PERTH MODERN SCHOOL
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Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST
UNIT 1

Section Two:
Calculator-assumed

Your Name Solutions

Your Teacher's Name ____________________

Time allowed for this section
Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Max	Question	Marks	Max
9		7	17		7
10		11	18		5
11		3	19		5
12		5	20		6
13		4	21		6
14		5	22		12
15		7	23		7
16		4	24		6

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free			50		35
Section Two: Calculator-assumed			100		65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2019*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(100 Marks)**

This section has **sixteen (16)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 100 minutes.

Question 9 **(7 marks)**

Western Australia requires its residents to register their motor vehicles and display vehicle registration plates. Prior to 2000, regular issue plates consisted of three letters chosen from the 26 in the alphabet followed by three digits chosen from 0, 1, 2, ..., 9.

For example

Z	A	L	3	9	7

How many such registration plates are possible if:

(a) There are no restrictions on the choice of letters and digits. (2 marks)

$$N^{\underline{o}} = 26^3 \times 10^3 \checkmark$$

$$= 17576000 \checkmark$$

(b) No letter and no digit may feature more than once in the plate. (2 marks)

$$N^{\underline{o}} = 26 \times 25 \times 24 \times 10 \times 9 \times 8 \checkmark$$

$$= 11232000 \checkmark$$

(c) There is no restriction on the digits other than the final digit must be one more than the digit before it and the letters must be consecutive letters of the alphabet featuring in alphabetical order ("reverse alphabetical" not permitted). (3 marks)

$$N^{\underline{o}} = 24 \times 1 \times 1 \checkmark \times 8 \times 1 \times 9 \checkmark$$

$$= 1728 \checkmark$$

Question 10 **(11 marks)**

(a) Determine the number of integers between 1 and 900 inclusive that are divisible by 7 or 8.

(3 marks)

Nº Divisible by 7 $= 900 \div 7$
$= 128$
Nº Divisible by 8 $= 900 \div 8$
$= 112$ ✓
Nº Divisible by 56 $= 900 \div 56$
(ie both) $= 16$ ✓

$\therefore$ Nº Req^d $= 128 + 112 - 16$
$= 224$ ✓

(b) Given that 8 students in a class are good debaters,

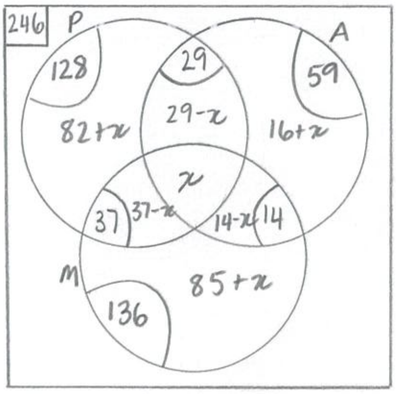
(i) in how many ways can a team of three be chosen from the 8 students? (1 mark)

$\binom{8}{3} = 56$ ✓

(ii) in how many ways can a team of three be chosen if one particular student must be chosen as captain and another student of the 8 cannot attend the debate? (2 marks)

$\binom{1}{1}\binom{1}{0}\binom{6}{2} = 15$ ✓ ✓

(c) There are 246 Year 11 students at Perth Modern School and each of them must study at least one of Physics, or Mathematical Applications or Mathematical Methods. There are 128 students who study Physics, 59 who study Mathematical Applications and 136 who study Mathematical Methods. Furthermore, 29 study Physics and Mathematical Applications, 37 study Physics and Mathematical Methods and 14 study Mathematical Applications and Methods. How many students study all three subjects? (2 marks)



$246 = 128 + 16 + x + 14 - x + 85 + x$

$x = 3$

✓ working (either)

$|P \cup A \cup M| = |P| + |A| + |M| - |P \cap A| - |P \cap M| - |A \cap M| + |P \cap A \cap M|$

$246 = 128 + 59 + 136 - 29 - 37 - 14 + x$

$246 = 243 + x$

$\therefore |P \cap A \cap M| = 3$ ✓ correct answer

(d) Annie has a street stall. She sells T-shirts in ten different colours but can only display six T-shirts at a time.

(i) In how many ways can she display six different coloured T-shirts in a line at the back of her stall? (1 mark)

$\underline{10}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5} = 151\ 200$ ✓ correct answer

$({}^{10}P_6)$

Annie's football team wear yellow and blue.

(ii) In how many ways can Annie display six different coloured T-shirts with a yellow T-shirt next to a blue T-shirt? (2 marks)

$\binom{2}{2}\binom{8}{4} \times 5! \times 2! = 16800$ ✓ correct answer

✓ choosing yellow and blue

Question 11 **(3 marks)**

There are n students in a class. Each of them writes a different integer from 1 to 49 on the board. No matter how they do this, there is always some pair of students whose numbers add to 50. What is the least possible value of n? Justify your answer.

If 25 students they could write from $1 \rightarrow 25$
Max total = 49 ✓✓

$\therefore$ 26 students as lowest pair would be $25 + 26 = 51$ ✓

Question 12 **(5 marks)**

(a) After noticing that $7^2 - 5^2 = 24 = 6 \times 4$ and that $19^2 - 17^2 = 72 = 6 \times 12$, a student made the conjecture that the difference of the squares of any two consecutive odd integers is always a multiple of 6. Is this conjecture true? Justify your answer. (2 marks)

No. Counterexample is $5^2 - 3^2$
$= 25 - 9$
$= 16$
which is not a multiple of 6 $\therefore$ Not True

(b) Use algebra to prove that the difference of the squares of any two consecutive odd integers is always even. (3 marks)

Let $n \in \mathbb{Z}$

Consecutive odd numbers are $2n+1$ and $2n+3$ ✓

Now $(2n+3)^2 - (2n+1)^2$ ✓

$= 4n^2 + 12n + 9 - (4n^2 + 4n + 1)$

$= 8n + 8$

$= 2(4n+4)$ ✓

$\therefore$ even $\forall n \in \mathbb{Z}$

Question 13 **(4 marks)**

Consider statement A: *If P then Q*

a) Suppose statement A is true. Is the **contrapositive** of A true or false? (1 mark)

True ✓

b) Write the **converse** of statement A. (1 mark)

If P then Q ✓

c) Suppose the converse of statement A is false. Is the **inverse** of statement A true or false? Explain your answer. (2 marks)

Inverse is false ✓

This is because the inverse is the contrapositive of the converse. ✓

Question 14 **(5 marks)**

Use the method of proof by contradiction to prove the following statement:

If p is an irrational number and q is a rational number, then pq is irrational.

Assume that p is irrational, q is rational and that pq is rational ✓

Then $p = \frac{a}{b}$ and $pq = \frac{c}{d}$ where $a,b,c,d \in \mathbb{Z}$ with $b,d \neq 0$ ✓

Hence $\frac{a}{b} \times q = \frac{c}{d}$

ie $q = \frac{cb}{da}$ ✓

But cb and da are integers implying that q is rational ✓

This contradicts the initial assumption

Hence if p is rational and q is rational then pq is irrational. ✓

Question ~~[illegible]~~ 15 **(7 marks)**

Use mathematical induction to prove that $3^n - 1 - 2n$ is divisible by 4 for all non-negative integers n.

Let $P(n)$ be the proposition '$3^n - 1 - 2n$ is divisible by 4' for all $n \in \mathbb{Z}^+$.

With $n = 1$, $3^n - 1 - 2n = 3^1 - 1 - 2 \times 1$ $= 0$ ✓ shows divisible by 4 when $n=1$.

which is divisible by 4. Hence $P(1)$ is true.

Now assume $P(k)$ is true for some $k \in \mathbb{Z}^+$. ✓ Assumes $P(k)$ is true.

Then $3^k - 1 - 2k = 4m$ for some $m \in \mathbb{Z}$, and so $3^k = 4m + 2k + 1$.

$$
\begin{aligned}
\text{Now } 3^{k+1} - 1 - 2(k+1) &= 3(3^k) - 1 - 2k - 2 \\
&= 3(3^k) - 2k - 3 \\
&= 3(4m + 2k + 1) - 2k - 3 \\
&= 12m + 6k + 3 - 2k - 3 \\
&= 12m + 4k \\
&= 4(3m + k)
\end{aligned}
$$

✓ considers expression with $n = k+1$

✓ rearranges

✓ writes as multiple of 4.

which is divisible by 4 since $3m + k$ is an integer.

So $P(k+1)$ is true. ✓ Observes that $P(k+1)$ must be true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$. ✓ Writes conclusion.

Question 16 **(4 marks)**

If M, N and P have position vectors 13**i** – **j**, 5**i** + **j** and 29**i** – 5**j** respectively, prove that the points M, N and P are collinear.

$\overrightarrow{MN} = 5\underset{\sim}{i} + \underset{\sim}{j} - (13\underset{\sim}{i} - \underset{\sim}{j})$
$= -8\underset{\sim}{i} + 2\underset{\sim}{j}$
$= -2(4\underset{\sim}{i} - \underset{\sim}{j})$ ✓

$\overrightarrow{NP} = 29\underset{\sim}{i} - 5\underset{\sim}{j} - (5\underset{\sim}{i} + \underset{\sim}{j})$
$= 24\underset{\sim}{i} - 6\underset{\sim}{j}$
$= 6(4\underset{\sim}{i} - \underset{\sim}{j})$ ✓

ie $\overrightarrow{MN} = \lambda \overrightarrow{NP}$ with common pt ✓

Hence collinear ✓

Question 17 **(7 marks)**

Given the vectors **a** = 2**i** + (x – 1)**j** and **b** = (x+2)**i** + 2**j** ;

a) determine the value(s) of x if **a** and **b** are parallel. (3 marks)

If parallel $\frac{x+2}{2} = \frac{2}{x-1}$ ✓✓

ie $(x-1)(x+2) = 4$

ie $x = 2, -3$ ✓

b) find two vectors parallel to **b** and of the same magnitude as **a** if x = -1. Give your answer in exact form. (4 marks)

If $x = -1$, $\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j}$

$|\underset{\sim}{a}| = \sqrt{2^2 + 2^2}$
$= \sqrt{8}$ ✓

$|\underset{\sim}{b}| = \sqrt{1^2 + 2^2}$
$= \sqrt{5}$ ✓

Vector 1 $= \frac{\sqrt{8}}{\sqrt{5}}(\underset{\sim}{i} + 2\underset{\sim}{j})$ ✓

Vector 2 $= -\frac{\sqrt{8}}{\sqrt{5}}(\underset{\sim}{i} + 2\underset{\sim}{j})$ ✓

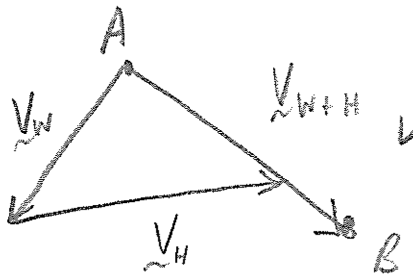
Question 18 **(5 marks)**

A helicopter, with a maximum speed through still air of 240 km/h, leaves its base at A to fly to a destination at B.

The position vector of B from A is $(155\mathbf{i} - 95\mathbf{j})$ km, and a steady wind of velocity $(-17\mathbf{i} - 22\mathbf{j})$ km/h is blowing over the area.

Find the velocity vector the helicopter pilot should set in order to fly directly from A to B.

$\underset{\sim}{V}_H = a\underset{\sim}{i} + b\underset{\sim}{j}$



✓

$\underset{\sim}{V}_W + \underset{\sim}{V}_H = -17\underset{\sim}{i} - 22\underset{\sim}{j} + a\underset{\sim}{i} + b\underset{\sim}{j}$

$= (a-17)\underset{\sim}{i} + (b-22)\underset{\sim}{j}$ ✓

$\underset{\sim}{V}_{W+H} = \lambda(155\underset{\sim}{i} - 95\underset{\sim}{j})$

$\therefore (a-17)\underset{\sim}{i} = 155\lambda\underset{\sim}{i}$

$(b-22)\underset{\sim}{j} = -95\lambda\underset{\sim}{j}$

ie $\dfrac{a-17}{b-22} = \dfrac{155}{-95}$

ie $a = -1.63b + 52.9$ ✓

Also $a^2 + b^2 = 240^2$ ✓

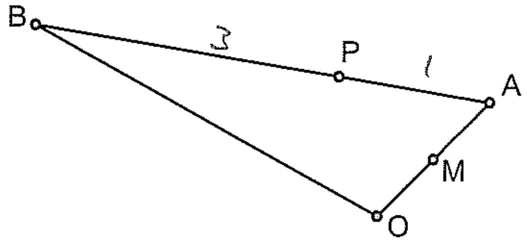
ie $(-1.63b + 52.9)^2 + b^2 = 240^2$

ie $b = -101.09$, $a = 217.68$ ✓

$\therefore$ Velocity Vector $= 217.67\underset{\sim}{i} - 101.09\underset{\sim}{j}$

Question 19 **(5 marks)**

In the triangle below, **a** = $\overrightarrow{OA}$, **b** = $\overrightarrow{OB}$, M is the midpoint of OA and P is a point on AB such that $|\overrightarrow{AP}| : |\overrightarrow{PB}| = 1:3$.



(a) Express each of the following in terms of **a** and /or **b**.

(i) $\overrightarrow{BA}$ $= \overrightarrow{BO} + \overrightarrow{OA}$ (1 mark)

$= \underset{\sim}{a} - \underset{\sim}{b}$ ✓

(ii) $\overrightarrow{OP}$ $= \overrightarrow{OB} + \frac{3}{4}\overrightarrow{BA}$ (1 mark)

$= \underset{\sim}{b} + \frac{3}{4}\underset{\sim}{a} - \frac{3}{4}\underset{\sim}{b}$

$= \frac{3}{4}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}$ ✓

(iii) $\overrightarrow{MP}$ (1 mark)

$= \frac{1}{2}\overrightarrow{OA} + \frac{1}{4}\overrightarrow{AB}$

$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{4}(-\underset{\sim}{a} + \underset{\sim}{b})$

$= \frac{1}{4}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}$ ✓

(b) If $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = -9\mathbf{i} + 4\mathbf{j}$, determine $|\overrightarrow{MP}|$. (2 marks)

$\overrightarrow{MP} = \frac{1}{4}\left[(\underset{\sim}{i} + 2\underset{\sim}{j}) + (-9\underset{\sim}{i} + 4\underset{\sim}{j})\right]$

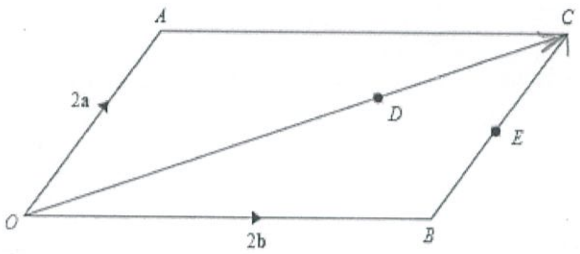
$= \frac{1}{4}(-8\underset{\sim}{i} + 6\underset{\sim}{j})$ ✓

$|\overrightarrow{MP}| = \sqrt{(-2)^2 + \left(\frac{3}{2}\right)^2}$

$= 2.5$ ✓

Question 20 **(~~5~~ 6 marks)**

The diagram below shows a parallelogram $OABC$. $\overrightarrow{OA} = 2a, \overrightarrow{OB} = 2b$. E is the midpoint of BC
Find the value of k where $\overrightarrow{OD} = k\overrightarrow{OC}$, so that A, D and E are on the same straight line.



$\overrightarrow{OC} = 2\tilde{a} + 2\tilde{b}$

$\overrightarrow{OD} = k(2\tilde{a} + 2\tilde{b})$

$\underline{\overrightarrow{AD}} = \overrightarrow{OD} - \overrightarrow{OA}$ ✓

$= (k-1)2\tilde{a} + k \cdot 2b$ ✓ Calculates expression for $\overrightarrow{AD}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = (1-k)(2a+2b)$

$\overrightarrow{EC} = \frac{1}{2}\overrightarrow{OA} = \tilde{a}$

$\underline{\overrightarrow{DE}} = \overrightarrow{DC} - \overrightarrow{EC} = (1-k)2\tilde{a} + (1-k)2\tilde{b} - \tilde{a}$

$= 2a - 2ka + (1-k)2b - \tilde{a}$

$= a - 2ka + (1-k)2b$

$= (1-2k)a + (1-k)2b$

$= (\frac{1}{2} - k)2a + (1-k)2b$ ✓✓

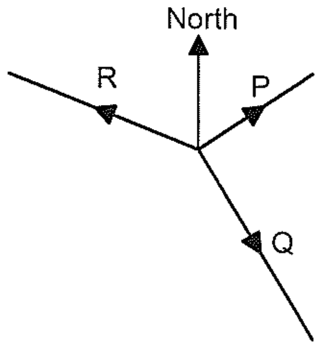
Calculates $\overrightarrow{DE}$

$AD \parallel DE \Rightarrow \dfrac{k-1}{\frac{1}{2}-k} = \dfrac{k}{1-k}$ ✓ Equating the ratio of a & b

Solve $k = \dfrac{2}{3}$. ✓ Solve for k.

Question 21 **(6 marks)**

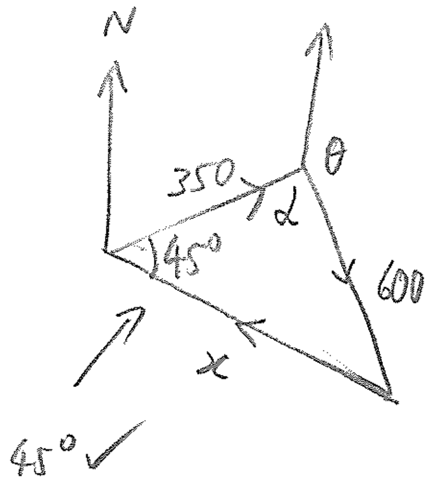
A body in equilibrium is acted on by three forces, as shown in the diagram (not to scale).



P is of magnitude 350N on a bearing of 060°, **Q** is of magnitude 600N on a bearing of θ, where $090° < \theta < 180°$, and **R** is of magnitude xN on a bearing of 285°.

Determine x and θ.

For equilibrium, $\underset{\sim}{P} + \underset{\sim}{Q} + \underset{\sim}{R} = 0$ ✓



By Cosine Rule

$600^2 = x^2 + 350^2 + 2 \times 350 \times x \cos 45°$ ✓

ie $x = 794.068$ N ✓

$\frac{\sin\alpha}{794.068} = \frac{\sin 45°}{600}$ ✓

ie $\alpha = 110.6$

$\therefore \theta = 129.4°$ ✓

Question 22 **(12 marks)**

From a fixed origin $(0,0)$, particle ***A*** has a position vector $12\mathbf{i}+61\mathbf{j}$ metres at 8am.

The particle is travelling with a constant velocity of $7\mathbf{i}-8\mathbf{j}$ metres per hour.

Calculate:

a) the position vector of the particle at 11am. (2 marks)

$$\underset{\sim}{r}_A = 12\underset{\sim}{i} + 61\underset{\sim}{j} + 3(7\underset{\sim}{i} - 8\underset{\sim}{j}) \checkmark$$
$$= 33\underset{\sim}{i} + 37\underset{\sim}{j} \checkmark$$

b) the distance travelled by the particle between 8am and 11 am. (2 marks)

$$dist = \sqrt{21^2 + 24^2}$$
$$= 31.89m$$

c) to the nearest minute, the first time when the particle is 50 metres from the origin. (4 marks)

$$\underset{\sim}{r}_A = 12\underset{\sim}{i} + 61\underset{\sim}{j} + \lambda(7\underset{\sim}{i} - 8\underset{\sim}{j}) \checkmark$$

$$\therefore 50 = \sqrt{(12+7\lambda)^2 + (61-8\lambda)^2} \checkmark$$

$$\text{ie } \lambda = 2.737 hrs \checkmark \therefore \text{Time} = 10:44am \checkmark$$

At 8am another particle "**B**" is located by position vector $57\mathbf{i}-29\mathbf{j}$ metres and is travelling with a constant velocity of $-2\mathbf{i}+10\mathbf{j}$ metres per hour.

d) Determine, to the nearest centimetre, the distance between the particles at 10am. (4 marks)

At 10am
$$\left.\begin{aligned} \underset{\sim}{r}_A &= 26\underset{\sim}{i} + 45\underset{\sim}{j} \\ \underset{\sim}{r}_B &= 53\underset{\sim}{i} - 9\underset{\sim}{j} \end{aligned}\right\} \checkmark$$

$$\therefore \overrightarrow{AB} = 27\underset{\sim}{i} - 54\underset{\sim}{j} \checkmark$$

$$\text{ie } |\overrightarrow{AB}| = \sqrt{27^2 + 54^2} \checkmark$$

$$= 60.37m \checkmark \text{ (to nearest cm)}$$

Question 23 **(~~6~~ 7 marks)**

In a coordinate plane, vertices $A(1,2)$, $B(2,-1)$, $C(8,1)$ and $D\,(7,4)$ form a quadrilateral $ABCD$.

(a) Use a **vector method** to prove that the quadrilateral $ABCD$ is a parallelogram.

(~~3~~ marks) 3.

$\overrightarrow{AB} = \begin{pmatrix} +1 \\ -3 \end{pmatrix}$, $\overrightarrow{DC} = \begin{pmatrix} +1 \\ -3 \end{pmatrix}$ ✓

$\overrightarrow{DC} = \overrightarrow{AB}$ $\therefore AB \parallel DC$ ✓

$|CD| = |AB|$ $\therefore AB \overset{\parallel}{=} DC$ ✓

$\therefore ABCD$ is a parallelogram.

(b) Is $ABCD$ a rectangle? Use a **vector method** to explain why or why not. (~~3~~ 4 marks)

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ ✓ Calculates a pair of adjacent vectors

$\overrightarrow{AB} \cdot \overrightarrow{CD} = 1 \times 6 - 3 \times 2 = 0$ ✓ Calculates Scalar product

$\therefore AB \perp CD$ & $ABCD$ is a parallelogram

✓ Conclude perpendicular from scalar product 0.

$\therefore ABCD$ is a rectangle.

✓ Conclude yes, it is a rectangle.

. **Question** 24 (6 marks)

Given that $\tilde{a}$ and $\tilde{b}$ are two non-zero vectors. Find the angle θ between $\tilde{a}$ and $\tilde{b}$, such that $\tilde{a} + 2\tilde{b}$ and $\tilde{a} - 4\tilde{b}$ are perpendicular, $\tilde{a} + 3\tilde{b}$ and $2\tilde{a} - 3\tilde{b}$ are perpendicular.

$$\begin{cases} \tilde{a}^2 - 2\tilde{a}\tilde{b} - 8\tilde{b}^2 = 0 & (1) \times 3 \\ 2a^2 + 3\tilde{a}\tilde{b} - 9\tilde{b}^2 = 0 & (2) \times 2 \end{cases}$$ ✓

$$\begin{cases} 3\tilde{a}^2 - 6\tilde{a}\tilde{b} - 24\tilde{b}^2 = 0 & (3) \\ 4a^2 + 6\tilde{a}\tilde{b} - 18\tilde{b}^2 = 0 & (4) \end{cases}$$

(3) + (4)

$$7\tilde{a}^2 - 42\tilde{b}^2 = 0$$

$$\underline{\tilde{a}^2 = 6\tilde{b}^2} \quad (*)$$ ✓

i.e $|a|^2 = 6|b|^2$

$$\underline{|a| = \sqrt{6}\,|b|}$$ ✓

In (1) : $2\tilde{a}\tilde{b} = a^2 - 8b^2$

$$\underline{ab} = \frac{a^2 - 8b^2}{2}$$

$$= \frac{6b^2 - 8b^2}{2}$$

$$= -\frac{2b^2}{2}$$

$$\underline{= -b^2}$$ ✓

$$\underline{\cos\theta} = \frac{a \cdot b}{|a||b|}$$

$$= \frac{-b^2}{\sqrt{6}|b||b|}$$

$$\underline{= -\frac{1}{\sqrt{6}}}$$ ✓

$$\underline{\underline{\theta = 114.09^\circ}}$$ ✓

Additional working space

Question number: ________

Additional working space

Question number: ________

Additional working space

Question number: ________