IE426 – Optimization models and applications

Fall 2015 – Homework #5

November 24, 2015

This homework accounts for 5% of the final grade. There are 20 points available. For all problems where an AMPL model is required, include the AMPL model file, the data file, and the optimal solution, shown clearly with the command (`display`) you have used to print it. This homework is due by Thursday, December 3 by 4pm. It can be delivered to mail boxes of the professor or TAs in Mohler 421 or during the project meeting.

1 Support Vector Machines(5pts)

Formulate and solve the classification problem via the linear support vector machines

$$\begin{aligned} \min_{\xi,\mathrm{w},\beta} \quad & \frac{1}{2}w^\top w + c\sum_{i=1}^{n}\xi_i \\ \text{s.t.} \quad & y_i(w^\top x_i + \beta) \geq 1 - \xi_i, \quad i = 1,\ldots,n \\ & \xi \geq 0, \qquad i = 1,\ldots,n. \end{aligned}$$

using the following training data (see the text file with the data for simple paste and copy purposes)

$$X =$$

$$\begin{bmatrix} -0.0192 & 0.4565 \\ -0.0302 & -0.8531 \\ -0.1170 & -0.9854 \\ 0.4454 & 0.3952 \\ -0.7989 & -0.2569 \\ 0.0935 & 0.7398 \\ 0.2654 & 0.3098 \\ 0.6040 & -0.0959 \\ -0.6324 & -0.9139 \\ 0.9770 & -0.4862 \\ 0.9260 & 0.0075 \\ 0.8055 & -0.0103 \\ 0.3007 & 0.9564 \\ -0.2771 & 0.1357 \\ 0.3782 & 0.6800 \\ -0.5911 & -0.1808 \\ -0.2501 & 0.4231 \\ -0.1130 & 0.8032 \\ 0.9353 & -0.2590 \\ -0.1272 & 0.9856 \\ -0.0244 & 0.7780 \\ 0.2476 & 0.7701 \\ 0.1555 & -0.8341 \\ -0.9507 & -1.0000 \\ -0.6986 & -0.0473 \\ -0.4293 & -0.9466 \\ -0.8917 & 0.2226 \\ 0.1545 & 0.4526 \\ -0.8230 & 0.7856 \\ -0.5885 & 0.5231 \\ -0.6578 & -0.7660 \\ -0.2009 & -0.7598 \\ 0.0965 & -0.8755 \\ 0.3399 & 0.8383 \\ -0.0519 & 0.9419 \end{bmatrix}$$

$$y =$$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Solve 3 instances of the problem: with penalty c=10000, c=100 and c=1. Formulate the appropriate QPs in AMPL, solve it using Cplex. Compare the solutions - report the support vectors, i.e. the points for which $y_i(w^\top x_i + \beta) = 1$ and the points violating classification constraints, i.e. for which $y_i(w^\top x_i + \beta) < 1$

2 Quadratic Integer Programming - Support Vector Machines (5pts)

Consider a version of the SVM problem, where you want to build a model by using at most k features of the data X. In other words, you want to select vector w, such that at most k elements of w are not zero.

$$
\begin{array}{rc}
\min_{\xi, \mathrm{w}, \beta} & \frac{1}{2}\|w\|^2 + c\sum_{i=1}^{n} \xi_i \\
\text{s.t} & y_i(w^\top x_i + \beta) \geq 1 - \xi_i, \quad i = 1, \dots, n \\
& \#\text{ of nonzeros}(w) \leq k, \\
& \xi \geq 0
\end{array}
$$

- Formulate this problem as a problem with integer variables, quadratic objective and linear constraints.
- Write down a convex relaxation of this problem.

3 Modeling (4 pts.)

Consider the problem of estimating sparse (with few nonzero entries) solution x which approximately satisfies the system $Ax = b$.

$$
\begin{array}{rl}
\min_{\mathrm{x}} & \|x\|_1 \\
\text{s.t.} & \|Ax - b\|_\infty \leq \epsilon
\end{array}
$$

where $\|v\|_\infty = \max_i |v_i|$ is an infinity norm of a vector v. Reformulate this problem as a linear programming problem.

4 Stochastic Programming (6 pts.)

You are planning a fundraising dinner for the Metropolitan Museum of Art in New York. You need to plan how many dinners to order from the caterer, but the number of people expected to attend is uncertain. Since the attendees are high profile celebrities and influential people you cannot insist on them committing in advance and also you cannot turn people away at the door.

The capacity of the space is B =1,400 people and per person cost of dinner is c =\$120. You are charging each attendee the price of f =\$250 per ticket, and obviously you sell as many tickets as the number of people who actually show up. If you order 140 dinners but 100 people turn up then you raise \$25,000-\$120*140=\$8,200. On the other hand, if we do not order enough dinners in advance, you'll have to pay \$550 for each dinner that is delivered on top of the original order.

The number of people you expect to show up is d and you estimated three possible scenarios:

- d =450 people 0.25;
- d =900 people 0.55;
- d =1,250 people 0.20,

so the problem is that of deciding, **today**, how many dinners to order to maximize the profit.

1. Assuming we know the future every time we solve the problem, what would be the *expected* (i.e., average) profit? To answer this question, you need to solve three optimization problems (one for each case). You can use AMPL to solve those three, but my suggestion is that they are extremely easy and you can solve them on the fly.

   Report the results of each optimization problem, and then the expected value (note: it must be a weighted average as there is a different probability for each event).

2. Consider the robust solution - maximize the minimum profit and consider the expected profit for that solution.

3. Assuming instead we know neither the future nor Stochastic Programming, we are tempted to just set the uncertain parameter at its (weighted) average and solve the corresponding problem. Formulate this *average* problem and solve it. Again, you probably won't need AMPL to solve it, but formulating this model is useful toward the model you'll have to write in point 3.

   Now, the objective function you get here doesn't really have any meaning. In fact, if you implement this decision, you will get a different profit (or even a loss) depending on the scenario. Compute the profit (or loss) you'd make when implementing that decision in each of the scenarios. What is the (weighted) average profit? Is it greater or smaller than the expected profit at point 1?

4. Formulate a Stochastic Programming model for this problem. Remember that you need to understand what the *stages* and the *scenarios* are for this problem, and then define the first stage and second stage variables.

5. Solve it with AMPL and determine the expected profit.

6. Compare all expected profits.

5 Quadratic Integer Programming - Support Vector Machines (practice problem only, do not turn in with homework)

The following formulation is the standard formulation of support vector machines. The second term accounts for the points that do not satisfy the constraints for separation. This term is called "hinge loss" and it depends linearly on the "amount of violation" for each constraint.

$$\min_{\xi,\mathrm{w},\beta} \frac{1}{2}\|w\|^2 + c\sum_{i=1}^{n} \max\{1 - y_i(w^\top x_i + \beta), 0\}$$

Now, imagine instead I want to count and minimize the *number* of points which violate the constraints (the number of "outliers"), hence I want to optimize

$$\begin{aligned} \min_{\xi,\mathrm{w},\beta} \quad & \tfrac{1}{2}\|w\|^2 + c(\#\text{ of nonzeros}(\xi)) \\ \text{s.t} \quad & y_i(w^\top x_i + \beta) \geq 1 - \xi_i, \quad i = 1,\dots,n \\ & \xi \geq 0 \end{aligned}$$

- Formulate this problem as a problem with integer variables, quadratic objective and linear constraints.

$$\begin{aligned} \min_{\xi,\mathrm{w},\beta} \quad & \tfrac{1}{2}\|w\|^2 + c\textstyle\sum_{z_i} \\ \text{s.t} \quad & y_i(w^\top x_i + \beta) \geq 1 - \xi_i, \quad i = 1,\dots,n \\ & \xi \leq M z_i \qquad\qquad \xi \geq 0, z_i \in \{0,1\} \end{aligned}$$

- Write down the convex relaxation of this problem

$$\begin{aligned} \min_{\xi,\mathrm{w},\beta} \quad & \tfrac{1}{2}\|w\|^2 + c\textstyle\sum_{z_i} \\ \text{s.t} \quad & y_i(w^\top x_i + \beta) \geq 1 - \xi_i, \quad i = 1,\dots,n \\ & \xi \leq M z_i \\ & \xi \geq 0, z_i \in [0,1] \end{aligned}$$

or equivalently

- Write down the convex relaxation of this problem

$$\begin{aligned} \min_{\xi,\mathrm{w},\beta} \quad & \tfrac{1}{2}\|w\|^2 + \tfrac{c}{M}\textstyle\sum_{\xi_i} \\ \text{s.t} \quad & y_i(w^\top x_i + \beta) \geq 1 - \xi_i, \quad i = 1,\dots,n \\ & \xi \geq 0. \end{aligned}$$

6 Stochastic Integer Programming (practice problem only, do not turn in with homework)

You have \$20,000 to invest. Stock XYZ sells at \$20 per share today. A European call option to buy 100 shares of stock XYZ at \$15 exactly six months from today sells for \$1000[1]. You can also raise additional funds which can be immediately invested, if desired, by selling these same call options. In addition, a 6-month (riskless zero-coupon) bond with \$100 face value sells for \$90. This means if you buy a bond today at \$90 you will get exactly \$100 in 6 months. You have decided to limit the number of call options that you buy or sell to at most 50.

You consider three scenarios for the price of stock XYZ six months from today: the price will be the same as today (scenario 1), the price will go up to \$40 (scenario 2), or drop to \$12 (scenario 3). Your best estimate is that each of these scenarios is equally likely.

The profit from buying one call option in scenario 1 is $\$100(20-15)-\$1000 = -\$500$ (loss), in scenario 2 the profit is $\$100(40 - 15) - \$1000 = \$1500$, and in scenario 3 the profit (loss) is $-\$1000$, as you will not exercise this option in 6 months.

- Formulate and solve a linear program to determine the portfolio of stocks, bonds, and options that maximizes expected profit, considering the expected value for the profits from each investment expected value.

- (ii) Suppose that the investor wants a profit of at least \$2000 in any of the three scenarios. Write a stochastic linear program that will maximize the investor's expected profit under this additional constraint.

7 Integer Programming (practice problem only, do not turn in with homework)

You currently own a portfolio of eight stocks. Using the Markowitz model, you computed the optimal mean/variance portfolio. The weights of these two portfolios are shown in the following table:

Stock	A	B	C	D	E	F	G	H
Your Portfolio	0.12	0.15	0.13	0.10	0.20	0.10	0.12	0.08
M/V Portfolio	0.02	0.05	0.25	0.06	0.18	0.10	0.22	0.12

You would like to rebalance your portfolio in order to be closer to the M/V portfolio. To avoid excessively high transaction costs, you decide to rebalance only three stocks from your portfolio. Let x_i denote the weight of stock i in your rebalanced portfolio. The objective is to minimize the quantity

$$|x_1 - 0.02| + |x_2 - 0.05| + |x_3 - 0.25| + \ldots + |x_8 - 0.12|$$

which measures how closely the rebalanced portfolio matches the M/V portfolio.

Formulate this problem as a mixed integer linear program.

[1] A European call option is the right to buy a stock at a future determined date at the agreed price

8 Integer Programming (practice problem only, do not turn in with homework)

Alice and Bob are planning their holiday movie watching schedule. There are 2 lists of movies, namely Alice's and Bob's favorites, in the order of preference:

- $A_1, \ldots, A_n$, and
- $B_1, \ldots, B_n$

Formulate the following constraints using variables:

- $x_i = 1$, if they see movie A_i, 0 otherwise;
- $y_i = 1$, if they see movie B_i, 0 otherwise.
- They cannot see three consecutive movies from either of the lists (e.g., A_1, A_2 and A_3 are three consecutive movies from Alice's list) (assume these list do not have any movies in common)
- They will not see Alice's first choice if they do not see Bob's first choice.
- Formulate

$$(x_1 + x_2 + x_3 \leq 2) \ OR \ (x_6 + x_7 + x_8 \geq 1) \ OR \ (x_9 + x_{10} \geq 1)$$

9 Linear Programming (practice problem only, do not turn in with homework)

You are managing a factory of a company which plans to make 3 products: widgets and gadgets and superwidgets. For the planning horizon of the next 12 months, the demand for widgets in month i is w_i, and for gadgets it is g_i. In each month we can hire up to 50 employees (for that month only), at the monthly wage of 2000 \$, and assign each to make either widgets, or gadgets or superwidgets, but no employee can switch. One employee can make 40 widgets, 45 gadgets, or 10 superwidgets in a month. Each superwidget is made using 2 widgets and 1 gadget. We can store widgets, gadgets and superwidgets: having one widget in the inventory for one month incurs 20 \$ cost, and having one gadget in the inventory for one month incurs 25 \$ cost, storing superwidgets costs \$50/month. The formulation of this problem was as follows:

Variables:

- mw_i, mg_i, msw_i: number of widgets, gadgets, or superwidgets made in month i, resp. $(i = 1, \ldots, 12)$.
- sw_i, sg_i, ssw_i: number of widgets, gadgets, or superwidgets stored from month i to month $i+1$, resp. $(i = 0, \ldots, 12)$.
- lsw_i: number of superwidgets sold in month i $(i = 1, \ldots, 12)$.

Objective:

- $\min \sum_{i=1}^{12}(2000(mw_i/40 + mg_i/45 + msw_i/10) + 20w_i + 25g_i + 50ssw_i - 500lsw_i)$.

  Constraints:

- $mw_i/40+mg_i/45+msw_i/10 \leq 50$ (not more than 50 employees per month can be hired).
- $sw_0 = 0$, $sg_i = 0$, $ssw_i = 0$.
- $sw_{i-1}+mw_i-2msw_i = w_i+sw_i$ (balance constraint for widgets in month i, considering that each superwidget we make uses 2 widgets.)
- $sg_{i-1} + mg_i - msw_i = g_i + sg_i$ (balance constraint for gadgets in month i, considering that each superwidget we make uses 1 gadget.)
- $ssw_{i-1} + msw_i = lsw_i + ssw_i$ (balance constraint for superwidgets in month i.)
- All variables nonnegative.

Now assume that you have solved this problem with nice linear programming software and found an optimal solution for the 12 month period and you are ready to start production. At the last moment a big boss from headquarters comes to the factory and says that she has a brilliant idea to also produce supergadgets, which can be produced from 2 gadgets and 1 widget by any employee at the rate of 10/month and the cost of \$60/month and can be sold for \$520. You formulate the new problem, which involves supergadgets:

After you formulate the problem you discover that your computer is not working, so you cannot run linear programming. However you think that maybe it would not make sense to produce any supergadgets anyway, as it would not be profitable. How can you check this, knowing your solution for the plan before supergadgets were introduced? By doing relatively simple computations can you inform the big boss if producing supergadgets will actually reduce the expenses/increase the profit in your factory? You can justify your answer using specific numbers provided, but it is better to explain a general approach which does not depend on the given numbers (note that the demands are not given as specific numbers, so it is better to give a general answer).