ISE426 – Optimization models and applications

Fall 2015 – Final exam, December 9, 2015

First name	
Last name	
Program	

You have three hours. This exam accounts for 25% of the final grade. There are 5 problems and 110 points available. Any result with 100 points and above will be considered full score. Read all problem descriptions carefully!! I suggest to read **all problems first** and make sure you understand what is required. Use a **readable** handwriting: it will be hard to grade your answers if they are not readable, let alone give them full points. For each model, clearly specify the meaning of each variable and of each constraint.

1 Convexity (10 pts.)

Which of the following four problems is convex (there is only one), explain why:

(1) $\min_{x,y} \quad \max\{x+2y, -2x-3y\}$
$\qquad x^2+3y \geq 1$

(2) $\max_{x,y} \quad \max\{x+2y, -2x-3y\}$
$\qquad x^2+3y \leq 1$

(3) $\min_{x,y} \quad \max\{x+2y, -2x-3y\}$
$\qquad |x|+3y \leq 1$

(4) $\min_{x,y} \quad \max\{x+2y, -2x-3y\}$
$\qquad |x|+3y \geq 1$

Solution and comments:

Recall that a problem is convex if the objective function is convex and is minimized and if the constraint functions are convex and the constraints are "$\leq$". Only then the functions are linear, the problem can be convex even if the objective function is maximized and the constraints are "$\geq$" or "$=$". We also learned and repeated many times that an absolute value function is convex and $\max$ function of several linear functions is convex. Neither of them are linear though. Hence

- Problem (1) is not convex because the constraint $x^2+3y \geq 1$ is not convex.
- Problem (2) is not convex because the convex objective functions is maximized, not minimized.
- Problem (3) is convex since the objective function is convex and is minimized and the constraint is convex and is "$\leq$".
- Problem (4) is not convex because the constraint $|x|+3y \geq 1$ is not convex.

Rewrite the convex problem from the above as a linear problem.

$$
\begin{array}{rrrl}
\min_{t,x,y} & t & & \\
\text{s.t.} & x & +2y & \leq t \\
 & -2x & -3y & \leq t \\
 & x & +3y & \leq 1 \\
 & -x & +3y & \leq 1
\end{array}
$$

Comments: This problem was a pretty basic repetition of what we did repeatedly in the course. Hence if not done correctly, a lot of points were taken off. Especially, when the reformulation contained binary variables - reformulating a convex problem should never include binary variables.

2 Duality (10 pts)

Write the dual problem of this LP problem.

If you are unable to solve the reformulation problem (and only in that case), write the dual of the following LP (which is not a correct answer to the reformulation problem, but is similar). Hint: pay attention to the four variables, x, y, t, and their position in each constraint:

$$\begin{array}{rrrl} \min_{t,x,y} & t & & \\ \text{s.t.} & x & +y & \leq t \\ & y & -x & \leq t \\ & 5x & +3y & \leq 2 \\ & -3x & +y & \leq 2 \end{array}$$

Solution and comments:

Many people did this correctly, but despite the hints some did not treat t as a primary variable but instead as a right hand side and this is a fundamental mistake. Additionally mistakes were made in handling the connections between the signs of the constraints and variables. Again, these are basic mistakes and cost a lot of points, as they represent the lack of grasp of the fundamentals of the course.

$$\begin{array}{rrrrrl} \max_u & & & u_3 & +u_4 & \\ \text{s.t.} & u_1 & -2u_2 & +u_3 & -u_4 & = 0 \\ & 2u_1 & -3u_2 & +3u_3 & +3u_4 & = 0 \\ & -u_1 & -u_2 & & & = 1 \\ & u_1 & u_2 & u_3 & u_4 & \leq 0 \end{array}$$

3 Integer Programming and Stochastic Programming (25 pts.)

Sophia is the owner of a small gym and is considering buying three new peaces of equipment: a treadmill \$1000, an elliptical trainer, \$800 and a stationary bike, \$600. But her budget is \$2000. Each peace of equipment can serve T, E and B customers per day, respectively.

- Assume that she has P customers that come daily and knows that each of her customers has the following preferences for the three peaces of equipment p_{it}, p_{ie} and p_{ib} respectively, for $i = 1, \ldots, P$ - being the index of a customer. Formulate the problem which she needs to solve to maximize overall happiness of her customers (if a customer does not get to use any of the new equipment, than their happiness is 0).

  Solution and comments:

  This is essentially the same as the facility location problem studied in class extensively. Let x_t, x_e and x_b be the binary variables that represent if the corresponding pieces of equipment are purchased (some people assumed more than one piece can be purchased and that is OK, then the variables are integer, not binary). Let y_{it}, y_{ie} and y_{ib} be the binary variables that indicate if a given customer i is using the corresponding peace of equipment. Then the formulation is

$$\begin{array}{ll} \max_{x,y} & \sum_{i=1}^{P}(p_{it}y_{it} + p_{ie}y_{ie} + p_{ib}y_{ib}) \\ \text{s.t.} & \sum_{i=1}^{P} y_{it} \leq Tx_t \\ & \sum_{i=1}^{P} y_{it} \leq Ex_e \\ & \sum_{i=1}^{P} y_{it} \leq Bx_b \\ & x_e + 0.8x_b + 0.6x_t \leq 2 \end{array}$$

One can use constraints $y_{ij} \leq x_j \; \forall i = 1, \dots P, j = t, e, b$, which give us a tighter formulation as we discussed in class. But both were considered correct. Some people added a constraint $\sum_j y_{ij} = 1$ for all i - this ensures that all customers get services by the new equipment. This constraint can make the problem infeasible, but I did not consider it incorrect. The biggest mistakes were - not linking x and y variables by any constraints, using nonlinear expressions such as $x * y$ in the objective function and treating p_{ij} are variables, while they are clearly part of the data.

- Now assume that she does not really know the customer preferences, but she used some predictive modeling and thinks that with probability $1/5$ her customers will have preferences p^1_{it}, p^1_{ie} and p^1_{ib} for $i = 1, \dots, P$, with probability $1/3$ they will have preferences p^2_{it}, p^2_{ie} and p^2_{ib} and with remaining probability the preference will be $p^3_{it}, p^3_{i.e.}$ and p^3_{ib}. Formulate the stochastic optimization problem that will optimize the expected satisfaction of the customers. Make sure to identify first and second stage variables.

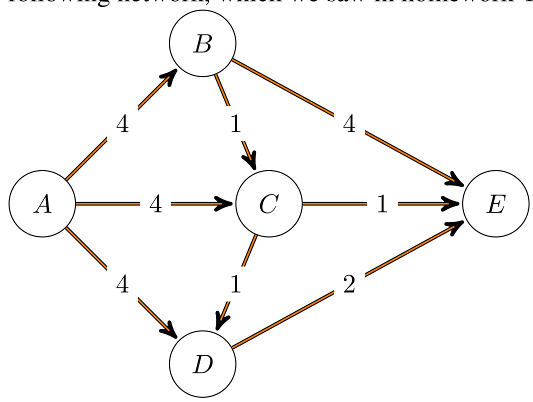
  Here the first stage variables are x_t, x_e and x_b since the purchase of equipment needs to be done before the customer preferences are observed, while y_{it}, y_{ie} and y_{ib} are the second stage variables which can change depending on the actual customer preferences, as they come in to the gym. Hence now we have variables y^k_{ij} for $i = 1, \dots P, j = t, e, b$ and $k = 1, \dots 3$.

$$
\begin{array}{ll}
\max_{x,y} & \frac{1}{5}\sum_{i=1}^P (p^1_{it}y^1_{it} + p^1_{ie}y^1_{ie} + p^1_{ib}y^1_{ib}) + \\
& \frac{1}{3}\sum_{i=1}^P (p^2_{it}y^2_{it} + p^2_{ie}y^2_{ie} + p^2_{ib}y^2_{ib}) + \\
& \frac{7}{15}\sum_{i=1}^P (p^3_{it}y^3_{it} + p^3_{ie}y^3_{ie} + p^3_{ib}y^3_{ib}) \\
\text{s.t.} & \sum_{i=1}^P y^k_{it} \leq Tx_t, \; k = 1, 2, 3 \\
& \sum_{i=1}^P y^k_{it} \leq Ex_e \; k = 1, 2, 3 \\
& \sum_{i=1}^P y^k_{it} \leq Bx_b \; k = 1, 2, 3 \\
& x_e + 0.8x_b + 0.6x_t \leq 2
\end{array}
$$

  If you did not introduce new variables y^k_{ij}, then around 8 points were taken off on this problem.

4 Flow problem and branch and bound (25 pts.)

Recall the infeasible flow problem of sending 9 units of flow from node A to node E in the following network, which we saw in homework 4 but with different capacities.



1. Formulate a problem that keeps all flow conservation constraints feasible (and sends 9 units of flow from A to E), but relaxes the capacity constraints in the following way: each arc can have its capacity doubled or not increased at all. Minimize the **number** of arcs for which the capacity is doubled.

   I am happy to say that most people did this correctly. y_{ij}, $(i, j) \in \mathcal{E}$ are binary variables representing the decision if arcs capacity is doubled. Let C_{ij} be the edge capacities.

$$\begin{array}{ll} \min_{x,y} & \sum_{(i,j)\in\mathcal{E}} y_{ij} \\ \text{s.t.} & \sum_{i\in V:(i,j)\in\mathcal{E}} x_{ij} - \sum_{i\in V:(j,i)\in\mathcal{E}} x_{ji} = 0, \ \forall j \in V, j \neq A, E \\ & \sum_{i\in V:(i,j)\in\mathcal{E}} x_{ij} - \sum_{i\in V:(j,i)\in\mathcal{E}} x_{ji} = 9, \ j = E \\ & x_{ij} \leq C_{ij} + C_{ij} y_{ij}, \ \forall (j,i) \in \mathcal{E} \end{array}$$

2. Formulate the problem that keeps all capacity constraints feasible, but relaxes the flow conservation at the intermediate nodes. Minimize the maximum violation of all of the flow conservation constraints.

$$\begin{array}{ll} \min_{x,y,z} & z \\ \text{s.t.} & \sum_{i\in V:(i,j)\in\mathcal{E}} x_{ij} - \sum_{i\in V:(j,i)\in\mathcal{E}} x_{ji} = y_j^+ - y_j^-, \ \forall j \in V, j \neq A, E \\ & \sum_{i\in V:(i,j)\in\mathcal{E}} x_{ij} - \sum_{i\in V:(j,i)\in\mathcal{E}} x_{ji} = 9, \ j = E \\ & x_{ij} \leq C_{ij} \ \forall (j,i) \in \mathcal{E} \\ & z \geq y_j^+, \ z \geq y_j^- \ \forall j \in V, j \neq A, E \end{array}$$

   Here, unfortunately there were quite a few people who used the objective function $\sum_j (y_j^+ + y_j^-)$, however, as you were asked to minimize the maximum violation of the flow constraints - this is not correct, this would minimize the total violation.

5 Integer Programming(15 pts)

The Congress of Neverland consists of 300 representatives. Each of them insulted exactly one other representative, so for each representative $i \in \{1, \dots, 300\}$ there is a representative $j_i \in \{1, \dots, 300\}$, $i \neq j_i$ who has been insulted by i. You are asked to prove that it is possible to select an Ethics Committee consisting of 100 representatives none of whom insulted anyone else on the Committee. Formulate this problem as a feasible set of a binary optimization problem - meaning that if you find a feasible solution to your formation, you will have the possible choice for the Ethics Committee (do not actually try to find such a solutions, although it exists).

Solution and comments:

This is an ”stable set” or ”node packing” problem which we covered in class. It has a very simple formulation. You had a similar problem as the max clique problem in the homework. There was a lot of misunderstanding with this problem. If you read carefully it is clear that whether one representative insults another is NOT a variable, is it already given. That is why for every i there is j_i which is a known index, which is the index of a representative insulted by i. Many people included binary variables for this data. This is wrong, however, if otherwise the problem was formulated sensibly, I deducted very few points in this case. A lot of formulations were still quite wrong loosing more points. The correct formation is as follows. Let x_i, be the binary variable that indicates if representative i is chosen to the ethics committee. Then if the following system is feasible

$$\begin{array}{c} x_i + x_{j_i} \leq 1 \\ \sum x_j \geq 100, \end{array}$$

then there exists an ethics committee of 100 representative that did not insult each other.

I think you can agree that this was a very easy formulation and is not very different from what was used in class and in a homework. You just needed to think clearly what was given.

6 Integer Programming (25 pts)

Four people are trying to cross a dark bridge. They only have one flash light and the bridge can only fit at most two people at a time. Each person walks with their own speed, s_i, $i = 1, \ldots, 4$. Formulate the problem of getting everyone across the bridge in the shortest possible time. Specifically, if persons 1 and 2 cross the bridge together with the flashlight, then they get to the other side in time $\max\{s_1, s_2\}$. Then one of them, say person 1 goes back with the flashlight taking s_1 to get back. Then two more persons can cross the bridge and so on. Clearly you will need 5 crossings in total. Consider the following variables:

- x_{ij} is 1 if person i is on the starting point of the bridge at the beginning of crossing j, and 0 otherwise (that is if they are on the ending end of the bridge).
- y_{ij} is 1 if person i is crossing the bridge during crossing j.
- Keep in mind that the person can only cross the bridge at a given crossing if they are on the appropriate end of the bridge.
- At most two people can cross the bridge at a time.

Solution and comments: This problem was indeed a challenge since it was not exactly like anything we did before, however, it used all the same logical constraints that we used in class (and quite a few of you used in your projects by now).

$$\begin{array}{ll}
\min_{x,y} & \sum_{j=1}^{5} \max_{i=1,\ldots,4}\{s_i y_{ij}\} \\
\text{s.t.} & y_{ij} \leq x_{ij} \text{ for } j = 1, 3, 5,\ \forall i = 1, \ldots, 4 \\
& y_{ij} \leq (1 - x_{ij}) \text{ for } j = 2, 4,\ \forall i = 1, \ldots, 4 \\
& x_{i,j+1} = x_{ij} - y_{ij},\ \text{ for } j = 1, 3, 5\ \forall i = 1, \ldots, 4 \\
& x_{i,j+1} = x_{ij} + y_{ij},\ \text{ for } j = 2, 4,\ \forall i = 1, \ldots, 4 \\
& x_{i,1} = 1\ \forall i = 1, \ldots, 4 \\
& x_{i,6} = 0\ \forall i = 1, \ldots, 4 \\
& \sum_{i=1}^{4} y_{ij} \leq 2\ \forall j = 1, \ldots, 5
\end{array}$$

To reformulate this using linear objective introduce a variables t_j and have

$$
\begin{array}{ll}
\min_{x,y,t} & \sum_{j=1}^{5} t_j \\
\text{s.t.} & t_j \geq s_i y_{ij},\ \forall i = 1, \ldots, 4 \\
& y_{ij} \leq x_{ij} \text{ for } j = 1, 3, 5,\ \forall i = 1, \ldots, 4 \\
& y_{ij} \leq 1 - x_{ij} \text{ for } j = 2, 4,\ \forall i = 1, \ldots, 4 \\
& x_{i,j+1} = x_{ij} - y_{ij},\ \text{for } j = 1, 3, 5\ \forall i = 1, \ldots, 4 \\
& x_{i,j+1} = x_{ij} + y_{ij},\ \text{for } j = 2, 4,\ \forall i = 1, \ldots, 4 \\
& x_{i,1} = 1\ \forall i = 1, \ldots, 4 \\
& x_{i,6} = 0\ \forall i = 1, \ldots, 4 \\
& \sum_{i=1}^{4} y_{ij} \leq 2\ \forall j = 1, \ldots, 5
\end{array}
$$

There are other possible formulation but the one above is the easiest to explain, in my opinion. We formulate constraints depending on the type of crossing - back or forth. The constraints $y_{ij} \leq x_{ij}$ and $y_{ij} \leq 1 - x_{ij}$ ensure that at each crossing only those can cross who are on the correct side of the bridge. During crossings 1,3 and 5 only people for who $x_{ij} = 1$ can cross and during crossings 2 and 4, only those with $x_{ij} = 0$ can cross. The constraints $x_{i,j+1} = x_{ij} - y_{ij}$ and $x_{i,j+1} = x_{ij} + y_{ij}$ make sure that x_{ij} represent the correct side of the bridge - if a person crosses the bridge their x variable has to change accordingly. Finally, everyone has to be on the starting side at the beginning of the first crossing and at the end side at the end of the 5th crossing, which we count as the beginning of the 6th, dummy crossing. The last constraint restrict number of people crossing at once to 2. We can restrict this number of 1 on crossing 2 and 4, but this is not necessary, as it will never lead to an optimal solution.

Some people got the correct formulation, quite a few got some of the constraints right but not all of them, so overall there is a good number of people that rose to the challenge!