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4-18. (a) Suppose using the coin 1 is $X_0 = 0$ and using the coin 2 is $X_1 = 1$,

Then the transition matrix is.

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix}, \text{ then } \begin{cases} \pi_0 = 0.6\pi_0 + 0.5\pi_1 \\ \pi_1 = 0.4\pi_0 + 0.5\pi_1 \\ \pi_0 + \pi_1 = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = \frac{5}{9} \\ \pi_1 = \frac{4}{9} \end{cases}$$

$\therefore$ the proportion of flips use coin 1 is $\pi_0 = \frac{5}{9}$.

(b) $p = \pi_1 + \pi_0 \times 0.4 + \pi_0 \times 0.6 \times 0.4 + \pi_0 \times 0.6^2 \times 0.4$

$= \frac{22}{25}$

$= 0.88$

4-20. proof:

For any irreducible and aperiodic M.C. with limited states, there is only one solution of $\begin{cases} \pi = P\pi \\ \sum \pi_j = 1 \end{cases}$

So, if $\pi_j = \frac{1}{M+1}$, $j = 0, 1, \ldots M$ is one of the solution, it is the unique one.

$$\begin{cases} \sum_{i=0}^{M} P_{ij} \pi_i = \pi_j, \quad j = 0, 1, \ldots M. & ① \\ \sum_{j=0}^{M} \pi_j = 1 & ② \end{cases}$$

put $\pi_j = \frac{1}{M+1}$ in ①: $\sum_{i=0}^{M} P_{ij} \cdot \pi_i = \frac{1}{M+1} \sum_{i=0}^{M} P_{ij} = \frac{1}{M+1} \cdot 1 = \pi_j$.

put $\pi_j = \frac{1}{M+1}$ in ②: $\frac{1}{M+1} \cdot (M+1) = \sum_{j=0}^{M} \pi_j = 1$,

which gives the solution.

4-25. Solution

Suppose $X_n = \{$ the number of pairs of shoes at front door $\mid n = 0, 1, \cdots K \}$.

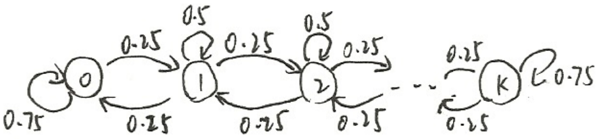
when $X_n = 1, 2, \cdots, K-1$

$$X_{n+1} = \begin{cases} X_n - 1 & \text{leave from F door, back to B door.} \\ X_n + 1 & \text{leave from B door, back to F door.} \\ X_n & \text{leave from F door, back to F door.} \\ X_n & \text{leave from B door, back to B door,} \end{cases}$$

each probability is equal to $\frac{1}{4}$.

when $X_n = 0$.

$$X_{n+1} = \begin{cases} 0 & F \to F;\ B \to B;\ F \to B; \\ 1 & B \to F. \end{cases}$$



$$P = \begin{bmatrix} 0.75 & 0.25 & 0 & 0 & \cdots & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 & \cdots & & 0 \\ 0 & 0.25 & 0.5 & \ddots & & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & & 0 & 0.5 & 0.25 \\ 0 & 0 & \cdots & \cdots & 0 & 0.25 & 0.75 \end{bmatrix}$$

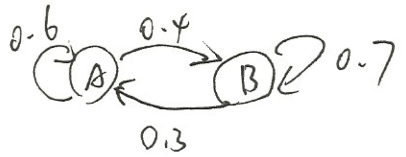
, it is a doubly stochastic Matrix,

so, $\pi_j = \frac{1}{k+1}$, proved in 4-20.

$\therefore$ leave in bare foot: $p = \frac{1}{2} \cdot \pi_0 + \frac{1}{2} \cdot \pi_k$

$= \frac{1}{k+1}$.

4-52. Solution:



$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}.$$

$$\begin{cases} \pi_A = 0.6\pi_A + 0.3\pi_B \\ \pi_B = 0.4\pi_A + 0.7\pi_B \\ \pi_A + \pi_B = 1 \end{cases} \Rightarrow \begin{cases} \pi_A = \frac{3}{7} \\ \pi_B = \frac{4}{7} \end{cases}$$

Salary $S = 8 \cdot \pi_B \cdot P_{BB} + 6 \times \pi_A \times P_{AA} + 12 \times (\pi_B \times P_{BA} + \pi_A \times P_{AB})$

$= \frac{62}{7}$

4-58. proof:

suppose Event $A = \{X_{n+1} = i+1\}$; $B = \{X_n = i\}$; $C = \{\lim_{m \to \infty} X_m = N\}$.

The probability we want is $p\{A \mid B \cap C\}$.

According to basic formula: $p(A \cap C \mid B) = p(A \mid B \cap C) \cdot p(C \mid B) = P(C \mid B \cap A) \cdot p(A \mid B)$

$\because$ this M.C. is time-irrelavent

$\therefore B \cap A = A$

$$\therefore P(A \mid B \cap C) = \frac{P(C \mid A) \cdot P(A \mid B)}{P(C \mid B)} = \frac{p_{(i+1)} \cdot p}{p_{(i)}} = \frac{\left[\frac{1-(q/p)^{i+1}}{1-(q/p)^N}\right] \cdot p}{\left[\frac{1-(q/p)^i}{1-(q/p)^N}\right]} = \frac{p\left[1-(q/p)^{i+1}\right]}{1-(q/p)^i}$$

when $p \neq \frac{1}{2}$.

$$P(A \mid B \cap C) = \frac{\left(\frac{i+1}{N}\right) \cdot \frac{1}{2}}{\left(\frac{i}{N}\right)} = \frac{i+1}{2i}, \text{ when } p = \frac{1}{2}.$$

4-60. solution:

(a) suppose p_i is the probability that state 3 is entered before state 4 given the initial state is i, $i=1,2,3$.

$$\begin{cases} p_1 = 0.4\,p_1 + 0.3\,p_2 + 0.2 \\ p_2 = 0.2\,p_1 + 0.2\,p_2 + 0.2 \end{cases} \Rightarrow p_1 = \frac{11}{21}$$

(b) suppose m_i is the mean number of transitions either state 3 or 4 is entered ~~as~~ given the initial state is i, $i=1,2,3$

$$\begin{cases} m_1 = 0.4(m_1+1) + 0.3(m_2+1) + 0.3(m_3+1) \\ m_2 = 0.2(m_1+1) + 0.2(m_1+1) + 0.6(m_3+1) \\ m_3 = 0 \end{cases} \Rightarrow m_1 = \frac{55}{21}$$

6) proof: ① It is irreducible

$EY_1 < 0 \Rightarrow P\{Y_1 = -k\} > 0,\ k>0,\ k\in\mathbb{Z} \Rightarrow$ state 0 is reachable from other states.

$P\{Y_1 = 1\} > 0 \Rightarrow$ any state is reachable from 0.

$\therefore$ This M.C. is irreducible.

②. It is aperiodic.

$EY_n < 0 \Rightarrow \exists X_k = -5,\ Y_{k+1} < 0,\ k = 0,1,2,\cdots$.

then $P\{X_{k+1} = \max\{X_k + Y_{k+1}, -5\} = -5 \mid X_k = -5\} > 0$.

$k = \gcd\{n>0 : pr(X_{k+1} = -5 \mid X_k = -5) > 0\} = 1$.

$\therefore$ M.C. is aperiodic.

③ To prove positive recurrence, use Lyapunov – Foster criterion,

$V(i) = i$.

$$E(V(X_1)|X_0=i) - V(i) = E\max\{Y_1, -i\} \xrightarrow[i\to\infty]{} EY_1 < 0$$

$\therefore \exists\ \varepsilon > 0$, and $i_0 \geq 0$ that

$$E(V(X_1)|X_0=i) - V(i) \leq -\varepsilon, \quad i > i_0$$

$$\therefore E(V(X_1)|X_0=i) - V(i) \leq E|Y_1| < \infty, \quad \text{for all } i.$$

It satisfy to choose $X_0 = \{0, 1, \cdots, i_0\}$.