#3 Homework, Name: Bolun Xu, ISE 429, page 1.

1) 4-70: solution:

(a) $P(X_{n+1} = i+1 \mid X_n = i) = \left(\frac{m-i}{m}\right)^2$ $\quad P_{1,0}=1; P_{m,m-1}=1;$

$P(X_{n+1} = i-1 \mid X_n = i) = \left(\frac{i}{m}\right)^2$ $\quad P_{00}=0; P_{mm}=0;$

$P(X_{n+1} = i \mid X_n = i) = \frac{2i(m-i)}{m^2}$, $i = 1, 2, \dots, m-1$

(b) $\pi_i = \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} = \frac{\binom{m}{i}^2}{\binom{2m}{m}}$

(c) ① To verify $\sum_i \pi_i P_{ij} = \pi_j$. is to verify:

$\pi_{j-1} P_{j-1,j} + \pi_j \cdot P_{jj} + \pi_{j+1} \cdot P_{j+1,j} = \pi_j$.

$\binom{2m}{m} \times \text{Left side} = \binom{m}{j-1}^2 \cdot \left(\frac{m-j+1}{m}\right)^2 + \binom{m}{j+1}^2 \cdot \left(\frac{j+1}{m}\right)^2 + \binom{m}{j}^2 \cdot \frac{2j(m-j)}{m^2} =$

$= \binom{m}{j}^2 \left(\frac{j}{m}\right)^2 + \binom{m}{j}^2 \frac{2j(m-j)}{m^2} + \binom{m}{j}^2 \left(\frac{m-j}{m}\right)^2 =$

$= \binom{m}{j}^2 \left(\frac{m+j-j}{m}\right)^2 = \binom{m}{j}^2 = \binom{2m}{m} \times \pi_j$,

So $\pi_j = \frac{\binom{m}{j}^2}{\binom{2m}{m}}$ is true.

② To verify $\pi_i P_{ij} = \pi_j P_{ji}$ is to verify:

$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$. --- (*)

$\pi_i P_{ii} \equiv \pi_i P_{ii}$

$\pi_i P_{i,i-1} = \pi_{i-1} P_{i-1,i}$. --- ($\Delta$)

plug in formulas from (a), (*) & (Δ) are true, so M.C. is time-reversible.

2) 4-76: Solution:

The following chess board has entry $\sum_j w_{i,j}$ in square i (note that i represents a board location), for a knight:

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

for a corner:

$$\pi_j = \frac{2}{\sum_i \sum_j w_{i,j}} = \frac{2}{336}$$

; mean return time is $\frac{1}{\pi_j} = 168$.

3) 6-37: Solution:

(a) n_i type i patients in hospital, for all $i=1,\dots,k$, with state vector $(n_1, n_2, \dots, n_k)$.

(b) $M/M/\infty$, so its time reversible.

(c) $N_i(t), t \geq 0$ are independent process, so it is a time-reversible continuous-time M.C.

HOMEWORK #3, Name, Bolun Xu, ISE 429, page 3.

3) 6-37:

(d) $$p(n_1,\cdots,n_k)=\prod_{j=1}^{k} e^{-\frac{\lambda_j}{\mu_j}}\frac{\left(\frac{\lambda_j}{\mu_j}\right)^{n_j}}{n_j!}.$$

(e). $$p^A(n_1,\cdots,n_k)=K\prod_{j=1}^{k} e^{-\frac{\lambda_j}{\mu_j}}\frac{\left(\frac{\lambda_j}{\mu_j}\right)^{n_j}}{n_j!},\ \cancel{(n_1,\cdots,n_k)\in A}.$$

$$K=\left[\sum_{(n_1,\cdots,n_k)\in A}\prod_{j=1}^{k} e^{-\frac{\lambda_j}{\mu_j}}\frac{\left(\frac{\lambda_j}{\mu_j}\right)^{n_j}}{n_j!}\right]^{-1}$$

$$A=\left\{(n_1,\cdots,n_k):\sum_{j=1}^{k} n_j w_j\le C\right\}.$$

(f) $$r_i=\sum_{(n_1,\cdots,n_i+1,\cdots,n_k)\in A}\lambda_i\, p^A(n_1,\cdots,n_k)$$

(g). The fraction of patients are permitted are

$$\frac{\sum_{i=1}^{k} r_i}{\sum_{i=1}^{k}\lambda_i}.$$

6-43: solution:

With the state being $i=(i_1,i_2,i_3)$ when that there are i_j customers at server j for $j=1,2,3$, the instantaneous transition rates of M.C. are:

$q(i,j,k),(i+1,j,k)=\lambda$; $q(i,j,k),(i-1,j+1,k)=\mu_1$, $i>0$;

$q(i,j,k),(i,j-1,k+1)=\mu_2$, $j>0$; $q(i,j,k),(i,j,k-1)=\mu_3$, $k>0$;

the conjectured instantaneous rates for the reversed chain are:

$q^*(i,j,k),(i,j,k+1)=\lambda$; $q^*(i,j,k),(i,j+1,k-1)=\mu_3$, $k>0$;

$q^*(i,j,k),(i+1,j-1,k)=\mu_2$, $j>0$; $q^*(i,j,k),(i-1,j,k)=\mu_1$, $i>0$;

As is proved in class, M/M/1 QUEUE SYSTEMS ARE independent.

so, $P(i,j,k)=(1-\rho_1)\rho_1^i\cdot(1-\rho_2)\rho_2^j\cdot(1-\rho_3)\rho_3^k$, $\rho_i=\frac{\lambda}{\mu_i}$, $i=1,2,3$.

This is easily verified if pluged in balance EQS.

3 ~~(a)~~: Solution:

(a) $\{N_i(t)\}$ is a renewal process.

$N(t) = \sum N_i(t)$

$$\lim \frac{N(t)}{t} = \sum_i \lim \frac{N_i(t)}{t} = \sum_i \frac{1}{\mu_i+\theta_i},$$

μ_i and θ_i are the mean of distribution F_i and H_i.

(b) For each skier, whether they are climbing up or skiing down. Constitutes an ~~alt~~ alternating renewal process, and so the limiting probability that skier i is climbing up is

$p_i = \frac{\mu_i}{\mu_i+\theta_i}$. From this we obtain.

$$\lim_{t\to\infty} P\{U(t)=k\} = \sum_S \left\{ \prod_{i\in S} p_i \prod_{i\in S^c} (1-p_i) \right\},$$

where the above sum is over all of the $\left[\begin{matrix} n \\ k \end{matrix} \right]$ subsets S of size k.

$$\text{So, } \lim_{t\to\infty} E[U(t)] = \sum_{k=0}^{n} k \cdot \lim_{t\to\infty} P\{U(t)=k\} =$$

$$= \sum_{k=0}^{n} k \cdot \sum_S \left\{ \prod_{i\in S} p_i \prod_{i\in S^c} (1-p_i) \right\},$$

the defination of set S is mentioned ~~above~~ above

HOMEWORK #3, Name: Bolun Xu, ISE 429, Page 6.

7-39. solution:

Let B be the length of a busy period. With S equal to the service time of the machine whose failure initiated the busy period, and T equal to the remaining life of the other machine at that moment, we obtain.

$$E[B] = \int E[B \mid S = s]\, g(s)\, ds.$$

Now, $E[B \mid S=s] =$

$$= E[B \mid S=s, T \leq s](1-e^{-\lambda s}) + E[B \mid S=s, T > s]\, e^{-\lambda s}$$

$$= (s + E[B])(1-e^{-\lambda s}) + s e^{-\lambda s}$$

$$= s + E[B](1-e^{-\lambda s})$$

Substituting back gives that:

$$E[B] = E[S] + E[B]\, E[1-e^{-\lambda s}]$$

So, $E[\text{idle}] = \dfrac{1}{1 + 2\lambda \cdot E[B]}$