ISE429. Homework 5

1) (weight 0.25) $\{X_t\}$ is a standard one-dimensional Brownian motion. ("Standard" means that variance parameter is 1, $X_0 = 0$, and the drift parameter is 0.) Find the following probabilities:
$P(X_2 > -2)$,
$P(X_1 > X_2)$,
$P(X_2 < X_1 < X_3)$,
$P(X_t = 0$ for some $t$ with $2 \leq t \leq 3)$,
$P(X_t < 4$ for all $t$ with $0 < t \leq 9)$,
$P(X_t < -10$ for all $t > 10)$.

*Comment.* You do not have to produce actual numbers. It is ok to leave answers in terms of intergals, without numerically evaluating them. But, it has to be an expression that can be numerically evaluated. (If you also produce the numbers, that is a useful thing to practice. But, it will not affect the grade.)

*Solution.* In all solutions, we denote by

$$\phi(x; \alpha) = \frac{1}{\sqrt{2\pi\alpha}} e^{-x^2/(2\alpha)}$$

the density of a normal distribution with zero mean and variance $\alpha > 0$; and for CDF and complementary CDF:

$$\Phi(x; \alpha) = \int_{-\infty}^{x} \phi(y; \alpha) dy, \qquad \bar{\Phi}(x; \alpha) = 1 - \Phi(x; \alpha).$$

Also we say $Z$ is $N(b, \alpha)$ to mean '$Z$ has normal distribution with mean $b$ and variance $\alpha$'.

$P(X_2 > -2) = \bar{\Phi}(-2; 2)$

$P(X_1 > X_2) = 1/2$, because $X_2 - X_1$ is $N(0, 1)$, independent of $X_1$

$P(X_2 < X_1 < X_3) = P(Y_2 = X_1 - X_2 > 0,\ Y_3 = X_2 - X_3 < -Y_2)$, where $Y_2, Y_3$ are independent $N(0, 1)$. Then,

$$P(Y_2 > 0,\ Y_3 < -Y_2) = \int_0^\infty \phi(y, 1)\Phi(-y, 1) dy.$$

Conditioning on $X_2$ and using symmetry of Brownian motion and reflection princile,

$$P(X_t = 0 \text{ for some } t \text{ with } 2 \leq t \leq 3) = 2P(X_2 \geq 0,\ \min_{2 \leq t \leq 3} [X_t - X_2] \leq -X_2) =$$

$$2 \int_0^\infty \phi(y, 2) 2\bar{\Phi}(y, 1) dy.$$

$P(X_t < 4 \text{ for all } t \text{ with } 0 < t \le 9) = 1 - 2\bar{\Phi}(4/\sqrt{9}, 1)$.

$P(X_t < -10 \text{ for all } t > 10) = 0$.

2) (weight 0.25) $\{X_t\}$ and $\{Y_t\}$ are independent standard one-dimensional Brownian motions. Find the following probability:
$P(X_t = Y_t \text{ for infinitely many } t > 10)$.

*Solution.* $Z_t = X_t - Y_t$ is a Brownian motion. (This needs to be proved, which is straightforward to do. We mentioned this in class as well.) It returns to any value, from any initial state, infinitely many times with prob. 1.
$P(Z_t = 0 \text{ for infinitely many } t > 10) = 1$.

3) (weight 0.25) $\{X_t\}$ is a standard one-dimensional Brownian motion. Find the density of $M = \max\{X_t \;:\; 0 \le t \le 10\}$ and compute its expectation and variance.

*Comment.*. You can leave answers in terms of intergrals, that can be evaluated numerically. (If you also produce the numbers, that is a useful thing to practice. But, it will not affect the grade.)

*Solution.*

$$P(M > x) = 2\bar{\Phi}(x/\sqrt{10}; 1) = 2\int_{x/\sqrt{10}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \quad x \ge 0.$$

Density:

$$f(x) = \frac{2}{\sqrt{2\pi \cdot 10}} e^{-x^2/(2\cdot 10)}, \quad x \ge 0.$$

$$EM = \int_0^{\infty} x f(x) dx = 20/\sqrt{20\pi},$$

$$E(M^2) = \int_0^{\infty} x^2 f(x) dx = 10,$$

$$Var\ M = E(M^2) - (EM)^2 = 10 - 20/\pi.$$

4) (weight 0.25) $\{X_t\}$ is a standard one-dimensional Brownian motion. Find the density of $T = \min\{t \;:\; X_t = 1\}$ and prove that $ET = \infty$.

*Solution.*

$$P(T \le s) = P(\max_{0 \le t \le s} X_t \ge 1) = 2\bar{\Phi}(1/\sqrt{s}, 1) = 2\int_{1/\sqrt{s}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

Density:

$$f(s) = \frac{2}{\sqrt{2\pi}} e^{-1/(2s)} (1/2) s^{-3/2}, \quad s \ge 0.$$

As $s \to \infty$, $f(s) \sim C s^{-3/2}$, that is $f(s)/s^{-3/2} \to C > 0$ Then,

$$ET = \int_0^{\infty} s f(s) ds = \infty.$$