ISE 426
Optimization models and applications

Lecture 10-11 — October 1, 2015

- basic feasible solutions
- simplex method
- connection with dual variables

Reminders:

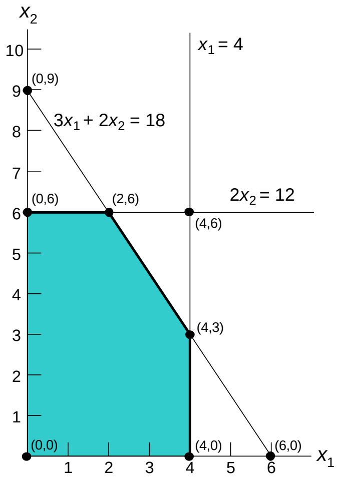
- Homework #2 is due in class 10/06. No late homework will be accepted!
- Quiz on 10/08, practice on 10/06.

Simplex Method

Example

$$\begin{array}{lrcrcr} \text{maximize} & 3x_1 & + & 5x_2 & & \\ \text{subject to} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

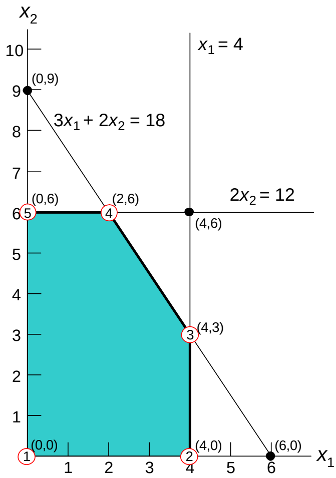
The lines are the **constraint boundaries**.



Corner-Point Solutions

5 **corner-point feasible (CPF) solutions**:

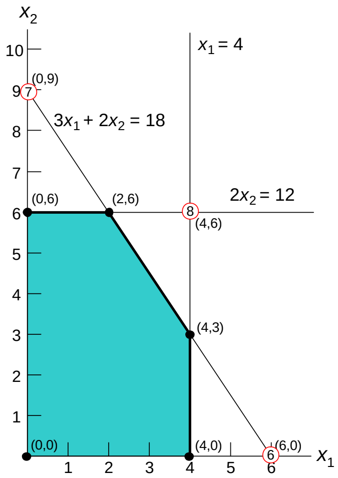
1. (0,0)
2. (4,0)
3. (4,3)
4. (2,6)
5. (0,6)



Corner-Point Solutions

3 **corner-point infeasible solutions**:

6. (6,0)
7. (0,9)
8. (4,6)

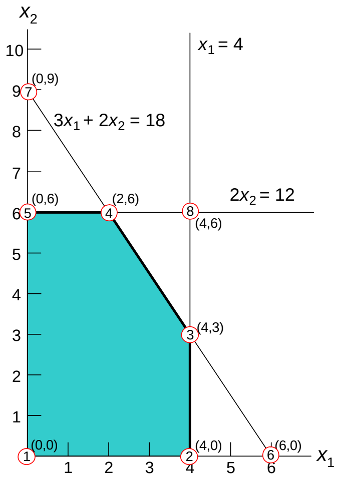


Corner-Point Solutions

Each corner-point solution (feasible or infeasible) lies at the intersection of two constraint boundaries.

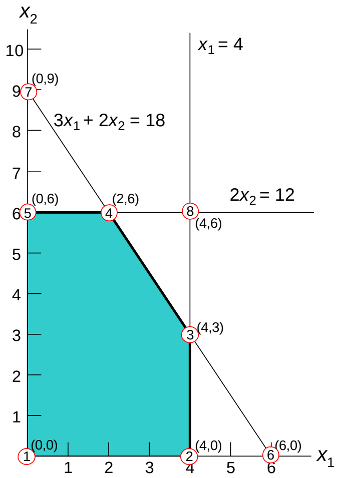
For an LP with n Decision Variables:

Each corner-point solution lies at the intersection of n constraint boundaries.



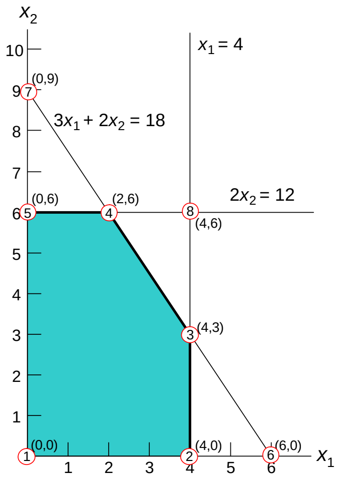
Adjacent CPFs

- For an LP with n decision variables, two CPF solutions are **adjacent** if they share $n-1$ constraint boundaries.
- Two adjacent CPF solutions are connected by an **edge** of the feasible region.



Adjacent CPFs

- For an LP with n decision variables, two CPF solutions are **adjacent** if they share $n-1$ constraint boundaries.
- Two adjacent CPF solutions are connected by an **edge** of the feasible region.
- (0,6) and (2,6) are adjacent.
- (2,6) and (4,3) are adjacent.

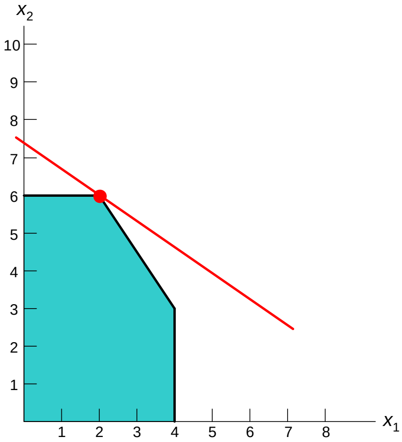


Why All the Fuss Over CPF Solutions?

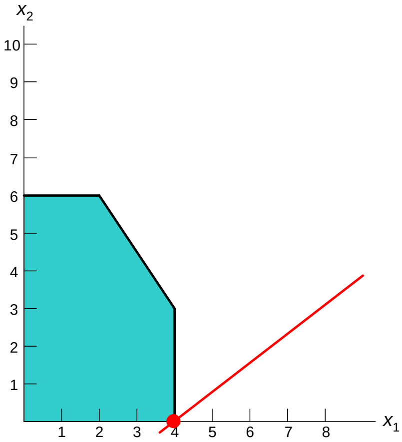
Important Property #1

If an LP has a single optimal solution, it is a CPF solution. If an LP has more than one optimal solution, at least two are CPF solutions.

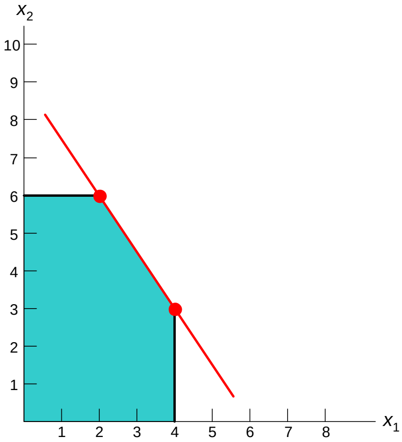
Why is it true?



Why is it true?



Why is it true?



How Many CPFs Are There?

- IP #1 means that we can focus only on CPF solutions and ignore the rest of the feasible region.
- There are an *infinite* number of feasible solutions
  - (assuming there are at least 2).
- There are a *finite* number of CPF solutions
  - (assuming feasible region is bounded and there are a finite number of constraints).
- That means we can focus on a *much smaller* set of possible answers.
- Can we just examine every CPF?

How Many CPFs? cont'd

- If there are n decision variables and m functional constraints, how many constraint boundaries are there?
- How many ways can we choose n constraint boundaries?
  - Answer:

$$\binom{m+n}{n} = \frac{(m+n)!}{m!n!}$$

- If $m = 50$ and $n = 50$, there are 10^{29} CPFs to examine
- If you could examine 1 billion CPFs per second, it would take you

$$3,170,979,198,376 \text{ years}$$

  to examine all of the CPFs.

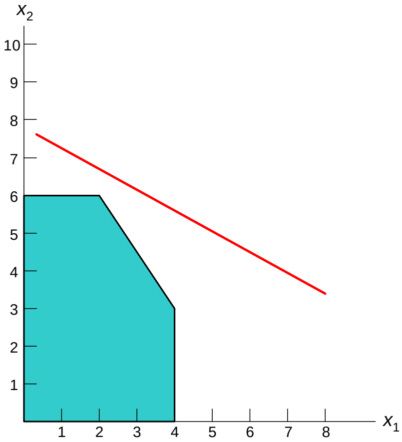
Why All the Fuss Over *Adjacent* CPF Solutions?

Important Property #2

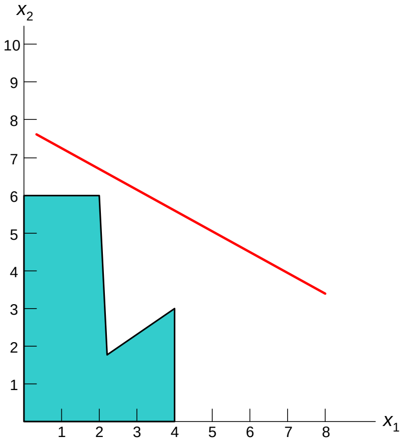
If a CPF solution has no *adjacent* CPF solutions that are better, then it must be an *optimal* solution.

In other words, if we find a CPF solution with no better neighbors, we can stop looking—there are no better solutions anywhere.

Why is this true?



Why is this true?



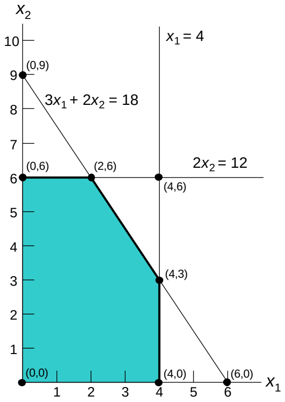
The Key Idea

Taken together, the two properties mean we can find an optimal solution by:

1. Starting at any CPF solution
2. Moving to a better adjacent CPF solution, if one exists
3. Continuing until the current CPF solution has no adjacent CPF solutions that are better

This is the essence of the simplex method.

The Dual



$$
\boxed{
\begin{array}{lrcrcrcr}
\text{minimize} & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{subject to} & u_1 & & & + & 3u_3 & \geq & 3 \\
 & & & 2u_2 & + & 2u_3 & \geq & 5 \\
 & u_1 & & & & & \geq & 0 \\
 & & & u_2 & & & \geq & 0 \\
 & & & & & u_3 & \geq & 0
\end{array}
}
$$

Example

$$\begin{array}{lrcrcl}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}$$

$$\begin{array}{lrcrcrcl}
\min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\
 & & & 2u_2 & + & 2u_3 & \geq & 5 \\
 & u_1 & & & & & \geq & 0 \\
 & & & u_2 & & & \geq & 0 \\
 & & & & & u_3 & \geq & 0
\end{array}$$

Consider optimal $(x_1, x_2) = (2, 6)$, compute dual from complementary slackness:

$$\begin{aligned}
&u_1(x_1 - 4) = 0 \\
&u_2(+2x_2 - 12) = 0 \\
&u_3(3x_1 + 2x_2 - 18) = 0 \\
&x_1(u_1 + 3u_3 - 3) = 0 \\
&x_2(2u_2 + 2u_3 - 5) = 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
&u_1(-2) = 0 \\
&u_2(0) = 0 \\
&u_3(0) = 0 \\
&2(u_1 + 3u_3 - 3) = 0 \\
&6(2u_2 + 2u_3 - 5) = 0
\end{aligned}$$

Example

$$\begin{array}{llllll} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{llllllll} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider optimal $(x_1, x_2) = (2, 6)$, compute dual from complementary slackness:

$$\begin{array}{l} u_1 = 0 \\ 3u_3 = 3 \\ 2u_2 + 2u_3 = 5 \end{array} \quad \Rightarrow \quad \begin{array}{l} u_1 = 0 \\ u_2 = \frac{3}{2} \\ u_3 = 1 \end{array}$$

Example

$$\begin{array}{llllll} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{llllllll} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (4, 3)$, compute dual from complementary slackness:

$$\begin{array}{l} u_1(x_1 - 4) = 0 \\ u_2(+2x_2 - 12) = 0 \\ u_3(3x_1 + 2x_2 - 18) = 0 \\ x_1(u_1 + 3u_3 - 3) = 0 \\ x_2(2u_2 + 2u_3 - 5) = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} u_1(0) = 0 \\ u_2(-4) = 0 \\ u_3(0) = 0 \\ 2(u_1 + 3u_3 - 3) = 0 \\ 6(2u_2 + 2u_3 - 5) = 0 \end{array}$$

Example

$$
\begin{array}{lrcrcl}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
& & & 2x_2 & \leq & 12 \\
& 3x_1 & + & 2x_2 & \leq & 18 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{lrcrcrcl}
\min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\
& & & 2u_2 & + & 2u_3 & \geq & 5 \\
& u_1 & & & & & \geq & 0 \\
& & & u_2 & & & \geq & 0 \\
& & & & & u_3 & \geq & 0
\end{array}
$$

Consider a feasible CPF $(x_1, x_2) = (4, 3)$, compute dual from complementary slackness:

$$
\begin{aligned}
u_2 &= 0 \\
u_1 + 3u_3 &= 3 \\
2u_3 &= 5
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
u_1 &= -\tfrac{9}{2} \\
u_2 &= 0 \\
u_3 &= \tfrac{5}{2}
\end{aligned}
$$

Example

$$\begin{array}{llllll}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}$$

$$\begin{array}{llllllll}
\min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\
\text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\
 & & & 2u_2 & + & 2u_3 & \geq & 5 \\
 & u_1 & & & & & \geq & 0 \\
 & & & u_2 & & & \geq & 0 \\
 & & & & & u_3 & \geq & 0
\end{array}$$

Consider a feasible CPF $(x_1, x_2) = (0, 6)$, compute dual complementary solution:

$$\begin{aligned}
u_1(x_1 - 4) &= 0 \\
u_2(+2x_2 - 12) &= 0 \\
u_3(3x_1 + 2x_2 - 18) &= 0 \\
x_1(u_1 + 3u_3 - 3) &= 0 \\
x_2(2u_2 + 2u_3 - 5) &= 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
u_1(-4) &= 0 \\
u_2(0) &= 0 \\
u_3(-6) &= 0 \\
0(u_1 + 3u_3 - 3) &= 0 \\
6(2u_2 + 2u_3 - 5) &= 0
\end{aligned}$$

Example

$$\begin{array}{lrcrcl} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcrcl} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (0, 6)$, compute dual complementary solution:

$$\begin{aligned} u_1 &= 0 \\ 2u_2 &= 2 \\ u_3 &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} u_1 &= 0 \\ u_2 &= \tfrac{5}{2} \\ u_3 &= 0 \\ u_1 + 3u_3 - 3 &= -3 \end{aligned}$$

Example

$$\begin{array}{lcccccc} \max & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{lcccccccc} \min & 4u_1 & + & 12u_2 & + & 18u_3 & & \\ \text{s.t.} & u_1 & & & + & 3u_3 & \geq & 3 \\ & & & 2u_2 & + & 2u_3 & \geq & 5 \\ & u_1 & & & & & \geq & 0 \\ & & & u_2 & & & \geq & 0 \\ & & & & & u_3 & \geq & 0 \end{array}$$

Consider a feasible CPF $(x_1, x_2) = (0, 6)$, compute dual complementary solution:

- $s_1 = u_1 + 3u_3 - 3 = -3$ is the reduced cost for the variable $x_1 \geq 0$ at the solution $x_1 = 0$.
- $s_1 < 0$ means that the objective function will improve if we allow $x > 0$ while we keep $2x_2 = 12$.
- Consider $x_1 = 1$, then $x_2 = 6$ then $3x_1 + 5x_2 = 3 * 0 + 5 * 6 + 3$.

Changes in Objective Function Coefficients

- Suppose x^* is the optimal solution for an LP.
- Z^* is its optimal objective value.
- Suppose that some objective function coefficient c_j changes.

$$
\begin{array}{lrcrcl}
\max Z = & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}
$$

- What if the 3 changed?

Changes in c, cont'd

$$
\begin{array}{llcccl}
\max Z = & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}
$$

- If 3 changed to 0, what would the new solution be?
- If 3 changed to 30, what would the new solution be?
- If the 3 changed to $3 \pm \delta$, where δ is tiny, what would the new solution be?

Example: Maximization

$$\begin{array}{llcl} \max Z = 3x_1 + 5x_2 & & & \\ \text{s.t.} \quad x_1 & & \leq & 4 \\ & 2x_2 & \leq & 12 \\ 3x_1 & + 2x_2 & \leq & 18 \\ x_1 & , \quad x_2 & \geq & 0 \end{array}$$

- Optimal solution is $(x_1^*, x_2^*) = (2, 6)$, $Z^* = 36$.
- Suppose the 3 increased to 7.
- Which of the following is true?
    1. Z^* will increase.
    2. Z^* will decrease.
    3. Z^* will stay the same.
    4. We can't say.

Example: Maximization, cont'd

$$
\boxed{
\begin{array}{lrcrl}
\max Z = & 3x_1 & + & 5x_2 & \\
\text{s.t.} & x_1 & & & \leq 4 \\
 & & & 2x_2 & \leq 12 \\
 & 3x_1 & + & 2x_2 & \leq 18 \\
 & x_1 & , & x_2 & \geq 0
\end{array}
}
$$

- $(x_1^*, x_2^*) = (2, 6)$, $Z^* = 36$.
- 3 increases to 7.
- The local rate of increase is derived from $x_1^* \delta = 2 \times 4 = 8$.
- Which of the following is true?
  1. Z^* will increase by *exactly* 8.
  2. Z^* will increase by *at most* 8.
  3. Z^* will increase by *at least* 8.
  4. Z^* will increase, but we don't know by how much.

Changes in Constraint Right-Hand Sides

$$\begin{array}{llcl}
\max Z = 3x_1 + 5x_2 & & & \\
\text{s.t.} & x_1 & \leq & 4 \\
 & 2x_2 & \leq & 12 \\
 & 3x_1 + 2x_2 & \leq & 18 \\
 & x_1 \;,\; x_2 & \geq & 0
\end{array}$$

- Suppose 12 increased.
- Would optimal solution change?
- Would optimal objective value change?
- Would optimal basis change?

Example

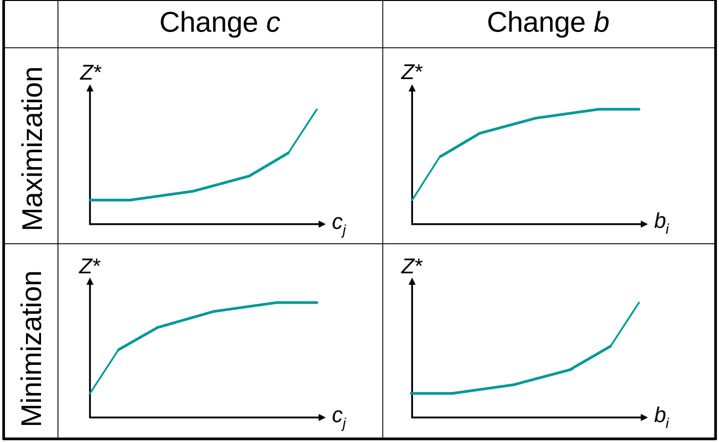
$$\begin{array}{llcl}
\max Z = 3x_1 + 5x_2 & & & \\
\text{s.t.} & x_1 & & \leq 4 \\
& & 2x_2 & \leq 12 \\
& 3x_1 + & 2x_2 & \leq 18 \\
& x_1 \quad , & x_2 & \geq 0
\end{array}$$

- Optimal solutions are $(x_1^*, x_2^*) = (2, 6)$, $(u_1^*, u_2^*, u_3^*) = (0, \frac{3}{2}, 1)$, $Z^* = 36$.
- Suppose the 12 increased to 16.
- Which of the following is true?
  1. Z^* will increase.
  2. Z^* will decrease.
  3. Z^* will stay the same.
  4. We can't say.

Example, cont'd

$$
\boxed{\begin{array}{lrcrcl}
\max Z = & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}}
$$

- $(x_1^*, x_2^*) = (2, 6)$, $(u_1^*, u_2^*, u_3^*) = (0, \frac{3}{2}, 1)$, $Z^* = 36$.
- 12 increases to 16.
- The local rate of increase come from the shadow price: $u_2^* \delta = \frac{3}{2} \times 4 = 6$.
- Which of the following is true?
  1. Z^* will increase by *exactly* 6.
  2. Z^* will increase by *at most* 6.
  3. Z^* will increase by *at least* 6.
  4. Z^* will increase, but we don't know by how much.


Change c
Change b
Maximization
Minimization
Z*
Z*
Z*
Z*
c_j
b_i
c_j
b_i

Relationship to Complementary Slackness

- Suppose there is slack in the ith primal constraint.
  - Increasing the RHS would not change the optimal solution.
  - By complementary slackness, u_i^* must equal 0 (in the dual).
  - Using the statement on the previous slides, the optimal objective function changes by u_i^*, or 0.
- Suppose there is no slack in the ith primal constraint.
  - Increasing the RHS *would* change the optimal solution.
  - u_i^* *probably* (!) is greater than 0.
  - Using the statement above, the optimal objective function changes by u_i^*.
- This agrees with our interpretation of the dual values as *shadow prices*.

Relationship to Complementary Slackness, cont'd

- Now suppose there is slack in the jth dual constraint.
  - By complementary slackness, $x_j^* = 0$ (in the primal).
  - If we increase c_j slightly, we'll still want to set $x_j^* = 0$.
  - We argued that for each unit increase in c_j, Z^* changes by x_j^* (if optimal basis stays the same).
  - So Z^* increases by 0 when c_j increases.
- Suppose there is no slack in the jth dual constraint (reduced cost is 0).
  - $x_j^* > 0$ (probably).
  - If we increase c_j by 1, the objective value will go up by x_j^*.