ISE426 – Optimization models and applications

Fall 2013 – Quiz #1, October 8, 2013

First name	
Last name	
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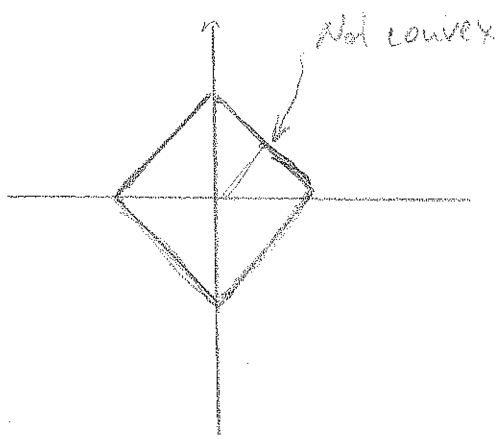
You have 75 minutes. There are two problems. This quiz accounts for 10% of the final grade. There are 40 points available. Please write clear and concise statements. Use a **readable** handwriting: it will be hard to grade your answers if they are not readable — let alone give them full points. Use the back of each sheet if you need more space, for example to draw graphs. Do not use calculators.

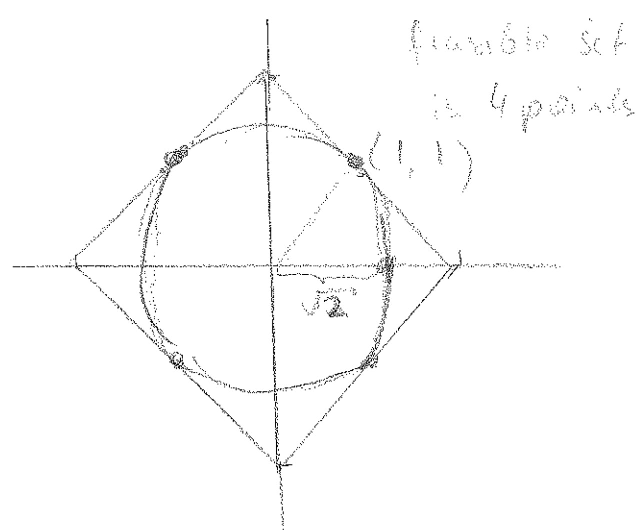
1 Convexity and relaxations (10 pts.)

The following two problems are not convex, explain why (4 pts.):

$$(1)\ \min\ x^2 + y^2 \quad |x| + |y| = 1 \qquad\qquad (2)\ \max\ x \quad |x| + |y| = 2,\ \ x^2 + y^2 \le 2$$

Feasible set





function $|x|+|y|$ is not linear in the "=" constraint

For each problem, find upper and lower bounds on the optimal value and explain how you know that these are indeed upper and lower bounds. Find the optimal solution graphically and check that it is between the upper and lower bounds. (6 pts.).

1. a) Point $(0,1)$ is feasible, hence the objective function value $0^2+1^2=1$ gives an <u>upper</u> bound on the optimal value

b) Consider relaxation by replacing constraint with $|x|+|y|\le 1$ $\Rightarrow$ optimal solution is $(0,0)$ $\Rightarrow$ 0 is a <u>lower</u> bound.

c) Optimal solution is $(\frac{1}{2}, \frac{1}{2})$ with value $0 \le \frac{1}{2} \le 1$.

2. a) Point $(1, 1)$ is feasible, hence value of obj $x=1$ gives a <u>lower</u> bound.

b) Relax constraint $|x|+|y|=2$ with $|x|+|y| \geq 2$. Solve the problem graphically. Solution is $(\sqrt{2}, 0)$ with value $\sqrt{2}$, which gives an <u>upper</u> bound

c) Points $(1,1)$ and $(1,-1)$ are optimal for original problem. The value 1 is between upper and lower bounds.

3 Linear Programming Model (20 pts.)

Sophia goes out to lunch 5 days a week. She is trying to decide on what she will be eating during a given week. Her options are: lo mein, hamburger, fried chicken, soup, pizza and a Subway sandwich. She can eat each of these as many times as she chooses. She eventually needs to have 5 meals. Her budget is $20 and she is trying to maximize her satisfaction. In the table below you see the cost per each type of meal and their satisfaction coefficients.

Food	Lo Mein	Hamb.	Fr. Chic.	Soup	Pizza	Sub.
satisfaction	6	5	4	4	3	3
cost	7	5	5	4	4	3

(a) Formulate an integer optimization problem to choose the meals to maximize Sophia satisfaction (note that each meal can be chosen more than once as long as the total is 5 meals). Create a linear programming relaxation. If you do this correctly, $(0, 2.5, 0, 0, 0, 2.5)$ would be an optimal solution to the relaxation. (6pts)

max $6x_1 + 5x_2 + 4x_3 + 4x_4 + 3x_5 + 3x_6$

$7x_1 + 5x_2 + 5x_3 + 4x_4 + 4x_5 + 3x_6 \leq 20$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 5$

$x_i \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$

Relaxation: replace integrality with

$x_i \geq 0$

(b) Write down the dual of the linear programming relaxation defined in part (a). Using the optimal solution of the relaxation given in part (a), compute the dual optimal solution from complementary slackness conditions and show that this solution is feasible (hence the primal solution that is given is optimal). (8 pts)

$$\min \; 20u_1 + u_2$$

$$7u_1 + u_2 \geq 6$$

$$5u_1 + u_2 \geq 5$$

$$5u_1 + u_2 \geq 4$$

$$4u_1 + u_2 \geq 1$$

$$4u_1 + u_2 \geq 3$$

$$3u_1 + u_2 \geq 3$$

$$u_1, u_2 \geq 0$$

Primal solution : $(0, 2.5, 0, 0, 0, 2.5) \Rightarrow$

by complementarity

$$5u_1 + u_2 = 5 \qquad \Rightarrow u_1 = 1, \quad u_2 = 0$$

$$3u_1 + u_2 = 3$$

(c) Consider the satisfaction coefficient of lo mein. How much higher does it have to be to change to optimal solution of the relaxation? Derive your answer from the feasibility of the complementary dual solution. (2pts)

$7u_1 + u_2 \geq C_1$ with $u_1 = 1, u_2 = 0$

$\Rightarrow C_1 \leq 7$ solution does not change.

(d) Consider the satisfaction coefficient of the soup. Show that it cannot get any higher ~~lower~~ without changing the optimal solution. (2 pts)

$4u_1 + u_2 \geq C_4$, $u_1 = 1, u_2 = 0$

$\Rightarrow C_4 \leq 4$ does not change solution, but $C_4 > 4$ does.

(e) $(0, 2.5, 0, 0, 0, 2.5)$ is not a feasible solution to the original problem since it is not integer. Does it give you a bound on the optimal solution of the original problem? Find an optimal solution of the original problem and prove that it is optimal using that bound. (2pts)

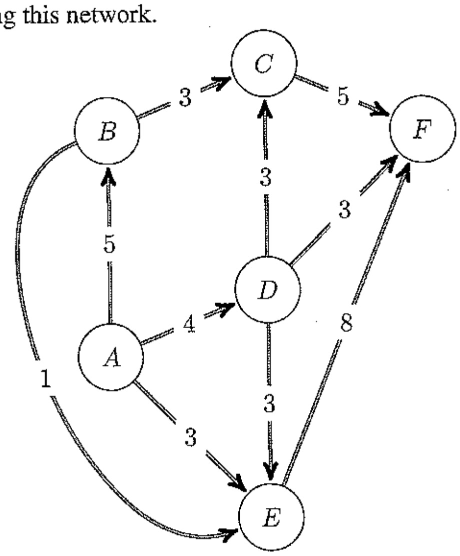
The solution $(0, 2.5, 0, 0, 0, 2.5)$ gives an upper bound on the optimal value, this bound is 20

Solution $(0, 0, 0, 5, 0, 0)$ is feasible for the original problem and has objective value of 20 $\Rightarrow$ it is optimal.

Conclusion: eat soup every day!

2 Flow problem (10 pts.)

Recall this graph problem from Homework #2. This time let the numbers on the arcs represent the cost of the flow along the arcs. Let the capacity of each arc equal 5. Consider the minimum cost flow problem sending 7 units of flow from source A to sink F using this network.



1. Formulate this min cost flow problem as a linear programming problem, using the formulations studied in this course, with the objective equal to the total cost of the flow.

min $X_{BE} + 5X_{AB} + 3X_{AE} + 4X_{AD} + 3X_{DE} + 3X_{DC} +$

$5X_{CF} + 3X_{DF} + 8X_{EF} + 3X_{BC}$

$X_{AB} + X_{AD} + X_{AE} = 7$ (A)

$X_{AB} - X_{BE} - X_{BC} = 0$ (B)

$X_{AE} + X_{DE} + X_{BE} - X_{EF} = 0$ (E)

$X_{AD} - X_{DE} - X_{DF} - X_{DC} = 0$ (D)

$X_{BC} + X_{DC} - X_{CF} = 0$ (C)

$X_{ij} \leq 5$ $\quad X_{ij} \geq 0$

extra sheet

$$\max\ 7u_A + 5\sum_{ij\in A} z_{ij}$$

$$u_A + u_B + z_{AB} \le 5$$

$$-u_B + u_E + z_{BE} \le 1$$

$$u_A + u_E + z_{AE} \le 3$$

$$u_A + u_D + z_{AD} \le 4$$

$$-u_D + u_E + z_{ED} \le 3$$

$$-u_D + u_C + z_{DC} \le 3$$

$$-u_C + u_F + z_{CF} \le 5$$

$$-u_D + u_F + z_{DF} \le 3$$

$$-u_E + u_F + z_{EF} \le 8$$

$$-u_B + u_C + z_{BC} \le 3$$

$$z_{ij} \ge 0 \quad (i,j)\in A$$

u_i - unrestricted $i \in V$