ISE 426
Optimization models and applications

Lecture 10 — September 30, 2014

- basic feasible solutions
- simplex method
- connection with dual variables

Reminders:

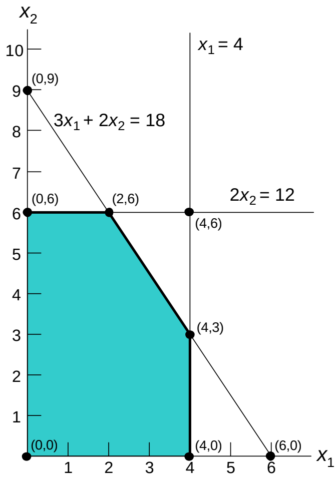
- Quiz on 10/14, practice on 10/09.

Simplex Method

Example

$$
\begin{array}{lrcrcl}
\text{maximize} & 3x_1 & + & 5x_2 & & \\
\text{subject to} & x_1 & & & \leq & 4 \\
& & & 2x_2 & \leq & 12 \\
& 3x_1 & + & 2x_2 & \leq & 18 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0
\end{array}
$$

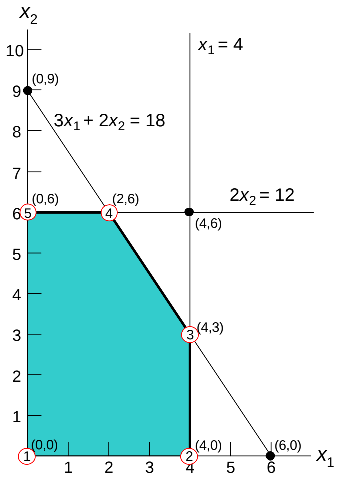
The lines are the **constraint boundaries**.



Corner-Point Solutions

5 **corner-point feasible (CPF) solutions**:

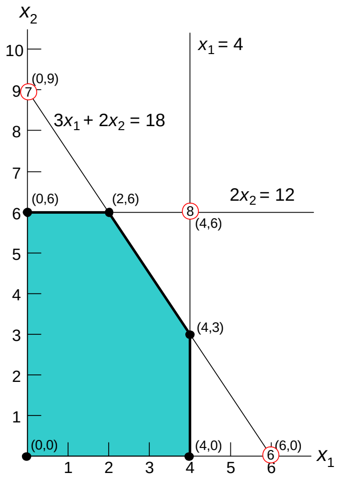
1. (0,0)
2. (4,0)
3. (4,3)
4. (2,6)
5. (0,6)



Corner-Point Solutions

3 **corner-point infeasible solutions**:

6. (6,0)
7. (0,9)
8. (4,6)

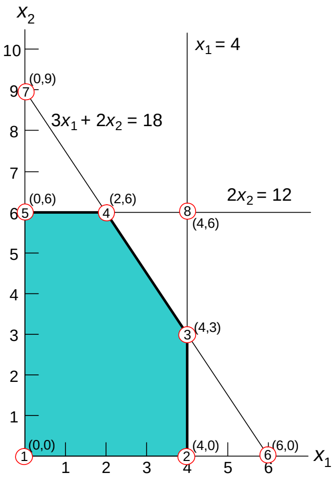


Corner-Point Solutions

Each corner-point solution (feasible or infeasible) lies at the intersection of two constraint boundaries.

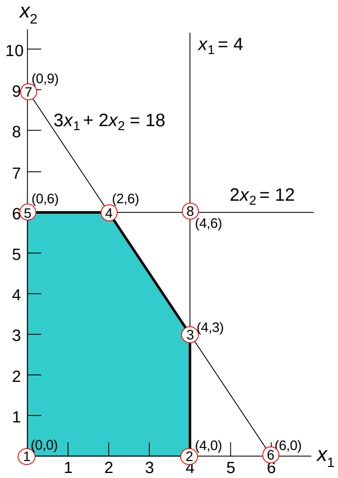
For an LP with n Decision Variables:

Each corner-point solution lies at the intersection of n constraint boundaries.



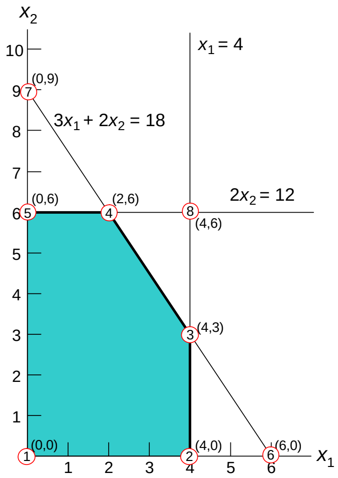
Adjacent CPFs

- For an LP with n decision variables, two CPF solutions are **adjacent** if they share $n - 1$ constraint boundaries.
- Two adjacent CPF solutions are connected by an **edge** of the feasible region.



Adjacent CPFs

- For an LP with n decision variables, two CPF solutions are **adjacent** if they share $n-1$ constraint boundaries.
- Two adjacent CPF solutions are connected by an **edge** of the feasible region.
- (0,6) and (2,6) are adjacent.
- (2,6) and (4,3) are adjacent.

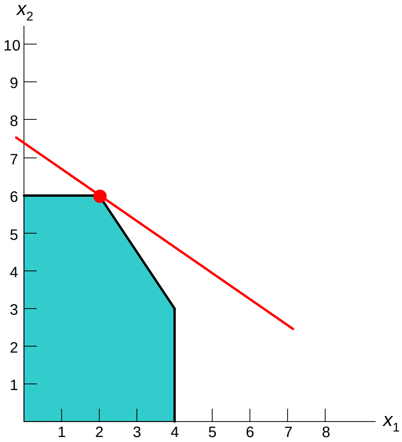


Why All the Fuss Over CPF Solutions?

Important Property #1

If an LP has a single optimal solution, it is a CPF solution. If an LP has more than one optimal solution, at least two are CPF solutions.

Why is it true?



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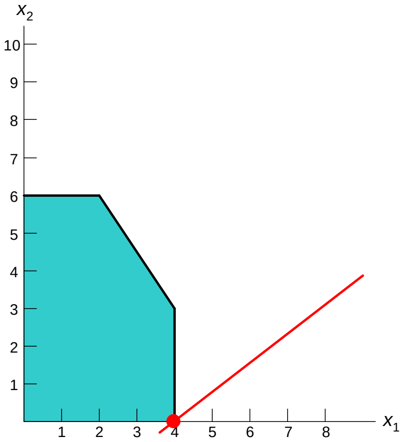
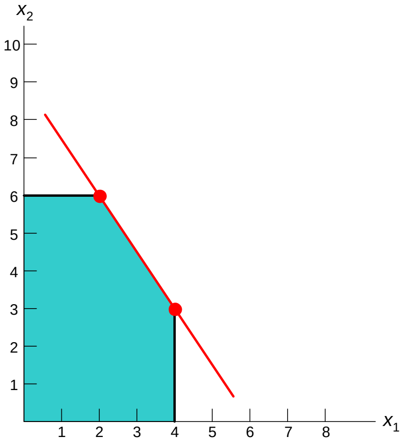


Why is it true?



How Many CPFs Are There?

- IP #1 means that we can focus only on CPF solutions and ignore the rest of the feasible region.
- There are an *infinite* number of feasible solutions
  - (assuming there are at least 2).
- There are a *finite* number of CPF solutions
  - (assuming feasible region is bounded and there are a finite number of constraints).
- That means we can focus on a *much smaller* set of possible answers.
- Can we just examine every CPF?

How Many CPFs? cont'd

- If there are n decision variables and m functional constraints, how many constraint boundaries are there?
- How many ways can we choose n constraint boundaries?
  - Answer:
$$\binom{m+n}{n} = \frac{(m+n)!}{m!n!}$$
- If $m = 50$ and $n = 50$, there are 10^{29} CPFs to examine
- If you could examine 1 billion CPFs per second, it would take you
$$3,170,979,198,376 \text{ years}$$
  to examine all of the CPFs.

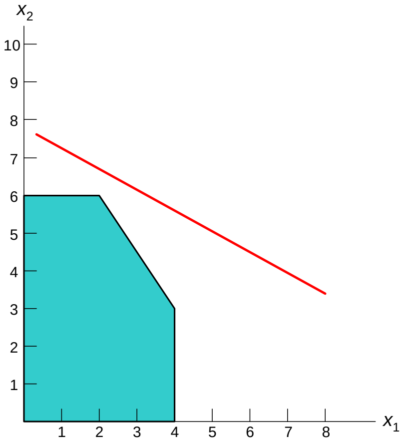
Why All the Fuss Over *Adjacent* CPF Solutions?

Important Property #2

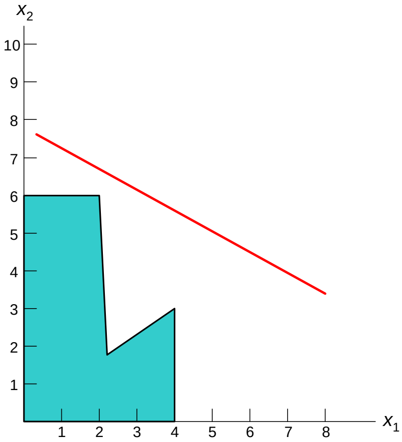
If a CPF solution has no *adjacent* CPF solutions that are better, then it must be an *optimal* solution.

In other words, if we find a CPF solution with no better neighbors, we can stop looking—there are no better solutions anywhere.

Why is this true?



Why is this true?



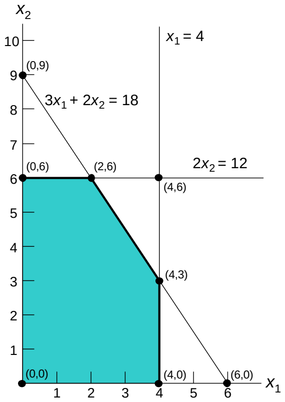
The Punchline

Taken together, the two properties mean we can find an optimal solution by:

1. Starting at any CPF solution
2. Moving to a better adjacent CPF solution, if one exists
3. Continuing until the current CPF solution has no adjacent CPF solutions that are better

This is the essence of the simplex method.

The Dual



$$
\begin{array}{lrcrcrcr}
\text{minimize} & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{subject to} & u_1 & & & + & 3u_3 & \geq & 3 \\
 & & & 2u_2 & + & 2u_3 & \geq & 5 \\
 & u_1 & & & & & \geq & 0 \\
 & & & u_2 & & & \geq & 0 \\
 & & & & & u_3 & \geq & 0
\end{array}
$$

Example

$$\begin{array}{lrcrcl}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
& & & 2x_2 & \leq & 12 \\
& 3x_1 & + & 2x_2 & \leq & 18 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0
\end{array}$$

$$\begin{array}{lrcrcrcl}
\min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\
& & & 2u_2 & + & 2u_3 & \geq & 5 \\
& u_1 & & & & & \geq & 0 \\
& & & u_2 & & & \geq & 0 \\
& & & & & u_3 & \geq & 0
\end{array}$$

Consider optimal $(x_1, x_2) = (2, 6)$, compute dual from complementary slackness:

$$\begin{array}{l}
u_1(x_1 - 4) = 0 \\
u_2(+2x_2 - 12) = 0 \\
u_3(3x_1 + 2x_2 - 18) = 0 \\
x_1(u_1 + 3u_3 - 3) = 0 \\
x_2(2u_2 + 2u_3 - 5) = 0
\end{array}
\quad \Rightarrow \quad
\begin{array}{l}
u_1(-2) = 0 \\
u_2(0) = 0 \\
u_3(0) = 0 \\
2(u_1 + 3u_3 - 3) = 0 \\
6(2u_2 + 2u_3 - 5) = 0
\end{array}$$

Example

$$\begin{array}{llllll} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{llllllll} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider optimal $(x_1, x_2) = (2, 6)$, compute dual from complementary slackness:

$$\begin{aligned} u_1 &= 0 \\ 3u_3 &= 3 \\ 2u_2 + 2u_3 &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_1 &= 0 \\ u_2 &= \tfrac{3}{2} \\ u_3 &= 1 \end{aligned}$$

Example

$$\begin{array}{llclcl}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
& & & 2x_2 & \leq & 12 \\
& 3x_1 & + & 2x_2 & \leq & 18 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0
\end{array}$$

$$\begin{array}{llclclcl}
\min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\
& & & 2u_2 & + & 2u_3 & \geq & 5 \\
& u_1 & & & & & \geq & 0 \\
& & & u_2 & & & \geq & 0 \\
& & & & & u_3 & \geq & 0
\end{array}$$

Consider a feasible CPF $(x_1, x_2) = (4, 3)$, compute dual from complementary slackness:

$$\begin{aligned}
u_1(x_1 - 4) &= 0 \\
u_2(+2x_2 - 12) &= 0 \\
u_3(3x_1 + 2x_2 - 18) &= 0 \\
x_1(u_1 + 3u_3 - 3) &= 0 \\
x_2(2u_2 + 2u_3 - 5) &= 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
u_1(0) &= 0 \\
u_2(-4) &= 0 \\
u_3(0) &= 0 \\
2(u_1 + 3u_3 - 3) &= 0 \\
6(2u_2 + 2u_3 - 5) &= 0
\end{aligned}$$

Example

$$\begin{array}{lccccc} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lccccccc} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (4, 3)$, compute dual from complementary slackness:

$$\begin{aligned} u_2 &= 0 \\ u_1 + 3u_3 &= 3 \\ 2u_3 &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_1 &= -\tfrac{9}{2} \\ u_2 &= 0 \\ u_3 &= \tfrac{5}{2} \end{aligned}$$