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NAME: _____

VCE®SPECIALIST MATHEMATICS

Units 3 & 4 Practice Written Examination 1

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK

Structure of book

<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
8	8	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are **NOT** permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and Answer Booklet of 13 pages.
- Formula Sheet.
- Working space is provided throughout the Question and Answer Booklet.

Instructions

- Write your **student name** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the space provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ ms}^{-2}$ where $g = 9.8$.

Question 1 (4 marks)

- a. Use mathematical induction to prove that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3} \text{ for all } n \in N. \quad 2 \text{ marks}$$

- b. Use proof by contrapositive to prove that when $n \in N$, if n^3 is odd then n is odd.

2 marks

Question 2 (9 marks)

- a. Find the Cartesian equation of the plane containing the points $(0, 1, -1)$, $(2, 0, 1)$ and $(3, -1, 1)$. 4 marks

- b. Find the point of intersection of the plane with the Cartesian equation $2x + 2y - z = 3$ and the line $\mathbf{r} = \mathbf{i} - \mathbf{j} + \mathbf{k} + t(\mathbf{2i} + \mathbf{k})$, $t \in \mathbb{R}$. 2 marks

- c. Find in the form $\arccos(q)$, where $-1 \leq q \leq 1$, the angle between the plane and the line in **Part b.** 3 marks

Question 3 (4 marks)

Find an antiderivative of $ax^2 e^{-ax}$.

Question 4 (3 marks)

Find the surface area generated when the curve defined by the parametric equations

$x = \cos(3t)$ and $y = \sin(3t)$, $\frac{\pi}{12} \leq t \leq \frac{\pi}{6}$, is rotated around the x -axis.

Question 5 (3 marks)

- a. The heights of students at a particular school are normally distributed with a mean of 160 cm and a variance of 36 cm^2 . Find, correct to three decimal places, the probability that the total height of nine students randomly chosen from the school is at least 1404 cm. 2 marks

- b. One year later, a random sample of 16 students is collected from the school. The mean height of the sample is found to be 170 cm. Assume that the heights of students at the school are still normally distributed with variance 36 cm^2 .

Find a 95% confidence interval for the mean height of students at the school. Give both endpoints correct to the nearest integer. 1 mark

Question 6 (5 marks)

- a. Solve $z^3 + 2iz^2 + 3z + 6i = 0$, $z \in \mathbb{C}$.

2 marks

- b. Find in **polar form** the cube roots of $4\sqrt{2} - 4\sqrt{2} i$.

3 marks

Question 7 (6 marks)

Consider the function $f(x) = 2 \arccos\left(\frac{1-x}{2}\right) - \frac{\pi}{2}$.

- a. Prove that the function has no turning points.

1 mark

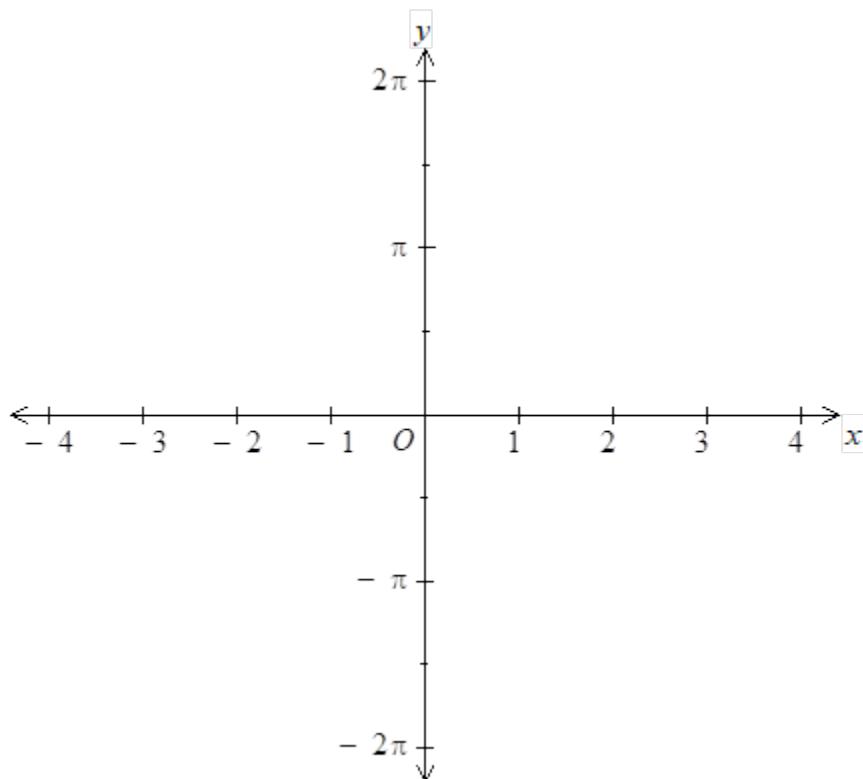
- b. State the coordinates of the point of inflection of the function.

1 mark

- c. Find the coordinates of the axis intercepts of the graph of $y = f(x)$.

2 marks

- d. Sketch the graph of $y = f(x)$ on the set of axes below. Label all endpoints with their coordinates. 2 marks



Working space

Question 8 (6 marks)

A reservoir contains 200 L of salt water. The concentration of salt in the reservoir is 0.1 kg/L. To reduce the amount of salt, pure water is pumped into the reservoir at a rate of 10 L/min and the contents of the reservoir are drained away at a rate of 5 L/min.

The amount of salt in the reservoir t minutes after the pure water starts being pumped in is equal to x kg.

- a. Show that $\frac{dx}{dt} = \frac{-x}{40+t}$. 1 mark

- b. Find the amount of salt $x = x(t)$ in the reservoir at time t . 4 marks

- c. Find the time at which the amount of salt in the reservoir is equal 10 kg. 1 mark

END OF EXAMINATION

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VCE® Specialist Mathematics

Practice Written Examination 1

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Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Specialist Mathematics Examination 1: Marking Scheme

1(a)

Let $S(n) = 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ be the conjecture.

- Base case: Try $n = 1$.

$$\text{LHS} = 1 \times 2 = 2. \quad \text{RHS} = \frac{1(1+1)(1+2)}{3} = 2.$$

Therefore $S(1)$ is true.

- Inductive hypothesis: Assume $S(k)$ is true for some $k \in N$.

1 mark

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

- Show that if $S(k)$ is true then it follows that $S(k+1)$ is true:

$$\underbrace{1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1)}_{\text{using the inductive hypothesis}} + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

factorise the numerator by identifying the common factor $(k+1)(k+2)$

$$= \frac{(k+3)(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3} \quad \text{which is } S(k+1).$$

- Since $S(1)$ is true and it follows that if $S(k)$ is true then $S(k+1)$ is true, it follows from the principle of mathematical induction that $S(n)$ for $n \in N$.

1 mark

1(b)	<p>Let P be the statement n^3 is odd. Let Q be the statement n is odd. Proving $P \Rightarrow Q$ is equivalent to proving the contrapositive $(\text{Not } Q) \Rightarrow (\text{Not } P)$: If n is even, then n^3 is not odd (even).</p> <ul style="list-style-type: none"> • Prove the contrapositive statement: If n is even, then n^3 is not odd (even). Let $n = 2m$ for some $m \in \mathbb{Z}$. Then $n^3 = (2m)^3 = 2(4m^3)$. Therefore n^3 is not odd (even). <p>• The contrapositive statement is true therefore the original statement is true.</p>	1 mark 1 mark
2(a)	<p>Given the points $A(0, 1, -1)$, $B(2, 0, 1)$ and $C(3, -1, 1)$ we may construct the following two vectors that lie in the plane:</p> $\vec{AB} = \vec{AO} + \vec{OB} = -\vec{OA} + \vec{OB} = \underset{\sim}{a} = 2\vec{i} - \vec{j} + 2\vec{k}.$ $\vec{AC} = \vec{AO} + \vec{OC} = -\vec{OA} + \vec{OC} = \underset{\sim}{b} = 3\vec{i} - 2\vec{j} + 2\vec{k}.$ <ul style="list-style-type: none"> • A vector perpendicular to the plane is given by $\underset{\sim}{a} \times \underset{\sim}{b}$: $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 3 & -2 & 2 \end{vmatrix} = 2\vec{i} + 2\vec{j} - \vec{k}.$ <ul style="list-style-type: none"> • Therefore, the equation of the plane is $2x + 2y - z = d$. • Substitute the point $(2, 0, 1)$ and solve for d: $d = 3$. <p>Answer: $2x + 2y - z = 3$.</p>	1 mark 1 mark 1 mark 1 mark
2(b)	<p>From $\underset{\sim}{r} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} + t(\underset{\sim}{2i} + \underset{\sim}{k})$: $x = 1 + 2t$, $y = -1$, $z = 1 + t$. Substitute into $2x + 2y - z = 3$:</p> $2(1 + 2t) - 2(1) - (1 + t) = 3 \Rightarrow 4t - t - 1 = 2 \Rightarrow t = 1.$ $\underset{\sim}{r}(1) = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} + (\underset{\sim}{2i} + \underset{\sim}{k}) = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}.$ <p>Answer: $(3, -1, 2)$.</p>	1 mark 1 mark

2(c)	<p>A normal vector \mathbf{n} to the plane $2x + 2y - z = 3$ is given by $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Let \mathbf{b} be a vector in the direction of the line: $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$. Let θ be the angle between the normal to the plane and the line:</p> $\cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{n}}{\ \mathbf{b}\ \ \mathbf{n}\ } = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}}$ $\theta = \frac{\pi}{2} - \alpha$ <p>where α is angle between the line and the plane</p> $\Rightarrow \alpha = \frac{\pi}{2} - \theta$ $\Rightarrow \cos(\alpha) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta).$ $\sin(\theta) = \sqrt{1 - \cos^2(\theta)} \quad (\text{since } 0 < \theta < 180^\circ) = \sqrt{1 - \frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}.$ <p>Answer: $\arccos\left(\frac{2}{\sqrt{5}}\right)$.</p>	1 mark 1 mark
3	<ul style="list-style-type: none"> Use integration by parts: $\int f'g dx = fg - \int fg' dx$. <p>Let $f' = e^{-ax} \Rightarrow f = -\frac{1}{a}e^{-ax}$ and $g = ax^2 \Rightarrow g' = 2ax$:</p> $fg = \frac{-e^{-ax}}{a} \times ax^2 = -x^2 e^{-ax}, \quad \int fg' dx = \int -2ax \frac{e^{-ax}}{a} dx = -\int 2xe^{-ax} dx.$ <p>Therefore $\int ax^2 e^{-ax} dx = -x^2 e^{-ax} + \int 2xe^{-ax} dx$.</p> <p>Use integration by parts on $\int 2xe^{-ax} dx$: $\int u'v dx = uv - \int uv' dx$.</p> <p>Let $u' = e^{-ax} \Rightarrow u = -\frac{1}{a}e^{-ax}$ and $v = 2x \Rightarrow v' = 2$:</p> $uv = \frac{-2xe^{-ax}}{a}, \quad \int uv' dx = \int -2 \frac{e^{-ax}}{a} dx = \frac{-2e^{-ax}}{a^2}.$	1 mark 1 mark 1 mark 1 mark

	<p>Therefore $\int 2xe^{-ax} dx = \frac{-2xe^{-ax}}{a} + \frac{2e^{-ax}}{a^2}$.</p> <p>Therefore:</p> $\int ax^2 e^{-ax} dx = -x^2 e^{-ax} - \frac{2xe^{-ax}}{a} + \frac{-2e^{-ax}}{a^2} + c = \frac{-e^{-ax}}{a^2}(a^2x^2 + 2ax + 2) + c.$ <p>(Arbitrary constant is not required, simplification is required)</p>	1 mark
4	$A = 2\pi \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$ <p>Substitute $\frac{dx}{dt} = -3\sin(3t)$ and $\frac{dy}{dt} = 3\cos(3t)$:</p> $A = 2\pi \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \sin(3t) \sqrt{9\sin^2(3t) + 9\cos^2(3t)} dt$ $= 2\pi \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 3\sin(3t) dt = \left[-2\pi \cos(3t) \right]_{\frac{\pi}{12}}^{\frac{\pi}{6}} = \pi\sqrt{2}.$ <p>Answer: $\pi\sqrt{2}$.</p>	1 mark
5(a)	<p>Let H be the random variable “<i>Height of a student</i>”.</p> <p>Let X be the random variable “<i>Sum of the heights of nine students</i>”:</p> $X = H_1 + H_2 + \dots + H_9.$ $\mu_X = 9 \times 160 = 1440. \quad \sigma_X^2 = 9 \times 36 = 324 \quad \Rightarrow \sigma_X = 18.$ $Z = \frac{X - \mu_X}{\sigma_X} = \frac{1404 - 1440}{18} = -2.$ <p>Use the symmetry of the normal curve:</p> $\Pr(X \geq 1404) = \Pr(Z > -2) = \Pr(Z \leq 2).$ <p>Use $\Pr(-2 < Z < 2) \approx 0.95$ and the symmetry of the normal curve:</p> $\Pr(Z \leq 2) = 1 - 0.025 = 0.975.$	1 mark

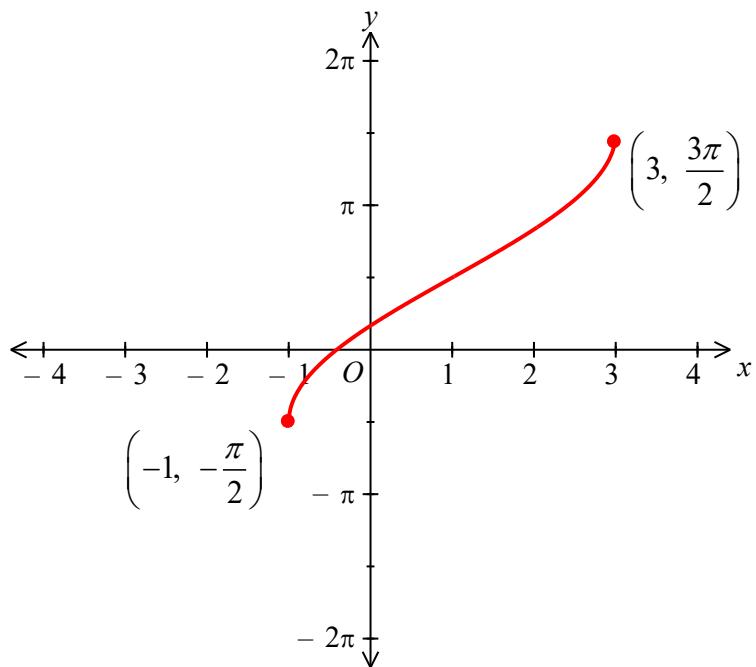
<p>5(b)</p> <p>Confidence interval endpoints: $\bar{x} \pm z_c \frac{\sigma}{\sqrt{n}}$.</p> <p>Substitute $\bar{x} = 170$, $\sigma = 6$, $n = 16$ and use the approximate critical value $z_c = 2$ for a 95% confidence interval:</p> $\bar{x} \pm \frac{2\sigma}{\sqrt{n}} = 170 \pm \frac{12}{4}.$ <p>Answer: [167, 173].</p>	1 mark
<p>6(a)</p> <p>Use ‘pair-pair’ grouping to factorise:</p> $(z^3 + 2iz^2) + (3z + 6i) = 0 \quad \Rightarrow z^2(z + 2i) + 3(z + 2i) = 0$ $\Rightarrow (z^2 + 3)(z + 2i) = 0$ $\Rightarrow (z + 2i)(z + \sqrt{3}i)(z - \sqrt{3}i) = 0.$ <p>Answer: $z = -2i$, $\pm\sqrt{3}i$.</p>	1 mark 1 mark 1 mark
<p>6(b)</p> <p>Let the cube roots be $z = r\text{cis}(\theta) \Rightarrow z^3 = r^3\text{cis}(3\theta)$.</p> $4\sqrt{2} - 4\sqrt{2}i = 8\text{cis}\left(-\frac{\pi}{4} + 2n\pi\right), n \in \mathbb{Z}.$ <p>The cube roots are found by solving</p> $z^3 = 4\sqrt{2} - 4\sqrt{2}i \quad \Rightarrow r^3\text{cis}(3\theta) = 8\text{cis}\left(-\frac{\pi}{4} + 2n\pi\right).$ <p>Equate moduli: $r^3 = 8 \Rightarrow r = 2$.</p> <p>Equate arguments: $3\theta = -\frac{\pi}{4} + 2n\pi \Rightarrow \theta = -\frac{\pi}{12} + \frac{2n\pi}{3}$.</p> <p>Substitute r and θ into $z = r\text{cis}(\theta)$ for three consecutive values of n.</p> <p>Answer:</p> $n = 0: z_1 = 2\text{cis}\left(\frac{-\pi}{12}\right). \quad n = 1: z_2 = 2\text{cis}\left(\frac{7\pi}{12}\right). \quad n = -1: z_3 = 2\text{cis}\left(-\frac{3\pi}{4}\right).$ <p>Note: The polar forms must use principle arguments.</p>	1 mark 1 mark

7(a) $f'(x) = \frac{1}{\sqrt{4-(1-x)^2}} = 0$ has no solution therefore there are no turning points.	1 mark
7(b) $y = a \arccos(bx+c) + d$ has a point of inflection where $bx+c=0$ and $y=d$. Therefore $f(x) = 2 \arccos\left(\frac{1-x}{2}\right) - \frac{\pi}{2}$ has a point of inflection where $\frac{1-x}{2} = 0 \Rightarrow x=1$ and $y = -\frac{\pi}{2}$. Answer: $\left(1, -\frac{\pi}{2}\right)$.	1 mark
Note: $\begin{aligned} f''(x) &= \frac{d}{dx} \left(\frac{1}{\sqrt{4-(1-x)^2}} \right) = \frac{d}{dx} (4-(1-x)^2)^{-\frac{1}{2}} \\ &= -\frac{1}{2} (4-(1-x)^2)^{-\frac{3}{2}} \times 2(1-x) = \frac{x-1}{(4-(1-x)^2)^{\frac{3}{2}}} = 0 \\ &\Rightarrow x=1. \end{aligned}$ <p>There is a potential point of inflection when $x=1$.</p> <p>Check for change in concavity:</p> $x < 1: f'' > 0. \quad x > 1: f'' < 0$ <p>Therefore, there is a change of concavity and so there is a point of inflection at $x=1$.</p> $f(1) = 2 \arccos(0) - \frac{\pi}{2} = \frac{\pi}{2}.$	

7(d)

$$\text{Domain: } -1 \leq \frac{1-x}{2} \leq 1 \Rightarrow -2 \leq 1-x \leq 2 \Rightarrow x \in [-1, 3].$$

Therefore, the coordinates of the endpoints of the graph are $(-1, -\frac{\pi}{2})$ and $(3, \frac{3\pi}{2})$.



**1 mark
for shape**
**1 mark
for
endpoints**

8(a)

Volume of salt water in reservoir at time t : $V = 200 + 10t - 5t = 200 + 5t$.

$$\text{Concentration of salt at time } t: c = \frac{x}{200 + 5t}.$$

Inflow of salt: 0.

$$\text{Outflow of salt: } 5c = 5\left(\frac{x}{200 + 5t}\right) = \frac{x}{40 + t}.$$

Therefore:

$$\frac{dx}{dt} = 0 - \frac{x}{40 + t} = \frac{-x}{40 + t}.$$

1 mark

8(b)	<p>$\frac{dx}{dt} = \frac{-x}{40+t}$ is a separable differential equation:</p> $\int -\frac{dx}{x} = \int \frac{dt}{40+t}$ $\Rightarrow -\log_e x = \log_e 40+t + K$ $\Rightarrow \log_e \left \frac{40+t}{x} \right = -K$ $\Rightarrow \left \frac{40+t}{x} \right = e^{-K} \quad \Rightarrow \frac{40+t}{x} = \pm e^{-K}$ $\Rightarrow \frac{40+t}{x} = A \text{ where } A = \pm e^{-K} \in R \setminus \{0\}$ $\Rightarrow x = \frac{A}{40+t}$ <p>Concentration of salt at $t = 0$ is 0.1 kg/L therefore $x = (0.1)(200) = 20$ at $t = 0$:</p> $20 = \frac{A}{40} \quad \Rightarrow A = 800.$ <p>Answer: $x = \frac{800}{40+t}$.</p>	1 mark 1 mark 1 mark 1 mark
8(c)	<p>Substitute $x = 10$ into $x = \frac{800}{40+t}$:</p> $10 = \frac{800}{40+t} \quad \Rightarrow t = 400.$ <p>Answer: 40 minutes.</p>	1 mark