Details $\Rightarrow$ Non convexity of first constraint: it's a nonlinear equality which causes the non-convex problem.

$\Rightarrow$ Non Convexity of fesible region: pick two arbitrary points, such $P_1 = (0, 1)$ and $P_2 (0, -1)$ which belong to fesible region, derive the convex combination of these two points for $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$

$\lambda_1 P_1 + \lambda_2 P_2 = (0, 0) \notin$ fesible region.

ISE426 – Optimization models and applications

Fall 2014 – Quiz #1, October 14, 2014

First name	
Last name	
Lehigh email	

You have 75 minutes. There are three problems. This quiz accounts for 10% of the final grade. There are 40 points available. Please write clear and concise statements. Use a **readable** handwriting: it will be hard to grade your answers if they are not readable — let alone give them full points. Use the back of each sheet if you need more space, for example to draw graphs. Do not use calculators.

1 Convexity and relaxations (8 pts.)

The following problem is not convex, explain why (4 pts.):

$$\begin{aligned} \min \quad & x \\ & |y| = 1 \\ & x^2 + y^2 \leq 5 \end{aligned}$$

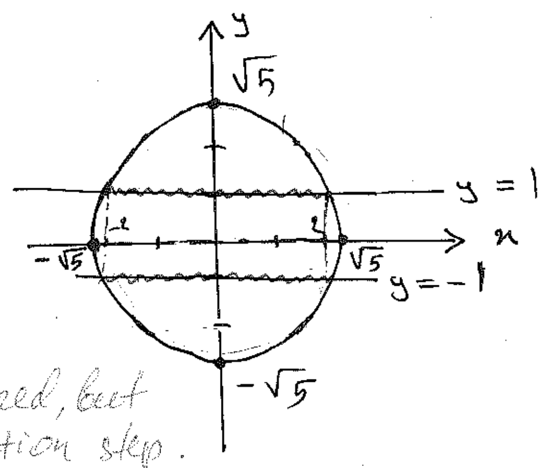
Solution:

Since objective function is a linear function it is convex, so we have to check the convexity of feasible region, first constraint is not convex for sure, but we have to check the intersection of both constraints. (It is possible that the intersection be convex, for example if we had $|y| = 1$ and $y \geq 0$ as our feasible region)

$\Rightarrow$ the feasible region of our problem is two parallel line segments $y = 1$ and $y = -1$,

for $x \in [-2, 2]$

which is obviously Non-convex,



This was not required, but is an extra explanation step.

2 Linear Programming Model (16 pts.)

Consider the following LP problem, which is a slight modification of the problem from your homework #2:

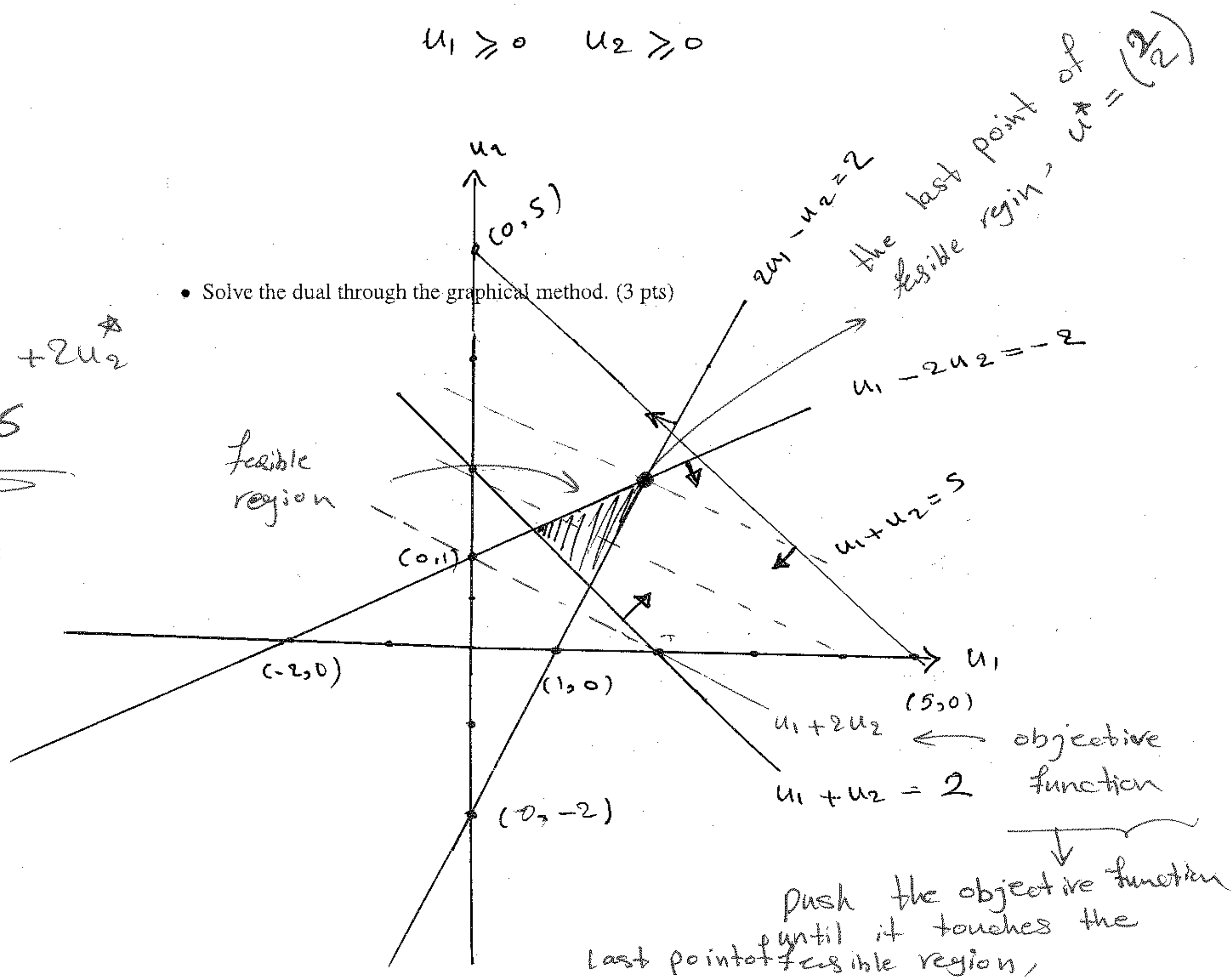
$$
\begin{array}{llllll}
\min & 2x_1 & +5x_2 & +2x_3 & -2x_4 & \\
\text{s.t.} & x_1 & +x_2 & +2x_3 & +x_4 & \geq 1 \longrightarrow u_1 \\
 & x_1 & +x_2 & -x_3 & -2x_4 & \geq 2 \longrightarrow u_2 \\
 & x_2, x_3 & \geq 0 & x_1, x_4 & \leq 0. &
\end{array}
$$

- Write the dual. (5 pts)

$$
\begin{array}{ll}
\max & u_1 + 2u_2 \\
\text{s.t.} & \\
 & u_1 + u_2 \geq 2 \longrightarrow x_1 \\
 & u_1 + u_2 \leq 5 \longrightarrow x_2 \\
 & 2u_1 - u_2 \leq 2 \longrightarrow x_3 \\
 & u_1 - 2u_2 \geq -2 \longrightarrow x_4 \\
 & u_1 \geq 0 \quad u_2 \geq 0
\end{array}
$$

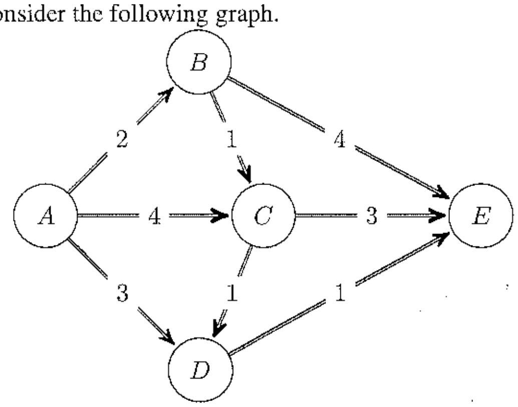
- Solve the dual through the graphical method. (3 pts)

$$z^* = u_1^* + 2u_2^* = 6$$



3 Flow problem (16 pts.)

Consider the following graph.



1. Formulate the shortest path problem for going from A to E as a linear programming problem, using the formulations studied in this course. Note that now the arcs are directed, hence you can only travel one way on each arc. (6 pts)

Primal Problem:

$$\min \; 2x_{AB} + 4x_{AC} + 3x_{AD} + 4x_{BE} + x_{BC} + 3x_{CE} + x_{CD} + x_{DE}$$

s.t.

$$x_{AB} + x_{AC} + x_{AD} = 1 \qquad u_A$$

$$-x_{AB} + x_{BE} + x_{BC} = 0 \qquad u_B$$

$$-x_{BC} - x_{AC} + x_{CE} + x_{CD} = 0 \qquad u_C$$

$$-x_{AD} - x_{CD} + x_{DE} = 0 \qquad u_D$$

$$-x_{BE} - x_{CE} - x_{DE} = -1 \qquad u_E$$

$$x_{ij} \geq 0$$

3. Find the shortest path for the problem, by simple observation. This gives you the primal optimal solution for your LP. Write it down. Using it write down complementarity conditions of the optimal dual solution. Observe that the dual optimal solution is not defined uniquely. But these complementarity conditions uniquely define the optimal value of the objective function of the dual. Demonstrate this. (2 pts)

$\Rightarrow$ optimal shortest path: $A \to D \to E$: $x^*_{AD} = x^*_{DE} = 1$

and $x^*_{ij} = 0$ if $(i,j) \neq (A,D)$ and $(i,j) \neq (D,E)$

complementary slackness in terms of primal variables x_{AD} and x_{DE}

$x^*_{AD}(u_A - u_D - 3) = 0 \to$ ① $u_A - u_D = 3$ $\xRightarrow{\text{by Summing}}$ $u_A - u_E = 4$

$x^*_{DE}(u_D - u_E - 1) = 0 \to$ ② $u_D - u_E = 1$

Since we have $x^*_{ij} = 0$ for other primal variables, we can not use other equations in complementary slackness corresponding to these primal variables, so we can use just equations ① and ②

So we have 2 Constraints and 5 variables which allows us to have multiple optimal solutions for Dual problem.

4. Consider the change in the length of the (B, E) edge. It is 4 in the original problem. How low can it get before the shortest path changes. Demonstrate this by showing that if the length of (B, E) gets any lower than this value, then the complementary dual solution will no longer be feasible (you can show this even if you do not compute the unique dual solution). (3 pts)

change the length of edge (B,E) to α, we have $u_B - u_E \leq \alpha$

equivalently we have, $u_B - u_A + u_A - u_E \leq \alpha$, by using

the optimal value of dual objective function $u_A - u_E = 4$

we will have $u_B - u_A \leq \alpha - 4 \Rightarrow \boxed{u_A - u_B \geq 4 - \alpha}$ *

on the other hand based the first equation in dual problem we have $\boxed{u_A - u_B \leq 2}$ **

By * and ** we have $\alpha \geq 2$, which means the cost of edge (B,E) can be at least 2.

3 Linear Programming Model (20 pts.)

Sophia goes out to lunch 5 days a week. She is trying to decide on what she will be eating during a given week. Her options are: lo mein, hamburger, fried chicken, soup, pizza and a Subway sandwich. She can eat each of these as many times as she chooses. She eventually needs to have 5 meals. Her budget is $20 and she is trying to maximize her satisfaction. In the table below you see the cost per each type of meal and their satisfaction coefficients.

Food	Lo Mein	Hamb.	Fr. Chic.	Soup	Pizza	Sub.
satisfaction	6	5	4	4	3	3
cost	7	5	5	4	4	3

(a) Formulate an integer optimization problem to choose the meals to maximize Sophia satisfaction (note that each meal can be chosen more than once as long as the total is 5 meals). Create a linear programming relaxation. If you do this correctly, $(0, 2.5, 0, 0, 0, 2.5)$ would be an optimal solution to the relaxation. (6pts)

max $6x_1 + 5x_2 + 4x_3 + 4x_4 + 3x_5 + 3x_6$

$7x_1 + 5x_2 + 5x_3 + 4x_4 + 4x_5 + 3x_6 \leq 20$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 5$

$x_i \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$

Relaxation : replace integrality with

$x_i \geq 0$

(b) Write down the dual of the linear programming relaxation defined in part (a). Using the optimal solution of the relaxation given in part (a), compute the dual optimal solution from complementary slackness conditions and show that this solution is feasible (hence the primal solution that is given is optimal). (8 pts)

$$\min \; 20u_1 + u_2$$

$$7u_1 + u_2 \geq 6$$

$$5u_1 + u_2 \geq 5$$

$$5u_1 + u_2 \geq 4$$

$$4u_1 + u_2 \geq 1$$

$$4u_1 + u_2 \geq 3$$

$$3u_1 + u_2 \geq 3$$

$$u_1, u_2 \geq 0$$

Primal solution : $(0, 2.5, 0, 0, 0, 2.5) \Rightarrow$

by complementarity

$$5u_1 + u_2 = 5$$
$$3u_1 + u_2 = 3$$
$$\Rightarrow u_1 = 1, \quad u_2 = 0$$

(c) Consider the satisfaction coefficient of lo mein. How much higher does it have to be to change to optimal solution of the relaxation? Derive your answer from the feasibility of the complementary dual solution. (2pts)

$7u_1 + u_2 \geq C_1$ with $u_1 = 1$, $u_2 = 0 \Rightarrow$

$\Rightarrow C_1 \leq 7$ solution does not change.

(d) Consider the satisfaction coefficient of the soup. Show that it cannot get any higher ~~lower~~ without changing the optimal solution. (2 pts)

$4u_1 + u_2 \geq C_4$, $u_1 = 1$, $u_2 = 0$

$\Rightarrow C_4 \leq 4$ does not change solution, but $C_4 > 4$ does.

(e) $(0, 2.5, 0, 0, 0, 2.5)$ is not a feasible solution to the original problem since it is not integer. Does it give you a bound on the optimal solution of the original problem? Find an optimal solution of the original problem and prove that it is optimal using that bound. (2pts)

The solution $(0, 2.5, 0, 0, 0, 2.5)$ gives an upper bound on the optimal value, this bound is 20

Solution $(0, 0, 0, 5, 0, 0)$ is feasible for the original problem and has objective value of $20 \Rightarrow$ it is optimal.

Conclusion: eat soup every day!