ISE426 Optimization models and applications, Fall 2014
Homework 3 Solution,

1 Goal programming

2. The nonpreemtive goal programming model is given by

$$
\begin{aligned}
\min\ & y_1^+ + y_2^- + y_3^+ + y_3^- + y_4^+ + y_5^- + y_6^+ - y_6^+ + y_7^+ \\
\text{s.t.}\ & \\
& -2x_3 + 3x_4 \leq y_1^+, \\
& x_1 + x_4 \geq 1 - y_2^-, \\
& x_1 - 2x_3 + 2x_4 = y_3^+ - y_3^-, \\
& x_2 + x_4 \leq y_4^+, \\
& x_1 + x_2 - x_3 + x_4 \geq 2 - y_5^-, \\
& 3x_1 - x_2 - x_3 - x_4 = y_6^+ - y_6^-, \\
& x_1 - x_2 + 2x_3 \leq 2 + y_7^+, \\
& x_1, x_2, x_3, x_4, y_1^+, y_2^-, y_3^+, y_3^-, y_4^+, y_5^-, y_6^+, y_6^-, y_7^+ \geq 0.
\end{aligned}
$$

3. To get a feasible model, we need to eliminate a portion of the constraints from consideration. Thus, a subset with maximum cardinality is obtained by minimizing the number of constraints needed to be relaxed in order to get a feasible model. This problem is formulated as a mixed integer program as follows

$$
\begin{aligned}
\min\ & z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 \\
\text{s.t.}\ & \\
& -2x_3 + 3x_4 \leq Mz_1, \\
& x_1 + x_4 \geq 1 - Mz_2, \\
& x_1 - 2x_3 + 2x_4 \leq Mz_3, \\
& x_1 - 2x_3 + 2x_4 \geq -Mz_3, \\
& x_2 + x_4 \leq Mz_4, \\
& x_1 + x_2 - x_3 + x_4 \geq 2 - Mz_5, \\
& 3x_1 - x_2 - x_3 - x_4 \leq Mz_6, \\
& 3x_1 - x_2 - x_3 - x_4 \geq -Mz_6, \\
& x_1 - x_2 + 2x_3 \leq 2 + Mz_7, \\
& x_1, x_2, x_3, x_4, y_1^+, y_2^-, y_3^+, y_3^-, y_4^+, y_5^-, y_6^+, y_6^-, y_7^+ \geq 0.
\end{aligned}
$$

4. Based on the results obtained from the previous part, the feasible model is given by

$$
\begin{aligned}
\min \;\; & 2x_1 + x_2 - 11x_3 + 5x_4 \\
\text{s.t.} \;\; & \\
& -2x_3 + 3x_4 \leq 0, \\
& x_1 + x_4 \geq 1, \\
& x_1 - 2x_3 + 2x_4 = 0, \\
& x_1 + x_2 - x_3 + x_4 \geq 2, \\
& 3x_1 - x_2 - x_3 - x_4 = 0, \\
& x_1 - x_2 + 2x_3 \leq 2, \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

5. According to the largest set of feasible constraints, the formulation of the preemptive model would be

Step 1:

$$
\begin{aligned}
\min \;\; & y_1^+ + y_2^- + y_3^+ + y_3^- + y_4^+ + y_5^- + y_6^+ + y_6^- + y_7^+ \\
\text{s.t.} \;\; & \\
& -2x_3 + 3x_4 \leq y_1^+, \\
& x_1 + x_4 \geq 1 - y_2^-, \\
& x_1 - 2x_3 + 2x_4 = y_3^+ - y_3^-, \\
& x_2 + x_4 \leq y_4^+, \\
& x_1 + x_2 - x_3 + x_4 \geq 2 - y_5^-, \\
& 3x_1 - x_2 - x_3 - x_4 = y_6^+ - y_6^-, \\
& x_1 - x_2 + 2x_3 \leq 2 + y_7^+, \\
& x_1, x_2, x_3, x_4, y_1^+ + y_2^- + y_3^+ + y_3^- + y_4^+ + y_5^- + y_6^+ + y_6^- + y_7^+ \geq 0.
\end{aligned}
$$

The optimum solution to this problem is given by $y_1^+ = y_2^- = y_3^+ = y_3^- = +y_5^- = y_6^+ = y_6^- = y_7^+ = 0$.

Step 2:

$$
\begin{aligned}
\min \;\; & y_4^+ \\
\text{s.t.} \;\; & \\
& -2x_3 + 3x_4 \leq 0, \\
& x_1 + x_4 \geq 1, \\
& x_1 - 2x_3 + 2x_4 = 0, \\
& x_2 + x_4 \leq y_4^+, \\
& x_1 + x_2 - x_3 + x_4 \geq 2, \\
& 3x_1 - x_2 - x_3 - x_4 = 0, \\
& x_1 - x_2 + 2x_3 \leq 2, \\
& x_1, x_2, x_3, x_4, y_4^+ \geq 0.
\end{aligned}
$$

2 Logic

1. Let x_2 denote the second clause in $a \vee (b \wedge \neg c)$. Then, the statement can be formulated as

$$\begin{aligned}
&x_a + x_2 \geq 1,\\
&x_2 \leq x_b,\\
&x_2 \leq 1 - x_c,\\
&x_2 + 1 \geq x_b + (1 - x_c),\\
&x_a, x_b, x_c, x_2 \in \{0, 1\}.
\end{aligned}$$

2. We aim to formulate the opposite of the above; that is $\neg(a \vee (b \wedge \neg c))$. This statement means neither a nor $b \wedge \neg c$ holds true. Then, using the same variables as in the previous part, we have

$$\begin{aligned}
&x_a = 0,\\
&x_2 = 0,\\
&x_2 \leq x_b,\\
&x_2 \leq 1 - x_c,\\
&x_2 + 1 \geq x_b + (1 - x_c),\\
&x_a, x_b, x_c, x_2 \in \{0, 1\}.
\end{aligned}$$

3. Notice that $\neg(a \wedge b)$ is equivalent to $\neg a \vee \neg b$. Now, letting x_2 represent the second clause in the statement, we can rephrase this implication in the following manner

$$\begin{aligned}
&1 - x_a \leq x_2,\\
&1 - x_b \leq x_2,\\
&x_c + x_d \geq x_2,\\
&x_c \leq x_2,\\
&x_d \leq x_2,\\
&x_a, x_b, x_c, x_d, x_2\{0, 1\}.
\end{aligned}$$

4. Let x_1 stand for $b \vee c$ and x_2 represent $(\neg a \wedge (b \vee c))$. Now, this implication can be represented by

$$\begin{aligned}
&x_2 = 1 - x_d,\\
&x_2 \leq 1 - x_a,\\
&x_2 \leq x_1,\\
&x_2 + 1 \geq x_1 + (1 - x_a),\\
&x_1 \leq x_b + x_c,\\
&x_1 \geq x_c,\\
&x_1 \geq x_b,\\
&x_1, x_2, x_a, x_b, x_c, x_d \in \{0, 1\}.
\end{aligned}$$

5. Suppose that x_1 denotes $a \vee b \vee c$ and x_2 serves as $\neg a \wedge \neg b$. Then, this implication is given by

$$\begin{aligned}
&x_1 \leq x_2,\\
&x_1 \leq x_a + x_b + x_c,\\
&x_1 \geq x_a,\\
&x_1 \geq x_b,\\
&x_1 \geq x_c,\\
&x_2 \leq 1 - x_a,\\
&x_2 \leq 1 - x_b,\\
&x_2 + 1 \geq (1 - x_a) + (1 - x_b),\\
&x_1, x_2, x_a, x_b, x_c \in \{0, 1\}.
\end{aligned}$$

6. Let z_1 correspond to $\sum_{i=1}^n x_i \leq k$ and z_2 concern $\sum_{i=1}^n y_i \geq l$. Then, $\sum_{i=1}^n x_i \leq k$ implies $z_1 = 1$ and $z_2 = 1$ implies $\sum_{i=1}^n y_i \geq l$. These implications are given by

$$\begin{aligned}
&\sum_{i=1}^n x_i \leq k + (n - k)(1 - z_1),\\
&\sum_{i=1}^n x_i \geq (k + 1) - (k + 1)z_1,\\
&\sum_{i=1}^n y_i \geq l - l(1 - z_2),\\
&z_1 \leq z_2\\
&x_i, y_i, z_1, z_2 \in \{0, 1\}.
\end{aligned}$$

7. Let z_1 denote $\sum_{i=1}^n x_i \leq 3$, z_2 represent $\sum_{i=1}^n y_i \geq 3$ and z_3 stand for $\sum_{i=1}^n y_i \geq 4$. Since $\neg(z_1 \wedge z_2)$ is equivalent to $\neg z_1 \vee \neg z_2$, we aim to rephrase $\neg z_1 \vee \neg z_2 \Rightarrow z_3$. This implication is given by

$$\begin{aligned}
&z_3 \geq 1 - z_1,\\
&z_3 \geq 1 - z_2,\\
&\sum_{i=1}^n x_i \leq 3 + (n - 3)(1 - z_1),\\
&\sum_{i=1}^n x_i \geq 4 - 4z_1,\\
&\sum_{i=1}^n y_i \geq 3 - 3(1 - z_2),\\
&\sum_{i=1}^n y_i \leq 2 + (n - 2)z_2,\\
&\sum_{i=1}^n y_i \geq 4 - 4(1 - z_3),\\
&x_i, y_i, z_1, z_2, z_3 \in \{0, 1\}.
\end{aligned}$$

8. In this statement, we want either of the clauses to be true and the other false. Let z_1 and z_2 denote the first and the second clause respectively. Then, this implication is equivalent to $z_1 \iff \neg z_2$.

Consequently, we have

$$
\begin{aligned}
& z_1 = 1 - z_2, \\
& \sum_{i=1}^{n} x_i \geq 5 - 5(1 - z_1), \\
& \sum_{i=1}^{n} y_i \geq 4 - 4(1 - z_2), \\
& \sum_{i=1}^{n} x_i \leq 4 + (n-4) z_1, \\
& \sum_{i=1}^{n} y_i \leq 3 + (n-3) z_2, \\
& x_i, y_i, z_1, z_2 \in \{0, 1\}.
\end{aligned}
$$

9 . Letting z_1,z_2 and z_3 denote the first, second and the third clauses respectively, we have $z_1 \vee z_2 \vee z_3$. This is equivalent to

$$
\begin{aligned}
& x_1 + x_2 + x_3 \leq 1 + 2(1 - z_1), \\
& x_1 + x_2 + x_3 \geq 2 - 2z_1, \\
& x_6 + x_7 + x_8 \geq 1 - (1 - z_2), \\
& x_6 + x_7 + x_8 \leq 3z_2, \\
& x_9 + x_{10} \geq 1 - (1 - z_3), \\
& x_9 + x_{10} \leq 2z_3, \\
& z_1 + z_2 + z_3 \geq 1, \\
& x_1, x_2, x_3, x_6, x_7, x_8, x_9, x_{10}, z_1, z_2, z_3 \in \{0, 1\}.
\end{aligned}
$$

3 Formulations

1. We can simply replace $|y_i|$ by $y_i^+ + y_i^-$ to get

$$
\begin{aligned}
& \sum_{i=1}^{n} y_i^+ + y_i^- \leq 1 \\
& y_i^+, y_i^- \geq 0, \quad i = 1, ..., n
\end{aligned}
$$

   add replcae y_i by $y_i^+ - y_i^-$ in the objective function. The care should be taken that at least either of y_i^+ or y_i^- has to be zero. This restriction is automatically satisfied in the solution of this linear program as the coefficient vectors of y_i^+ and y_i^- are linearly dependent.

2. To linearize this problem, we replace each term $|y_i|$ by $y_i^+ + y_i^-$ and add $y_i = y_i^+ - y_i^-$ to the constraints where $y_i^+, y_i^- \geq 0$. Nevertheless, note that y_i^+ and y_i^+ cannot take positive values simultaneously. To impose this restriction, a set of auxiliary binary variables is defined as follows

$$
z_i = \begin{cases} 1 & \text{if } y_i^+ \text{ takes a positive value} \\ 0 & \text{Otherwise} \end{cases}
$$

Now, an equivalent formulation is given by

$$\begin{aligned}
&\sum_{i=1}^{n} y_i^+ + y_i^- \geq 1 \\
&y_i = y_i^+ - y_i^- \\
&-M \leq y_i \leq M \\
&y_i^+ \leq M z_i \\
&y_i^- \leq M(1 - z_i) \\
&y_i^+, y_i^- \geq 0, \quad z_i \in \{0, 1\}, \; i = 1, ..., n
\end{aligned}$$

3. Since the variables are pure integer, we can simply apply the formulation given in the previous part to impose this restriction. To be more precise, we append the model with

$$\sum_{i=1}^{n} |y_i - y_i^*| \geq 1.$$

Then, we end up with

$$\begin{aligned}
\max\ & c^T y \\
\text{s.t.}\ & \\
& Ay \leq b \\
& -M \leq y_i \leq M \ \forall i \\
& \sum_{i=1}^{n} |y_i - y_i^*| \geq 1 \\
& y \in \mathbb{Z}^n
\end{aligned}$$

Now, we need to resort to auxiliary binary variables to replace the absolute value function by a linear term in the same way as in the previous part.

Here is an alternative solution. In a way, this problem implies that $y*$ has to be violated by either of $Ay \leq b$ or $-M \leq y_i \leq M$. One way to impose this constraint is to define an auxiliary binary variable z so that

$$z = \begin{cases} 1 & \text{if } y = y^* \\ 0 & \text{Otherwise} \end{cases} \tag{1}$$

The relationship between y and z makes sense when we append the model with

$$-M(1 - z) \leq y_i - y_i^* \leq M(1 - z) \ \ \forall i,$$

Now, the model has to be infeasible as long as $y = y*$. Thus, we only need to consider an additional constraint which will be in conflict with the existing constraints when $z = 1$ (i.e., when $y = y^*$). In light of this wording, the problem can be casted into the following formulation

$$
\begin{aligned}
&\max\ c^T y \\
&\quad \text{s.t.} \\
&\qquad Ay \le b \\
&\qquad -M \le y_i \le M\ \forall i \\
&\qquad -M(1-z) \le y_i - y_i^* \le M(1-z)\ \forall i \\
&\qquad y_1 \ge (\epsilon + M)z - M(1-z) \\
&\qquad y \in \mathbb{Z}^n, \quad z \in \{0,1\}
\end{aligned}
$$

where ϵ stands for a positive value sufficiently close to zero. Note that $y = y^*$ implies $z = 1$ and then the two constraints

$$
\begin{aligned}
y_i &\le M \\
y_i &\ge \epsilon + M
\end{aligned}
$$

are in conflict.

You may also make a slight change in the model. For instance, you may replace the second and the third constraint sets by

$$
-(M - y_i^*)(1-z) \le y_i - y_i^* \le (M - y_i^*)(1-z) \quad \forall i.
$$

Now, the new formulation is given by

$$
\begin{aligned}
&\max\ c^T y \\
&\quad \text{s.t.} \\
&\qquad Ay \le b \\
&\qquad -(M - y_i^*)(1-z) \le y_i - y_i^* \le (M - y_i^*)(1-z)\ \forall i \\
&\qquad a_1^T y \ge b_1 + \epsilon z - U(1-z) \\
&\qquad y \in \mathbb{Z}^n, \quad z \in \{0,1\}
\end{aligned}
$$

where a_1 denotes the first row of A and U stands for a sufficiently large parameter.