ISE426 Optimization models and applications, Fall 2015
Homework 4 Solution,

1. Integer Programming/Scheduling Formulation (5 pts.)

Define binary variable x_{ij} as following,

$$x_{ij} = \begin{cases} 1 & \text{if job } j \text{ is processed on processor } i \\ 0 & \text{else} \end{cases}$$

ILP formulation of this problem is as following,

$$\begin{aligned} \min\ & C \\ s.t. & \sum_{j=1}^{m} x_{ij} t_j \le C, && \forall i = \{1,..,n\}, \\ & \sum_{i=1}^{n} x_{ij} = 1, \\ & x_{ij} \in \{0,1\}, && \forall i \in \{1,..,n\},\ \forall j \in \{1,..,m\}, \end{aligned}$$

□ *Interpretation:* The set of first constraint implies $C = \max_{i=1,..,n}\{\sum_{j=1}^{m} x_{ij}t_j\}$, so C is the maximum completion time of all processors which are processing different number of jobs, so in terms of minimizing the time of completion of all given tasks, the objective function has to minimize C .

Clearly the second constraint is enforcing each job to be assigned to just “one” processor.

2. Integer Programming/Matching Formulation(5 pts.)

1. Define binary variable x_{ij} as following,

$$x_{ij} = \begin{cases} 1 & \text{if socks } i \text{ is worn with socks } j \text{ in same day} \\ 0 & \text{else} \end{cases}$$

ILP formulation of this problem is as following,

$$\begin{aligned} \min\ & \sum_{i=1}^{20}\sum_{j=1}^{20} e_{ij}x_{ij}, \\ & \sum_{j\neq i}^{20} x_{ij} = 1, && \forall i \in \{1,2,..,20\}, \\ & x_{ij} = x_{ji}, && \forall \{i,j\} \in \{1,2,..,20\}, \\ & x_{ij} \in \{0,1\}, && \forall \{i,j\} \in \{1,2,..,20\}, \end{aligned}$$

The set of first constraint implies that socks i can be worn just one day with another socks.

2. In this part we need to define another two binary variables y_{ij} and z_{ij} as following,

$$y_{ij} = \begin{cases} 1 & \text{if a short is worn with socks } i \text{ and } j \text{ in same day} \\ 0 & \text{else} \end{cases}$$

$$z_{ij} = \begin{cases} 1 & \text{if a long pant is worn with socks } i \text{ and } j \text{ in same day} \\ 0 & \text{else} \end{cases}$$

ILP formulation of this problem is as following,

$$\begin{aligned}
\min \; & \sum_{i=1}^{20}\sum_{j=1}^{20}(s_{ij}y_{ij} + l_{ij}z_{ij}), \\
& \sum_{j\neq i}^{20} x_{ij} = 1, \qquad (1) \qquad \forall i \in \{1,2,..,20\}, \\
& \sum_{i=1}^{20}\sum_{j=i+1}^{20} y_{ij} = 5, \qquad (2) \\
& \sum_{i=1}^{20}\sum_{j=i+1}^{20} z_{ij} = 5, \qquad (3) \\
& y_{ij} + z_{ij} = x_{ij}, \quad (4) \qquad \forall \{i,j\} \in \{1,2,..,20\}, \\
& x_{ij} = x_{ji},\ y_{ij} = y_{ji},\ z_{ij} = z_{ji} \quad \forall \{i,j\} \in \{1,2,..,20\}, \\
& x_{ij} \in \{0,1\}, \qquad \forall \{i,j\} \in \{1,2,..,20\},
\end{aligned}$$

The set of first constraints (2) and (3) imply this request that Bob wears shorts on five days, and long pants on five other days.

The set of constraint (4) implies that a short or a long pant can be worn with socks i and j if socks i and j are worn in the same day.

3. Graph optimization (5 pts.)

1. Consider this social network as a graph $G = (V, E)$ where V, the node set, denotes the set of people and E, the edge set, indicates whether or not there is a connection. In other words, the edge (i, j) belongs to E if and only if the people i and j are mutual friends. In the words of graph theory, these two nodes are referred to as adjacent. Now, suppose that the graph nodes are labeled with $1, 2, ..., n$ where n stands for the number of people. Then, the following definition is in order.

$$x_i = \begin{cases} 1 & \text{if i is chosen as a member of group} \\ 0 & \text{Otherwise} \end{cases}$$

Consequently, a group S of mutual friends (adjacent nodes) consists of a collection of people (nodes) $i \in S$ where

$$\begin{aligned} x_i + x_j \leq 1 \quad & \forall (i,j) \notin E, \\ x_i \in \{0,1\} \quad & \forall i \in 1, ..., n, \end{aligned}$$

and S is a subset of $\{1, 2, ..., n\}$. In fact, the constraint $x_i + x_j \leq 1$ implies that either of i or j or both has to be excluded from the group if they are not connected. Now, the problem of finding the largest group is formulated as

$$\max \sum_{i=1}^{n} x_i$$
$$\text{s.t.}$$
$$x_i + x_j \leq 1 \quad \forall (i,j) \notin E$$
$$x_i \in \{0,1\} \quad \forall i = 1,...,n$$

2. Assume that x^* is an n-dimensional incidence vector which denotes the optimal solution to the problem we just formulated in part (a). As we deal with a binary integer program, we only need to append the model with

$$\sum_{i \notin B} x_i \geq 1$$

to find the next maximum social group, where $B = \{i \,|x_i^* = 1,\ i = 1,...,n\}$ denotes the set of members in the current optimal solution and the operator $|.|$ stands for the size of this group. In a way, this constraint implies that we are looking for another social group which includes at least one new person. Hence, we have

$$\max \sum_{i=1}^{n} x_i$$
$$\text{s.t.}$$
$$x_i + x_j \leq 1 \quad \forall (i,j) \notin E$$
$$\sum_{i \notin B} x_i \geq 1$$
$$x_i \in \{0,1\} \quad \forall i = 1,...,n$$

4. Flow problem and goal programming (5 pts.)

1.

$$x_{AB} + x_{AC} + x_{AD} = 8 \tag{1}$$
$$x_{AB} = x_{BC} + x_{BE} \tag{2}$$
$$x_{AC+BC} = x_{CD} + x_{CE} \tag{3}$$
$$x_{AD} + x_{CD} = x_{DE} \tag{4}$$
$$x_{AB} \leq 2 \tag{5}$$
$$x_{AC} \leq 4 \tag{6}$$
$$x_{AD} \leq 3 \tag{7}$$
$$x_{BC} \leq 1 \tag{8}$$
$$x_{BE} \leq 3 \tag{9}$$
$$x_{CD} \leq 1 \tag{10}$$
$$x_{CE} \leq 1 \tag{11}$$
$$x_{DE} \leq 2 \tag{12}$$
$$x_{AB}, x_{AC}, x_{AD}, x_{BC}, x_{BE}, x_{CD}, x_{CE}, x_{DE} \geq 0 \tag{13}$$

Where constraints 1-4 are flow conservation constraints and 5-12 imply capacity constraints.

The min cut is $x_{AB} + x_{CE} + x_{DE} \leq 5 < 8$; Hence, the problem is infeasible.

2. Stage 1:

$$
\begin{aligned}
\min \quad & y_4^+ + y_5^+ + y_6^+ + y_7^+ + y_8^+ + y_9^+ + y_{10}^+ + y_{11}^+ \\
s.t. \quad & x_{AB} + x_{AC} + x_{AD} = 8 \\
& x_{AB} = x_{BC} + x_{BE} + y_1^+ - y_1^- \\
& x_{AC}{+}_{BC} = x_{CD} + x_{CE} + y_2^+ - y_2^- \\
& x_{AD} + x_{CD} = x_{DE} + y_3^+ - y_3^- \\
& x_{AB} \leq 2 + y_4^+ \\
& x_{AC} \leq 4 + y_5^+ \\
& x_{AD} \leq 3 + y_6^+ \\
& x_{BC} \leq 1 + y_7^+ \\
& x_{BE} \leq 3 + y_8^+ \\
& x_{CD} \leq 1 + y_9^+ \\
& x_{CE} \leq 1 + y_{10}^+ \\
& x_{DE} \leq 2 + y_{11}^+ \\
& x_{AB}, x_{AC}, x_{AD}, x_{BC}, x_{BE}, x_{CD}, x_{CE}, x_{DE} \geq 0 \\
& y_i^+ \geq 0 \qquad \text{for } i = 1, ..., 11 \\
& y_i^- \geq 0 \qquad \text{for } i = 1, ..., 3
\end{aligned}
$$

The optimal solution of this problem is $y_i^{+*} = 0$ for i = 4 ,..., 11

Now by substitution of this optimal solution in the model of stage 1, we will have

Stage 2:

$$
\begin{aligned}
\min \quad & y_1^+ + y_1^- + y_2^+ + y_2^- + y_3^+ + y_3^- \\
s.t. \quad & x_{AB} + x_{AC} + x_{AD} = 8 \\
& x_{AB} = x_{BC} + x_{BE} + y_1^+ - y_1^- \\
& x_{AC} + _{BC} = x_{CD} + x_{CE} + y_2^+ - y_2^- \\
& x_{AD} + x_{CD} = x_{DE} + y_3^+ - y_3^- \\
& x_{AB} \le 2 + y_4^+ \\
& x_{AC} \le 4 + y_5^+ \\
& x_{AD} \le 3 + y_6^+ \\
& x_{BC} \le 1 + y_7^+ \\
& x_{BE} \le 3 + y_8^+ \\
& x_{CD} \le 1 + y_9^+ \\
& x_{CE} \le 1 + y_{10}^+ \\
& x_{DE} \le 2 + y_{11}^+ \\
& x_{AB}, x_{AC}, x_{AD}, x_{BC}, x_{BE}, x_{CD}, x_{CE}, x_{DE} \ge 0 \\
& y_i^+ = 0 \qquad \text{for } i = 4, ..., 11 \\
& y_i^+, y_i^- \ge 0 \qquad \text{for } i = 1, ..., 3
\end{aligned}
$$

3. Stage 1:

$$
\begin{aligned}
\min \quad & y_1^+ + y_1^- + y_2^+ + y_2^- + y_3^+ + y_3^- \\
s.t. \quad & x_{AB} + x_{AC} + x_{AD} = 8 \\
& x_{AB} = x_{BC} + x_{BE} + y_1^+ - y_1^- \\
& x_{AC} + _{BC} = x_{CD} + x_{CE} + y_2^+ - y_2^- \\
& x_{AD} + x_{CD} = x_{DE} + y_3^+ - y_3^- \\
& x_{AB} \le 2 + y_4^+ \\
& x_{AC} \le 4 + y_5^+ \\
& x_{AD} \le 3 + y_6^+ \\
& x_{BC} \le 1 + y_7^+ \\
& x_{BE} \le 3 + y_8^+ \\
& x_{CD} \le 1 + y_9^+ \\
& x_{CE} \le 1 + y_{10}^+ \\
& x_{DE} \le 2 + y_{11}^+ \\
& x_{AB}, x_{AC}, x_{AD}, x_{BC}, x_{BE}, x_{CD}, x_{CE}, x_{DE} \ge 0 \\
& y_i^+ \ge 0 \qquad \text{for } i = 1, ..., 11 \\
& y_i^- \ge 0 \qquad \text{for } i = 1, ..., 3
\end{aligned}
$$

The optimal solution of this problem is $y_i^{+*} = 0$ for i = 1 ,..., 3 and $y_i^{-*} = 0$ for i = 1 ,..., 3. Now by substitution of this optimal solution in the model of stage 1, we will have

Stage 2:

$$\begin{aligned}
\min \quad & y_4^+ + y_5^+ + y_6^+ + y_7^+ + y_8^+ + y_9^+ + y_{10}^+ + y_{11}^+ \\
s.t. \quad & x_{AB} + x_{AC} + x_{AD} = 8 \\
& x_{AB} = x_{BC} + x_{BE} + y_1^+ - y_1^- \\
& x_{AC+BC} = x_{CD} + x_{CE} + y_2^+ - y_2^- \\
& x_{AD} + x_{CD} = x_{DE} + y_3^+ - y_3^- \\
& x_{AB} \leq 2 + y_4^+ \\
& x_{AC} \leq 4 + y_5^+ \\
& x_{AD} \leq 3 + y_6^+ \\
& x_{BC} \leq 1 + y_7^+ \\
& x_{BE} \leq 3 + y_8^+ \\
& x_{CD} \leq 1 + y_9^+ \\
& x_{CE} \leq 1 + y_{10}^+ \\
& x_{DE} \leq 2 + y_{11}^+ \\
& x_{AB}, x_{AC}, x_{AD}, x_{BC}, x_{BE}, x_{CD}, x_{CE}, x_{DE} \geq 0 \\
& y_i^+ = 0 \qquad \text{for } i = 1, ..., 3 \\
& y_i^- = 0 \qquad \text{for } i = 1, ..., 3 \\
& y_i^+ \geq 0 \qquad \text{for } i = 4, ..., 11
\end{aligned}$$

4.

The solution of part 2 is

$x_{AB} = 2, x_{AC} = 3, x_{AD} = 3, x_{BC} = 0, x_{BE} = 2, x_{CD} = 0, x_{CE} = 1, x_{DE} = 2, y_2^+ = 2, y_3^+ = 1$

The solution of part 3 is

$x_{AB} = 2, x_{AC} = 3, x_{AD} = 3, x_{BC} = 0, x_{BE} = 2, x_{CD} = 2, x_{CE} = 1, x_{DE} = 5, y_{11}^+ = 3$

The value of objective function for both of these two problems is 3, but for the first solution, the constraints 3 and 4 (both of them are flow conservation constraints) are violated and for the second problem the capacity constraint 12 is violated.

5. Let

$$z_i = \begin{cases} 1 & \text{if corresponding capacity constraint is violated.} \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{array}{ll}
\min & z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 \\
s.t. & x_{AB} + x_{AC} + x_{AD} = 8 \\
 & x_{AB} = x_{BC} + x_{BE} \\
 & x_{AC} +_{BC} = x_{CD} + x_{CE} \\
 & x_{AD} + x_{CD} = x_{DE} \\
 & x_{AB} = 2 + 2z_1 \\
 & x_{AC} = 4 + 4z_2 \\
 & x_{AD} = 3 + 3z_3 \\
 & x_{BC} = 1 + z_4 \\
 & x_{BE} = 3 + 3z_5 \\
 & x_{CD} = 1 + z_6 \\
 & x_{CE} = 1 + z_7 \\
 & x_{DE} = 2 + 2z_8 \\
 & x_{AB}, x_{AC}, x_{AD}, x_{BC}, x_{BE}, x_{CD}, x_{CE}, x_{DE} \geq 0 \\
 & z_i \in \{0, 1\} \qquad \text{for } i = 1, ..., 8
\end{array}
$$