ISE429, Fall 2015. Test 1

1) (weight 0.25)

$\{X_n\}$ is a discrete-time Markov chain, with $m+1$ states ($m \geq 1$) and the transition probability matrix P being

$$P = \begin{pmatrix} q_0 & q_1 & q_2 & \cdots & q_m \\ q_m & q_0 & q_1 & \cdots & q_{m-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ q_2 & q_3 & q_4 & \cdots & q_1 \\ q_1 & q_2 & q_3 & \cdots & q_0 \end{pmatrix}$$

with $q_0 > 0$ and $q_m > 0$. Find $\lim_{n\to\infty} P^n$.

Solution. Imagine the states $0, 1, \ldots, m$ arranged clock-wise on a circle. (So, state m is a "neighbor" of 0.) Then condition $q_m > 0$ means that from any state i, with non-zero probability, the process can move to the counter-clock-wise neighbor state. So, the chain is irreducible. It is then positive recurrent (because finite). It is aperiodic, because with non-zero probability q_0 it returns to same state in one step. Its stationary distribution is uniform, $\pi_i = 1/(m+1)$, $\forall i$, because P is doubly stochastic (or simply because of the symmetry). Therefore, $\lim_{n\to\infty} P^n$ is the square matrix with all elements equal $1/(m+1)$. $\square$

2) (weight 0.40)

Consider the Markov process on the integers $\mathcal{X} = \{\ldots, -2, -1, 0, 1, 2, ...\}$, such that $P_{k,k-1} = 0.67, \quad P_{k,k+2} = 0.33$ for all $k \in \mathcal{X}$.
(a) Is this chain irreducible? aperiodic? (weight 0.10)
(b) Is this chain transient? null-recurrent? positive recurrent? (weight 0.30)

Solution. Irreducibility is obvious – clearly any state communicates with any other. The chain has period 3 – to return to any state, there should be exactly twice more jumps to the left than to the right.

The process can be viewed as $X_{n+1} = X_n + Y_n$, where Y_n, $n = 0, 1, 2, \ldots$, are i.i.d. with expected value $(-1) \cdot 0.67 + 2 \cdot 0.33 = -0.01 < 0$. By the Strong Law of Large Numbers, with probability 1,

$$(X_n - X_0)/n = (Y_0 + \ldots + Y_{n-1})/n \to -0.01,$$

and then $X_n \to -\infty$. This chain is transient. $\square$

3) (weight 0.35)

Let X_n, with time $n = 0, 1, \ldots$, be the simple random walk on "numbers" (states) $\{0, 1, ..., 10\}$, arranged in a "circle". That is, from state i the process goes to $i - 1$ or $i + 1$ with equal probability $1/2$. Here, by convention, if $i = 0$ then $i - 1 = 10$, and if $i = 10$ then $i + 1 = 0$. The process starts at $X_0 = 3$. What is the probability of the following event A: After leaving state 3, the process visits state 9 before it visits state 6 or returns to state 3.

Solution. For the event A to occur, the process must first go to state 2, which happens with probability 1/2. After that, it must hit 9 before it hits 3; using gambler's ruin model (with absorbtion at 3 and 9), this probability is 1/5. (1 is the "distance" from 2 to 3, and 5 is the "distance" from 9 to 3.)
$\mathbb{P}(A) = (1/2)(1/5) = 1/10$. $\square$