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Theorem 1 [Lyapunov-Foster criterion for positive recurrence]. *Suppose, X_n, $n = 0, 1, 2, \ldots$, is an irreducible Markov chain with countable state space $\mathcal{X}$. Assume that there exists a non-negative finction $V = V(i), i \in \mathcal{X}$, a finite subset $\mathcal{X}_0 \subset \mathcal{X}$, and $\epsilon > 0$, such that:*
(a) $\forall i \in \mathcal{X}_0$: $\quad E[V(X_1)|X_0 = i] < \infty$,
(b) $\forall i \in \overline{\mathcal{X}} = \mathcal{X} \setminus \mathcal{X}_0$: $\quad E(V(X_1)|X_0 = i) \le V(i) - \epsilon$.
Then $\{X_n\}$ is positive recurrent.

Proof. Fix initial state $X_0 = m$. Then, for any time $n \ge 1$, for some fixed $C > 0$

$$EV(X_n) - EV(X_{n-1}) = \sum_i p_{mi}^{n-1}[E(V(X_n)|X_{n-1} = i) - V(i)] \le$$

$$\le [\sum_{i \in \mathcal{X}_0} p_{mi}^{n-1}]C + [1 - \sum_{i \in \mathcal{X}_0} p_{mi}^{n-1}](-\epsilon).$$

Then,

$$(1/k)[EV(X_k) - EV(X_0)] = (1/k)\sum_{n=1}^{k} E[V(X_n) - EV(X_{n-1})] \le$$

$$\le [\sum_{i \in \mathcal{X}_0} (1/k) \sum_{n=1}^{k} p_{mi}^{n-1}]C + [1 - \sum_{i \in \mathcal{X}_0} (1/k) \sum_{n=1}^{k} p_{mi}^{n-1}](-\epsilon). \tag{1}$$

If the Markov chain is not positive recurrent, then

$$(1/k)\sum_{n=1}^{k} p_{mi}^{n-1} \to \pi_i = 0.$$

Then, as $k \to \infty$ the right-hand side of (1) converges to $(-\epsilon)$. This means that $EV(X_k) \to -\infty$. Impossible. The contradiction proves the result. $\square$