ISE429. Test 2

1 (weight 0.35)

Random variables $Y_1, Y_2, \ldots$ are independent, identically distributed, each has the exponential distribution with mean 2. Let $T = \min\{n \mid Y_1 + Y_2 + \ldots + Y_n > 9\}$. Find $\mathbb{P}\{Y_1 + Y_2 + \ldots + Y_T + Y_{T+1} + Y_{T+2} > 10\}$.

Solution.

Since all Y_n are independent and have exponential distribution with mean 2, the points Y_1, Y_1+Y_2, $Y_1+Y_2+Y_3$, ..., form a Poisson process with rate $1/2$. $\mathbb{P}\{Y_1+Y_2+\ldots+Y_T+Y_{T+1}+Y_{T+2} > 10\}$ is the probability that this Poisson process has 0 or 1 or 2 points in the interval $(9, 10]$. The distribution of the random number of points in $(9, 10]$ is Poisson with mean $(1/2)\cdot 1 = 1/2$. Therefore,

$$\mathbb{P}\{Y_1 + Y_2 + \ldots + Y_T + Y_{T+1} + Y_{T+2} > 10\} = e^{-1/2} + (1/2)e^{-1/2} + (1/2)^2 e^{-1/2}/2!.$$

$\square$

2) (weight 0.30)

(a) Consider a continuous time Markov chain $\{X(t)\}$ with $m+1$ states ($m \geq 1$), $\{0, 1, \ldots, m\}$. Let $\lambda > \mu > 0$. The transition rates are: $q_{i,i+1} = \lambda$ and $q_{i+1,i} = \mu$ for $i = 0, \ldots, m-1$; $q_{m,0} = \lambda$ and $q_{0,m} = \mu$. All other $q_{i,j} = 0$. Does this Markov chain have a stationary distribution? Is it unique? If so, what is it? Is this Markov chain reversible w.r.t. its stationary distribution? What are the transition rates of the time-reversed (stationary) Markov chain?
(b) Same Markov chain as in (a), except $q_{m,0} = q_{0,m} = 0$. Does this Markov chain have a stationary distribution? Is it unique? If so, what is it? Is this Markov chain reversible w.r.t. its stationary distribution? What are the transition rates of the time-reversed (stationary) Markov chain?

Solution.

(a) Imagine the states $0, 1, \ldots, m$ arranged clock-wise on a circle. The stationary distribution is uniform: $\pi_i = 1/(m+1)$ for all i. This is by symmetry. Or can be verified directly from balance equations. It is unique because the chain is irreducible. The chain is *not* reversible w.r.t. its stationary distribution, because the detailed balance conditions do not hold. The transition rates of the time reversed stationary chain are: $\hat{q}_{i,j} = q_{j,i}$.

(b) Here the "circle" is broken – there are no direct transitions between 0 and m. This is birth-death process, the unique stationary distribution has the form $\pi_i = \pi_0(\lambda/\mu)^i$, with π_0 normalized to make $\sum_i \pi_i = 1$. This process is reversible. The transition rates of the time reversed stationary chain in this case are, of course, the same as for original chain: $\hat{q}_{i,j} = q_{i,j}$.
□

3) (weight 0.35)

There are two types of light bulbs that you use in your desk lamp, 1 and 2. Type 1 is cheaper and lasts exactly 1 unit of time (say, month). Type 2 is more expensive and lasts exactly $\sqrt{3}$ units of time. Consider two replacement strategies. Assume that a replacement takes zero time.

(a) You start with type 1, then replace with type 2, then type 1, and so on. $Y(t)$ is the residual (excess) time of the current bulb (whatever type it happens to be) at time t. What is the limiting fraction of time that $Y \geq 1/2$? Namely, what is

$$\phi = \lim_{t\to\infty} (1/t) \int_0^t \mathbb{P}\{Y(s) \geq 1/2\} ds.$$

Does the limit

$$\psi = \lim_{t\to\infty} \mathbb{P}\{Y(t) \geq 1/2\}$$

exist, and if so, what is it?

(b) You start with type 1. When it is time to replace a bulb, you replace it with the same type with probability 9/10, and change the type with prob. 1/10. $Y(t)$ is the residual (excess) time of the current bulb (whatever type it happens to be) at time t . What is the limiting fraction of time that $Y \geq 1/2$? Namely, what is

$$\phi = \lim_{t\to\infty} (1/t) \int_0^t \mathbb{P}\{Y(s) \geq 1/2\} ds.$$

Does the limit

$$\psi = \lim_{t\to\infty} \mathbb{P}\{Y(t) \geq 1/2\}$$

exist, and if so, what is it?

Comment: You do not need to worry about direct integrability. But, have to substantiate everything else you do.

Solution.

(a) View the process as regenerative with the renewal points being 0 and then the time instants when type 2 is replaced by type 1. If T is renewal time, then it is non-random, $T = 1 + \sqrt{3}$, so $ET = 1 + \sqrt{3}$. The total (non-random) time during one renewal cycle, when condition $Y \geq 1/2$ holds is $(1 - 1/2) + (\sqrt{3} - 1/2) = \sqrt{3}$. By the theorem for the limiting fraction of time, we have

$$\phi = \sqrt{3}/(1 + \sqrt{3}).$$

Clearly, the limit ψ does not exist, because the process follows deterministic cycles. For example, $\mathbb{P}\{Y(t) \geq 1/2\} = 1$ for $t = (1 + \sqrt{3})n + 1/4$, $n = 0, 1, \ldots$; and $\mathbb{P}\{Y(t) \geq 1/2\} = 0$ for $t = (1 + \sqrt{3})n + 3/4$, $n = 0, 1, \ldots$.

(b) View the process as regenerative with the renewal points being 0 and then the time instants when type 2 is replaced by type 1. A renewal cycle consists of the random (geometrically distributed with mean $1/0.1 = 10$) number of type 1, followed by the random (geometrically distributed with mean $1/0.1 = 10$) number of type 2. So, $ET = 10 \cdot 1 + 10 \cdot \sqrt{3}$. The total average time during one renewal cycle, when condition $Y \geq 1/2$ holds is $10(1 - 1/2) + 10(\sqrt{3} - 1/2) = 10\sqrt{3}$. By the theorem for the limiting fraction of time, we have

$$\phi = \sqrt{3}/(1 + \sqrt{3}).$$

The distribution of T is non-lattice, because T can takes values, for example, $1 + \sqrt{3}$ and $2+\sqrt{3}$ with positive probabilities; and $(1+\sqrt{3})/(2+\sqrt{3})$ is not a rational number. (The direct integrability of function $h(\cdot)$ can be checked too.) Therefore, by the key renewal theorem, the limit ψ exists and is equal to ϕ. $\square$