Perth Modern School

WA Exams Practice Paper A, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4
Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section
Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	13	13	100	97	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free 35% (53 Marks)

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 (5 marks)

Consider $g(z) = z^3 + 9z^2 + 28z + 20$, $z \in \mathbb{C}$. Factorise $g(z)$ over $\mathbb{C}$.

$z(-1) = 0 \;\Rightarrow\; z+1$ is a factor.

$$\frac{z^3 + 9z^2 + 28z + 20}{z+1} = z^2 + 8z + 20$$

$$(z+4)^2 - 16 + 20 = 0$$
$$(z+4)^2 = -4$$

$$z + 4 = \pm 2i^2$$
$z = -4 \pm 2i^2 \;\Rightarrow\; (z+4+2i)$ and $(z+4-2i)$

$$z^3 + 9z^2 + 28z + 20 = (z+1)(z+4+2i)(z+4-2i)$$

Question 2 **(8 marks)**

(a) Determine $\int \frac{x-4}{x^2-x}\,dx$. (4 marks)

$$\frac{x-4}{x^2-x} = \frac{x-4}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$Ax - A + Bx = x - 4 \Rightarrow A = 4,\ B = -3$$

$$\int \frac{x-4}{x^2-x}\,dx = \int \frac{4}{x} - \frac{3}{x-1}\,dx$$

$$= 4\ln|x| - 3\ln|x-1| + c$$

(b) Evaluate $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin(2x)\cos^2(2x)\,dx$. (4 marks)

$$u = \cos 2x \Rightarrow du = -2\sin 2x$$

$$x = \frac{\pi}{2}, u = -1;\ x = \frac{3\pi}{4}, u = 0$$

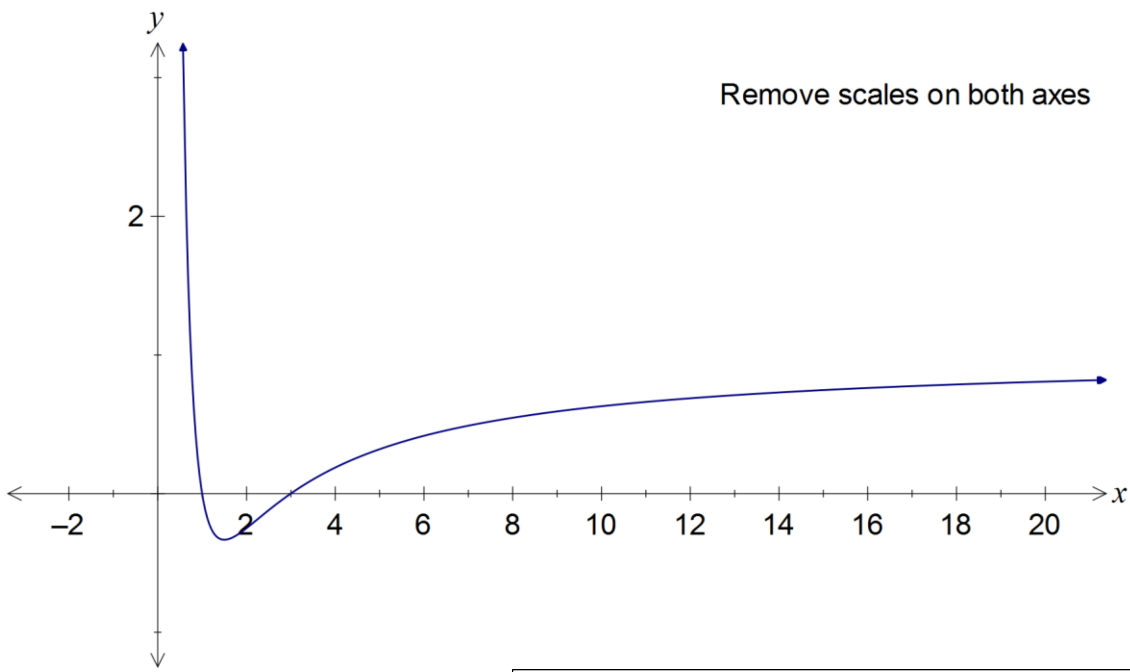
$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin(2x)\cos^2(2x)\,dx = \int_{-1}^{0} -\frac{1}{2}u^2\,du$$

$$= \left[-\frac{u^3}{6}\right]_{-1}^{0} = -\frac{1}{6}$$

Question 3 **(6 marks)**

The function $f(x)$ is defined for $x > 0$ by $f(x) = \dfrac{(x-1)(x-3)}{x^2}$.

Sketch the graph of $y = f(x)$ on the axes below.



Marks:
- roots
- turning point (3)
- y=1 as asymptote
- shape

$$f(x) = \frac{(x-1)(x-3)}{x^2}, \quad f(x) = 0 \ \Rightarrow \ x = 1,\ 3$$

$$f(x) = \frac{x^2 - 4x + 3}{x^2}$$

$$f'(x) = \frac{(2x-4)(x^2) - \left(x^2 - 4x + 3\right)(2x)}{x^4}$$

$$= \frac{2x^3 - 4x^2 - 2x^3 + 8x^2 - 6x}{x^4}$$

$$= \frac{4x-6}{x^3}$$

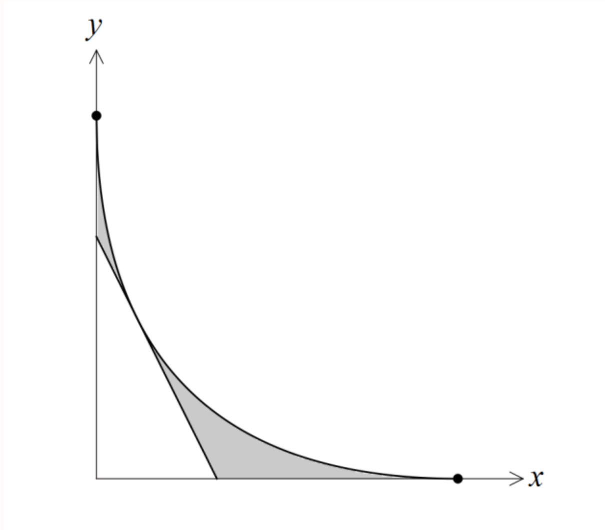
$$f'(x) = 0 \ \Rightarrow \ x = 1.5$$

$$f(1.5) = \frac{0.5 \times -1.5}{1.5^2} = -\frac{1}{3}$$

$$x \to \infty, \ y \to \frac{x^2}{x^2} \to 1$$

Question 4 **(8 marks)**

A curve, defined implicitly by $\sqrt{y}+\sqrt{x}=3$ for $0 \le x \le 9$ and $0 \le y \le 9$, and the tangent to the curve when $x=1$, is shown below.



(a) Determine an expression for $\frac{dy}{dx}$. (2 marks)

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx}+\frac{1}{2\sqrt{x}}=0$$

$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

(b) Determine the equation of the tangent to the curve when $x=1$. (2 marks)

$$\sqrt{y}+\sqrt{1}=3 \ \Rightarrow \ y=4$$

$$\frac{dy}{dx}=-\frac{\sqrt{4}}{\sqrt{1}}=-2$$

$$y-4=-2(x-1)$$

$$y=-2x+6$$

(c) Determine the shaded area shown in the first quadrant, trapped between the curve and the tangent. (4 marks)

$\sqrt{y}+\sqrt{x}=3 \;\Rightarrow\; y=9-6\sqrt{x}+x \quad (0\le x\le 9)$

$y=-2x+6 \;\Rightarrow\; y=0 \;\Rightarrow\; x=3$

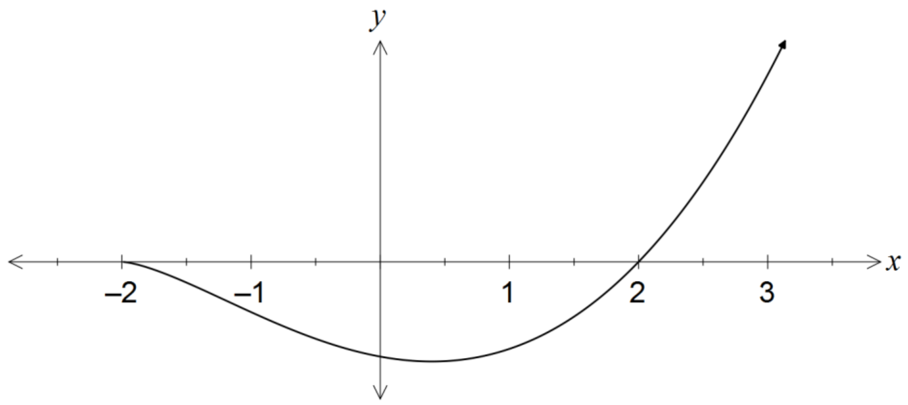
$\int_0^3 \left(\left(9-6\sqrt{x}+x\right)-\left(6-2x\right)\right)\,dx+\int_3^9\left(9-6\sqrt{x}+x\right)\,dx$

$=\left(\dfrac{45}{2}-12\sqrt{3}\right)+\left(12\sqrt{3}-18\right)$

$=\dfrac{9}{2}$ sq u

Question 5 **(11 marks)**

Part of the graph with equation $y=(x^2-4)\sqrt{x+2}$ is shown below, where x and y are measured in centimetres.



(a) A particle moves along the curve so that when it reaches $x=2$ cm, its y-coordinate is increasing at a rate of 12 cm per second. Determine the rate at which its x-coordinate is increasing at this instant. (4 marks)

$$x=2,\ \frac{dy}{dt}=12,\ \frac{dx}{dt}=?$$

$$\frac{dy}{dx}=(2x)\left(\sqrt{x+2}\right)+\left((x^2-4)\right)\left(\frac{1}{2\sqrt{x+2}}\right)\Bigg|_{x=2}$$

$$=4\times 2+0$$
$$=8$$

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}\ \Rightarrow\ 12=8\times\frac{dx}{dt}\ \Rightarrow\ \frac{dx}{dt}=1.5 \text{ cm/s}$$

(b) Show that the area of the region between the x-axis and the curve can be expressed in the form $\frac{2^n}{35}$ cm^2, and state the value of n. (7 marks)

$$(x^2-4)\sqrt{x+2} = (x-2)(x+2)\sqrt{x+2}$$
$$= (x-2)(x+2)^{1.5}$$

$$u = x+2 \;\Rightarrow\; du = dx$$
$$x=-2, u=0;\; x=2, u=4$$

$$\int_{-2}^{2}(x^2-4)\sqrt{x+2}\;dx = \int_0^4 (u-4)u^{1.5}du$$
$$= \int_0^4 u^{2.5} - 4u^{1.5}du$$

$$= \left[\frac{2u^{3.5}}{7} - \frac{8u^{2.5}}{5}\right]_0^4$$

$$= \frac{2^8}{7} - \frac{2^8}{5}$$

$$= \left(\frac{1}{7} - \frac{1}{5}\right)\times 2^8$$
$$= -\frac{2}{35}\times 2^8$$

Area is $\frac{2^9}{35}$ cm^2 $\Rightarrow$ $n = 9$

Question 6 **(8 marks)**

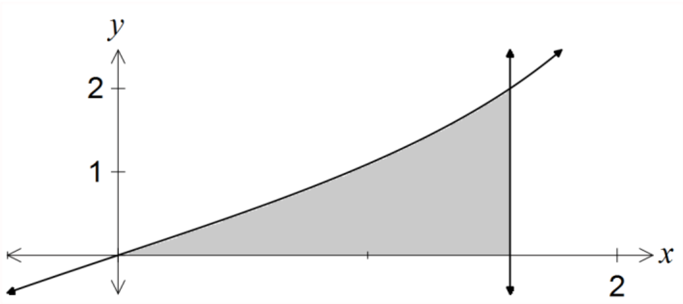
(a) Determine $\int \frac{1}{4}\sin^2\left(\frac{x}{4}\right)\,dx$. (3 marks)

$$\int \frac{1}{4}\sin^2\left(\frac{x}{4}\right)\,dx = \frac{1}{8}\int 1-\cos\left(\frac{x}{2}\right)\,dx$$

$$= \frac{1}{8}\left(x - 2\sin\left(\frac{x}{2}\right)\right) + c$$

$$= \frac{x}{8} - \frac{1}{4}\sin\left(\frac{x}{2}\right) + c$$

(b) The shaded region in the diagram below, bounded by the x-axis, the line $x = \frac{\pi}{2}$ and part of the graph of $y = 2\tan\left(\frac{x}{2}\right)$, is rotated around the x-axis to generate a solid. Determine the exact volume of the solid formed. (5 marks)



$$V = \int_0^{\frac{\pi}{2}} \pi \times 4\tan^2\left(\frac{x}{2}\right)\,dx$$

$$= \pi\int_0^{\frac{\pi}{2}} 4\sec^2\left(\frac{x}{2}\right) - 4\,dx$$

$$= \pi\left[8\tan\left(\frac{x}{2}\right) - 4x\right]_0^{\frac{\pi}{2}}$$

$$= \pi[8-2\pi] - \pi[0-0]$$
$$= 8\pi - 2\pi^2$$

Question 7 **(7 marks)**

The rate of change of the distance, x centimetres, between two particles t seconds after an experiment began, is given by the equation $t\dfrac{dx}{dt} = x\ln(t)$, where $t > 0$ and $x > 0$.

(a) Using the substitution $u = \ln(t)$, or otherwise, determine $\displaystyle\int \frac{\ln(t)}{t}\,dt$. (3 marks)

$$u = \ln(t) \;\Rightarrow\; du = \frac{1}{t}dt$$

$$\int u\;du = \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}\left(\ln(t)\right)^2 + c$$

(b) Determine an expression for x in terms of t, if the distance between the two particles after 1 second is 0.5 cm. (4 marks)

$$\int \frac{1}{x}dx = \int \frac{1}{t}\ln(t)\;dt$$

$$\ln(x) = \frac{1}{2}\left(\ln(t)\right)^2 + c \quad (x > 0)$$

$$x = ke^{\frac{(\ln(t))^2}{2}}$$

$$x(1) = \tfrac{1}{2} \;\Rightarrow\; k = \tfrac{1}{2}$$

$$x = \frac{1}{2}e^{\frac{(\ln(t))^2}{2}}$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

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