ISE426 Optimization models and applications, Fall 2014
Homework 2 Solution,

1. Max flow and duality (5 pts.)

1. The Corresponding primal model of the problem is as follows:

$$
\begin{aligned}
\max \quad & x_{CF}+x_{DF}+x_{EF} \\
x_{AB}-x_{BD}-x_{BC} &= 0 \quad \mathbf{u_1} \\
x_{BC}+x_{DC}-x_{CF} &= 0 \quad \mathbf{u_2} \\
x_{AD}+x_{BD}-x_{DC}-x_{DF}-x_{DE} &= 0 \quad \mathbf{u_3} \\
x_{AE}+x_{DE}-x_{EF} &= 0 \quad \mathbf{u_4} \\
x_{AB} &\le 5 \quad \mathbf{v_1} \\
x_{AD} &\le 4 \quad \mathbf{v_2} \\
x_{AE} &\le 4 \quad \mathbf{v_3} \\
x_{BD} &\le 5 \quad \mathbf{v_4} \\
x_{DE} &\le 3 \quad \mathbf{v_5} \\
x_{BC} &\le 2 \quad \mathbf{v_6} \\
x_{DC} &\le 2 \quad \mathbf{v_7} \\
x_{DF} &\le 3 \quad \mathbf{v_8} \\
x_{EF} &\le 8 \quad \mathbf{v_9} \\
x_{CF} &\le 5 \quad \mathbf{v_{10}} \\
x_{AB},x_{AD},x_{AE},x_{BD},x_{DE},x_{BC},x_{DC},x_{DF},x_{EF},x_{CF} &\ge 0
\end{aligned}
\tag{1}
$$

2. Dual problem:

$$
\begin{aligned}
\min \quad & 5v_1+4v_2+4v_3+5v_4+3v_5+2v_6+2v_7+3v_8+8v_9+5v_{10} \\
u_1+v_1 &\ge 0 \quad \mathbf{x_{AB}} \\
u_3+v_2 &\ge 0 \quad \mathbf{x_{AD}} \\
u_4+v_3 &\ge 0 \quad \mathbf{x_{AE}} \\
-u_1+u_2+v_6 &\ge 0 \quad \mathbf{x_{BC}} \\
-u_3+u_4+v_5 &\ge 0 \quad \mathbf{x_{DE}} \\
-u_1+u_3+v_4 &\ge 0 \quad \mathbf{x_{BD}} \\
u_2-u_3+v_7 &\ge 0 \quad \mathbf{x_{DC}} \\
-u_3+v_8 &\ge 1 \quad \mathbf{x_{DF}} \\
-u_4+v_9 &\ge 1 \quad \mathbf{x_{EF}} \\
-u_2+v_{10} &\ge 1 \quad \mathbf{x_{CF}} \\
v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9,v_{10} &\ge 0 \\
u_1,u_2,u_3,u_4 & \text{ unrestricted.}
\end{aligned}
\tag{2}
$$

3. An AMPL model that solves the max-flow problem 1 and the output is the following:

```
var x_AB >=0;
var x_AD >=0;
var x_AE >=0;
var x_BD >=0;
var x_DE >=0;
var x_BC >=0;
var x_DC >=0;
var x_DF >=0;
```

```
var x_EF >=0;
var x_CF >=0;

maximize flow: x_CF+x_DF+x_EF;

subject to nodeB: x_AB-x_BD-x_BC=0;
subject to nodeC: x_BC+x_DC-x_CF=0;
subject to nodeD: x_AD+x_BD-x_DC-x_DF-x_DE=0;
subject to nodeE: x_AE+x_DE-x_EF=0;

subject to bound_c1: x_AB<=5 ;
subject to bound_c2: x_AD<=4;
subject to bound_c3: x_AE<=4;
subject to bound_c4: x_BD<=5;
subject to bound_c5: x_DE<=3;
subject to bound_c6: x_BC<=2;
subject to bound_c7: x_DC<=2 ;
subject to bound_c8: x_DF<=3;
subject to bound_c9: x_EF<=8;
subject to bound_c10: x_CF<=5;


#option solver cplex;
solve;
display x_AB,x_AD,x_AE,x_BD,x_DE,x_BC,x_DC,x_DF,x_EF,x_CF;

display nodeB.dual;
display nodeC.dual;
display nodeD.dual;
display nodeE.dual;
display bound_c1.dual;
display bound_c2.dual;
display bound_c3.dual;
display bound_c4.dual;
display bound_c5.dual;
display bound_c6.dual;
display bound_c7.dual;
display bound_c8.dual;
display bound_c9.dual;
display bound_c10.dual;

display flow;


CPLEX 12.5.0.0: optimal solution; objective 13
0 dual simplex iterations (0 in phase I)
x_AB = 5
x_AD = 4
x_AE = 4
x_BD = 4
x_DE = 3
```

```
x_BC = 1
x_DC = 2
x_DF = 3
x_EF = 7
x_CF = 3

nodeB.dual = -1
nodeC.dual = -1
nodeD.dual = -1
nodeE.dual = -1
bound_c1.dual = 1
bound_c2.dual = 1
bound_c3.dual = 1
bound_c4.dual = 0
bound_c5.dual = 0
bound_c6.dual = 0
bound_c7.dual = 0
bound_c8.dual = 0
bound_c9.dual = 0
bound_c10.dual = 0

flow = 13
```

□ An AMPL model for the dual problem (2) with the output is as follows:

```
var u1 ;
var u2 ;
var u3 ;
var u4 ;

var v1>=0;
var v2>=0;
var v3>=0;
var v4>=0;
var v5>=0;
var v6>=0;
var v7>=0;
var v8>=0;
var v9>=0;
var v10>=0;


minimize f_dual: 5*v1+4*v2+4*v3+5*v4+3*v5+2*v6+2*v7+3*v8+8*v9+5*v10 ;

subject to dual_c1: u1+v1>=0;
subject to dual_c2: u3+v2>=0;
subject to dual_c3: u4+v3>=0;
subject to dual_c4: -u1+u2+v6>=0;
subject to dual_c5: -u3+u4+v5>=0;
subject to dual_c6: -u1+u3+v4>=0;
subject to dual_c7: u2-u3+v7>=0;
```

```
subject to dual_c8: -u3+v8>=1;
subject to dual_c9: -u4+v9>=1;
subject to dual_c10: -u2+v10>=1;

#option solver cplex;
solve;

display u1,u2,u3,u4,v1,v2,v3,v4,v5,v6,v7,v8,v9,v10;

display dual_c1.dual;
display dual_c2.dual;
display dual_c3.dual;
display dual_c4.dual;
display dual_c5.dual;
display dual_c6.dual;
display dual_c7.dual;
display dual_c8.dual;
display dual_c9.dual;
display dual_c10.dual;

display f_dual;


CPLEX 12.5.0.0: optimal solution; objective 13
3 dual simplex iterations (0 in phase I)
u1 = -1
u2 = -1
u3 = -1
u4 = -1
v1 = 1
v2 = 1
v3 = 1
v4 = 0
v5 = 0
v6 = 0
v7 = 0
v8 = 0
v9 = 0
v10 = 0

dual_c1.dual = 5
dual_c2.dual = 4
dual_c3.dual = 4
dual_c4.dual = 2
dual_c5.dual = 3
dual_c6.dual = 3
dual_c7.dual = 2
dual_c8.dual = 2
dual_c9.dual = 7
dual_c10.dual = 4
```

```
f_dual = 13
```

For the dual solutions from the primal problem (max-flow) formulation, we have

$$(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}) = (-1, -1, -1, -1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)$$

For the (primal) solutions of the dual formulation in (2), we again have

$$(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}) = (-1, -1, -1, -1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)$$

so they exactly coincide.
For the primal solution of the primal formulation (max-flow), we have

$$(x_{AB}, x_{AD}, x_{AE}, x_{BD}, x_{DE}, x_{BC}, x_{DC}, x_{DF}, x_{EF}, x_{CF}) = (5, 4, 4, 4, 3, 1, 2, 3, 7, 3)$$

For the dual solution of the dual formulation in (2), we have

$$(x_{AB}, x_{AD}, x_{AE}, x_{BD}, x_{DE}, x_{BC}, x_{DC}, x_{DF}, x_{EF}, x_{CF}) = (5, 4, 4, 2, 3, 3, 2, 2, 7, 4)$$

We see that the solutions are different, which are, of course, both optimal. This shows that the max-flow problem can have more than one optimal solution.

4. The optimal solution of the min-cut is obtained as follows: In the dual optimal solution for the max-flow problem, examine the non-zero dual variables which correspond to the bound constraints. Above the only nonzero dual variables are $(v_1, v_2, v_3) = (1, 1, 1)$ and these dual variables correspond to the bound constraints $x_{AB} \leq 5$, $x_{AD} \leq 4$ and $x_{AE} \leq 4$ respectively. We conclude that we obtain a min-cut when we remove the arcs (or edges) x_{AB} and x_{AD} andx_{AE}.

5. Notice that in the max-flow problem, the flow which exits the source node A must be equal to the flow which enters the sink node F. So we can always solve a max-flow problem by maximizing the either objective, in this case $x_{CF} + x_{DF} + x_{EF}$ or $x_{AB} + x_{AD} + x_{AE}$. When we use the objective $x_{AB} + x_{AD} + x_{AE}$, the new dual problem becomes the following:

$$\begin{array}{rrl}
\min & 5v_1 + 4v_2 + 4v_3 + 5v_4 + 3v_5 + 2v_6 + 2v_7 + 3v_8 + 8v_9 + 5v_{10} & \\
& u_1 + v_1 \geq 1 & \mathbf{x_{AB}} \\
& u_3 + v_2 \geq 1 & \mathbf{x_{AD}} \\
& u_4 + v_3 \geq 1 & \mathbf{x_{AE}} \\
& -u_1 + u_2 + v_6 \geq 0 & \mathbf{x_{BC}} \\
& -u_3 + u_4 + v_5 \geq 0 & \mathbf{x_{DE}} \\
& -u_1 + u_3 + v_4 \geq 0 & \mathbf{x_{BD}} \\
& u_2 - u_3 + v_7 \geq 0 & \mathbf{x_{DC}} \\
& -u_3 + v_8 \geq 0 & \mathbf{x_{DF}} \\
& -u_4 + v_9 \geq 0 & \mathbf{x_{EF}} \\
& -u_2 + v_{10} \geq 0 & \mathbf{x_{CF}} \\
& v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10} \geq 0 & \\
& u_1, u_2, u_3, u_4 \text{ unrestricted.} &
\end{array} \tag{3}$$

The relation between the dual problems (2) and (3) is more tricky. We need to show that by solving the dual problems (2) or (3), we get the same objective values. Suppose that $(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ is an optimal solution for (2). Then it is a routine check to verify that $(u_1 - 1, u_2 - 1, u_3 - 1, u_4 - 1, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ satisfies the constraints in the system (3) (check this yourself). Note also that since the objective functions in (2) and (3) only involve the variables $v's$ and since the solutions for $(u_1, u_2, u_3, u_4, v_1, v_2, \cdots, v_{10})$ and $(u_1 - 1, u_2 - 1, u_3 - 1, u_4 - 1, v_1, v_2, \cdots, v_{10})$ have exactly the same $v's$, the objective values in (2) and (3) will be the same. Similarly it is easy to see that when (3) has an optimal solution $(u_1, u_2, u_3, u_4, v_1, v_2, \cdots, v_{10})$, then $(u_1 + 1, u_2 + 1, u_3 + 1, u_4 + 1, v_1, v_2, \cdots, v_{10})$ is a solution for (2) with the same objective value. This argument shows that by solving (2) or (**??**), we obtain the same objective values.

2. Shortest path and duality (5 pts.)

1. The Corresponding primal model of the problem is as follows:

$$
\begin{aligned}
\min \quad & 3x_{AB} + 10x_{AD} + 4x_{AE} + 4x_{BC} + 4x_{BD} + 4x_{ED} + 2x_{EF} + 5x_{DC} + x_{DG} + x_{DF} + 5x_{CG} + 10x_{GF} \\
& +3x_{BA} + 10x_{DA} + 4x_{EA} + 4x_{CB} + 4x_{DB} + 4x_{DE} + 2x_{FE} + 5x_{CD} + x_{GD} + x_{FD} + 5x_{GC} + 10x_{FG} \\
& x_{AB} + x_{AD} + x_{AE} - x_{BA} - x_{DA} - x_{EA} = 1 \quad \mathbf{u_1} \\
& x_{BA} + x_{BD} + x_{BC} - x_{AB} - x_{DB} - x_{CB} = 0 \quad \mathbf{u_2} \\
& x_{CB} + x_{CD} + x_{CG} - x_{BC} - x_{DC} - x_{GC} = 0 \quad \mathbf{u_3} \\
& x_{DA} + x_{DB} + x_{DC} + x_{DE} + x_{DF} + x_{DG} - x_{AD} - x_{BD} - x_{CD} - x_{ED} - x_{FD} - x_{GD} = 0 \quad \mathbf{u_4} \\
& x_{EA} + x_{ED} + x_{EF} - x_{AE} - x_{DE} - x_{FE} = 0 \quad \mathbf{u_5} \\
& x_{FE} + x_{FD} + x_{FG} - x_{EF} - x_{DF} - x_{GF} = 0 \quad \mathbf{u_6} \\
& x_{GC} + x_{GD} + x_{GF} - x_{CG} - x_{DG} - x_{FG} = -1 \quad \mathbf{u_7} \\
& x_{AB}, x_{AD}, x_{AE}, x_{BC}, x_{BD}, x_{DC}, x_{DG}, x_{DF}, x_{EF}, x_{CG}, x_{FG} \geq 0, \\
& x_{BA}, x_{DA}, x_{EA}, x_{CB}, x_{DB}, x_{CD}, x_{GD}, x_{FD}, x_{FE}, x_{GC}, x_{GF} \geq 0.
\end{aligned}
\tag{4}
$$

2. Dual problem:

$$
\begin{array}{rl}
\max \quad u_1 - u_7 & \\
u_1 - u_2 \leq 3 & \mathbf{x_{AB}} \\
u_2 - u_1 \leq 3 & \mathbf{x_{BA}} \\
u_1 - u_4 \leq 10 & \mathbf{x_{AD}} \\
u_4 - u_1 \leq 10 & \mathbf{x_{DA}} \\
u_1 - u_5 \leq 4 & \mathbf{x_{AE}} \\
u_5 - u_1 \leq 4 & \mathbf{x_{EA}} \\
u_2 - u_4 \leq 4 & \mathbf{x_{BD}} \\
u_4 - u_2 \leq 4 & \mathbf{x_{DB}} \\
u_5 - u_4 \leq 4 & \mathbf{x_{ED}} \\
u_4 - u_5 \leq 4 & \mathbf{x_{DE}} \\
u_2 - u_3 \leq 4 & \mathbf{x_{BC}} \\
u_3 - u_2 \leq 4 & \mathbf{x_{CB}} \\
u_4 - u_3 \leq 5 & \mathbf{x_{DC}} \\
u_3 - u_4 \leq 5 & \mathbf{x_{CD}} \\
u_4 - u_7 \leq 1 & \mathbf{x_{DG}} \\
u_7 - u_4 \leq 1 & \mathbf{x_{GD}} \\
u_4 - u_6 \leq 1 & \mathbf{x_{DF}} \\
u_6 - u_4 \leq 1 & \mathbf{x_{FD}} \\
u_5 - u_6 \leq 2 & \mathbf{x_{EF}} \\
u_6 - u_5 \leq 2 & \mathbf{x_{FE}} \\
u_3 - u_5 \leq 5 & \mathbf{x_{CG}} \\
u_5 - u_3 \leq 5 & \mathbf{x_{GC}} \\
u_6 - u_7 \leq 10 & \mathbf{x_{FG}} \\
u_7 - u_6 \leq 10 & \mathbf{x_{GF}} \\
u_1, u_2, u_3, u_4, u_5, u_6, u_7 \text{ unrestricted.} &
\end{array}
\tag{5}
$$

3. An AMPL model that solves the max-flow problem 4 and the output is the following:

```
var x_AB >=0;
var x_BA >=0;
var x_AD >=0;
var x_DA >=0;
var x_AE >=0;
var x_EA >=0;
var x_BC >=0;
var x_CB >=0;
var x_BD >=0;
var x_DB >=0;
var x_DC >=0;
var x_CD >=0;
var x_DG >=0;
var x_GD >=0;
var x_DF >=0;
var x_FD >=0;
var x_ED >=0;
var x_DE >=0;
var x_EF >=0;
var x_FE >=0;
```

```
var x_CG >=0;
var x_GC >=0;
var x_FG >=0;
var x_GF >=0;

minimize cost: 3*x_AB+10*x_AD+4*x_AE+4*x_BC+4*x_BD+4*x_ED+2*x_EF+5*x_DC+x_DG+x_DF+5*x_CG+10*x_GF
              +3*x_BA+10*x_DA+4*x_EA+4*x_CB+4*x_DB+4*x_DE+2*x_FE+5*x_CD+x_GD+x_FD+5*x_GC+10*x_FG;

subject to nodeA:  x_AB+x_AD+x_AE-x_BA-x_DA-x_EA=1;
subject to nodeB:  x_BA+x_BD+x_BC-x_AB-x_DB-x_CB=0;
subject to nodeC:  x_CB+x_CD+x_CG-x_BC-x_DC-x_GC=0;
subject to nodeD:  x_DA+x_DB+x_DC+x_DE+x_DF+x_DG-x_AD-x_BD-x_CD-x_ED-x_FD-x_GD=0;
subject to nodeE:  x_EA+x_ED+x_EF-x_AE-x_DE-x_FE=0;
subject to nodeF:  x_FE+x_FD+x_FG-x_EF-x_DF-x_GF=0;
subject to nodeG:  x_GC+x_GD+x_GF-x_CG-x_DG-x_FG=-1;

#option solver cplex;
solve;
display x_AB,x_AD,x_AE,x_BC,x_BD,x_DC,x_DG,x_DF,x_ED,x_EF,x_CG,x_FG,
        x_BA,x_DA,x_EA,x_CB,x_DB,x_CD,x_GD,x_FD,x_DE,x_FE,x_GC,x_GF;

display nodeA.dual;
display nodeB.dual;
display nodeC.dual;
display nodeD.dual;
display nodeE.dual;
display nodeF.dual;
display nodeG.dual;

display cost;


CPLEX 12.5.0.0: optimal solution; objective 8
0 dual simplex iterations (0 in phase I)
x_AB =1
x_BA =0
x_AD =0
x_DA =0
x_AE =0
x_EA =0
x_BC =0
x_CB =0
x_BD =1
x_DB =0
x_DC =0
x_CD =0
x_DG =1
x_GD =0
x_DF =0
x_FD =0
x_ED =0
```

```
x_DE =0
x_EF =0
x_FE =0
x_CG =0
x_GC =0
x_FG =0
x_GF =0

nodeA.dual = 7
nodeB.dual = 4
nodeC.dual = 0
nodeD.dual = 0
nodeE.dual = 3
nodeF.dual = 1
nodeG.dual = -1

cost = 8
```

☐ An AMPL model for the dual problem (5) with the output is as follows:

```
var u1 ;
var u2 ;
var u3 ;
var u4 ;
var u5 ;
var u6 ;
var u7 ;

maximize f_dual: u1-u7 ;

subject to dual_c1: u1-u2<=3;
subject to dual_c2: u2-u1<=3;
subject to dual_c3: u1-u4<=10;
subject to dual_c4: u4-u1<=10;
subject to dual_c5: u1-u5<=4;
subject to dual_c6: u5-u1<=4;
subject to dual_c7: u2-u4<=4;
subject to dual_c8: u4-u2<=4;
subject to dual_c9: u5-u4<=4;
subject to dual_c10: u4-u5<=4;
subject to dual_c11: u2-u3<=4;
subject to dual_c12: u3-u2<=4;
subject to dual_c13: u4-u3<=5;
subject to dual_c14: u3-u4<=5;
subject to dual_c15: u4-u7<=1;
subject to dual_c16: u7-u4<=1;
subject to dual_c17: u4-u6<=1;
subject to dual_c18: u6-u4<=1;
subject to dual_c19: u5-u6<=2;
subject to dual_c20: u6-u5<=2;
subject to dual_c21: u3-u5<=5;
```

```
subject to dual_c22: u5-u3<=5;
subject to dual_c23: u6-u7<=10;
subject to dual_c24: u7-u6<=10;

#option solver cplex;
solve;

display u1,u2,u3,u4,u5,u6,u7;

display dual_c1.dual;
display dual_c2.dual;
display dual_c3.dual;
display dual_c4.dual;
display dual_c5.dual;
display dual_c6.dual;
display dual_c7.dual;
display dual_c8.dual;
display dual_c9.dual;
display dual_c10.dual;
display dual_c11.dual;
display dual_c12.dual;
display dual_c13.dual;
display dual_c14.dual;
display dual_c15.dual;
display dual_c16.dual;
display dual_c17.dual;
display dual_c18.dual;
display dual_c19.dual;
display dual_c20.dual;
display dual_c21.dual;
display dual_c22.dual;
display dual_c23.dual;
display dual_c24.dual;

display f_dual;


CPLEX 12.5.0.0: optimal solution; objective 8
3 dual simplex iterations (0 in phase I)
u1 = 7
u2 = 4
u3 = 0
u4 = 0
u5 = 3
u6 = 1
u7 = -1

dual_c1.dual = 1
dual_c2.dual = 0
dual_c3.dual = 0
dual_c4.dual = 0
```

```
dual_c5.dual = 0
dual_c6.dual = 0
dual_c7.dual = 1
dual_c8.dual = 0
dual_c9.dual = 0
dual_c10.dual = 0
dual_c11.dual = 0
dual_c12.dual = 0
dual_c13.dual = 0
dual_c14.dual = 0
dual_c15.dual = 1
dual_c16.dual = 0
dual_c17.dual = 0
dual_c18.dual = 0
dual_c19.dual = 0
dual_c20.dual = 0
dual_c21.dual = 0
dual_c22.dual = 0
dual_c23.dual = 0
dual_c24.dual = 0

f_dual = 8
```

4. The only nonzero primal variables are x_{AB}, x_{BD} and x_{DG}, so we need just check the complementary slackness for these primal variables and their corresponding dual constraints as the following,

$$x_{AB}(u_1 - u_2 - 3) = 0,$$
$$x_{BD}(u_2 - u_4 - 4) = 0,$$
$$x_{DG}(u_4 - u_7 - 1) = 0.$$

clearly, by using the value of dual variables $u_1 = 7$, $u_2 = 4$, $u_4 = 0$ and $u_7 = -1$, we can see that above equalities are satisfied.

5. The first solution of shortest path which is obtained by solving the primal model is $A \to B \to D \to G$ or equivalently $x_{AB} = x_{BD} = x_{DG} = 1$. In order to find the second solution we can add a new constraint $x_{AB}+x_{BD}+x_{DG} < 3$, in this way we are enforcing our model to find another optimal solution(if there is any).

Note: If the problem has a unique solution, by adding this new constraint, since we are blocking the only optimal solution we can not get the that anymore.

Complementary slackness and duality

1)Unique primal-dual solutions

a) A feasible solution can be simply found as we have only two functional constraints. As a few, $x = (0, 1, 0, 0)^T$ with objective $z_{primal} = 4$ and $x = (0, 0, 1, -1)^T$ with $z_{primal} = 4$ can be mentioned.

□

b) Resorting to the canonical format of an LP, the dual problem can be represented as

$$\begin{aligned} &\max\ u_2 \\ &\ \text{s.t.} \\ &\quad u_1 + u_2 \geq 2, \\ &\quad u_1 + u_2 \leq 4, \\ &\quad 2u_1 - u_2 \leq 2, \\ &\quad u_1 - 2u_2 \geq -2, \\ &\quad u_1, u_2 \geq 0. \end{aligned}$$

c) A feasible solution to the dual problem would be $u = (1, 1)^T$ with $z_{dual} = 1$. Recall from LP duality that for a minimization problem, a feasible solution to its associated dual gives a lower bound for the optimal objective value while a feasible solution to the primal problem gives an upper bound. This simple example also supports this fact as $z_{dual} = 1 \leq z_{primal} = 4$. $\square$

d) As the feasible region lives in a 2-dimensional space, we can simply draw the feasible region and objective contours to find the optimum solution. Doing so, the optimum solution would be $u = (2, 2)^T$ with $z_{opt} = 2$. $\square$

e) As $u = (2, 2)^T$ is the dual optimal solution, it satisfies the complementary conditions as follows

$$\begin{cases} u_1(x_1 + x_2 + 2x_3 + x_4) = 0 \\ u_2(x_1 + x_2 - x_3 - 2x_4 - 1) = 0 \end{cases} \tag{6}$$

Since $u_1, u_2 > 0$, we would have

$$\begin{aligned} u_1(x_1 + x_2 + 2x_3 + x_4) = 0 \quad &\Rightarrow \quad x_1 + x_2 + 2x_3 + x_4 = 0 \\ u_2(x_1 + x_2 - x_3 - 2x_4 - 1) = 0 \quad &\Rightarrow \quad x_1 + x_2 - x_3 - 2x_4 = 1 \end{aligned}$$

On the other hand, except for the first dual constraint, all the dual constraints are binding at $u = (2, 2)^T$. All this implies that

$$x_1(u_1 + u_2 - 2) = 0 \quad \Rightarrow \quad x_1 = 0$$

Notice that the information on the other dual constraints would not be helpful here as they already satisfy the complementary conditions.

Now, based on the information given so far, we would get

$$\begin{cases} x_2 + 2x_3 + x_4 = 0 & \Rightarrow x_4 = -\frac{2}{3} + x_2 \\ x_2 - x_3 - 2x_4 = 1 & \Rightarrow x_3 = \frac{1}{3} - x_2 \end{cases} \tag{7}$$

Obviously, this system of equations has infinitely many solutions. Thus, the optimal set would be $S = \{(x_1, x_2, x_3, x_4) | x_1 = 0,\ x_3 = \frac{1}{3} - x_2,\ x_4 = -\frac{2}{3} + x_2,\ x_2, x_3 \geq 0,\ x_4 \leq 0\}$. $\square$

2) Multiple primal solutions

a) In this case, the dual problem will change to

$$
\begin{aligned}
\max \quad & -u_2 \\
\text{s.t.} \quad & \\
& u_1 + u_2 \geq 2, \\
& u_1 + u_2 \leq 4, \\
& 2u_1 - u_2 \leq 2, \\
& u_1 - 2u_2 \geq -2, \\
& u_1, u_2 \geq 0.
\end{aligned}
$$

In fact, only the dual objective function is affected. $\square$

b) As the feasible region is the same as in part (1-b), we only need to evaluate the three feasible corners to find the maximum objective value. Doing so, we end up with $u = (\frac{4}{3}, \frac{2}{3})^T$ as the optimum solution. $\square$

c) By the complementary conditions, we would get

$$
\begin{cases} u_1(x_1 + x_2 + 2x_3 + x_4) = 0 \quad \Rightarrow \quad x_1 + x_2 + 2x_3 + x_4 = 0 \\ u_2(x_1 + x_2 - x_3 - 2x_4 + 1) = 0 \quad \Rightarrow \quad x_1 + x_2 - x_3 - 2x_4 = -1 \end{cases} \tag{8}
$$

Further, since the second and the fourth dual constraints are not binding at $u = (\frac{4}{3}, \frac{2}{3})^T$, we have

$$
\begin{aligned}
x_2(4 - u_1 - u_2) = 0 \quad &\Rightarrow \quad x_2 = 0, \\
x_4(u_1 - 2u_2 + 2) = 0 \quad &\Rightarrow \quad x_4 = 0,
\end{aligned}
$$

and consequently, to get the primal optimal solutions, we need to solve

$$
\begin{cases} x_1 + 2x_3 = 0 \\ x_1 - x_3 = -1 \end{cases}
$$

This system of equations has a single solution. As a result, the only optimal solution to this problem would be $x = (-\frac{2}{3}, 0, \frac{1}{3}, 0)^T$. $\square$

Linear programming This problem can be simply rewritten as a linear program as follows

$$
\begin{aligned}
\min \quad & x \\
\text{s.t.} \quad & \\
& x \geq a_i \quad i = 1, ..., n
\end{aligned}
$$

From the objective function and the constraints, it can be interpreted that the optimal solution equals $x = \max\{a_1, a_2, ..., a_n\}$.

The dual associated with this problem is formulated as

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} a_i y_i \\
\text{s.t.} \quad & \\
& \sum_{i=1}^{n} y_i = 1 \\
& y_i \geq 0 \quad i = 1, ..., n
\end{aligned}
$$

Now, by the complementary conditions, we would get

$$y_i(x - a_i) = 0 \quad i = 1, ..., n \tag{9}$$

Assume with no loss of generality that $i = arg\max_i a_i$; that is $x_{optimal} = a_i$ and i is unique. Then, by the complementary conditions, $y_j = 0$ for all $j \neq i$ so that $a_j < a_i$. Thus, by the dual constraint, it follows that $y_i = 1$ and the dual problem has only one optimal solution.

In case that the tie is broken in $i = arg\max_i a_i$, it can be simply shown by a similar reasoning that the dual problem has multiple optimal solutions.