3 Integer Programming (10 pts.)

Consider a graph $G = (V, E)$, in Figure 1, and a cost C_{ij} for each edge $\{i, j\} \in E$. Suppose you want to find the subset S of V with at least k nodes, such that the cost of all edges, that link a node in S with a node outside of S is minimized. For example, if S is the set of four dark nodes is the graph in Figure 1, then the total cost of all edges connecting S to other nodes is $C_{12} + C_{14} + C_{15} + C_{23} + C_{34} + C_{36}$

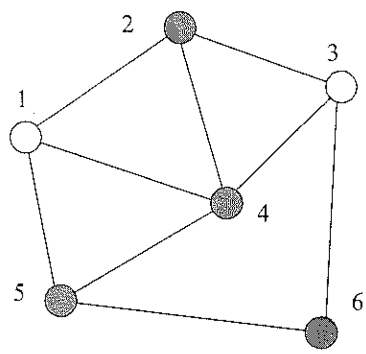

Figure 1:

1. Consider binary variables that indicate if a node is in S or not. Consider also binary variables that indicate if an edge is connecting a node in S with a node outside of S. Now write down conditions between these types of variables, which ensure logical implications: for all $\{i, j\} \in E$, if $i \in S$ and $j \in V/S$ then edge $\{i, j\}$ connects node in S with a node outside S. (5pts)

Define the auxiliary binary variables as follows:

$$x_i = \begin{cases} 1 & \text{if node } i \text{ belongs to the set } S \\ 0 & \text{Otherwise} \end{cases} \quad \text{for } i = 1, \ldots, |V|$$

Note: $|\cdot|$ denotes the size of a set.

$$y_{ij} \begin{cases} 1 & \text{if exactly one out of the two node } i \text{ and } j \text{ belongs to } S \\ 0 & \text{Otherwise} \end{cases}$$

This problem requires that $x_i + x_j = 1 \longrightarrow y_{ij} = 1$

This relationship can be established by imposing $\longrightarrow$ see the back of the page

$$\begin{cases} y_{ij} \geq x_i - x_j & (1) \\ y_{ij} \geq x_j - x_i & (2) \end{cases}$$

$x_i, x_j, y_{ij} \in \{0, 1\}$

It can be interpreted from the constraints that

if $x_i + x_j = 0$, then by $(3) \longrightarrow y_{ij} \leq x_i + x_j = 0$

we also get (1), (2), (4) $\begin{cases} y_{ij} \geq 0 \\ y_{ij} \leq 2 \quad \text{Redundant} \end{cases}$

In a similar way, we have

if $x_i + x_j = 2 \Rightarrow$ by (4) $2 + y_{ij} \leq 2 \Rightarrow y_{ij} = 0$

by (1), (2), (3) $\begin{cases} y_{ij} \geq 0 \\ y_{ij} \leq 2 \quad \text{Redundant} \end{cases}$

if $x_i + x_j = 1 \Rightarrow \begin{cases} y_{ij} \geq 1 \\ y_{ij} \geq -1 \quad \text{Redundant} \\ y_{ij} \leq 1 \end{cases}$

2. Write the full formulation of the problem of selecting at least k nodes so that the edge cost is minimized. (5pts).

By the binary variables and logical conditions defined in the previous part, the formulation is given by

$$\text{Min} \sum_{(i,j)\in E} c_{ij}\, y_{ij}$$

s.t.

$$y_{ij} \geq x_i - x_j \quad \forall (i,j)\in E$$

$$y_{ij} \geq x_j - x_i \quad \forall (i,j)\in E$$

$$\sum_{i\in V} x_i \geq k$$

$$x_i \in \{0,1\} \quad \forall i \in V$$

$$y_{ij} \in \{0,1\} \quad \forall (i,j)\in E$$