ISE429. Homework 4

1) (weight 0.15) You do three independent rolls of a dice. X is the value of the first roll and Y is the sum of all three rolls. What is $E(Y|X)$ and $E(X|Y)$?

Solution. The there dice values, X, Y, Z, are i.i.d. uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$; $EX = EU = EV = 3.5$. $Y = X + U + V$. Then, $E(Y|X) = X + 2 \cdot 3.5 = 7$, and by symmetry, $E(X|Y) = E(X|X + U + V) = (1/3)E(X + U + V|X + U + V) = (1/3)Y$. $\Box$

2) (weight 0.15) Suppose, $\{X_1, X_2, \ldots\}$ is a sequence of i.i.d. random variables. Suppose, for some fixed number t, $m(t) = Ee^{tX_1} < \infty$. Let $S_n = X_1 + \ldots + X_n$, $n \geq 1$. Show that $\{M_n = e^{tS_n}/m(t)^n\}$ is a martingale w.r.t. filtration $\{\mathcal{F}_n = \mathcal{F}_{X_1,\ldots,X_n}\}$.

Solution. $E[M_{n+1}|\mathcal{F}_n] = M_n E[e^{tX_{n+1}}/m(t)|\mathcal{F}_n] = M_n E[e^{tX_{n+1}}/m(t)] = M_n$, $n \geq 1$. Other conditions are easily checked as well. $\Box$

3) (weight 0.25) Consider a random walk $\{X_n,\ n = 0, 1, \ldots\}$ on the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, with transition probabilities $P_{i,i+1} = p$ and $P_{i,i-1} = q = 1 - p$, where $0 < p < 1$, $p \neq 1/2$. Suppose, this random walk starts from state $X_0 = a$, such that $0 < a < N$. Let T be the first time the random walk reaches 0 or N. What is $P\{X_T = 0\}$? What is $\lim_{N\to\infty} P\{X_T = 0\}$?
Comment: You have to substantiate all steps and conclusions you make.
Hint: Consider $M_n = (q/p)^{X_n}$.

Solution. Denote $\beta = q/p$; if $p > 1/2$, $\beta < 1$; if $p < 1/2$, $\beta > 1$. Then $M_n = \beta^{X_n}$. $\{M_n\}$ is a martingale. Indeed, $E[M_{n+1}|\mathcal{F}_n] = M_n \cdot (\beta \cdot p + \beta^{-1} \cdot q) = M_n$, $n \geq 0$. Other conditions are easily checked as well. T is a stopping time, and as we learned in class, for some $C > 0$ and $0 < \rho < 1$,

$$P\{T > n\} \leq C\rho^n.$$

(Because T is the random time to reach a "boundary" – subset of states – of a finite irreducible Markov chain.) Using this, we can verify conditions of and apply Opt. Sampling Th. 2:

$$EM_T = EM_0.$$

Denoting $z = P\{X_T = 0\}$, we obtain

$$z\beta^0 + (1 - z)\beta^N = \beta^a$$

Solve for z:

$$z = [\beta^a - \beta^N]/[1 - \beta^N].$$

$\lim_{N\to\infty} P\{X_T = 0\} = 1$ if $p < 1/2$.
$\lim_{N\to\infty} P\{X_T = 0\} = (q/p)^a$ if $p > 1/2$. $\Box$

4) (weight 0.25) $\{X_n\}$ is an irreducible discrete-time Markov chain on the finite state space $\mathcal{X}$. The matrix of transition probabilities is $P = (p_{xy})$. Let A be a fixed subset $A \subset \mathcal{X}$.

There is a function $f(x)$ defined for $x \in A$, and a function $g(x)$ defined on $\mathcal{X} \setminus A$. Suppose, there is a function $u(x)$, defined on $x \in \mathcal{X}$, which satisfies conditions:

$$u(x) = f(x), \quad x \in A,$$

$$[Pu](x) = u(x) + g(x), \quad x \in \mathcal{X} \setminus A,$$

where we used notation

$$[Pu](x) \equiv \sum_{y \in \mathcal{X}} p_{x,y} u(y).$$

Let $T = \min\{n \mid X_n \in A\}$. (Note, it is possible that $T = 0$.) Let $T_n = \min\{T, n\}$.
(a) Show that

$$M_n = u(X_{T_n}) - \sum_{j=0}^{T_n - 1} g(X_j), \quad n = 0, 1, 2, \ldots,$$

is a martingale w.r.t. filtration $\{\mathcal{F}_n = \mathcal{F}_{X_0,\ldots,X_n}\}$. (In the above display, the sum is 0 when $T_n = 0$.)
(b) Show that

$$u(x) = E\left[f(X_T) - \sum_{j=0}^{T-1} g(X_j) \mid X_0 = x \right].$$

Comment: You have to substantiate all steps and conclusions you make.

Solution.

(a) We have

$$M_{n+1} - M_n = I\{T > n\}(M_{n+1} - M_n) = I\{T > n\}(u(X_{n+1}) - g(X_n) - u(X_n)).$$

Then

$$\begin{aligned} E[M_{n+1} - M_n \mid \mathcal{F}_n] &= I\{T > n\} E[u(X_{n+1}) - g(X_n) - u(X_n) \mid \mathcal{F}_n] = \\ & I\{T > n\}\left[E[u(X_{n+1}) \mid \mathcal{F}_n] - g(X_n) - u(X_n)\right] = \\ &= I\{T > n\}\left[g(X_n) + u(X_n) - g(X_n) - u(X_n)\right] = 0, \end{aligned}$$

so

$$E[M_{n+1} \mid \mathcal{F}_n] = M_n.$$

Other conditions, $\mathcal{F}_n$-measurability of M_n and $E|M_n| < \infty$, are easily verified.

(b) Consider the process with a fixed initial state $X_0 = x$. T is a stopping time. Martingale $\{M_n\}$ and stopping time T satisfy conditions of Opt. Sampling Th. 2. Indeed,

$$|M_T| \leq \max_y u(y) + (\max_y g(y))T,$$

and $ET < \infty$ because we learned in class that for some $C > 0$ and $0 < \rho < 1$, $P\{T > n\} \leq C\rho^n$. (Because T is the random time to reach a "boundary" – subset of states – of a finite irreducible Markov chain.) This implies $E|M_T| < \infty$. Also,

$$E|M_n| I\{T > n\} \leq \max_y u(y) + (\max_y g(y)) n P\{T > n\} \to 0, \ n \to \infty.$$

By Opt. Sampling Th. 2, $u(x) = EM_0 = EM_T$, which gives the desired equation. $\square$

5) (weight 0.20) **Wald's identity.** *Suppose,* $\{X_1, X_2, \ldots\}$ *is a sequence of i.i.d. random variables, such that* $E|X_1| < \infty$, $EX_1 = \mu$. *Suppose,* $T \geq 1$ *is a stopping time w.r.t. filtration* $\{\mathcal{F}_n = \mathcal{F}_{X_1,\ldots,X_n}\}$, *such that* $ET < \infty$. *Then* $E(X_1 + \ldots + X_T) = \mu ET$.
You need to fill in some steps of the proof (highlighted in bold below), that we did not do in class.
Proof.
Step 1. Without loss of generality, it is sufficient to prove the result for the case $EX_1 = \mu = 0$. **Explain why.** For the rest of the proof we assume that.
Step 2. Define $Y = |X_1| + \ldots + |X_T|$. Then $EY < \infty$.
Step 3. Define $T_n = \min\{T, n\}$, $M_n = X_1 + \ldots + X_{T_n}$. Then, $\{M_n\}$ is uniformly integrable.
Step 4. $\{M_n\}$ is a martingale w.r.t. $\{\mathcal{F}_n\}$. **Prove this.**
Step 5. $E(X_1 + \ldots + X_T) = EM_T = 0$. **Prove this.**
$\square$

Solution. $\{M_n\}$ is a martingale:

$$E[M_{n+1} - M_n \mid \mathcal{F}_n] = E[I\{T > n\}X_{n+1} \mid \mathcal{F}_n] = I\{T > n\}E[X_{n+1} \mid \mathcal{F}_n] = 0,$$

and other conditions, $\mathcal{F}_n$-measurability of M_n and $E|M_n| < \infty$, are easily verified. It is a uniformly integrable martingale, and $E|M_T| \leq EY < \infty$. Opt. Sampling Th. 3 implies that $EM_T = EM_1 = EX_1 = 0$. $\square$