ISE 429 #5 HW Name: Bolun Xu. page 1

1): $P(X_2 > -2) = P(X_2 < 2) = P\left(\frac{X_2}{\sqrt{2}} < \sqrt{2}\right) = \Phi(\sqrt{2}) = 0.9207$

2: $P(X_1 > X_2) = P(X_2 - X_1 < 0) = P(X_1 < 0) = \Phi(0) = \frac{1}{2}$

3: $P(X_2 < X_1 < X_3) = P(X_1 > X_2, X_3 > X_1)$

$= P(X_2 - X_1 < 0,\ X_3 - X_2 > -(X_1 - X_2))$

$= \int_{-\infty}^{0} p(X_2 - X_1 = t)\, p(X_3 - X_2 > -t)\, dt$

$= \int_{-\infty}^{0} \phi(t)\, \Phi(t)\, dt.$

4 $P(X_t = 0$ for some t with $2 \le t \le 3)$

$= 2P(X_2 \ge 0, X_3 \le 0, X_0 = 0) + 2P(X_2 \le 0, X_3 \ge 0, X_0 = 0)$

$= 4P(X_2 \ge 0, X_3 \le 0, X_0 = 0)$

$= 4\int_0^\infty \frac{1}{\sqrt{4\pi}} e^{-x^2/4} dx \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$

$P(X_t < 4 \text{ for all } t \text{ with } 0 < t \leq 9)$.

$= 1 - P(X_t \geq 4 \text{ for } \exists t, 0 < t \leq 9)$

$= 1 - P\{\max_{0 \leq t \leq 9} X_t \geq 4\}$

$= 1 - 2P\{X_9 \geq 4\} = 2\Phi(\frac{4}{3}) - 1$

$P(X_t < -10 \text{ for all } t > 10)$ $\because P(\exists X_t = 0, t \to \infty) = 1$, so

$= 0$.

2) $\because Y_t \sim N(0, t)$

$X_t \sim N(0, t)$

$\therefore Y_t + X_t \sim N(0, 2t)$

$\because Y_0, X_0 = 0$

$\therefore Z_t = (Y_t + X_t)$ is BRM

$\therefore P\{Z_t = 0 \text{ for infinitely many } t > 10\} = 1$

3). $\{X_t\}$ is $BRM(0,1)$

$$P(M \geq a) = 2\left(1 - \Phi\left(\frac{a-0}{\sqrt{10}}\right)\right) = 2\int_{\frac{x}{\sqrt{10}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$P(M = a) = \frac{-d\, P(M \geq a)}{da}.$$

Leibniz integral rule:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\, dt\right) = f(x, b(x))\, b'(x) - f(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x} dt.$$

$$\underline{\underline{P(M=x)}} = \frac{-2}{\sqrt{2\pi}}\left(0 - e^{-\frac{x^2}{20}} \cdot \frac{1}{\sqrt{10}} + 0\right) = \frac{2}{\sqrt{20\pi}} e^{-\frac{x^2}{20}} \quad (x \geq 0)$$

$$EM = \int_0^{\infty} x\, P(M=x)\, dx = \frac{2}{\sqrt{20\pi}} \int_0^{\infty} x \cdot e^{-\frac{x^2}{20}} dx.$$

$$= \frac{20 \cdot 20}{\sqrt{20\pi}} \cdot \frac{1}{2}$$

$$= \frac{20}{\sqrt{20\pi}}$$

$$EM^2 = \int_0^{\infty} x^2\, P(M=x)\, dx = 10 \quad \therefore \mathrm{Var}M = EM^2 - (EM)^2$$

$$\underline{\underline{= 10 - \frac{20}{\pi}}}.$$

4). $P(T\le s) = P(\max_{0\le t\le s} X_t \ge 1) = 2\left(1-\Phi\left(\frac{1}{\sqrt{s}}\right)\right)$

$$= 2\int_{\frac{1}{\sqrt{s}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy.$$

use Leibniz integral rule.

$$f(s) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2s}} s^{-\frac{3}{2}}.$$

$$ET = \int_0^\infty s f(s)\, ds = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{\frac{1}{2s}} s^{-\frac{1}{2}} ds = \infty.$$