Details ⇒ Non convexity of first constraint: it's a nonlinear equality which causes the non-convex problem.

⇒ Non Convexity of fesible region: pick two arbitrary points, such $P_1 = (0,1)$ and $P_2(0,-1)$ which belong to fesible region, derive the convex combination of these two points for $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$

ISE426 – Optimization models and applications

Fall 2014 – Quiz #1, October 14, 2014

$\lambda_1 P_1 + \lambda_2 P_2$

$= (0,0)$

$\notin$ fesible region.

First name	
Last name	
Lehigh email	

You have 75 minutes. There are three problems. This quiz accounts for 10% of the final grade. There are 40 points available. Please write clear and concise statements. Use a **readable** handwriting: it will be hard to grade your answers if they are not readable — let alone give them full points. Use the back of each sheet if you need more space, for example to draw graphs. Do not use calculators.

1 Convexity and relaxations (8 pts.)

The following problem is not convex, explain why (4 pts.):

$$\begin{aligned} \min \quad & x \\ & |y| = 1 \\ & x^2 + y^2 \leq 5 \end{aligned}$$

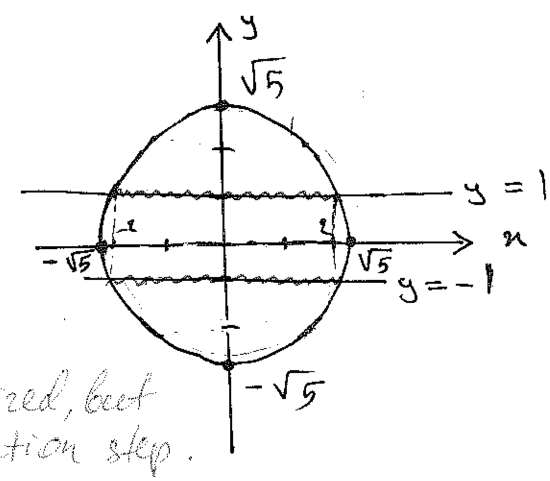
Solution:

Since objective function is a liner function it is convex, so we have to check the convexity of fesible region, first constraint is not convex for sure, but we have to check the intersection of both constraints. (It is possible that the intersection be convex, for example if we had $|y|=1$ and $y \geq 0$ as our fesible region)

⇒ the fesible region of our problem is two parallel line segments $y=1$ and $y=-1$,

for $x \in [-2, 2]$

which is obviously Non-convex,



This was not required, but is an extra explanation step.

Find upper and lower bounds on the optimal value and explain how you know that these are indeed upper and lower bounds. Find the optimal solution graphically and check that it is between the upper and lower bounds. (4 pts.).

lower bound: Consider following relaxed problem: (eliminate first constraint you can also consider $|y| \leq 1$)

$$\min x$$

the optimal solution of relaxed problem is $(-\sqrt{5}, 0)$ which gives a lower bound for original problem. $\Rightarrow$ $LB = -\sqrt{5}$

upper bound: any feasible solution of our original problem can be an upper bound for it, for example point $(0,1)$ is a feasible solution, so we will have $UB = 0$

(you can have different upper bound)

optimal solution of "original problem":

As shown in figue, optimal value of original problem is $x^* = -2$,

(by putting $y^2 = 1$ in second constraint $x^2 \leq 4$

$\Rightarrow |x| \leq 2 \Rightarrow -2 \leq x \leq 2 \Rightarrow \min x = -2$)

which is between lower bound and upper bound,

$$-\sqrt{5} \leq -2 \leq 0$$

$$LB \leq z^* \leq UB$$

2 Linear Programming Model (16 pts.)

Consider the following LP problem, which is a slight modification of the problem from your homework #2:

$$
\begin{array}{llllll}
\min & 2x_1 & +5x_2 & +2x_3 & -2x_4 & \\
\text{s.t.} & x_1 & +x_2 & +2x_3 & +x_4 & \geq 1 \longrightarrow u_1 \\
 & x_1 & +x_2 & -x_3 & -2x_4 & \geq 2 \longrightarrow u_2 \\
 & x_2, x_3 & \geq 0 & x_1, x_4 & \leq 0. &
\end{array}
$$

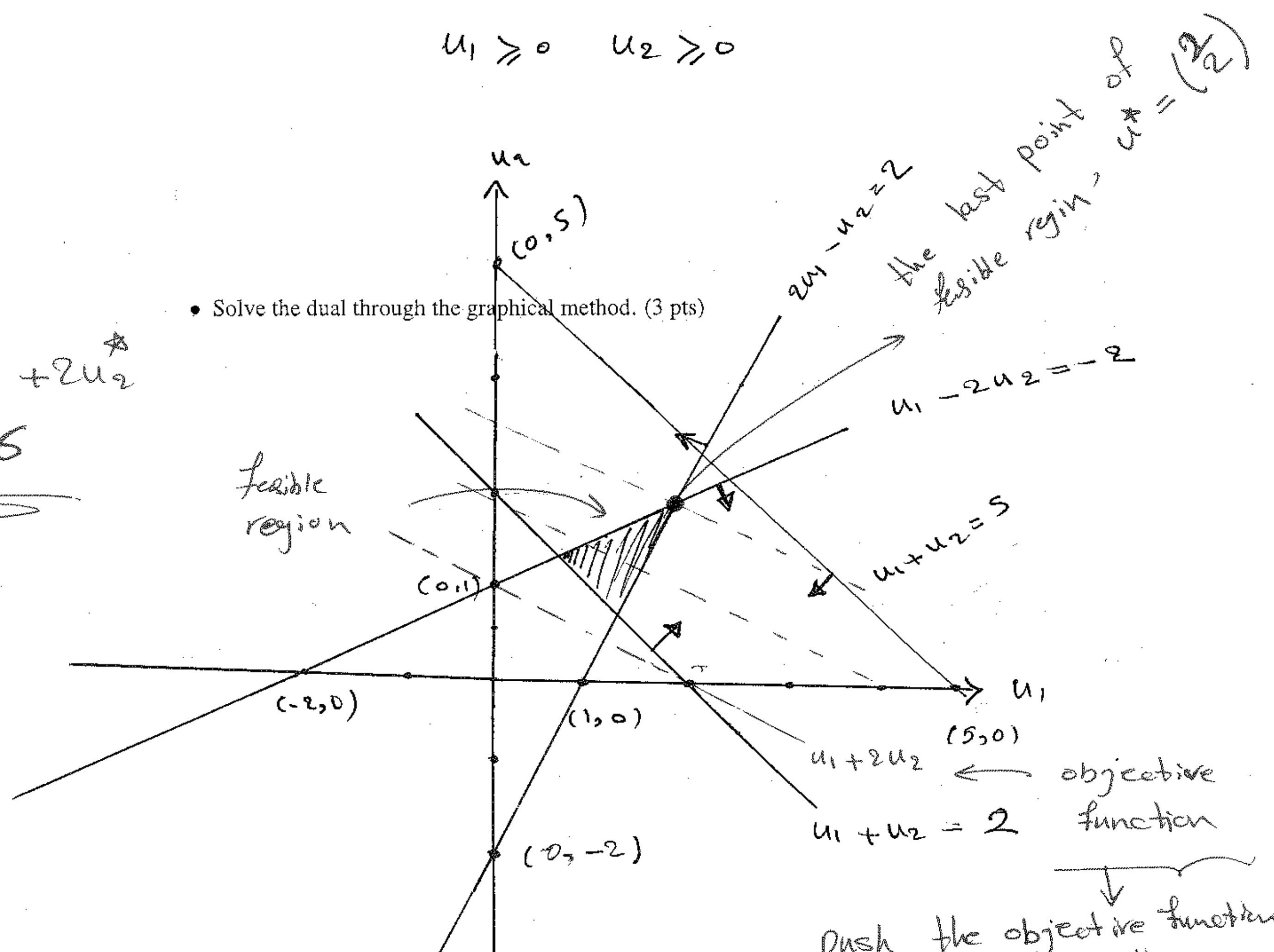
- Write the dual. (5 pts)

$$
\begin{array}{ll}
\max & u_1 + 2u_2 = \cancel{8} \\
\text{s.t.} & \\
 & u_1 + u_2 \geq 2 \longrightarrow x_1 \\
 & u_1 + u_2 \leq 5 \longrightarrow x_2 \\
 & 2u_1 - u_2 \leq 2 \longrightarrow x_3 \\
 & u_1 - 2u_2 \geq -2 \longrightarrow x_4 \\
 & u_1 \geq 0 \quad u_2 \geq 0
\end{array}
$$

- Solve the dual through the graphical method. (3 pts)

$Z^* = u_1^* + 2u_2^*$

$= 6$



push the objective function until it touches the last point of feasible region.

optimal solution of dual $\Rightarrow$ $u^* = \begin{cases} u_1^* = 2 \\ u_2^* = 2 \\ u_3^* = 0 \\ u_4^* = 0 \end{cases}$

- After finding the optimal value of the dual variables, use complementary slackness to find the optimal value of the primal variables. (4 pts)

* Complementary Slackness

$$(1) \quad u_1 \, (x_1 + x_2 + 2x_3 + x_4 - 1) = 0$$

$$(2) \quad u_2 \, (x_1 + x_2 - x_3 - 2x_4 - 2) = 0$$

$$(3) \quad x_1 \, (u_1 + u_2 - 2) = 0 \xrightarrow{\text{use } u^*} x_1 (2) = 0 \Rightarrow x_1^* = 0$$

$$(4) \quad x_2 \, (u_1 + u_2 - 5) = 0 \xrightarrow{\text{use } u^*} x_2 (2) = 0 \Rightarrow x_2^* = 0$$

$$(5) \quad x_3 \, (2u_1 - u_2 - 2) = 0 \xrightarrow{\text{use } u^*} x_3 \text{ can be zero or not}$$

$$(6) \quad x_4 \, (u_1 - 2u_2 + 2) = 0 \xrightarrow{\text{use } u^*} x_4 \text{ can be zero or not}$$

By using u^* and $x_1^* = x_2^* = 0$ in (1) and (2) we will have:

$$\begin{aligned} 2x_3 + x_4 &= 1 \\ -x_3 - 2x_4 &= 2 \end{aligned} \Rightarrow \begin{cases} x_3^* = \frac{4}{3} \\ x_4^* = -5/3 \end{cases}$$

- Find the range of the right hand side coefficient of the first constraint (which is "1" in the original problem) for which the dual solution that you found remains optimal. Find the answer by checking for which values of this coefficient the primal complementary solution remains feasible. (4 pts)

Since $u_1^* \neq 0$ and $u_2^* \neq 0$, if we change the RHS of first constraint to "b", in terms of keeping optimality, we need to satisfy (1) and (2) in Complementary Slackness, and since $u_1^* \neq 0$ and $u_2^* \neq 0$ we need to select "b" such that

$$x_1 + x_2 + 2x_3 + x_4 = b$$

$$x_1 + x_2 - x_3 - 2x_4 = 2$$

Based on u^*, since $u_1^* + u_2^* \neq 2$, by (3) we have $x_1^* = 0$ and similarly since $u_1^* + u_2^* \neq 5$, by (4) we have $x_2^* = 0$.

So by using two last equalitis we have:

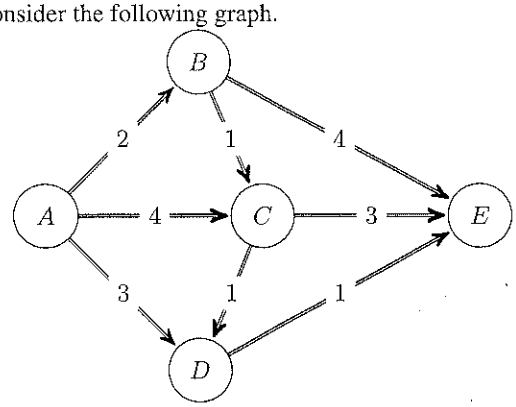
$$x_3 = \frac{2}{3}(b+1) \quad \text{and} \quad x_4 = -\left(\frac{b+4}{3}\right)$$

and since $x_3 \geq 0$ and $x_4 \geq 0$ we have

$$\left.\begin{aligned} x_3 &= \frac{2}{3}(b+1) \geq 0 \Rightarrow b \geq -1 \\ x_4 &= -\left(\frac{b+4}{3}\right) \geq 0 \Rightarrow b \geq -4 \end{aligned}\right\} \Rightarrow \boxed{b \geq -1}$$

3 Flow problem (16 pts.)

Consider the following graph.



1. Formulate the shortest path problem for going from A to E as a linear programming problem, using the formulations studied in this course. Note that now the arcs are directed, hence you can only travel one way on each arc. (6 pts)

Primal Problem:

$$\min \; 2x_{AB} + 4x_{AC} + 3x_{AD} + 4x_{BE} + x_{BC} + 3x_{CE} + x_{CD} + x_{DE}$$

s.t.

$$x_{AB} + x_{AC} + x_{AD} = 1 \qquad u_A$$

$$-x_{AB} + x_{BE} + x_{BC} = 0 \qquad u_B$$

$$-x_{BC} - x_{AC} + x_{CE} + x_{CD} = 0 \qquad u_C$$

$$-x_{AD} - x_{CD} + x_{DE} = 0 \qquad u_D$$

$$-x_{BE} - x_{CE} - x_{DE} = -1 \qquad u_E$$

$$x_{ij} \geq 0$$

2. Write the dual of the above shortest path problem. (5 pts)

Dual Problem:

$$\max \quad u_A - u_E$$

$$u_A - u_B \leq 2$$

$$u_A - u_C \leq 4$$

$$u_A - u_D \leq 3 \quad \leftarrow x_{AD}$$

$$u_B - u_E \leq 4$$

$$u_B - u_C \leq 1$$

$$u_C - u_E \leq 3$$

$$u_C - u_D \leq 1$$

$$u_D - u_E \leq 1 \quad \leftarrow x_{DE}$$

u_i unrestricted in sign.

3. Find the shortest path for the problem, by simple observation. This gives you the primal optimal solution for your LP. Write it down. Using it write down complementarity conditions of the optimal dual solution. Observe that the dual optimal solution is not defined uniquely. But these complementarity conditions uniquely define the optimal value of the objective function of the dual. Demonstrate this. (2 pts)

$\Rightarrow$ optimal shortest path: $A \to D \to E$: $x^*_{AD} = x^*_{DE} = 1$

and $x^*_{ij} = 0$ if $(i,j) \neq (A,D)$ and $(i,j) \neq (D,E)$

complementary slackness in terms of primal variables x_{AD} and x_{DE}

$x^*_{AD}\,(u_A - u_D - 3) = 0 \rightarrow$ ① $u_A - u_D = 3$ $\xRightarrow{\text{by Summing}}$ $u_A - u_E = 4$

$x^*_{DE}\,(u_D - u_E - 1) = 0 \rightarrow$ ② $u_D - u_E = 1$

Since we have $x^*_{ij} = 0$ for other primal variables, we can not use other equations in complementary slackness corresponding to these primal variables, so we can use just equations ① and ②

So we have 2 Constraints and 5 variables which allows us to have multiple optimal solutions for Dual problem.

4. Consider the change in the length of the (B, E) edge. It is 4 in the original problem. How low can it get before the shortest path changes. Demonstrate this by showing that if the length of (B, E) gets any lower than this value, then the complementary dual solution will no longer be feasible (you can show this even if you do not compute the unique dual solution). (3 pts)

change the length of edge (B,E) to α, we have $u_B - u_E \leq \alpha$

equivalently we have, $u_B - u_A + u_A - u_E \leq \alpha$, by using the optimal value of dual objective function $u_A - u_E = 4$

we will have $u_B - u_A \leq \alpha - 4 \Rightarrow \boxed{u_A - u_B \geq 4 - \alpha}$ *

on the other hand based the first equation in dual problem we have $\boxed{u_A - u_B \leq 2}$ **

By * and ** we have $\alpha \geq 2$, which means the cost of edge (B,E) can be at least 2.