ISE 429 Test #2 Name: Bolun Xu.

ISE429. Test 2

1 (weight 0.35)

Random variables $Y_1, Y_2, \ldots$ are independent, identically distributed, each has the exponential distribution with mean 2. Let $T = \min\{n \mid Y_1 + Y_2 + \ldots + Y_n > 9\}$. Find $\mathbb{P}\{Y_1 + Y_2 + \ldots + Y_T + Y_{T+1} + Y_{T+2} > 10\}$.

solution: Define $S_n = Y_1 + \cdots + Y_n$,

we can formulate the model as a poisson process,

since Y_n are independent, identically distributed and each has the exponetial distribution, with $\frac{1}{\lambda} = 2$.

So, the problem becomes to:

$$P\{S_{T+2} > 10 \mid S_{T-1} \leq 9, S_T > 9\}$$

$$= P\{N(10) < T+2 \mid N(9) \geq T-1, N(9) < T\}.$$

$$= P\{N(10) < T+2 \mid N(9) = T-1\}.$$

$$= P\{N(1) < 3\}.$$

$$= P\{S_3 > 1\}$$

$$= 1 - P\{S_3 \leq 1\}.$$

$\lambda = \frac{1}{2}$

$$= 1 - \sum_{j=3}^{\infty} e^{-\frac{1}{2}} \cdot \frac{(\frac{1}{2})^j}{j!}$$

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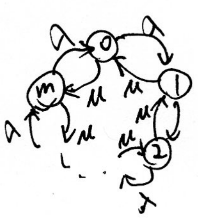
2) (weight 0.30)

(a) Consider a continuous time Markov chain $\{X(t)\}$ with $m+1$ states ($m \geq 1$), $\{0, 1, \ldots, m\}$. Let $\lambda > \mu > 0$. The transition rates are: $q_{i,i+1} = \lambda$ and $q_{i+1,i} = \mu$ for $i = 0, \ldots, m-1$; $q_{m,0} = \lambda$ and $q_{0,m} = \mu$. All other $q_{i,j} = 0$. Does this Markov chain have a stationary distribution? Is it unique? If so, what is it? Is this Markov chain reversible w.r.t. its stationary distribution? What are the transition rates of the time-reversed (stationary) Markov chain?
(b) Same Markov chain as in (a), except $q_{m,0} = q_{0,m} = 0$. Does this Markov chain have a stationary distribution? Is it unique? If so, what is it? Is this Markov chain reversible w.r.t. its stationary distribution? What are the transition rates of the time-reversed (stationary) Markov chain?

Solution: (a) Obviously, it is pos. REC. M.C.,

① so it has a stationary distribution ② and it's unique.



we get

STATE	Rate at which leave = rate at which enter
0	$(\lambda+\mu)P_0 = \mu P_1 + \lambda P_m$
1	$(\lambda+\mu)P_1 = \mu P_2 + \lambda P_0$
2	$(\lambda+\mu)P_2 = \mu P_3 + \lambda P_1$
⋮	⋮
m	$(\lambda+\mu)P_m = \mu P_0 + \lambda P_{m-1}$

$\because \sum_{n=0}^{m} P_n = 1$

③ $\therefore P_i = \frac{1}{m+1}, \quad i = 0, \cdots, m$

④. obviously, it is not time-reversible M.C

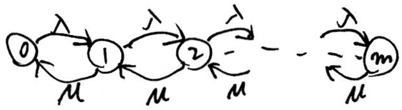
Since $P_0 q_{01} \neq P_1 q_{10}$. But when we reverse this M.C., Its stationary distribution does not change after similar calculation above.

⑤ $P_i q^*_{i,i+1} = P_{i+1} \cdot q_{i+1,i} \Rightarrow \begin{cases} q^*_{i,i+1} = \mu, \quad q^*_{i+1,i} = \lambda \text{ for } i = 0, \cdots, m-1 \\ q^*_{m,0} = \mu \text{ and } q^*_{0,m} = \lambda \end{cases}$

(b) obviously, it is POS. REC. M.C.

① So it has a stationary distribution, ② and it is unique.

③



We obtain

STATE	Rate at which leave = rate at which enter.
0	$\lambda P_0 = \mu P_1$
1	$(\lambda+\mu)P_1 = \mu P_2 + \lambda P_0$
2	$(\lambda+\mu)P_2 = \mu P_3 + \lambda P_1$
⋮	⋮
m	$\mu P_m = \lambda P_{m-1}$

$$\therefore\ P_i = \left(\frac{\lambda}{\mu}\right)^i P_0,\quad i = 0 \cdots m,$$

$$\because\ \sum_{n=0}^{m} P_n = 1$$

$$\therefore\ P_0\left[1 + \frac{\lambda}{\mu} + \cdots + \left(\frac{\lambda}{\mu}\right)^m\right] = 1$$

$$\therefore\ P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{m+1}},\quad P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{m+1}},\quad n = 0, 1 \cdots m.$$

④. As is listed above, $\lambda P_i = \mu P_{i+1} \Leftrightarrow$

$$\left.\begin{array}{l} \Leftrightarrow P_i \cdot q_{i,i+1} = P_{i+1} \cdot q_{i+1,i} \\ \sum_{n=0}^{m} P_n = 1 \end{array}\right\} \Rightarrow \text{It is time-reversible M.C.}$$

⑤ Since it is a time-reversible M.C., its transition rates of the time-reversed M.C. remain the same as above, so does the distribution.

3) (weight 0.35)

There are two types of light bulbs that you use in your desk lamp, 1 and 2. Type 1 is cheaper and lasts exactly 1 unit of time (say, month). Type 2 is more expensive and lasts exactly $\sqrt{3}$ units of time. Consider two replacement strategies. Assume that a replacement takes zero time.

(a) You start with type 1, then replace with type 2, then type 1, and so on. $Y(t)$ is the residual (excess) time of the current bulb (whatever type it happens to be) at time t. What is the limiting fraction of time that $Y \geq 1/2$? Namely, what is

$$\phi = \lim_{t\to\infty} (1/t) \int_0^t \mathbb{P}\{Y(s) \geq 1/2\} ds.$$

Does the limit

$$\psi = \lim_{t\to\infty} \mathbb{P}\{Y(t) \geq 1/2\}$$

exist, and if so, what is it?

(b) You start with type 1. When it is time to replace a bulb, you replace it with the same type with probability 9/10, and change the type with prob. 1/10. $Y(t)$ is the residual (excess) time of the current bulb (whatever type it happens to be) at time t . What is the limiting fraction of time that $Y \geq 1/2$? Namely, what is

$$\phi = \lim_{t\to\infty} (1/t) \int_0^t \mathbb{P}\{Y(s) \geq 1/2\} ds.$$

Does the limit

$$\psi = \lim_{t\to\infty} \mathbb{P}\{Y(t) \geq 1/2\}$$

exist, and if so, what is it?

Comment: You do not need to worry about direct integrability. But, have to substantiate everything else you do.

Solution: (a) In this process there exists a time $T_1 = \sqrt{3}+1$, such that the Continuation of the process beyond T_1 is a probabilistic replica of the whole process start at 0.

So, this process is a regenerative process. Define T_n as cycle time in sequence n.

$\psi = \lim_{t\to\infty} P\{Y(t) \geq \frac{1}{2}\}$ exist.

$$\psi = \lim_{t\to\infty} P\{Y(t) \geq \tfrac{1}{2}\} = \frac{E[\text{amount of time that } Y(t) \geq \frac{1}{2} \text{ in Cycle } T_1]}{E[T_n]} =$$

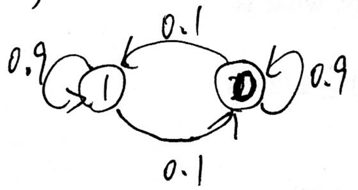
3

$$= \frac{[1-\frac{1}{2}]+[\sqrt{3}-\frac{1}{2}]}{\sqrt{3}+1} = \frac{\sqrt{3}}{\sqrt{3}+1} = \frac{3-\sqrt{3}}{2}$$

(b) Assume state 0 as the state that bulb 1 is in use.

state 1 as the state that bulb 2 is in use.

So,



It is a pos. REC. Markov Process.

$$\begin{cases} \pi_0 = 0.9\pi_0 + 0.1\pi_1 \\ \pi_1 = 0.1\pi_0 + 0.9\pi_1 \\ \pi_0 + \pi_1 = 1 \end{cases} \Rightarrow \pi_0 = \pi_1 = 0.5$$

As mentioned in the book, pos. REC. Markov process is a regenerative process.

So, $\psi = \lim_{t\to\infty} P\{Y(t) \geq \frac{1}{2}\}$ Exist.

$$\psi = \lim_{t\to\infty} P\{Y(t) \geq \tfrac{1}{2}\} = \frac{\pi_0(T_0 - \frac{1}{2}) + \pi_1 (T_1 - \frac{1}{2})}{\pi_0 \cdot T_0 + \pi_1 \cdot T_1} = \frac{\frac{1}{2}(1-\frac{1}{2}) + \frac{1}{2}(\sqrt{3}-\frac{1}{2})}{\frac{1}{2}\cdot 1 + \frac{1}{2}\sqrt{3}} =$$

$$= \frac{\sqrt{3}}{\sqrt{3}+1} = \frac{3-\sqrt{3}}{2}$$