IE426 – Optimization models and applications

Fall 2012 – Quiz #2, November 15, 2012

First name	
Last name	
Lehigh email	

You have 75 minutes. This quiz accounts for 10% of the final grade. There are 40 points available. Please write clear and concise statements. Use a **readable** handwriting: it will be hard to grade your answers if they are not readable — let alone give them full points. Use the back of each sheet if you need more space, for example to draw graphs. Do not use calculators. For each model, clearly specify the meaning of each variable and of each constraint.

1 Reformulate as a Linear Programming Problem (8pts.)

Consider the following problem, where the vectors $c^1, \ldots, c^k, b$ and the matrix A are given:

$$\begin{array}{rc} \min & \sum_{i=1}^k |(c^i)^T x| \\ st. & Ax \le b \end{array}$$

The objective function is convex but not linear, since it is the sum of absolute values. Reformulate this problem as a linear programming problem using techniques studied in class introducing appropriate extra variables.

$$\begin{array}{rc} \min & \sum_{i=1}^k y_i \\ & y_i \ge (c^i)^T x, \\ st. & y_i \ge -(c^i)^T x, \\ & Ax \le b \end{array}$$

2 Mixed Integer/Goal Programming (14 pts.)

The following LP is infeasible

$$\begin{array}{rrrl} \min & 3x_1 & +5x_2 & \\ & -x_1 & +2x_2 & \le -2 \\ & -2x_1 & +x_2 & \ge 1 \\ & x_1 & -x_2 & \ge -4 \\ & x_1 & & \in [0,5] \\ & & x_2 & \in [0,5] \end{array}$$

1. Write a nonpreemptive goal programming formulation to minimize the constraint violation. Do not solve! (6pts)

$$\begin{array}{llll}
\min & y_1 & +y_2 & +y_3 \\
& -x_1 & +2x_2 & \le -2+y_1 \\
& -2x_1 & +x_2 & \ge 1-y_2 \\
& x_1 & -x_2 & \ge -4-y_3 \\
& x_1 & & \in [0,5] \\
& & x_2 & \in [0,5] \\
& & y_i & \ge 0,\ i=1,2,3.
\end{array}$$

2. In nonpreemptive goal programming we minimize the total sum of the constraint violation. You can think of it as minimizing the cost of violating constraints if each time you violate a constraint by one unit you pay one unit of cost. Now imagine that each time you violate one of the constraints you pay a fixed cost of 20 on top of the per-unit cost of 1. Notice that for the fixed cost we do not care by how much these constraints are violated, as long as they are. Formulate the problem of minimizing the total cost of violating the constraints. (6 pts).

$$\begin{array}{llll}
\min & y_1 & +y_2 & +y_3+20(z_1+z_2+z_3) \\
& -x_1 & +2x_2 & \le -2+y_1 \\
& -2x_1 & +x_2 & \ge 1-y_2 \\
& x_1 & -x_2 & \ge -4-y_3 \\
& x_1 & & \in [0,5] \\
& & x_2 & \in [0,5] \\
& & y_i & \le M_i z_i,\ i=1,2,3, \\
& & y_i & \ge 0,\ i=1,2,3.
\end{array}$$

3. Write the good values for the "big" M constants (2pts).

   Here you need to compute what is the largest value each y_i can take, given that $x_i \in [0,5]$

$$\begin{array}{rcr}
M_1 & = & 12, \\
M_2 & = & 11, \\
M_3 & = & 1.
\end{array}$$

3 Integer Programming and Logic (12pts)

Formulate the following sets of constraints via mixed integer programming

1. $\max\{(a_1x_1+a_2x_2+\ldots,a_nx_n),(b_1x_1+b_2x_2+\ldots,b_nx_n)\} \ge 0$.

$$\begin{array}{ll}
a^\top x - b^\top x & \le My \\
a^\top x & \ge -M(1-y) \\
b^\top x & \ge -My \\
y & \in \{0,1\}.
\end{array}$$

2. If $a_1x_1+a_2x_2+\ldots,a_nx_n \ge 0$ then $y=1$, for binary y.

$$\begin{array}{ll}
a^\top x & \le -\epsilon + My \\
y & \in \{0,1\}.
\end{array}$$

3. $\sum_{i=1}^n x_i = 1,\ 0 \le x \le 1$, at most k elements of x are allowed to be positive for some given $k < n$.

$$\begin{array}{ll}
\sum_{i=1}^n x_i & = 1, \\
x_i & \le y_i, \\
\sum y_i \le k, & \\
x_i & \in [0,1],\ \forall i., \\
y_i & \in \{0,1\},\ \forall i.
\end{array}$$

4 Branch and Bound (6 pts)

Briefly but accurately give answers to the following questions.

1. The process of a Branch and Bound algorithm is represented by a tree. What are the nodes of that tree? What does each node represent and how nodes that are connected by an edge are related to each other?

   Each node in the tree represent a linear programming problem, which is a relaxation of an MIP, which is a subproblem of the original problem. It provides a lower bound for all solutions in that subproblem. Two nodes are connected by an edge when one problem is a subproblem of the other obtained by putting a restriction on one of the variables.

2. What is the largest number of nodes and a smallest number of nodes a B&B tree can have for a problem with n binary variables.

   Smallest is 1, largest is 2^n.

3. What should happen for the tree to have the smallest number of nodes?

   The LP relaxation of the original problem should have an integer solution, or alternatively if we have a feasible integer solution whose objective value is the same as the lower bound obtained by optimizing the LP relaxation of the main problem.