HW5 Solutions

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1 SVM

AMPL:

```
param n := 35;
param m := 2;
set I := 1..n;
set J := 1..m;
param y{i in I};
param x{i in I,j in J};
param c;
var xi{i in I} >= 0;
var w1; var w2; var beta;
minimize obj: 1/2*(w1*w1+w2*w2) + c*sum{i in I} xi[i];
subject to con{i in I}: y[i]*(w1*x[i,1]+w2*x[i,2]+beta) >= 1-xi[i]; data;
param c := 10000;
param x : 1 2:=
1 -0.0192 0.4565
2 -0.0302 -0.8531
3 -0.1170 -0.9854
4 0.4454 0.3952
5 -0.7989 -0.2569
6 0.0935 0.7398
7 0.2654 0.3098
8 0.6040 -0.0959
9 -0.6324 -0.9139
10 0.9770 -0.4862
11 0.9260 0.0075
12 0.8055 -0.0103
13 0.3007 0.9564
14 -0.2771 0.1357
15 0.3782 0.6800
16 -0.5911 -0.1808
17 -0.2501 0.4231
18 -0.1130 0.8032
```

19 0.9353 -0.2590
20 -0.1272 0.9856
21 -0.0244 0.7780
22 0.2476 0.7701
23 0.1555 -0.8341
24 -0.9507 -1.0000
25 -0.6986 -0.0473
26 -0.4293 -0.9466
27 -0.8917 0.2226
28 0.1545 0.4526
29 -0.8230 0.7856
30 -0.5885 0.5231
31 -0.6578 -0.7660
32 -0.2009 -0.7598
33 0.0965 -0.8755
34 0.3399 0.8383
35 -0.0519 0.9419;
param y :=
1 1
2 -1
3 -1
4 1
5 1
6 1
7 1
8 -1
9 -1
10 -1
11 -1
12 -1
13 1
14 1
15 1
16 1
17 1
18 1
19 -1
20 1
21 1
22 1
23 -1
24 -1
25 1
26 -1
27 1
28 1

29 1
30 1
31 -1
32 -1
33 -1
34 1
35 1;

Modify c to see different results. 2

when c = 10000,

result:

ampl: model svm.mod;

ampl: option solver MINOS;

ampl: solve; MINOS 5.5: optimal solution found. 37 iterations, objective 8.483095223

Nonlin evals: obj = 38, grad = 37.

ampl: display xi;

xi [*] :=

1 0 5 0 9 0 13 0 17 0 21 0 25 0 29 0 33 0
2 0 6 0 10 0 14 0 18 0 22 0 26 0 30 0 34 0
3 0 7 0 11 0 15 0 19 0 23 0 27 0 31 0 35 0
4 0 8 0 12 0 16 0 20 0 24 0 28 0 32 0 ;

ampl: display w1;

w1 = -1.93193

ampl: display w2;

w2 = 3.63783

ampl: display beta;

beta = 0.515755

when c = 100,

result: ampl:

model svm.mod;

ampl: option solver MINOS;

ampl: solve;

MINOS 5.5:

optimal solution found. 38 iterations, objective 8.483095223

Nonlin evals: obj = 39, grad = 38.

ampl: display xi; xi [*] :=

1 0 5 0 9 0 13 0 17 0 21 0 25 0 29 0 33 0
2 0 6 0 10 0 14 0 18 0 22 0 26 0 30 0 34
0 3 0 7 0 11 0 15 0 19 0 23 0 27 0 31 0 35 0
4 0 8 0 12 0 16 0 20 0 24 0 28 0 32 0 ;

ampl: display w1;

w1 = -1.93193

ampl: display w2;

w2 = 3.63783

ampl: display beta;

beta = 0.515755

For c = 10000
and c = 100,
there is no constraint violations and there are 3 support vectors:
x8 , x16 , x31 .
when c = 1,
result: ampl: model svm.mod; 3
ampl: option sovler MINOS;
ampl: solve; MINOS 5.5:
optimal solution found. 30 iterations, objective 6.056335781
Nonlin evals: obj = 35, grad = 34.
ampl: display xi;
xi [*] :=

```
1 0 8 0.207215 15 0 22 0 29 0
2 0 9 0 16 0.311718 23 0 30 0
3 0 10 0 17 0 24 0.243217 31 0.388117
4 0.399453 11 0 18 0 25 0 32 0
5 0.200319 12 0.127322 19 0 26 0 33 0
6 0 13 0 20 0 27 0 34 0
7 0.349366 14 0 21 0 28 0 35 0
```

ampl: display w1;
w1 = -1.40852
ampl: display w2;
w2 = 2.38229
ampl: display beta;
beta = 0.286423
For c = 1, there are 3 support vectors x9 , x11 , x14 and 8 constraints violations which are 4 , 5 , 7 , 8 , 12 , 16 , 24 , 31

NOTE: you could use CPLEX as well, as it now solves convex quadratic problems.

2 Modeling (5pts)

The problem

$$\begin{aligned} \min_{\text{x}} \quad & \|x\|_1 \\ \text{s.t.} \quad & \|Ax-b\|_\infty \le \epsilon \end{aligned}$$

is equivalent to

$$\begin{aligned} \min_{\text{x}} \quad & \sum_i |x|_i \\ \text{s.t.} \quad & \max_j \{a_j^T x - b_j\} \le \epsilon \end{aligned}$$

which is in turn equivalent to

$$\begin{aligned}
\min_{\mathrm{x,z}} \quad & \sum y_i \\
\text{s.t.} \quad & a_j^T x - b_j \leq \epsilon \ \forall j = 1, \ldots, m \\
& y_i \geq x_i \\
& y_i \geq -x_i \ \forall i = 1, \ldots, n
\end{aligned} \tag{1}$$

3 Integer Programming

You currently own a portfolio of eight stocks. Using the Markowitz model, you computed the optimal mean/variance portfolio. The weights of these two portfolios are shown in the following table:

Stock	A	B	C	D	E	F	G	H
Your Portfolio	0.12	0.15	0.13	0.10	0.20	0.10	0.12	0.08
M/V Portfolio	0.02	0.05	0.25	0.06	0.18	0.10	0.22	0.12

You would like to rebalance your portfolio in order to be closer to the M/V portfolio. To avoid excessively high transaction costs, you decide to rebalance only three stocks from your portfolio. Let x_i denote the weight of stock i in your rebalanced portfolio. The objective is to minimize the quantity

$$|x_1 - 0.02| + |x_2 - 0.05| + |x_3 - 0.25| + \ldots + |x_8 - 0.12|$$

which measures how closely the rebalanced portfolio matches the M/V portfolio.

Formulate this problem as a mixed integer linear program.

Let a_i denote the weight of the i-th stock in the MV portfolio, and b_i denote the weight of the ith stock in your portfolio.

$$\begin{aligned}
\min_{\mathrm{z,x,t}} \quad & t_1 + t_2 + \ldots + t_8 \\
\text{s.t} \quad & x_i - a_i \leq t_i \quad i = 1, \ldots, 8 \\
& a_i - x_i \leq t_i \quad i = 1, \ldots, 8 \\
& x_i - b_i \leq z_i \quad i = 1, \ldots, 8 \\
& b_i - x_i \leq z_i \quad i = 1, \ldots, 8 \\
& \textstyle\sum_i z_i \leq 3 \\
& {}_i x \geq 0, t_i \geq 0, z_i \in \{0, 1\} \quad i = 1, \ldots, 8.
\end{aligned}$$

4 Quadratic Integer Programming - Support Vector Machines (practice problem only, do not turn in with homework)

The following formulation is the standard formulation of support vector machines. The second term accounts for the points that do not satisfy the constraints for separation. This term is called "hinge loss" and it depends linearly on the "amount of violation" for each constraint.

$$\min_{\xi,\mathrm{w},\beta}\frac{1}{2}\|w\|^2 + c\sum_{i=1}^{n}\max\{1-y_i(w^\top x_i+\beta),0\}$$

Now, imagine instead I want to count and minimize the *number* of points which violate the constraints (the number of "outliers"), hence I want to optimize

$$\begin{array}{rc} \min_{\xi,\mathrm{w},\beta} & \frac{1}{2}\|w\|^2 + c(\#\text{ of nonzeros}(\xi)) \\ \text{s.t} & y_i(w^\top x_i+\beta)\geq 1-\xi_i, \quad i=1,\dots,n \\ & \xi\geq 0 \end{array}$$

- Formulate this problem as a problem with integer variables, quadratic objective and linear constraints.

$$\begin{array}{rcc} \min_{\xi,\mathrm{w},\beta} & \frac{1}{2}\|w\|^2 + c\sum_i z_i & \\ \text{s.t} & y_i(w^\top x_i+\beta)\geq 1-\xi_i, \quad i=1,\dots,n & \\ & \xi\leq Mz_i & \xi\geq 0, z_i\in\{0,1\} \end{array}$$

- Write down the convex relaxation of this problem

$$\begin{array}{rc} \min_{\xi,\mathrm{w},\beta} & \frac{1}{2}\|w\|^2 + c\sum_i z_i \\ \text{s.t} & y_i(w^\top x_i+\beta)\geq 1-\xi_i, \quad i=1,\dots,n \\ & \xi\leq Mz_i \\ & \xi\geq 0, z_i\in[0,1] \end{array}$$

or equivalently

- Write down the convex relaxation of this problem

$$\begin{array}{rc} \min_{\xi,\mathrm{w},\beta} & \frac{1}{2}\|w\|^2 + \frac{c}{M}\sum_i \xi_i \\ \text{s.t} & y_i(w^\top x_i+\beta)\geq 1-\xi_i, \quad i=1,\dots,n \\ & \xi\geq 0. \end{array}$$