ISE426 – Optimization models and applications

Fall 2015, Homework #1. Due Sept 17, 2015, in class.

September 11, 2015

This homework accounts for 5% of the final grade. There are 20 points available. In this and future homework assignments, problems named “Modeling” do not require you to formulate Linear Programming (LP) model. However, for problems named “Linear Programming,” all correct solutions are LP models, therefore formulations with either nonlinear constraints or integer or binary variables will be penalized. This homework is a good training for quiz #1, so you would want to work independently.

1 Convexity and relaxations (5 pts.)

For each of the following problems, determine if they are convex or not, by looking at the constraints and the objective (i.e., it is not necessary to draw the feasible set on the Cartesian plane), and motivate your answer:

$$
(3)\ \min\ \begin{array}{l} x^2 + 10y^4 - \ln(z) \\ x^2 + y^2 + 3z^2 \le 1 \\ 3y^2 \le 125 \\ y - 2z^2 \ge 0 \end{array}
\qquad
(4)\ \min\ \begin{array}{l} (x+y)^2 \\ x - y = 0 \\ e^x \le 1 \end{array}
$$

$$
(1)\ \min\ \begin{array}{l} 2x^2 + 2y^2 + xy \\ 15x + 50y \ge 17 \\ 0.001x - y = 0 \\ x^2 + y^2 = 1/3 \end{array}
\qquad
(2)\ \min\ \begin{array}{l} 3x^2 + 3y^2 + 5xy \\ 50x + 50y \le 530 \\ 5x - y \le 3 \\ 2x^2 + 2y^2 + 3xy \le 5 \end{array}
$$

$$
(5)\ \min\ \begin{array}{l} x^2 + 3y + 5z \\ e^x + e^y \le 1 \\ x^2 + z^2 + 2xy \le 1 \end{array}
\qquad
(6)\ \min\ \begin{array}{l} 2x^2 + 2y^2 + 5xy \\ x^2 + y^4 \le 1 \\ x + y \ge 0.1 \\ 3x + 2y = 5 \end{array}
$$

2 Local and global minima (5 pts.)

Consider the following problem:

$$
\begin{array}{lrl} \max & y + x & \\ & 3x + y & \le 6 \\ & x + y & \ge 1 \\ & x + 3y & \le 6 \\ & x, y & \in \mathbb{Z} \end{array}
$$

1. Is the problem convex?
2. Is the solution $(x, y) = (1, 3)$ feasible?
3. Is the solution $(x, y) = (1.5, 1.5)$ feasible?
4. What is the value of the objective function that corresponds to each of the previous two solutions?
5. Are these two numbers lower bounds, upper bounds, none, or both?
6. Eliminate the last constraint (integrality of x and y). Is the new problem convex?
7. Are the two solutions above feasible for the relaxation?
8. Are two objective function values lower bounds, upper bounds, none, or both, for the relaxation?
9. Solve the relaxation by the graphical method. What does the optimal value of the relaxation give you in terms of the original problem?

3 Linear Programming (5 pts.)

Your boss plans to throw a big office Christmas party and wants to serve his specialty cocktail where he mixes Smirnoff Mojito Mix, Smirnoff Vodka and lime juice. He hires you as the optimization expert to figure out the proportions in which to mix the three components to minimize his total cost. Your Christmas bonus depends on your success! He has the following requirements for the cocktail:

- 1. The total percentage of alcohol in the cocktail should be between 15% and 25%
- 2. The total percentage of sugar should be at most 10%.

Assume that vodka costs C_V dollars per fl.oz., mojito mix costs C_M dollars per fl.oz. and lime juice costs C_L per fl.oz. The percentage of alcohol in vodka is 40%, the percentage of alcohol in the mojito mix is 15%. Lime juice contains no alcohol. The percentage of sugar in the mojito mix is 20%, and in lime juice is 4%, while in vodka the content of sugar is negligeble.

Formulate a linear programming problem to find the optimal mixing proportions for an acceptable cocktail. Explain your variables and constraints.

4 Knapsack problem (5 pts.)

Consider the knapsack problem similar to the one described in our lecture notes, but in which we maximize the profit while bounding the total weight

$$\mathbf{P} : \max \quad \begin{array}{l} \sum_{i=1}^{n} p_i x_i \\ \sum_{i=1}^{n} w_i x_i \leq W \\ x_i \in \{0, 1\} \quad \forall i = 1, 2, \ldots, n \end{array}$$

Suppose there are $n = 9$ objects and $W = 4$.

i	1	2	3	4	5	6	7	8	9
p_i	30	13	6	7	21	35	10	31	20
w_i	2	1	1	1	2	3	3	3	2

1. Apply greedy heuristic (the method we used in class, by sorting items according to the price/weight ratio) to find a feasible solution for this problem. Show that you cannot improve this solution by swapping only one object for any other, i.e. moving one of the "1"s to a different position. Yet observe that this solution is not a global optimum. Does it give you an upper bound or a lower bound on the optimal value? (Note that you are maximizing.)

2. Solve the linear programming relaxation of this problem. What solution did you obtain? Does it give you an upper bound or a lower bound to the optimal value of the original problem.

3. Find the global optimal solution. Can you prove that it is the global optimal solution by just looking and upper and lower bounds you obtained so far?