ISE 339 HW#2 Name: Bolun Xu.

chapter 4:

63. solution: $S = (I - P_T)^{-1} = \begin{bmatrix} 2.2069 & 1.3793 & 0.6207 \\ 0.9655 & 3.1034 & 0.8966 \\ 1.3103 & 2.0690 & 1.9310 \end{bmatrix}$

$S_{13} = 0.6207, \; S_{23} = 0.8966, \; S_{33} = 1.9310.$

$f_{ij} = \frac{S_{ij} - \delta_{ij}}{S_{ij}} \qquad f_{13} = \frac{S_{13}}{S_{33}} = 0.3214, \; f_{23} = \frac{S_{23}}{S_{33}} = 0.4643, \; f_{33} = \frac{S_{33} - 1}{S_{33}} = 0.4821$

66. (a) $\pi_0 = \frac{1}{4} + \frac{3}{4}\pi_0^2 \Rightarrow \pi_0 = \frac{1}{3}$

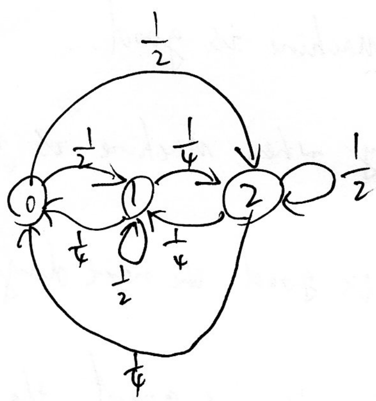
(b) $\mu = 1 \Rightarrow \pi_0 = 1$

(c) $\pi_0 = \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^3 \Rightarrow 2\pi_0^3 - 3\pi_0 + 1 = 0.$

$\Downarrow$

$\pi_0 = \frac{-1+\sqrt{3}}{2} = 0.366$

73.



$P_T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \Rightarrow \begin{cases} \pi_0 = \frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 \\ \pi_1 = \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 \\ 1 = \pi_0 + \pi_1 + \pi_2 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} \pi_0 = 0.2 \\ \pi_1 = 0.4 \\ \pi_2 = 0.4 \end{cases} \Rightarrow \begin{cases} \pi_0 p_{01} = \pi_1 p_{10} \\ \pi_0 p_{02} = \pi_2 p_{20} \\ \pi_1 p_{12} = \pi_2 p_{21} \end{cases} \Rightarrow$ M.C. is time-reversible

Markov Desicion Process additional problem.

Solution: Let ✓ be "machine is working".

X be "machine is not working".

Buck

	✓	X	
✓	0.4	0.6	x_1
X	0.6	0.4	x_2

$P_T = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}$.

Bill

	✓	X	
✓	0.6	0.4	y_1
X	0.5	0.5	y_2

$P_T = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix}$

Suppose $x_1 = \begin{cases} 0 & \text{~~Buck stay at home when~~ o.w.} \\ 1 & \text{Buck stay at home next day when machine is good.} \end{cases}$

$x_2 = \begin{cases} 1 & \text{Buck stay at home next day when machine is not good.} \\ 0 & \text{o.w.} \end{cases}$

$y_1 = \begin{cases} 1 & \text{Bill stay at home when machine is good. the next day} \\ 0 & \text{o.w.} \end{cases}$

$y_2 = \begin{cases} 1 & \text{Bill stay at home when machine is not good. the next day} \\ 0 & \text{o.w.} \end{cases}$

object function; Max $60 \times 20 \times \pi_0$

Constraints: $\pi_0 = 0.4x_1 + 0.6y_1 + 0.6x_2 + 0.5y_2$;

$\pi_1 = 0.6x_1 + 0.4x_2 + 0.4y_1 + 0.5y_2$;

$\pi_1 + \pi_0 = 1$;

$x_1 + y_1 = 1$;

$x_2 + y_2 = 1$;

$\pi_1, \pi_0 \geq 0$; $\boxed{x_i, y_j \text{ is binary}, i=1,2, j=1,2;}$

But this is not linear program, so relax it to linear; $\boxed{x_i, y_j \geq 0}$ Replace it and solve the problem. We get $y_1 = 1, x_2 = 1$, $\pi_0 = 0.6$.

we get $y_1 = 1, x_2 = 1, \pi_0 = 0.6$, so $P_T = \begin{pmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{pmatrix}$

$$\text{Max } 60 \cdot 20 \cdot \pi_0 = 720 \text{ (dollars)}$$

The optimal policy is: "When ~~[illegible]~~ machine is good, Bill stay at home the next day, when machine is not good, Buck stay at home the next day." The expected average profit is \$720 per day.

Infinite State Markov Chain

Solution:

$$P_{M\times M} = \begin{pmatrix} 0.65 & 0.15 & 0.1 & 0.1 & 0 & 0 & \cdots & 0 \\ 0.55 & 0.1 & 0.15 & 0.1 & 0.1 & 0 & \cdots & 0 \\ 0.2 & 0.35 & 0.1 & 0.15 & 0.1 & 0.1 & 0 \cdots & 0 \\ 0 & 0.2 & 0.35 & 0.1 & 0.15 & 0.1 & 0.1 \ddots & \vdots \\ \vdots & 0 & 0.2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0.1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0.15 & 0.1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0.35 & 0.1 & 0.15 \\ 0 & 0 & 0 & \cdots & \cdots & 0.2 & 0.35 & 0.1 \end{pmatrix}$$

$M = 50$

$$b^T = (0, 0, \cdots\cdots 1)$$

$$[I - P^T]\pi = b \Rightarrow \pi^T =$$

1	2	3	---	41	42	43	44	45	46	47	48	49	5[illegible]
0.1198	0.05554	0.05724	---	0.003492	-	-	-	-	-	-	-	-	0.00[illegible]

- Finding $\min_{0.9M \le n \le M} \left\{ \frac{\pi_n}{\pi_{n-1}} \right\}$.

$M=50 \Rightarrow \begin{cases} 0.9M = 45 \\ M = 50. \end{cases}$

$\pi =$

45	0.00258
46	0.002399
47	0.002205
48	0.002108
49	0.00175
50	0.002281

ratio $= \frac{\pi_k}{\pi_{k-1}} =$

45	0.9266
46	0.9297
47	0.9190
48	0.9562
49	0.8301 ⟵ min = η
50.	1.3035

$$\sum_{j=0}^{M-1} \pi_j + \frac{\pi_M}{1-\eta} = 1$$

plug in $\eta = 0.8301$, and calutate again, We obtain:

$\pi^T =$

1	2	3	...	45	46	...	50
0.1199	0.05554			0.002552	0.002372	...	0.002256

$\pi =$

45	0.002552
46	0.002372
47	0.002180
48	0.002085
49	0.001731
50	0.002256.

ratio $= \frac{\pi_k}{\pi_{k-1}} =$

45	0.9266
46	0.9297
47	0.9190
48	0.9562
49	0.8301 ⟵ min = η (nochange)
50	1.3035

$n^* = 49$

So, $\pi_n = \pi_{n^*} \cdot \eta^{n-n^*} = 0.001731 \cdot 0.8301^{n-49}$ for $n \ge n^* = 49$

$i = 1 \sim 100$, in this case, divide all probability estimates by 0.9952.

* Data in this case will be posted at Applendix. $\sum_i \pi_i = 0.9952$

Applendix for Infinite M.C. Problem

First round calculation:

π =

1	0. 121316
2	0. 056155
3	0. 057877
4	0. 060428
5	0. 051921
6	0. 048921
7	0. 045694
8	0. 042088
9	0. 039104
10	0. 036273
11	0. 033617
12	0. 031179
13	0. 028912
14	0. 026809
15	0. 02486
16	0. 023052
17	0. 021376
18	0. 019822
19	0. 01838
20	0. 017044
21	0. 015805
22	0. 014655
23	0. 01359
24	0. 012602
25	0. 011685
26	0. 010836
27	0. 010048
28	0. 009317
29	0. 00864
30	0. 008011
31	0. 007429
32	0. 006889
33	0. 006388
34	0. 005923
35	0. 005493
36	0. 005093
37	0. 004723
38	0. 00438
39	0. 004061
40	0. 003766
41	0. 003492
42	0. 003238
43	0. 003002
44	0. 002785
45	0. 00258
46	0. 002399
47	0. 002205
48	0. 002108
49	0. 00175
50	0. 002281

Second round calculation:

π =

1	0. 119979
2	0. 055536
3	0. 057239
4	0. 059762
5	0. 051348
6	0. 048382
7	0. 04519
8	0. 041624
9	0. 038673
10	0. 035873
11	0. 033246
12	0. 030835
13	0. 028593
14	0. 026513
15	0. 024586
16	0. 022798
17	0. 02114
18	0. 019603
19	0. 018178
20	0. 016856
21	0. 01563
22	0. 014494
23	0. 01344
24	0. 012463
25	0. 011557
26	0. 010716
27	0. 009937
28	0. 009214
29	0. 008544
30	0. 007923
31	0. 007347
32	0. 006813
33	0. 006317
34	0. 005858
35	0. 005432
36	0. 005037
37	0. 004671
38	0. 004331
39	0. 004016
40	0. 003724
41	0. 003453
42	0. 003202
43	0. 002969
44	0. 002754
45	0. 002552
46	0. 002372
47	0. 00218
48	0. 002085
49	0. 001731
50	0. 002256

i=100, π i =

1	0.119979	51	0.001192
2	0.055536	52	0.00099
3	0.057239	53	0.000822
4	0.059762	54	0.000682
5	0.051348	55	0.000566
6	0.048382	56	0.00047
7	0.04519	57	0.00039
8	0.041624	58	0.000324
9	0.038673	59	0.000269
10	0.035873	60	0.000223
11	0.033246	61	0.000185
12	0.030835	62	0.000154
13	0.028593	63	0.000128
14	0.026513	64	0.000106
15	0.024586	65	8.79E-05
16	0.022798	66	7.3E-05
17	0.02114	67	6.06E-05
18	0.019603	68	5.03E-05
19	0.018178	69	4.18E-05
20	0.016856	70	3.47E-05
21	0.01563	71	2.88E-05
22	0.014494	72	2.39E-05
23	0.01344	73	1.98E-05
24	0.012463	74	1.65E-05
25	0.011557	75	1.37E-05
26	0.010716	76	1.13E-05
27	0.009937	77	9.41E-06
28	0.009214	78	7.81E-06
29	0.008544	79	6.49E-06
30	0.007923	80	5.38E-06
31	0.007347	81	4.47E-06
32	0.006813	82	3.71E-06
33	0.006317	83	3.08E-06
34	0.005858	84	2.56E-06
35	0.005432	85	2.12E-06
36	0.005037	86	1.76E-06
37	0.004671	87	1.46E-06
38	0.004331	88	1.21E-06
39	0.004016	89	1.01E-06
40	0.003724	90	8.36E-07
41	0.003453	91	6.94E-07
42	0.003202	92	5.76E-07
43	0.002969	93	4.78E-07
44	0.002754	94	3.97E-07
45	0.002552	95	3.3E-07
46	0.002372	96	2.74E-07
47	0.00218	97	2.27E-07
48	0.002085	98	1.89E-07
49	0.001731	99	1.56E-07
50	0.001437	100	1.3E-07
		Σ π i=	0.995177

Absorbing state Markov Chain problem:

Solution: ① NO GRP:

$$P_T = \begin{array}{cc} \text{Hospital} & \text{Home} \end{array} \begin{pmatrix} 0.991 & 0.003 \\ 0.025 & 0.969 \end{pmatrix} \qquad F=(I-P_T)^{-1} = \begin{array}{cc} \text{Hos.} & \text{Home} \end{array} \begin{pmatrix} 151.9608 & 14.7059 \\ 122.5490 & 44.1176 \end{pmatrix}$$

obviously, any patients must come to hospital first, and then they will decide the next step. So, at first, a patient must be at hospital.

$$E_1(\text{Money}) = 151.9608 \times \$655 + 14.7059 \times \$226 = 102857.86 \text{ dollars.}$$

② With GRP:

$$P_T = \begin{array}{cccc} \text{GRP} & \text{HOS} & \text{HOME (GRP)} & \text{HOME (Direct)} \end{array} \begin{pmatrix} 0.854 & 0.028 & 0.112 & 0 \\ 0.013 & 0.978 & 0 & 0.003 \\ 0.025 & 0 & 0.969 & 0 \\ 0 & 0.025 & 0 & 0.969 \end{pmatrix}$$

$$F = (I-P_T)^{-1} = \begin{array}{cccc} \text{GRP} & \text{HOS} & \text{HOME (GRP)} & \text{HOME (Direct)} \end{array} \begin{pmatrix} 26.9632 & 38.5569 & 97.4153 & 3.7313 \\ 17.9014 & 76.6696 & 64.6761 & 7.4196 \\ 21.7445 & 31.0943 & 110.8188 & 3.0091 \\ 14.4366 & 61.8303 & 52.1581 & 38.2416 \end{pmatrix}$$

$$E_2(\text{Money}) = 76.6696 \times \$655 + 17.9014 \times \$680 + (64.6761 + 7.4196) \times \$226$$
$$= \$78685.17$$

(a) $\because E_2 < E_1$ $\therefore$ GRP does save money.

(b) ① $E_1(\text{Months in Hos.}) = 151.96$

② $E_2(\text{Months in Hos.}) = 76.67$.

Markov "Census"-type Model problem. :

Solution:

$$\begin{array}{c|cccc} & V & R & K & \text{Leave} \\ \hline V & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ R & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ K & 0 & \frac{2}{3} & \frac{1}{3} & 0. \\ L & 0 & 0 & 0 & 1 \end{array}$$

Goal $N = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$

$$F = (I - P_T)^{-1} = \begin{array}{c} \begin{matrix} V & R & K \end{matrix} \\ \begin{pmatrix} 3 & 3 & 3 \\ 3 & 6 & 4.5 \\ 3 & 6 & 6 \end{pmatrix} \end{array}$$

$$F^{-1^T} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$H = F^{-1^T} N = \begin{pmatrix} \frac{5}{3} \\ 0 \\ 0 \end{pmatrix}$$

Answer: Each month, Federation introduces $\$ \frac{5}{3}$ billion into Vulcan.