1 Convexity and relaxations (10 pts.)

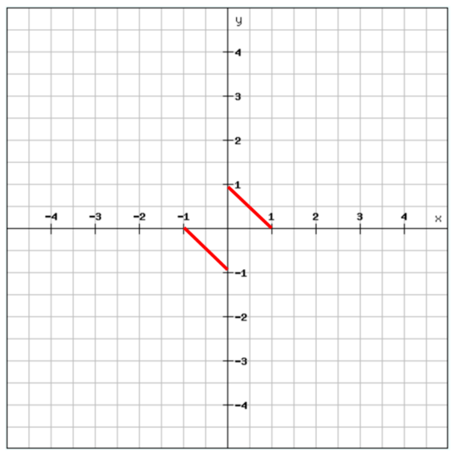
(a) The following two problems are not convex, explain why (4 pts.):

$$(1)\min \quad x^2+y^2 \qquad\qquad (2)\max \quad x$$
$$|x+y|=1 \qquad\qquad |x+y|=1$$
$$-1\le x\le 1 \qquad\qquad x^2+y^2\le 1$$
$$-1\le y\le 1$$

ANSWER

$|x+y|=1$ creates non-convexity since it is a non-linear constraint. The objective functions are convex functions. The other constraints are convex since they are linear inequalities.

The feasible region is provided in the figure:



(b) For each problem, find upper and lower bounds on the optimal value and explain how you know that these are indeed upper and lower bounds. Can you find an optimal solution just from the lower and upper bounds? If yes, explain, if not, find the optimal solution graphically or by simple observation and check that it is between the upper and lower bounds. (6 pts.).

ANSWER

<u>(Problem 1)</u>

For a minimization type problem, any feasible solution will give an upperbound value. For instance, (1,0) is a feasible solution. Hence, its corresponding objective function value, $1^2+0=1$, is an upperbound.

We should look at the optimal objective function value in order to find a lowerbound. For instance, relax the constraint $|x+y| = 1$ into $|x+y| \leq 1$. Then, the optimal solution of the relaxed problem becomes (0,0) with the objective function value 0. Hence, 0 is a lowerbound.

We know that the optimal value of this problem will be between 0 and 1 (ie, between lowerbound and upperbound). However, we cannot find the optimal solution just from the lower and upper bounds since there are many values between 0 and 1. We can find the solution graphically, (0.5, 0.5) and (-0.5, -0.5) are the optimal solutions. They have the same the objective function value 0.5. It is obvious that 0.5 is between 0 and 1.

(Problem 2)

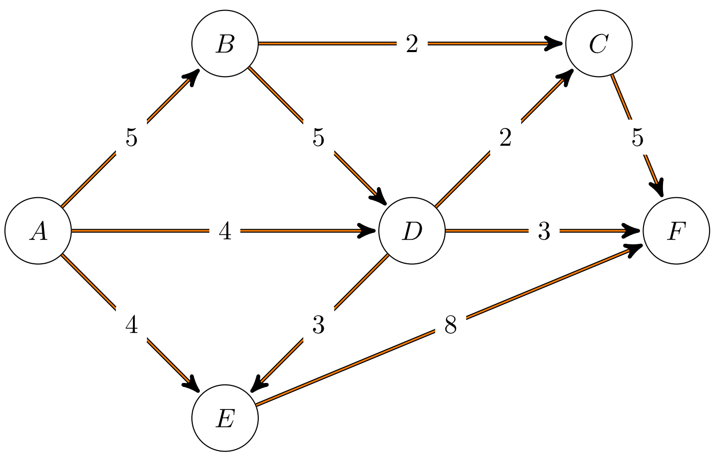
For a maximization type problem, any feasible solution will give a lowerbound value. For instance, (1,0) is a feasible solution. Hence, its corresponding objective function value, 1, is a lowerbound.

We should look at the optimal objective function value in order to find an upperbound. For instance, relax the constraint $|x+y| = 1$ into $|x+y| \leq 1$. Then, the optimal solution of the relaxed problem becomes (1,0) with the objective function value 1. Hence, 1 is an upperbound.

We know that the optimal value of this problem will be between 1 and 1 (ie, between lowerbound and upperbound). Hence (1,0) is the optimal solution with the objective function value 1. We can find the optimal solution just from the lower and upper bounds for this case.

2 Max flow and duality (10 pts.)

Consider the problem of sending the largest possible amount of flow from source A to sink F using the following network, where the numbers on the arcs represent the capacity of the arcs.



(a) Formulate this max flow problem as a linear programming problem, using the formulations studied in this course. (5 pts.).

ANSWER

Variables:

x_{ij}: amount of flow from node i to node j. $\quad i \in A, B, .., F, \quad j \in A, B, .., F$

Mathematical Model:

$$
\begin{aligned}
&\max x_{AB} + x_{AD} + x_{AE} \\
&\text{s.t.} \\
&\quad x_{AB} = x_{BC} + x_{BD} \\
&\quad x_{BC} + x_{DC} = x_{CF} \\
&\quad x_{AD} + x_{BD} = x_{DC} + x_{DE} + x_{DF} \\
&\quad x_{AE} + x_{DE} = x_{EF} \\
&\quad x_{AB} \leq 5 \\
&\quad x_{AD} \leq 4 \\
&\quad x_{AE} \leq 4 \\
&\quad x_{BC} \leq 2 \\
&\quad x_{BD} \leq 5 \\
&\quad x_{CF} \leq 5 \\
&\quad x_{DC} \leq 2 \\
&\quad x_{DE} \leq 3 \\
&\quad x_{DF} \leq 3 \\
&\quad x_{EF} \leq 8 \\
&\quad x_{ij} \geq 0 \quad i \in A, B, .., F, \quad j \in A, B, .., F
\end{aligned}
$$

Equivalently, the objective function can be written as: max $x_{CF} + x_{DF} + x_{EF}$

(b) Write the dual of the above max flow problem (5 pts.).

Dual Variables : $u_1, u_2, u_3, u_4, z_{AB}, z_{AD}, ..., z_{EF}$

Dual Problem :

$$
\begin{aligned}
&\min 5z_{AB} + 4z_{AD} + 4z_{AE} + 2z_{BC} + 5z_{BD} + 5z_{CF} + 2z_{DC} + 3z_{DE} + 3z_{DF} + 8z_{EF} \\
&\text{s.t.} \\
&\quad u_1 + z_{AB} \geq 1 \\
&\quad u_3 + z_{AD} \geq 1 \\
&\quad u_4 + z_{AE} \geq 1 \\
&\quad - u_1 + u_2 + z_{BC} \geq 0 \\
&\quad - u_1 + u_3 + z_{BD} \geq 0 \\
&\quad - u_2 + z_{CF} \geq 0 \\
&\quad u_2 - u_3 + z_{DC} \geq 0 \\
&\quad - u_3 + u_4 + z_{DE} \geq 0 \\
&\quad - u_3 + z_{DF} \geq 0 \\
&\quad - u_4 + z_{EF} \geq 0 \\
&\quad z_{ij} \geq 0 \quad i \in A, B, .., F, \quad j \in A, B, .., F \\
&\quad u_1, u_2, u_3, u_4 : unrestricted
\end{aligned}
$$

3 Linear/Integer programming model, relaxations, duality, upper and lower bounds (20 pts.)

Kyra is applying to several high schools. She has 8 high schools to choose from, each high school has a ranking (higher is better) and each of them requires several days to complete the applications. Below in the table are the list of high schools and the their ranking and the time their applications require. She has 10 days in total. She wants to maximize the total ranking of all schools she applies to.

School	#1	#2	#3	#4	#5	#6	#7	#8
rating	4	5	3	2	4	3	3	2
# of days	5	9	4	3	2	1	2	1

Part (a)

Part a1. Formulate an optimization problem to choose the schools that she can apply to within 10 days and whose total rating is maximized.

Variables:

x_i: 1 if Kyra chooses school i, 0 otherwise. $i \in 1, 2, ..., 8$

Mathematical Model:

$$\begin{aligned} \max\ & 4x_1 + 5x_2 + 3x_3 + 2x_4 + 4x_5 + 3x_6 + 3x_7 + 2x_8 \\ \text{s.t.}\ & \\ & 5x_1 + 9x_2 + 4x_3 + 3x_4 + 2x_5 + x_6 + 2x_7 + x_8 \leq 10 \\ & x_i \in \{0, 1\} \quad \forall i \in \{1, 2, ..., 8\} \end{aligned}$$

Part a2. Explain why the resulting problem is not a convex problems.

We have non-convexity due to integrality constraint:

$x_i \in \{0, 1\} \quad \forall i \in \{1, 2, ..., 8\}$.

Part a3. Create a linear programming relaxation and solve it (by applying greedy method used in homework and in class for knapsack problems).

LP Relaxation model (only relax the integrality constraint):

$$\begin{aligned} \max\ & 4x_1 + 5x_2 + 3x_3 + 2x_4 + 4x_5 + 3x_6 + 3x_7 + 2x_8 \\ \text{s.t.}\ & \\ & 5x_1 + 9x_2 + 4x_3 + 3x_4 + 2x_5 + x_6 + 2x_7 + x_8 \leq 10 \\ & 0 \leq x_i \leq 1 \quad \forall i \in \{1, 2, ..., 8\} \end{aligned}$$

Note that you should apply the greedy method to the LP Relaxation Problem, not the original problem. The greedy method solution will give the optimal solution for the LP problem. The optimal soln is $(0.8, 0, 0, 0, 1, 1, 1, 1)$ with objective value 15.2.

Part a4. What does the solution of the relaxation give you?

This is a maximization-type problem. The optimal value of the relaxed problem is upperbound. 15.2 is the upperbound for the problem in Part a.1.

Part a5. Can you use this solution to generate a feasible solution of the original problem?

Since $x_1 = 0.8$ is fractional element, you should take this issue into consideration for creating a feasible solution. For instance $(0, 0, 0, 0, 1, 1, 1, 1)$ is a feasible solution with the corresponding objective 12. Also $(0, 0, 1, 0, 1, 1, 1, 1)$ is another feasible solution with the objective value 15.

Part a6. Can you generate an optimal solution to the original problem? Justify your answers - prove that the solution you obtain is optimal by using the optimal value obtained from the relaxation.

$(0, 0, 1, 0, 1, 1, 1, 1)$ is a feasible solution for the original problem with the objective function value is 15 (ie, 15 is lowerbound). The upperbound is 15.2. The objective function value of the original problem MUST BE integer. This is because, the objective function coefficients are integer and all variables are integer in the original problem. Hence, we cannot obtain a better solution that 15. Then, 15 is both lower and upperbound. Hence it is the optimal value.

Part (b) There is an additional constraint, which says that Kyra can only apply to one of the first three high schools (#1, #2 or # 3). Add this constraint to your formulation. What is the optimal solution of the new problem? Can you prove that it is an optimal solution? (2pts)

Add the constraint $x_1 + x_2 + x_3 \leq 1$ to the model in part a.1. 15 still satisfies the constraints, hence it is a lowerbound (because it is a maximization-type problem). 15.2 is still the upperbound since it corresponds to the relaxation of the problem. The same argument applies about the integrality of the objective function value. 15 is both lower and upperbound. Hence it is the optimal value.

(c) Consider the LP relaxation of the problem defined in Part (a). Write down the dual of the linear programming relaxation. (Hint: do not forget the upper bounds on the primal variables). Compute the dual optimal solution from complementary slackness conditions and show that this solution is feasible (hence the primal solution you computed by hand is optimal). (7 pts)

Let's write the model in a more open form. The variables near the constraints correspond to the associated dual variables.

Primal Problem (The LP relaxation in Part a.3.)

$$
\begin{aligned}
\max\ & 4x_1 + 5x_2 + 3x_3 + 2x_4 + 4x_5 + 3x_6 + 3x_7 + 2x_8 \\
\text{s.t.}\ & \\
& 5x_1 + 9x_2 + 4x_3 + 3x_4 + 2x_5 + x_6 + 2x_7 + x_8 \leq 10 \qquad (y) \\
& x_1 \leq 1 \qquad (u_1) \\
& x_2 \leq 1 \qquad (u_2) \\
& x_3 \leq 1 \qquad (u_3) \\
& x_4 \leq 1 \qquad (u_4) \\
& x_5 \leq 1 \qquad (u_5) \\
& x_6 \leq 1 \qquad (u_6) \\
& x_7 \leq 1 \qquad (u_7) \\
& x_8 \leq 1 \qquad (u_8) \\
& x_i \geq 0 \quad \forall i \in \{1, 2, ..., 8\}
\end{aligned}
$$

Dual Problem:

$$
\begin{aligned}
\min\ & 10y + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 \\
\text{s.t.}\ & \\
& 5y + u_1 \geq 4 \\
& 9y + u_2 \geq 5 \\
& 4y + u_3 \geq 3 \\
& 3y + u_4 \geq 2 \\
& 2y + u_5 \geq 4 \\
& y + u_6 \geq 3 \\
& 2y + u_7 \geq 3 \\
& y + u_8 \geq 2 \\
& y, u_i \geq 0 \quad \forall i \in \{1, 2, ..., 8\}
\end{aligned}
$$

We know that $x = (0.8, 0, 0, 0, 1, 1, 1, 1)$ is the optimal solution for primal problem.

Complementary Slackness Conditions:

$$
\begin{aligned}
y(5x_1 + 9x_2 + 4x_3 + 3x_4 + 2x_5 + x_6 + 2x_7 + x_8 - 10) = 0 \\
u_i(x_i - 1) = 0 \quad \forall i \in \{1, 2, ..., 8\} \\
x_1(5y + u_1 - 4) = 0 \\
x_2(9y + u_2 - 5) = 0 \\
x_3(4y + u_3 - 3) = 0 \\
x_4(3y + u_4 - 2) = 0 \\
x_5(2y + u_5 - 4) = 0 \\
x_6(y + u_6 - 3) = 0 \\
x_7(2y + u_7 - 3) = 0 \\
x_8(y + u_8 - 2) = 0
\end{aligned}
$$

It gives us: $y = 0.8, u_1 = u_2 = u_3 = u_4 = 0, u_5 = 2.4, u_7 = 1.4, u_8 = 1.2$

You can also check the results from primal objective function equality condition.With the corresponding optimal primal decision variables (x's) and dual variables (y and u's), we must have $4x_1 + 5x_2 + 3x_3 + 2x_4 + 4x_5 + 3x_6 + 3x_7 + 2x_8 = 10y + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 = 15.2$

(d) Consider the LP relaxation in part (a). How much higher the ranking of the 4th school should be for Kyra to consider applying there. Derive your answer from complementarity conditions and feasibility of the dual solution. (3 pts)

Consider the dual $y = 0.8$ and $u_4 = 0$. Let A be the ranking of the 4th school. The dual constraint:

$$3y + u_4 \geq A \ \Rightarrow 2.4 \geq A$$

The CS condition:

$$x_4(3y + u_4 - A) = 0 \ \Rightarrow x_4(2.4 - A) = 0$$

If $A > 2.4$, then Kyra will consider applying 4th school.