ISE 426
Optimization models and applications

Lecture 8 — September 23, 2014

Duality

Reading:

- W.&V. Sections 6.5–6.7, pages 295-308
- H.&L. Section 6.1–6.4, pages 151-169

First: playing with equations and inequalities

- Trivial: if $a \le b$ and $c \le d$, then $\boxed{a + c \le b + d}$
- Similarly, if $a = b$ and $c = d$, then $\boxed{a + c = b + d}$
- And: if $a \le b$ and $c = d$, then $\boxed{a + c \le b + d}$
- Also, if $a \le b$, for any $k \ge 0$ we have $\boxed{ka \le kb}$

$\Rightarrow$ We can mix inequalities (and equations)! Example:

$$\begin{aligned} a_1x_1 + a_2x_2 \ldots + a_nx_n &\le a_0 \\ b_1x_1 + b_2x_2 \ldots + b_nx_n &\le b_0 \\ c_1x_1 + c_2x_2 \ldots + c_nx_n &\le c_0 \end{aligned}$$

and three numbers, $p, q, r \ge 0$, the following is true:

$$\begin{aligned} &p(a_1x_1 + a_2x_2 \ldots + a_nx_n) + \\ &q(b_1x_1 + b_2x_2 \ldots + b_nx_n) + \\ &r(c_1x_1 + c_2x_2 \ldots + c_nx_n) \qquad \le pa_0 + qb_0 + rc_0 \end{aligned}$$

That is, a nonnegative combination of constraints is valid.

Second: non-negative variables and linear functions

Consider $f(x_1, x_2) = 5x_1 + 3x_2$, and suppose $x_1, x_2 \geq 0$.
What is $\leq f(x_1, x_2)$ for any $x_1, x_2 \geq 0$?

- $g_1(x_1, x_2) = 0$
- $g_2(x_1, x_2) = x_1$
- $g_3(x_1, x_2) = 5x_1$
- $g_4(x_1, x_2) = 5.00001x_1$
- $g_5(x_1, x_2) = 4x_1 + 2x_2$
- $g_6(x_1, x_2) = 2x_1 + 9x_2$
- $g_7(x_1, x_2) = 3x_1 + 5x_2$

What can we conclude?

Second: non-negative variables and linear functions

Consider $f(x_1, x_2) = 5x_1 + 3x_2$, and suppose $x_1, x_2 \geq 0$.
What is $\leq f(x_1, x_2)$ for any $x_1, x_2 \geq 0$?

- $g_1(x_1, x_2) = 0$
- $g_2(x_1, x_2) = x_1$
- $g_3(x_1, x_2) = 5x_1$
- $g_4(x_1, x_2) = 5.00001x_1$
- $g_5(x_1, x_2) = 4x_1 + 2x_2$
- $g_6(x_1, x_2) = 2x_1 + 9x_2$
- $g_7(x_1, x_2) = 3x_1 + 5x_2$

What can we conclude?
For $x_1, x_2 \geq 0$, a function $g(x_1, x_2) = ax_1 + bx_2$ is lower than $f(x_1, x_2)$ only if $a \leq 5$ and $b \leq 3$.

Lower bounds of an LP problem

Consider the following **minimization** problem:

$$
\begin{array}{lll}
\min & 5x_1 + 4x_2 & \\
 & 2x_1 + x_2 & \geq 1 \\
 & x_1 + 2x_2 & \geq 1 \\
 & x_1, x_2 \geq 0 &
\end{array}
$$

- Solve this problem graphically
- Using constraints can we derive lower bounds for the optimal value?

Lower bounds of an LP problem

$$\begin{aligned} \min \quad & 5x_1 + 4x_2 & \\ & 2x_1 + x_2 & \geq 1 \\ & x_1 + 2x_2 & \geq 1 \\ & x_1, x_2 \geq 0 & \end{aligned}$$

- The optimal solution $x^* = (1/3, 1/3)$, $f(x^*) = 5x_1^* + 4x_2^* = 3$.
- $2x_1 + x_2 \geq 1$, and $x_1, x_2 \geq 0$ hence $5x_1^* + 4x_2^* \geq 1$.
- $2x_1 + x_2 \geq 1$, and $x_1 + 2x_2 \geq 1$ hence $5x_1^* + 4x_2^* \geq 2$.
- $2(2x_1 + x_2) \geq 2$, and $x_1 + 2x_2 \geq 1$ hence $5x_1^* + 4x_2^* \geq 3$.
- Once we know that $5x_1^* + 4x_2^* \geq 3$ and that $x = (1/3, 1/3)$ is feasible, then we know it is optimal.

Lower bounds of an LP problem, another example

Consider the following **minimization** problem:

$$\begin{array}{lll} \min & 3x_1 + 4x_2 & \\ & 5x_1 + 6x_2 & \geq 7 \\ & 8x_1 + 9x_2 & \geq 10 \\ & 11x_1 + 12x_2 & \geq 13 \\ & x_1, x_2 \geq 0 & \end{array}$$

We have a lower bound if we prove $3x_1 + 4x_2 \geq K$, for some K, by showing that

$$ax_1 + bx_2 \geq K$$

with $a \leq 3$ and $b \leq 4$.

Finding lower bounds using constraints

We can use nonnegative combinations of constraints?

For example: 0.25(first constraint) + 0.2(second constraint) =

$$\begin{array}{llcl} 0.25(5x_1 + 6x_2) & +0.2(8x_1 + 9x_2) & \geq & 0.25 \cdot 7 + 0.2 \cdot 10 \\ (1.25 + 1.6)x_1 & +(1.5 + 1.8)x_2 & \geq & 1.75 + 2.0 \\ 2.85x_1 & +3.3x_2 & \geq & 3.75 \end{array}$$

- $2.85 \leq 3$ and $3.3 \leq 4$

$\Rightarrow$ $2.85x_1 + 3.3x_2$ is a lower bound of $3x_1 + 4x_2$ for $x_1, x_2 \geq 0$

$\Rightarrow$ 3.75 is a lower bound:

$$3x_1 + 4x_2 \geq 2.85x_1 + 3.3x_2 \geq 3.75$$

So what kind of inequalities can we get?

- We have three constraints to work with
- We need three numbers $u_1, u_2, u_3 \geq 0$
- With u_1, u_2, u_3, we construct a new constraint $ax_1 + bx_2 \geq c$:

$$\begin{aligned} u_1(5x_1 + 6x_2) + u_2(8x_1 + 9x_2) + u_3(11x_1 + 12x_2) &\geq 7u_1 + 10u_2 + 13u_3 \\ \underbrace{(5u_1 + 8u_2 + 11u_3)}_{a} x_1 + \underbrace{(6u_1 + 9u_2 + 12u_3)}_{b} x_2 &\geq \underbrace{7u_1 + 10u_2 + 13u_3}_{c} \end{aligned}$$

- The new constraint is valid for any $u_1, u_2, u_3 \geq 0$
- We want $ax_1 + bx_2$ to be always below $3x_1 + 4x_2$

... (so that $c = 7u_1 + 10u_2 + 13u_3$ is a valid **lower bound**)

$\Rightarrow$ We must have $a \leq 3$ and $b \leq 4$. Hence,

$$\begin{aligned} 5u_1 + 8u_2 + 11u_3 &\leq 3 \\ 6u_1 + 9u_2 + 12u_3 &\leq 4 \end{aligned}$$

To recap

For any u_1, u_2, u_3 such that

$$\begin{aligned} &5u_1 + 8u_2 + 11u_3 \leq 3 \\ &6u_1 + 9u_2 + 12u_3 \leq 4 \\ &u_1, u_2, u_3 \geq 0 \end{aligned}$$

$7u_1 + 10u_2 + 13u_3$ is a lower bound of $3x_1 + 4x_2$. Examples:

- $(u_1, u_2, u_3) = (0, 0, 0)$. Lower bound: 0
- $(u_1, u_2, u_3) = (0.2, 0.1, 0.1)$. Lower bound: 3.7
- $(u_1, u_2, u_3) = (0.1, 0.2, 0)$. Lower bound: 2.7
- $(u_1, u_2, u_3) = (0.5, 0, 0.04)$. Lower bound: 4.

Should we try all possible combinations? Is there a better way?

The dual problem

Yes, there is. We like large lower bounds, so we want to maximize that $7u_1 + 10u_2 + 13u_3$.

$$\begin{array}{ll} \max & 7u_1 + 10u_2 + 13u_3 \\ & 5u_1 + 8u_2 + 11u_3 \quad \leq 3 \\ & 6u_1 + 9u_2 + 12u_3 \quad \leq 4 \\ & u_1, u_2, u_3 \geq 0 \end{array}$$

- Any feasible solution to this problem provides a lower bound to the original problem.
- An optimal solution to this problem provides a **good** lower bound to the original problem.

Primal problem, dual problem

Primal

$$\begin{array}{lll} \min & 3x_1 + 4x_2 & \\ & 5x_1 + 6x_2 & \geq 7 \\ & 8x_1 + 9x_2 & \geq 10 \\ & 11x_1 + 12x_2 & \geq 13 \\ & x_1, x_2 \geq 0 & \end{array}$$

Dual

$$\begin{array}{lll} \max & 7u_1 + 10u_2 + 13u_3 & \\ & 5u_1 + 8u_2 + 11u_3 & \leq 3 \\ & 6u_1 + 9u_2 + 12u_3 & \leq 4 \\ & u_1, u_2, u_3 \geq 0 & \end{array}$$

In general:

$$\begin{array}{ll} \min & c^\top x \\ & Ax \geq b \\ & x \geq 0 \end{array} \qquad\qquad \begin{array}{ll} \max & b^\top u \\ & A^\top u \leq c \\ & u \geq 0 \end{array}$$

The **primal** has n variables and m constraints

$\Rightarrow$ The **dual** has m variables and n constraints

Primal problem, dual problem

Primal	Dual
$$\begin{array}{lll} \min & 5x_1 + 4x_2 & \\ & 2x_1 + x_2 & \geq 1 \\ & x_1 + 2x_2 & \geq 1 \\ & x_1, x_2 \geq 0 & \end{array}$$	$$\begin{array}{lll} \max & u_1 + u_2 & \\ & u_1 + u_2 & \leq 5 \\ & 2u_1 + u_2 & \leq 4 \\ & u_1, u_2 \geq 0 & \end{array}$$

How to construct the dual of an LP

Variable	Constraint
Constraint	Variable
Minimize	Maximize
Variable ≥ 0	Constraint $\leq$
Variable ≤ 0	Constraint $\geq$
Var. Unrestricted	Constraint $=$
Constraint $\leq$	Variable ≤ 0
Constraint $\geq$	Variable ≥ 0
Constraint $=$	Var. Unrestricted

Examples

$$
\begin{array}{llll}
\min & 12x_1 & -47x_2 & \\
\text{s.t.} & 25x_1 & -36x_2 & \leq 97 \\
& 38x_1 & +89x_2 & \geq 10 \\
& x_1 \geq 0 & &
\end{array}
\quad \rightarrow \quad
\begin{array}{llll}
\max & 97u_1 & +10u_2 & \\
\text{s.t.} & 25u_1 & +38u_2 & \leq \ 12 \\
& -36u_1 & +89u_2 & = -47 \\
& u_1 \leq 0 & u_2 \geq 0 &
\end{array}
$$

$$
\begin{array}{ll}
\max & 2x_1 + x_2 \\
\text{s.t.} & -x_1 + x_2 = 9 \\
& 9x_1 - x_2 \geq 11 \\
& 7x_1 + 2x_2 \leq 3 \\
& -x_1 + 7x_2 \geq 1 \\
& x_1 \leq 0, x_2 \geq 0
\end{array}
\quad \rightarrow \quad
\begin{array}{ll}
\min & 9u_1 + 11u_2 + 3u_3 + u_4 \\
\text{s.t.} & -u_1 + 9u_2 + 7u_3 - u_4 \ \leq 2 \\
& u_1 - u_2 + 2u_3 + 7u_4 \ \geq 1 \\
& u_2, u_4 \leq 0; u_3 \geq 0
\end{array}
$$

What is the dual of the dual?

$$\begin{array}{ll} \min & c^\top x \\ & Ax \geq b \\ & x \geq 0 \end{array} \quad \rightarrow \quad \begin{array}{ll} \max & b^\top u \\ & A^\top u \leq c \\ & u \geq 0 \end{array} \quad \rightarrow \quad \begin{array}{ll} \min & c^\top x \\ & Ax \geq b \\ & x \geq 0 \end{array}$$

- The dual of the dual is the primal problem
- An LP and its dual are said to form a **primal-dual pair**

Properties of duality in LP

Weak duality: Given a primal $\min\{c^\top x : Ax \geq b, x \geq 0\}$ and its dual $\max\{b^\top u : A^\top u \leq c, u \geq 0\}$,

$$b^\top \bar{u} \leq c^\top \bar{x}$$

for any $\bar{x}$ and $\bar{u}$ feasible for their respective problems.

- Proof: $c^\top \bar{x} \geq (A^\top \bar{u})^\top \bar{x} = \bar{u}^\top A\bar{x} \geq \bar{u}^\top b$
- $u^\top b$ is a LB, our main purpose when constructing the dual.

Strong duality[1]: If a problem $\min\{c^\top x : Ax \geq b, x \geq 0\}$ is bounded and its dual $\max\{b^\top u : A^\top u \leq c, u \geq 0\}$ is bounded, their optimal solutions $\bar{x}$ and $\bar{u}$ coincide in value:

$$c^\top \bar{x} = b^\top \bar{u}$$

[1] The proof of this is much more complicated, but beautiful nonetheless.

Properties of duality in LP (cont.)

Consequence: solving the dual or the primal doesn't matter: we get the same objective function value.

What if the primal (or the dual) is infeasible or unbounded?

Four cases:

- Primal bounded, dual bounded;
- Primal infeasible, dual infeasible;
- Primal unbounded ($c^\top x = -\infty$), dual infeasible;
- Primal infeasible, dual unbounded ($b^\top u = +\infty$).

		Dual		
		bounded	unbounded	infeasible
Primal	bounded	Possible	–	–
	unbounded	–	–	Possible
	infeasible	–	Possible	Possible

Complementary slackness

- Given a primal-dual pair, now we know how to solve one and get the optimal objective function of the other.

e.g. Solve primal $\Rightarrow$ get optimal obj.f. $c^\top \bar{x}$, an optimal solution $\bar{x}$, and the optimal dual obj.f. $b^\top \bar{u}$. **How do we get $\bar{u}$?**

Complementary Slackness: If the primal problem

$$\min\{c^\top x : \sum_{i=1}^n a_{ji} x_i \geq b_j \ \forall j = 1, 2 \ldots, m, x \geq 0\}$$

is bounded and admits optimum $\bar{x}$, and its dual

$$\max\{b^\top u : \sum_{j=1}^m a_{ji} u_j \leq c_i \ \forall i = 1, 2 \ldots, n, u \geq 0\}$$

is bounded and admits optimal solution $\bar{u}$, then

$$\bar{u}_i(\sum_{i=1}^n a_{ji}\bar{x}_i - b_j) = 0 \quad \forall i = 1, 2 \ldots, m;$$
$$\bar{x}_j(\sum_{j=1}^m a_{ji}\bar{u}_j - c_j) = 0 \quad \forall j = 1, 2 \ldots, n$$

So if we solve the primal and get $\bar{x}$, we can get $\bar{u}$ by solving a system of equations.

Example

$$\begin{array}{ll} \min & 3x_1 + 4x_2 \\ & 5x_1 + 6x_2 \quad \geq 7 \\ & 8x_1 + 9x_2 \quad \geq 10 \\ & 11x_1 + 12x_2 \quad \geq 13 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{ll} \max & 7u_1 + 10u_2 + 13u_3 \\ & 5u_1 + 8u_2 + 11u_3 \quad \leq 3 \\ & 6u_1 + 9u_2 + 12u_3 \quad \leq 4 \\ & u_1, u_2, u_3 \geq 0 \end{array}$$

Solve the dual (with AMPL+CPLEX): get $(u_1, u_2, u_3) = (0.6, 0, 0)$.
Find (x_1, x_2) with complementary slackness:

$$\begin{array}{l} u_1(5x_1 + 6x_2 - 7) = 0 \\ u_2(8x_1 + 9x_2 - 10) = 0 \\ u_3(11x_1 + 12x_2 - 13) = 0 \\ x_1(5u_1 + 8u_2 + 11u_3 - 3) = 0 \\ x_2(6u_1 + 9u_2 + 12u_3 - 4) = 0 \end{array} \Rightarrow \begin{array}{l} 0.6(5x_1 + 6x_2 - 7) = 0 \\ 0(8x_1 + 9x_2 - 10) = 0 \\ 0(11x_1 + 12x_2 - 13) = 0 \\ x_1(5 \cdot 0.6 + 8 \cdot 0 + 11 \cdot 0 - 3) = 0 \\ x_2(6 \cdot 0.6 + 9 \cdot 0 + 12 \cdot 0 - 4) = 0 \end{array}$$

$$\begin{array}{l} 5x_1 + 6x_2 = 7 \\ x_1 \cdot 0 = 0 \\ x_2 \cdot (-0.4) = 0 \end{array} \Rightarrow \begin{array}{l} 5x_1 + 6x_2 = 7 \\ x_1 \cdot 0 = 0 \\ x_2 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = \frac{7}{5} \\ x_2 = 0 \end{array}$$

An example: the shortest path problem

Given

- a digraph $G = (V, A)$,
- a function $c : A \to \mathbb{R}_+$, and
- two nodes s and t of V,

find a subset $P = \{(s, i_1), (i_1, i_2) \ldots, (i_k, t)\}$ of A forming a path from s to t whose length, $c_{si_1} + c_{i_1 i_2} \ldots c_{i_k t}$, is minimum.

- Countless applications, e.g. GPS navigation systems.
- We can formulate it as a special case of min-cost-flow:

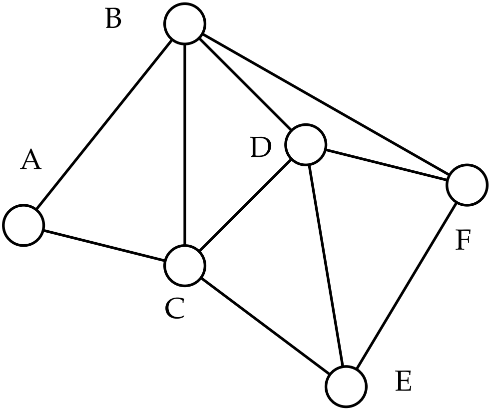
$$
\begin{array}{rll}
\min & \sum_{(i,j)\in A} c_{ij} x_{ij} & \\
\text{s.t.} & \sum_{j\in V:(i,j)\in A} x_{ij} - \sum_{j\in V:(j,i)\in A} x_{ji} = b_i & \forall i \in V \\
& x_{ij} \geq 0 & \forall (i,j) \in A
\end{array}
$$

$$
\text{where } b_i = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}
$$

An example: the shortest path problem

For simplicity, the graph below is undirected, but we can assume for each edge there are two oppositely oriented arcs.

Suppose the problem is to compute the shortest path A $\to$ F.



The shortest path problem: primal

$$
\begin{array}{llll}
\min & c_{AB}x_{AB} + c_{BA}x_{BA} + \ldots + c_{EF}x_{EF} & +c_{FE}x_{FE} & \\
 & x_{AB} + x_{AC} & -x_{BA} - x_{CA} & = 1 \\
 & x_{BA} + x_{BC} + x_{BD} + x_{BF} & -x_{AB} - x_{CB} - x_{DB} - x_{FB} & = 0 \\
 & x_{CA} + x_{CB} + x_{CD} + x_{CE} & -x_{AC} - x_{BC} - x_{DC} - x_{EC} & = 0 \\
 & x_{DB} + x_{DC} + x_{DE} + x_{DF} & -x_{BD} - x_{CD} - x_{ED} - x_{FD} & = 0 \\
 & x_{EC} + x_{ED} + x_{EF} & -x_{CE} - x_{DE} - x_{FE} & = 0 \\
 & x_{FB} + x_{FD} + x_{FE} & -x_{BF} - x_{DF} - x_{DF} & = -1 \\
 & x_{AB}, x_{BA} \ldots, x_{EF}, x_{FE} \geq 0 & &
\end{array}
$$

- We can express this as $\min\{c^\top x : Ax = b, x \geq 0\}$
- A is the adjacency matrix of G
- $|V|$ constraints, $|A|$ variables
- All constraints are equalities

The shortest path problem: dual

$$
\begin{aligned}
\max \quad & u_A - u_F \\
& u_A - u_B \le c_{AB} \\
& u_B - u_A \le c_{BA} \\
& u_A - u_C \le c_{AC} \\
& u_C - u_A \le c_{CA} \\
& \vdots \\
& u_E - u_F \le c_{EF} \\
& u_F - u_E \le c_{FE}
\end{aligned}
$$

- This is $\max\{b^\top u : A^\top u \le c\}$
- $A^\top$ is the **transposed** adjacency matrix of G
- $|V|$ variables, $|A|$ constraints
- All variables are unrestricted in sign