PERTH MODERN SCHOOL

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INDEPENDENT PUBLIC SCHOOL

Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST
UNIT 1

Section One:
Calculator-free

Your Name Solutions

Your Teacher's Name ____________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Max	Question	Mark	Max
1		8	5		7
2		5	6		3
3		9	7		3
4		8	8		7

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free			50		35
Section Two: Calculator-assumed			100		65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2019*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section One: Calculator-free **(50 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(8 marks)**

(a) Simplify $\frac{(n+3)!}{(n+1)!}$ (2 marks)

$$= \frac{(n+3)(n+2)(n+1)!}{(n+1)!} \checkmark$$

$$= n^2 + 5n + 6 \checkmark$$

(b) Evaluate ${}^{36}P_{29} \div {}^{34}P_{30}$ (3 marks)

$$= \frac{36!}{7!} \div \frac{34!}{4!} \checkmark$$

$$= \frac{36 \times 35 \times 34!}{7 \times 6 \times 5 \times 4!} \times \frac{4!}{34!} \checkmark$$

$$= 6 \checkmark$$

(c) Show that $n! \cdot (n-1) = (n^2 - n) \cdot (n-1) \cdot (n-2)!$ For $n \geq 2$. (3 marks)

$$n!(n-1) = n \cdot (n-1)(n-2)!(n-1) \checkmark$$

$$= (n^2 - n)(n-1)(n-2)! \checkmark\checkmark$$

OR

$$(n^2 - n)(n-1)(n-2)! = n(n-1)(n-1)(n-2)! \checkmark\checkmark$$

$$= n!(n-1)! \checkmark$$

Question 2 **(5 marks)**

(a) Consider the following statement:

For all positive integers m and n, n^2 is not a multiple of m^3.

Write whether the statement is true or false, and prove or disprove it accordingly. (3 marks)

False. ✓ As a counter example, let $m=2$ and $n=4$ ✓
then $n^2=16$ and $m^3=8$
so n^2 is a multiple of m^3 ✓

(b) A student was asked to prove the following statement:

An integer n is the sum of 2 odd primes p and q if and only if n is even and at least 6.

The student wrote the following proof:

Proof:
Assume that n is the sum of two odd primes p and q. Then $p = 2k + 1$ and $q = 2\ell + 1$ where k and ℓ are integers greater than or equal to 1. So

$$\begin{aligned} n &= p + q \\ &= 2k + 1 + 2\ell + 1 \\ &= 2(k + \ell) + 2 \\ &= 2(k + \ell + 1) \end{aligned}$$

which is even since $k + \ell + 1$ is an integer. Moreover, $n \geq 6$ since $k + \ell + 1 \geq 3$.
Hence n is even and at least 6.

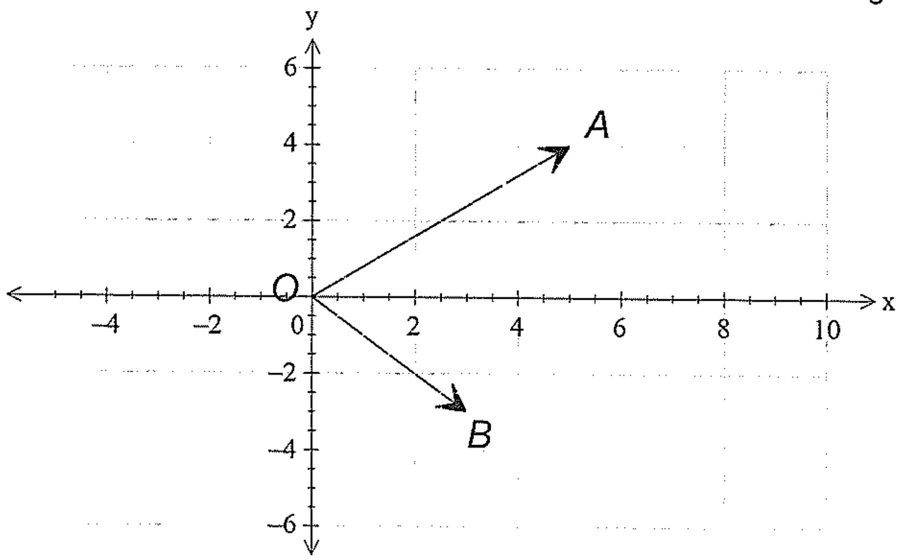
QED

Explain why this is insufficient as a proof of the given statement. (2 marks)

The statement contains the phrase "if and only if" ✓
but the student has proved only the "forward" direction and not the backward direction ✓

Question 3 **(9 marks)**

The diagram shows the position vectors of points A and B in relation to some origin O.



(a) Express $\overrightarrow{OB}$ in vector component form (ie x**i** + y**j**). (1 marks)

$$\overrightarrow{OB} = 3\underset{\sim}{i} - 3\underset{\sim}{j}$$

(b) On the axes provided, accurately draw and label the vector **r** = $\overrightarrow{OA} + \overrightarrow{OB}$. (2 marks)

$$\overrightarrow{OA} + \overrightarrow{OB} = (5\underset{\sim}{i} + 4\underset{\sim}{j}) + (3\underset{\sim}{i} - 3\underset{\sim}{j}) \checkmark$$
$$= 8\underset{\sim}{i} + \underset{\sim}{j} \checkmark$$

(c) If the unit vector **j** is pointing due North, give the direction of $\overrightarrow{OB}$ as a bearing. (2 marks)

$$\overrightarrow{OB} = 3\underset{\sim}{i} - 3\underset{\sim}{j}$$
$$\therefore \tan\theta = \frac{-3}{3} = -1 \checkmark$$
$$\therefore \theta = 135^\circ$$
Bearing is 135° ✓

(d) Write down the position vector of point C if $\overrightarrow{CA} = 8\mathbf{i} + \mathbf{j}$. (2 marks)

$$\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA}$$
$$\text{ie } 8\underset{\sim}{i} + \underset{\sim}{j} = -(x\underset{\sim}{i} + y\underset{\sim}{j}) + 5\underset{\sim}{i} + 4\underset{\sim}{j} \checkmark$$
$$\therefore \overrightarrow{OC} = -3\underset{\sim}{i} + 3\underset{\sim}{j} \checkmark$$

(e) Calculate the exact value of $|\overrightarrow{OB}|$ in simplified surd form. (2 marks)

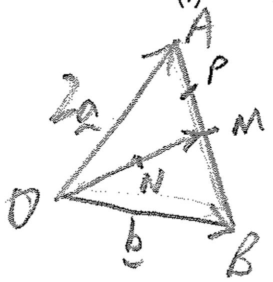
$$|\overrightarrow{OB}| = \sqrt{3^2 + (-3)^2} \checkmark$$
$$= \sqrt{18}$$
$$= 3\sqrt{2} \checkmark$$

Question 4 **(8 marks)**

In a triangle OAB, $\overrightarrow{OA} = 2a, \overrightarrow{OB} = b$. M is the midpoint of AB, N is the midpoint of OM and P is the midpoint of AM.

(a) Express the following vectors in terms of a and b.

(i) $\overrightarrow{AM}$ (2 marks)



$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} \checkmark$$
$$= \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$$
$$= \frac{1}{2}(b - 2a)$$
$$= \frac{1}{2}b - a \checkmark$$

(ii) $\overrightarrow{ON}$ (2 marks)

$$\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OM}$$
$$= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{AM}) \checkmark$$
$$= \frac{1}{2}(2a + \frac{1}{2}b - a)$$
$$= \frac{1}{2}a + \frac{1}{4}b \checkmark$$

(b) Use a **vector method** to show that NP and OA are parallel. (4 marks)

$$\overrightarrow{NM} = \overrightarrow{ON}$$
$$= \frac{1}{2}a + \frac{1}{4}b \checkmark$$
$$\overrightarrow{MP} = -\overrightarrow{PM}$$
$$= -\frac{1}{2}\overrightarrow{AM}$$
$$= -\frac{1}{2}(\frac{1}{2}b - a)$$
$$= \frac{1}{2}a - \frac{1}{4}b \checkmark$$
$$\overrightarrow{NP} = \overrightarrow{NM} + \overrightarrow{MP}$$
$$= \frac{1}{2}a + \frac{1}{4}b + \frac{1}{2}a - \frac{1}{4}b$$
$$= a \checkmark$$

$\therefore \overrightarrow{OA} = 2\overrightarrow{NP} \checkmark$

Hence

$\overrightarrow{NP} \parallel \overrightarrow{OA}$

Question 5 **(7 marks)**

(a) Write $0.15\dot{3}$ in the form $\frac{a}{b}$ where are a and b are integers with highest common factor 1. (3 marks)

Let $x = 0.15\dot{3}$

$1000x = 153.\dot{3}$

$100x = 15.\dot{3}$

$\therefore 900x = 138$ ✓

ie $x = \frac{138}{900}$ ✓

$= \frac{23}{50}$ ✓

(b) The conjecture that 2^n *will always have a factor of 2* is not true.

Give a counterexample to show that it is not true and then modify the statement " 2^n *will always have a factor of 2* " so that it is always true. (2 marks)

Not true since if $n=\frac{1}{2}$, $2^n=\sqrt{2}$ which is not divisible by 2 ✓ Counter example

'For all positive integers n, 2^n has a factor of 2.

(c) Use an algebraic argument to prove that the mean of three consecutive even numbers will always be even. (2 marks)

Let a, b, c be three consecutive even numbers. Then $\exists\, k \in \mathbb{Z}$ s.t. $a=2k$ and hence $b=2k+2$ and $c=2k+4$ ✓

The mean of a,b,c is $\frac{a+b+c}{3} = \frac{2k+2k+2+2k+4}{3}$

$= \frac{6k+6}{3}$

$= 2(k+1)$ ✓

which is even since $k+1 \in \mathbb{Z}$

Question 6 **(3 marks)**

Show that the numbers in any row of Pascal's triangle are symmetrical.

Each position is given by: $^nC_r = \frac{n!}{r!(n-r)!}$ ✓

$= \frac{n!}{(n-r)!\,r!}$

$= \frac{n!}{(n-r)!\,(n-n+r)!}$ ✓

$= \frac{n!}{(n-r)!\,(n-(n-r))!}$

$= {}^nC_{n-r}$ ✓

Question 7 **(3 marks)**

In order to ensure there is never a tied vote, a particular council always has an odd number of members for its committees and subcommittees. Develop a formula to find the number of different subcommittees possible for a committee of any size. (Note: Assume the size of a subcommittee can range from one member up to as many as the whole committee.)

Each member is either in or out of the subcommittee ✓

$\therefore$ No odd subcommittees $= \frac{2^n}{2}$ ← total No. ✓

$= 2^{n-1}$ ✓

Question 8 **(7 marks)**

Let $S_n = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots + \frac{1}{(2n-1)(2n+1)}$. Use mathematical induction to prove that

$$S_n = \frac{n}{2n+1}$$

for all positive integers n.

Let $P(n)$ be the propoition '$S_n = \frac{n}{2n+1}$' for all $n \in \mathbb{Z}^+$.

In $P(1)$, $LHS = \frac{1}{(2\times1-1)(2\times1+1)} = \frac{1}{3}$ and $RHS = \frac{1}{2\times1+1} = \frac{1}{3}$

✓ Evaluates LHS & RHS separately

Hence LHS = RHS, and so $P(1)$ is true ✓ shows P(1) is true

Now assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$ with $k \geq 1$.

Then $\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$. ✓ assumes P(k) is true

We now need to prove $P(k+1)$.

LHS of $P(k+1)$ $= \frac{1}{3} + \frac{1}{15} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$ ✓ substitutes $\frac{k}{2k+1}$ into LHS of P(k+1)

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$ ✓ rearranges

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{(2k+3)}$

$= \frac{k+1}{2(k+1)+1}$ ✓ obtains RHS of P(k+1)

$=$ RHS of $P(k+1)$

So $P(k+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

✓ writes conclusion.