Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section One: Calculator-free

SOLUTIONS

Students Name ______________________________

Teacher Name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

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Question 1 **(7 marks)**

Let $f(x) = 15 - 4x - 6x^2 - 4x^3 - x^4$.

(a) The curve $y = f(x)$ cuts the horizontal axis at $x = 1$. State, with reasons, whether the function is increasing, decreasing or neither at this point. (2 marks)

Solution
$f'(x) = -4 - 12x - 12x^2 - 4x^3, \quad f'(1) = -4 - 12 - 12 - 4 = -32$ Since the gradient at this point is negative, then the function is decreasing.
Specific behaviours
✓ indicates that $f'(1) < 0$ ✓ uses sign of derivative to deduce function is decreasing

(b) Determine $f''(0)$ and use this value to describe the concavity of the curve $y = f(x)$ where it crosses the vertical axis. (2 marks)

Solution
$f''(x) = -12 - 24x - 12x^2, \quad f''(0) = -12$ The curve is concave down at this point.
Specific behaviours
✓ correctly evaluates $f''(0)$ ✓ states concavity

(c) Does the curve $y = f(x)$ have any points of inflection? If it does, determine the coordinates of their location. If not, justify your answer. (3 marks)

Solution	
No, the curve does not have any points of inflection. $f''(x) = -12(x^2 + 2x + 1) = -12(x+1)^2, \quad f''(x) = 0 \Rightarrow x = -1$ Possible point of inflection at $x = -1$, so test for inflection:	
$f''(-1.1) = -12(-1.1+1)^2 < 0$ $f''(-0.9) = -12(-0.9+1)^2 < 0$ As curve is concave down on either side of $x = -1$, then not a point of inflection.	$f'''(x) = -24 - 24x$ $f'''(-1) = 0$ Since both second and third derivatives are zero at $x = -1$ then not a point of inflection.
Specific behaviours	
✓ states no, with reasonable attempt to justify ✓ locates point where $f''(x) = 0$ ✓ checks concavity either side of point, uses third derivative test, or other valid reasoning	

Question 2 **(6 marks)**

(a) Determine $f'(-2)$ when $f(x) = 2(3x+5)^3$. (3 marks)

Solution
$f'(x) = 2(3)(3)(3x+5)^2$ $= 18(3x+5)^2$ $f'(-2) = 18(-1)^2$ $= 18$
Specific behaviours
✓ recognises the need to use chain rule ✓ obtains correct derivative ✓ obtains correct value

(b) Determine $g(2)$ when $g'(x) = 12e^{3x-3}$ and $g(1) = 7$. (3 marks)

Solution	Alternative Solution
$g(2) = g(1) + \int_1^2 g'(x)\,dx$ $= 7 + \int_1^2 12e^{3x-3}\,dx$ $= 7 + [4e^{3x-3}]_1^2$ $= 7 + 4e^3 - 4e^0$ $= 3 + 4e^3$	$g(x) = 4e^{3x-3} + c$ $g(1) = 7 \to 4e^0 + c = 7$ $c = 3$ Therefore, $g(x) = 4e^{3x-3} + 3$ $g(2) = 4e^3 + 3$
Specific behaviours	**Specific behaviours**
✓ indicates total change is integral of rate of change ✓ obtains correct antiderivative ✓ obtains correct value	✓ correct antiderivative ✓ correct constant c ✓ obtains correct value

Question 3 **(8 marks)**

The function f is defined for $x > 0$ by $f(x) = \frac{e^{4x-1}}{x}$, and $f''(x) = \frac{2(8x^2-4x+1)e^{4x-1}}{x^3}$.

(a) Determine the coordinates and nature of all stationary points of the graph of $y = f(x)$. Justify your answer. (6 marks)

Solution
$f'(x) = \frac{(4e^{4x-1})(x)-(1)(e^{4x-1})}{x^2} = \frac{e^{4x-1}(4x-1)}{x^2}$ $f'(x) = 0 \rightarrow e^{4x-1}(4x-1) = 0 \rightarrow x = {}^1/_4$ $f''({}^1/_4) = \frac{2\left(\frac{1}{2}-1+1\right)e^0}{({}^1/_4)^3} = 64$ $f''({}^1/_4) > 0 \rightarrow \text{stationary point is a minimum}$ $f({}^1/_4) = \frac{e^0}{{}^1/_4} = 4$ The only stationary point of the graph is a minimum at $({}^1/_4, 4)$.
Specific behaviours
✓ attempts to use quotient rule   ✓ correctly obtains $f'(x)$ ✓ uses $f'(x) = 0$ to determine x-coordinate of stationary point   ✓ determine sign of second derivative at stationary point   ✓ correctly identifies nature of stationary point   ✓ correct coordinates of stationary point

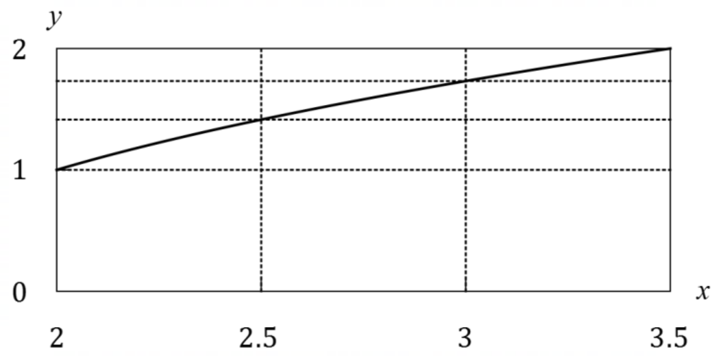
(b) Show that the graph of $y = f(x)$ has no points of inflection. (2 marks)

Solution
For a point of inflection to exist, $f''(x) = 0 \rightarrow 8x^2 - 4x + 1 = 0$.   But for this quadratic, $b^2 - 4ac = (-4)^2 - 4(8)(1) = -16$, and so this equation has no solutions as the discriminant is less than zero. Thus, the graph has no points of inflection.
Specific behaviours
✓ uses $f''(x) = 0$ to obtain quadratic   ✓ uses quadratic to explain why no points of inflection

Question 4 **(8 marks)**

The graph of $y = \sqrt{2x-3}$ between $x = 2$ and $x = 3.5$ is shown at right.

Approximate values for $\sqrt{2}$ and $\sqrt{3}$ are 1.41 and 1.73 respectively.



(a) Use the areas of the rectangles shown to explain why $2.07 < \int_2^{3.5} \sqrt{2x-3}\,dx < 2.57$. (3 marks)

Solution
The value of the integral is the area under the curve between 2 and 3.5. The area of the inscribed rectangles is $\frac{1}{2}(1 + 1.41 + 1.73) = 2.07$, an underestimate. The area of the circumscribed rectangles is $\frac{1}{2}(1.41 + 1.73 + 2) = 2.57$, an overestimate. Hence the value of the integral must lie between these two.
Specific behaviours
✓ derives area approximation using inscribed rectangles ✓ derives area approximation using circumscribed rectangles ✓ explains inequality

(b) Evaluate $\int_2^{3.5} \sqrt{2x-3}\,dx$. (3 marks)

Solution
$\int_2^{3.5} (2x-3)^{\frac{1}{2}}\,dx = \left[\frac{1}{3}(2x-3)^{\frac{3}{2}}\right]_2^{3.5}$ $= \frac{1}{3}(4)^{\frac{3}{2}} - \frac{1}{3}(1)^{\frac{3}{2}}$ $= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$
Specific behaviours
✓ obtains $(2x-3)^{1.5}$ term in antiderivative ✓ obtains correct antiderivative ✓ substitutes both bounds and simplifies

(c) Evaluate $\int_2^{3.5} \left(1 - \sqrt{2x-3}\right) dx$. (2 marks)

Solution	Alternative Solution
$\int_2^{3.5} \left(1 - \sqrt{2x-3}\right) dx = \int_2^{3.5} 1\,dx - \int_2^{3.5} \sqrt{2x-3}\,dx$ $= \frac{3}{2} - \frac{7}{3} = -\frac{5}{6}$	$\int_2^{3.5} \left(1 - \sqrt{2x-3}\right) = \left[x - \frac{1}{3}(2x-3)^{\frac{3}{2}}\right]_2^{3.5}$ $= \frac{5}{6} - \frac{5}{3} = -\frac{5}{6}$
Specific behaviours	**Specific behaviours**
✓ uses linearity ✓ correct value	✓ correct integral ✓ correct value

Question 5 **(8 marks)**

When a seed is randomly selected from a packet and grown, the probability that it yields a white flower is $0.35 = \frac{7}{20}$.

(a) Explain why this context is suitable for modelling with a Bernoulli random variable and state the mean of the Bernoulli distribution. (2 marks)

Solution
There is a two-outcome situation (seed yields a white flower or it doesn't). $\mu = 0.35$
Specific behaviours
✓ indicates a two-outcome situation ✓ correct mean

(b) When several Bernoulli trials are repeated, the total number of successes can be modelled with a binomial random variable provided the trials meet two conditions. Briefly describe these conditions. (2 marks)

Solution
(i) The trials are independent of each other. (ii) The probability of success is the same for each trial.
Specific behaviours
✓ states trials independent ✓ states trails have same probability

(c) Nine seeds are randomly selected and grown. Write an expression for the probability that seven or eight of these seeds will yield a white flower. (2 marks)

Solution
$p = \binom{9}{7}(0.35)^7(0.65)^2 + \binom{9}{8}(0.35)^8(0.65)$
Specific behaviours
✓ one correct term of expression ✓ correct expression

(d) When a gardener wants to be at least 99% certain of obtaining one or more white flowers, the number of seeds n that must be selected and grown will be the solution of the inequality $b^n \leq a$. State, with justification, the value of the constant a and the value of the constant b. (2 marks)

Solution
$P(X \geq 1) = 1 - P(X = 0)$ $= 1 - 0.65^n$ Hence $1 - 0.65^n \geq 0.99, \quad 0.65^n \leq 0.01 \quad \rightarrow \quad a = 0.01, b = 0.65$
Specific behaviours
✓ indicates correct expression for $P(X \geq 1)$ ✓ correct values for a and b

Question 6 **(8 marks)**

Let $f(x) = e^{-2x}(\sin 2x - \cos 2x)$.

(a) Determine $f'(x)$, simplifying your answer. (3 marks)

Solution
$f'(x) = (-2e^{-2x})(\sin 2x - \cos 2x) + (e^{-2x})(2\cos 2x + 2\sin 2x)$ $= 4e^{-2x}\cos 2x$
Specific behaviours
✓ correctly applies product rule ✓ correctly differentiates trig terms ✓ simplifies to obtain correct derivative

(b) Use **differentiation** and your previous answer to show that

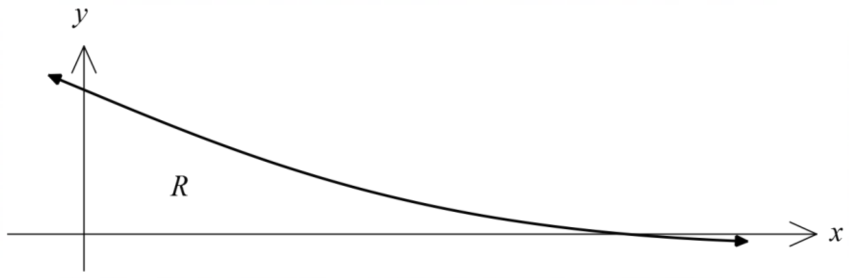
$$\int \left(e^{-2x}\cos 2x\right) dx = \frac{1}{4}e^{-2x}(\sin 2x - \cos 2x) + c,$$

where c is a constant. (2 marks)

Solution
Derivative of LHS (using derivative of integral of a function is original function): $\frac{d}{dx}\left(\int \left(e^{-2x}\cos 2x\right) dx\right) = e^{-2x}\cos 2x$ Derivative of RHS (using part (a)): $\frac{d}{dx}\left(\frac{1}{4}e^{-2x}(\sin 2x - \cos 2x) + c\right) = \frac{1}{4} \times 4e^{-2x}\cos 2x = e^{-2x}\cos 2x$ Hence LHS=RHS.
Specific behaviours
✓ differentiates LHS ✓ differentiates RHS and simplifies to equal derivative of LHS Note: 1 mark maximum if integration is used instead of differentiation

(c) The graph of $y = e^{-2x}\cos 2x$ is shown below. Determine the area of the region R, bounded by the curve, the x-axis and the y-axis.

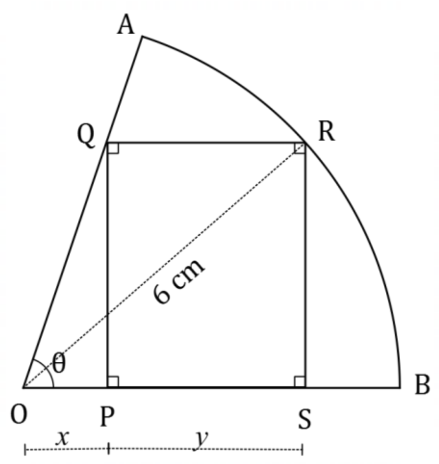
(3 marks)



Solution
$\cos 2x = 0 \Rightarrow 2x = \frac{\pi}{2}, x = \frac{\pi}{4}$ $\int_0^{\frac{\pi}{4}} e^{-2x}\cos 2x\,dx = \left[\frac{e^{-2x}}{4}(\sin 2x - \cos 2x)\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{e^{-\frac{\pi}{2}}}{4}\left(\sin\frac{\pi}{2} - \cos\frac{\pi}{2}\right)\right) - \left(\frac{e^0}{4}(\sin 0 - \cos 0)\right)$ $= \frac{e^{-\frac{\pi}{2}}}{4} + \frac{1}{4} = \frac{e^{-\frac{\pi}{2}} + 1}{4}$
Specific behaviours
✓ forms integral with correct bounds   ✓ writes antiderivative and substitutes bounds   ✓ simplifies

Question 7 **(7 marks)**

The diagram shows the vertices of rectangle $PQRS$ lying on sector OAB that subtends an angle of θ in a circle of radius 6 cm, and where $\tan\theta = 4$. Let $OP = x$ cm and $PS = y$ cm.



(a) Show that the perimeter of the rectangle is given by $p = 6x + 4\sqrt{9-4x^2}$ cm. (3 marks)

Solution
$PQ = OP\tan\theta = 4x, \qquad RS = 4x$ $OS^2 + SR^2 = OR^2$ $(x+y)^2 + (4x)^2 = 6^2$ $(x+y)^2 = 36 - 16x^2$ $x + y = 2\sqrt{9-4x^2}$ $\therefore PS = y = 2\sqrt{9-4x^2} - x$ $p = 2(PQ + PS)$ $= 2\left(4x + 2\sqrt{9-4x^2} - x\right)$ $= 6x + 4\sqrt{9-4x^2}$
Specific behaviours
✓ derives expression for PQ ✓ derives expression for PS ✓ derives expression for perimeter

(b) Determine the maximum perimeter of rectangle $PQRS$. (4 marks)

Solution
Derivative of p with respect to x: $$\frac{dp}{dx} = 6 + 4\left(\frac{1}{2}\right)(9-4x^2)^{-\frac{1}{2}}(-8x)$$ $$= 6 - \frac{16x}{\sqrt{9-4x^2}}$$ Derivative will be zero when $\frac{dp}{dx} = 0$. Hence $$16x = 6\sqrt{9-4x^2}$$ $$8x = 3\sqrt{9-4x^2}$$ Squaring both sides: $$64x^2 = 9(9-4x^2)$$ $$64\mathrm{x}^2 = 81 - 36\mathrm{x}^2$$ $$100x^2 = 81$$ $$x = \frac{9}{10}\ \mathrm{cm}$$ Finally determine perimeter: $$p = 6\left(\frac{9}{10}\right) + 4\sqrt{9 - 4\left(\frac{9}{10}\right)^2}$$ $$= \frac{54}{10} + 4\sqrt{\frac{9\times 100 - 4\times 9\times 9}{100}}$$ $$= \frac{54}{10} + \frac{4\times 3\sqrt{100-36}}{10}$$ $$= \frac{54}{10} + \frac{12\times 8}{10}$$ $$= \frac{150}{10}$$ $$= 15\ \mathrm{cm}$$
Specific behaviours
✓ obtains derivative ✓ equates derivative to zero ✓ obtains correct value of x ✓ obtains correct perimeter

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________