IE426 – Optimization models and applications

Fall 2012 – Final exam, December 12, 2012

December 10, 2012

First name	Jia
Last name	Lu.
Lehigh email	jil612@lehigh.edu

You have three hours. This exam accounts for 25% of the final grade. There are 100 points available. Please write clear and concise statements. Use a **readable** handwriting: it will be hard to grade your answers if they are not readable, let alone give them full points. Use the back of each sheet if you need more space, for example to draw graphs. For each model, clearly specify the meaning of each variable and of each constraint.

1 Duality (20 pts.)

Consider the following minimization problem.

$$\begin{array}{llll} \min & x_1 & +2x_2 & +5x_3 & \\ \text{s.t.} & x_1 & & +x_3 & =2 \\ & & x_2 & +x_3 & =1 \\ & \multicolumn{3}{l}{x_1, x_2 \le 0; x_3 \ge 0} & \end{array}$$

1. Write its dual (6 pts.);
2. Solve the dual with the graphical method (4 pts.);
3. Find the optimal solution of the primal using that of the dual (8 pts.).

Solution: (1) max $2u_1 + u_2$

s.t. $u_1 \ge 1$

$u_2 \ge 2$

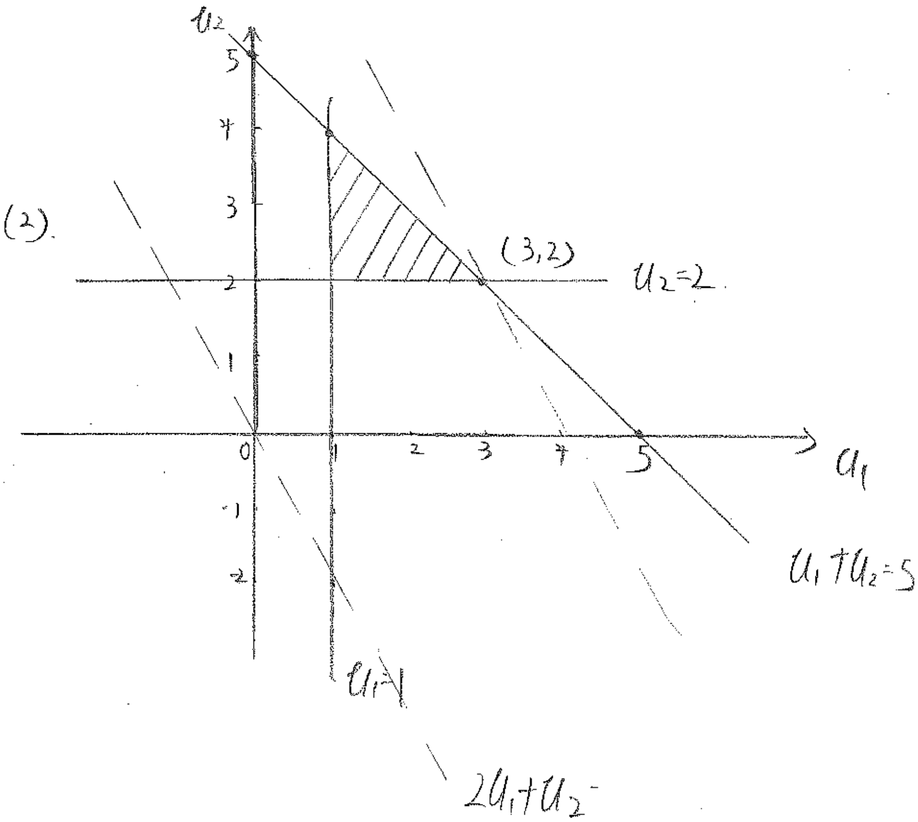
$u_1 + u_2 \le 5$

u_1, u_2 unrestricted

100/100

(cont'd)

(2).



From the graph we can know the optimal solution is $u_1=3, u_2=2$. the optimal solution is 8

(3). Using Complimentary Slackness.

$$\begin{cases} \bar{u}_1 \cdot (\bar{x}_1+\bar{x}_3-2)=0. \\ \bar{u}_2 \cdot (\bar{x}_2+\bar{x}_3-1)=0. \\ \bar{x}_1 \cdot (\bar{u}_1-1)=0 \\ \bar{x}_2 \cdot (\bar{u}_2-2)=0. \\ \bar{x}_3 \cdot (\bar{u}_1+\bar{u}_2-5)=0 \end{cases} \Longrightarrow \begin{cases} \bar{x}_1=0 \\ \bar{x}_2=-1 \\ \bar{x}_3=2 \end{cases}$$

✓

The optimal value is $0+2(-1)+5\cdot2=8$.

Hence the dual is correct

And the optimal solution of the primal problem is $x_1=0, x_2=-1, x_3=2$

2 Integer Programming (20 pts.)

My daughter Kyra is trying to arrange a group of her friends into two soccer teams to play each other. Each player has skill level, s_i, $i = 1, \ldots, n$.

- Formulate a integer linear optimization problem to separate the friends into two teams so that the teams are as evenly matched as possible (that is to minimize the different between the total skill levels for each team).

Solution: $\min \left|\sum X_i S_i - \sum (1-X_i) \cdot S_i\right| \quad i=1,\cdots,n$

$$\text{s.t.} \quad \sum X_i = \frac{n}{2}$$

$$X_i \in \{0,1\}.$$

$$X_i = \begin{cases} 1 & \text{If } i \text{ player is picked into the first team} \\ 0 & \text{otherwise} \end{cases}$$

This is not an linear problem so we neet transform:

$$\min \; y$$
$$y \geq \sum X_i \cdot S_i - \sum (1-X_i) \cdot S_i$$
$$y \geq -\left(\sum X_i S_i - \sum (1-X_i) \cdot S_i\right)$$
$$\sum X_i = \frac{n}{2} \quad X_i \in \{0,1\}.$$

- Assume that there are some kids who insist on being on the same team with their friends. The pairs of such friends are given by the set $E = \{(i,j)\}$. Formulate the same optimization problem as above, but making sure that all pairs of friends are kept together.

Solution: $\min \; y$

$$\begin{aligned}
\text{s.t.} \quad & y \geq \sum X_i S_i - \sum (1-X_i) S_i && i=1,\cdots,n \\
& y \geq -\left(\sum X_i S_i - \sum (1-X_i) S_i\right) && i=1,\cdots,n \\
& X_i = X_j && i=1,2,\cdots,n,\ j=1,2,\cdots,n \\
& X_i \in \{0,1\} && i \neq j \\
& \sum X_i = \frac{n}{2} && i=1,2,\cdots,n
\end{aligned}$$

3 Integer Programming (10 pts.)

Consider a graph $G = (V, E)$, in Figure 1, and a cost c_i for each node $i \in V$. Suppose you want to find the subset S of V of minimum total cost such that every edge in E is *covered* by at least one node in S, i.e., for every edge $\{i, j\} \in E$, either $i \in S$ or $j \in S$ or both. As an example, for the graph in Figure 1, the dark nodes constitute a feasible solution for this problem.

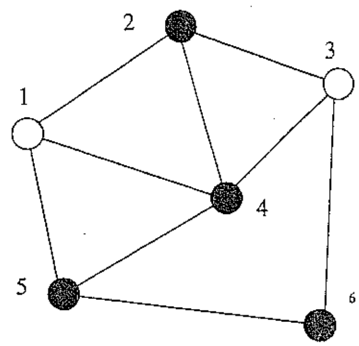


Figure 1:

Formulate this problem as an integer linear programming problem.

Solution: $\min C_i \cdot X_i$ $\quad i \in V;$

$\text{s.t.} \quad X_i + X_j \geq 1 \quad (i,j) \in E,\ i \in (V);\ j \in (V)\ i \neq j.$

$X_i = \begin{cases} 1 & \text{If picking } i \text{ node} \\ 0 & \text{otherwise} \end{cases}$

In this case.

$\min C_i \cdot X_i \quad i = 1, 2, 3, 4, 5, 6$

$\text{s.t.} \quad X_1 + X_2 \geq 1$

$X_1 + X_4 \geq 1$

$X_1 + X_5 \geq 1$

$X_2 + X_3 \geq 1$

$X_2 + X_4 \geq 1$

$X_3 + X_4 \geq 1$

$X_3 + X_6 \geq 1$

$X_4 + X_5 \geq 1$

$X_5 + X_6 \geq 1$

4 Stochastic Integer Programming (30 pts.)

Your company is considering to open a few grocery stores, each possibly with a pharmacy, in Pennsylvania. The stores and pharmacies are intended to serve an area with a set C of n cities, and there is a set L of m potential locations.

You are given the distance d_{ij} between any $i \in C$ and any $j \in L$. The cost for building a grocery store at location $j \in L$ is c_j, and the additional cost for adding a pharmacy is f_j.

A pharmacy cannot be built at j unless a grocery store is also built at j. The total budget is B, and the company aims at minimizing the sum of two functions: the first is the sum, among all cities i, of the distance between i and the closest grocery, which can be thought of as the grocery *assigned* to i; the second is analogous to the first and is related to pharmacies.

- Formulate this as an integer linear programming problem.

Solution: $\min \sum (d_{ij} \cdot y_{ij} + d_{ij} \cdot z_{ij})$

$$
\begin{aligned}
\text{s.t. } & \sum c_j \cdot x_j + \sum f_j \cdot p_j \le B & j \in L \\
& p_j \le x_j & j \in L \\
& y_{ij} \le x_j & j \in L \quad i \in C \\
& z_{ij} \le p_j & j \in L \quad i \in C \\
& \sum_{j \in L} z_{ij} = 1 & \forall i \in C \\
& \sum_{j \in L} y_{ij} = 1 & \forall i \in C \\
& x_j \in \{0,1\};\ p_j \in \{0,1\};\ z_{ij} \in \{0,1\};\ y_{ij} \in \{0,1\}
\end{aligned}
$$

$y_{ij} = \begin{cases} 1 & \text{If } j \text{ grocery assigned to } i \\ 0 & \text{other wise} \end{cases}$

$z_{ij} = \begin{cases} 1 & \text{If } j \text{ pharmacy assigned to } i \\ 0 & \text{otherwise} \end{cases}$

$x_j = \begin{cases} 1 & \text{If building a grocery at location } j \\ 0 & \text{otherwise} \end{cases}$

$p_j = \begin{cases} 1 & \text{If building a pharmacy at location } j \\ 0 & \text{otherwise} \end{cases}$

✓

- Now assume that you build all the groceries first. While you were building them the construction cost may have gone up 20% or down 20%, or it may have remained the same. The three scenarios are equally likely. The change in the construction costs affects only the cost of construction of the pharmacies (but not the groceries). Formulate the new problem as a stochastic integer programming problem. Identify first and second stage variables.

(cont'd)

First stage variables: X_j, y_{ij}

Solution: second stage variables: $p_{1j}, p_{2j}, p_{3j}, z_{ij-1}, z_{ij-2}, z_{ij-3}$

scenario 1: goes down 20%.

scenario 2: remain the same

scenario 3: goes up 20%

$p_{1j} = \begin{cases} 1 & \text{If pharmacy builded at } j \text{ in scenario 1} \\ 0 & \text{otherwise} \end{cases}$

$p_{2j} = \begin{cases} 1 & \text{If pharmacy builded at } j \text{ in scenario 2} \\ 0 & \text{otherwise} \end{cases}$

$p_{3j} = \begin{cases} 1 & \text{If pharmacy builded at } j \text{ in scenario 3} \\ 0 & \text{otherwise} \end{cases}$

$z_{ij-1} = \begin{cases} 1 & \text{If } j \text{ pharmacy assigned to } i \text{ in scenario 1} \\ 0 & \text{otherwise} \end{cases}$

$z_{ij-2} = \begin{cases} 1 & \text{If } j \text{ pharmacy assigned to } i \text{ in scenario 2} \\ 0 & \text{otherwise} \end{cases}$

$z_{ij-3} = \begin{cases} 1 & \text{If } j \text{ pharmacy assigned to } i \text{ in scenario 3} \\ 0 & \text{otherwise} \end{cases}$

$$\min. \ \sum \left[d_{ij} \cdot y_{ij} + \frac{1}{3}(d_{ij} \cdot z_{ij-1} + d_{ij} \cdot z_{ij-2} + d_{ij} \cdot z_{ij-3}) \right]$$

s.t.

$$\sum c_j \cdot X_j + 0.8 \sum f_j \, p_{1j} \leq B$$

$$\sum c_j \cdot X_j + \sum f_j \, p_{2j} \leq B$$

$$\sum c_j \cdot X_j + 1.2 \sum f_j \, p_{3j} \leq B$$

$p_{1j} \leq X_j; \quad j \in L \qquad \sum_{j \in L} z_{ij-1} = 1 \quad \forall i \in C$

$p_{2j} \leq X_j; \quad j \in L \qquad \sum_{j \in L} z_{ij-2} = 1 \quad \forall i \in C$

✓ $p_{3j} \leq X_j; \quad j \in L \qquad \sum_{j \in L} z_{ij-3} = 1. \quad \forall i \in C$

$z_{ij-1} \leq p_{1j}; \quad j \in L, i \in C \qquad X_j \in \{0,1\},$

$z_{ij-2} \leq p_{2j}, \quad i \in C, j \in L \qquad p_{1j}, p_{2j}, p_{3j} \in \{0,1\}.$

$z_{ij-3} \leq p_{3j}; \quad i \in C, j \in L \qquad y_{ij} \in \{0,1\}.$

$y_{ij} \leq X_j, \quad i \in C, j \in L.$

$\sum y_{ij} = 1; \quad \forall i \in C. \qquad z_{ij-1}, z_{ij-2}, z_{ij-3} \in \{0,1\}$

5 Support Vector Machines (20 pts.)

The convex quadratic formulation of support vector machines

$$\begin{aligned} \min_{\xi,\mathrm{w},\beta} \quad & \frac{1}{2}w^\top w + c\sum_{i=1}^{n}\xi_i \\ \text{s.t.} \quad & y_i(w^\top x_i + \beta) \geq 1 - \xi_i, \quad i = 1,\ldots,n \\ & \xi \geq 0, \qquad i = 1,\ldots,n. \end{aligned} \tag{1}$$

can be rewritten as an unconstrained nonsmooth convex optimization problem

$$\min_{\xi,\mathrm{w},\beta}\frac{1}{2}w^\top w + c\sum_{i=1}^{n}\max\{1 - y_i(w^\top x_i + \beta), 0\}$$

Consider the following modification of the problem

$$\min_{\xi,\mathrm{w},\beta}\frac{1}{2}w^\top w + c\sum_{i=1}^{n}(\max\{1 - y_i(w^\top x_i + \beta), 0\})^2$$

Rewrite this as a convex quadratic problem (in the spirit of (1)).

Solution: Let $\xi_i = \max\{1 - y_i(w^\top x_i + \beta), 0\}$

So $\min_{\xi,w,\beta} \frac{1}{2}w^\top w + c\sum_{i=1}^{n}\xi_i$

s. s.t $\xi_i \geq 1 - y_i(w^\top x_i + \beta)$

$\xi_i \geq 0$

Let $p = (p_1, p_2, \cdots, p_n)$

So the formalation can be transformed in

$\min_{p,w,\beta} \frac{1}{2}w^\top w + c\,p^\top p$.

$p_i \geq 1 - y_i(w^\top x_i + \beta)$

$p_i \geq 0 \quad i = 1, 2, \cdots, n$