ISE-429
HW3-sol

70. (a) $P_{i,i+1} = \frac{(m-i)^2}{m^2}, \quad P_{i,i-1} = \frac{i^2}{m^2}$

$$P_{i,i} = \frac{2i(m-i)}{m^2}$$

(b) Since, in the limit, the set of m balls in urn 1 is equally likely to be any subset of m balls, it is intuitively clear that

$$\pi_i = \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} = \frac{\binom{m}{i}^2}{\binom{2m}{m}}$$

(c) We must verify that, with the π_i given in (b),

$$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$$

That is, we must verify that

$$(m-i)\binom{m}{i} = (i+1)\binom{m}{i+1}$$

which is immediate.

76. We can view this problem as a graph with 64 nodes where there is an arc between 2 nodes if a knight can go from one node to another in a *single move*. The weights on each are equal to 1. It is easy to check that $\sum_i \sum_j w_{ij} = 336$, and for a corner node i, $\sum_j w_{ij} = 2$. Hence, from Example 7b, for one of the 4 corner nodes i, $\prod_i = 2/336$, and thus the mean time to return, which equals $1/r_i$, is $336/2 = 168$.

37. (a) The state is $(n_1, \ldots, n_k)$ if there are n_i type i patients in the hospital, for all $i = 1, \ldots, k$.

(b) It is a $M/M/\infty$ birth and death process, and thus time reversible.

(c) Because $N_i(t), t \geq 0$ are independent processes for $i = 1, \ldots, k$, the vector process is a time reversible continuous time Markov chain.

(d)

$$P(n_1, \ldots, n_k) = \prod_{i=1}^{k} e^{-\lambda i/\mu_i} (\lambda_i/\mu_i)^{n_i}/n_i!$$

(e) As a truncation of a time reversible continuous time Markov chain, it has stationary probabilites

$$P(n_1, \ldots, n_k) = K \prod_{i=1}^{k} e^{-\lambda_i/\mu_i} (\lambda_i/\mu_i)^{n_i}/n_i!, \quad (n_1, \ldots, n_k) \in A$$

where $A = \{(n_1, \ldots, n_k) : \sum_{i=1}^{k} n_i w_i \leq C\}$, and K is such that

$$K \sum_{(n_1, \ldots, n_k) \in A} \prod_{i=1}^{k} e^{-\lambda_i/\mu_i} (\lambda_i/\mu_i)^{n_i}/ni! = 1$$

(f) With r_i equal to the rate at which type i patients are admitted,

$$r_i = \sum_{(n_1, \ldots, n_i+1, \ldots, n_k) \in A} \lambda_i P(n_1, \ldots, n_k)$$

(g) $\sum_{i=1}^{k} r_i / \sum_{i=1}^{k} \lambda_i$

43. We make the conjecture that the reverse chain is a system of same type, except that the Poisson arrivals at rate λ arrive at server 3, then go to server 2, then to server 1, and then depart the system. With the state being $\mathbf{i} = (i_1, i_2, i_3)$ when that there are i_j customers at server j for $j = 1, 2, 3$, the instantaneous transition rates of the chain are

$$
\begin{aligned}
q_{(i,j,k),(i+1,j,k)} &= \lambda \\
q_{(i,j,k),(i-1,j+1,k)} &= \mu_1, \quad i > 0 \\
q_{(i,j,k),(i,j-1,k+1)} &= \mu_2, \quad j > 0 \\
q_{(i,j,k),(i,j,k-1)} &= \mu_3, \quad k > 0
\end{aligned}
$$

whereas the conjectured instantaneous rates for the reversed chain are

$$
\begin{aligned}
q^*_{(i,j,k),(i,j,k+1)} &= \lambda \\
q^*_{(i,j,k),(i,j+1,k-1)} &= \mu_3, \quad k > 0 \\
q^*_{(i,j,k),(i+1,j-1,k)} &= \mu_2, \quad j > 0 \\
q_{(i,j,k),(i-1,j,k)} &= \mu_1, \quad i > 0
\end{aligned}
$$

The conjecture is correct if we can find probabilities $P(i, j, k)$ that satisfy the reverse time equations when the preceding are the instantaneous rates for the reversed chain , and it is easy to check that

$$P(i, j, k) = K \left(\frac{\lambda}{\mu_1}\right)^i \left(\frac{\lambda}{\mu_2}\right)^j \left(\frac{\lambda}{\mu_3}\right)^k$$

satisfy.

36. (a) If we let $N_i(t)$ denote the number of times person i has skied down by time t, then $\{N_i(t)\}$ is a (delayed) renewal process. As $N(t) = \sum N_i(t)$, we have

$$\lim \frac{N(t)}{t} = \sum_i \lim \frac{N_i(t)}{t} = \sum_i \frac{1}{\mu_i + \theta_i}$$

where μ_i and θ_i are respectively the mean of the distributions F_i and G_i.

(b) For each skier, whether they are climbing up or skiing down constitutes an alternating renewal process, and so the limiting probability that skier i is climbing up is $p_i = \mu_i/(\mu_i + \theta_i)$. From this we obtain

$$\lim P\{U(t) = k\} = \sum_S \left\{\prod_{i \in S} p_i \prod_{i \in S^c} (1 - p_i)\right\}$$

where the above sum is over all of the $\begin{bmatrix} n \\ k \end{bmatrix}$ subsets S of size k.

39. Let B be the length of a busy period. With S equal to the service time of the machine whose failure initiated the busy period, and T equal to the remaining life of the other machine at that moment, we obtain

$$E[B] = \int E[B|S = s]g(s)ds$$

Now,

$$\begin{aligned} E[B|S = s] &= E[B|S = s, T \le s](1 - e^{-\lambda s}) \\ &\quad + E[B|S = s, T > s]e^{-\lambda s} \\ &= (s + E[B])(1 - e^{-\lambda s}) + se^{-\lambda s} \\ &= s + E[B](1 - e^{-\lambda s}) \end{aligned}$$

Substituting back gives

$$E[B] = E[S] + E[B]E[1 - e^{-\lambda s}]$$

or

$$E[B] = \frac{E[S]}{E[e^{-\lambda s}]}$$

Hence,

$$E[\text{idle}] = \frac{1/(2\lambda)}{1/(2\lambda) + E[B]}$$

Capital S

Process: X(t) = the number of machines working. Renewal point is a time instant when a transition from X=2 to X=1 occurs. Busy period B starts at a renewal point and ends when there is a transition back to X=2. Renewal time consists of a busy period B followed by an idle period U; B and U are independent. The quantity we are interested in is E[U]/E[B+U].