ISE429. Homework 2

Unless stated otherwise, the problems are from the Ross textbook (11th edition):

1) (weight 0.20) 5-12 (a,b,c).

2) (weight 0.15) 6-9

3) (weight 0.15) 6-15. *Comment: If Markov chain is in a stationary regime, with stationary distribution $\{\pi_j\}$, then the time-average rate at which transtions $i \to j$ occur is $\pi_i q_{ij}$.*

4) (weight 0.15) 6-22

5) (weight 0.15) 6-23

6) (weight 0.20) Consider the continuous-time birth-death (Markov) process with state space $\mathcal{X} = \{0, 1, 2, 3, \ldots\}$. The transition rate to the "left" ("death" rate) is always equal to 1. The transition rate to the "right" (birth rate) depends on state n and is equal to λ_n.
(a) Suppose $\lambda_n = (1/2) + 10e^{-n},\ n = 0, 1, 2, \ldots$. Is this Markov chain transient?
(b) Suppose $\lambda_n = 1 - (1/2)e^{-n},\ n = 0, 1, 2, \ldots$. Is this Markov chain positive recurrent?

Solution. This Markov chain (MC) is of course irreducible.
(a) This Markov chain is positive recurrent; and therefore is not transient. We have $\lambda_n \to 1/2$ as $n \to \infty$; so, $\lambda_n < 3/4$ for large n; so $\prod_{n=0}^{k} \lambda_n \le C(3/4)^k$, for all $k \ge K$, for some constanst C, K. We have

$$\sum_{k=0}^{\infty} \prod_{0}^{k} \lambda_n < \infty,$$

which implies positive recurrence.
(b) This Markov chain is not positive recurrent. We have

$$\prod_{n=0}^{k} \lambda_n = \prod_{n=0}^{k} (1 - (1/2)e^{-n}) \to \prod_{n=0}^{\infty} (1 - (1/2)e^{-n}) > 0, \quad k \to \infty.$$

(You can check this by showing that $\log \prod_{n=0}^{k} \lambda_n \to \sum_{n=0}^{\infty} \log(1 - (1/2)e^{-n}) > -\infty$.) Then,

$$\sum_{k=0}^{\infty} \prod_{0}^{k} \lambda_n = \infty,$$

so the process cannot be positive recurrent. (By the way, do you think it is transient or null-recurrent?) $\square$