ISE429. Homework 1

Unless stated otherwise, the problems are from the Ross textbook (11th edition):

1) (weight 0.15) 4-18. Here "proportion" means "limiting proportion (fraction of time)"

2) (weight 0.20) 4-20 and 4-25

3) (weight 0.15) 4-52

4) (weight 0.15) 4-58

5) (weight 0.15) 4-60

Solution. Modify the Markov chain to make states 3 and 4 absorbing. Use matrix expressions that we derived in class.

6) (weight 0.20) Discrete time Markov chain $\{X_n,\ n = 0, 1, 2, \ldots\}$ takes values in state space $\mathcal{X} = \{-5, -4, -3, -2, -1, 0, 1, 2, \ldots\}$. It has the following structure:

$$X_n = \max\{X_{n-1} + Y_n, -5\}, \quad n = 1, 2, \ldots,$$

where $Y_n,\ n = 1, 2, \ldots$ are i.i.d. (independent identically distributed) random variables, taking integer values, and such that

$$E|Y_1| < \infty, \qquad EY_1 < 0, \quad P\{Y_1 = 1\} > 0.$$

This Markov chain is irreducible (explain why) and aperiodic (explain why). Is it positive recurrent? *Comment: You must give rigorous proofs.*

Solution. First of all, let us relabel the states, so that state i becomes $i + 5$. Then, the state space becomes $\mathcal{X} = \{0, 1, 2, \ldots\}$, and the structure of the Markov chain is:

$$X_n = \max\{X_{n-1} + Y_n, 0\}, \quad n = 1, 2, \ldots.$$

Since $EY_1 < 0$, there exists a positive integer m, such that $P\{Y_1 = -m\} > 0$. This means that state 0 is reachable from any other. Since $P\{Y_1 = 1\} > 0$, any state is reachable from 0. This shows irreducibility. The one-step transition probability from 0 to itself: $P_{00} > 0$; this means the chain period is 1. To prove positive recurrence, use Lyapunov-Foster criterion with $V(i) = i$.

$$E(V(X_1)|X_0 = i) - V(i) = E\max\{Y_1, -i\} \to EY_1 < 0, \quad \text{as } i \to \infty.$$

This mean that exists $\epsilon > 0$ and $i_0 \geq 0$ such that

$$E(V(X_1)|X_0 = i) - V(i) \leq -\epsilon \ \text{ for } i > i_0.$$

And, clearly,

$$E(V(X_1)|X_0 = i) - V(i) \leq E|Y_1| < \infty \ \text{ for all } i.$$

It suffices to choose $\mathcal{X}_0 = \{0, 1, \ldots, i_0\}$. $\square$