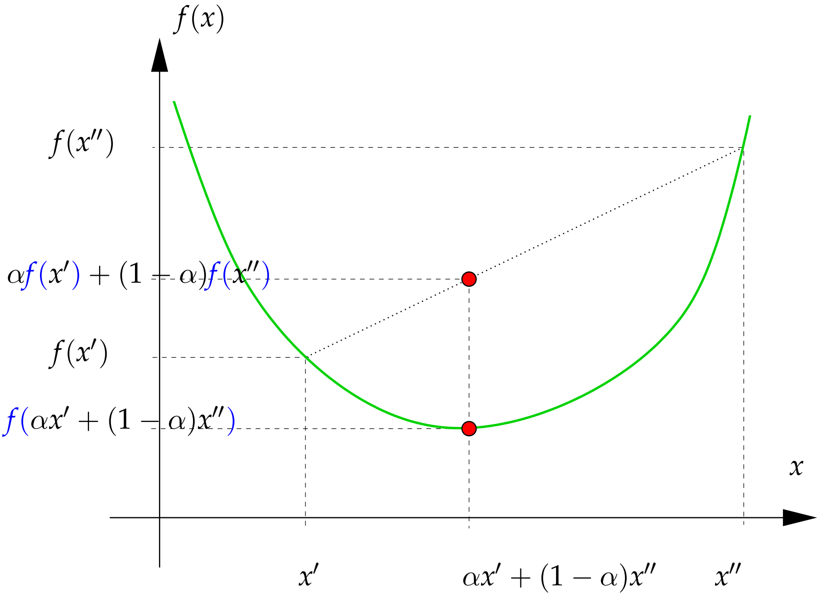
$f(x)$
$f(x'')$
$\alpha f(x') + (1-\alpha) f(x'')$
$f(x')$
$f(\alpha x' + (1-\alpha)x'')$
x
x'
$\alpha x' + (1-\alpha)x''$
x''

Convex functions

Def.: A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if, for any two points x' and $x'' \in \mathbb{R}^n$ and for any $\alpha \in [0, 1]$

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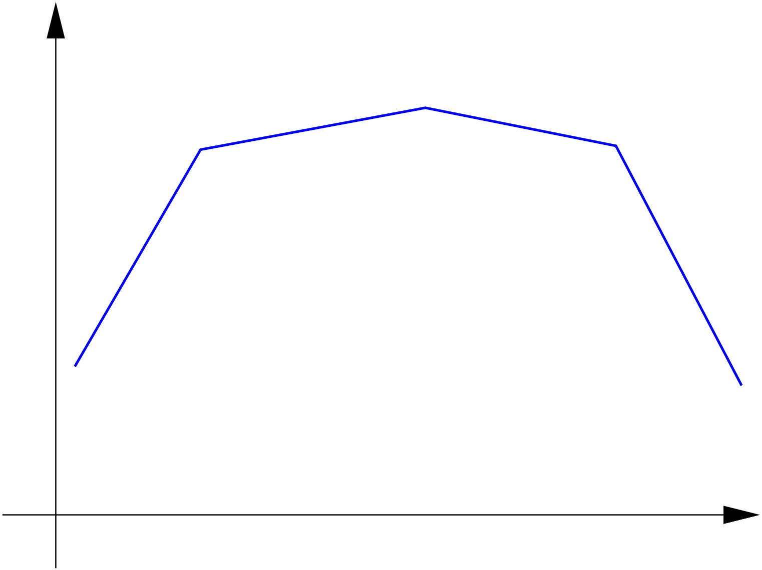
- The sum of convex functions is a convex function
- Multiplying a convex function by a positive scalar gives a convex function
- **linear** functions $\sum_{i=1}^{k} a_i x_i$ are convex, irrespective of the sign of a_i's.

Definition



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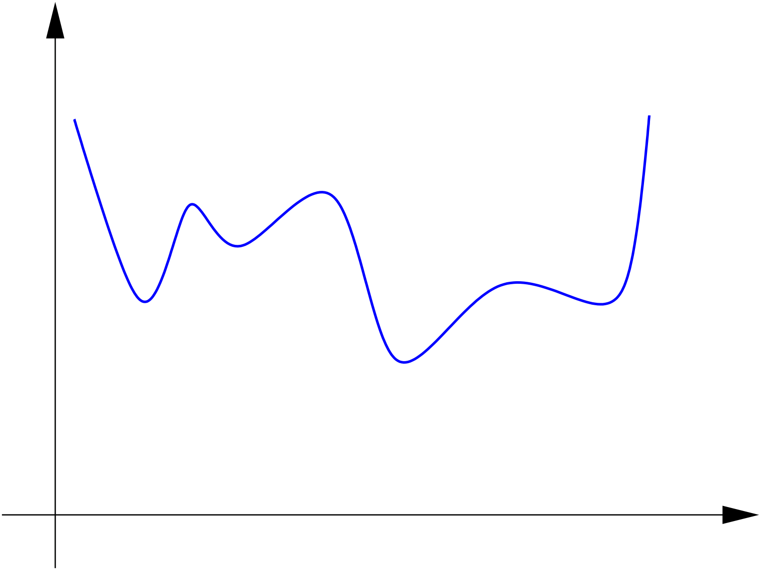

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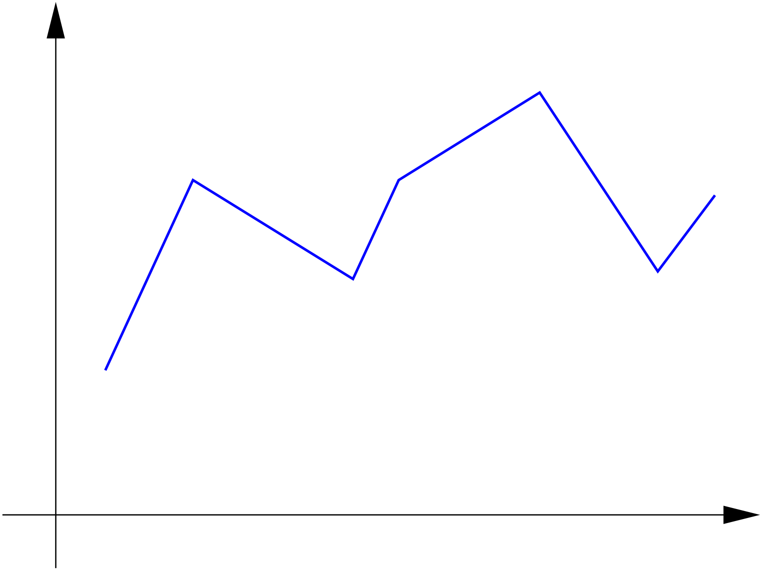

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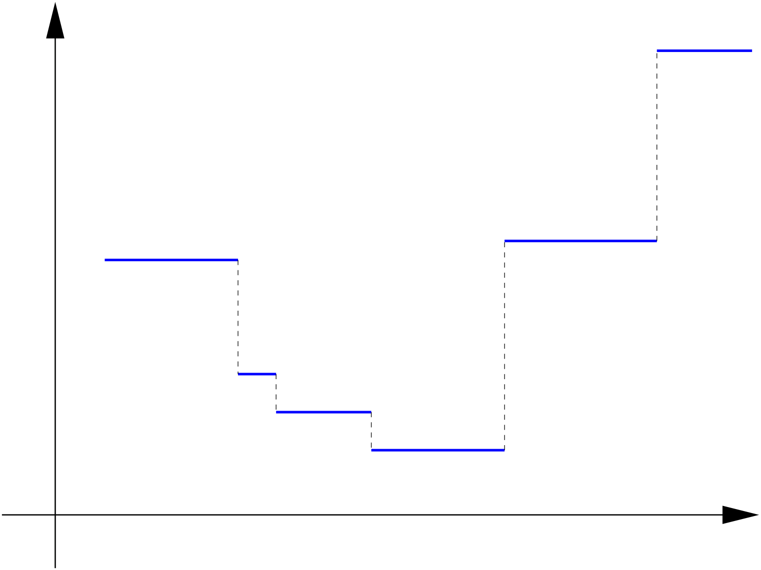


Local and global optima

A vector $x^l \in \mathbb{R}^n$ is a local optimum if

- there is a *neighbourhood*[1] N of x^l with no better point than x^l:

$$\forall x \in N, f_0(x) \geq f_0(x^l)$$

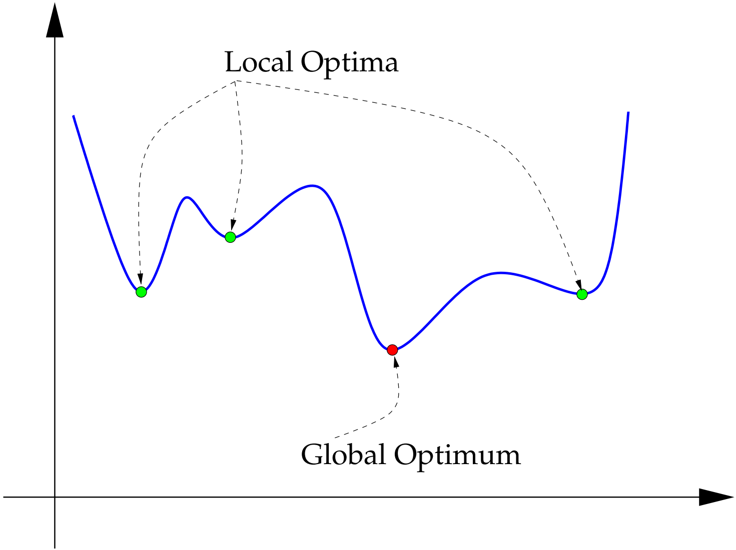
A vector $x^g \in \mathbb{R}^n$ is a global optimum if

- x^g
- there is no x better than x^g, i.e.,

$$f_0(x) \geq f_0(x^g) \qquad \forall x$$

[1] A *neighbourhood* of x^l can be defined as $N = \{x : ||x - x^l||_2 \leq \epsilon\}$ for some ϵ.

Local optima, global optima



Positive (Semi)Definite Matrices

A square $n \times n$ matrix A is **Positive Definite** (PD) (denoted with $A \succ 0$) if, for any n-vector $x \neq 0$, the following holds:

$$x^\top A x \geq 0$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j =$$

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$$\begin{array}{llll}
a_{11}x_1^2 & +a_{12}x_1x_2 & +\dots & +a_{1n}x_1x_n + \\
a_{21}x_2x_1 & +a_{22}x_2^2 & +\dots & +a_{2n}x_2x_n + \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}x_nx_1 & +a_{n2}x_nx_1 & +\dots & +a_{nn}x_n^2
\end{array}$$

$$> 0$$

Some linear algebra

- A **minor** of a $m \times n$ matrix A is the determinant of a (square) submatrix of A obtained by removing some rows and columns of A.
- For a square matrix B, a **principal minor** of B is obtained by removing the same row and column indices from B
- For a square matrix B, a **leading principal minor** of B is a principal minor obtained by removing the last k rows and (the same) columns from B

For example, $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 0 \end{pmatrix}$.

Leading principal minors: (1), $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 8 \end{pmatrix}$

Positive (Semi)Definite Hessian Matrices

A symmetric matrix $A \in R^{n \times n}$ is **Positive Semidefinite** (PSD) (denote it with $A \succeq 0$) if all principal minors of A are nonnegative.

A symmetric matrix $A \in R^{n \times n}$ is **Positive Semidefinite** (PSD) if and only if $A = BB^\top$ for some $B \in R^{n \times n}$.

A symmetric matrix $A \in R^{n \times n}$ is **Positive Semidefinite** (PSD) if for all $i = 1, \ldots, n$ $a_{ii} \geq \sum_{j=1, j \neq i}^{n} |a_{ij}|$.

A symmetric matrix $A \in R^{n \times n}$ is **Positive Semidefinite** (PSD) if all its eigenvalues are nonnegative.

Positive (Semi)Definite Hessian Matrices

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, consider its **Hessian**:

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** on set $\Omega \in R^n$ if and only if the Hessian $H_f(x) \succeq 0$ for all $x \in \Omega$.

Examples

- The function $f(x) = x$ is convex
- The function $f(x_1, x_2) = x_1 + x_2$ is convex
- The function $f(x_1, x_2) = x_1^2 + x_2$ is convex
- The function $f(x_1, x_2) = 5x_1^2 + 3x_2^2$ is convex
- The function $f(x_1, x_2) = x_1^2 + x_2^2 - x_1x_2$ is convex

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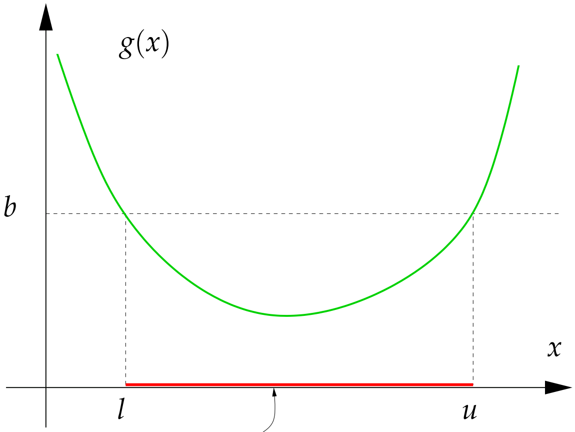
- The function $f(x_1, x_2) = x_1^2 + x_2^2 + 5x_1x_2$ is nonconvex
- The function $f(x_1, x_2) = x_1^2 - x_2^2$ is nonconvex
- The function $f(x_1, x_2) = x_1x_2$ is nonconvex
- The function $f(x) = \sin x$, for $x \in [0, 2\pi]$ is nonconvex
- The function $f(x) = -x^2$ is nonconvex

Convex constraints

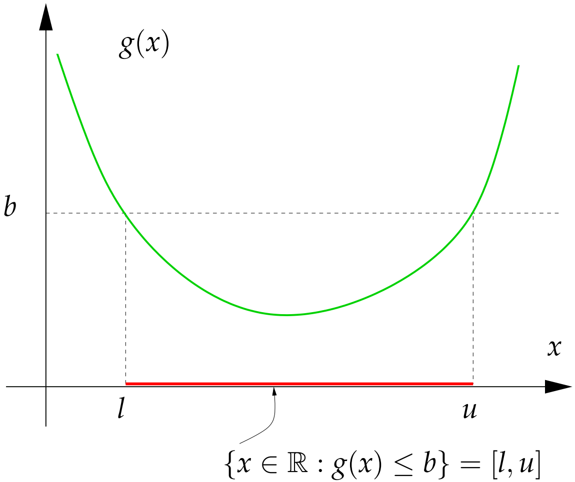
- A constraint $g(x) \leq b$, with $g : \mathbb{R}^n \to \mathbb{R}$, defines a subset S of $\mathbb{R}^n$, that is,
$$S = \{x \in \mathbb{R}^n : g(x) \leq b\}$$
- The constraint $g(x) \leq b$ is convex if the set S is convex.
- If the *function* $g(x)$ is convex, the *constraint* $g(x) \leq b$ is convex.

Convex constraints



$$\{x \in \mathbb{R} : g(x) \leq b\} = [l, u]$$

Convex constraints



Note: if the *function* $g(x)$ is convex, the *constraint* $g(x) \geq b$ may be nonconvex!

Convex and concave functions

Def.: A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if $-f(x)$ is convex.

- concave functions are useful with **maximization** problems:

$$\begin{aligned} \max \quad & f(x) \\ & g_i(x) \leq 0 \quad \forall i = 1, 2 \ldots, m \\ & h_i(x) = 0 \quad \forall i = m+1, m+2 \ldots, m+q \\ & x \in \mathbb{R}^n \end{aligned}$$

  is a convex problem if $f(x)$ is concave, all $g_i(x)$ are convex, and all $h_i(x)$ are affine.

- concave functions are also useful for $\geq$ constraints:

$$g_i(x) \geq 0$$

  is a convex constraint if $g_i(x)$ is concave.

For what convex $g(x)$ the *constraint* $g(x) \geq b$ is always convex?

For what convex $g(x)$ the *constraint* $g(x) \geq b$ is always convex?

- linear constraints $\sum_{i=1}^{k} a_i x_i \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b$ are convex

The general optimization problem

Consider a vector $x \in \mathbb{R}^n$ of variables.
An optimization problem can be expressed as:

$$
\begin{aligned}
\mathbf{P}: \qquad \text{minimize} \quad & f_0(x) \\
\text{s.t.} \quad & f_1(x) \leq b_1 \\
& f_2(x) \leq b_2 \\
& \vdots \\
& f_m(x) \leq b_m
\end{aligned}
$$

Feasible solutions, local and global optima

Define $F = \{x \in \mathbb{R}^n : f_1(x) \leq b_1, f_2(x) \leq b_2, \ldots, f_m(x) \leq b_m\}$, that is, F is the feasible set of an optimization problem.
All points $x \in F$ are called feasible solutions.

A vector $x^l \in \mathbb{R}^n$ is a local optimum if

- $x^l \in F$
- there is a *neighbourhood*[2] N of x^l with no better point than x^l:

$$\forall x \in N \cap F, f_0(x) \geq f_0(x^l)$$

A vector $x^g \in \mathbb{R}^n$ is a global optimum if

- $x^g \in F$
- there is no $x \in F$ better than x^g, i.e.,

$$f_0(x) \geq f_0(x^g) \qquad \forall x \in F$$

[2] A *neighbourhood* of x^l can be defined as $N = \{x : ||x - x^l||_2 \leq \epsilon\}$ for some ϵ.

Convex problems

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- the objective function is convex
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Convex optimization problems are *easy*:

> **If a problem P is convex, a local optimum $x^\star$ of P is also a global optimum of P.**

(Hint) When modeling an optimization problem, it would be good if we found a convex problem.

Examples of convex problems

$$
\mathbf{P}: \quad \begin{array}{rl} \min & x_1^2 + 2x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 \leq 1 \\ & 0 \leq x_1 \leq 2 \\ & 1 \leq x_2 \leq 5 \\ & x_2 \in \mathbb{Z} \end{array}
$$

Examples of ”hidden” convex problems

$$
\begin{array}{llr}
\mathbf{P}: & \min & x_1 - 2x_2^2 \\
& \text{s.t.} & x_1^2 + x_2^2 \le 1 \\
& & x_2 = 0 \\
& & 0 \le x_1 \le 5
\end{array}
$$

Your first Optimization model

Variables	r: radius of the can's base
	h: height of the can
Objective	$2\pi rh + 2\pi r^2$ (minimize)
Constraints	$\pi r^2 h = V$
	$h > 0$
	$r > 0$

Relaxations

Relaxation of an Optimization problem

Consider an optimization problem

$$
\mathbf{P}: \qquad \begin{array}{rl} \min & f_0(x) \\ \text{s.t.} & f_1(x) \leq b_1 \\ & f_2(x) \leq b_2 \\ & \vdots \\ & f_m(x) \leq b_m, \end{array}
$$

Let us denote F the set of points x that satisfy all constraints:

$$
F = \{x \in \mathbb{R}^n : \begin{array}{l} f_1(x) \leq b_1, \\ f_2(x) \leq b_2, \\ \vdots \\ f_m(x) \leq b_m\} \end{array}
$$

So we can denote $\mathbf{P}: \min\{f_0(x) : x \in F\}$ for short.

Relaxation of an Optimization problem

Consider a problem $\mathbf{P} : \min\{f_0(x) : x \in F\}$.

A problem $\mathbf{P}' : \min\{f_0'(x) : x \in F'\}$ is a **relaxation** of $\mathbf{P}$ if:

- $F' \supseteq F$
- $f_0'(x) \leq f_0(x)$ for all $x \in F$.[3]

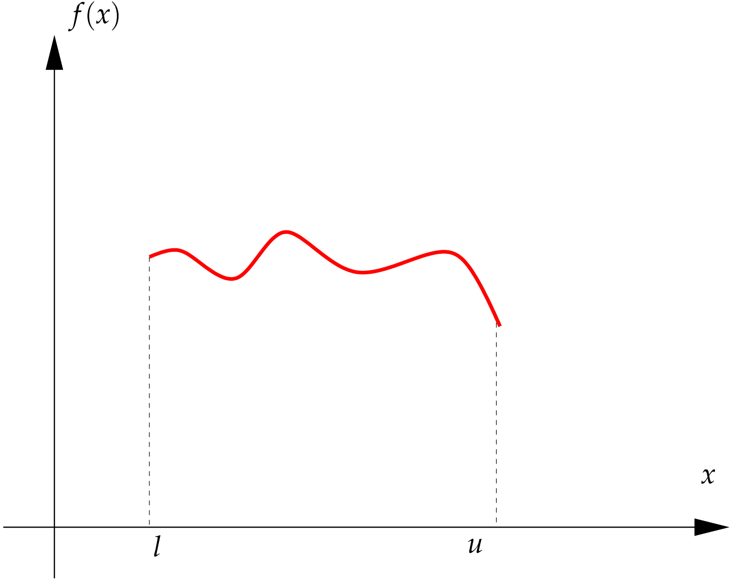
If $\mathbf{P}'$ is a relaxation of a problem $\mathbf{P}$, then the global optimum of $\mathbf{P}'$ is $\leq$ the global optimum of $\mathbf{P}$.

[3]We don't care what $f_0'(x)$ is outside of F.

Examples

- $\min\{f(x) : -1 \leq x \leq 1\}$ is a relaxation of $\min\{f(x) : x = 0\}$
- $\min\{f(x) : -1 \leq x \leq 1\}$ is a r. of $\min\{f(x) : 0 \leq x \leq 1\}$
- $\min\{f(x) : -1 \leq x \leq 1\}$ is not a r. of $\min\{f(x) : -2 \leq x \leq 1\}$
- $\min\{f(x) : g(x) \leq b\}$ is a r. of $\min\{f(x) : g(x) \leq b - 1\}$
- $\min\{f(x) - 1 : g(x) \leq b\}$ is a r. of $\min\{f(x) : g(x) \leq b\}$

Relaxation of an Optimization problem



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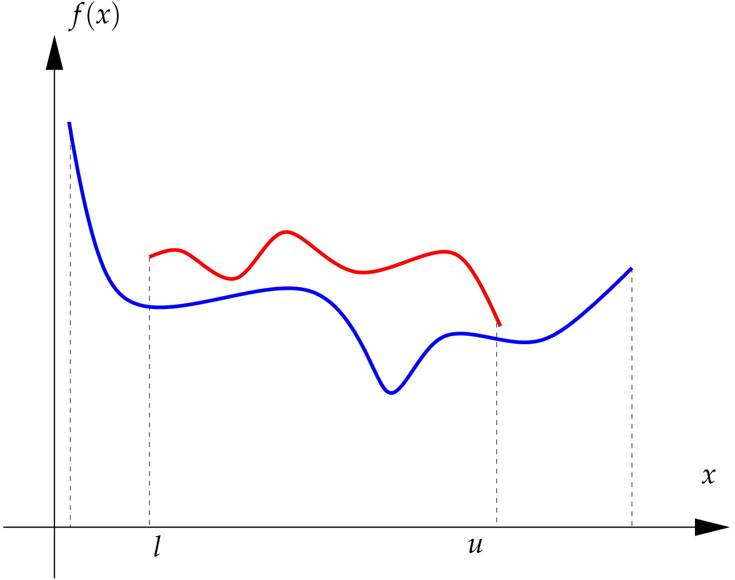

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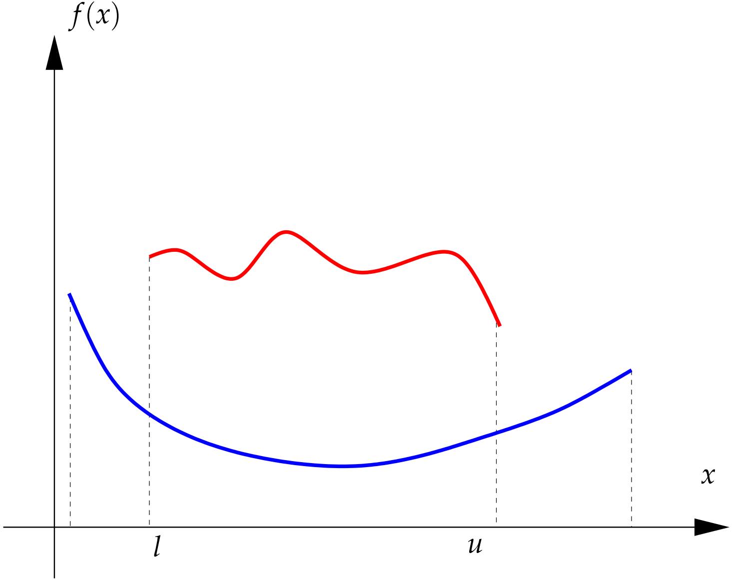

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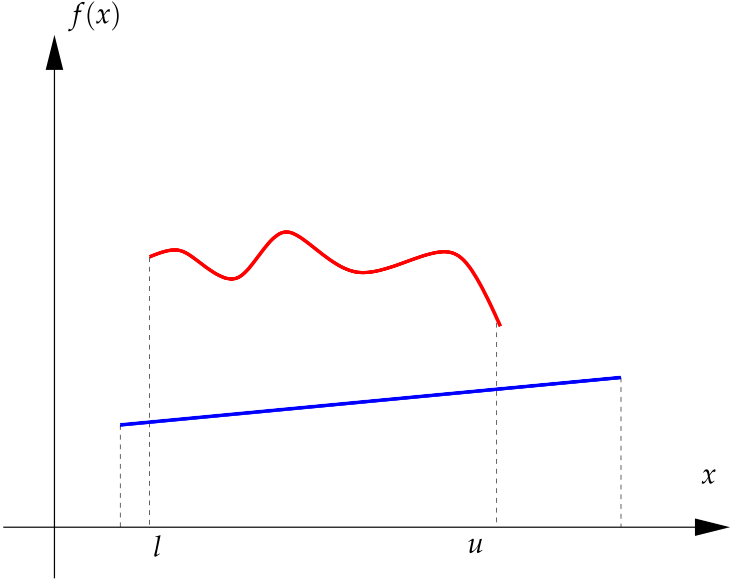

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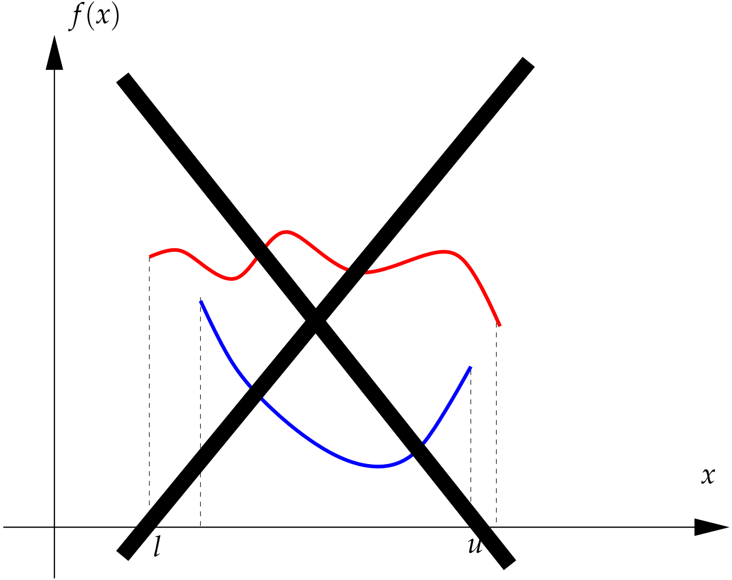

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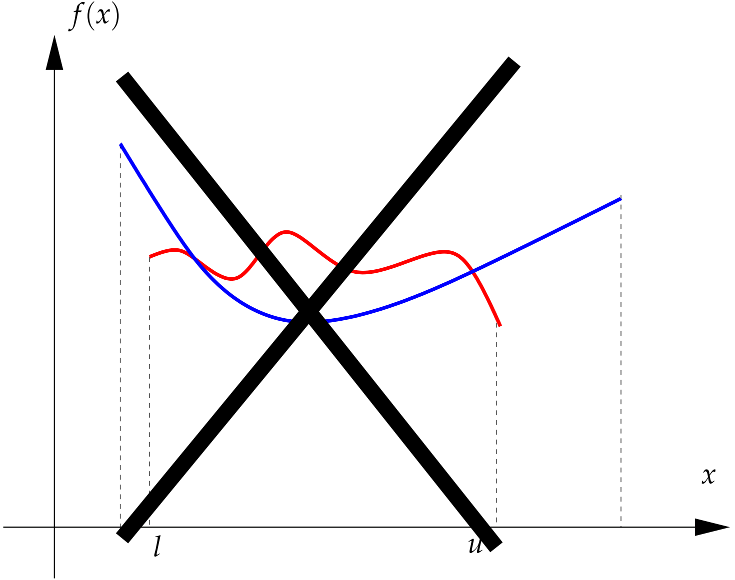

Relaxation of an Optimization problem

Consider again a problem
$\mathbf{P}$: $\min\{f_0(x) : f_1(x) \le b_1, f_2(x) \le b_1, \ldots, f_m(x) \le b_m\}$, or
$\mathbf{P}$: $\min\{f_0(x) : x \in F\}$ for short.

- deleting a constraint from $\mathbf{P}$ provides a relaxation of $\mathbf{P}$.
- adding a constraint $f_{m+1}(x) \le b_{m+1}$ to a problem $\mathbf{P}$ does the opposite:

$$F'' = \{x \in \mathbb{R}^n : \begin{array}{l} f_1(x) \le b_1, \\ f_2(x) \le b_2, \\ \ldots, \\ f_m(x) \le b_m, \\ f_{m+1}(x) \le b_{m+1}\} \subseteq F \end{array}$$

and therefore

$$\min\{f_0(x) : x \in F''\} \quad \ge \quad \min\{f_0(x) : x \in F\}$$

$\boxed{\textbf{Upper \& Lower bounds}}$

Lower and upper bounds

Consider an optimization problem $\mathbf{P} : \min\{f_0(x) : x \in F\}$:

- for any feasible solution $x \in F$, the corresponding objective function value $f_0(x)$ is an upper bound.
- the most interesting upper bounds are the local optima.
- a lower bound of $\mathbf{P}$ is instead a value z such that

$$z \leq \min\{f_0(x) : x \in F\}.$$

Upper vs. Lower bounds

Situation #1:

You: "We found a solution that will only cost 372,000 $."

Boss: "Ok, that sounds good."

Upper vs. Lower bounds

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Situation #2:

You: “We found a solution that will only cost 372,000 \$.”

Boss: “That’s too much, find something better.”

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You: "We found a solution that will only cost 372,000 $."

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You: "We found another solution that costs 354,000 $."

Boss: "Can't you do better than that?"

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Boss: "That's too much, find something better."

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You: "We found another solution that costs 354,000 $."

Boss: "Can't you do better than that?"

You: "I can try again, but here's the proof that we can't go below 351,500."

Boss: "Ok then, that's a good solution."

What relaxations are for

- If $\mathbf{P}'$ is a relaxation of a problem $\mathbf{P}$, then the global optimum of $\mathbf{P}'$ is $\leq$ the global optimum of $\mathbf{P}$.
- Hence, any relaxation $\mathbf{P}'$ of $\mathbf{P}$ provides a lower bound on $\mathbf{P}$.

$\Rightarrow$ If a problem $\mathbf{P}$ is difficult but a relaxation $\mathbf{P}'$ of $\mathbf{P}$ is easier to solve than $\mathbf{P}$ itself, we can still try and solve $\mathbf{P}'$: (i) we get a lower bound and (ii) the solution of $\mathbf{P}'$ may help solve $\mathbf{P}$.

To recap: the Knapsack problem

At a flea market in Rome, you spot n objects (old pictures, a vessel, rusty medals…) that you could re-sell in your antique shop for about double the price.

- You want these objects to pay for your flight ticket to Rome, which cost C.
- Also, you don't want a heavy backpack, so you want to buy the objects that will load it as little as possible.

How do you solve this problem?

The Knapsack problem

Each object $i = 1, 2, \dots, n$ has a price p_i and a weight w_i.

- Variables: one variable x_i for each $i = 1, 2 \dots, n$. This is a "yes/no" variable: either you take the i-th object or not.

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- Objective function: the total weight

Your first (non-trivial) optimization model

$$\begin{array}{ll} \min & \sum_{i=1}^{n} w_i x_i \\ & \sum_{i=1}^{n} p_i x_i \geq C \\ & x_i \in \{0,1\} \quad \forall i = 1, 2, \ldots, n \end{array}$$

Is it convex?

Relaxing the Knapsack problem

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This relaxation would give us $x_i = 0$ for all $i = 1, 2, \ldots, n$, and a lower bound of $\sum_{i=1}^{n} w_i x_i = 0$. Not so great…

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- Relaxing integrality of the variables gives a relaxation where we admit fractions of objects.

$\approx$ we pulverized all objects and took some spoonfuls of each

- It doesn't make sense, but it's a relaxation, and it **does** give us a better lower bound.