IE426 – Optimization models and applications

Fall 2012 – Quiz #1, October 16, 2012

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You have 75 minutes. There are three problems. This quiz accounts for 10% of the final grade. There are 40 points available. Please write clear and concise statements. Use a **readable** handwriting: it will be hard to grade your answers if they are not readable — let alone give them full points. Use the back of each sheet if you need more space, for example to draw graphs. Do not use calculators.

1 Convexity and relaxations (16 pts.)

For each of the following problems, determine if they are convex or not, and why (8 pts.):

$$\begin{array}{llll}
(1)\,\min & xy & (2)\,\max & x^2 \\
& x^2+y^2 \le 1 & & x^2+y^2 \le 1 \\
& x+y=1 & & x+y=1 \\
\\
(3)\,\min & x^2 & (4)\,\min & x^2 \\
& x^2+y^2 \le 1 & & x^2+y^2 \ge 1 \\
& x+y=1 & & x+y=1 \\
\\
(3)\,\min & z^2 & (6)\,\max & x^2+5y^2-5yx \\
& x^2+y^2 \le 1 & & x-y=1 \\
& x+y=1 & & e^{x+y} \le 5
\end{array}$$

(Continue here)

For each of the nonconvex problems, find a feasible solution. For each problem explain whether the objective value at a feasible solution provides an upper bound or a lower bound on the optimal solution for this problems? (4 pts.).

(1), (2), (4), (6) non-convex.

(1) (1,0) with objective 0 is an upper bound (also a lower bound because optimal objective is 0).

(2) (1,0) with objective 1 is a lower bound. (also an upper bound because optimal objective is 1).

(4) (1,0) with objective 1 is an upper bound.

(6) (1,0) with objective 1 is a lower bound.

(1) $\min \; xy$

$x^2+y^2 \leq 1$

$x+y=1$] linear constraint

Hessian for x^2+y^2

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is PSD (positive semi-definite)

$\Rightarrow$ convex constraint.

Hessian for xy

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ not PSD

non-convex constraint.

$\boxed{\text{non-convex problem}}$

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(2) $\max \; x^2$

$x^2+y^2 \leq 1$

$x+y=1$

$\rightarrow$

$\min \; -x^2$

$x^2+y^2 \leq 1$

$x+y=1$

Hessian for $-x^2$

$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$ not PSD

$\boxed{\text{non-convex problem}}$

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(3) $\min \; x^2 \leftarrow$ convex function

$x^2+y^2 \leq 1$

$x+y=1$] convex constraints

$\boxed{\text{convex problem}}$

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(4) $\min \; x^2$

$x^2+y^2 \geq 1 \longleftrightarrow -(x^2+y^2) \leq -1$

$x+y=1$

Hessian for $-(x^2+y^2)$

$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ not PSD

$\boxed{\text{non-convex problem.}}$

(5) $\min \; z^2$ ← convex function

$$\left.\begin{aligned} x^2 + y^2 &\leq 1 \\ x + y &= 1 \end{aligned}\right\}$$ convex constraints

$\boxed{\text{Convex problem}}$

(6) $\max \; x^2 + 5y^2 - 5yx$

$x - y = 1$

$e^{x+y} \leq 5$

$\longrightarrow$

$\min \; -(x^2 + 5y^2 - 5yx)$

$x - y = 1$

$e^{x+y} \leq 5$ ← convex constraint

non-convex function

$\boxed{\text{non-convex problem}}$

(Continue here)

For one of the nonconvex problems (your choice) create a convex relaxation whose optimal solution is the same as the optimal solution of the original nonconvex problem. NOTE: if the objective function is not convex then a convex relaxation should have a convex objective which is guaranteed to be at least as good as (that is "not greater than" in the minimization case and "not smaller than" in the maximization case) the original objective on the feasible set. Prove that you found an optimal solution by comparing upper and lower bounds. (4 pts.)

In (4) ignore the non-convex constraint $x^2+y^2 \geq 1$ so

we have $\boxed{\begin{array}{l} \min \; x^2 \\ x+y=1 \end{array}}$ with optimal point $(0,1)$ with objective 0.

convex problem

Since optimal objective value is also 0 for the original problem, this convex relaxation yields the optimal objective value

2 Linear Programming Model (24 pts.)

Sophia is visiting Universal Studios park in Orlando and she has only 2 hours left of the day. She wants to enjoy herself as much as possible in the remaining time. There are eight rides she can go on: The Hulk, Forbidden Journey, Rock It, E.T. , Mummy, Jurassic River Adventure, MIB and Dragon Challenge. She will go on each at most once and she will have to wait in line. Below is the table of rides with their waiting times and the level of fun Sophia expects from these rides.

Ride	Hulk	Forb. Jrn.	Rock It	E.T.	Mummy	Jur. Riv. Adv.	MIB	Drag. Chall.
fun	5	5	3	2	4	3	3	2
time	50	90	40	30	20	10	20	10

(a) Formulate an optimization problem to choose the rides that will maximize Sophia's fun within two hours. Is this a convex problems? If not, create a linear programming relaxation and solve it (by applying greedy method used in homework and in class for knapsack problems). What does the solution of the relaxation give you? Can you use this solution to generate a feasible solution of the original problem? Can you generate an optimal solution to the original problem? Justify your answers - prove that you the solution you obtain is optimal by using the upper bound of the optimal value obtained from the relaxation. (10pts)

2) Linear programming Model

a) $\max \; 5x_1 + 5x_2 + 3x_3 + 2x_4 + 4x_5 + 3x_6 + 3x_7 + 2x_8$

s.t. $40x_1 + 50x_2 + 40x_3 + 30x_4 + 20x_5 + 10x_6 + 20x_7 + 10x_8 \leq 120$

$x_1, x_2, \dots, x_8 \in \{0, 1\}$

The problem is not convex because the variables are integer.

Convex relaxation LP is obtained when we relax the integer constraints as $0 \leq x_i \leq 1 \quad i = 1, \dots, 8$.

Using the greedy method, we have the ratios

$r_1 = 5/40$

$r_2 = 5/50$

$r_3 = 3/40$

$r_4 = 2/30$

$r_5 = 4/20$

$r_6 = 3/10$

$r_7 = 3/20$

$r_8 = 2/10$

order these ratios:

$r_6 \geq r_8 \geq r_5 \geq r_7 \geq r_1 \geq r_3 \geq r_2 \geq r_4$

$x_6 = 1$

$x_8 = 1$

$x_5 = 1$

$x_7 = 1$

$x_1 = 1$

$x_3 = 1/4$

$\Rightarrow$ capacity (time) filled

optimal LP relaxation $x = [1, 0, 1/4, 0, 1, 1, 1, 1]$

with objective 17.75.

Using the greed method to find a feasible solution we set $x = [1, 0, 0, 0, 1, 1, 1, 1]$ with objective 17. Since the relaxation give 17.5 as the upper bound, we know optimal objective value (for the original problem cannot exceed 17.5). So the point $x = [1, 0, 0, 0, 1, 1, 1, 1]$ must be optimal.

(b) There is an additional constraint, which says that Sophia can only go on two of the four rides: The Hulk, Forbidden Journey, Rock It, Dragon Challenge, because they a fast roller coasters and she will get dizzy if she goes on more than two. Add this constraint to your formulation. What is the optimal solution of the new problem? Can you prove that it is an optimal solution? (2pts)

$x_1 + x_2 + x_3 + x_5 \leq 2$

Since optimal solution satisfies this constraint, it is still optimal for the new problem.

(c) On her way to the Dragon Challenge ride Sophia has spent 10 minutes getting some butterbeer, will she still be able go to all the rides that she planned? Will those rides present an optimal solution? Justify your answers. (2pts)

Note that the optimal solution

$x = [1, 0, 0, 0, 1, 1, 1, 1]$ takes 110 minutes. So after 10 minutes spent, the solution will be still a feasible solution.

(d) For the rest of the problem, consider the LP relaxation in part (a) (that is without the additional constraint). Write down the dual of the linear programming relaxation. (Hint: do not forget the upper bounds on the primal variables). Compute the dual optimal solution from complementary slackness conditions and show that this solution is feasible (hence the primal solution you computed by hand is optimal). (8 pts)

(d) <u>Dual:</u>

$$\min \quad 120u_1 + u_2 + u_3 + \dots u_9$$

s.t.

$$50u_1 + u_2 \geq 5$$
$$90u_1 + u_3 \geq 5$$
$$40u_1 + u_4 \geq 3$$
$$30u_1 + u_5 \geq 2$$
$$20u_1 + u_6 \geq 4$$
$$10u_1 + u_7 \geq 3$$
$$20u_1 + u_8 \geq 3$$
$$10u_1 + u_9 \geq 2$$
$$u_1, \dots u_9 \geq 0$$

<u>Complementary slackness:</u>

$$(50u_1 + u_2 - 5)x_1 = 0$$
$$(90u_1 + u_3 - 5)x_2 = 0$$
$$(40u_1 + u_4 - 3)x_3 = 0$$
$$(30u_1 + u_5 - 2)x_4 = 0$$
$$(20u_1 + u_6 - 4)x_5 = 0$$
$$(10u_1 + u_7 - 3)x_6 = 0$$
$$(20u_1 + u_8 - 3)x_7 = 0$$
$$(10u_1 + u_9 - 2)x_8 = 0$$

$$\left[120 - (5x_1 + 90x_2 + \dots 10x_8)\right]u_1 = 0$$

$$(1 - x_1)u_2 = 0$$
$$(1 - x_2)u_3 = 0$$
$$(1 - x_3)u_4 = 0$$
$$\vdots$$
$$(1 - x_8)u_9 = 0$$

$x = [1, 0, 1/4, 0, 1, 1, 1, 1]$ gives

$$(1 - x_2) u_3 = 0 \Rightarrow u_3 = 0$$

$$(1 - x_3) u_4 = 0 \Rightarrow u_4 = 0$$

$$(1 - x_4) u_5 = 0 \Rightarrow u_5 = 0$$

Also we have

$$\left.\begin{array}{l} 50 u_1 + u_2 = 5 \\ 40 u_1 + u_4 = 3 \\ 20 u_1 + u_6 = 4 \\ 10 u_1 + u_7 = 2 \\ 20 u_1 + u_8 = 3 \\ 10 u_1 + u_9 = 2 \end{array}\right] \Rightarrow u_1 = 3/40$$

$u_2 = 5/4$

$u_3 = 0$

$u_4 = 0$

$u_5 = 0$

$u_6 = 5/2$

$u_7 = 5/4$

$u_8 = 3/2$

$u_9 = 5/4$

(e) How much shorter the line to Forbidden Journey should be for Sophia to consider going on that ride. Derive your answer from complementarity conditions and feasibility of the dual solution. (2 pts)

Note that for $x_2 > 0$ it must be that the corresponding dual constraint

$$(90 u_1 + u_3 - 5) = 0$$

Suppose we decrease 90 to B so that for $x_2 > 0$ we must have $B u_1 + u_3 - 5 = 0$

with current dual optimal solution $u_1 = 3/40$
$u_3 = 0$

$$\Rightarrow \frac{B \cdot 3}{40} = 5 \Rightarrow B = 200/3$$

Namely as soon as the time for Forbidden Journey becomes $\frac{200}{3}$, x_2 can become positive (which means this ride will be eligible)

So the ride should be $90 - \frac{200}{3} = \frac{70}{3} = 23.33$ minutes shorter than its current length.