ISE426 – Optimization models and applications

Fall 2014, Homework #1. Due Sept 16, 2014, in class.

September 9, 2014

This homework accounts for 5% of the final grade. There are 20 points available. In this and future homework assignments, problems named "Modeling" do not require you to formulate Linear Programming (LP) model. However, for problems named "Linear Programming," all correct solutions are LP models, therefore formulations with either nonlinear constraints or integer or binary variables will be penalized. This homework is a good training for quiz #1, so you would want to work independently.

1 Convexity and relaxations (5 pts.)

For each of the following problems, determine if they are convex or not, **only** by looking at the functions one by one (i.e., it is not necessary to draw the feasible set on the Cartesian plane), and motivate your answer:

$$(3)\ \begin{array}{ll} \min & x^2 + 10y^4 - \ln(z) \\ & x^2 + y^2 + 3z^2 \leq 1 \\ & 3y^2 \leq 125 \\ & y - 2z^2 \geq 0 \end{array}$$

$$(4)\ \begin{array}{ll} \min & (x+y)^2 \\ & x - y = 0 \\ & e^x \leq 1 \end{array}$$

$$(1)\ \begin{array}{ll} \min & 2x^2 + 2y^2 + xy \\ & 15x + 50y \geq 17 \\ & 0.001x - y = 0 \\ & x^2 + y^2 = 1/3 \end{array}$$

$$(2)\ \begin{array}{ll} \min & 3x^2 + 3y^2 + 5xy \\ & 50x + 50y \leq 530 \\ & 5x - y \leq 3 \\ & 2x^2 + 2y^2 + 3xy \leq 5 \end{array}$$

$$(5)\ \begin{array}{ll} \min & x^2 + 3y + 5z \\ & e^x + e^y \leq 1 \\ & x^2 + z^2 + 2xy \geq 1 \end{array}$$

$$(6)\ \begin{array}{ll} \max & 2x^2 + 2y^2 + 3xy \\ & x^2 + y^4 \leq 1 \\ & x + y \geq 0.1 \\ & 3x + 2y = 5 \end{array}$$

2 Local and global minima (5 pts.)

Consider the following problem:

$$\begin{array}{lrl} \max & -20x + 24y & \\ & -x + y & \leq 4 \\ & 2x + y & \leq 5 \\ & x + y & \geq 3 \\ & x, y & \in \mathbb{Z} \end{array}$$

1. Is the problem convex?
2. Is the solution $(x, y) = (0, 4)$ feasible?
3. Is the solution $(x, y) = (1\frac{1}{3}, 4\frac{1}{3})$ feasible?
4. What is the value of the objective function that corresponds to each of the previous two solutions?
5. Are these two numbers lower bounds, upper bounds, none, or both?
6. Eliminate the last constraint (integrality of x and y). Is the new problem convex?
7. Are the two solutions above feasible for the relaxation?
8. Is any of them optimal for the relaxation?
9. Can you conclude that $(x, y) = (0, 4)$ is optimal for the original problem from the objective function value at this point and the objective value at $(x, y) = (1\frac{1}{3}, 4\frac{1}{3})$?

3 Linear Programming Model (5 pts.)

In Professor Snape's potions class Harry and Ron need to mix an *comfort-amnesia* potion, which can be obtained by mixing three ingredients: ground *forget-me stone*, which costs C_1 galleons per ounce, *liquid morpheus*, which costs C_2 galleons per ounce, and *essence of sirens*, costing C_3 galleons per ounce. Each of the three ingredients contain three main components in different proportions. The three components are: *memory eraser*, *relaxant* and *hallucinogen*. The proportions of memory eraser, relaxant and hallucinogen in the forget-me stone are 60%-20%-20%, in the liquid morpheus they are 30%-30%-40% and in the essence of sirens it is 40%-20%-40%. The resulting mixture should contain between M_1 and M_2 percentage of memory eraser, between R_1 and R_2 of relaxant and between H_1 and H_2 of hallucinogen. Snape is an unpleasant and stingy professor, so Harry and Ron better make a correct potion while minimizing the total cost of the used ingredients. Formulate this as a Linear Programming problem. Do not solve!

4 Knapsack problem (5 pts.)

Consider the knapsack problem as described in our lecture notes

$$\begin{aligned} \mathbf{P}: \min \quad & \textstyle\sum_{i=1}^{n} w_i x_i \\ & \textstyle\sum_{i=1}^{n} p_i x_i \geq C \\ & x_i \in \{0,1\} \quad \forall i = 1, 2, \ldots, n \end{aligned}$$

Suppose there are $n = 9$ objects and $C = 51$.

i	1	2	3	4	5	6	7	8	9
p_i	30	13	6	11	21	35	10	31	20
w_i	2	1	1	1	2	3	3	3	2

1. Consider solution $(0, 0, 0, 0, 0, 0, 0, 1, 1)$. Show that you cannot improve this solution by swapping only one object for any other, i.e. moving one of the "1"s to a different position. Yet observe that this solution is not a global optimum.

2. Apply the greedy method (by ordering the variables according to their price/weight ratio) to solve the LP relaxation of this problem. Compute the the lower bound on the original problem.

3. Using the solution to the relaxation find a feasible solution for the original problem. Compare the objective value to the lower bound. s the resulting solution - a global optimal solution?

4. Find the global optimal solution. Can you prove that it is the global optimal solution?