Morning

MORNING Class

ISE426 – Optimization models and applications

Fall 2014 – Quiz #2, November 11, 2014

First name	
Last name	
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You have 75 minutes. There are three problems. This quiz accounts for 10% of the final grade. There are 40 points available. Please write clear and concise statements. Use a **readable** handwriting: it will be hard to grade your answers if they are not readable — let alone give them full points. Use the back of each sheet if you need more space, for example to draw graphs. Do not use calculators.

1 Reformulation using binary variables (8 pts.)

Consider a set of vectors $x \in R^n$ described by the following conditions.

$$\min_i \{x_1, x_2, x_3, \ldots, x_n\} \leq 1$$

Describe this set by using a set of linear constraints and binary variables, as we did in homework and class. In other words, you only should use linear constraints that can involve continuous and/or binary variables, and all feasible solutions for this set of constraints should give x that is feasible for the the above set and vice versa.

Define following Binary variable y_i as
then it implies

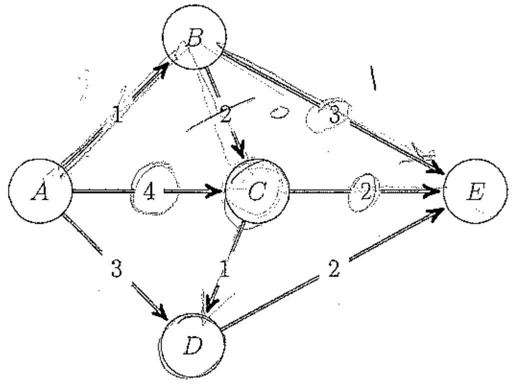
$$y_i = \begin{cases} 1 & x_i \leq 1 \quad \text{for } i = 1, \ldots, n \\ 0 & \text{else.} \end{cases}$$

So we will have following model:

$$\begin{cases} x_i \leq 1 + M(1 - y_i) \\ \sum_{i=1}^{n} y_i \geq 1 \quad y_i \in \{0, 1\} \\ \text{for } i = 1, \ldots, n \end{cases}$$

in this way we can grantee that at least one of constraints $x_i \leq 1$ is held which implies $\min_i \{x_1, x_2, \ldots, x_n\} \leq 1$. *

2 Flow problem and goal programming (22 pts.)



1. Formulate the problem of sending 8 units of flow from A to E as a linear programming problem, using the formulations studied in this course. The numbers on the arcs are the capacities. Do not use an objective function - the problem is infeasible. Show this by finding the min cut whose value is smaller than 8. (4 pts)

Define variable x_{ij} as the flow of node i to node j.

⇒ Sending 8 units from A to E:

$$\begin{cases} x_{AB} + x_{AC} + x_{AD} = 8 & \text{(node A)} \\ x_{BE} + x_{CE} + x_{DE} = 8 & \text{(node E)} \end{cases}$$

⇒ flow constraints

$$\begin{cases} x_{AB} - x_{BC} - x_{BE} = 0 & \text{(node B)} \\ x_{AC} + x_{BC} - x_{CD} - x_{CE} = 0 & \text{(node C)} \\ x_{AD} + x_{CD} - x_{DE} = 0 & \text{(node D)} \end{cases}$$

⇒ Capacity constraints

$$\begin{cases} 0 \leq x_{AB} \leq 1 & 0 \leq x_{AC} \leq 4 & 0 \leq x_{AD} \leq 3 \\ 0 \leq x_{BC} \leq 2 & 0 \leq x_{CD} \leq 1 & 0 \leq x_{BE} \leq 3 \\ 0 \leq x_{CE} \leq 2 & 0 \leq x_{DE} \leq 2 & \end{cases}$$

⇒ clearly Min cut value is 5, which is less than 8!

2. Assume that the condition that 8 units of flow have to leave A is not relaxable (it has to be satisfied). You have two groups of other constraints - the flow conservation constraints and the capacity constraints.

   Consider preemptive goal programming formulation where you first want to satisfy the capacity constraints and only after that you care to satisfy the flow conservation constraints. Show the formulation that you need to solve for this goal programming. (5pts)

we will have :

Step 1) $\min\ y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}$

(I)
$$x_{AB} + x_{AC} + x_{AD} = 8$$
$$x_{BE} + x_{CE} + x_{DE} = 8 + y_1^+ - y_1^-$$
$$x_{AB} - x_{BC} - x_{BE} = y_2^+ - y_2^- \quad \text{node B}$$
$$x_{AC} + x_{BC} - x_{CD} - x_{CE} = y_3^+ - y_3^- \quad \text{node C}$$
$$x_{AD} + x_{CD} - x_{DE} = y_4^+ - y_4^- \quad \text{node D}$$

$x_{AB} \leq 1 + y_5$ $\quad x_{AC} \leq 4 + y_6$

$x_{AD} \leq 3 + y_7$ $\quad x_{BC} \leq 2 + y_8$

$x_{CD} \leq 1 + y_9$ $\quad x_{BE} \leq 3 + y_{10}$

$x_{CE} \leq 2 + y_{11}$ $\quad x_{DE} \leq 2 + y_{12}$

where $x_{ij} \geq 0$

$y_i^+ ; y_i^- \geq 0 \quad i = 1..4$

$y_j \geq 0 \quad j = 5 \ldots 12$

Step 2) fix $y_5 = y_6 = \cdots = y_{12} = 0$ as the solution of step 1 and consider $\underbrace{\text{above constraints}}$ with following objective function:

$$\min\ y_1^+ + y_1^- + y_2^+ + y_2^- + y_3^+ + y_3^- + y_4^+ + y_4^-$$

Constraints (I)

3. Now, consider preemptive goal programming where you switch the priorities of the constraints (flow conservation is more important than capacities). Show the formulation. (4pts)

Step 1 $\Rightarrow$ $\text{Min } y_1^+ + y_1^- + y_2^+ + y_2^- + y_3^+ + y_3^- + y_4^+ + y_4^-$

s.t Constraints in (I)

Step 2 $\Rightarrow$ fix $y_1^+ = y_1^- = \cdots = y_4^- = 0$ as solution of step 1

and apply constraints in (I) by using following objective function

$$\text{Min } y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}$$

4. Can you figure out the solutions for each of the preemptive approaches and show how they are different from each other? (4pts)

"Part 2"

Based on the formulation in part 2, we can get $y_5 = \cdots = y_{12} = 0$, which gives us zero, as the optimal value of objective function, but in terms of getting a feasible solution we need to violate some flow constraints which have the lower priority in Part 2, so we have to assign $y_1^- = 3$ and then we need to put $y_3^+ + y_4^+ = 3$ such $y_3^+ = 1$ and $y_4^+ = 2$ "or" $y_3^+ = 2$ and $y_4^+ = 1$ (remaining y_i^+ are zero)

"Part 3" $\Rightarrow$ Based on step one we have $y_1^+ = y_1^- = \cdots = y_4^- = 0$ and in terms of finding feasible solution we can change capacity constraints in different ways such for example $y_{11} = 2$, $y_{DE} = 1$ and put others equal to zero (you just need to change the right hand sides of capacity constraints 3 units).

5. Now, formulate a problem that keeps all flow constraints feasible (and sends 8 units of flow from A to E), but relaxes the capacity constraints in the following way: each arc can have its capacity doubled or not increased at all. Minimize the **number** of arcs for which the capacity is doubled. (5pts)

Define binary variable y_{ij} corresponding to flow x_{ij} such,

$$y_{ij} = \begin{cases} 1 & \text{if } x_{ij} \text{ is doubled} \\ 0 & \text{else} \end{cases}$$

$$\min \; y_{AB} + y_{AC} + y_{AD} + y_{BC} + y_{CD} + y_{BE} + y_{CE} + y_{DE}$$

$$x_{AB} + x_{AC} + x_{AD} = 8$$

$$x_{BE} + x_{CE} + x_{DE} = 8$$

$$x_{AB} - x_{BC} - x_{BE} = 0$$

$$x_{AC} + x_{BC} - x_{CD} - x_{CE} = 0$$

$$x_{AD} + x_{CD} - x_{DE} = 0$$

$$0 \leq x_{AB} \leq 1 + y_{AB}$$

$$0 \leq x_{AC} \leq 4 + 4y_{AC}$$

$$0 \leq x_{AD} \leq 3 + 3y_{AD}$$

$$0 \leq x_{BC} \leq 2 + 2y_{BC}$$

$$0 \leq x_{CD} \leq 1 + y_{CD}$$

$$0 \leq x_{BE} \leq 3 + 3y_{BE}$$

$$0 \leq x_{CE} \leq 2 + 2y_{CE}$$

$$0 \leq x_{DE} \leq 2 + 2y_{DE} \qquad y_{ij} \in \{0,1\} \text{ for } \{i,j\} \in E$$

3 Integer Programming (10 pts.)

Consider a graph $G = (V, E)$, in Figure 1, and a cost C_{ij} for each edge $\{i, j\} \in E$. Suppose you want to find the subset S of V with at least k nodes, such that the cost of all edges, that link two nodes in (have both ends in) S is minimized. For example, if S is the set of four dark nodes is the graph in Figure 1, then the total cost of all edges connecting nodes in S is $C_{24} + C_{45} + C_{56}$

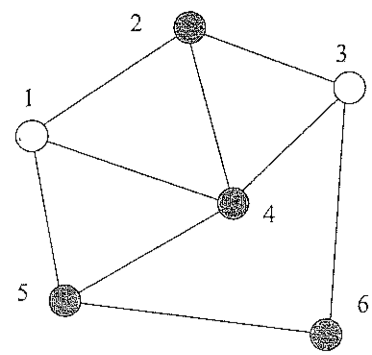


Figure 1:

1. Consider binary variables that indicate if a node is in S or not. Consider also binary variables that indicate if an edge is connecting two nodes in S. Now write down conditions between these types of variables, which ensure logical implications: for all $\{i, j\} \in E$, if $i \in S$ and $j \in S$ then edge $\{i, j\}$ connects two nodes in S. (5pts)

Define binary variables x_i and y_{ij} as following:

$$x_i = \begin{cases} 1 & \text{if node } i \text{ is in } S \\ 0 & \text{else} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if edge } (i,j) \text{ connecting two nodes } i \text{ and } j \text{ which are in } S \\ 0 & \text{else} \end{cases}$$

we need to satisfy $\Rightarrow$ $\begin{cases} \text{if } x_i + x_j = 2 \text{ then } y_{ij} = 1 \quad i \neq j \\ \end{cases}$

2. Write the full formulation of the problem of selecting at least k nodes so that the edge cost is minimized. (5pts).

$$\text{Min} \sum_{\{i,j\} \in E} c_{ij} \, y_{ij}$$

$$y_{ij} \geq x_i + x_j - 1 \qquad i \neq j \qquad \forall i, j \in V$$

$$\sum_{i \in V} x_i \geq k$$

$$x_{ij} \in \{0,1\}$$

$$y_{ij} \in \{0,1\}$$