ISE 426
Optimization models and applications

Lecture 19 — November 6, 2014

- Minimum Spanning Tree
- Network Design

Announcements:

- Case studies will be assigned after the quiz. Form into groups of 4 or 5.

The Minimum Spanning Tree Problem

- Given a graph $G = (V, E)$ and a function $c : E \to \mathbb{R}_+$, find a subgraph of G (that is, a subset of E) such that all (pairs of) nodes of V are connected by at least one path.
- This is a problem very well known in Optimization as the Minimum Spanning Tree (MST) problem
- It can be solved very easily by first sorting the edges according to non-decreasing c. Two algorithms:

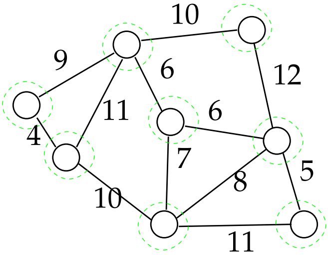
Prim: Create a tree starting from a single node (of your choice), including one edge at a time.

Kruskal: Include one edge at a time in that order, ensuring no cycles are created in the graph.

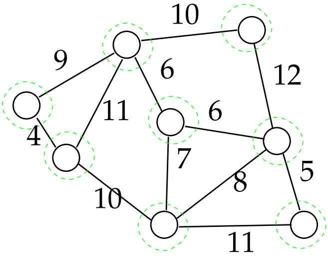
How Kruskal works

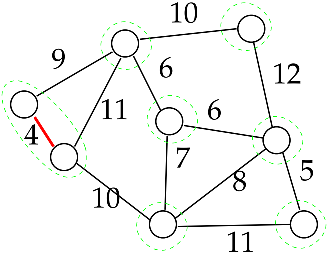
Sort the edges by weight from smallest to largest. Add one edge at a time, as long as they do not create a cycle. If an edge creates a cycle - skip it.

How Kruskal works

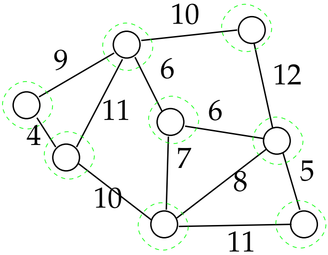


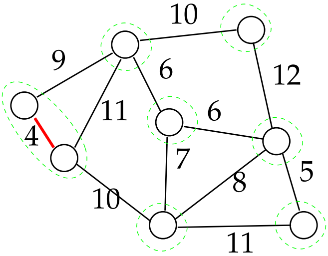
How Kruskal works

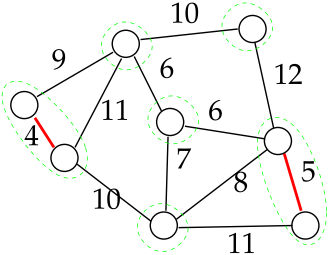




How Kruskal works







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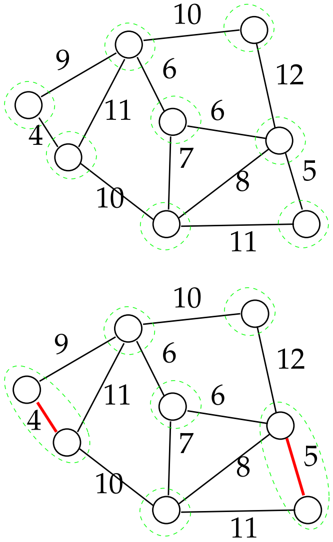
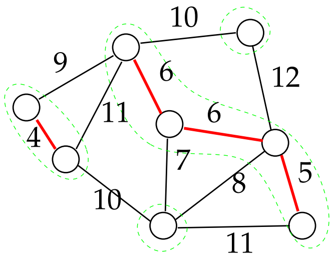
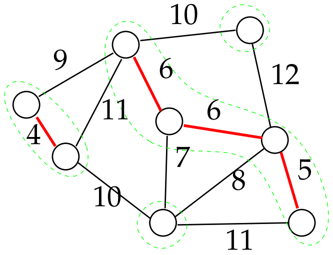
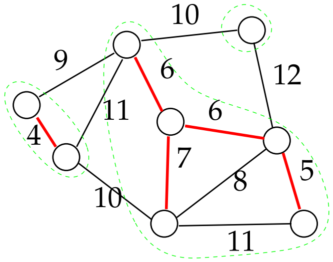


How Kruskal works (cont.)

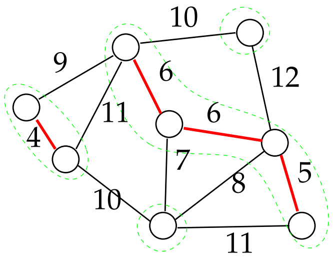


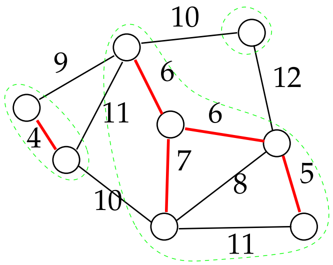
How Kruskal works (cont.)

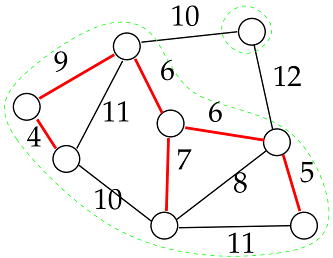




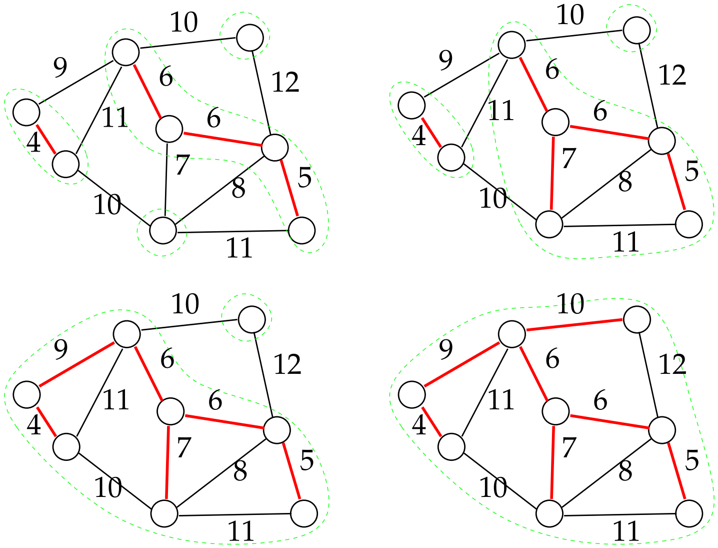
How Kruskal works (cont.)







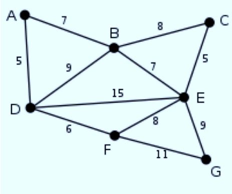
How Kruskal works (cont.)



How Prim works

Pick any node a. Let the current set $S = \{a\}$. Consider all nodes connectedto S by one edge. Pick a node whose is closest to any node in S. Add this node to S, procceed in the same manner until all nodes are added.

How Prim works



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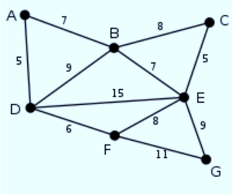
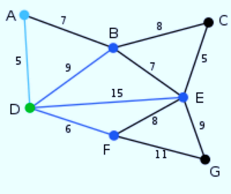
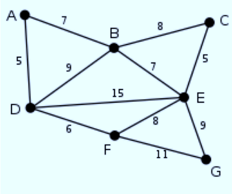
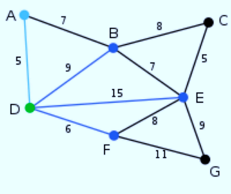
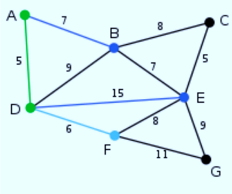




How Prim works







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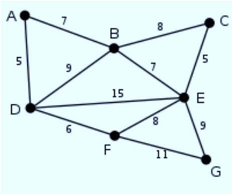
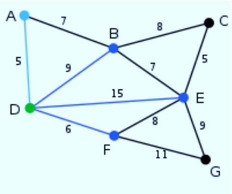
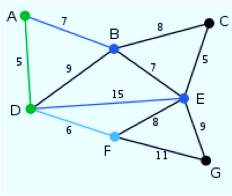
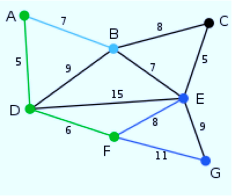
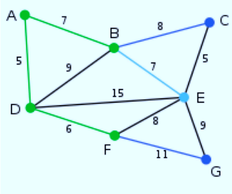




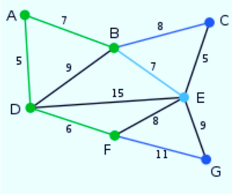


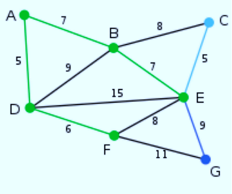


How Prim works (cont.)

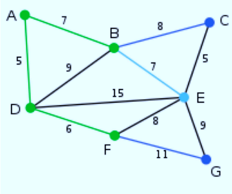


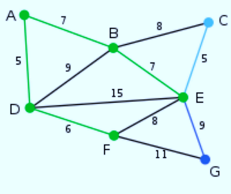
How Prim works (cont.)

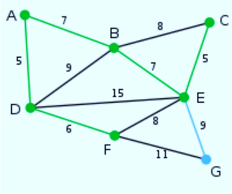




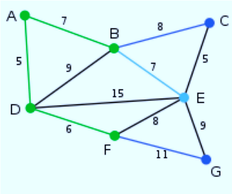
How Prim works (cont.)

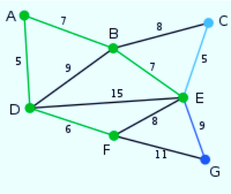


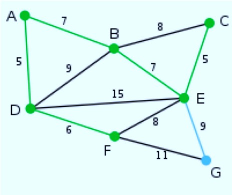


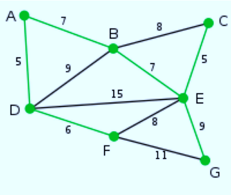


How Prim works (cont.)









How do we model this as an Optimization model?

Given a graph $G = (V, E)$, determine a subset $S \subseteq E$ such that

- S is a tree (how do you write this constraint?)
- The edges in S *span* the whole graph

i.e. The edges in S connect any two nodes of V

We need a subset of $E \Rightarrow$ a binary variable x_{ij} for each $\{i, j\} \in E$.

Graph Theory tells us that any spanning tree for a graph of n nodes has exactly $n - 1$ edges.

$\Rightarrow$ Does that mean that

$$\sum_{\{i,j\} \in E} x_{ij} = |V| - 1$$

is enough?

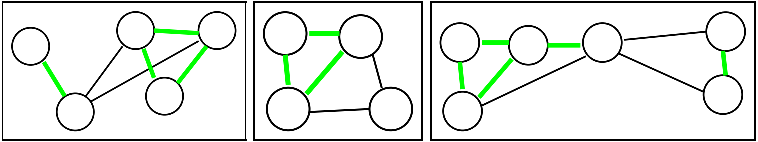
Too good to be true

$$
\begin{array}{ll}
\min & \sum_{\{i,j\}\in E} c_{ij} x_{ij} \\
 & \sum_{\{i,j\}\in E} x_{ij} = |V| - 1 \\
 & x_{ij} \in \{0,1\} \quad \forall \{i,j\} \in E
\end{array}
$$

Too good to be true

$$\begin{array}{ll} \min & \sum_{\{i,j\}\in E} c_{ij}x_{ij} \\ & \sum_{\{i,j\}\in E} x_{ij} = |V| - 1 \\ & x_{ij} \in \{0,1\} \quad \forall \{i,j\} \in E \end{array}$$

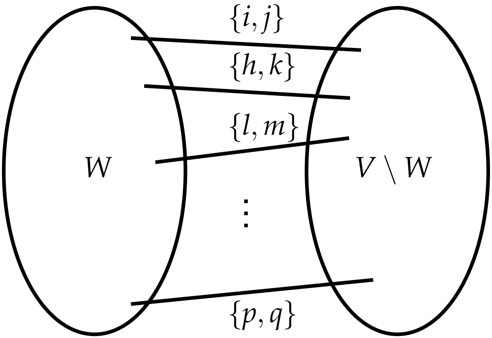
Optimal solutions of this model can be (green edges $\in S$):



None of them is a tree. None of them spans the graph.

Formulation #1: cuts

- "There is a path between any two nodes of G". That is,
- "There is no node pair that is not connected". Or,
- "If we split the set of nodes V in two parts W and $V \setminus W$, they must be connected."



Formulation #1: cuts (cont'd)

- Partition V into two subsets: W and $V \setminus W$.
- Would you accept a solution that contains no edge between W and $V \setminus W$? No!

$$
\begin{array}{lll}
\min & \sum_{\{i,j\} \in E} c_{ij} x_{ij} & \\
 & \sum_{\{i,j\} \in E : i \in S, j \in V \setminus S} x_{ij} \geq 1 & \forall S \subset V : S \neq \emptyset \\
 & x_{ij} \in \{0, 1\} & \forall \{i, j\} \in E
\end{array}
$$

This is a "weak" formulation and gives bad lower bounds.

Formulation #2: flow

Spanning $\equiv$ "There is a path between any two nodes s, t of G".

- Ensuring this would avoid those wrong solutions, as they would violate at least one of the *connection* constraints.
- We could use two sets of variables:

  flow: define a path between any two nodes in V
  edge: one if the edge is used by at least one flow, zero otherwise

- We need a set of flow variables for each pair (s, t) of nodes
- Binary variables x_{ij}: one if there is (even just one unit of) flow on $\{i, j\}$, doesn't matter if $i \to j$ or $j \to i$

Formulation #2: flow

- All pairs of nodes: $n(n-1)$
- Enough to connect a node r of our choice to all other nodes

$\Rightarrow$ Ensure that there be $n-1$ paths: from r to all other nodes

- $s \to t \quad \equiv \quad (s \to r) + (r \to t)$
- Use flow variables for each node $k \in V \setminus \{r\}$ for path $r \to k$
- Include in S the edges of E that host at least one path

i.e. If an edge $\{i, j\}$ is used by a path, then $x_{ij} = 1$

Formulation #2: flow

Variables:

- f_{ij}^k flow variables (binary): flow in both directions $i \to j$ and $j \to i$ on edge $\{i, j\}$ for a path that is going from r to k
- x_{ij} binary variables for any edge $\{i, j\} \in E$: 1 if $\{i, j\}$ is included in the solution, 0 otherwise

Constraints:

1. Conservation of flow
2. No flow on edge $\{i, j\}$ for any destination k if it is not included in the solution

Formulation #2: flow

$$
\begin{array}{lll}
\min & \sum_{\{i,j\}\in E} c_{ij}x_{ij} & \\
 & \sum_{j:\{i,j\}\in E}(f_{ij}^k - f_{ji}^k) = 0 & \forall i \in V \setminus \{k\}, \forall k \in V \setminus \{r\} \\
 & \sum_{j:\{r,j\}\in E} f_{rj}^k = 1 & \forall k \in V \setminus \{r\} \\
 & f_{ij}^k \leq x_{ij} & \forall \{i,j\} \in E, \forall k \in V \setminus \{r\} \\
 & f_{ji}^k \leq x_{ij} & \forall \{i,j\} \in E, \forall k \in V \setminus \{r\} \\
 & x_{ij} \in \{0,1\} & \forall \{i,j\} \in E, \forall k \in V \setminus \{r\} \\
 & f_{ij}^k \in [0,1] & \forall \{i,j\} \in E
\end{array}
$$

Q.: Can we replace $f_{ij}^k \leq x_{ij}$ and $f_{ji}^k \leq x_{ij}$ with

$$f_{ij}^k + f_{ji}^k \leq x_{ij}$$

Network design

We want to design a telecommunication network that connects a set of cities using their railway network (built long ago and with some pipes along the rails that are basically unused), so as to spend nothing on digging costs.

- The railway network can be defined as a graph $G = (V, E)$.
- Installing a unit of network capacity on edge $\{i, j\} \in E$ (for instance, a fiber optics cable) costs c_{ij}

  (it depends on the distance between cities i and j)
- Once installed, it carries up to U Mb/s of data traffic in both directions (U Mb/s $i \to j$ and U Mb/s $j \to i$)
- For any ordered pair (k, l) of cities in a set P, there is a planned amount of data traffic from k to l denoted as d_{kl}.

$\Rightarrow$ Determine how much capacity to install on all edges of G in order to satisfy all traffic demands $(k, l) \in P$ while minimizing total installation cost.

Network design

$$
\begin{array}{lll}
\min & \sum_{\{i,j\} \in E} c_{ij} y_{ij} & \\
& \sum_{j \in V : \{i,j\} \in E} (x_{ij}^{kl} - x_{ji}^{kl}) = 0 & \forall i \in V : k \neq i \neq l, \forall (k,l) \in V^2 \\
& \sum_{j \in V : \{k,j\} \in E} (x_{kj}^{kl} - x_{jk}^{kl}) = 1 & \forall (k,l) \in P \\
& \sum_{(k,l) \in P} d_{kl} x_{ij}^{kl} \leq U y_{ij} & \forall \{i,j\} \in E \\
& \sum_{(k,l) \in P} d_{kl} x_{ji}^{kl} \leq U y_{ij} & \forall \{i,j\} \in E \\
& x_{ij}^{kl}, x_{ji}^{kl} \in \{0,1\} & \forall \{i,j\} \in E, \forall (k,l) \in P \\
& y_{ij} \in \mathbb{Z} & \forall \{i,j\} \in E
\end{array}
$$