ISE-429
HW2-sol-B

12. (a) $P\{X_1 < X_2 < X_3\}$

$$\begin{aligned}
&= P\{X_1 = \min(X_1, X_2, X_3)\} \\
&\quad P\{X_2 < X_3 | X_1 = \min(X_1, X_2, X_3)\} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} P\{X_2 < X_3 | X_1 \\
&= \min(X_1, X_2, X_3)\} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_2 + \lambda_3}
\end{aligned}$$

where the final equality follows by the lack of memory property.

(b) $P\{X_2 < X_3 | X_1 = \max(X_1, X_2, X_3)\}$

$$\begin{aligned}
&= \frac{P\{X_2 < X_3 < X_1\}}{P\{X_2 < X_3 < X_1\} + P\{X_3 < X_2 < X_1\}} \\
&= \frac{\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_3}{\lambda_1 + \lambda_3}}{\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_3}{\lambda_1 + \lambda_3} + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_1 + \lambda_2}} \\
&= \frac{1/(\lambda_1 + \lambda_3)}{1/(\lambda_1 + \lambda_3) + 1/(\lambda_1 + \lambda_2)}
\end{aligned}$$

(c) $\dfrac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \dfrac{1}{\lambda_2 + \lambda_3} + \dfrac{1}{\lambda_3}$

9. Since the death rate is constant, it follows that as long as the system is nonempty, the number of deaths in any interval of length t will be a Poisson random variable with mean μt. Hence,

$$P_{ij}(t) = e^{-\mu t}(\mu t)^{i-j}/(i-j)!, \quad 0 < j \le i$$

$$P_{i,0}(t) = \sum_{k=i}^{\infty} e^{-\mu t}(\mu t)^k/k!$$

15. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \quad \lambda_i = 0, \quad i \geq 4$$
$$\mu_1 = 2, \quad \mu_2 = \mu_3 = 4$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0, \quad P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0, \quad P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143}$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0, \; P_2 = \frac{3}{4}P_1 = \left[\frac{3}{4}\right]^2 P_0, \; P_3 = \left[\frac{3}{4}\right]^3 P_0$$

and $P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}$$

22. The number in the system is a birth and death process with parameters

$$\lambda_n = \lambda/(n+1), \quad n \geq 0$$
$$\mu_n = \mu, \quad n \geq 1$$

From Equation (5.3),

$$1/P_0 = 1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n/n! = e^{\lambda/\mu}$$

and

$$P_n = P_0(\lambda/\mu)^n/n! = e^{-\lambda/\mu}(\lambda/\mu)^n/n!, \quad n \geq 0$$

23. Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10},\ \lambda_1 = \frac{2}{10},\ \lambda_2 = \frac{1}{10},\ \lambda_i = 0, \quad i \geq 3$$
$$\mu_1 = \frac{1}{8},\ \mu_2 = \frac{2}{8},\ \mu_3 = \frac{2}{8}$$

The balance equations reduce to

$$P_1 = \frac{3/10}{1/8} P_0 = \frac{12}{5} P_0$$
$$P_2 = \frac{2/10}{2/8} P_1 = \frac{4}{5} P_1 = \frac{48}{25} P_0$$
$$P_3 = \frac{1/10}{2/8} P_2 = \frac{4}{10} P_3 = \frac{192}{250} P_0$$

Hence, using $\sum_0^3 P_i = 1$ yields

$$P_0 = \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right]^{-1} = \frac{250}{1522}$$

(a) Average number not in use

$$= P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}$$

(b) Proportion of time both repairmen are busy

$$= P_2 + P_3 = \frac{672}{1522} = \frac{336}{761}$$