ISE426 Optimization Models and Applications, Fall 2015
17 September, Homework #1 Solution (Updated),

ANSWER 1. **Convexity and Relaxations (5 pts.)**

(1) Non-convex; (2) Convex; (3) Convex; (4) Convex; (5) Non-convex; (6) Non-convex.

Before going through the details, consider following notes about Convexity:

– Consider a **minimization-type** optimization problem (i.e., we are trying to minimize the objective function value). This problem is a convex optimization problem if the objective function is a convex function and the feasible region is a convex set.

– The feasible region is a convex set if all constraints (in "less than or equal to" form) are convex.

– A function is convex if and only if its Hessian is Positive Semidefinite.

– A symmetric matrix is Positive Semidefinite if and only if all principal minors are nonnegative.

– If the function $g(x)$ is convex, the constraint $g(x) \leq b$ is convex.

– Some common convex functions:
- All Linear functions,
- $f(x) := x^{2n}$, for even n,
- $f(x) := e^x$ and $f(x) := -\ln(x)$.

– Positive multiple of a convex function is convex.

– Sum of two convex functions is convex.

– A **Non**-linear equality constraint is always **Non**-convex.

Now, consider the following tables for details of convexity of each problems.

Problem1	**Function**	**Convexity**	**Description**
Obj.	$2x^2 + 2y^2 + xy$	Yes	Positive definite Hessian
Con1.	$15x + 50y \geq 17$	Yes	Linear inequality
Con2.	$0.001x - y = 0$	Yes	Linear equality
Con3.	$x^2 + y^2 = 1/3$	No	Non-linear equality

Problem 1 is NOT Convex. (i.e., Problem 2 is NOT a convex optimization problem)

Problem2	**Function**	**Convexity**	**Description**
Obj.	$3x^2 + 2y^2 + 5xy$	Yes	Positive definite Hessian
Con1.	$50x + 50y \leq 530$	Yes	Linear inequality
Con2.	$5x - y \leq 3$	Yes	Linear inequality
Con3.	$2x^2 + 2y^2 + 3xy \leq 5$	Yes	Positive definite Hessian

Problem 2 is Convex. (i.e., Problem 2 is a convex optimization problem)

Problem3	**Function**	**Convexity**	**Description**
Obj.	$x^2 + 10y^4 - \ln(z)$	Yes	Sum of the convex functions
Con1.	$x^2 + y^2 + 3z^2 \leq 1$	Yes	Sum of the convex functions
Con2.	$3y^2 \leq 125$	Yes	Quadratic function
Con3.	$-y + 2z^2 \leq 0$	Yes	Sum of the convex functions

Problem 3 is Convex.

Problem4	**Function**	**Convexity**	**Description**
Obj.	$(x + y)^2$	Yes	Positive Definite Hessian
Con1.	$x - y = 0$	Yes	Linear equality
Con2.	$e^x \leq 1$	Yes	e^x is a Convex function

Problem 4 is Convex.

Problem5	**Function**	**Convexity**	**Description**
Obj.	$x^2 + 3y + 5z$	Yes	Sum of the convex functions
Con1.	$e^x + e^y \leq 1$	Yes	Sum of the convex functions
Con2.	$x^2 + z^2 + 2xy \leq 1$	No	Not Positive Definite Hessian

Problem 5 is NOT Convex.

Problem6	**Function**	**Convexity**	**Description**
Obj.	$2x^2 + 2y^2 + 5xy$	No	Not Positive Definite Hessian
Con1.	$x^2 + y^4 \leq 1$	Yes	Sum of convex functions
Con2.	$x + y \geq 0.1$	Yes	Linear inequality
Con3.	$3x + 2y = 5$	Yes	Linear equality

Problem 6 is NOT Convex.

ANSWER 2. **Local and Global Minima (5 pts.)**

Consider the following problem:

$$\begin{aligned} \max y + x \\ 3x + y \leq 6 \\ x + y \geq 1 \\ x + 3y \leq 6 \\ x, y \in \mathbb{Z}. \end{aligned}$$

(1) No. This problem is NOT convex. Because the feasible region is not a convex set due to the integrality constraint (i.e, $x, y \in \mathbb{Z}$.).

(2) No. The solution $(x, y) = (1, 3)$ is infeasible because the 3rd constraint is not satisfied at that point.

(3) No. The solution $(x, y) = (1.5, 1.5)$ is NOT feasible. The integrality constraint is violated at that point.

(4) The value of the objective function that corresponds to each of the previous two solutions are 4 and 3, respectively.

(5) The first objective value 4 is neither upper nor lower bound. However, the second objective value 3 is upper bound because $(x, y) = (1.5, 1.5)$ is the optimal solution for LP relaxation of the problem (i.e, by relaxing the integer constraints)

(6) Yes. After eliminating the integrality constraint, the new problem is a convex optimization problem. Because the objective function and constraints are all linear.

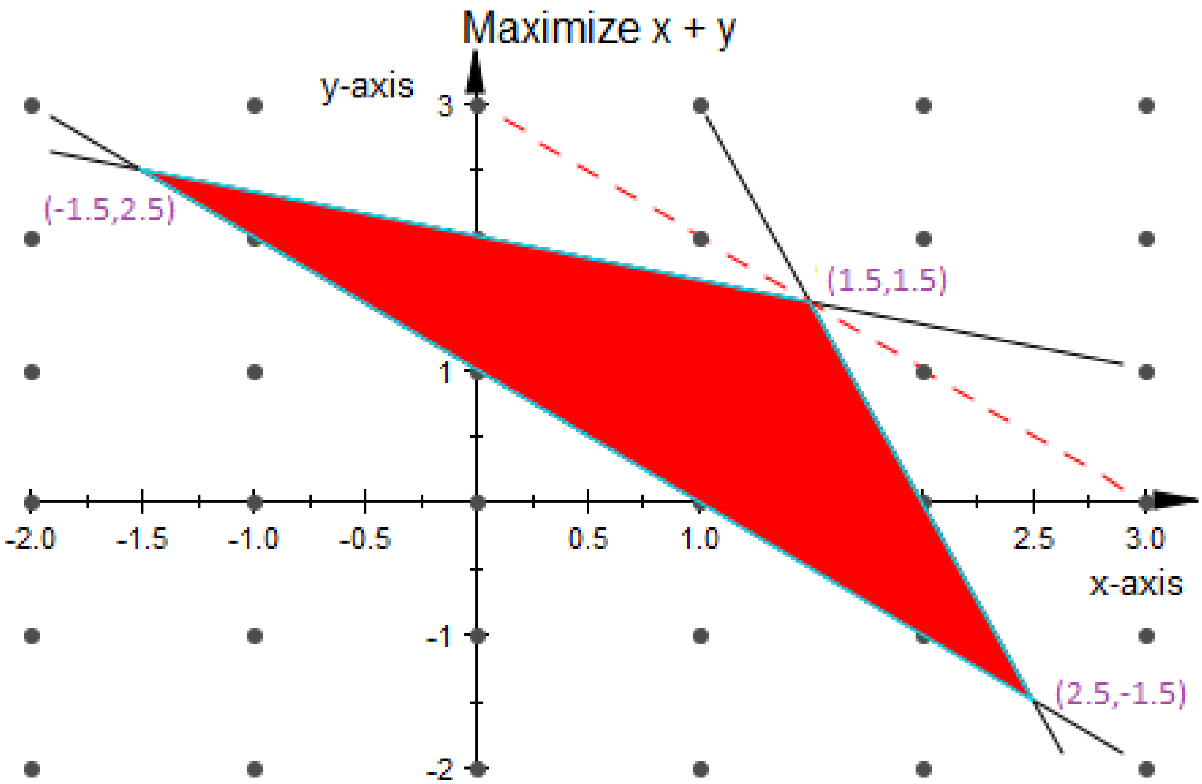
(7) The point $(x, y) = (1, 3)$ is still not feasible for the relaxation. The point $(x, y) = (1.5, 1.5)$ is feasible for the relaxation.

(8) Since the point $(x, y) = (1, 3)$ is still not feasible for the relaxation, so the objective value 4 coming from it is neither a lower nor upper bound. The point $(x, y) = (1.5, 1.5)$ is feasible for the relaxation, hence is automatically a lower bound for the relaxation (since it is a maximization-type problem).

As discussed above, the point $(x, y) = (1.5, 1.5)$ happens to be optimal for the relaxation. So, based on this information, 3 is an upper bound for the relaxation.

(Side note: As stated in part (5), 3 was also upperbound for the original problem since it is the optimal value for the relaxed problem)

(9) See the following figure. The optimal solution is $(x, y) = (1.5, 1.5)$ with objective function value $f(x, y) = 3$. The optimal value of the relaxation give us an upper bound of the original problem.


Maximize x + y
y-axis
3
1
-1
-2
-2.0
-1.5
-1.0
-0.5
0.5
1.0
2.5
3.0
x-axis
(-1.5,2.5)
(1.5,1.5)
(2.5,-1.5)

ANSWER 3. **Linear programming model (5 pts.)**

The problem asks to determine the optimal amount of each component with the aim of minimizing the cost of the cocktail. The components are Smirnoff Mojito Mix, Smirnoff Vodka and Lime Juice. Therefore, our decision variables are as follows:

x_1 : the percentage quantity of Smirnoff Mojito Mix in fluid ounce (fl. oz.) in the cocktail
x_2 : the percentage quantity of Smirnoff Vodka in fluid ounce (fl. oz.) in the cocktail
x_3 : the percentage quantity of Lime Juice in fluid ounce (fl. oz.) in the cocktail

As stated in the question:

C_M denotes the cost per ounce associated with the component Smirnoff Mojito Mix.

C_V denotes the cost per ounce associated with the component Smirnoff Vodka.

C_L denotes the cost per ounce associated with the component Lime Juice.

Then, the objective function is as follows:

$$\min \; C_M x_1 + C_V x_2 + C_L x_3 \tag{1}$$

The problem requires that the percentage of each component (i.e., Smirnoff Mojito Mix, Smirnoff Vodka and Lime Juice) obtained from the desired combination lie within a specified bound. All these requirements are reflected in the following constraints:

Constraints for Alcohol Bounds:

$$\begin{aligned}(0.15)x_1 + (0.4)x_2 + (0)x_3 \geq 0.15\\ (0.15)x_1 + (0.4)x_2 + (0)x_3 \leq 0.25\end{aligned}$$

Constraint for Sugar Bounds:

$$(0.2)x_1 + (0)x_2 + (0.04)x_3 \leq 0.1$$

Overall, the model is as follows:

$$\begin{aligned}\min\ & C_M x_1 + C_V x_2 + C_L x_3\\ \text{s.t.}\ & (0.15)x_1 + (0.4)x_2 + (0)x_3 \geq 0.15\\ & (0.15)x_1 + (0.4)x_2 + (0)x_3 \leq 0.25\\ & (0.2)x_1 + (0)x_2 + (0.04)x_3 \leq 0.1\\ & x_1 + x_2 + x_3 = 1\\ & x_1, x_2.x_3 \geq 0\end{aligned}$$

ANSWER 4. **Knapsack problem (5 pts.)**

(1) Compute the ratios $\frac{p_i}{w_i}$ in the following table, sort them in descending order, and add the objects according to the order until no more objects can be added without violating the constraint. The solution given by the greedy heuristic method is $x = (1, 1, 0, 1, 0, 0, 0, 0, 0)$ with the corresponding objective value 50. Hence $f(x) = 50$.

i	1	2	3	4	5	6	7	8	9
p_i	30	13	6	7	21	35	10	31	20
w_i	2	1	1	1	2	3	3	3	2
p_i/w_i	15	13	6	7	10.5	11.7	3.3	10.3	10

Now we found a feasible solution $x = (1, 1, 0, 1, 0, 0, 0, 0, 0)$. We cannot improve this solution by swapping only one object for any other. For instance, let's say we are swapping object 1 and 9. Hence we are taking object 9 and leaving object 1: $\hat{x} = (0, 1, 0, 1, 0, 0, 0, 0, 1)$ and $f(\hat{x}) = 40$. Hence, the objective function value did not get better due to swapping object 1 and 9.

The swapping operation should be considered for other swapping cases as well. But we should point out that only 1 object is swapped in each case and capacity constraint should be satisfied for every case.

The heuristic solution is feasible for the original knapsack problem, so it gives a lower bound (for a max problem).

Solve this problem either in AMPL or simply by enumeration of possible solutions. We get an global optimal solution $x* = (1, 0, 0, 0, 1, 0, 0, 0, 0)$ with optimal objective function value $f(x*) = 51$.

(2) Now, we solve the LP relaxation. It gives us the optimal solution $x** = (1, 1, 0, 0, 0, \frac{1}{3}, 0, 0, 0)$ with optimal objective function value $f(x**) = 54.66$. Since $f(x**) = 54.66$ is the optimal objective of the relaxation, it gives an upper bound for the original problem (for a max problem).

(3) Based on the previous parts, the lowerbound is 50 and the upperbound is 54.66. Hence we know that the optimal objective value is between 50 and 54.66. By only looking at these lower and upper bounds, it is not possible to prove that the optimal objective function value for the original knapsack problem is $f(x) = 51$.