# ISE 426
# Optimization models and applications

Lecture 5 — September 10, 2015

- Graphical solutions of LPs
- Production planning problem
- Shortest path problem
- Optimization on graphs

Reading:
WV 3.10, 7.1 and 8.2

Homework #1 will be out after class! It is due **Thursday, September 17.** Check the CourseSite.

# Example: Transportation problem

- A large manufacturing company produces liquid nytrogen in five plants spread out in East Pennsylvania
- Each plant has a monthly production capacity

| Plant | $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Capacity | $p_i$ | 120 | 95 | 150 | 120 | 140 |

- It has seven retailers in the same area
- Each retailer has a monthly demand to be satisfied

| Retailer | $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Demand | $d_j$ | 55 | 72 | 80 | 110 | 85 | 30 | 78 |

- transportation between any plant $i$ and any retailer $j$ has a cost of $c_{ij}$ dollars per volume unit of nytrogen
- $c_{ij}$ is **constant** and depends on the distance between $i$ and $j$

$\Rightarrow$ find how much nitrogen to be transported from each plant to each retailer

- ... while minimizing the total transportation cost

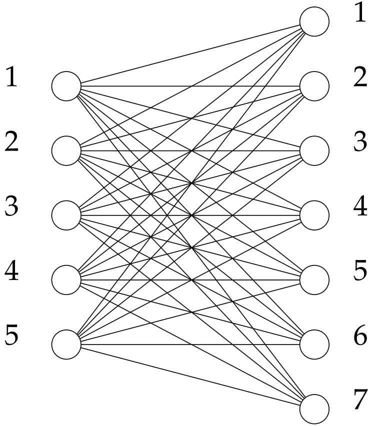
# Transportation model

Variables: qty from plant $i$ to retailer $j$: $x_{ij}$ (non-negative)

Constraints:

1. capacity: $\sum_{j=1}^{7} x_{ij} \leq p_i \quad \forall i$
2. demand: $\sum_{i=1}^{5} x_{ij} \geq d_j \quad \forall j$

Objective function: total transportation cost, $\sum_{i=1}^{5} \sum_{j=1}^{7} c_{ij} x_{ij}$



# Example: Production planning

A small firm produces plastic for the car industry.

- At the beginning of the year, it knows exactly the demand $d_i$ of plastic for every month $i$.
- It also has a maximum production capacity of $P$

  and an inventory capacity of $C$.
- The inventory is empty on 01/01

  and has to be empty again on 12/31
- production has a monthly cost $c_i$

What do we produce at each month to minimize total production cost while satisfying demand?

# Production planning. What variables?

- How much to produce each month $i$: $x_i, \ i = 1, 2 \ldots, 12$
- Anything else?

# Production planning. What variables?

- How much to produce each month $i$: $x_i, \ i = 1, 2 \ldots, 12$
- Anything else?
- The inventory level at the beginning of each month (demand is satisfied by part of production **and** part of inventory): $y_i, \ i = 0, 1, 2 \ldots, 12$
- Why 0? Because we need to know (actually we need to constrain!) the inventory on 01/01 and on 12/31.

# Production planning. What constraints?

- Production capacity constraint: $x_i \leq P \quad \forall i = 1, 2 \ldots, 12$

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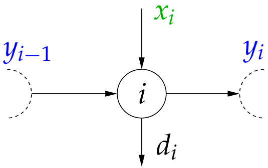
- Production capacity constraint: $x_i \leq P \quad \forall i = 1, 2 \ldots, 12$
- Inventory capacity constraint: $y_i \leq C \quad \forall i = 1, 2 \ldots, 11$
- Beginning/end of year: $y_0 = 0, \quad y_{12} = 0$

# Production planning. What constraints?

- Production capacity constraint: $x_i \leq P \quad \forall i = 1, 2 \ldots, 12$
- Inventory capacity constraint: $y_i \leq C \quad \forall i = 1, 2 \ldots, 11$
- Beginning/end of year: $y_0 = 0, \quad y_{12} = 0$
- What goes in must go out…



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- Production capacity constraint: $x_i \leq P \quad \forall i = 1, 2 \ldots, 12$
- Inventory capacity constraint: $y_i \leq C \quad \forall i = 1, 2 \ldots, 11$
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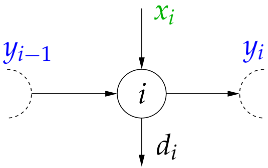
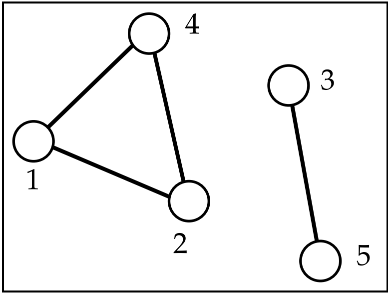



$$\Rightarrow \quad x_i + y_{i-1} = d_i + y_i \quad \forall i = 1, 2 \ldots, 12$$

# Optimization on graphs

A (**undirected**) graph is defined as

- a set $V$ of nodes (or *vertices*)
- a set $E$ of *edges*
- each edge is a **subset** containing two nodes of $V$

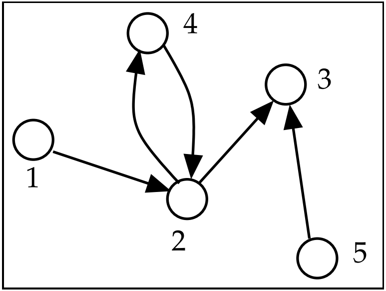



$$V = \{1, 2, 3, 4, 5\}$$
$$E = \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 5\}\}$$

# Directed graphs

A **directed** graph (or *digraph*) is defined as

- a set $V$ of nodes (or *vertices*)
- a set $A$ of *arcs*
- each arc is an **ordered pair** of nodes of $V$




$$V = \{1, 2, 3, 4, 5\}$$
$$A = \{(1, 2), (2, 3), (4, 2), (2, 4), (5, 3)\}$$

# Graphs are **useful**!

…because they can model

- road networks
- gas/oil pipelines
- telecommunication networks
- electronic circuits

Optimization problems often arise in the management of network-like structures.

$\Rightarrow$ Variables, constraints, obj. f. related to nodes and edges/arcs.

# The shortest path problem

Given

- a directed graph $G = (V, A)$,
- a function $c : A \to \mathbb{R}_+$, and
- two nodes $s$ and $t$ of $V$,

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find a subset $P = \{(s, i_1), (i_1, i_2) \ldots, (i_k, t)\}$ of $A$ forming a path from $s$ to $t$ whose length, $c_{si_1} + c_{i_1 i_2} + \ldots + c_{i_k t}$, is minimum.

- Countless applications, e.g. GPS navigation systems.

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- Countless applications, e.g. GPS navigation systems.
- Variables - $x_{ij} = \begin{cases} 1 & \text{if travel from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$

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- Variables - $x_{ij} = \begin{cases} 1 & \text{if travel from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$

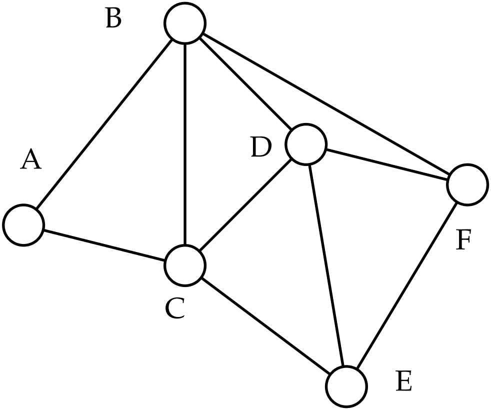
$$\begin{array}{rll} \min & \sum_{(i,j)\in A} c_{ij} x_{ij} & \\ \text{s.t.} & \sum_{j\in V:(i,j)\in A} x_{ij} - \sum_{j\in V:(j,i)\in A} x_{ji} = b_i & \forall i \in V \\ & x_{ij} \geq 0 & \forall (i,j) \in A \end{array}$$

where $b_i = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}$

# An example: the shortest path problem

For simplicity, the graph below is undirected, but we can assume for each edge there are two oppositely oriented arcs.

Suppose the problem is to compute the shortest path A $\to$ F.



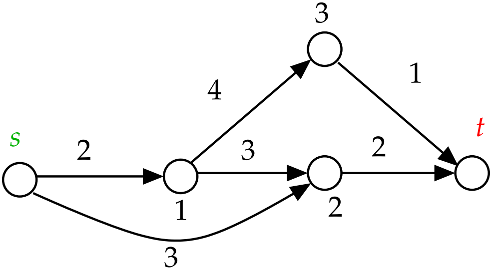
# The shortest path problem: primal

$$
\begin{array}{llll}
\min & c_{AB}x_{AB} + c_{BA}x_{BA} + \ldots + c_{EF}x_{EF} & +c_{FE}x_{FE} & \\
 & x_{AB} + x_{AC} & -x_{BA} - x_{CA} & = 1 \\
 & x_{BA} + x_{BC} + x_{BD} + x_{BF} & -x_{AB} - x_{CB} - x_{DB} - x_{FB} & = 0 \\
 & x_{CA} + x_{CB} + x_{CD} + x_{CE} & -x_{AC} - x_{BC} - x_{DC} - x_{EC} & = 0 \\
 & x_{DB} + x_{DC} + x_{DE} + x_{DF} & -x_{BD} - x_{CD} - x_{ED} - x_{FD} & = 0 \\
 & x_{EC} + x_{ED} + x_{EF} & -x_{CE} - x_{DE} - x_{FE} & = 0 \\
 & x_{FB} + x_{FD} + x_{FE} & -x_{BF} - x_{DF} - x_{DF} & = -1 \\
 & x_{AB}, x_{BA} \ldots, x_{EF}, x_{FE} \geq 0 & &
\end{array}
$$

- We can express this as $\min\{c^\top x : Ax = b, x \geq 0\}$
- $A$ is the adjacency matrix of $G$
- $|V|$ constraints, $|A|$ variables
- All constraints are equalities

# Problem 1: oil pipeline[2]

An oil pipeline pumps oil from an oil well $s$ to an oil refinery $t$.




Each pipe has its own monthly capacity (in mega-barrels[1]).
Assuming for now an infinite supply of oil at $s$,

⇒ maximize the amount of oil arriving at $t$ each month

- while not exceeding pipe capacities

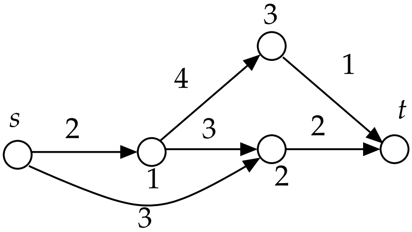
[1]One mega-barrel = $10^6$ barrels

[2]Winston&Venkataramanan, page 420.

# Max-Flow

- The Maximum Flow problem is classical in Optimization
- Can be solved with a very simple and neat algorithm
- We'll see a Linear Programming **model** of this problem
- Variables: oil flowing between each node pair
- Constraints: Oil is conserved at intermediate nodes;
  There is a maximum capacity on each pipe.
- Objective function: The total oil arriving at $t$
  (note: this is exactly the amount of oil that left $s$)

# Max-Flow




- Variables: oil flowing on each arc:
  $x_{s1}, x_{s2}, x_{12}, x_{13}, x_{2t}, x_{3t}$
- Constraints: oil is conserved at intermediate nodes
  i.e. what enters node 1 exits node 1: $x_{s1} = x_{12} + x_{13}$
  what enters node 2 exits node 2: $x_{s2} + x_{12} = x_{2t}$
  what enters node 3 exits node 3: $x_{13} = x_{3t}$
- Constraints: there is a maximum capacity on each pipe,
  $x_{s1} \leq 2$, $x_{s2} \leq 3$, $x_{12} \leq 3$, $x_{13} \leq 4$, $x_{2t} \leq 2$, $x_{3t} \leq 1$
- Objective function: The total oil at node $t$: $x_{2t} + x_{3t}$
- this should be the same oil that left $s$: $x_{s1} + x_{s2}$

# Max-Flow: the model

$$
\begin{aligned}
\max \quad & x_{2t} + x_{3t} \\
x_{s1} &= x_{12} + x_{13} \\
x_{s2} + x_{12} &= x_{2t} \\
x_{13} &= x_{3t} \\
0 \le x_{s1} &\le 2 \\
0 \le x_{s2} &\le 3 \\
0 \le x_{12} &\le 3 \\
0 \le x_{13} &\le 4 \\
0 \le x_{2t} &\le 2 \\
0 \le x_{3t} &\le 1
\end{aligned}
$$

Could re-write objective function as $\max \ x_{s1} + x_{s2}$

# Max-Flow: the general model

Consider a digraph $G = (V, A)$:

- two nodes $s$ and $t$ of $V$ act as *source* and *destination*.
- each arc $(i, j) \in A$ has a capacity $c_{ij}$

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  In general, $x_{ij}$ can be $\neq x_{ji}$ (opposite flow)

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| | | |
|---|---|---|
| At an interm. node, $\forall i \in V : s \neq i \neq t$ | what enters | will leave |

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| At an interm. node, | what enters | | will leave |
|---|---|---|---|
| $\forall i \in V : s \neq i \neq t$ | $\sum_{j \in V : (j,i) \in A} x_{ji}$ | $=$ | $\sum_{j \in V : (i,j) \in A} x_{ij}$ |

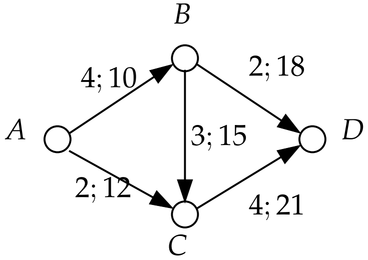
- Constraints: min and max flow, $0 \leq x_{ij} \leq c_{ij} \forall (i, j) \in A$

# Max-Flow: the general model

- Objective Function: flow entering $t$, i.e., $\sum_{j \in V:(j,t) \in A} x_{jt}$
- (same that leaves $s$, i.e. $\sum_{j \in V:(s,j) \in A} x_{sj}$)

$$
\begin{array}{rll}
\max & \sum_{j \in V:(j,t) \in A} x_{jt} & \\
\text{s.t.} & \sum_{j \in V:(j,i) \in A} x_{ji} = \sum_{j \in V:(i,j) \in A} x_{ij} & \forall i \in V : s \neq i \neq t \\
& 0 \leq x_{ij} \leq c_{ij} & \forall (i,j) \in A
\end{array}
$$

# Problem 2: another oil pipeline




- Oil company X now uses company Y's pipeline
- Each month, X wants to pump 5 mega-barrels from A to D,
- Y reserves part of the capacity and charges a cost/flow

A→B allows $\leq$ 4 mega-barrels, cost of 10k$/mega-barrel
A→C allows $\leq$ 2 at a cost of 12
B→C allows $\leq$ 3, cost: 15
B→D allows $\leq$ 2, cost: 18
C→D allows $\leq$ 4, cost: 21

⇒ Can the company send oil from A to D?

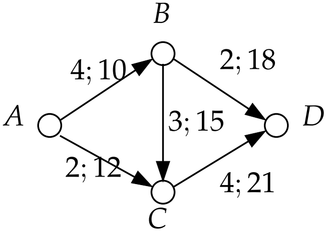
- How to do it at minimum cost?

# Minimum cost flow

Another classical problem in Optimization

- Variables: same as in the Max-Flow problem, i.e., quantity of oil flowing on each arc (i.e. between each node pair)
- Constraints: same as in the Max-Flow problem, plus:
- ⋆ the flow leaving node $A$ ($=$ entering node $D$) has to be equal to 5 mega-barrels
- Objective function: total flow cost

# Min-Cost-Flow




- Variables: oil flowing on each arc:
  $x_{AB}, x_{AC}, x_{BC}, x_{BD}, x_{CD}$
- Constraints: oil does not evaporate at interm. nodes
  i.e. what enters B exits B: $x_{AB} = x_{BC} + x_{BD}$
  what enters C exits C: $x_{AC} + x_{BC} = x_{CD}$
- Constraints: there is a maximum capacity on each pipe,
  $x_{AB} \leq 4, x_{AC} \leq 2, x_{BC} \leq 3, x_{BD} \leq 2, x_{CD} \leq 4$
- Constraint: required flow must leave $A$, i.e., $x_{AB} + x_{AC} = 5$
- Objective function: The total pumping cost:
  $10x_{AB} + 12x_{AC} + 15x_{BC} + 18x_{BD} + 21x_{CD}$

# Min-Cost-Flow: the model

$$\min \quad 10x_{AB} + 12x_{AC} + 15x_{BC} + 18x_{BD} + 21x_{CD}$$

$$\begin{aligned}
x_{AB} &= x_{BC} + x_{BD} \\
x_{AC} + x_{BC} &= x_{CD} \\
x_{AB} + x_{AC} &= 5
\end{aligned}$$

$$\begin{aligned}
&0 \leq x_{AB} \leq 4 \\
&0 \leq x_{AC} \leq 2 \\
&0 \leq x_{BC} \leq 3 \\
&0 \leq x_{BD} \leq 2 \\
&0 \leq x_{CD} \leq 4
\end{aligned}$$

# Min-Cost-Flow: the general model

Consider a digraph $G = (V, A)$:

- two nodes $s$ and $t$ of $V$ act as *source* and *destination*.
- each arc $(i, j) \in A$ has a capacity $c_{ij}$ and a cost $d_{ij}$
- a *required flow* $r$
- Variables: flow on each arc $(i, j)$, call it $x_{ij}$
- Constraints: conservation of flow at intermediate node $i$:

| At an interm. node, | what enters | | must leave |
|---|---|---|---|
| $\forall i \in V : s \neq i \neq t$ | $\sum_{j \in V:(j,i) \in A} x_{ji}$ | $=$ | $\sum_{j \in V:(i,j) \in A} x_{ij}$ |

- Constraints: min and max flow, $0 \leq x_{ij} \leq c_{ij}\ \forall (i, j) \in A$
- Constraint: required flow leaves $s$, i.e., $\sum_{j \in V:(s,j) \in A} x_{sj} = r$

# Min-Cost-Flow: the general model

- Objective Function: cost of flow, i.e., $\sum_{(i,j)\in A} d_{ij}x_{ij}$

$$
\begin{array}{rll}
\min & \sum_{(i,j)\in A} d_{ij}x_{ij} & \\
\text{s.t.} & \sum_{j\in V:(j,i)\in A} x_{ji} = \sum_{j\in V:(i,j)\in A} x_{ij} & \forall i \in V : s \neq i \neq t \\
& \sum_{j\in V:(s,j)\in A} x_{sj} = r & \\
& 0 \leq x_{ij} \leq c_{ij} & \forall (i,j) \in A
\end{array}
$$

# Min-Cost-Flow: the general model with multiple sources and sinks

Consider a digraph $G = (V, A)$:

- Sets of nodes $s \in S \subseteq V$ and $t \in T \subseteq V$ act as *sources* and *destinations*.
- each arc $(i, j) \in A$ has a capacity $c_{ij}$ and a cost $d_{ij}$
- Each destination $t \in T$ has a *required flow* demand $d_t$
- Variables: flow on each arc $(i, j)$, call it $x_{ij}$
- Constraints: conservation of flow at intermediate node $i$:

| At an interm. node, | what enters | | must leave |
|---|---|---|---|
| $\forall i \in V / \{S \cup T\}$ | $\sum_{j \in V:(j,i) \in A} x_{ji}$ | $=$ | $\sum_{j \in V:(i,j) \in A} x_{ij}$ |

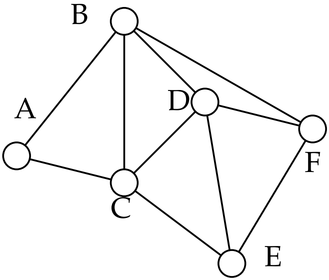
- Constraints: min and max flow, $0 \leq x_{ij} \leq c_{ij} \; \forall (i, j) \in A$
- Constraints: required flow demand for each $t \in T$, i.e., $\sum_{j \in V:(j,t) \in A} x_{jt} - \sum_{j \in V:(t,j) \in A} x_{tj} = d_t$

# What about the shortest path problem?

$$
\begin{array}{lll}
\min & \sum_{(i,j)\in A} c_{ij}x_{ij} & \\
\text{s.t.} & \sum_{j\in V:(i,j)\in A} x_{ij} - \sum_{j\in V:(j,i)\in A} x_{ji} = b_i & \forall i \in V \\
& x_{ij} \geq 0 & \forall (i,j) \in A
\end{array}
$$

$$
\text{where } b_i = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}
$$

The problem is to compute the shortest path A $\rightarrow$ F.



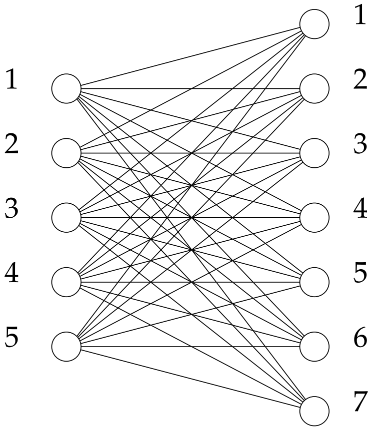
# Transportation model?

Variables: qty of product from producer $i$ to location $j$: $x_{ij}$ (non-negative)

Constraints:

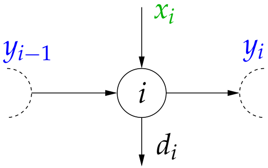
1. capacity: $\sum_{j=1}^{7} x_{ij} \leq p_i \quad \forall i$
2. demand: $\sum_{i=1}^{5} x_{ij} \geq d_j \quad \forall j$

Objective function: total transportation cost, $\sum_{i=1}^{5} \sum_{j=1}^{7} c_{ij} x_{ij}$



# Production planning?

- Production capacity constraint: $x_i \leq P \quad \forall i = 1, 2 \ldots, 12$
- Beginning/end of year: $y_0 = 0, \quad y_{12} = 0$
- What goes in must go out…

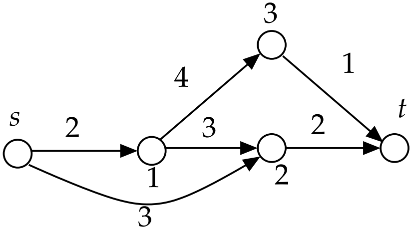



$$\Rightarrow \quad x_i + y_{i-1} = d_i + y_i \;\; \forall i = 1, 2 \ldots, 12$$

# Generalizations

- multiple sources (have a negative flow balance), multiple destinations (positive flow balance)
- non-zero lower bounds on flows
- nonlinear flow costs
- multiple types of flow on the same arc

# Back to Max-Flow

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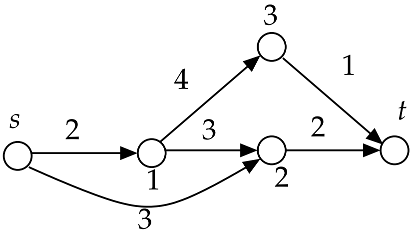
& 0 \leq x_{ij} \leq c_{ij} & \forall (i,j) \in A
\end{array}
$$

# Min-Cut



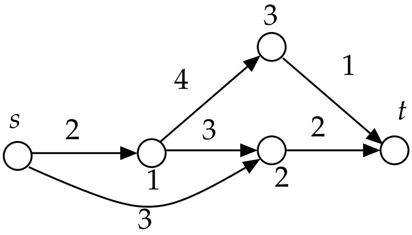

- What is the set of edges of the smallest capacity which, if cut, completely cuts off $s$ from $t$?
- Useful applications: finding bottleneck in networks, damming systems of rivers, military...

# Min-Cut

- Variables for each node $u_i = \{0, 1\}$: 0 if $i$ is cut off from $t$, 1 if $i$ is not cut off from $t$.
- Variables for each arc $z_{ij} = \{0, 1\}$: 1 if arc $(i, j)$ is in the cut, 0 otherwise.
- Objective Function: the capacity of the cut $\sum_{(i,j)\in A} c_{ij} z_{ij}$

$$\begin{array}{rll} \max & \sum_{(i,j)\in A} c_{ij} z_{ij} & \\ \text{s.t.} & z_{ij} \geq u_j - u_i & \\ & u_s = 0 & \\ & u_t = 1 & \\ & 0 \leq z_{ij} & \forall (i,j) \in A \\ & 0 \leq u_i & \forall i \in V \end{array}$$

# Min-Cut = Max-Flow




- **THEOREM:** (Ford, Fullkerson, 1956) The capacity of the minimum cut equals the value of the maximum flow
- **NOTE:** The capacity of any cut is gearter or equal to the value of any feasible flow in the network.

**Can this be done for general LP problems?**