ISE426 Optimization models and applications, Fall 2015
Homework 3 Solution,

1 Goal programming

Part 2. The nonpreemtive goal programming model is given by

$$
\begin{aligned}
\min \quad & y_1^+ + y_2^- + y_3^+ + y_3^- + y_4^+ + y_5^- + y_6^+ - y_6^+ + y_7^- \\
\text{s.t.} \quad & \\
& -2x_3 + 3x_4 \leq y_1^+, \\
& x_1 + x_4 \geq 1 - y_2^-, \\
& x_1 - 2x_3 + 2x_4 = y_3^+ - y_3^-, \\
& x_2 + x_4 \leq y_4^+, \\
& x_1 + x_2 - x_3 + x_4 \geq 1 - y_5^-, \\
& 3x_1 - x_2 - x_3 - x_4 = y_6^+ - y_6^-, \\
& x_1 - x_2 + 2x_3 \geq 2 - y_7^-, \\
& x_1, x_2, x_3, x_4, y_1^+, y_2^-, y_3^+, y_3^-, y_4^+, y_5^-, y_6^+, y_6^-, y_7^- \geq 0.
\end{aligned}
$$

Part 3. To get a feasible model, we need to eliminate a portion of the constraints from consideration. Thus, a subset with maximum cardinality is obtained by minimizing the number of constraints needed to be relaxed in order to get a feasible model.

$$
z_i = \begin{cases} 1 & \text{if} \quad \text{i-th constraint is violated} \\ 0 & \text{Otherwise} \end{cases}
$$

Hence we are minimizing this value

This problem is formulated as a mixed integer program as follows :

$$
\begin{aligned}
\min \quad & z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 \\
\text{s.t.} \quad & \\
& -2x_3 + 3x_4 \leq Mz_1, \\
& x_1 + x_4 \geq 1 - Mz_2, \\
& x_1 - 2x_3 + 2x_4 \leq Mz_3, \\
& x_1 - 2x_3 + 2x_4 \geq -Mz_3, \\
& x_2 + x_4 \leq Mz_4, \\
& x_1 + x_2 - x_3 + x_4 \geq 1 - Mz_5, \\
& 3x_1 - x_2 - x_3 - x_4 \leq Mz_6, \\
& 3x_1 - x_2 - x_3 - x_4 \geq -Mz_6, \\
& x_1 - x_2 + 2x_3 \geq 2 - Mz_7, \\
& x_1, x_2, x_3, x_4 \geq 0 \\
& z_1, z_2, z_3, z_4, z_5, z_6, z_7 \in \{0, 1\}.
\end{aligned}
$$

where M is a sufficiently large number.

Alternatively, you can maximize the number of the feasible constraint, it is also true. In that case you should define:

$$t_i = \begin{cases} 1 & \text{if} \quad \text{i-th constraint is feasible} \\ 0 & \text{Otherwise} \end{cases}$$

Apparently, $t_i = 1 - z_i$ and we are maximizing $t_1 + t_2 + ... + t_7$

$$\begin{aligned} \max \quad & t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 \\ \text{s.t.} \quad & \\ & -2x_3 + 3x_4 \leq M(1 - t_1), \\ & x_1 + x_4 \geq 1 - M(1 - t_2), \\ & x_1 - 2x_3 + 2x_4 \leq M(1 - t_3), \\ & x_1 - 2x_3 + 2x_4 \geq -M(1 - t_3), \\ & x_2 + x_4 \leq M(1 - t_4), \\ & x_1 + x_2 - x_3 + x_4 \geq 1 - M(1 - t_5), \\ & 3x_1 - x_2 - x_3 - x_4 \leq M(1 - t_6), \\ & 3x_1 - x_2 - x_3 - x_4 \geq -M(1 - t_6), \\ & x_1 - x_2 + 2x_3 \geq 2 - M(1 - t_7), \\ & x_1, x_2, x_3, x_4 \geq 0 \\ & t_1, ..., t_7 \in \{0, 1\}. \end{aligned}$$

Part 4. Based on the results obtained from the previous part, the 4th constraint creates infeasibility. (We solved the model in Part 3 and see that $t_4 = 0$ and $t_i = 1 \quad \forall i \neq 4$)

Hence, we remove 4th constraint from our original problem and obtain a feasible problem.

The feasible model (where we consider only the selected constraints) is given by

$$\begin{aligned} \min \quad & 3x_1 + 2x_2 - 1x_3 + 25x_4 \\ \text{s.t.} \quad & \\ & -2x_3 + 3x_4 \leq 0, \\ & x_1 + x_4 \geq 1, \\ & x_1 - 2x_3 + 2x_4 = 0, \\ & x_1 + x_2 - x_3 + x_4 \geq 1, \\ & 3x_1 - x_2 - x_3 - x_4 = 0, \\ & x_1 - x_2 + 2x_3 \geq 2, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Part 5. According to the largest set of feasible constraints, the formulation of the preemptive model would be

Step 1:

$$\begin{aligned}
\min \quad & y_1^+ + y_2^- + y_3^+ + y_3^- + y_5^- + y_6^+ + y_6^- + y_7^- \\
\text{s.t.} \quad & \\
& -2x_3 + 3x_4 \leq y_1^+, \\
& x_1 + x_4 \geq 1 - y_2^-, \\
& x_1 - 2x_3 + 2x_4 = y_3^+ - y_3^-, \\
& x_2 + x_4 \leq y_4^+, \\
& x_1 + x_2 - x_3 + x_4 \geq 1 - y_5^-, \\
& 3x_1 - x_2 - x_3 - x_4 = y_6^+ - y_6^-, \\
& x_1 - x_2 + 2x_3 \geq 2 - y_7^-, \\
& x_1, x_2, x_3, x_4, y_1^+ + y_2^- + y_3^+ + y_3^- + y_4^+ + y_5^- + y_6^+ + y_6^- + y_7^- \geq 0.
\end{aligned}$$

The optimum solution to this problem is given by $y_1^+ = y_2^- = y_3^+ = y_3^- = +y_5^- = y_6^+ = y_6^- = y_7^- = 0$. Hence, the problematic one is y_4^+. We will work on that on step 2.

Step 2:

$$\begin{aligned}
\min \quad & y_4^+ \\
\text{s.t.} \quad & \\
& -2x_3 + 3x_4 \leq 0, \\
& x_1 + x_4 \geq 1, \\
& x_1 - 2x_3 + 2x_4 = 0, \\
& x_2 + x_4 \leq y_4^+, \\
& x_1 + x_2 - x_3 + x_4 \geq 1, \\
& 3x_1 - x_2 - x_3 - x_4 = 0, \\
& x_1 - x_2 + 2x_3 \geq 2, \\
& x_1, x_2, x_3, x_4, y_4^+ \geq 0.
\end{aligned}$$

The result will give $y_4^+ = 1.227272727$.

2 Logic

Note: all variables are assumed **binary** unless otherwise specified.

Part 1. $b \vee \neg c \vee (\neg d \wedge a)$

Answer: Let x_1 denote $(\neg d \wedge a)$

$$\begin{aligned}
x_b + (1 - x_c) + x_1 \geq & \quad 1 \\
x_1 + 1 \geq & \quad (1 - x_d) + x_a \\
x_1 \leq & \quad (1 - x_d) \\
x_1 \leq & \quad x_a
\end{aligned}$$

Part 2. The opposite of $b \vee \neg c \vee (\neg d \wedge a)$

Answer: the opposite is

$$\begin{aligned}\neg(b \vee \neg c \vee (\neg d \wedge a)) &= (\neg b) \wedge (\neg(\neg c \vee (\neg d \wedge a))) \\ &= (\neg b) \wedge ((c \wedge \neg(\neg d \wedge a))) \\ &= \neg b \wedge c \wedge (d \vee \neg a)\end{aligned}$$

Let x_1 denote $(\neg d \wedge a)$ again. The statement is true if and only if

$$\begin{aligned}(1 - x_b) = && 1 \\ x_c = && 1 \\ x_1 = && 1 \\ x_1 \leq && x_d + (1 - x_a) \\ x_1 \geq && x_d \\ x_1 \geq && (1 - x_a)\end{aligned}$$

Part 3. $\neg c \wedge d \Rightarrow (a \vee b)$

Let x_1 denote $(\neg c \wedge d)$ and let x_2 denote $(a \vee b)$

Answer: the statement can be formulated as

$$\begin{aligned}x_1 \leq && x_2 \\ x_1 \leq && (1 - x_c) \\ x_1 \leq && (x_d) \\ x_1 + 1 \geq && (1 - x_c) + x_d \\ x_2 \leq && x_a + x_b \\ x_2 \geq && x_a \\ x_2 \geq && x_b\end{aligned}$$

Part 4. $(a \wedge b) \vee (a \wedge c) \Leftrightarrow d \wedge c$

Let x_1 denote $(a \wedge b)$, let x_2 denote $(a \wedge c)$. Let x_t denote $(d \wedge c)$ Also, let x_r denote $(a \wedge b) \vee (a \wedge c)$.

Answer: the statement can be formulated as

$$\begin{aligned}x_r = && x_t \\ x_r \leq && x_1 + x_2 \\ x_1 \leq && x_r \\ x_2 \leq && x_r \\ x_1 \leq && x_a \\ x_1 \leq && x_b \\ x_1 + 1 \geq && x_a + x_b \\ x_2 \leq && x_a \\ x_2 \leq && x_c \\ x_2 + 1 \geq && x_a + x_c \\ x_t \leq && x_d \\ x_t \leq && x_c \\ x_t + 1 \geq && x_d + x_c\end{aligned}$$

Part 5. *Answer:*

Let z_1 denote $2x_1 + x_2 + 3x_3 \leq 5$. Let z_2 denote $2x_6 - x_7 + x_8 \geq 3$. Let z_3 denote $x_9 + x_{10} \geq 1$. Hence we have

$$z_1 = \begin{cases} 1 & \text{if} \quad 2x_1 + x_2 + 3x_3 \leq 5 \\ 0 & \text{Otherwise} \end{cases}$$

Obviously, it means:

$$z_1 = \begin{cases} 1 & \text{if} \quad 2x_1 + x_2 + 3x_3 \leq 5 \\ 0 & \quad 2x_1 + x_2 + 3x_3 > 5 \end{cases}$$

We should write in inequality form for LP formulation. Consider a very small positive number ε. Hence, now we have:

$$z_1 = \begin{cases} 1 & \text{if} \quad 2x_1 + x_2 + 3x_3 \leq 5 \\ 0 & \quad 2x_1 + x_2 + 3x_3 \geq 5 + \varepsilon \end{cases}$$

Write the other variables z_2 and z_3 accordingly. Let z_R denote $2x_1+x_2+3x_3 \leq 5$ AND $2x_6-x_7+x_8 \geq 3$. The statement can be formulated as follows:

$$\begin{aligned}
2x_1 + x_2 + 3x_3 \leq &\quad 5 + M_1(1 - z_1) \\
2x_1 + x_2 + 3x_3 \geq &\quad 5 + \varepsilon - M_1 z_1 \\
2x_6 - x_7 + x_8 \geq &\quad 3 - M_2(1 - z_2) \\
2x_6 - x_7 + x_8 \leq &\quad 3 - \varepsilon + M_2 z_2 \\
x_9 + x_{10} \geq &\quad 1 - M_3(1 - z_3) \\
x_9 + x_{10} \leq &\quad 1 - \varepsilon + M_3 z_3 \\
z_1 + z_2 \leq &\quad 1 + z_R \\
z_1 \geq &\quad z_R \\
z_2 \geq &\quad z_R \\
z_R \leq &\quad z_3
\end{aligned}$$

The values M_1, M_2, M_3 are large enough numbers.

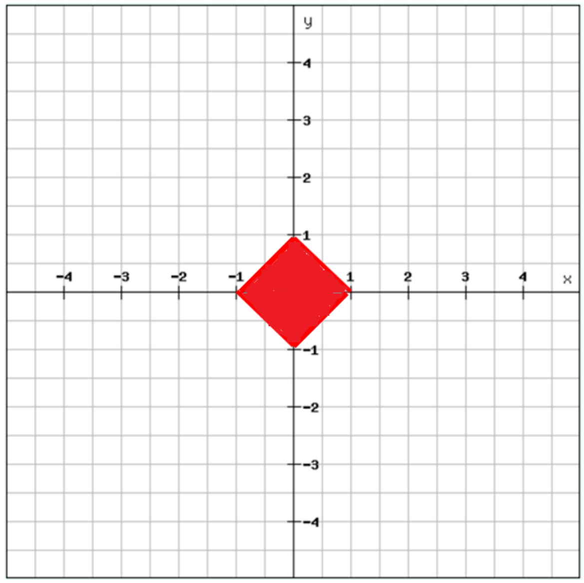
3 Formulations

Part 1. Formulate $\mathcal{S} = \{x + y \in \mathbb{R} : |x| + |y| \leq 1\}$ as a linear programming problem.

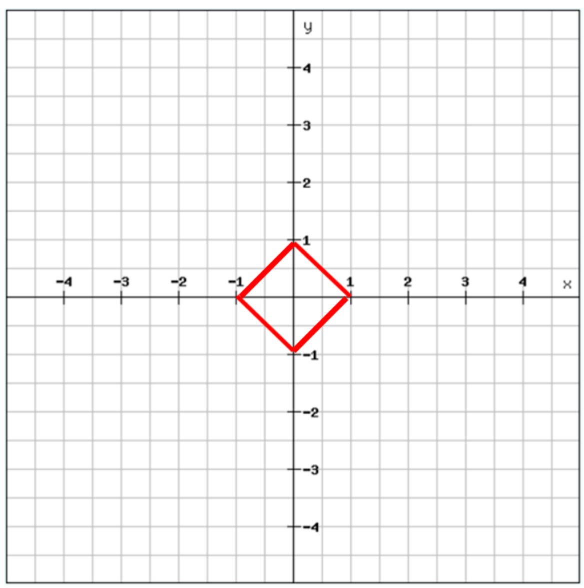
The feasible region is as follows:

Answer: with the objective arbitrarily specified, the LP is formulated as

$$\begin{aligned}
\min \quad & 0 \\
\text{s.t.} \quad & x + y \leq 1 \\
& x - y \leq 1 \\
& -x - y \leq 1 \\
& -x + y \leq 1
\end{aligned}$$



Feasible set at Part (1)



Feasible set at Part (2)

Part 2. The feasible region is as follows:

Answer:

Here, z_i variables correspond to the lines accordingly.

$$z_1 = \begin{cases} 1 & \text{if } x + y = 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$z_2 = \begin{cases} 1 & \text{if } x - y = 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$z_3 = \begin{cases} 1 & \text{if } -x + y = 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$z_4 = \begin{cases} 1 & \text{if } -x - y = 1 \\ 0 & \text{Otherwise} \end{cases}$$

In order to denote the feasible region, at least one of these lines should be utilized. Hence, $z_1 + z_2 + z_3 + z_4 \geq 1$.

M corresponds to a very big number (theoretically it is infinity). Consider M= 999. (Note that the choice of M should be carefully handled, we usually try to find the smallest M that work for the corresponding formulation)

The objective arbitrarily specified, The MIP is formulated as follows:

$$\begin{array}{ll}
\min & 0 \\
\text{s.t.} & x + y \leq 1 + M(1 - z_1) \\
& x + y \geq 1 - M(1 - z_1) \\
& x - y \leq 1 + M(1 - z_2) \\
& x - y \geq 1 - M(1 - z_2) \\
& -x + y \leq 1 + M(1 - z_3) \\
& -x + y \geq 1 - M(1 - z_3) \\
& -x - y \leq 1 + M(1 - z_4) \\
& -x - y \geq 1 - M(1 - z_4) \\
& z_1 + z_2 + z_3 + z_4 \geq 1 \\
& -1 \leq x \leq 1 \\
& -1 \leq y \leq 1 \\
& z \in \{0, 1\}^4
\end{array}$$

Part 3. The key part of the reformulation is $x \neq \bar{x}$, which can first be formulated as $\sum_{i=1}^{n} |x_i - \bar{x}_i| \geq 1$. However since this is a binary problem we can separate the indices. We know $\bar{x}$ vector since it is a parameter.

Let set $\mathcal{A}$ be the set of indices where $\bar{x}_i = 0$ Similarly, let set $\mathcal{B}$ includes indices where $\bar{x}_i = 1$.

We know $\mathcal{A} \cup \mathcal{B} = \{1, 2, ..., n\}$

Hence, $\forall i \in \mathcal{A}$, we have $\bar{x}_i = 0$

$\forall i \in \mathcal{B}$, we have $\bar{x}_i = 1$

To satisfy $x \neq \bar{x}$ constraint, we should have at least one of those conditions: $\sum_{i \in \mathcal{A}} x_i \geq 1$
$\sum_{i \in \mathcal{A}} x_i < |\mathcal{B}|$

$|\mathcal{B}|$ corresponds to the cardinality of set $\mathcal{B}$

Let y_1 and y_2 denote the corresponding conditions.

$$y_1 = \begin{cases} 1 & \text{if } \sum_{i \in \mathcal{A}} x_i \geq 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$y_2 = \begin{cases} 1 & \text{if } \sum_{i \in \mathcal{A}} x_i < |\mathcal{B}| \\ 0 & \text{Otherwise} \end{cases}$$

At least y_1 or y_2 should be 1. (should hold). Hence $y_1 + y_2 \geq 1$

Then we can formulate the problem as follow:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & \textstyle\sum_{i \in \mathcal{A}} x_i \geq 1 + M(1 - y_1) \\ & \textstyle\sum_{i \in \mathcal{B}} x_i \leq |\mathcal{B}| - 1 + M(1 - y_2) \\ & y_1 + y_2 \geq 1 \\ & x \in \{0,1\}^n, \\ & y_1, y_2 \in \{0,1\} \end{aligned}$$

where M is a sufficiently large number.

4 Branch and Bound

This is a maximization-type problem. Hence the optimal solution of the LP relaxation will give us upperbound. The optimal solution of the LP can be found by utilizing greedy method as we always do.

The optimal solution of LP problem:

$x = (1, 1, 0.8, 1, 0, 1)$ and the corresponding optimal objective value is 21.2.

We branch on the variable x_3 since it is fractional. Look at the figure that gives the branch-and-bound process.

We have an upperbound by LP relaxation, 21.2. We also have a lowerbound due to Node 2 since it is a feasible solution. We MUST have an integer optimal value since the objective coefficients of our original problem are all integer. Hence, 21 is the biggest integer we can get and we got this value by Node 2. Hence Node 2 gives us the optimal solution.

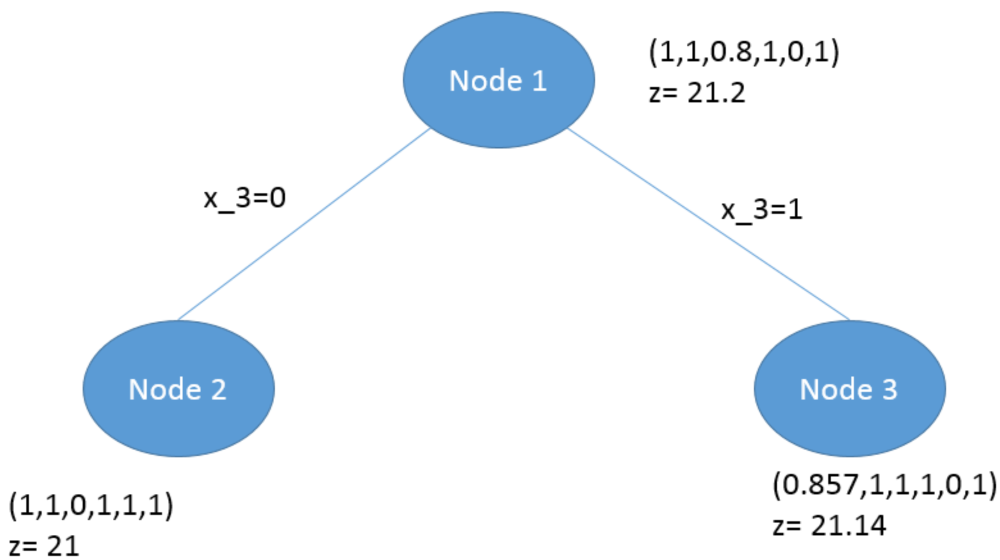


Figure 1: Illustration of Question 4.