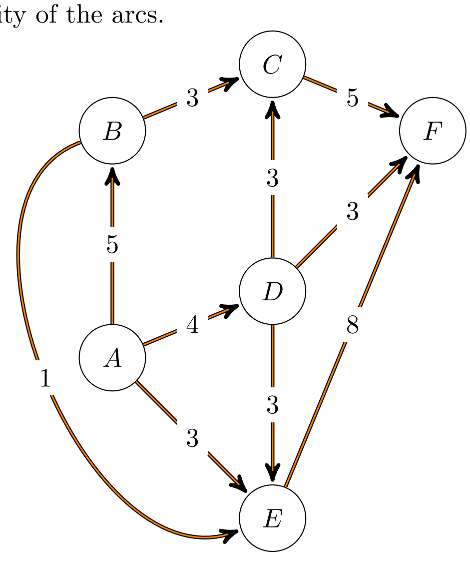
ISE426 – Optimization models and applications

Fall 2015 – Homework #2

This homework accounts for 5% of the final grade. It is due on Thursday, October 6, in class. There are 20 points available. For all problems where an AMPL model is required, include the model file, the data file (they can be the same file), and the optimal solution, shown with the command (e.g. `display`) used to print it.

1 Max Flow and Duality (5 pts.)

Consider the problem of sending the largest possible amount of flow from source A to sink F using the following network, where the numbers on the arcs represent the capacity of the arcs.



1. Formulate this max-flow problem as a linear programming problem, using the formulations studied in this course, with the objective equal to the total flow that goes into F.

2. Write the dual of the above max-flow problem.

3. Solve both problems with AMPL, and for each print the values of the variables and the values of the dual variables (if a problem has a constraint `c1`, its dual value can be displayed with the command "`display c1.dual;`"). Verify that the numbers correspond between primal and dual, and that the optimal solution of the primal and the dual is in accordance with duality.

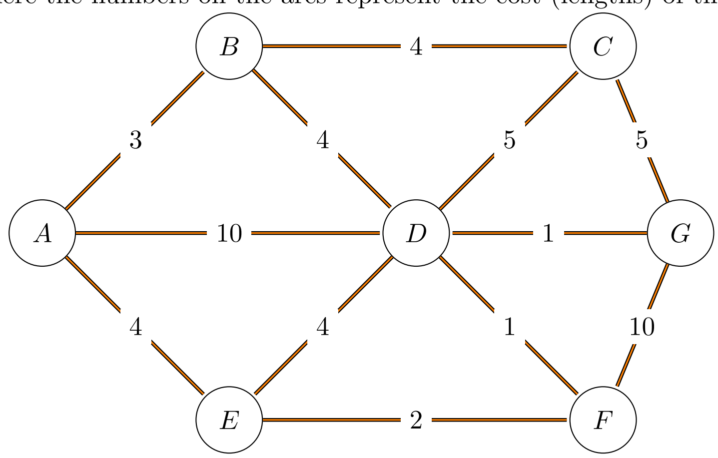
4. Show the minimum cut that results from solving the dual problems.

5. Consider replacing the objective function in the max-flow problem with the total flow that flows out of A, which gives an equivalent formulation of

the max-flow problem. How does the dual formulation change? Explain why it is equivalent to the first dual problem.

2 Shortest path and duality (5 pts.)

Consider the shortest path problems from A to G using the following network, where the numbers on the arcs represent the cost (lengths) of the arcs.



1. Formulate this shortest path problem as a linear programming problem (without imposing the binary conditions of the variables)

2. Write the dual of the above shortest path problem.

3. Solve both problems with AMPL, and for each print the values of the variables and the values of the dual variables (if a problem has a constraint `c1`, its dual value can be displayed with the command "`display c1.dual;`").

4. Recall that by complementary slackness either a primal variable will be zero or the dual constraint that corresponds to it holds as equality. Verify this with the optimal primal and dual solutions for the shortest path. Observe that the nonzero primal variables correspond to the edges that lie along the shortest path.

5. Observe that there are two shortest paths, hence multiple optimal solutions. Which one do you obtain from AMPL? Can you make AMPL produce the other optimal path, by maybe modifying the problem a little bit?

3 Complementary slackness and duality(7 pts.)

Consider the following LP problem:

$$\begin{array}{rllll} \min & x_1 & +3x_2 & +x_3 & -x_4 \\ \text{s.t.} & x_1 & +x_2 & +x_3 & +x_4 \ \geq 0 \\ & x_1 & +x_2 & -x_3 & -x_4 \ \geq 1 \\ & x_2, x_3 & \geq 0 & x_1, x_4 & \leq 0. \end{array}$$

1. Unique primal-dual solutions.

   - Find a feasible solution (by trying a few guesses) and compute its associated value of the objective function, z_{primal}.

- Write the dual.
- Find a feasible solution to the dual (just pick one) and compute its associated value of the objective function, z_{dual}, and check, according to weak duality, that it is not larger than z_{primal}.
- Solve the dual through the graphical method.
- After finding the optimal value of the dual variables, use complementary slackness to find the optimal value of the primal variables.
- Solve both primal and dual problems in AMPL. Print out the solutions.

2. Multiple primal solutions.

   - Change the right hand side of the second constraint to -1. How would the dual change?
   - Solve the new dual problem graphically.
   - Again use complementarity to derive the primal solution. Do you get a unique primal solution or do you get more than one? Can you find all primal optimal solutions? (Hint: all feasible solutions that satisfy complementarity are optimal).
   - Solve both primal and dual problems in AMPL. Print out the solutions.
   - For the primal problem which of the multiple solutions do you obtain? Can you modify the primal problem a little to obtain a different solution from the optimal set?

4 Linear Programming (3 pts.)

Consider the following Linear Programming problem:

$$\begin{array}{rllllr} \min & c_1x_1 & +c_2x_2 & \dots & +c_nx_n & \\ \text{s.t.} & a_1x_1 & +a_2x_2 & \dots & +a_nx_n & \geq b \\ & x_1, & x_2, & \dots, & x_n & \geq 0 \end{array}$$

Write the dual of this problem. Find the simple formula for the optimal solution of the dual. From duality theory and complementarity slackness give a simple formula for the primal optimal solution.