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$T > 0$ is the random *renewal time*; $\mu = ET > 0$; $\mu < \infty$.

The distribution of T is called a *lattice distribution* if for some constant $a > 0$,

$$P\{T = ak \ \text{ for some } \ k = 1, 2, \ldots\} = 1.$$

The distribution of T is called a *non-lattice distribution* if it is not lattice. The following condition is sufficient for the distribution of T to be non-lattice: $T = T_1 + T_2$, where T_1 has continuous distribution and T_1 and T_2 are independent.

Renewal process: $N(t) =$ the number of renewal points in $(0, t]$, assuming that 0 is a renewal point. (Note that point 0 does not count into $N(t)$.)

Renewal function: $m(t) = EN(t)$.

Definition of a *directly Riemann integrable* non-negative function $g(t), t \geq 0$, see e.g. in [FELLER, Vol. 2, page 362]. Direct integrability implies, in particular, that $\int_0^\infty g(t)dt < \infty$.
Either of the following conditions implies direct integrability:
(a) $g(t)$ is monotone non-increasing and $\int_0^\infty g(t)dt < \infty$;
(b) $g(t)$ is a bounded piece-wise continuous (with finite number of pieces) on finite interval $[0, b]$, and $g(t) = 0$ for $t > b$.
(In all your assignments you can always assume that, for any B, the function

$$h(t) = P\{X(t) \in B, \ t < T \mid X(0) = x_*\}$$

is directly integrable, so no need to verify this.)

Theorem 1 [Key Renewal Theorem]. *Suppose renewal time T has non-lattice distribution. Then, for any directly Riemann integrable function $g(t), \ t \geq 0$,*

$$\int_0^t g(s)d[-m(t-s)] \to \frac{1}{\mu} \int_0^\infty g(s)ds, \quad t \to \infty.$$

The following Elementary Renewal Theorem is a weaker property. (It follows from the Key Renewal Theorem.)

Theorem 2 [Elementary Renewal Theorem].

$$\frac{m(t)}{t} \to \frac{1}{\mu}, \quad t \to \infty.$$