Baldivis Secondary College

WA Exams Practice Paper C, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2
Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________________

Your name ______________________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(8 marks)**

(a) For each of the following statements, state whether they are always true or sometimes false. Support each answer with an example.

(i) If $P \Rightarrow Q$, then if follows that $Q \Rightarrow P$. (2 marks)

False

If $x = 2$ then $x^2 = 4$ but if $x^2 = 4$ then $x = \pm 2$

(ii) If $P \Leftrightarrow Q$, then if follows that $Q \Rightarrow P$ and $P \Rightarrow Q$. (2 marks)

True

If $2x = 6$ then $x = 3$

(iii) If $P \Rightarrow Q$, then if follows that $\overline{P} \Rightarrow \overline{Q}$. ($\overline{P}$ *is the negation of* P) (2 marks)

False

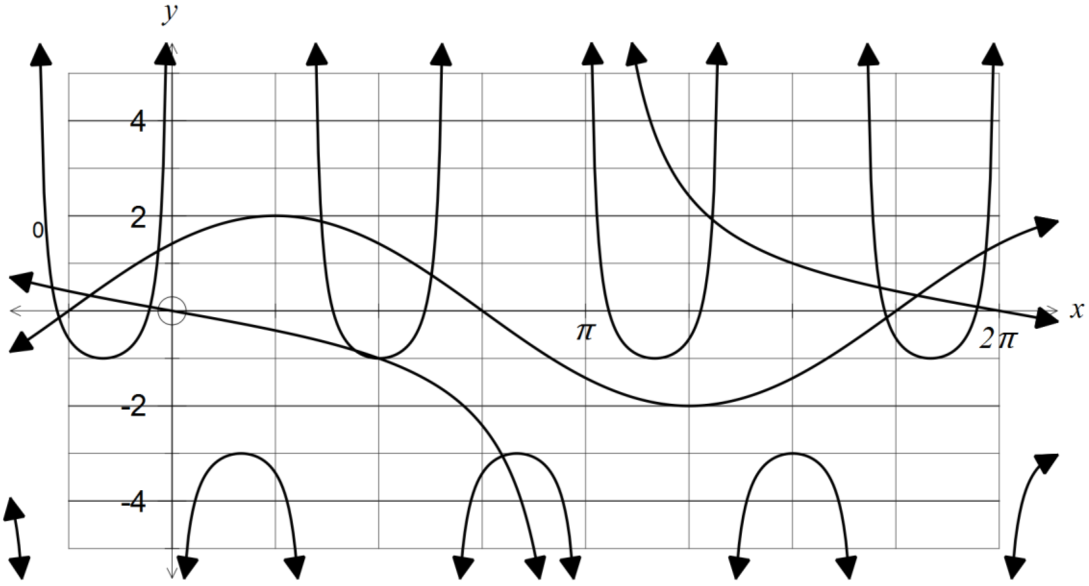
If $x = 2$ then $x^2 = 4$ but if $x \neq 2$, x^2 can still be 4 if $x = -2$

(b) If $B \Rightarrow A$ is a true statement, circle all of the following that are **always** true. (2 marks)

$B \Leftrightarrow A$ $\overline{B} \Rightarrow \overline{A}$ $\overline{A} \Rightarrow B$ $\boxed{\overline{A} \Rightarrow \overline{B}}$ $A \Leftarrow \overline{B}$ $B \Rightarrow \overline{A}$

Question 9 **(6 marks)**

The graphs of the functions $y = a \cdot \cos(x + b)$, $y = \operatorname{cosec}(c \cdot x) + d$, $y = e \cdot \tan(f \cdot x)$ are shown below, where a, b, c, d, e and f are real constants.



State the values of constants a, b, c, d, e and f.

$a = 2$

$b = -\frac{\pi}{4}$

$c = -3$

$d = -2$

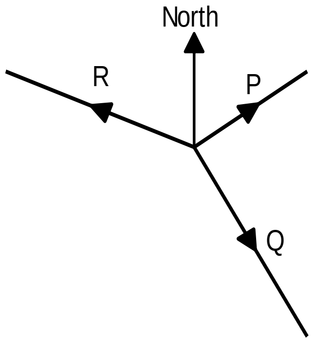
$e = -1$

$f = 0.5$

(alternative: $e = 1$, $f = -0.5$)

Question 10 **(6 marks)**

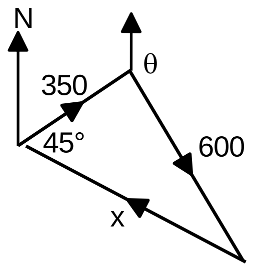
A body in equilibrium is acted on by three forces, as shown in the diagram (not to scale).



P is of magnitude 350N on a bearing of 060°, **Q** is of magnitude 600N on a bearing of θ, where $090° < \theta < 180°$, and **R** is of magnitude x N on a bearing of 285°.

Determine x and θ.

For equilibrium, **P** + **Q** + **R** = 0



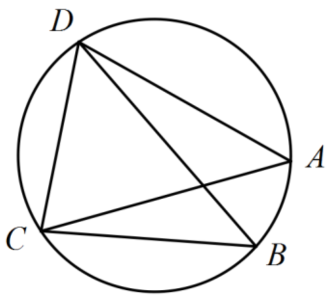
$$600^2 = 350^2 + x^2 - 2 \times 350 \times x \cos 45$$
$$x = 794.068 \text{ N}$$

$$\frac{\sin \alpha}{794.068} = \frac{\sin 45}{600}$$
$$\alpha = 110.6$$

$$\theta = 360 - 120 - 110.6$$
$$= 129.4°$$

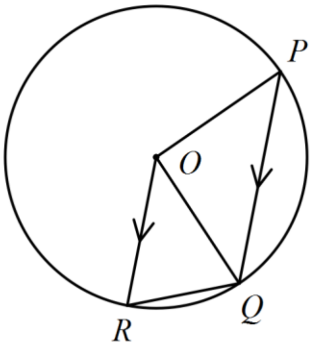
Question 11 **(9 marks)**

(a) A, B, C and D lie on a circle with diameter BD. If $AC = AD$ and $\angle ADB = 20°$, determine the size of angle BDC. (3 marks)



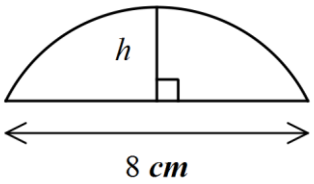
$$\begin{aligned}\angle BCA &= \angle BDA \\ &= 20 \\ \angle ADC &= \angle ACD \\ &= 90 - \angle BCA \\ &= 70 \\ \angle BDC &= 70 - 20 \\ &= 50°\end{aligned}$$

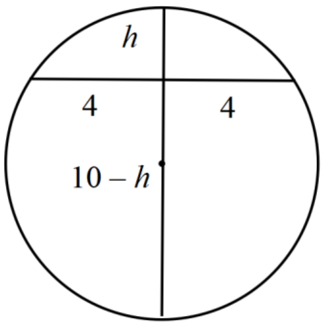
(b) P, Q and R lie on circle with centre O. If PQ is parallel to OR and $\angle ROP = 136°$, determine the size of angle ORQ. (3 marks)



$$\begin{aligned}\angle OPQ &= 180 - 136 \\ &= 44 \\ &= \angle OQP \\ &= \angle QOR \\ \angle ORQ &= \frac{180 - 44}{2} \\ &= 68°\end{aligned}$$

(c) A segment of a circle of radius 5 cm is shown (not to scale). Given that the width of the segment is 8 cm and the height is h cm, determine h. (3 marks)





Using the intersecting chord theorem:

$$4 \times 4 = h(10 - h)$$

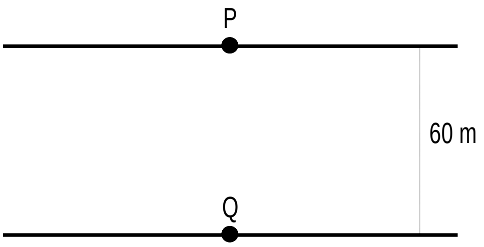
$$h^2 - 10h + 16 = 0$$

$$(h - 2)(h - 8) = 0$$

$$h = 2, h = 8 \text{ cm}$$

Question 12 **(8 marks)**

P and Q are two points directly across from each other on opposite banks of a river. P and Q are 60 m apart and a steady current flows along the river at 1.5 ms^{-1}.



(a) A boat leaves Q at a speed of 3.6 ms^{-1}, and steers at an angle of 90º to the river bank. Determine how far downstream of P the boat is when it reaches the opposite bank.

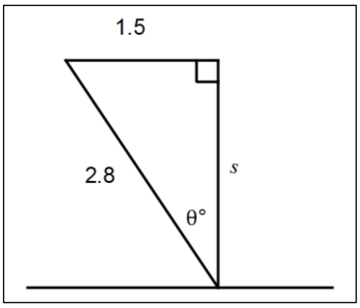
(3 marks)

Time taken to reach bank: $60 \div 3.6 = 16.\bar{6}$ seconds.

Distance downstream: $16.\bar{6} \times 1.5 = 25$ metres.

(b) A boat with a speed of 2.8 ms^{-1} must travel directly from P to Q.

Determine the angle the path of the boat must make with the river bank, to the nearest degree, and the time the boat will take, to the nearest tenth of a second. (5 marks)



$\theta = \sin^{-1}\dfrac{1.5}{2.8}$

$= 32.39$

$\approx 32°$

$s = \sqrt{2.8^2 - 1.5^2}$

$= 2.36431$

Steer at $90 - 32 = 58°$ to the bank

$t = 60 \div 2.36431$

$= 25.377$

≈ 25.4 seconds

Question 13 **(7 marks)**

Five matrices are given by $\begin{bmatrix}2\\3\end{bmatrix}$, $\begin{bmatrix}4 & 10\\6 & 15\end{bmatrix}$, $\begin{bmatrix}2 & 3\\5 & 8\end{bmatrix}$, $\begin{bmatrix}2 & 5\end{bmatrix}$ and $\begin{bmatrix}3\\2\end{bmatrix}$.

(a) If the five matrices are A, B, C, D and E (although not in that order) and it is known that the inverse of B exists and that $ED = A$, match each of the above matrices with A, B, C, D and E. (3 marks)

$$B = \begin{bmatrix}2 & 3\\5 & 8\end{bmatrix}$$

$$\begin{bmatrix}2\\3\end{bmatrix}\begin{bmatrix}2 & 5\end{bmatrix} = \begin{bmatrix}4 & 10\\6 & 15\end{bmatrix}$$

$$E = \begin{bmatrix}2\\3\end{bmatrix} \quad D = \begin{bmatrix}2 & 5\end{bmatrix} \quad A = \begin{bmatrix}4 & 10\\6 & 15\end{bmatrix}$$

$$C = \begin{bmatrix}3\\2\end{bmatrix}$$

(b) Find X where $\begin{bmatrix}2 & 3\\-2 & -1\end{bmatrix}X + \begin{bmatrix}2\\-1\end{bmatrix} = X + \begin{bmatrix}7\\1\end{bmatrix}$. (4 marks)

$$\begin{aligned}
PX + Q &= X + R\\
PX - X &= R - Q\\
(P - I)X &= R - Q\\
X &= (P - I)^{-1}(R - Q)
\end{aligned}$$

$$X = \begin{bmatrix}1 & 3\\-2 & -2\end{bmatrix}^{-1}\begin{bmatrix}5\\2\end{bmatrix}$$

$$X = \begin{bmatrix}-4\\3\end{bmatrix}$$

Question 14 **(7 marks)**

(a) Let $f(x) = x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4$, for all real x.

(i) Write the contrapositive of the statement $f(x) \geq 0 \Rightarrow x \geq 0$. (1 mark)

$$\boxed{x < 0 \Rightarrow f(x) < 0}$$

(ii) Hence, or otherwise, prove that if $f(x) \geq 0$ then $x \geq 0$. (2 marks)

If $x < 0$, then $x^5 < 0$, $-4x^4 < 0$, $3x^3 < 0$, $-x^2 < 0$ and $3x < 0$.

Hence the sum of all the terms in $f(x)$ must be less than 0 and so $f(x) < 0$, and the truth of the contrapositive implies the truth of the original statement.

(b) Prove by contradiction that $\dfrac{\sqrt{3}-\sqrt{5}}{4}$ is irrational. (4 marks)

Assume that $\dfrac{\sqrt{3}-\sqrt{5}}{4} = a$ where a is rational.

Multiply both sides by four and square:

$3 + 5 - 2\sqrt{15} = 16a^2 \Rightarrow \sqrt{15} = 4 - 8a^2$.

If a is a rational, then $4 - 8a^2$ must also be rational.

But we know that $\sqrt{15}$ is irrational, which contradicts our assumption that a is rational.

Hence $\dfrac{\sqrt{3}-\sqrt{5}}{4}$ is irrational.

Question 15 **(8 marks)**

(a) A bag contains 3 green, 5 red and 4 blue balls.

(i) If six balls are removed at random from a bag, what fraction of all possible selections will have two balls of each colour? (2 marks)

$$\frac{{}^3C_2 \times {}^5C_2 \times {}^4C_2}{{}^{12}C_6} = \frac{3 \times 10 \times 6}{924}$$
$$= \frac{15}{77}$$

(ii) What is the least number of balls that should be removed from the ball to be certain that the selection will contain at least three balls of the same colour? Justify your answer. (2 marks)

To be sure of getting no more than two of any colour, the most balls we can take is 2 G, 2 R and 2 B. If we take just one more, then we are bound to have at least three of the same colour.

So minimum number is 7.

(b) In a particular restaurant customers may order any combination of chips, peas and gravy to accompany their main course. Records show that 45% of customers choose gravy, 59% choose chips, 24% choose peas, 19% choose peas and chips, 15% choose gravy and peas, 25% choose gravy and chips and 15% choose all three. What percentage of customers choose to order nothing? (2 marks)

$$n(C \cup P \cup G) = 45 + 59 + 24 - 19 - 15 - 25 + 15$$
$$= 84$$

$$100 - 84 = 16\%$$

(c) Four letters are chosen from those in the word MATHS. How many of the resulting arrangements start with M or end with S? (2 marks)

$$1 \times 4! + 1 \times 4! - 1 \times 1 \times 3! = 24 + 24 - 6$$
$$= 42$$

Question 16 **(8 marks)**

(a) Find the value(s) of x for which the matrix $\begin{bmatrix} x & 1 \\ -1 & x+2 \end{bmatrix}$ is singular. (2 marks)

$$x^2 + 2x + 1 = 0$$
$$(x+1)(x+1) = 0$$
$$x = -1$$

(b) Express the two simultaneous equations $y = 2x - 1$ and $4x - 2y = 3$ as a matrix equation. (1 mark)

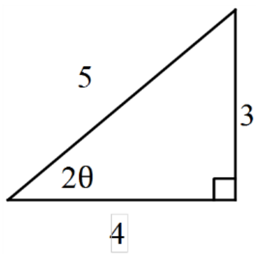
$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(c) State how many solutions exist to the matrix equation in (b), and explain your answer geometrically. (2 marks)

No solutions, as matrix is singular.

The two equations represent parallel lines that never intersect.

(d) Determine the exact coordinates of the image of the point (10, 5) when it is reflected in a line through the origin, inclined at an angle of θ to the x-axis, where $\theta = \frac{1}{2}\tan^{-1}\left(\frac{3}{4}\right)$. (3 marks)



$$\begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$

Image at (11, 2)

Question 17 **(9 marks)**

(a) Consider the true statement 'if a triangle has two congruent sides, then it has two congruent angles'.

(i) Write the contrapositive of the statement and explain whether or not the contrapositive is also true. (2 marks)

If a triangle does not have two congruent angles then it does not have two congruent sides.

True – Contrapositive is always true.

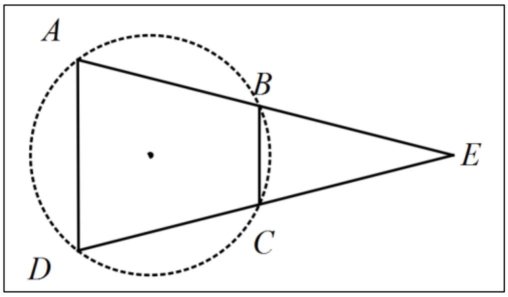
(ii) Write the inverse of the statement and explain whether or not the inverse is also true. (2 marks)

If a triangle is does not have two congruent sides, then it does not have two congruent angles.

True – properties of an isosceles triangle.

(b) The sides AB and DC of a quadrilateral are produced to meet at E. If $AB = CE = 8$ cm, $BE = 6$ cm and $CD = 2.5$ cm

(i) show that $ABCD$ is concylic. (3 marks)



$$AE = 6 + 8 = 14$$
$$DE = 2.5 + 8 = 10.5$$

$$\begin{aligned} AE.BE &= 14 \times 6 \\ &= 84 \\ &= 10.5 \times 8 \\ &= DE.CE \end{aligned}$$

Hence by the converse of the intersecting chord theorem, $ABCD$ is concylic.

(ii) prove that $\angle BAD = \angle BCE$. (2 marks)

$\angle BAD = 180 - \angle DCB$ (concylic)

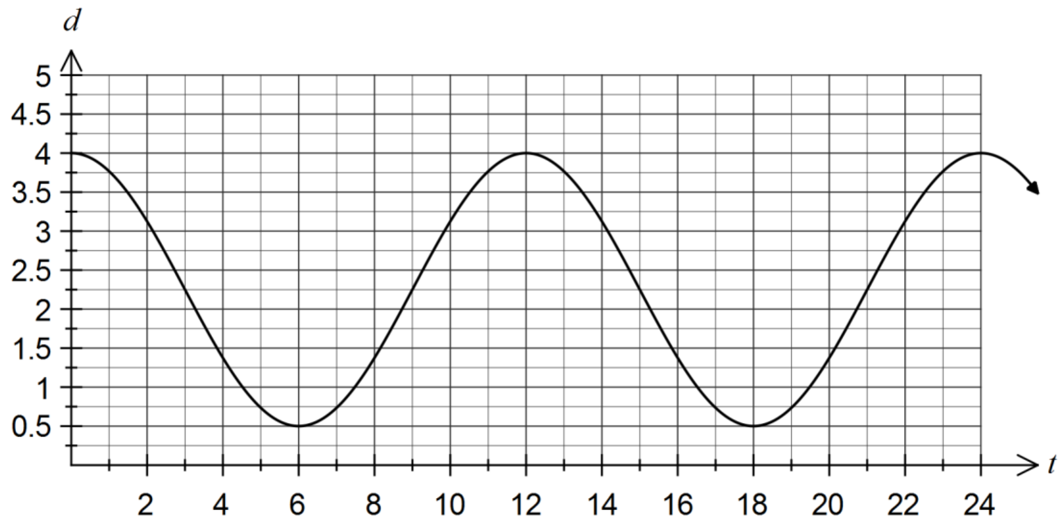
$= 180 - (180 - \angle BCE)$ (straight angle)

$= \angle BCE$

Question 18 **(7 marks)**

The depth of water in Harbour A, d metres, at t hours after midnight, is given by the formula $d = 2.25 + 1.75\cos\left(\frac{\pi t}{6}\right)$.

(a) Draw the graph of the depth of water in Harbour A for $0 \le t \le 24$. (3 marks)



(b) A ship requires a depth of at least 2.75 m to use the harbour. Determine how many hours a day the ship can use the harbour, giving your answer to the nearest minute. (2 marks)

$$t = 2.4466 \times 4$$
$$= 9.7864$$

$$0.7864 \times 60 = 47.2$$

9 hours and 47 minutes

(c) At another harbour, B, the depth of water varied with the same period and amplitude as Harbour A, but the first high tide of the day occurred two and a half hours later than at Harbour A, at which time the depth of water was 6.3 meters.

Determine a suitable formula for the depth of water in this harbour at time t hours after midnight. (2 marks)

$$d = 2.25 + 1.75\cos\left(\frac{\pi(t-2.5)}{6}\right) + 2.3$$
$$= 4.55 + 1.75\cos\left(\frac{\pi(t-2.5)}{6}\right)$$

Question 19 **(8 marks)**

(a) Determine the exact coordinates of the image of point A with coordinates $(-4, 6)$ after an anti-clockwise rotation of 120° about the origin. (2 marks)

$$\begin{bmatrix} \cos(120°) & -\sin(120°) \\ \sin(120°) & \cos(120°) \end{bmatrix}\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} -3\sqrt{3}+2 \\ -2\sqrt{3}-3 \end{bmatrix}$$

$A'(-3\sqrt{3}+2,\ -2\sqrt{3}-3)$

(b) A plane figure OBC of area 35 cm^2 is transformed geometrically by matrix $P = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, then by matrix $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and finally by matrix $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(i) Describe geometrically the effect of transformation matrix Q. (1 mark)

A reflection in the x-axis.

(ii) Determine the single matrix that represents the combined effect of the three transformations in the order given. (2 marks)

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

(iii) Calculate the area of the plane figure OBC after the three transformations. (1 mark)

$|RQP| = 4 \Rightarrow$ new area $= 4 \times 35 = 140$ cm^2

(c) The matrix M is such that $M^2 = -I$. Determine a possible matrix M and describe its geometric effect. (2 marks)

$$M^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Rotation of 90º *(clockwise or anticlockwise)*.

Question 20 **(7 marks)**

Let $P(n)=\dfrac{n^5}{5}+\dfrac{n^3}{3}+\dfrac{7n}{15}$.

(a) Evaluate $P(1)$ and $P(4)$. (1 mark)

$P(1)=1$

$P(4)=228$

(b) Prove by induction that $P(n)$ is always an integer, when n is a positive integer. (6 marks)

$P(1)=1 \Rightarrow P(n)$ is an integer when $n=1$

Assume that $P(k)$ is an integer, where k is a positive integer, so that $\dfrac{k^5}{5}+\dfrac{k^3}{3}+\dfrac{7k}{15}=I$

k+1 is the next consecutive integer after k.

$$P(k+1)=\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$$
$$P(k+1)=\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^3+3k^2+3k+1}{3}+\frac{7k+7}{15}$$
$$P(k+1)=\frac{k^5}{5}+k^4+\frac{7k^3}{3}+3k^2+\frac{37k}{15}+1$$
$$P(k+1)=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+k^4+2k^3+3k^2+2k+1$$
$$P(k+1)=I+k^4+2k^3+3k^2+2k+1$$

Hence, if $P(k)$ is an integer, then $P(k+1)$ is also an integer, as both I and k are integers.

Since $P(1)$ is an integer, then $P(2)$ must be an integer, and so on for all positive integer n.

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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