Final '12

1. In a factorial design with three factors, the following runs were performed: $a = 10$, $b = 12$, $c = 15$, $abc = 20$.

a) What is the defining relation for this design and what is its resolution?

The defining relation is $C = AB$ (or, equivalently, $I = ABC$).

b) Find the estimates for the main factor effects assuming all interactions are negligible.

$$A = \frac{abc + a - b - c}{2} = 1.5$$
$$B = \frac{abc + b - a - c}{2} = 3.5$$
$$C = \frac{abc + c - a - b}{2} = 6.5$$

c) Suppose additional four runs have been performed: $(1) = 5$, $ab = 8$, $ac = 12$, $bc = 10$. What are the estimates of main factor effects from these four runs (assuming interactions are negligible)?

$$A = \frac{ab + ac - bc - (1)}{2} = 2.5$$
$$B = \frac{ab + bc - ac - (1)}{2} = 0.5$$
$$C = \frac{ac + bc - ab - (1)}{2} = 4.5$$

d) Combine the two sets of runs to find the estimates of the main effects and two-factor interactions, if possible.

$$A = \frac{1.5 + 2.5}{2} = 2, \quad BC = \frac{1.5 - 2.5}{2} = -0.5$$
$$B = \frac{3.5 + 0.5}{2} = 2, \quad AC = \frac{3.5 - 0.5}{2} = 1.5$$

$$C = \frac{6.5 + 4.5}{2} = 5.5, \quad AB = \frac{6.5 - 4.5}{2} = 1$$

e) If the two sets of runs, each being a block, are combined in a single design, is it possible to estimate the effect ABC? Explain your answer.

No, ABC will be confounded with blocks.

2. A single factor experiment was performed for 5 levels of the factor with 4 replicates. The ANOVA analysis revealed that $MS_E = 25$.

a) How many mutually orthogonal contrasts can be written?

The number of orthogonal contrasts is $5 - 1 = 4$.

b) Suppose two contrasts are $\Gamma_1 = \mu_1 - \mu_2$ and $\Gamma_2 = \mu_1 + \mu_2 - 2\mu_3$. In case these are orthogonal, construct additional contrasts so that all of them are mutually orthogonal.

$\Gamma_3 = \mu_1 + \mu_2 + \mu_3 - 3\mu_4, \qquad \Gamma_4 = \mu_1 + \mu_2 + \mu_3 + \mu_4 - 4\mu_5.$

c) Find the standard errors for the contrasts Γ_1, Γ_2 and also for the contrasts constructed in (b).

$$S_{C_1} = \sqrt{\frac{MS_E}{n}(1+1)} = \sqrt{\frac{25}{4}(1+1)} = \frac{5\sqrt{2}}{2}$$
$$S_{C_2} = \sqrt{\frac{MS_E}{n}(1+1+4)} = \sqrt{\frac{25}{4}(6)} = \frac{5\sqrt{6}}{2}$$
$$S_{C_3} = \sqrt{\frac{MS_E}{n}(1+1+1+9)} = \sqrt{\frac{25}{4}(12)} = 5\sqrt{3}$$
$$S_{C_4} = \sqrt{\frac{MS_E}{n}(1+1+1+1+16)} = \sqrt{\frac{25}{4}(20)} = 5\sqrt{5}$$

d) Suppose it was found that $\bar{y}_{1\cdot} = 20$, $\bar{y}_{2\cdot} = 25$, $\bar{y}_{3\cdot} = 30$, $\bar{y}_{4\cdot} = 35$, $\bar{y}_{5\cdot} = 40$. Using Sheffe's method, which of the contrasts from (c) would be considered significantly different from zero at $\alpha = 0.05$?

$$|C_1| = 5 < S_{0.05,1} = S_{C_1}\sqrt{4 \cdot F_{0.05,4,15}} = 12.37$$
$$|C_2| = 15 < S_{0.05,2} = S_{C_2}\sqrt{4 \cdot F_{0.05,4,15}} = 21.42$$
$$|C_3| = 30 < S_{0.05,3} = S_{C_3}\sqrt{4 \cdot F_{0.05,4,15}} = 30.30$$
$$|C_4| = 50 > S_{0.05,4} = S_{C_4}\sqrt{4 \cdot F_{0.05,4,15}} = 39.12$$

Thus we conclude that the contrast Γ_4 is significantly different from zero at 0.05 significance level.

3. A single factor experiment was performed so that each replicate constituted a single block. The following results of the ANOVA analysis are known: $SS_{\text{Treatments}} = 1000$, $SS_{\text{Blocks}} = 300$, $SS_T = 1500$, $MS_{\text{Treatments}} = 200$, $MS_{\text{Blocks}} = 150$.

a) What was the value of SS_E?

$$SS_E = SS_T - SS_{\text{Blocks}} - SS_{\text{Treatments}} = 200.$$

b) What was the number of levels for the main factor in the experiment?

$a - 1 = \frac{SS_{\text{Treatments}}}{MS_{\text{Treatments}}} = 5$. Therefore $a = 6$.

c) What was the number of blocks?

$b - 1 = \frac{SS_{\text{Blocks}}}{MS_{\text{Blocks}}} = 2$. Therefore $b = 3$.

d) What was the value of MS_E?

$$MS_E = \frac{SS_E}{(a-1)(b-1)} = \frac{200}{10} = 20.$$

e) What was the value of the test statistic for the F-test on the significance of the main factor?

$$F_0 = \frac{MS_{\text{Treatments}}}{MS_E} = \frac{200}{20} = 10.$$

4. An experimenter needs to perform a screening experiment for 4 factors and finds out that only four runs are possible.

a) What is the fractional factorial design that needs to be performed?

$$2^{4-2}$$

b) How long are alias chains going to be?

Their length will be $\frac{16}{4} = 4$

c) What is the maximum resolution such a design can achieve?

Resolution II.

d) Suggest a set of generators so that the design has maximum possible resolution and minimum aberration.

We need to select two generators such that the minimum length of them and their product is as large as possible (easy to see it can't be larger than 2) and the number of them with the minimal length is also minimized. We can achieve that by choosing ABC and BCD as independent generators so that their product is AD. Then the abberations is (3,3,2).

e) Write down all alias relations for the design from (d).

$$\begin{aligned} I &= ABC = BCD = AD \\ A &= BC = ABCD = D \\ B &= AC = CD = ABD \\ C &= AB = BD = ACD \end{aligned}$$

f) What runs need to be performed for the design from (d)?

One possible set of runs (obtained by starting from a full design for factors A and B and using $C = AB$ and $D = A$) is c, ad, b, $abcd$.

5. Using a response surface design, the fitted second order model of the form

$$\hat{y} = 20 - 10x_1 + 10x_2 - 2x_1^2 - 7x_2^2 + 4x_1x_2$$

for the response y was obtained.

a) Write down the matrix $\mathbf{B}$ for this model.

$$\mathbf{B} = \begin{pmatrix} -2 & 2 \\ 2 & -7 \end{pmatrix}$$

b) Find the stationary point.

$$\mathbf{x}_s = -\frac{1}{2}\mathbf{B}^{-1}\mathbf{b} = -\frac{1}{2} \cdot \frac{1}{10} \begin{pmatrix} -7 & -2 \\ -2 & -2 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ 10 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ 0 \end{pmatrix}$$

c) Find the value $\hat{y}_s$ of the predicted response at the stationary point.

$$\hat{y}_s = 20 + \frac{1}{2} \begin{pmatrix} -\frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} -10 \\ 10 \end{pmatrix} = 32.5$$

d) Find the eigenvalues of $\mathbf{B}$.

We have $\begin{vmatrix} -2-\lambda & 2 \\ 2 & -7-\lambda \end{vmatrix} = 0$, i.e. $\lambda^2 + 9\lambda + 10 = 0$ implying that $\lambda_1 = -7.70$, $\lambda_2 = -1.30$.

e) Write the model in the canonical form. What is the type of the stationary point?

$\hat{y} = 32.5 - 1.30w_1^2 - 7.70w_2^2$ and therefore the stationary point is a maximum.

6. A factorial experiment with three factors is conducted. Factor A is fixed, and factors B and C are random. The number of levels is 3, 2 and 4 for A, B and C, respectively. There are 3 replicates. Assume the restricted model.

a) Write down the expression for $E(MS_A)$?

$$E(MS_A) = bcn\frac{\sum_{i=1}^{a}\tau_i^2}{a-1} + cn\sigma_{\tau\beta}^2 + bn\sigma_{\tau\gamma}^2 + n\sigma_{\tau\beta\gamma}^2 + \sigma^2 = 12\sum_{i=1}^{a}\tau_i^2 + 12\sigma_{\tau\beta}^2 + 6\sigma_{\tau\gamma}^2 + 3\sigma_{\tau\beta\gamma}^2 + \sigma^2$$

b) Write down the expression for $E(MS_B)$?

$$E(MS_B) = acn\sigma_\beta^2 + an\sigma_{\beta\gamma}^2 + \sigma^2 = 36\sigma_\beta^2 + 9\sigma_{\beta\gamma}^2 + \sigma^2$$

c) Write down the expression for $E(MS_{AB})$?

$$E(MS_{AB}) = cn\sigma_{\tau\beta}^2 + n\sigma_{\tau\beta\gamma}^2 + \sigma^2 = 12\sigma_{\tau\beta}^2 + 3\sigma_{\tau\beta\gamma}^2 + \sigma^2$$

d) Propose an F-test for the significance of factor A.

$$F_0 = \frac{MS_A + MS_{ABC}}{MS_{AB} + MS_{AC}}.$$

Note that an exact F-test can't be found in this case.

e) Propose an F-test for the significance of factor B.

$$F_0 = \frac{MS_B}{MS_{BC}}.$$