ISE 426
Optimization models and applications

Lecture 21 — November 18, 2014

- Intro to Stochastic Programming (SP)

Reading:

- Book by Kall & Wallace (pdf – Chapter 1 up to 1.6)
- J.R. Birge, F. Louveaux, Stochastic Programming

Decision problems under uncertainty

In all problems we've seen so far, we assumed 100% knowledge of the parameters (production capacity, customer demand, etc.)

In this context, a global solution to an Optimization problem is known to be the best possible thing to do

However, perfect knowledge of the parameters is unrealistic:

- We don't know what our competitor, customer, even co-worker or boss, will decide
- Nature doesn't usually tell us in advance what it will do: e.g., weather is never certain
- Parameters are often estimated, i.e., given with a level of accuracy $< 100\%$

Decision problems under uncertainty

Important: both the parameters and the variables are unknown in advance. However,

- the model's variables are something **we** decide

i.e. **we** find the right ones if we have the right tools

- the model's parameters are **not** under our control: if we treated them as variables, we'd find the ideal (and unrealistic) situation

e.g. the competitor goes bankrupt AND the temperature stays good all winter AND our employees decide to cut their salary AND we win the lottery AND…

Example: the uncertain knapsack problem

At a flea market in Rome, you spot n objects (old pictures, a vessel, rusty medals…) that you could re-sell in your antique shop for about double the price.

- You want these objects to pay for your flight ticket to Rome, which cost C.
- Also, your backpack can carry all of them, but you don't want it heavy, so you want to buy the objects that will load your backpack as little as possible.

How do you solve this problem?

Example: the uncertain knapsack problem

Each object $i = 1, 2, \ldots, n$ has a price p_i and a weight w_i.

- Variables: one variable x_i for each $i = 1, 2 \ldots, n$. This is a "yes/no" variable, i.e., either you take the i-th object or not.
- Constraint: total revenue must be at least C

  The revenue for the i-th item can be between $0.8p_i$ and $1.1p_i$, or $\alpha_i p_i$, with $0.8 \leq \alpha_i \leq 1.1$
- Objective function: the total weight

Example: the uncertain knapsack problem

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} w_i x_i \\
& \sum_{i=1}^{n} \alpha_i p_i x_i \geq C \\
& x_i \in \{0,1\} \quad \forall i = 1,2,\ldots,n
\end{aligned}
$$

- If α_i were variables, our solver would find an optimal solution where $\alpha_i = 1.1$ for all i.
- α_i is **not** a variable – it's an unknown parameter!
- Robust optimization - worst case scenario: simply set all α_is to 0.8 and solve the problem.
- We may look at each object and see the usual value of α_i for that object.

How to handle uncertainty?

Handling all sources of uncertainty is almost impossible, and we better model and simplify them.

- scenarios: enumerate a few “situations” that may happen and solve a problem for each of them

⇒ each scenario gives us some insight on what goes wrong and how to tackle it

- stages: some decisions have to be made now, some others may be made later
- however, the later decisions are influenced by what we decide now (and the events between “now” and “later”)

⇒ We have to model that influence, too – it’s as if we are anticipating the later decisions by taking into account what happens in the meantime

Farmer excercise

SP solves optimization problems with stochastic info.

- You grow **wheat**, **corn**, and **sugar beet** on 500 acres of land
- You should decide how many acres to use for each crop
- Planting an acre costs \$150, \$230, and \$260, respectively
- Need at least 200t of wheat and 240t of corn for cattle feed
- Eccess production is sold at \$170/t and \$150/t, resp.
- If less is produced, it is bought at \$238/t and \$210/t, resp.
- Sugar beet sells at \$36/t up to 6000t, and at \$10/t above that quota

The **average** yields of crop are:

- 2.5t/acre for wheat
- 3t/acre for corn
- 20t/acre for beet

Depending on how good the weather is, these yields may decrease or increase by 20%. How to solve this problem?

Model

Variables:

x_1: acres for growing wheat

x_2: acres for growing corn

x_3: acres for growing (sugar) beet

w_1: tons of wheat exceeding 200t (to be sold)

w_2: tons of corn exceeding 240t (to be sold)

w_3: tons of beet below 6,000t (to be sold at 36$/t)

w_4: tons of beet above 6,000t (to be sold at 10$/t)

y_1: tons of wheat to be bought (when $x_1 < 200$t)

y_2: tons of corn to be bought (when $x_2 < 240$t)

Objective function (to be maximized):

$$\begin{array}{lr} 170w_1 + 150w_2 + 36w_3 + 10w_4 & \text{crop sale} \\ -(150x_1 + 230x_2 + 260x_3) & \text{planting costs} \\ -(238y_1 + 210y_2) & \text{purchased wheat/corn} \end{array}$$

Model

Constraints:

$$
\begin{aligned}
&x_1 + x_2 + x_3 \leq 500 && \text{total area}\\
&w_1 \leq 2.5x_1 - 200 + y_1 && \text{excess wheat}\\
&w_2 \leq 3x_2 - 240 + y_2 && \text{excess corn}\\
&20x_3 = w_3 + w_4 && \text{total beet}\\
&y_1 \geq 200 - 2.5x_1 && \text{purchased wheat}\\
&y_2 \geq 240 - 3x_2 && \text{purchased corn}\\
&w_3 \leq 6,000 && \text{bound on the quota}\\
&x_1, x_2, x_3, w_1, w_2, w_3, w_4, y_1, y_2 \geq 0
\end{aligned}
$$

Optimal solution: $(x_1, x_2, x_3) = (120, 80, 300)$, with $w_1 = 100$, $w_3 = 6,000$, and a total profit of 118,600$.

Model under uncertainty

Scenario #1: yields below average (-20%).
Constraints (only those that change):

$$
\begin{array}{lr}
w_1 \leq 2x_1 - 200 + y_1 & \text{excess wheat} \\
w_2 \leq 2.4x_2 - 240 + y_2 & \text{excess corn} \\
16x_3 = w_3 + w_4 & \text{total beet} \\
y_1 \geq 200 - 2x_1 & \text{purchased wheat} \\
y_2 \geq 240 - 2.4x_2 & \text{purchased corn}
\end{array}
$$

Optimal solution: $(x_1, x_2, x_3) = (100, 25, 375)$, with $w_3 = 6,000$, $y_2 = 180$, and a total profit of 59,950\$.

Scenario #2: average yields. Same solution as before (118,600\$).

Model under uncertainty

Scenario #3: yields above average (+20%).
Constraints (only those that change):

$$
\begin{array}{lr}
w_1 \leq 3x_1 - 200 + y_1 & \text{excess wheat} \\
w_2 \leq 3.6x_2 - 240 + y_2 & \text{excess corn} \\
24x_3 = w_3 + w_4 & \text{total beet} \\
y_1 \geq 200 - 3x_1 & \text{purchased wheat} \\
y_2 \geq 240 - 3.6x_2 & \text{purchased corn}
\end{array}
$$

Optimal solution: $(x_1, x_2, x_3) = (183.3, 66.7, 250)$, with $w_1 = 350$, $w_3 = 6,000$, and a total profit of 167,666.67\$.

Which one is the right solution?

The pessimistic would choose the solution of scenario #1.

- At least we're sure we'll make at least 59,950$.
- If the weather turns out to be OK, the "pessimistic" choice gives us a return of 86,600$.
- If the weather is great, we'd make 113,250$.

The neither lucky nor unlucky would go for scenario #2.

- If the weather is bad, we make 55,120$.
- If the weather is OK, we make 118,600$.
- In the best case, we make 148,000$.

Those feeling lucky would choose the solution of scenario #3.

- If the weather is bad, we make 47,700$.
- If the weather is OK, we make 107,683.33$.
- In the best case, we make 167,666.67$.

Stochastic approach

We have nine values for the profit, and don't really know which one makes more sense.

Suppose that each scenario has the same probability: $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.
How can we get a solution that considers both three scenarios?

- The decision we want to make **now**, i.e., the area for wheat, corn, and beet (x_1, x_2, and x_3) are still variables
- For those quantities that depend on Nature, and that we have no control on, there is one variable for each scenario:

e.g. for w_1 there are three new variables w_{11}, w_{12}, and w_{13}

- each w_{1j} is the exceeding wheat if scenario j is realized
- these are second stage variables, in that their value is a consequence of the decisions of Nature, but they still influence our decision today.

How does the model change?

It clearly gets more complicated...
Variables (j is a scenario, $j \in \{1, 2, 3\}$):

- x_1: acres for growing wheat
- x_2: acres for growing corn
- x_3: acres for growing (sugar) beet
- w_{1j}: tons of wheat exceeding 200t (to be sold)
- w_{2j}: tons of corn exceeding 240t (to be sold)
- w_{3j}: tons of beet below 6,000t (to be sold at 36\$/t)
- w_{4j}: tons of beet above 6,000t (to be sold at 10\$/t)
- y_{1j}: tons of wheat to be bought (when $x_1 < 200$t)
- y_{2j}: tons of corn to be bought (when $x_2 < 240$t)

Model

Objective function (to be maximized):

$$
\begin{array}{ll}
 & -(150x_1 + 230x_2 + 260x_3) \\
+\frac{1}{3} & \left[(170w_{11} + 150w_{21} + 36w_{31} + 10w_{41}) - (238y_{11} + 210y_{21})\right] \\
+\frac{1}{3} & \left[(170w_{12} + 150w_{22} + 36w_{32} + 10w_{42}) - (238y_{12} + 210y_{22})\right] \\
+\frac{1}{3} & \left[(170w_{13} + 150w_{23} + 36w_{33} + 10w_{43}) - (238y_{13} + 210y_{23})\right]
\end{array}
$$

Constraints:

$$
\begin{array}{lr}
x_1 + x_2 + x_3 \leq 500 & \text{total area} \\
w_{11} \leq 2x_1 - 200 + y_{11} & \text{excess wheat} \\
w_{12} \leq 2.5x_1 - 200 + y_{12} & \\
w_{13} \leq 3x_1 - 200 + y_{13} & \\
w_{21} \leq 2.4x_2 - 240 + y_{21} & \text{excess corn} \\
w_{22} \leq 3x_2 - 240 + y_{22} & \\
w_{23} \leq 3.6x_2 - 240 + y_{23} &
\end{array}
$$

Stochastic Programming model (cont'd)

Constraints:

$$
\begin{aligned}
16x_3 &= w_{31} + w_{41} && \text{total beet} \\
20x_3 &= w_{32} + w_{42} && \\
24x_3 &= w_{33} + w_{43} && \\
y_{11} &\geq 200 - 2x_1 && \text{purchased wheat} \\
y_{12} &\geq 200 - 2.5x_1 && \\
y_{13} &\geq 200 - 3x_1 && \\
y_{21} &\geq 240 - 2.4x_2 && \text{purchased corn} \\
y_{22} &\geq 240 - 3x_2 && \\
y_{23} &\geq 240 - 3.6x_2 && \\
w_{3j} &\leq 6,000 && \forall j \in \{1,2,3\} \\
x_1, x_2, x_3 &\geq 0 && \\
w_{1j}, w_{2j}, w_{3j}, w_{4j}, y_{1j}, y_{2j} &\geq 0 && \forall j \in \{1,2,3\}
\end{aligned}
$$

Solution

$(x_1, x_2, x_3) = (170, 80, 250)$. Expected profit: 108,390\$.

scenario	w_1	w_2	w_3	w_4	y_1	y_2	profit
1	140	0	4,000	0	0	48	48,820\$
2	225	0	5,000	0	0	0	109,350\$
3	310	48	6,000	0	0	0	167,000\$

What do 108,390\$ mean? If the farmer repeated this choice for the next n years, under the same conditions he would make, on average, 108,390\$ a year.

Why is it not equal to 118,600\$, the profit in scenario 2? Because we considered uncertainty.

Averages and forecasts

What if we averaged the (deterministic) profits from the three scenarios?

$$\frac{59,950\$ + 118,600\$ + 167,666.67\$}{3} \approx 115,406\$,$$

yet another number. That would be the average profit over n years if we knew in advance, each year, how the weather would be and choose x_1, x_2, x_3 consequently.

The difference $115,406\$ - 108,390\$ = 7,016\$$ is called *Expected Value of Perfect Information* (EVPI).

- It's what we "lose" for being realistic…
- It's what we'd gain from **knowing** the weather before planting.

Averages and forecasts, 2

If we were lazy and considered **only** scenario #2, we could just sit and watch the average over n years. We'd have a profit of

- $55,120\$$ in bad years
- $118,600\$$ in OK years
- $148,000\$$ in very good years

Given that they occur with **known** probability $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$, we'd observe an average profit of

$$\frac{55120\$ + 118,600\$ + 148,000\$}{3} \approx 107,240\$,$$

which is clearly below $115,406\$$ because we bet on OK weather every year instead of using perfect knowledge of the weather.

The difference $108,390\$ - 107,240\$ = 1,150\$$ is the *value of the stochastic solution* (VSS). It tells you how much you gain if you use Stochastic Programming.

To recap

- Case #1: We know all parameters (the weather every year). We have perfect information and make the decision with a single, deterministic optimization model. We make a lot of money (in our dreams). [115,406$]
- Case #2: We don't know the parameters, but we pick the average value and solve a deterministic model. The solution is optimal only assuming those values of the parameters, which won't occur all the time. We won't make a lot of money. [107,240$]
- Case #3: We don't know the parameters, but we formulate a model that considers all possible events and their impact on our solution. [108,390$]

Note: we did make an assumption though. We assumed to know the **probabilities** of the three scenarios.

A more prudent farmer - robust optimization

…maximizes the minimum return instead of the expected value. Objective function (to be maximized):

$$
\begin{aligned}
& -(150x_1 + 230x_2 + 260x_3) \\
+\min \quad & \big[(170w_{11} + 150w_{21} + 36w_{31} + 10w_{41}) - (238y_{11} + 210y_{21}), \\
& (170w_{12} + 150w_{22} + 36w_{32} + 10w_{42}) - (238y_{12} + 210y_{22}), \\
& (170w_{13} + 150w_{23} + 36w_{33} + 10w_{43}) - (238y_{13} + 210y_{23})\big]
\end{aligned}
$$

Optimal solution: $(x_1, x_2, x_3) = (100, 25, 375)$, with $w_3 = 6,000$, $y_2 = 180$, and a total profit of 59,950$.

- Incidentally, the same solution as assuming scenario #1.
- Not always the case: here we are looking at the worst-case scenario as it would arise from the second stage variables,
- and it turns out that the bad weather scenario is the worst-case scenario (obvious)

A quick comparison of RO and SP

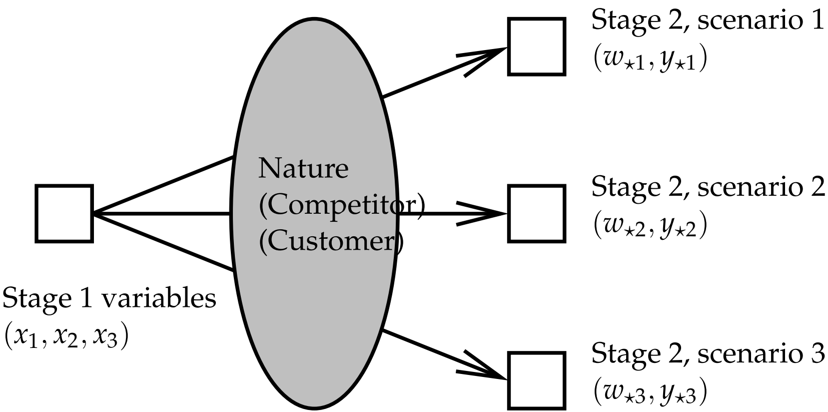
Robust Optimization:

- Robust optimization considers a set of possible parameter values and optimizes the **worst-case scenario**
- Gives a guarantee that the outcome will not be worse then estimted.
- No need for probability (distribution) estimates.
- Can be too pessimistic if all possible values are considered.
- Models may get more complex, but not larger.

Stochastic Porgramming:

- Optimizes average case.
- Needs probability estimtes.
- Not as pessimistic.
- Usually does not complicate type of models, but makes them much larger.

Scenarios and stages



Review: EVPI

The *Expected Value of Perfect Information* is the difference between what you'd make **with no uncertainty** and what you expect to make – the solution of an SP.

- Bad weather? Plant $(100, 25, 375)$ and make a 59,950$ profit
- OK weather? Plant $(120, 80, 300)$, make a 118,600$ profit
- Good weather? Plant $(183.3, 66.7, 250)$, make 167,666.67$
- The three scenarios have probability of $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

$\Rightarrow$ On average, we make $\frac{1}{3}59,950\$ + \frac{1}{3}118,600\$ + \frac{1}{3}167,666.67\$ = 115,406\$$

- SP tells us we make 108,390$

$\Rightarrow$ $\text{EVPI} = 115,406\$ - 108,390\$ = 7,016\$$

Review: VSS

Value of Stochastic Solution: the difference between the solution of an SP program and the expected value of the objective function when we fix parameters to **average** values and use the corresponding optimal solution.

- Assume weather will be OK all the time

⇒ Plant $(120, 80, 300)$, no matter what

- Bad weather? Make a 55,120$ profit
- OK weather? Make a 118,600$ profit
- Good weather? Make 148,000$
- The three scenarios have probability of $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

⇒ On average, we make
$\frac{1}{3}55,120 + \frac{1}{3}118,600 + \frac{1}{3}148,000\$ = 107,240\$$

- SP tells us we make 108,390$

⇒ $\text{VSS} = 108,390\$ - 107,240 = 1,150\$$

From deterministic…

Consider a set S of scenarios, a set of n first stage variables x and a set of p second stage (or recourse) variables, y.
We have to turn the *deterministic* model,

$$
\begin{array}{lllll}
\min & c_1 x_1 & \ldots + c_n x_n & + d_1 y_1 & \ldots + d_p y_p \\
 & a_{11} x_1 & \ldots + a_{1n} x_n & + b_{11} y_1 & \ldots + b_{1p} y_p \leq f_1 \\
 & a_{21} x_1 & \ldots + a_{2n} x_n & + b_{21} y_1 & \ldots + b_{2p} y_p \leq f_2 \\
 & \vdots & & & \\
 & a_{m1} x_1 & \ldots + a_{mn} x_n & + b_{m1} y_1 & \ldots + b_{mp} y_p \leq f_m,
\end{array}
$$

into a **stochastic** due to uncertain parameters (c, d, a, b, f).

- First stage variables are decisions to be made now, regardless of the scenario that will actually be realized.
- Recourse variables represent decisions to be made after part of the uncertainty is revealed.

… to Stochastic models

Introduce a set of recourse variables $y_1^s \dots y_p^s$ for each $s \in S$.

Consider an uncertain value of the parameter: instead of the (c, d, a, b, f) we have one $(c^s, d^s, a^s, b^s, f^s)$ for each $s \in S$.

If every $s \in S$ has probability p_s, the **expected value** of the objective function is

$$\sum_{s \in S} p_s (c_1^s x_1 \dots + c_n^s x_n + d_1^s y_1^s \dots + d_p^s y_p^s)$$

And we rewrite all constraints as

$$\begin{array}{llllll}
a_{11}^s x_1 & \dots + a_{1n}^s x_n & + b_{11}^s y_1^s & \dots + b_{1p}^s y_p^s \leq f_1^s & & \forall s \in S \\
a_{21}^s x_1 & \dots + a_{2n}^s x_n & + b_{21}^s y_1^s & \dots + b_{2p}^s y_p^s \leq f_2^s & & \forall s \in S \\
\vdots & & & & & \\
a_{m1}^s x_1 & \dots + a_{mn}^s x_n & + b_{m1}^s y_1^s & \dots + b_{mp}^s y_p^s \leq f_m^s & & \forall s \in S
\end{array}$$

Example: A facility location problem

A company wants to open a few malls, choosing from a set I of potential locations, to serve a set J of customers (towns).

- Each mall i, if open, has a capacity of p_i
- There is a transportation cost d_{ij} between $i \in I$ and $j \in J$
- Building a mall at i costs c_i
- Each town $j \in J$ has a demand f_j to be satisfied by **one** of the (open!) facilities
- A demand not served costs the company g per unit

Deterministic model

Variables:

- x_i: open a mall at i
- y_{ij}: mall i serves town j
- z_i unsatisfied demand for mall i

Constraints:

- Town j is served by mall i if mall i is open: $y_{ij} \leq x_i$, for all malls $i \in I$ and towns $j \in J$
- Town $j \in J$ is served by one mall: $\sum_{i \in I} y_{ij} = 1$ for all $j \in J$
- Definition of variable z_i, i.e., demand that mall i does not satisfy: $z_i \geq \sum_{j \in J} f_j y_{ij} - p_i$
- $z_i \geq 0 \forall i \in I; \quad x_i \in \{0, 1\} \forall i \in I; \quad y_{ij} \in \{0, 1\} \forall i \in I, j \in J$

Objective function: $\sum_{i \in I} c_i x_i + g \sum_{i \in I} z_i + \sum_{i \in I} \sum_{j \in J} d_{ij} y_{ij}$

Deterministic model

$$
\begin{array}{lll}
\min & \sum_{i\in I} c_i x_i + g\sum_{i\in I} z_i + \sum_{i\in I}\sum_{j\in J} d_{ij} y_{ij} & \\
 & y_{ij} \le x_i & \forall i \in I, \forall j \in J \\
 & z_i \ge \sum_{j\in J} f_j y_{ij} - p_i & \forall i \in I \\
 & z_i \ge 0, \quad x_i \in \{0,1\} & \forall i \in I \\
 & y_{ij} \in \{0,1\} & \forall i \in I, j \in J
\end{array}
$$

Suppose the demand f_i is not known, and is assumed of three types (*scenarios*): f_i^A, f_i^B, and f_i^C, with probabilities p^A, p^B, and p^C.

- x_i are first-stage variables: a decision to be made now
- z_i are recourse variables: they depend on actual demand
- y_{ij} could be both! Let's assume they are chosen by the population according to demand $\Rightarrow$ recourse variables

Stochastic model I

$$
\begin{array}{lll}
\min & \sum_{i \in I} c_i x_i + & \\
& +p^A (g \sum_{i \in I} z_i^A + \sum_{i \in I} \sum_{j \in J} d_{ij} y_{ij}^A) & \\
& +p^B (g \sum_{i \in I} z_i^B + \sum_{i \in I} \sum_{j \in J} d_{ij} y_{ij}^B) & \\
& +p^C (g \sum_{i \in I} z_i^C + \sum_{i \in I} \sum_{j \in J} d_{ij} y_{ij}^C) & \\
& y_{ij}^A \leq x_i & \forall i \in I, \forall j \in J \\
& y_{ij}^B \leq x_i & \forall i \in I, \forall j \in J \\
& y_{ij}^C \leq x_i & \forall i \in I, \forall j \in J \\
& \sum_{i \in I} y_{ij}^A = 1 & \forall j \in J \\
& \sum_{i \in I} y_{ij}^B = 1 & \forall j \in J \\
& \sum_{i \in I} y_{ij}^C = 1 & \forall j \in J \\
& z_i^A \geq \sum_{j \in J} f_j^A y_{ij}^A - p_i & \forall i \in I \\
& z_i^B \geq \sum_{j \in J} f_j^B y_{ij}^B - p_i & \forall i \in I \\
& z_i^C \geq \sum_{j \in J} f_j^C y_{ij}^C - p_i & \forall i \in I \\
& z_i^A, z_i^B, z_i^C \geq 0, \quad x_i \in \{0, 1\} & \forall i \in I \\
& y_{ij}^A, y_{ij}^B, y_{ij}^C \in \{0, 1\} & \forall i \in I, j \in J
\end{array}
$$

Stochastic model II

Case II: in another problem, y_{ij} may correspond to a road from i to j (i.e. to be decided now)

$$
\begin{array}{lll}
\min & \sum_{i \in I} c_i x_i + \sum_{i \in I} \sum_{j \in J} d_{ij} y_{ij} + & \\
& + p^A (g \sum_{i \in I} z_i^A) + & \\
& + p^B (g \sum_{i \in I} z_i^B) + & \\
& + p^C (g \sum_{i \in I} z_i^C) & \\
& y_{ij} \leq x_i & \forall i \in I, \forall j \in J \\
& \sum_{i \in I} y_{ij} = 1 & \forall j \in J \\
& z_i^A \geq \sum_{j \in J} f_j^A y_{ij} - p_i & \forall i \in I \\
& z_i^B \geq \sum_{j \in J} f_j^B y_{ij} - p_i & \forall i \in I \\
& z_i^C \geq \sum_{j \in J} f_j^C y_{ij} - p_i & \forall i \in I \\
& z_i^A, z_i^B, z_i^C \geq 0, \quad x_i \in \{0, 1\} & \forall i \in I \\
& y_{ij} \in \{0, 1\} & \forall i \in I, j \in J
\end{array}
$$