Number Representation

Qcamp 2019, Experimental Session 20/06/2019

1 Introduction

Philosophically speaking, what is a number? Conventionally we assign ten *unique symbols* such as 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to represent a quantity or basic counting units (for eg. like our favourite number 42). Using these ten symbols to represent a number is called the decimal representation. There are many other ways to represent a number, just like how an object can have many names in a different language.

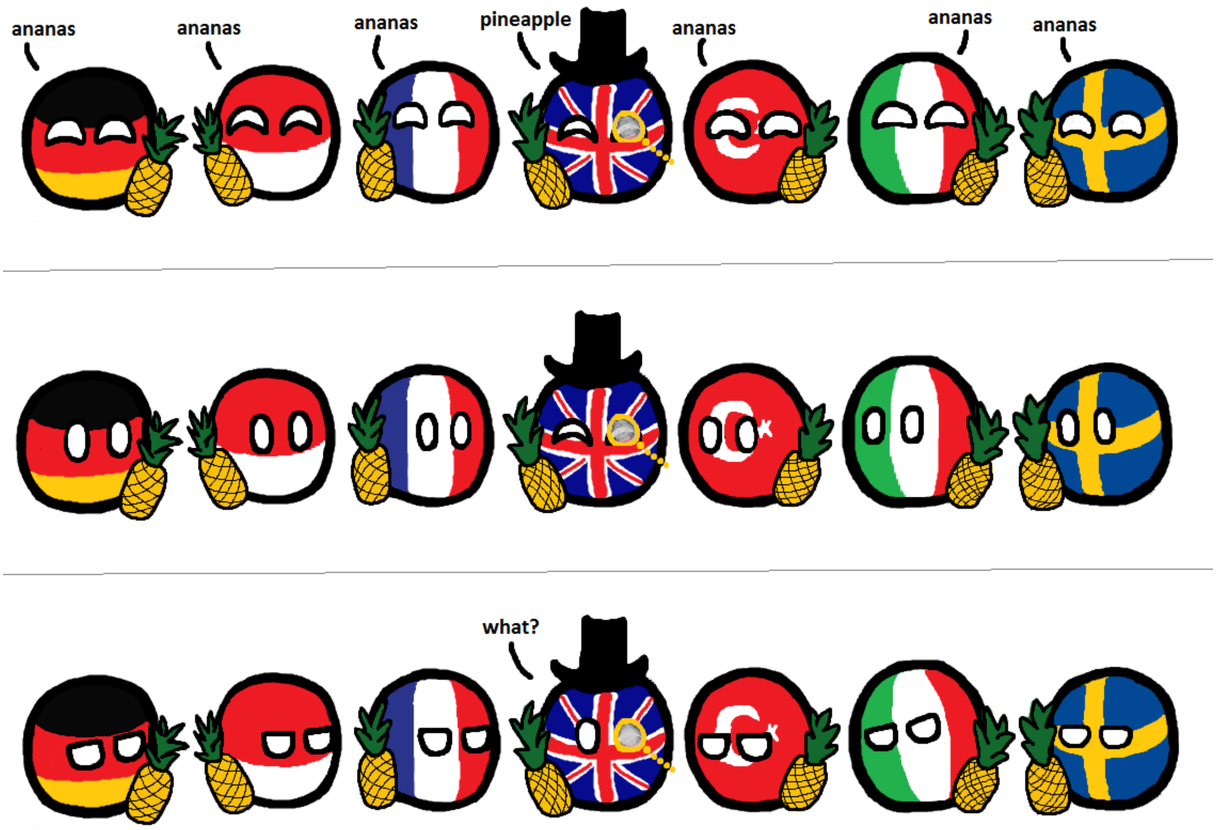

Figure 1: 'ananas' and 'pineapple'. Obtained from https://imgur.com/gallery/1sjgf

The aim of the exercise is to gain an appreciation for the following statements:

1. *"There are 10 types of people in this world; those who know binary and those who don't."*
2. *"In some context, dead beef is equivalent to 3735928559."*

1.1 Overview of Number Representation

As a starter, we shall start with a simple question: why do we normally write "ten" as 10? This is because the conventional numeral system we use daily is called the decimal representation, and if we start from zero (0), by the time we reached nine (9), we would have ran out of symbols to write since there are only ten of them. To write the next number, we simply use a combination of two digits in the form '10'. The first digit tells us we have one quantity of ten and zero for the basic unit. This is why we also call the position of the '1' in '10' the tens' place, because it signifies the number of tens we have. If we write '42', what we mean is that we have four quantities of ten and two for the basic unit

Mathematically, we have decomposed the number ten into $1 \times 10^1 + 0 \times 10^0$ and the number forty-two into $4 \times 10^1 + 2 \times 10^0$. The position of the symbol has a certain **weight**, so for example the symbol in the tens' place has a weight of ten. If we add another digit to the front, like in '100', then the '1' here is in the hundreds' place and has a weight of hundred. Another point to note is that the number of unique symbols we use to represent numbers under a certain representation is called the **base or radix** of the representation.

For the purposes of today, we will be using the binary representation (radix = 2) and hexadecimal representation (radix = 16). **Numbers represented in binary is prefixed by "0b" and numbers represented in hexadecimal is prefixed by "0x". If not stated explicitly by the conventions above, the number representation is decimal (radix = 10).**

Binary representation is convenient for computers because the basic counting units are just on/off switches in some sense. In fact, this is why it is a lie if someone tells you, you can only count up to 10 with your 2 hands (assuming you have 10 fingers. If you have 4 fingers only, then you probably serve nice chicken wings! [1]) On the other hand, hexadecimal allows quick reading of binary strings (which we will tell you how later in this worksheet).

1.2 Converting to Decimal (As Easy As Doing Multiplication!)

To put into practice, if we have a string of symbols how do we translate it into the decimal representation? We just apply the following:

$$\text{Number} = \sum_{n=} (\text{symbol value}) \times (\text{radix})^{\text{n}} \tag{1}$$

where n is the position of the symbol from the right and n starts from 0.

The symbols used in each number system in order of increasing value is as follows:

1. Decimal: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
2. Binary: 0, 1
3. Hexadecimal: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f

By applying Equation 1, we can convert easily from a number basis to decimal system. For eg., starting from the left of the binary representation of the number $0\text{b}1010$:

$$0\text{b}\mathbf{1110} = (\mathbf{1} \times 2^3) + (\mathbf{1} \times 2^2) + (\mathbf{1} \times 2^1) + (\mathbf{0} \times 2^0) = 14$$

[1] This statement is not sponsored by any fast food outlet in any way.

Or the hexadecimal representation of the number 0x2face:

$$\begin{aligned}
0\text{x}\textbf{2face} =&(0\text{x}\textbf{2} \times 16^4) + (0\text{x}\textbf{f} \times 16^3) + (0\text{x}\textbf{a} \times 16^2) + (0\text{x}\textbf{c} \times 16^1) + (0\text{x}\textbf{e} \times 16^0) \\
=&(\textbf{2} \times 16^4) + (\textbf{15} \times 16^3) + (\textbf{10} \times 16^2) + (\textbf{12} \times 16^1) + (\textbf{14} \times 16^0) \\
=&(\textbf{2} \times 65536) + (\textbf{15} \times 4096) + (\textbf{10} \times 256) + (\textbf{12} \times 16) + (\textbf{14} \times 1) \\
=&195278
\end{aligned}$$

1.3 Conversion between Binary and Hexadecimal (As Easy As Cutting Strings!)

One important observation to make is this: 16 is simply 2^4. This means that each hex symbol has a corresponding 4 bits(symbol) long binary representation. This is summarised in the table below:

0	1	2	3	4	5	6	7
0000	0001	0010	0011	0100	0101	0110	0111
8	9	a	b	c	d	e	f
1000	1001	1010	1011	1100	1101	1110	1111

Table 1: Hexadecimal Symbol and Corresponding 4-bit Binary Representation

With this in mind, this means that a number represented by a string of n hexadecimal symbols is represented by a string of $4n$ bit (if we include leading zeroes of course) and vice-versa. Let's look at converting 0x2face to its binary representation:

$$0\text{x2face} : (0010)(1111)(1010)(1011)(1110) = 0\text{b}00101111101010111110$$

Converting from binary to hexadecimal is easy as well! Just cut the binary strings up into 4-bit chunks starting from the right and add 0's to the start when necessary[2]. For example, let's convert 0b011101111101101 to hexadecimal:

$$0\text{b}11101111101101 : (00)11 \;\; 1011 \;\; 1110 \;\; 1101 = 0\text{x3BED}$$

1.4 Conversion from Decimal to a Arbitrary Representation (Divide and Conquer!)

To convert from decimal representation to others is not as straightforward, but can still be easily done. All you need is to apply the Euclidean algorithm for finding the greatest common divisor[3]:

1. Divide the number you are converting (q_0) by the new representation's radix (b).
2. Get the quotient q_1 and the remainder r_0. ($q_0 = q_1 * b + r_0$)
3. Divide q_1 by b to obtain r_1, do the same for q_2 and repeat until $q_i = 0$.
4. String all your r's together, from r_i to r_0 in order to form the answer ($r_{i+1}r_i \ldots r_1r_0$).

[2]This is also commonly termed as padding.

[3]Refer to this if unclear about the algorithm https://en.wikipedia.org/wiki/Euclidean_algorithm

Let's convert 47802 to its hexadecimal (radix = 16) representation:

$$\begin{aligned}47802 =& 2987 \times 16 + \mathbf{10}\\ =& ((186 \times 16 + \mathbf{11}) \times 16) + \mathbf{10}\\ =& (((11 \times 16 + \mathbf{10}) \times 16 + \mathbf{11}) \times 16) + \mathbf{10}\\ =& ((((0 \times 16 + \mathbf{11}) \times 16 + \mathbf{10}) \times 16 + \mathbf{11}) \times 16) + \mathbf{10}\end{aligned}$$

The bold numbers are what we are interested in, hence $47802 = 0$xBABA.

2 Assignment

Task 1a [1 pt] So by right, what number (in decimal representation) can you count up to using your 2 hands (assuming you have 10 fingers) and if you use a binary number representation?

decimal : 45
binary : 1023

Task 1b [1 pt] Stretch both your hands out with your open palms facing away from you, fingers facing up. Put down your ring, middle and index fingers on both hands. Assuming a finger that is up represents 1 and a finger that is down represents 0, and the lowest weighted position is on your right little finger (2^0, what number (in decimal representation) are you representing with your hands?

561

Task 2 [1 pt] What is 0b1111111011101101 in hexadecimal? Do your working on this paper to show how you arrived at the answer.

0xfeed

Task 3 [2 pt] Convert the number 48879 to its binary representation and hexadecimal. (Hint: The order in which you attempt this question makes life easier.)

0b1011111011101111
0xbeef

Task 4 [2 pts] Convert 0xDEAD to its binary representation and decimal representation. Show your working for the conversion to decimal representation.

57005

Task 5 [3 pts] One day you were given the task of creating questions on number representation in a handout that would be given to students during a workshop on experimental quantum physics. You ran out of ideas so you took out a pack of poker cards and was inspired to create the following number representation of radix 13, with the symbols in order of increasing value: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. You have decided to call this the "poker number representation" and is prefixed by "0p" [4]
To confuse the students even further, 0p2 does not correspond to the decimal value of 2, but corresponds to the decimal value of 1. More examples include: 0pA in decimal is 0, 0pQ in decimal is 11, and 0p10 is actually one symbol representing the decimal value of 9.

Convert the Royal Flush hand 0p10JQKA to its **hexadecimal representation**. (Show your working so we can award partial credit if the final answer is not correct)

0x5db1

[4] At the point of writing, we are not sure whether such a basis actually exist or not, and whether or not it has already been named something else or given a convention.