ISE-429
HW1-sol-B

18. If the state at time n is the nth coin to be flipped then a sequence of consecutive states constitutes a two-state Markov chain with transition probabilities

$$P_{1,1} = .6 = 1 - P_{1,2}, \quad P_{2,1} = .5 = P_{2,2}$$

(a) The stationary probabilities satisfy

$$\pi_1 = .6\pi_1 + .5\pi_2$$
$$\pi_1 + \pi_2 = 1$$

Solving yields that $\pi_1 = 5/9, \pi_2 = 4/9$. So the proportion of flips that use coin 1 is 5/9.

(b) $P_{1,2}^4 = .44440$

20. If $\sum_{i=0}^{m} P_{ij} = 1$ for all j, then $r_j = 1/(M+1)$ satisfies

$$r_j = \sum_{i=0}^{m} r_i P_{ij}, \sum_{0}^{m} r_j = 1$$

Hence, by uniqueness these are the limiting probabilities.

25. Letting X_n denote the number of pairs of shoes at the door the runner departs from at the beginning of day n, then $\{X_n\}$ is a Markov chain with transition probabilities

$$\begin{aligned}
P_{i,i} &= 1/4, \quad 0 < i < k \\
P_{i,i-1} &= 1/4, \quad 0 < i < k \\
P_{i,k-i} &= 1/4, \quad 0 < i < k \\
P_{i,k-i+1} &= 1/4, \quad 0 < i < k
\end{aligned}$$

The first equation refers to the situation where the runner returns to the same door she left from and then chooses that door the next day; the second to the situation where the runner returns to the opposite door from which she left from and then chooses the original door the next day; and so on. (When some of the four cases above refer to the same transition probability, they should be added together. For instance, if $i = 4, k = 8$, then the preceding states that $P_{i,i} = 1/4 = P_{i,k-i}$. Thus, in this case, $P_{4,4} = 1/2$.) Also,

$$\begin{aligned}
P_{0,0} &= 1/2 \\
P_{0,k} &= 1/2 \\
P_{k,k} &= 1/4 \\
P_{k,0} &= 1/4 \\
P_{k,1} &= 1/4 \\
P_{k,k-1} &= 1/4
\end{aligned}$$

It is now easy to check that this Markov chain is doubly stochastic—that is, the column sums of the transition probability matrix are all 1—and so the long-run proportions are equal. Hence, the proportion of time the runner runs barefooted is $1/(k+1)$.

52. Let the state be the successive zonal pickup locations. Then $P_{A,A} = .6$, $P_{B,A} = .3$. The long-run proportions of pickups that are from each zone are

$$\pi_A = .6\pi_A + .3\pi_B = .6\pi_A + .3(1 - \pi_A)$$

Therefore, $\pi_A = 3/7$, $\pi_B = 4/7$. Let X denote the profit in a trip. Conditioning on the location of the pickup gives

$$\begin{aligned} E[X] &= \frac{3}{7}E[X|A] + \frac{4}{7}E[X|B] \\ &= \frac{3}{7}[.6(6) + .4(12)] + \frac{4}{7}[.3(12) + .7(8)] \\ &= 62/7 \end{aligned}$$

58. Using the hint, we see that the desired probability is

$$P\{X_{n+1} = i + 1 | X_n = i\}$$
$$\frac{P\{lim\ X_m = N | X_n = i, X_n + 1 = i + 1\}}{P\{lim\ X_m = N | X_n = 1\}}$$
$$= \frac{p^P i + 1}{P_i}$$

and the result follows from Equation (4.74).