# ISE 426
# Optimization models and applications

Lecture 17 — October 30, 2014

- "Good" and "bad" formulations
- Branch&bound for MILP
- Examples of B&B

Reading:

- Hillier & Lieberman, Chapter 13, 13.4 to 13.5
- Winston & Venkataramanan, Chapter 9
- Winston, Chapter 9

# Relaxations and efficiency

Integer programming problems:

$$
\begin{array}{lllllll}
(IP) & \min & c_1x_1 & +c_2x_2 & \dots & +c_nx_n & \\
& & a_{11}x_1 & +a_{12}x_2 & \dots & +a_{1n}x_n & \leq b_1 \\
& & a_{21}x_1 & +a_{22}x_2 & \dots & +a_{2n}x_n & \leq b_2 \\
& & \vdots & & & & \\
& & a_{m1}x_1 & +a_{m2}x_2 & \dots & +a_{mn}x_n & \leq b_m \\
& & & & & x_i \in \mathbb{Z} & \forall i \in J \subseteq \{1, 2 \dots, n\}
\end{array}
$$

or, for short,

$$
\begin{array}{lll}
(IP) & \min & c^\top x \\
& & Ax \leq b \\
& & x_i \in \mathbb{Z} \quad \forall i \in J \subseteq N,
\end{array}
$$

can be solved using their LP relaxation:

$$
\begin{array}{lll}
(LP) & \min & c^\top x \\
& & Ax \leq b.
\end{array}
$$

A global optimum $z$ of $(LP)$ is a **lower bound** for $(IP)$.

# Relaxations and efficiency

If an optimal solution $x^\star$ of $(LP)$ is feasible for $(IP)$, i.e., for all $i \in J$ we have $x_i^\star \in \mathbb{Z}$, we're done!

This is **not** the case, usually…

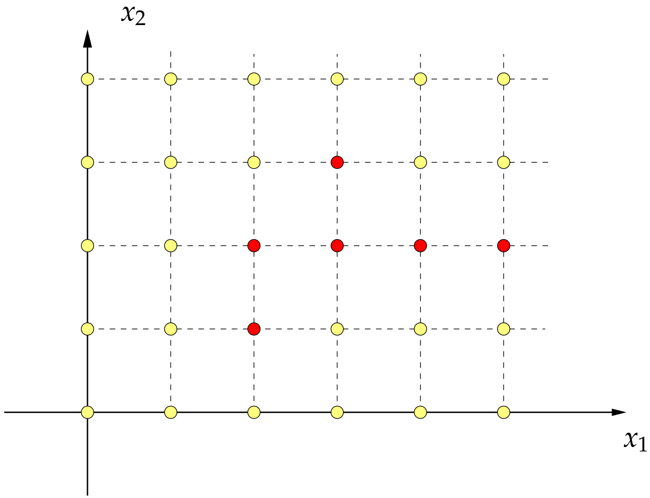
What do we know about the optimal solutions of $(LP)$? They are all *vertices* of the polyhedron

$$\{x \in \mathbb{R}^n : A^\top x \leq b\}$$

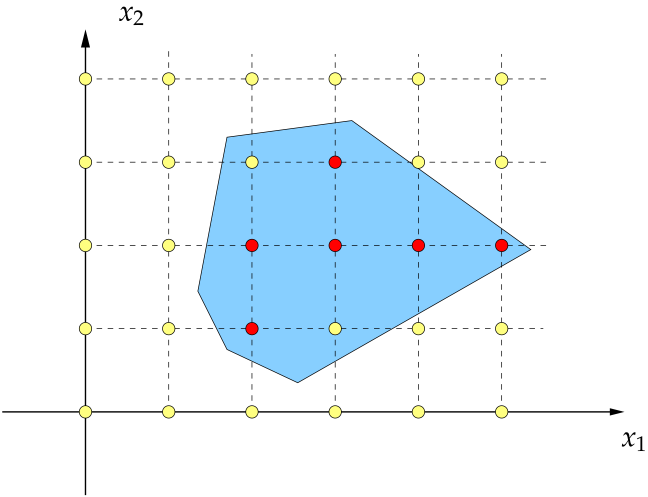
Therefore, it would be just great if all vertices of $(LP)$ were feasible for $(IP)$. Solving IPs would amount to solving LPs, which are a lot easier.

A good model may not achieve just that, but it can help a lot.

# Relaxations, the geometrical standpoint



# Relaxations, the geometrical standpoint



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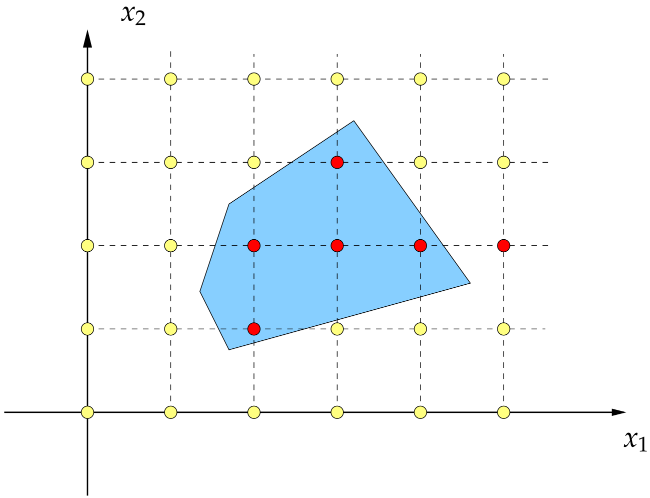

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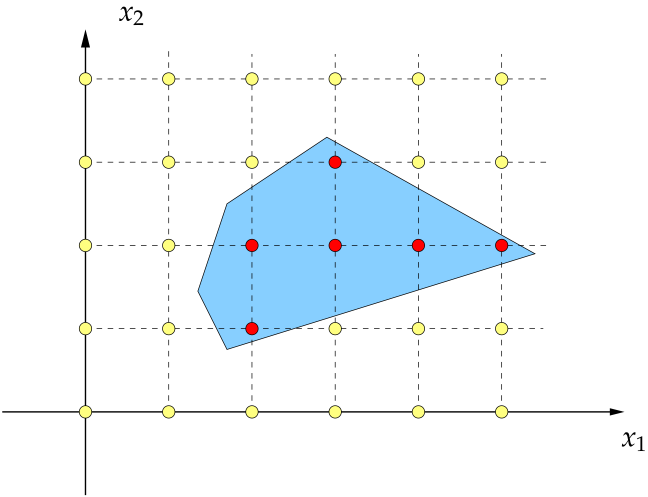

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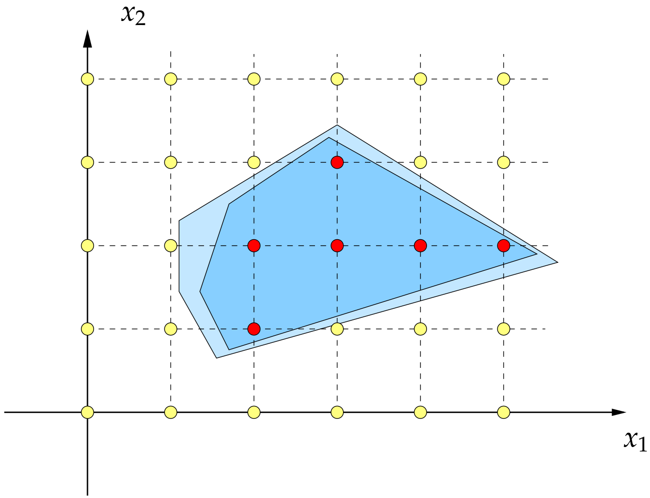

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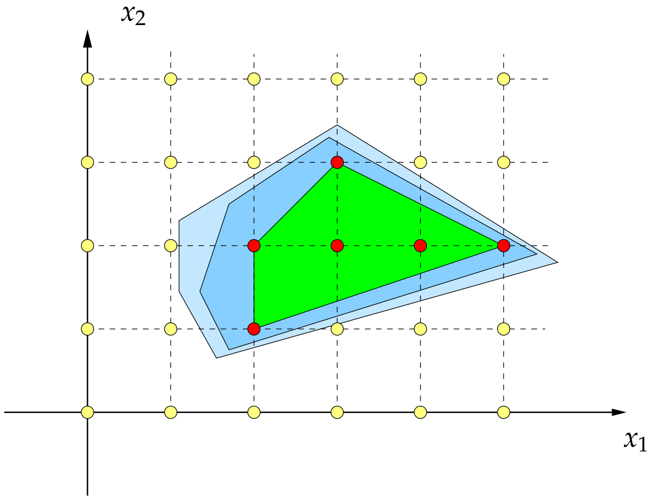

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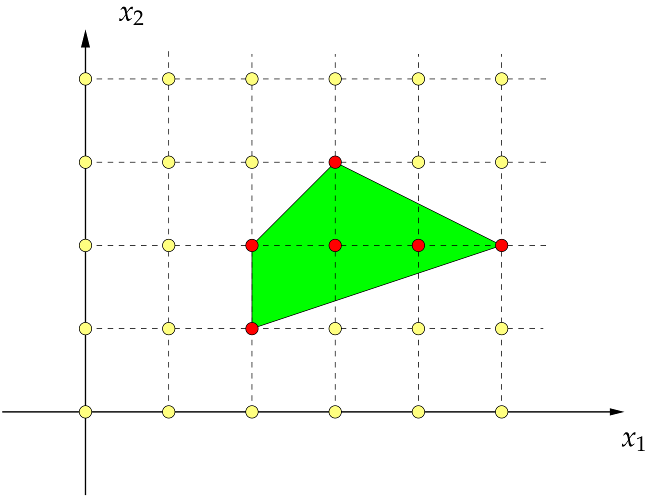


# Relaxations: the clique inequality

Two models for one problem have the same feasible set and global optima, but may be solved differently:

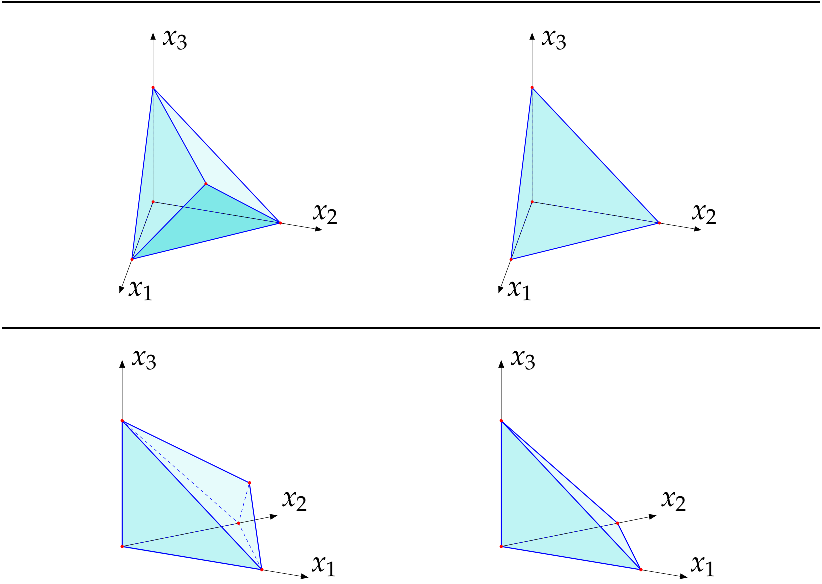
$$\left.\begin{array}{rl} P_1 : \min & -7x_1 - 8x_2 - 9x_3 \\ \text{s.t.} & x_1 + x_2 \le 1 \\ & x_1 + x_3 \le 1 \\ & x_2 + x_3 \le 1 \\ & x_1, x_2, x_3 \in \{0,1\} \end{array}\right\} \equiv \left\{\begin{array}{rl} P_2 : \min & -7x_1 - 8x_2 - 9x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 \le 1 \\ & x_1, x_2, x_3 \in \{0,1\} \end{array}\right.$$

Consider relaxations $R_1, R_2$ of $P_1, P_2$ with $x_i \in [0,1]$.

- $R_1$: optimal soln. $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, obj.f. $-12$: lower bound for $P_1, P_2$
- $R_2$: optimal solution $(0,0,1)$, obj.f. $-9$: lower bound for $P_2$ and $P_1$, and **feasible** for $P_2$ and $P_1$!
- $\Rightarrow$ optimum of $P_1, P_2$: $-9$, and $P_2$ is a better model than $P_1$

# Relaxations: the clique inequality

| $R_1$ | $R_2$ |
|---|---|



# Good vs. bad models: Uncapacitated Facility Location

A set $J$ of retailers has to be served by a set $S$ of plants, yet to be built. We don't know where the plants will be, but there is a set $I$ of potential sites, and there is

- a cost $f_i$ for building plant $i \in I$
- a (transportation) cost $c_{ij}$ from plant $i$ to retailer $j$

Each retailer will be served by exactly one plant . Choose a subset $S$ of $I$ such that the total cost is minimized.

Variables:

- $x_i, i \in I$: 1 if plant $i$ is built, 0 otherwise
- $y_{ij}$ assigns retailer $j$ to plant $i$: 1 if $i$ serves retailer $j$, 0 otherwise

# Good vs. bad models: Uncapacitated Facility Location

Objective function:

$$\sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij}$$

Constraints:

- for each customer, one facility: $\sum_{i \in I} y_{ij} = 1$
- customers go to mall $i$ if it's there:

$$\sum_{j \in J} y_{ij} \le |J| x_i \quad \forall i \in I$$

**or**

$$y_{ij} \le x_i \quad \forall i \in I, j \in J$$

# Good vs. bad models: Uncapacitated Facility Location

$(A)$

$$\begin{aligned} \min \quad & \sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \\ & \sum_{i \in I} y_{ij} = 1 \quad \forall i \in I \\ & \sum_{j \in J} y_{ij} \leq |J| x_i \quad \forall i \in I \\ & x_i, y_{ij} \in \{0, 1\} \end{aligned}$$

$(B)$

$$\begin{aligned} \min \quad & \sum_{i \in I} f_i x_i + \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \\ & \sum_{i \in I} y_{ij} = 1 \quad \forall i \in I \\ & y_{ij} \leq x_i \quad \forall i \in I, j \in J \\ & x_i, y_{ij} \in \{0, 1\} \end{aligned}$$

(A) and (B) are **equivalent**. However, for $|I| = |J| = 40$,

(A) takes 14 hours[1]

(B) takes 2 seconds

[1] On AMPL with an old version of CPLEX.

# How do we solve an Integer Programming problem?

$$
\begin{array}{llrl}
(P) & \min & cx & \\
 & & Ax & \geq b \\
 & & x_i & \in \mathbb{Z}, i \in J \subseteq \{1, 2 \ldots, n\}
\end{array}
$$

- Relaxing integrality gives an LP problem (easy)
- Solving LP gives a lower bound
- But the solution $x^\star$ may have fractional component $x_i^\star \notin \mathbb{Z}$, with $i \in J$ (for example, 3.31), infeasible for $(P)$.

⇒ **Divide** the solution set, **partition** the problem into two new subproblems, $P_1$ and $P_2$, with

$$P_1 : x_i \leq \lfloor x_i^\star \rfloor \qquad P_2 : x_i \geq \lceil x_i^\star \rceil$$

($P_1 : x_i \leq 3$ and $P_2 : x_i \geq 4$) and recursively solve $P_1$ and $P_2$.

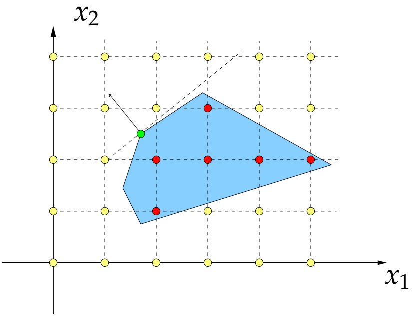
- Good: **no** feasible solution of $P_1$ or $P_2$ has $x_i = 3.31$

# The Branch&Bound - Devide and concour

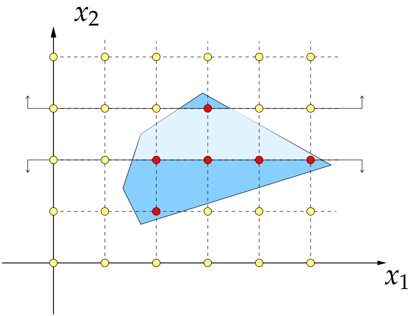
- If we solve the LP relaxation of $P_1$ and find a fractional point, we can recursively branch on $P_1$ and obtain two new subproblems $P_3$ and $P_4$.
- In principle, we have to branch on any node $P_k$ unless its LP relaxation returns a feasible solution or it is infeasible.

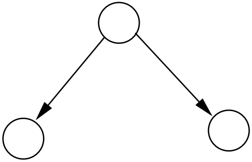
# Example: minimize $11x_1 - 10x_2$

$lb = -9.2$
$ub = ?$



# Example: minimize $11x_1 - 10x_2$



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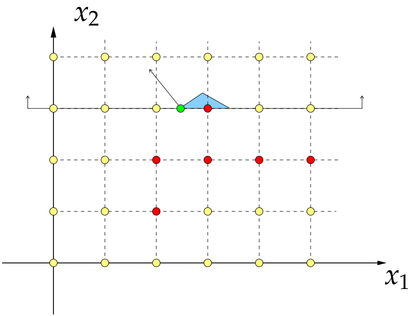


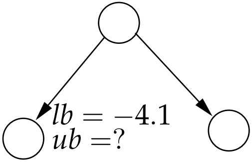

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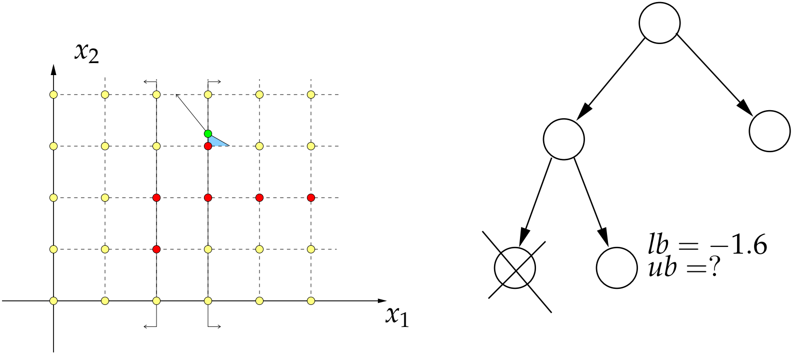

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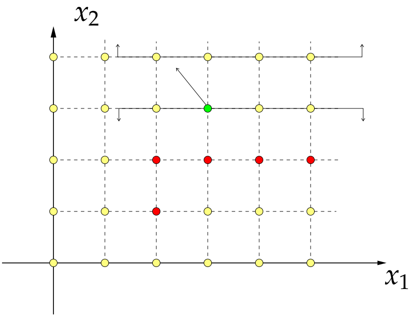


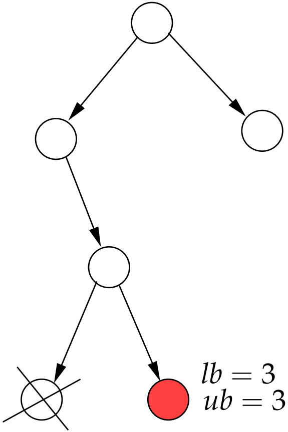

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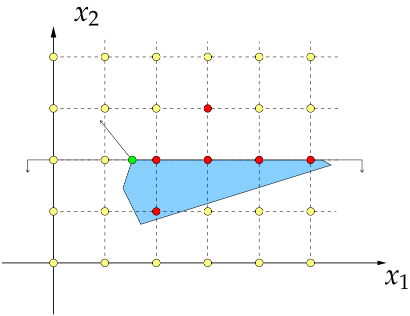


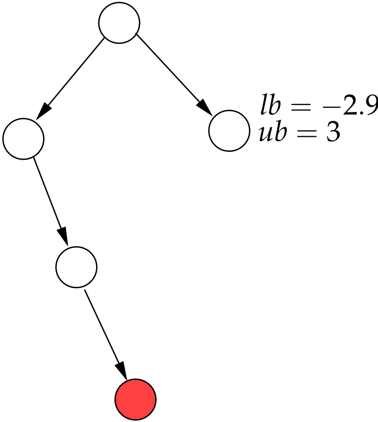

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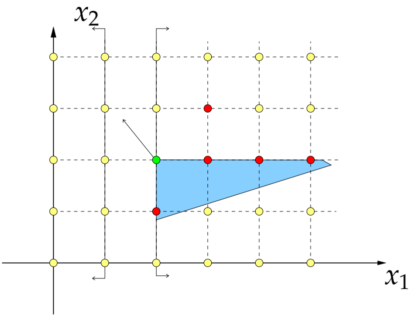


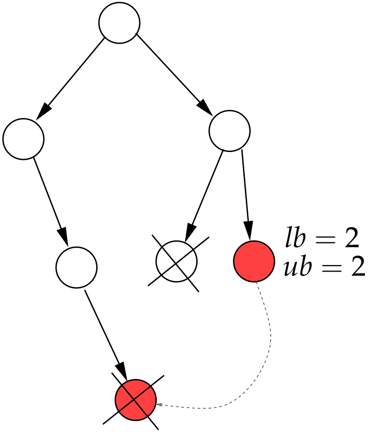

# The **Bound** in Branch&Bound

In practice, the upper bound is very useful! Suppose you just found an **upper** bound of 194.

- $P_3$ has a **lower** bound of 146, $P_4$ of 203
- 203 is a lower bound for $P_4$ $\Rightarrow$ any feasible solution of $P_4$ has objective function value worse than 203 (i.e., $\geq 203$)
- We already have something better (194) $\Rightarrow$ discard $P_4$

How to find upper bounds?

# Example: Knapsack problem

$$
\begin{array}{rrrrrrl}
\min & 3x_1 & +5x_2 & +8x_3 & +6x_4 & +12x_5 & \\
(IP_0) & 10x_1 & +7x_2 & +7x_3 & +5x_4 & +3x_5 & \geq 22 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \in \{0,1\}
\end{array}
$$

The general Knapsack problem is:

$$
\begin{array}{rl}
\min & \sum_{i=1}^{n} w_i x_i \\
 & \sum_{i=1}^{n} c_i x_i \geq C \\
 & x_i \in \{0,1\} \qquad \forall i = 1, 2 \dots, n
\end{array}
$$

Suppose that variables are sorted in non-decreasing order w.r.t. weight/value[2]. The LP relaxation is solved as follows:

1. Set $R := C$
2. **for** $i : 1, 2 \dots, n$
3. **if** $c_i \leq R$, **then** $x_i := 1$; $R := R - c_i$
4. **else** $x_i := R/c_i$; **stop**

[2] Those with a small weight/value ratio are more likely to be chosen.

# Solve the LP relaxation

$$\begin{array}{rrrrrrl}
\min & 3x_1 & +5x_2 & +8x_3 & +6x_4 & +12x_5 & \\
(LP_0) & 10x_1 & +7x_2 & +7x_3 & +5x_4 & +3x_5 & \geq 22 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \in [0,1]
\end{array}$$

- Solve this by the greedy method.
- Solution: $x_0^* = (1, 1, \frac{5}{7}, 0, 0)$, $V_0^* = 11\frac{4}{7}$.
- $V_0^*$ - the lower bound for the problem $IP_0$.
- Split the problem into two cases: $X_3 = 1$ ($IP_1$)and $X_3 = 0$ ($IP_2$) and consider the LP relaxations $LP_1$ and $LP_2$.

# Consider subproblem

Solve $LP_1$ problem $X_3 = 1$ (we will consider $LP_2$ later)

$$
\begin{array}{lrrrrrl}
\min & 3x_1 & +5x_2 & +8 & +6x_4 & +12x_5 & \\
(LP_1) & 10x_1 & +7x_2 & +7 & +5x_4 & +3x_5 & \geq 22 \\
 & x_1, & x_2, & & x_4, & x_5 & \in [0,1]
\end{array}
$$

- Solve this by the greedy method.
- Solution: $x_1^* = (1, \frac{5}{7}, 1, 0, 0)$, $V_1^* = 14\frac{4}{7}$.
- $V_1^*$ - the lower bound for the problem $IP_1$.
- Split the problem into two cases: $X_2 = 1$ and $X_2 = 0$ and consider the LP relaxations $LP_3$ and $LP_4$.

# Consider subproblem

Solve $LP_3$, $X_2 = 1$, $X_3 = 1$.

$$
\begin{array}{rrrrrrl}
\min & 3x_1 & +5 & +8 & +6x_4 & +12x_5 & \\
(LP_1) & 10x_1 & +7 & +7 & +5x_4 & +3x_5 & \geq 22 \\
 & x_1, & & & x_4, & x_5 & \in [0,1]
\end{array}
$$

- Solve this by the greedy method.
- Solution: $x_3^* = (\frac{8}{10}, 1, 1, 0, 0)$, $V_3^* = 15\frac{2}{10}$.
- $V_3^*$ - the lower bound for the problem $IP_3$.
- Split the problem into two cases: $X_2 = 1$ and $X_2 = 0$ and consider the LP relaxations $L_3$ and $L_4$.

# Consider subproblem

Solve $LP_3$, $X_2 = 0$, $X_3 = 1$.

$$\begin{array}{rrrrrl} \min & 3x_1 & +8 & +6x_4 & +12x_5 & \\ (LP_1) & 10x_1 & +7 & +5x_4 & +3x_5 & \geq 22 \\ & x_1, & & x_4, & x_5 & \in [0,1] \end{array}$$

- Solve this by the greedy method.
- Solution: $x_4^* = (1,0,1,1,0)$, $V_4^* = 17$.
- $V_4^*$ - the upper bound for the whole IP problem! Also is the optimal solution to $IP_4$.

# Example: Knapsack problem

Solve $LP_2$ problem $X_3 = 0$

$$
\begin{array}{rrrrrl}
\min & 3x_1 & +5x_2 & +6x_4 & +12x_5 & \\
(LP_1) & 10x_1 & +7x_2 & +5x_4 & +3x_5 & \geq 22 \\
 & x_1, & x_2, & x_4, & x_5 & \in [0,1]
\end{array}
$$

- Solve this by the greedy method.
- Solution: $x_2^* = (1,1,0,1,0)$, $V_2^* = 14$.
- $V_2^*$ is the new upper bound for the entire problem!! Also optimal solution to $IP_2$.
- Key observation: lower bound for $IP_3$ is bigger than upper bound for the entre problem. DONE!!!!!

# Branch&Bound

- $z^{\text{ub}} = +\infty$
- $\mathcal{L} \leftarrow \{P\}$
- **while** $\mathcal{L} \neq \emptyset$

  Choose $P'$ from $\mathcal{L}$ and set $\mathcal{L} = \mathcal{L} \setminus \{P'\}$

  Relax $P' \rightarrow$ obtain $LP'$

  solve $LP'$, obtain solution $x^{LP'}$ and lower bound $z^{LP'}$

  look for solution feasible for $P'$, obj. $z^{P'}$

  **if** $z^{P'} < z^{\text{ub}}$, set $z^{\text{ub}} \leftarrow z^{P'}$

  **if** $z^{LP'} < z^{\text{ub}}$ and $x^{LP'}$ infeasible for $P$

  choose $x_i : x_i^{LP'} \notin \mathbb{Z}$

  create $P'' : x_i :\leq \lfloor x_i^{LP'} \rfloor$

  create $P''' : x_i :\geq \lceil x_i^{LP'} \rceil$

  $\mathcal{L} \leftarrow \mathcal{L} \cup \{P'', P'''\}$