ISE 426
Optimization models and applications

Lecture 20 — November 13, 2014

- Bin Packing Problem
- Cutting Stock Problem
- Column generation

The bin packing problem

- Given a set of N bins of volume V and n_i objects of volumes v_i, $i = 1, \ldots, n$.
- We want to pack the objects into bins using as few bins as possible.
- Consider $V = 11$, $n = 3$ from which $n_1 = 20$ objects have $v_i = 5$, $n_2 = 10$ objects have $v_i = 4$ and $n_3 = 9$ objects have $v_i = 2$;
- Let us try a gready heristic, we get: 10 bins with two $(5, 5)$ objects, 5 bins with $(4, 4, 2)$ objects and 1 bin with $(2, 2, 2, 2)$ objects.
- Clearly 9 bins with $(5, 4, 2)$, 5 bins with $(5, 5)$ and one bin with $(5, 4)$ is better.

How do we model this as an Optimization model?

- y_i is a binary variable that indicates if a bin i is being used.
- x_{ij} is an integer variable that indicates how many objects of size j has been assigned to bin i.
- Clearly $x_{ij} \leq My_i$.

$$
\begin{array}{ll}
\min & \sum_{i=1}^{N} y_i \\
& \sum_{j=1}^{n} v_j x_{ij} \leq V y_i \\
& \sum_{i=1}^{N} x_{ij} = n_j \quad \forall j = 1 \ldots, n \\
& x_{ij} \in \mathbf{Z} \quad \forall i = 1, \ldots, N, \; j = 1 \ldots, n \\
& y_i \in \{0, 1\} \quad \forall i = 1, \ldots, N
\end{array}
$$

Weak formulation, too much symmetry, each bin is the same.
May help to add constraints

$$
y_1 \geq y_2 \geq y_3 \geq \ldots \geq y_N
$$

(Use the first bin first)

Formulation #2: extended formulation

- Consider all sets s_k of objects (patterns) that can fit into one bin.
- That is, $(5,5)$, $(5,4,2)$ $(4,4,2)$, $(4,2,2,2)$, $(2,2,2,2,2)$. (We do not need to consider $(5,4)$ because it is dominated by $(5,4,2)$)
- Let S be the set of all feasible patterns s_k, $|S| = K$.
- x_k is an integer variable indicating how many bins are filled with pattern s_k, $s_k \in S$.
- Let a_{jk} be the number of times object of size v_j appears in s_k, for instance for $s_k = (4,4,2)$, for $v_j = 4$, $a_{jk} = 2$ and for $v_j = 2$, $a_{jk} = 1$.

$$
\begin{array}{ll}
\min & \sum_{k=1}^{K} x_k \\
 & \sum_{k=1}^{K} a_{jk} x_k \geq n_j \; \forall j = 1, \ldots, n \\
 & x_k \in \mathbf{Z} \quad \forall k = 1, \ldots, K
\end{array}
$$

This is a "strong" formulation. The LP relaxation gives very good lower bounds.

Relaxation and another interpretation - cutting stock problem

$$\begin{array}{ll} \min & \sum_{k=1}^{K} x_k \\ & \sum_{k=1}^{K} a_{jk} x_k \geq n_j \quad \forall j = 1, \ldots, n \\ & x_k \geq 0 \quad \forall k = 1, \ldots, K \end{array}$$

- Given very long (infinite) roll of paper (or steel) of width V we need to cut paper into pieces of length n_j and width v_j, $j = 1, \ldots, n$.
- We are allowed to cut correct width, but smaller length and "glue" different lengths to obtain the right one.
- We want to use as little paper as possible.

Dual formulation

$$\begin{array}{ll} \max & \sum_{j=1}^{n} n_j y_j \\ & \sum_{j=1}^{n} a_{jk} y_j \leq 1 \; \forall k = 1, \ldots, K \\ & y_j \geq 0 \; \forall j = 1, \ldots, n \end{array}$$

- Each pattern corresponds to variable x_k, which in turn, corresponds to a constraint

$$\sum_{j=1}^{n} a_{jk} y_j \leq 1.$$

- Remember that given a basic feasible solutions we have a lot of $x_k = 0$.
- If the dual constraint for a given k is not feasible, then the corresponding x_k should be, possibly, nonzero.
- Column generation technique - generate k's for which

$$\sum_{j=1}^{n} a_{jk} y_j > 1$$

Column generation

$$\begin{array}{ll} \max & \sum_{j=1}^{n} n_j y_i \\ & \sum_{j=1}^{n} a_{jk} y_j \leq 1 \ \forall k = 1, \ldots, K \\ & y_j \geq 0 \ \forall j = 1, \ldots, n \end{array}$$

- Start with a few patterns and variables x_k (the rest of $x_k = 0$.
- Solve the primal problem with only those patterns.
- Compute the corrsponding dual solution
- Column generation technique - generate k's for which

$$\sum_{j=1}^{n} a_{jk} y_j > 1$$

and include them into the primal problem.

Column generation

$$\sum_{j=1}^{n} a_{jk} y_j > 1$$

How do we find new k? Find k for which $\sum_{j=1}^{n} a_{jk} y_j$ is the largest.

$$\begin{array}{ll} \max & \sum_{j=1}^{n} a_{jk} y_j \\ & \sum_{j=1}^{n} a_{jk} \leq V \ \forall k = 1, \ldots, K \\ & a_{jk} \in \mathbf{Z} \ \forall j = 1, \ldots, n \end{array}$$

The knapsack problems!!

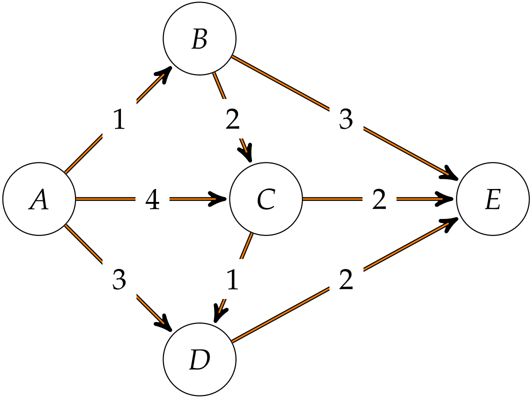
Reformulation using binary variables

Consider a set of vectors $x \in R^n$ described by the following conditions.

$$\min_i \{x_1, x_2, x_3, \ldots, x_n\} \leq 1$$

Describe this set by using a set of linear constraints and binary variables, as we did in homework and class. In other words, you only should use linear constraints that can involve continuous and/or binary variables, and all feasible solutions for this set of constraints should give x that is feasible for the the above set and vice versa.

Flow problem and goal programming (22 pts.)



Formulate the problem of sending 8 units of flow from A to E as a linear programming problem, using the formulations studied in this course. The numbers on the arcs are the capacities. Do not use an objective function - the problem is infeasible. Show this by finding the min cut whose value is smaller than 8. (4 pts)

Integer programming (22 pts.)

Consider a graph $G = (V, E)$, in Figure 1, and a cost C_{ij} for each edge $\{i, j\} \in E$. Suppose you want to find the subset S of V with at least k nodes, such that the cost of all edges, that link two nodes in (have both ends in) S is minimized. For example, if S is the set of four dark nodes is the graph in Figure 1, then the total cost of all edges connecting nodes in S is $C_{24} + C_{45} + C_{56}$

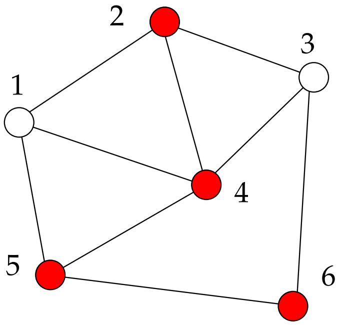


Figure: