ISE426 Optimization models and applications, Fall 2014
Homework 1 Solution,

1. **Convexity and Relaxations (5 pts.)**

(1) Non-convex; (2) Convex; (3) Convex; (4) Convex; (5) Non-convex; (6) Non- convex.

Before going through the details, consider following notes about Convexity,

– A (minimization) optimization problem is convex if the objective function is convex and all constraints (in "less than or equal to" form) are convex.

– A function is convex if and only if its Hessian is Positive Semidefinite.

– A symmetric matrix is Positive Semidefinite if and only if all principal minors are nonnegative.

– If the function $g(x)$ is convex, the constraint $g(x) \leq b$ is convex.

– Some common convex functions:
- All Linear functions,
- $f(x) := x^{2n}$, for even n,
- $f(x) := e^x$ and $f(x) := -\ln(x)$.

– Positive multiple of a convex function is convex.

– Sum of two convex functions is convex.

– A **Non**-linear equality constraint is always **Non**-convex.

Now, consider the following tables for details of convexity of each problems.

Problem1	**Function**	**Convexity**	**Description**
Obj.	$2x^2 + 2y^2 + xy$	Yes	Positive definite Hessian
Con1.	$15x + 50y \geq 17$	Yes	Linear inequality
Con2.	$0.001x - y = 0$	Yes	Linear equality
Con3.	$x^2 + y^2 = 1/3$	No	Non-linear equality

Problem 1 is NOT Convex.

Problem2	**Function**	**Convexity**	**Description**
Obj.	$3x^2 + 2y^2 + 5xy$	Yes	Positive definite Hessian
Con1.	$50x + 50y \leq 530$	Yes	Linear inequality
Con2.	$5x - y \leq 3$	Yes	Linear inequality
Con3.	$2x^2 + 2y^2 + 3xy \leq 5$	Yes	Positive definite Hessian

Problem 2 is Convex.

Problem3	Function	Convexity	Description
Obj.	$x^2 + 10y^4 - \ln(z)$	Yes	Sum of the convex functions
Con1.	$x^2 + y^2 + 3z^2 \leq 1$	Yes	Sum of the convex functions
Con2.	$3y^2 \leq 125$	Yes	Quadratic function
Con3.	$-y + 2z^2 \leq 0$	Yes	Sum of the convex functions

Problem 3 is Convex.

Problem4	Function	Convexity	Description
Obj.	$(x + y)^2$	Yes	Positive Definite Hessian
Con1.	$x - y = 0$	Yes	Linear equality
Con2.	$e^x \leq 1$	Yes	e^x is a Convex function

Problem 4 is Convex.

Problem5	Function	Convexity	Description
Obj.	$x^2 + 3y + 5z$	Yes	Sum of the convex functions
Con1.	$e^x + e^y \leq 1$	Yes	Sum of the convex functions
Con2.	$-x^2 - z^2 - 2xy \leq -1$	No	Not Positive Definite Hessian

Problem 5 is NOT Convex.

Problem6	Function	Convexity	Description
Obj.	$2x^2 + 2y^2 + 3xy$	No	Note, it's a MAX problem
Con1.	$x^2 + y^4 \leq 1$	Yes	Sum of convex functions
Con2.	$x + y \geq 0.1$	Yes	Linear inequality
Con3.	$3x + 2y = 5$	Yes	Linear equality

Problem 6 is NOT Convex.

2. **Local and Global Minima (5 pts.)**
Consider following problem,

$$\begin{aligned} \max\ & -20x + 24y \\ & -x + y \leq 4, \\ & 2x + y \leq 5, \\ & x + y \geq 3, \\ & x, y \in \mathbb{Z}. \end{aligned}$$

(1) No. This problem is NOT convex because of the integer variables.

(2) Yes. The solution $(x, y) = (0, 4)$ is feasible because it satisfies all constraints.

(3) No. The solution $(x, y) = (\frac{1}{3}, 4\frac{1}{3})$ is infeasible because of the integrality constraint.

(4) $f(0, 4) = 96$ and $f(\frac{1}{3}, 4\frac{1}{3}) = 97.33$.

(5) Any feasible solution of a Maximization problem is a **lower bound** for the optimal solution, so since $(x, y) = (0, 4)$ is a feasible solution, the value of $f(0, 4) = 96$ is a **lower bound** of the optimal

value of our problem. On the other hand, since $(x, y) = (\frac{1}{3}, 4\frac{1}{3})$ is not feasible, $f(\frac{1}{3}, 4\frac{1}{3}) = 97.33$ it can not be a lower bound, it can be just the **upper bound** of the optimal value of our problem if it is the optimal solution of the relaxed problem (after linear relaxation).

(6) Yes. After eliminating the integrality constraint, the new problem is convex, since the objective function and constraints are all linear.

(7) Yes. Both of them are feasible for the relaxed problem.

(8) Yes. The point $(x, y) = (\frac{1}{3}, 4\frac{1}{3})$ is the maximizer of the relaxed problem. Note that now we can say that $f(\frac{1}{3}, 4\frac{1}{3}) = 97.33$ is an **upper bound** for the optimal value of the original problem.

(9) So based on the value of lower bound and upper bound of the optimal value of the original problem, we have,

$$96 \leq z_{\text{opt}} \leq 97.33,$$

By considering these bounds and the integrality of x and y we can conclude that the the optimal value of the original problem has to be equal to its lower bound, $z_{\text{opt}} = 96$. This is because, the objective function has to be an integer number divisible by 4 and from the upper bound of 97.33 we conclude that the actually upper bound is 96, which equals the lower bound.

3. **Linear programming model (5 pts.)**

The problem asks to determine the optimal amount of each ingredient with the aim of minimizing the cost. One way to arrive at this aim is to define the decision variables as

$$\begin{aligned}
&x_1 : \text{the quantity of forget-me stone in ounce} \\
&x_2 : \text{the quantity of liquid morpheus in ounce} \\
&x_3 : \text{the quantity of essence of sirens in ounce}
\end{aligned}$$

Then, the objective function would be

$$\min \ c_1x_1 + c_2x_2 + c_3x_3, \tag{1}$$

where c_i denotes the cost per ounce associated with the ingredient i.

The problem requires that the percentage of each component (i.e., memory eraser, relaxant and hallocinogen) obtained from the desired combination lie within a specified bound. All these requirements are reflected in the following constraints

Constraint on memory eraser:

$$\begin{aligned}
\tfrac{60x_1+30x_2+40x_3}{x_1+x_2+x_3} \leq M_2 \quad &\Rightarrow \quad (60 - M_2)x_1 + (30 - M_2)x_2 + (40 - M_2)x_3 \leq 0 \\
\tfrac{60x_1+30x_2+40x_3}{x_1+x_2+x_3} \geq M_1 \quad &\Rightarrow \quad (M_1 - 60)x_1 + (M_1 - 30)x_2 + (M_1 - 40)x_3 \leq 0
\end{aligned}$$

Constraint on relaxant:

$$\begin{aligned}
\tfrac{20x_1+30x_2+20x_3}{x_1+x_2+x_3} \leq R_2 \quad &\Rightarrow \quad (20 - R_2)x_1 + (30 - R_2)x_2 + (20 - R_2)x_3 \leq 0 \\
\tfrac{20x_1+30x_2+20x_3}{x_1+x_2+x_3} \geq R_1 \quad &\Rightarrow \quad (R_1 - 20)x_1 + (R_1 - 30)x_2 + (R_1 - 20)x_3 \leq 0
\end{aligned}$$

Constraint on hallocinogen:

$$\frac{20x_1+30x_2+40x_3}{x_1+x_2+x_3} \leq H_2 \quad \Rightarrow \quad (20-H_2)x_1+(30-H_2)x_2+(40-H_2)x_3 \leq 0$$
$$\frac{20x_1+30x_2+40x_3}{x_1+x_2+x_3} \geq H_1 \quad \Rightarrow \quad (H_1-20)x_1+(H_1-30)x_2+(H_1-40)x_3 \leq 0$$

In the sequel, this problem can be casted into

$$\begin{aligned}
\min\ & c_1x_1+c_2x_2+c_3x_3 \\
\text{s.t. } & (60-M_2)x_1+(30-M_2)x_2+(40-M_2)x_3 \leq 0 \\
& (M_1-60)x_1+(M_1-30)x_2+(M_1-40)x_3 \leq 0 \\
& (20-R_2)x_1+(30-R_2)x_2+(20-R_2)x_3 \leq 0 \\
& (R_1-20)x_1+(R_1-30)x_2+(R_1-20)x_3 \leq 0 \\
& (20-H_2)x_1+(30-H_2)x_2+(40-H_2)x_3 \leq 0 \\
& (H_1-20)x_1+(H_1-30)x_2+(H_1-40)x_3 \leq 0 \\
& x_1, x_2.x_3 \geq 0
\end{aligned}$$

4. **Knapsack problem (5 pts.)**
1) Suppose that you keep Object 9 and swap Object 8 for something else. Then, the only possible object which Object 8 can be swapped for is Object 6 since otherwise, the constraint will be violated. Nevertheless, this decision does not make any difference in the objective value. Now suppose that you keep Object 8. Then, you are able to exchange Object 9 for Objects 1, 5 and 6 without violating the constraint. However, none of these replacements improves the objective value.

2) By ordering the decision variables according to their price/weight ratios, we get

i	1	2	6	4	5	8	9	3	7
p_i	30	13	35	11	21	31	20	6	10
w_i	2	1	3	1	2	3	2	1	3
p_i/w_i	15	13	11.6	11	10.5	10.33	10	6	3.33

Then, by applying the greedy algorithm, we would get

$$\begin{aligned}
x_1 &= 1 \quad \Rightarrow \quad \text{The updated right hand side} = 51-30 = 21 \\
x_2 &= 1 \quad \Rightarrow \quad \text{The updated right hand side} = 21-13 = 8 \\
x_6 &= \tfrac{8}{35}
\end{aligned}$$

where the objective value equals 3.69. Since this is a fractional solution (infeasible to the Knapsack problem), it gives a lower bound to the optimal objective value of the Knapsack problem; that is $Z_l = 3.69$.

3) Using the solution obtained from the LP relaxation, a feasible solution can be simply found. Toward this end, we round up or round down the values to the nearest integer solutions and then check for feasibility. In case that the constraint is violated, we keep searching through the objects left and add those with minimum weights. In our case, we would have

$$x = (1,1,0,0,0,\tfrac{8}{35},0,0,0) \text{ Integer rounding } \Rightarrow \quad x = (1,1,0,0,0,0,0,0,0) \tag{2}$$

However, this solution is still infeasible. By searching through the objects left, we can realize the Object 4 can be added to our collection and thus we would get $x = (1,1,0,1,0,0,0,0,0)$ with the objective value of 4. This is a feasible solution to the knapsack problem and thus it gives an upper bound; that is $Z_u = 4$.

Notice that the objective coefficents in the problem are all integer values. Hence, the optimal objective value of the knapsack problem (Z_{opt}) is expected to be an integer value. Due to the fact that $Z_l \leq Z_{opt} \leq Z_u$, we can conclude that x is indeed optimal.

4) Please see part (3).