MAT-MEK4270 Oblig 1 besvarelser

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1 Problem 1.2.3

We have the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where we want to check that the function $u(t,x,y) = e^{\imath(k_x x + k_y y - \omega t)}$ satisfies the wave equation. Now, we have that $\omega = c\sqrt{k_x^2 + k_y^2}$. Thus, using this fact, we simply insert the solution into the equation to check both sides are satified. We therefore get that:

$$\frac{\partial}{\partial t} u(t,x,y) = -\imath\omega e^{\imath(k_x x + k_y y - \omega t)} = -\imath\omega u(t,x,y).$$

Applying the partial derivative with respect to t again on $-\imath\omega u(t,x,y)$, we therefore easily get:

$$\frac{\partial}{\partial t}(-\imath\omega u(t,x,y)) = (-\imath\omega)^2 u(t,x,y) = -\omega^2 u(t,x,y).$$

With $\nabla^2 u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u$ we therefore get:

$$\frac{\partial}{\partial x} u(t,x,y) = \imath k_x e^{\imath(k_x x + k_y y - \omega t)} = \imath k_x u(t,x,y)$$

which further gives us that:

$$\frac{\partial}{\partial x}(\imath k_x u(t,x,y)) = (\imath k_x)^2 u(t,x,y) = -k_x^2 u(t,x,y).$$

Doing this for the partial derivative with respect to y gives us then $\frac{\partial^2}{\partial y^2} u(t,x,y) = -k_y^2 u(t,x,y)$. Thus, we have that $\frac{\partial^2}{\partial t^2} u(t,x,y) = -\omega^2 u(t,x,y)$ and also:

$$\nabla^2 u(t,x,y) = -k_x^2 u(t,x,y) + (-k_y^2 u(t,x,y)) = -(k_x^2 + k_y^2) u(t,x,y)$$

which therefore gives us $c^2 \nabla^2 u(t,x,y) = -c^2(k_x^2 + k_y^2) u(t,x,y)$. But, by what ω was defined as, we have that $\omega^2 = c^2(k_x^2 + k_y^2)$. Hence, using this we get:

$$\frac{\partial^2}{\partial t^2} u(t,x,y) = -\omega^2 u(t,x,y) = -c^2(k_x^2 + k_y^2) u(t,x,y) = c^2 \nabla^2 u(t,x,y)$$

where we had that $u(t,x,y) = e^{\imath(k_x x + k_y y - \omega t)}$.

2 Problem 1.2.4

Now we assume that $m_x = m_y$ such that $k_x = k_y = k$. Ans we have $u_{ij}^n = e^{\imath(kh(i+j) - \bar{\omega} n \Delta t)}$. We also presume that the CFL-number is $C = \frac{1}{\sqrt{2}}$ while by definition we also have that $C = \frac{c\Delta t}{h}$. Now, for the exact ω in this case, we have that $\omega = c\sqrt{k_x^2 + k_y^2} = c\sqrt{k^2 + k^2} = \sqrt{2}ck$.So, looking at the discretized version of the partial differential equation we have the following:

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{(i+1)j}^n - 2u_{ij}^n + u_{(i-1)j}^n}{h^2} + \frac{u_{i(j+1)}^n - 2u_{ij}^n + u_{i(j-1)}^n}{h^2} \right).$$

Inserting for $u_{ij}^n = e^{\imath(kh(i+j) - \bar{\omega} n \Delta t)}$ and then dividing by $e^{\imath(kh(i+j) - \bar{\omega} n \Delta t)}$, we therefore end up with:

$$\frac{e^{-\imath\bar{\omega}\Delta t} - 2 + e^{\imath\bar{\omega}\Delta t}}{\Delta t^2} = c^2 \left(\frac{e^{\imath kh} - 2 + e^{\imath kh}}{h^2} + \frac{e^{\imath kh} - 2 + e^{\imath kh}}{h^2} \right).$$

Now, we use the identity from complex numbers that $\cos(x) = \frac{e^{\imath x} + e^{-\imath x}}{2}$ which therefore implies that $2\cos(x) = e^{\imath x} + e^{-\imath x}$. inserting this identity therefore gives us:

$$\frac{2\cos(\bar{\omega}\Delta t) - 2}{\Delta t^2} = c^2 \frac{4\cos(kh) - 4}{h^2} \implies \cos(\bar{\omega}\Delta t) - 1 = \frac{c^2 \Delta t^2}{h^2}(2\cos(kh) - 2).$$

Now, we use another trigonometric identity. This time we have that $\cos(2x) = \cos^2(x) - \sin^2(x)$, and also $\sin^2(x) + \cos^2(x) = 1$. So, the left hand side becomes:

$$\cos(\bar{\omega}\Delta t) - 1 = \cos^2\left(\frac{\bar{\omega}\Delta t}{2}\right) - \sin^2\left(\frac{\bar{\omega}\Delta t}{2}\right) - \left(\sin^2\left(\frac{\bar{\omega}\Delta t}{2}\right) + \cos^2\left(\frac{\bar{\omega}\Delta t}{2}\right)\right) = -2\sin^2\left(\frac{\bar{\omega}\Delta t}{2}\right).$$

Doing the same on the right-hand side and also using that $C = \frac{c\Delta t}{h}$ we get:

$$\frac{c^2\Delta t^2}{h^2}(2\cos(kh) - 2) = 2C^2(\cos(kh) - 1) = 2C^2\left(-2\sin^2\left(\frac{kh}{2}\right)\right) = -4C^2\sin^2\left(\frac{kh}{2}\right).$$

Now, we are almost there. Using these two equalites for both sides we therefore have the equation:

$$-2\sin^2\left(\frac{\bar{\omega}\Delta t}{2}\right) = -4C^2\sin^2\left(\frac{kh}{2}\right) \implies \sin\left(\frac{\bar{\omega}\Delta t}{2}\right) = \sqrt{2}C\sin\left(\frac{kh}{2}\right) \implies \sin\left(\frac{\bar{\omega}\Delta t}{2}\right) = \sin\left(\frac{kh}{2}\right)$$

where we have used the assumption that $C = \frac{1}{\sqrt{2}}$. Taking the inverse sine function and getting $\bar{\omega}$ alone we therefore get:

$$\bar{\omega} = \frac{2}{\Delta t}\sin^{-1}\left(\sin\left(\frac{kh}{2}\right)\right) = \frac{2}{\Delta t}\frac{kh}{2} = \frac{kh}{\Delta t} = \frac{\omega h}{\sqrt{2}c\Delta t} = \frac{1}{\sqrt{2}}\frac{h}{c\Delta t}\omega = C\frac{1}{C}\omega = \omega$$

where we have used that $\omega = \sqrt{2}kc$ to get the expression for k through ω and c. Thus it is shown that the numerical dispersion $\bar{\omega}$ is equal to the exact ω.