ISE 426
Optimization models and applications

Lecture 20 — November 10, 2015

- Bin Packing Problem
- Cutting Stock Problem
- Column generation

The bin packing problem

- Given a set of N bins of volume V and n_i objects of volumes v_i, $i = 1, \ldots, n$.
- We want to pack the objects into bins using as few bins as possible.
- Consider $V = 11$, $n = 3$ from which $n_1 = 20$ objects have $v_1 = 5$, $n_2 = 10$ objects have $v_2 = 4$ and $n_3 = 9$ objects have $v_3 = 2$;
- Let us try a gready heristic, we get: 10 bins with two $(5, 5)$ objects, 5 bins with $(4, 4, 2)$ objects and 1 bin with $(2, 2, 2, 2)$ objects.
- Clearly 9 bins with $(5, 4, 2)$, 5 bins with $(5, 5)$ and one bin with $(5, 4)$ is better.

How do we model this as an Optimization model?

- y_i is a binary variable that indicates if a bin i is being used.
- x_{ij} is an integer variable that indicates how many objects of size j has been assigned to bin i.
- Clearly $x_{ij} \leq My_i$.

$$
\begin{array}{ll}
\min & \sum_{i=1}^{N} y_i \\
& \sum_{j=1}^{n} v_j x_{ij} \leq V y_i \\
& \sum_{i=1}^{N} x_{ij} = n_j \quad \forall j = 1 \ldots, n \\
& x_{ij} \in \mathbf{Z} \quad \forall i = 1, \ldots, N, \ j = 1 \ldots, n \\
& y_i \in \{0, 1\} \quad \forall i = 1, \ldots, N
\end{array}
$$

Weak formulation, too much symmetry, each bin is the same.
May help to add constraints

$$y_1 \geq y_2 \geq y_3 \geq \ldots \geq y_N$$

(Use the first bin first)

Formulation #2: extended formulation

- Consider all sets s_k of objects (patterns) that can fit into one bin.
- That is, $(5,5)$, $(5,4,2)$ $(4,4,2)$, $(4,2,2,2)$, $(2,2,2,2,2)$. (We do not need to consider $(5,4)$ because it is dominated by $(5,4,2)$)
- Let S be the set of all feasible patterns s_k, $|S| = K$.
- x_k is an integer variable indicating how many bins are filled with pattern s_k, $s_k \in S$.
- Let a_{jk} be the number of times object of size v_j appears in s_k, for instance for $s_k = (4,4,2)$, for $v_j = 4$, $a_{jk} = 2$ and for $v_j = 2$, $a_{jk} = 1$.

$$
\begin{array}{ll}
\min & \sum_{k=1}^{K} x_k \\
& \sum_{k=1}^{K} a_{jk} x_k \geq n_j \ \forall j = 1, \ldots, n \\
& x_k \in \mathbf{Z} \quad \forall k = 1, \ldots, K
\end{array}
$$

This is a "strong" formulation. The LP relaxation gives very good lower bounds.

Relaxation and another interpretation - cutting stock problem

$$\begin{array}{ll} \min & \sum_{k=1}^{K} x_k \\ & \sum_{k=1}^{K} a_{jk} x_k \geq n_j \quad \forall j = 1, \ldots, n \\ & x_k \geq 0 \quad \forall k = 1, \ldots, K \end{array}$$

- Given very long (infinite) roll of paper (or steel) of width V we need to cut paper into pieces of length n_j and width v_j, $j = 1, \ldots, n$.
- We are allowed to cut correct width, but smaller length and "glue" different lengths to obtain the right one.
- We want to use as little paper as possible.

Dual formulation

$$\begin{array}{ll} \max & \sum_{j=1}^{n} n_j y_j \\ & \sum_{j=1}^{n} a_{jk} y_j \leq 1 \ \forall k = 1, \ldots, K \\ & y_j \geq 0 \ \forall j = 1, \ldots, n \end{array}$$

- Each pattern corresponds to variable x_k, which in turn, corresponds to a constraint

$$\sum_{j=1}^{n} a_{jk} y_j \leq 1.$$

- Remember that given a basic feasible solutions we have a lot of $x_k = 0$.
- If the dual constraint for a given k is not feasible, then the corresponding x_k should be, possibly, nonzero.
- Column generation technique - generate k's for which

$$\sum_{j=1}^{n} a_{jk} y_j > 1$$

Column generation

$$\begin{array}{ll} \max & \sum_{j=1}^{n} n_j y_i \\ & \sum_{j=1}^{n} a_{jk} y_j \leq 1 \ \forall k = 1, \ldots, K \\ & y_j \geq 0 \ \forall j = 1, \ldots, n \end{array}$$

- Start with a few patterns and variables x_k (the rest of $x_k = 0$).
- Solve the primal problem with only those patterns.
- Compute the corresponding dual solution
- Column generation technique - generate k's for which

$$\sum_{j=1}^{n} a_{jk} y_j > 1$$

and include them into the primal problem.

Column generation

$$\sum_{j=1}^{n} a_{jk} y_j > 1$$

How do we find new k? Find k for which $\sum_{j=1}^{n} a_{jk} y_j$ is the largest.

$$\begin{array}{ll} \max & \sum_{j=1}^{n} a_{jk} y_j \\ & \sum_{j=1}^{n} v_j a_{jk} \leq V \ \forall k = 1, \ldots, K \\ & a_{jk} \in \mathbf{Z} \ \forall j = 1, \ldots, n \end{array}$$

The knapsack problems!!

Practice problems for the quiz

Reformulate using constraints of a Linear Programming problem or, if necessary, Mixed Integer Linear Programming Problem.

1. $\max\{|x|, |y|\} \leq 1$.(6pts)
2. $\max\{|x|, |y|\} \geq 1$.(6pts)

Formulation, Mixed Integer/Goal Programming

Kyra is organizing a large dinner party. There are k tables, each sitting n people. There are m men attending and w women. She needs to assign seats at the tables in such a way that the number of men and women at each table does not differ by more than two. Formulate this as a feasible set of an integer linear programming problem.

Is the above problem *always* feasible? Explain.

Write an integer linear optimization problem that minimizes the number of tables that violate the condition on the maximum difference between the number of men and women.

Reformulation using binary variables

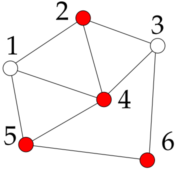
Consider a set of vectors $x \in R^n$ described by the following conditions.

$$\max_i \{x_1, x_2, x_3, \ldots, x_n\} \geq 1$$

Describe this set by using a set of linear constraints and binary variables, as we did in homework and class. In other words, you only should use linear constraints that can involve continuous and/or binary variables, and all feasible solutions for this set of constraints should give x that is feasible for the the above set and vice versa.

Integer Programming

Consider a graph $G = (V, E)$, and a cost C_{ij} for each edge $\{i, j\} \in E$. Suppose you want to find the subset S of V with at least k nodes, such that the cost of all edges, that link a node in S with a node outside of S is minimized. For example, if S is the set of four dark nodes is the graph below, then the total cost is $C_{12} + C_{14} + C_{15} + C_{23} + C_{34} + C_{36}$



1. Consider binary variables that indicate if a node is in S or not. Consider also binary variables that indicate if an edge is connecting a node in S with a node outside of S. Now write down conditions between these types of variables, which ensure logical implications: for all $\{i, j\} \in E$, if $i \in S$ and $j \in V/S$ then edge $\{i, j\}$ connects node in S with a node outside S.
2. Write the full formulation of the problem of selecting at least k nodes so that the edge cost is minimized.