ISE429. Test 2

1 (weight 0.35)

Random variables $Y_1, Y_2, \ldots$ are independent, identically distributed, each has the exponential distribution with mean 2. Let $T = \min\{n \mid Y_1 + Y_2 + \ldots + Y_n > 9\}$. Find $\mathbb{P}\{Y_1 + Y_2 + \ldots + Y_T + Y_{T+1} + Y_{T+2} > 10\}$.

2) (weight 0.30)

(a) Consider a continuous time Markov chain $\{X(t)\}$ with $m+1$ states $(m \geq 1)$, $\{0, 1, \ldots, m\}$. Let $\lambda > \mu > 0$. The transition rates are: $q_{i,i+1} = \lambda$ and $q_{i+1,i} = \mu$ for $i = 0, \ldots, m-1$; $q_{m,0} = \lambda$ and $q_{0,m} = \mu$. All other $q_{i,j} = 0$. Does this Markov chain have a stationary distribution? Is it unique? If so, what is it? Is this Markov chain reversible w.r.t. its stationary distribution? What are the transition rates of the time-reversed (stationary) Markov chain?

(b) Same Markov chain as in (a), except $q_{m,0} = q_{0,m} = 0$. Does this Markov chain have a stationary distribution? Is it unique? If so, what is it? Is this Markov chain reversible w.r.t. its stationary distribution? What are the transition rates of the time-reversed (stationary) Markov chain?

3) (weight 0.35)

There are two types of light bulbs that you use in your desk lamp, 1 and 2. Type 1 is cheaper and lasts exactly 1 unit of time (say, month). Type 2 is more expensive and lasts exactly $\sqrt{3}$ units of time. Consider two replacement strategies. Assume that a replacement takes zero time.

(a) You start with type 1, then replace with type 2, then type 1, and so on. $Y(t)$ is the residual (excess) time of the current bulb (whatever type it happens to be) at time t. What is the limiting fraction of time that $Y \geq 1/2$? Namely, what is

$$\phi = \lim_{t\to\infty} (1/t) \int_0^t \mathbb{P}\{Y(s) \geq 1/2\} ds.$$

Does the limit

$$\psi = \lim_{t\to\infty} \mathbb{P}\{Y(t) \geq 1/2\}$$

exist, and if so, what is it?

(b) You start with type 1. When it is time to replace a bulb, you replace it with the same type with probability 9/10, and change the type with prob. 1/10. $Y(t)$ is the residual (excess) time of the current bulb (whatever type it happens to be) at time t . What is the limiting fraction of time that $Y \geq 1/2$? Namely, what is

$$\phi = \lim_{t\to\infty} (1/t) \int_0^t \mathbb{P}\{Y(s) \geq 1/2\} ds.$$

Does the limit

$$\psi = \lim_{t\to\infty} \mathbb{P}\{Y(t) \geq 1/2\}$$

exist, and if so, what is it?

Comment: You do not need to worry about direct integrability. But, have to substantiate everything else you do.