ISE-429, Fall 2015. **Final Exam.** December 13, 2015.

General requirement. In problems 2-4 you have to derive and *substantiate* answers which have to be "computable expressions", even though you do not have to compute them. For example, $x = (5 - 2 + 7)^{1/2+9}$ or $x = \int_{-3}^{6} \exp(z^5)dz$ are "computable expressions". The important thing is to show clearly how you derive those expressions.

1

Consider a discrete time countable Markov chain, with state space and transition probabilities shown on Fig. 0. ("Right" and "left" branches of the graph are infinite; the branch going up is finite.) This Markov chain is obviously irreducible. Is it transient? or null recurrent? or positive recurrent? (You have to substantiate the answer.)

Solution

For an irreducible Markov chain, the existence of a stationary distribution is equivalent to positive recurrence. This chain has the following stationary distribution:

$$\pi(x) = Cf(x),$$

$$f(0) = 1,$$

$$f(1,1) = (1/3)/(3/5), \qquad f(1,i) = f(1,1)(2/3)^{i-1},\ i \geq 2,$$

$$f(2,1) = (1/3)/(4/5), \qquad f(2,i) = f(2,1)(1/4)^{i-1},\ i \geq 2,$$

$$f(3,1) = (1/3)/(1/8), \qquad f(3,i) = f(3,1)7^{i-1},\ i = 2, 3.$$

Clearly, $\sum_x f(x) < \infty$, so

$$C = [\sum_x f(x)]^{-1}.$$

Detailed balance equations for $\{\pi(x)\}$ are easily verified, so this is a stationary distribution. The MC is positive recurrent (and reversible w.r.t. the stationary distribution).

Alternatively, Lyapunov-Foster could be used as well. $\square$

2

X_n, $n = 0, 1, 2, \ldots$, is a simple random walk on the integers $\mathcal{X} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, starting at $X_0 = 3$. (The probabilities of jumping left and right are 1/2.) Let $T = \min\{n \mid X_n = 0 \text{ or } X_n = 10\}$. Find $\mathbb{E}[X^2_{2T-2}]$.

Comment: If you want, you can use the following fact without proof. (Although, you should be able to prove it as well.) Suppose Y is a non-negative integer-valued random variable, and it is such that for any integer $n \geq 0$, $\mathbb{P}\{Y > n\} \leq C\rho^n$, with some fixed $C > 0$ and $0 < \rho < 1$. Then for any integer $k \geq 0$, $\mathbb{E}Y^k < \infty$.

Solution

Consider the filtration $\{\mathcal{F}_n = \mathcal{F}_{X_0,\ldots,X_n},\ n = 0, 1, 2, \ldots\}$. We proved in class that: T is a stopping time; $\mathbb{E}T = 3 \cdot (10 - 3) = 21$; the process

$$M_n = X_n^2 - n, \quad n = 0, 1, 2, \ldots,$$

is a martingale w.r.t. filtration $\{\mathcal{F}_n\}$.

Therefore, if we can verify that $T_1 = 2T - 2$ is a stopping time w.r.t. filtration $\{\mathcal{F}_n\}$, and T_1 and $\{M_n\}$ satisfy conditions of the Optional Sampling Theorem 2, we will have

$$\mathbb{E}M_{T_1} = \mathbb{E}[X^2_{T_1}] - \mathbb{E}T_1 = \mathbb{E}M_0 = 3^2 = 9,$$

and then the answer is

$$\mathbb{E}[X^2_{2T-2}] = \mathbb{E}[X^2_{T_1}] = \mathbb{E}M_0 + \mathbb{E}T_1 = \mathbb{E}M_0 + \mathbb{E}(2T - 2) = 9 + 2 \cdot 21 - 2 = 49.$$

Proof that T_1 is a stopping time. Clearly, $T \geq 3$. Then $T_1 \geq 4$. For any $n \geq 4$, event $\{T_1 = n\}$ is equal to $\{T = (n+2)/2\}$. But $(n+2)/2 \leq n$ (for $n \geq 4$), and T is stopping time, so clearly $\{T_1 = n\}$ is $\mathcal{F}_n$-measurable (i.e. only depends on $X_0, \ldots, X_n$). So, T_1 is a stopping time.

Proof of conditions of Optional Sampling Theorem 2.
(a) Obviously, $T_1 = 2T - 2$ is finite w.p.1, because T is.
(b) We proved in class that for some fixed $C > 0$ and $0 < \rho < 1$,

$$\mathbb{P}\{T > n\} \leq C\rho^n, \quad n = 0, 1, 2, \ldots. \tag{1}$$

Now,

$$|M_{T_1}| = |X^2_{T_1} - T_1| \leq (3 + T_1)^2 + T_1 = (3 + (2T - 2))^2 + (2T - 2).$$

From the Comment and (1), we see that $\mathbb{E}|M_{T_1}| < \infty$.
(c) We have

$$|M_n| = |X_n^2 - n| \leq (3 + n)^2 + n.$$

Using this and (1),

$$\mathbb{E}[|M_n I\{T_1 > n\}] \leq [(3 + n)^2 + n]\mathbb{P}\{T_1 > n\} = [(3 + n)^2 + n]\mathbb{P}\{T > n/2 + 1\} \to 0, \quad n \to \infty.$$

□

3

Consider the continuous time Markov chain $\{X_t\}$ with state space and transition rates shown in Fig. 1. Suppose $X_0 = 5$. Let $T_2 \geq 0$ denote the random time when the process enters state 0 for the *second* time. (We say that the process *enters* state 0 at time t if at that time it makes a transition to 0 from some other state.) Find $\mathbb{E}T_2$.

Solution

$T_2 = T_1 + (T_2 - T_1)$, where T_1 is the time when the process enters 0 for the first time. Since the time the process spends in each state is exponentially distributed,

$$\mathbb{E}T_1 = 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10.$$

After the process enters state 0 at T_1, it spends average time $1/(1 + 11)$ in 0, and then goes to the left "loop" or right "loop" with probabilities $1/12$ and $11/12$, respectively. Therefore,

$$\mathbb{E}(T_2 - T_1) = 1/(1+11) + (1/12) \cdot (1/2 + 1/3 + 1/4 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10) +$$

$$(11/12) \cdot (1/12 + 1/13 + 1/14).$$

$\mathbb{E}T_2 = \mathbb{E}T_1 + \mathbb{E}(T_2 - T_1)$. $\square$

4

Let $W_t = (W_t^1, W_t^2)$ be a 2-dimensional standard Brownian motion. (Starts at $(0,0)$; variance parameter is 1.)
Find the probability that in the time interval $[0,1]$, W_t does not "hit" set G, consisting of two rays, as shown in Fig. 2. In other words, find

$$\mathbb{P}\{W_t \notin G, \quad 0 \leq t \leq 1\}.$$

Solution

Brownian motion is spherically symmetric; i.e. if we rotate the system of coordinates and consider the same process in new coordinates, it is also a Brownian motion (with the same variance parameter). So, the answer to the problem will not change if we rotate the set G by 45 degrees counter-clockwise about the origin, in other words replace G with G_2 shown in Fig.3. Thus, we need to find

$$\mathbb{P}\{W_t \notin G_2, \quad 0 \leq t \leq 1\} = \mathbb{P}\{\max_{[0,1]} W_t^1 < 1/\sqrt{2} \text{ and } \max_{[0,1]} W_t^2 < 1/\sqrt{2}\}$$

Using independence of W_t^1 and W_t^2, and applying reflection principle to each of them, we get

$$\mathbb{P}\{W_t \notin G_2, \quad 0 \leq t \leq 1\} = [1 - 2\bar{\Phi}(1/\sqrt{2})]^2,$$

where $\bar{\Phi}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$. $\square$