ISE 426
Optimization models and applications

Lecture 17 — October 29, 2015

- "Good" and "bad" formulations
- Branch&bound for MILP
- Examples of B&B

Reading:

- Hillier & Lieberman, Chapter 13, 13.4 to 13.5
- Winston & Venkataramanan, Chapter 9
- Winston, Chapter 9

Good vs. bad models: Uncapacitated Facility Location

A set J of retailers has to be served by a set S of plants, yet to be built. We don't know where the plants will be, but there is a set I of potential sites, and there is

- a cost f_i for building plant $i \in I$
- a (transportation) cost c_{ij} from plant i to retailer j

Each retailer will be served by exactly one plant . Choose a subset S of I such that the total cost is minimized.

Variables:

- $x_i, i \in I$: 1 if plant i is built, 0 otherwise
- y_{ij} assigns retailer j to plant i: 1 if i serves retailer j, 0 otherwise

Good vs. bad models: Uncapacitated Facility Location

Objective function:

$$\sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij}$$

Constraints:

- for each customer, one facility: $\sum_{i \in I} y_{ij} = 1$
- customers go to mall i if it's there:

$$\sum_{j \in J} y_{ij} \leq |J| x_i \quad \forall i \in I$$

or

$$y_{ij} \leq x_i \quad \forall i \in I, j \in J$$

Good vs. bad models: Uncapacitated Facility Location

(A)

$$\begin{aligned} \min \quad & \sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \\ & \sum_{i \in I} y_{ij} = 1 \quad \forall i \in I \\ & \sum_{j \in J} y_{ij} \le |J| x_i \quad \forall i \in I \\ & x_i, y_{ij} \in \{0,1\} \end{aligned}$$

(B)

$$\begin{aligned} \min \quad & \sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \\ & \sum_{i \in I} y_{ij} = 1 \quad \forall i \in I \\ & y_{ij} \le x_i \quad \forall i \in I, j \in J \\ & x_i, y_{ij} \in \{0,1\} \end{aligned}$$

(A) and (B) are **equivalent**. However, for $|I| = |J| = 40$,

(A) takes 14 hours[1]

(B) takes 2 seconds

[1]On AMPL with an old version of CPLEX.

How do we solve an Integer Programming problem?

$$\begin{array}{lll} (P) & \min & cx \\ & Ax & \geq b \\ & x_i & \in \mathbb{Z}, i \in J \subseteq \{1, 2 \ldots, n\} \end{array}$$

- Relaxing integrality gives an LP problem (easy)
- Solving LP gives a lower bound
- But the solution $x^\star$ may have fractional component $x_i^\star \notin \mathbb{Z}$, with $i \in J$ (for example, 3.31), infeasible for (P).

⇒ **Divide** the solution set, **partition** the problem into two new subproblems, P_1 and P_2, with

$$P_1 : x_i \leq \lfloor x_i^\star \rfloor \qquad P_2 : x_i \geq \lceil x_i^\star \rceil$$

($P_1 : x_i \leq 3$ and $P_2 : x_i \geq 4$) and recursively solve P_1 and P_2.

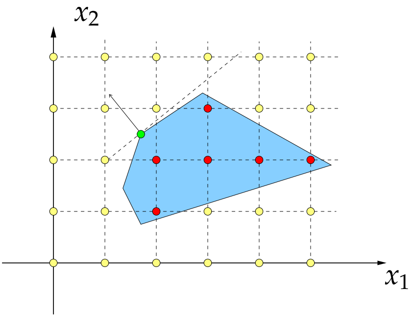
- Good: **no** feasible solution of P_1 or P_2 has $x_i = 3.31$

The Branch&Bound - Devide and concour

- If we solve the LP relaxation of P_1 and find a fractional point, we can recursively branch on P_1 and obtain two new subproblems P_3 and P_4.
- In principle, we have to branch on any node P_k unless its LP relaxation returns a feasible solution or it is infeasible.

Example: minimize $11x_1 - 10x_2$

$lb = -9.2$
$ub = ?$



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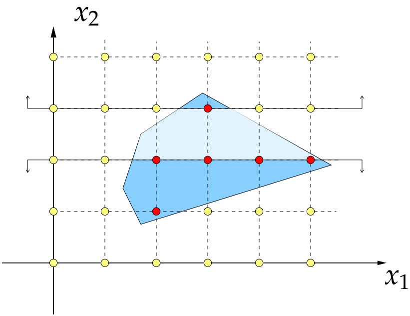

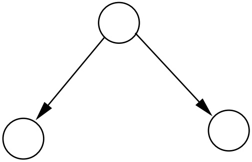

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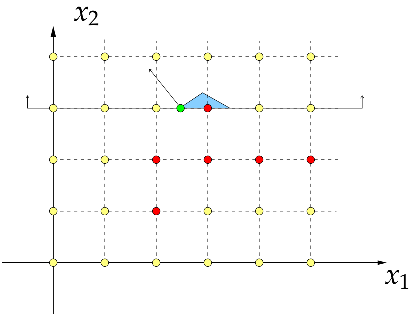

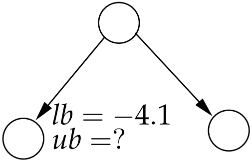

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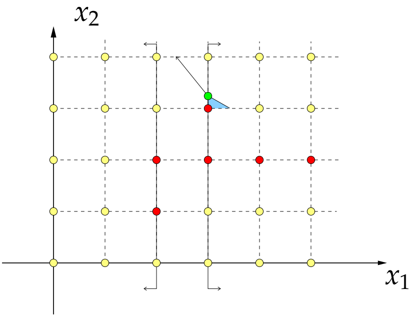

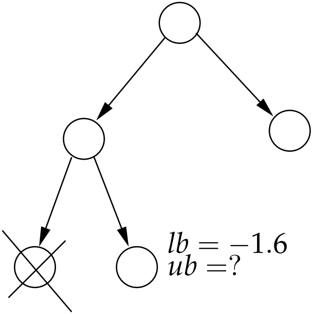

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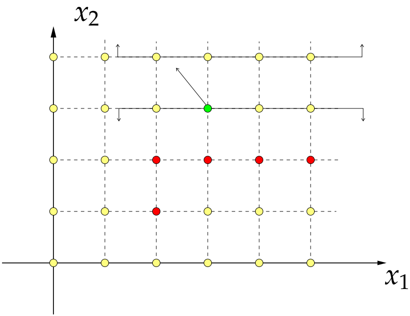

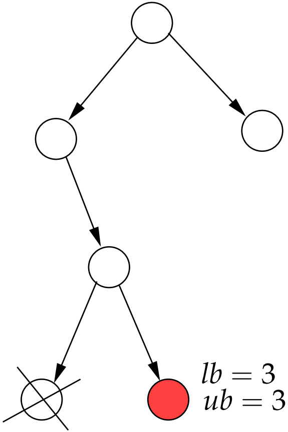

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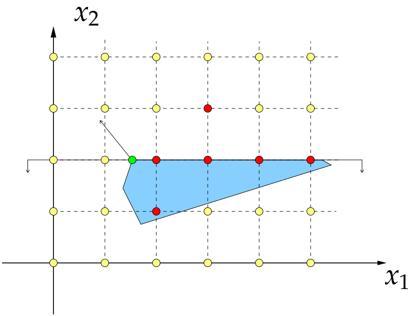

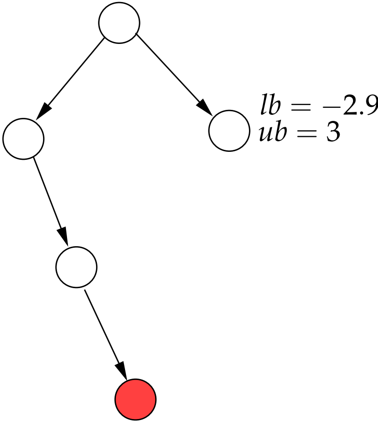

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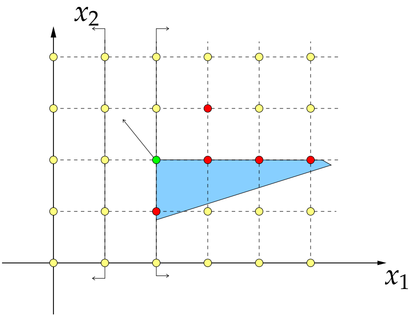

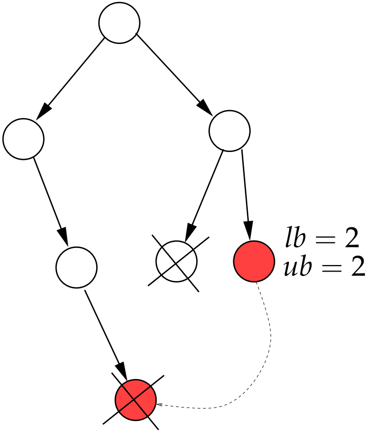

The **Bound** in Branch&Bound

In practice, the upper bound is very useful! Suppose you just found an **upper** bound of 194.

- P_3 has a **lower** bound of 146, P_4 of 203
- 203 is a lower bound for P_4 $\Rightarrow$ any feasible solution of P_4 has objective function value worse than 203 (i.e., ≥ 203)
- We already have something better (194) $\Rightarrow$ discard P_4

How to find upper bounds?

Example: Knapsack problem

$$\begin{array}{rrrrrrl}
\min & 3x_1 & +5x_2 & +8x_3 & +6x_4 & +12x_5 & \\
(IP_0) & 10x_1 & +7x_2 & +7x_3 & +5x_4 & +3x_5 & \geq 22 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \in \{0,1\}
\end{array}$$

The general Knapsack problem is:

$$\begin{array}{ll}
\min & \sum_{i=1}^{n} w_i x_i \\
 & \sum_{i=1}^{n} c_i x_i \geq C \\
 & x_i \in \{0,1\} \qquad \forall i = 1, 2 \ldots, n
\end{array}$$

Suppose that variables are sorted in non-decreasing order w.r.t. weight/value[2]. The LP relaxation is solved as follows:

1. Set $R := C$
2. **for** $i : 1, 2 \ldots, n$
3. **if** $c_i \leq R$, **then** $x_i := 1$; $R := R - c_i$
4. **else** $x_i := R/c_i$; **stop**

[2] Those with a small weight/value ratio are more likely to be chosen.

Solve the LP relaxation

$$
\begin{array}{rrrrrrl}
\min & 3x_1 & +5x_2 & +8x_3 & +6x_4 & +12x_5 & \\
(LP_0) & 10x_1 & +7x_2 & +7x_3 & +5x_4 & +3x_5 & \geq 22 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \in [0,1]
\end{array}
$$

- Solve this by the greedy method.
- Solution: $x_0^* = (1, 1, \frac{5}{7}, 0, 0)$, $V_0^* = 13\frac{5}{7}$.
- V_0^* - the lower bound for the problem IP_0.
- Split the problem into two cases: $X_3 = 1$ $(IP_1)$and $X_3 = 0$ (IP_2) and consider the LP relaxations LP_1 and LP_2.

Consider subproblem

Solve LP_1 problem $X_3 = 1$ (we will consider LP_2 later)

$$\begin{array}{rrrrrrl} \min & 3x_1 & +5x_2 & +8 & +6x_4 & +12x_5 & \\ (LP_1) & 10x_1 & +7x_2 & +7 & +5x_4 & +3x_5 & \geq 22 \\ & x_1, & x_2, & & x_4, & x_5 & \in [0,1] \end{array}$$

- Solve this by the greedy method.
- Solution: $x_1^* = (1, \frac{5}{7}, 1, 0, 0)$, $V_1^* = 14\frac{4}{7}$.
- V_1^* - the lower bound for the problem IP_1.
- Split the problem into two cases: $X_2 = 1$ and $X_2 = 0$ and consider the LP relaxations LP_3 and LP_4.

Consider subproblem

Solve LP_3, $X_2 = 1$, $X_3 = 1$.

$$\begin{array}{rrrrrrl} \min & 3x_1 & +5 & +8 & +6x_4 & +12x_5 & \\ (LP_1) & 10x_1 & +7 & +7 & +5x_4 & +3x_5 & \geq 22 \\ & x_1, & & & x_4, & x_5 & \in [0,1] \end{array}$$

- Solve this by the greedy method.
- Solution: $x_3^* = (\frac{8}{10}, 1, 1, 0, 0)$, $V_3^* = 15\frac{2}{10}$.
- V_3^* - the lower bound for the problem IP_3.
- Split the problem into two cases: $X_2 = 1$ and $X_2 = 0$ and consider the LP relaxations L_3 and L_4.

Consider subproblem

Solve LP_3, $X_2 = 0$, $X_3 = 1$.

$$\begin{array}{rccccl}
\min & 3x_1 & +8 & +6x_4 & +12x_5 & \\
(LP_1) & 10x_1 & +7 & +5x_4 & +3x_5 & \geq 22 \\
 & x_1, & & x_4, & x_5 & \in [0,1]
\end{array}$$

- Solve this by the greedy method.
- Solution: $x_4^* = (1,0,1,1,0)$, $V_4^* = 17$.
- V_4^* - the upper bound for the whole IP problem! Also is the optimal solution to IP_4.

Example: Knapsack problem

Solve LP_2 problem $X_3 = 0$

$$\begin{array}{rrrrrl} \min & 3x_1 & +5x_2 & +6x_4 & +12x_5 & \\ (LP_1) & 10x_1 & +7x_2 & +5x_4 & +3x_5 & \geq 22 \\ & x_1, & x_2, & x_4, & x_5 & \in [0,1] \end{array}$$

- Solve this by the greedy method.
- Solution: $x_2^* = (1,1,0,1,0)$, $V_2^* = 14$.
- V_2^* is the new upper bound for the entire problem!! Also optimal solution to IP_2.
- Key observation: lower bound for IP_3 is bigger than upper bound for the entre problem. DONE!!!!!

Branch&Bound

- $z^{\text{ub}} = +\infty$
- $\mathcal{L} \leftarrow \{P\}$
- **while** $\mathcal{L} \neq \emptyset$

    Choose P' from $\mathcal{L}$ and set $\mathcal{L} = \mathcal{L} \setminus \{P'\}$

    Relax $P' \rightarrow$ obtain LP'

    solve LP', obtain solution $x^{LP'}$ and lower bound $z^{LP'}$

    look for solution feasible for P', obj. $z^{P'}$

    if $z^{P'} < z^{\text{ub}}$, set $z^{\text{ub}} \leftarrow z^{P'}$

    if $z^{LP'} < z^{\text{ub}}$ and $x^{LP'}$ infeasible for P

        choose $x_i : x_i^{LP'} \notin \mathbb{Z}$

        create $P'' : x_i :\leq \lfloor x_i^{LP'} \rfloor$

        create $P''' : x_i :\geq \lceil x_i^{LP'} \rceil$

        $\mathcal{L} \leftarrow \mathcal{L} \cup \{P'', P'''\}$