ISE 426
Optimization models and applications

Lecture 18 — November 4, 2014

- The Traveling Salesperson Problem (TSP)
- The Quadratic Assignment Problem (QAP)
- Piecewise linear functions

The Traveling Salesperson Problem (TSP)

A salesperson has to visit n cities and then return home.

- She/he would like to spend as little as possible time/gas.
- Any pair of cities (i, j) is connected by a road, and the distance between them is denoted as d_{ij}.

A very well-known Optimization problem, with applications in the VLSI (chip manufacturing) industry:

- visit n points on a printed circuit board to punch each of them with a laser.

$\Rightarrow$ minimize time spent moving the robotic arm from point to point, i.e., minimize the total distance travelled: a TSP!

Other less obvious applications: machine scheduling with set up costs (e.g. painting cars).

http://www.math.uwaterloo.ca/tsp

Formulation(s)

- Let's define the set of cities $V = \{1, 2 \ldots, n\}$.
- Variables: x_{ij}, binary; 1 if $i \to j$ in the *tour*, 0 otherwise

$\Rightarrow$ $n(n-1)$ variables, one for each $(i,j) \in V^2 : i \neq j$

- Objective function: The total distance travelled,

$$\sum_{i \in V} \sum_{j \in V : i \neq j} d_{ij} x_{ij}$$

- Constraints: **every city** i is visited (once), that is, only one arc leaves i:

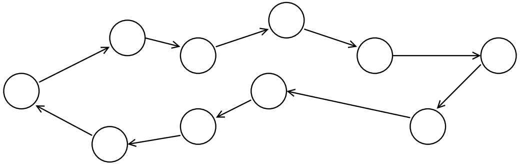
$$\sum_{j \in V : i \neq j} x_{ij} = 1 \qquad \forall i \in V$$

and only one arc enters i:

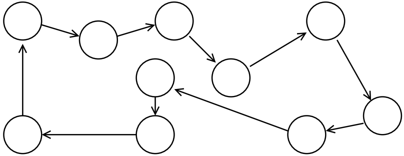
$$\sum_{j \in V : i \neq j} x_{ji} = 1 \qquad \forall i \in V$$

- Is that it?

“Feasible” solutions



OK



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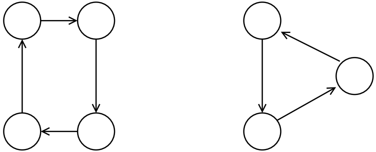


Oops…

Subtours

- An optimal solution to this IP model may be infeasible!
- We need to ensure that no solution is **a union of subtours**

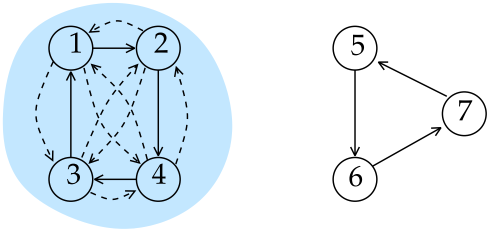
$\Rightarrow$ **Subtour elimination constraints**:

Every subset of $m < n$ nodes cannot have m connections:

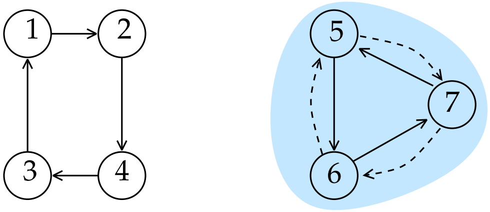
$$\sum_{i \in S, j \in S : i \neq j} x_{ij} \leq |S| - 1 \qquad \forall S \subset V : S \neq \emptyset$$

- How many such constraints are there?
- As many as the (proper, non-empty) subsets of V: $2^n - 2$
- For $n = 30$, that means a billion or so: $2^{30} = 1,073,741,824$.

Example: eliminate a subtour



$$x_{12} + x_{13} + x_{14} +$$
$$x_{21} + x_{23} + x_{24} +$$
$$x_{31} + x_{32} + x_{34} +$$
$$x_{41} + x_{42} + x_{43} \leq 3$$



$$x_{56} + x_{57} +$$
$$x_{65} + x_{67} +$$
$$x_{75} + x_{76} \leq 2$$

After adding these inequalities, the new solution may have subtours, but surely not these two.

Subtours

$\Rightarrow$ Do not add all of them, especially in real-world problems (where n is usually bigger than 30)

- **Iteratively** add those found to be violated:

  repeat

  solve LP

  find violated subtour elimination constraint(s)

  add them to LP

  until no subtour elimination constraint is found

  if solution is fractional, branch

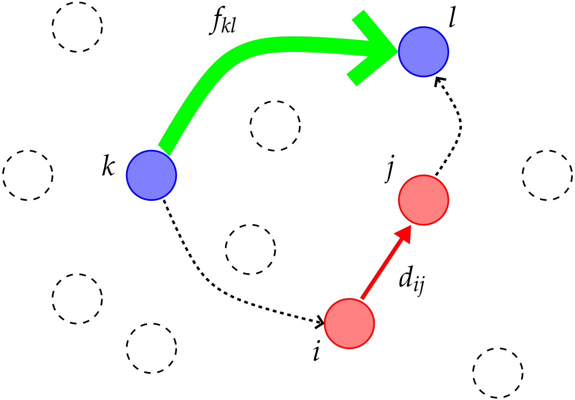
- This is called Branch&Cut (solves the $n = 85,900$ problem)

The Quadratic Assignment Problem (QAP)

- Consider n locations
- and n activities (e.g. stages of an industrial process)
- Any two locations i and j have a **distance** d_{ij}
- There is a **demand** f_{kl} between activities k and l
- Cost of satisfying each demand: proportional to
  - f_{kl} and
  - the distance between the two locations assigned to k and l

$\Rightarrow$ Assign each activity to a location such that the total assignment cost is minimum

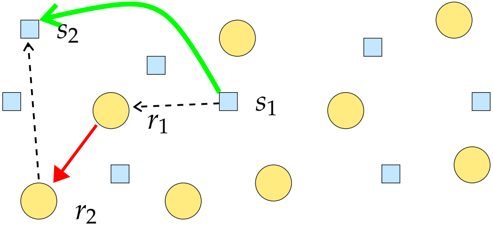
A single demand



QAP is a very general problem

It can be used when multiplicative costs factors are involved, for example:

- A set S of n students, a set R of n dorm rooms
- Each pair of students (k, l) has a friendship level f_{kl} (how often they visit each other).
- The distance between rooms i and j is d_{ij}



$\Rightarrow$ formulate this problem as a QAP:

- Assign students to rooms so that the total time spent walking between dorm rooms in minimized.
- Once we solve the QAP, construct solution to our problem

Formulation

Define $N := \{1, 2 \ldots, n\}$.

Variables: x_{ij}. 1 if student i assigned to room j, 0 otherwise

Constraints: Each student i is assigned to **exactly** one room:

$$\sum_{j \in N} x_{ij} = 1 \qquad \forall i \in N$$

Viceversa, each room j hosts **exactly** one student:

$$\sum_{i \in N} x_{ij} = 1 \qquad \forall j \in N$$

Formulation

- Objective function: For each pair (k, l) of students, frequency of visits f_{kl} is weighted with by distance between the rooms associated with k and l
- Suppose student k is assigned to room i (i.e. $x_{ki} = 1$) and student l is assigned to room j (i.e. $x_{lj} = 1$)
- The contribution to the cost: is f_{kl} multiplied by distance d_{ij}, but only if $x_{ki} = 1 \wedge x_{lj} = 1$, i.e., $x_{ki}x_{lj} = 1$ (nonlinear!).

$\Rightarrow$ The cost associated with each pair of students (k, l) is

$$(\star) \qquad \sum_{i \in N} \sum_{j \in N} f_{kl} d_{ij} x_{ki} x_{lj}$$

- The overall cost is therefore the sum of $(\star)$ on all pairs (k, l):

$$\sum_{k \in N} \sum_{l \in N} \sum_{i \in N} \sum_{j \in N} f_{kl} d_{ij} x_{ki} x_{lj}$$

Formulation

- The objective function is nonlinear, but we know what to do...
- Introduce a new variable y_{kilj} defined as $x_{ki}x_{lj}$

$\Rightarrow$ y_{kilj} are binary too, and are subject to the constraints:

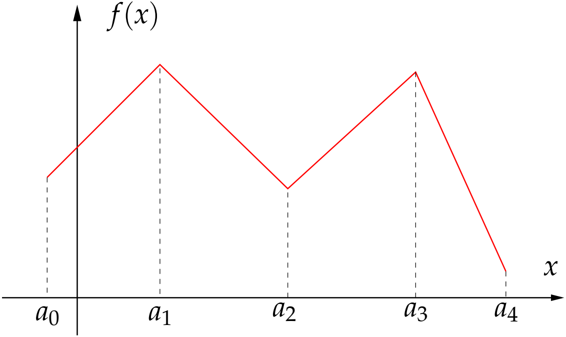
$$\begin{aligned} y_{kilj} &\leq x_{lj} \\ y_{kilj} &\leq x_{ki} \\ y_{kilj} &\geq x_{ki} + x_{lj} - 1 \end{aligned}$$

Formulation

$$
\begin{array}{lll}
\min & \sum_{k\in N}\sum_{l\in N}\sum_{i\in N}\sum_{j\in N} & f_{kl} d_{ij} \color{red}{y_{kilj}} \\
& \sum_{j\in N} x_{ij} = 1 & \forall i \in N \\
& \sum_{i\in N} x_{ij} = 1 & \forall j \in N \\
& y_{kilj} \le x_{ki} & \forall (k,i,l,j) \in N^4 \\
& y_{kilj} \le x_{lj} & \forall (k,i,l,j) \in N^4 \\
& y_{kilj} \ge x_{ki} + x_{lj} - 1 & \forall (k,i,l,j) \in N^4 \\
& y_{kilj} \in \{0,1\} & \forall (k,i,l,j) \in N^4 \\
& x_{ij} \in \{0,1\} & \forall (i,j) \in N^2
\end{array}
$$

Piecewise Linear functions

Consider a univariate, **piecewise linear** function $f(x)$ made of n linear pieces.



- it can be modeled with linear constraints
- but the function is not convex, hence we need a MILP model this time

A model for piecewise linear functions

Variable x needs to be modeled depending on the a_i's.

- If $a_2 \leq x \leq a_3$, we want $f(x)$ to be between $f(a_2)$ and $f(a_3)$
- If $x = \lambda a_2 + (1 - \lambda) a_3$, with $0 \leq \lambda \leq 1$, then $f(x)$ must be $\lambda f(a_2) + (1 - \lambda) f(a_3)$
- In general, use variables λ_i: $x = \sum_{i=0}^{n} \lambda_i a_i$, where
  - only two λ_i's are non zero, **and**
  - they sum up to one, **and**
  - they are consecutive

e.g. to model x exactly at the midpoint between a_2 and a_3, we need $\lambda_2 = \lambda_3 = \frac{1}{2}$ and $\lambda_0 = \lambda_1 = \lambda_4 = 0$

A model for piecewise linear functions

OK, but how do we ensure the "only two" and the "consecutive" things?

$\Rightarrow$ with binary variables!

- Define one binary variable y_i for each linear piece:
- There is only one nonzero y_i
- y_i is 1 if x is between a_{i-1} and a_i
- That is, we want
  - if $\lambda_0 > 0$, then $y_1 = 1$
  - if $\lambda_i > 0$ with $i = 1, 2 \ldots, n-1$, then $y_i = 1$ **or** $y_{i+1} = 1$
  - if $\lambda_n > 0$, then $y_n = 1$

A model for piecewise linear functions

Introduce a new variable φ for $f(x)$. We have:

$$\begin{aligned}
\varphi = &\quad \textstyle\sum_{i=0}^n \lambda_i f(a_i) \\
x = &\quad \textstyle\sum_{i=0}^n \lambda_i a_i \\
\textstyle\sum_{i=0}^n \lambda_i = &\quad 1 \\
\textstyle\sum_{i=1}^n y_i = &\quad 1 \\
\lambda_0 \leq &\quad y_1 \\
\lambda_n \leq &\quad y_n \\
\lambda_i \leq &\quad y_i + y_{i+1} && \forall i = 1, 2 \ldots, n-1 \\
\lambda_i \in &\quad [0, 1] && \forall i = 0, 2 \ldots, n \\
y_i \in &\quad \{0, 1\} && \forall i = 1, 2 \ldots, n
\end{aligned}$$