ISE426 Optimization models and applications, Fall 2014
Homework 4 Solution,

1. Integer Programming/Scheduling Formulation (5 pts.)

Define binary variable x_{ij} as following,

$$x_{ij} = \begin{cases} 1 & \text{if job } j \text{ is processed on processor } i \\ 0 & \text{else} \end{cases}$$

ILP formulation of this problem is as following,

$$\begin{aligned} &\min \ C \\ &s.t. \sum_{j=1}^{m} x_{ij} t_j \leq C, && \forall i = \{1,..,n\}, \\ &\sum_{i=1}^{n} x_{ij} = 1, \\ &x_{ij} \in \{0,1\}, \quad \forall i \in \{1,..,n\}, \ \forall j \in \{1,..,m\}, \end{aligned}$$

□ *Interpretation:* The set of first constraint implies $C = \max_{i=1,..,n}\{\sum_{j=1}^{m} x_{ij} t_j\}$, so C is the maximum completion time of all processors which are processing different number of jobs, so in terms of minimizing the time of completion of all given tasks, the objective function has to minimize C .

Clearly the second constraint is enforcing each job to be assigned to just "one" processor.

2. Integer Programming/Matching Formulation(5 pts.)

1. Define binary variable x_{ij} as following,

$$x_{ij} = \begin{cases} 1 & \text{if socks } i \text{ is worn with socks } j \text{ in same day} \\ 0 & \text{else} \end{cases}$$

ILP formulation of this problem is as following,

$$\begin{aligned} \min \ & \sum_{i=1}^{20} \sum_{j=1}^{20} e_{ij} x_{ij}, \\ & \sum_{j \neq i}^{20} x_{ij} = 1, && \forall i \in \{1,2,..,20\}, \\ & x_{ij} = x_{ji}, && \forall \{i,j\} \in \{1,2,..,20\}, \\ & x_{ij} \in \{0,1\}, && \forall \{i,j\} \in \{1,2,..,20\}, \end{aligned}$$

The set of first constraint implies that socks i can be worn just one day with another socks.

2. In this part we need to define another two binary variables y_{ij} and z_{ij} as following,

$$y_{ij} = \begin{cases} 1 & \text{if a short is worn with socks } i \text{ and } j \text{ in same day} \\ 0 & \text{else} \end{cases}$$

$$z_{ij} = \begin{cases} 1 & \text{if a long pant is worn with socks } i \text{ and } j \text{ in same day} \\ 0 & \text{else} \end{cases}$$

ILP formulation of this problem is as following,

$$\begin{aligned}
\min \quad & \sum_{i=1}^{20}\sum_{j=1}^{20}(s_{ij}y_{ij} + l_{ij}z_{ij}), \\
& \sum_{j\neq i}^{20} x_{ij} = 1, \qquad (1) \qquad\qquad \forall i \in \{1,2,..,20\}, \\
& \sum_{i=1}^{20}\sum_{j=i+1}^{20} y_{ij} = 5, \qquad (2) \\
& \sum_{i=1}^{20}\sum_{j=i+1}^{20} z_{ij} = 5, \qquad (3) \\
& y_{ij} + z_{ij} = x_{ij}, \quad (4) \qquad\qquad \forall\{i,j\} \in \{1,2,..,20\}, \\
& x_{ij} = x_{ji},\ y_{ij} = y_{ji},\ z_{ij} = z_{ji} \quad \forall\{i,j\} \in \{1,2,..,20\}, \\
& x_{ij} \in \{0,1\}, \qquad\qquad\qquad \forall\{i,j\} \in \{1,2,..,20\},
\end{aligned}$$

The set of first constraints (2) and (3) imply this request that Bob wears shorts on five days, and long pants on five other days.

The set of constraint (4) implies that a short or a long pant can be worn with socks i and j if socks i and j are worn in the same day.

3. Graph optimization (5 pts.)

(a) Consider this social network as a graph $G = (V, E)$ where V, the node set, denotes the set of people and E, the edge set, indicates whether or not there is a connection. In other words, the edge (i, j) belongs to E if and only if the people i and j are mutual friends. In the words of graph theory, these two nodes are referred to as adjacent. Now, suppose that the graph nodes are labeled with $1, 2, ..., n$ where n stands for the number of people. Then, the following definition is in order.

$$x_i = \begin{cases} 1 & \text{if i is chosen as a member of group} \\ 0 & \text{Otherwise} \end{cases}$$

Consequently, a group S of mutual friends (adjacent nodes) consists of a collection of people (nodes) $i \in S$ where

$$\begin{aligned}
& x_i + x_j \leq 1 \quad \forall (i,j) \notin E, \\
& x_i \in \{0,1\} \quad \forall i \in 1, ..., n,
\end{aligned}$$

and S is a subset of $\{1, 2, ..., n\}$. In fact, the constraint $x_i + x_j \leq 1$ implies that either of i or j or both has to be excluded from the group if they are not connected. Now, the problem of finding the largest group is formulated as

$$
\begin{aligned}
\max \ & \sum_{i=1}^{n} x_i \\
\text{s.t.} \ & \\
& x_i + x_j \le 1 \quad \forall (i,j) \notin E \\
& x_i \in \{0,1\} \quad \forall i = 1,...,n
\end{aligned}
$$

(b) Assume that x^* is an n-dimensional incidence vector which denotes the optimal solution to the problem we just formulated in part (a). As we deal with a binary integer program, we only need to append the model with

$$\sum_{i \in B} x_i \le |B| - 1$$

to find the next maximum social group, where $B = \{i \,|x_i^* = 1,\ i = 1,...,n\}$ denotes the set of members in the current optimal solution and the operator $|.|$ stands for the size of this group. In a way, this constraint implies that we are looking for another social group which excludes at least one member of the current social group. Hence, we have

$$
\begin{aligned}
\max \ & \sum_{i=1}^{n} x_i \\
\text{s.t.} \ & \\
& x_i + x_j \le 1 \quad \forall (i,j) \notin E \\
& \sum_{i \in B} x_i \le |B| - 1 \\
& x_i \in \{0,1\} \quad \forall i = 1,...,n
\end{aligned}
$$

4. Branch and Bound (5 pts.)

At the root node, we have to solve the linear program

$$
\begin{aligned}
& \max \ 6x_1 + 5x_2 + 4x_3 + 4x_4 + 3x_5 + 3x_6 \\
& \text{s.t.} \ 7x_1 + 5x_2 + 5x_3 + 4x_4 + 4x_5 + 3x_6 \le 23 \\
& 0 \le x_i \le 1 \quad i = 1,...,6
\end{aligned}
$$

Notice that the coefficients of the variables in the objective function are in a decreasing order. Then, using the greedy algorithm which you learnt for the knapsack problem, we get the optimum solution $(1, 1, 0.8, 1, 0, 1)$ with the objective value 21.2. Thus, 21.2 serves as an upper bound for the optimal value of the integer program; that is $Z_u = 21.2$. Now, since x_3 is fractional, we choose to branch on x_3 to introduce two new subproblems (nodes) to the branch and bound tree.

Suppose that using a heuristic approach for node selection, we decide to visit the node with $x_3 = 1$ which contains the subproblem

$$
\begin{aligned}
& \max \ 6x_1 + 5x_2 + 4x_4 + 3x_5 + 3x_6 + 4 \\
& \text{s.t.} \ 7x_1 + 5x_2 + 4x_4 + 4x_5 + 3x_6 \le 18 \\
& 0 \le x_i \le 1 \quad i = 1,...,6
\end{aligned}
$$

Relying on the same greedy approach as in above, we obtain the optimum solution $(\frac{6}{7}, 1, 1, 1, 0, 1)$ with the objective value 21.14. This value does not affect the upper bound nor the lower bound. As x_1 is fractional, we branch on x_1 to create two new subproblems one of which is imposed by $x_3 = 1$ and $x_1 = 1$ and the other by $x_3 = 1$ and $x_1 = 0$.

At this point, we have three open nodes. Suppose that we choose to visit the node with $x_3 = 0$ which contains the subproblem

$$\begin{aligned}
&\max\ 6x_1 + 5x_2 + 4x_4 + 3x_5 + 3x_6 \\
&\text{s.t. } 7x_1 + 5x_2 + 4x_4 + 4x_5 + 3x_6 \leq 23 \\
&0 \leq x_i \leq 1 \quad i = 1, ..., 6
\end{aligned}$$

The greedy approach gives the optimum solution $(1, 1, 0, 1, 1, 1)$ with the objective value 21. Since this is an integer solution, the lower bound is updated to $Z_l = 21$. Recall that $Z_u = 21.2$. Then, since the objective coefficients are integer and there is no integer value between 21 and 21.2, we can close the two remaining nodes and conclude that the solution $(1, 1, 0, 1, 1, 1)$ is indeed optimal.