Neap

Trial Examination 2020

VCE Mathematical Methods Units 3&4

Written Examination 2

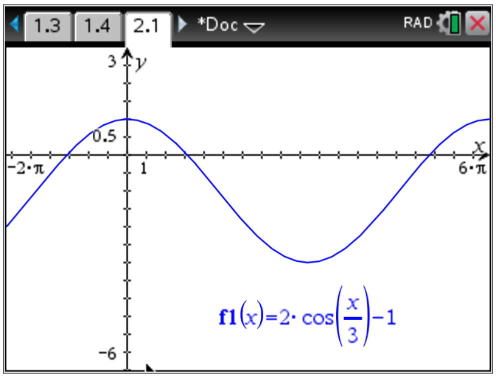
Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	B	C	D	**E**
2	**A**	B	C	D	E
3	A	**B**	C	D	E
4	A	**B**	C	D	E
5	**A**	B	C	D	E
6	A	B	**C**	D	E
7	**A**	B	C	D	E
8	**A**	B	C	D	E
9	**A**	B	C	D	E
10	A	B	C	D	**E**

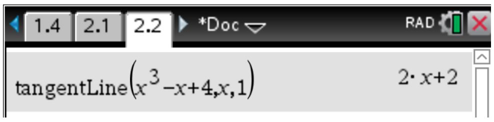
11	A	B	C	**D**	E
12	A	B	C	D	**E**
13	A	**B**	C	D	E
14	A	B	**C**	D	E
15	A	B	**C**	D	E
16	A	B	**C**	D	E
17	A	B	C	**D**	E
18	A	B	**C**	D	E
19	A	B	**C**	D	E
20	A	**B**	C	D	E

Question 1 **E**



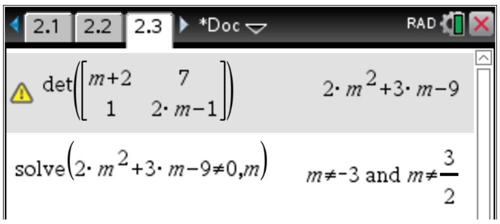
$$\text{period} = \frac{2\pi}{n}$$
$$= \frac{2\pi}{\frac{1}{3}}$$
$$= 6\pi$$

Question 2 **A**



Question 3 **B**

$$\begin{bmatrix} m+2 & 7 \\ 1 & 2m-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m+3 \\ 5 \end{bmatrix}$$



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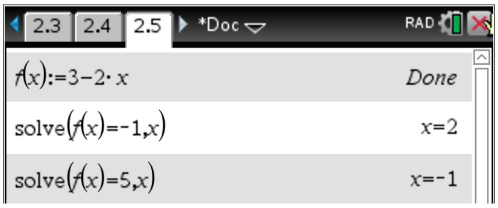
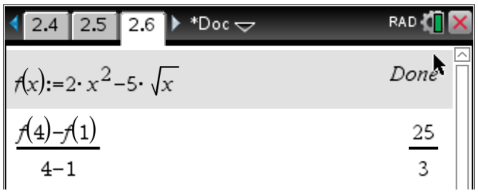
Question 4 **B**



Question 5 **A**



Question 6 **C**

$kx^2 - kx + \frac{1}{4} = 0$

$\Delta = b^2 - 4ac$

$\quad = k^2 - k$

$k^2 - k = 0$ for one solution

$k(k-1) = 0$

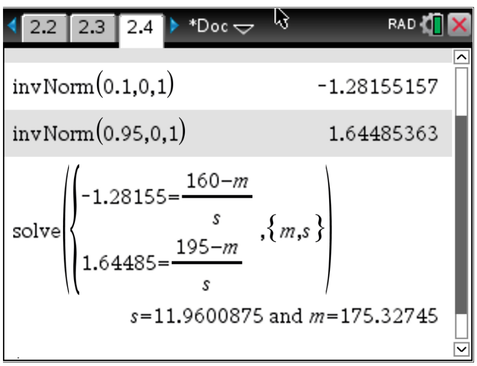
$k = 0$ or $k = 1$

However, when $k = 0$, the quadratic reduces to $\frac{1}{4} = 0$ and therefore $k \neq 0$.

$k = 1$ only

Question 7 **A**

$z = \frac{x - \bar{x}}{\mu}$



Question 8 A

$g(x) = 3f(2x - 4) + 1$

Transformations:

Dilation factor of 3 from x-axis: $(1, 3) \to (1, 9)$

Dilation factor of $\frac{1}{2}$ from y-axis: $(1, 9) \to \left(\frac{1}{2}, 9\right)$

Translate 2 units right: $(1, 9) \to \left(\frac{5}{2}, 9\right)$

Translate 1 unit down: $\left(\frac{5}{2}, 9\right) \to \left(\frac{5}{2}, 10\right)$

Question 9 A

Let f and g be two functions such that $f(x + 1) = x$ and $g(x + 2) = f(x)$.

$f(x + 1) = x \to f(x) = x - 1$

$g(x + 2) = f(x) = x - 1 \to g(x) = x - 3$

$$\begin{aligned} f(g(x)) &= (x - 3) - 1 \\ &= x - 4 \end{aligned}$$

Question 10 E

$p(x) := 2 \cdot x^3 - a \cdot x^2 - 9 \cdot x$ Done

$\text{solve}\left(p\left(\frac{-a}{2}\right) = 0, a\right)$ $a = -3$ or $a = 0$ or $a = 3$

Question 11 D

Let S = success, M = miss and A = any (success or miss).

Pr(at least 3 consecutive) = Pr($SSSAA$) + Pr($MSSSA$) + Pr ($AMSSS$)

Pr(at least 3 consecutive) = $\left(\frac{4}{5}\right)^3 \times 1 \times 1 + \left(\frac{1}{5}\right) \times \left(\frac{4}{5}\right)^3 \times 1 + 1 \times \frac{1}{5} \times \left(\frac{4}{5}\right)^3 = 0.7168$

Question 12 E

Let $a = \Pr(A)$ and $b = \Pr(B)$.

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$0.76 = a + b - ab$

$\Pr(A \cap B) = ab = 0.24$

2.6 3.1 3.2 *Doc RAD

$\text{solve}(a \cdot b = 0.24 \text{ and } a + b - a \cdot b = 0.76, a, b)$

$a = 0.4$ and $b = 0.6$ or $a = 0.6$ and $b = 0.4$

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Question 13 **B**

The derivative function f' is a positive cubic and matches the positive quartic in **B**.

Question 14 **C**

$2 - x > 0 \rightarrow x < 2$

$\log_e(2 - x) \neq 0 \rightarrow x \neq 1$

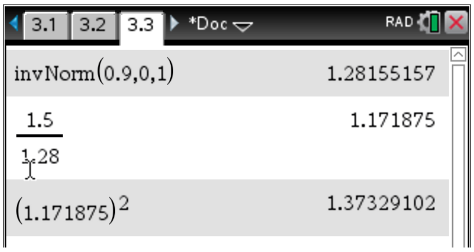
$\therefore x \in (-\infty, 2)\backslash\{1\}$

Question 15 **C**

$$z = \frac{x - \mu}{\sigma}$$

$$\mu = 0 \rightarrow \sigma = \frac{x}{z}$$

$$\text{Var}(X) = \sigma^2$$

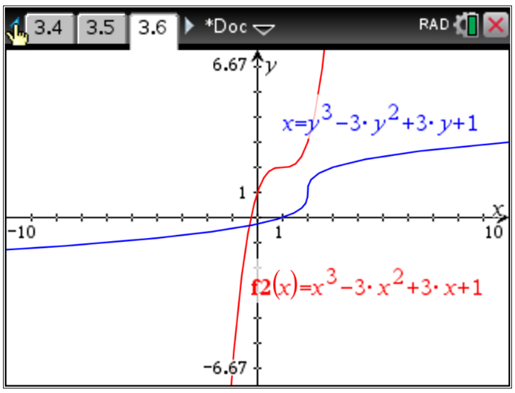


Question 16 **C**

$(x - 1)^3 = x^3 - 3x^2 + 3x - 1$

$\therefore g(x) = (x - 1)^3 + 2$

Thus the inverse will exist.



Question 17 **D**

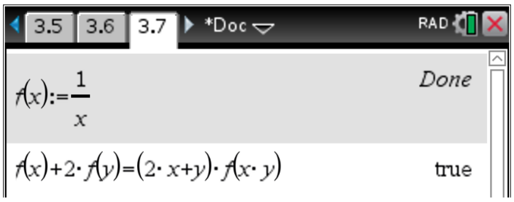
If $f(x) = \frac{1}{x}$:

$$\begin{aligned}
\text{LHS} &= f(x) + 2f(y) \\
&= \frac{1}{x} + 2 \times \frac{1}{y} \\
&= \frac{1}{x} + \frac{2}{y} \\
&= \frac{2x + y}{xy}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= (2x + y)f(xy) \\
&= (2x + y)\frac{1}{xy} \\
&= \frac{2x + y}{xy}
\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

$\therefore f(x) + 2f(y) = (2x + y)f(xy)$

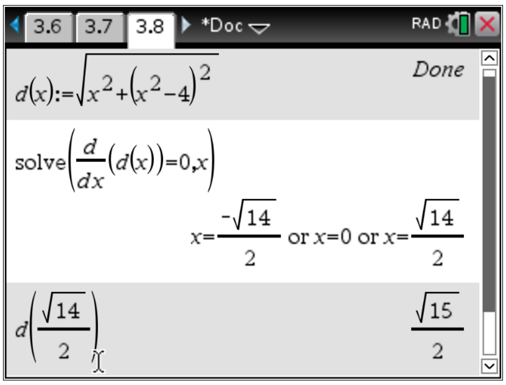


Question 18 **C**

$$\int_3^{10} f(x) + 1\,dx = \int_3^8 f(x)dx + \int_8^{10} f(x)dx + \int_3^{10} 1\,dx$$

$$\begin{aligned}
&= 10 - 4 + 7 \\
&= 13
\end{aligned}$$

Question 19 **C**



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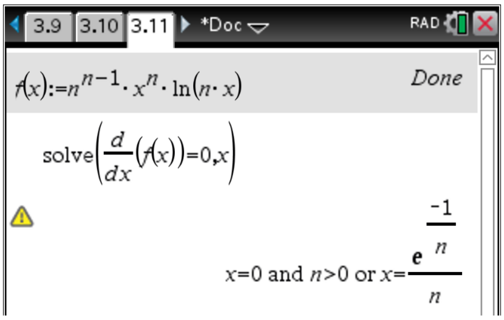
Question 20 **B**



However $f(x)$ is undefined for $x = 0$ and therefore there is only one possible stationary point.

SECTION B

Question 1 (7 marks)

a.

$f(x)$	$x^3-6\cdot x^2+8\cdot x$
solve$(f(x)=0,x)$	$x=0$ or $x=2$ or $x=4$

(0, 0), (2, 0), (4, 0) A1

b. Let $f(x)=g(x)$.

$$x^3-6x^2+8x=-ax$$

$$x^2-6x+8=-a \quad (1)$$

Let $f'(x)=g'(x)$.

$$3x^2-12x+8=-a \quad (2)$$ M1

Equate (1) and (2).

$$3x^2-12x+8=x^2-6x+8$$

$$2x^2-6x=0$$

$$2x(x-3)=0$$

$x=0$ or $x=3$ M1

$\therefore p=3$, as $p>0$.

Substitute $p=3$ into (1):

$$(3)^2-6(3)+8=-a$$ M1

$$a=-1$$

c. Let $d(x)$ = vertical distance between $f(x)$ and $g(x)$.

$d(x) = f(x) - g(x)$

$= x^3 - (6x^2 + 9x)$ M1

$d'(x) = 3x^2 - 12x + 9$

Let $d'(x) = 0$.

$x = 1$ or $x = 3$ M1

$d(1) = 4$ A1

maximum vertical distance = 4

$\frac{d}{dx}(d(x))$	$3\cdot x^2-12\cdot x+9$
$\text{solve}(3\cdot x^2-12\cdot x+9=0,x)$	$x=1$ or $x=3$
$d(x)$	$x^3-6\cdot x^2+9\cdot x$
$d(1)$	4

Question 2 (18 marks)

a. i.

1.4 | 1.5 | 2.1 | *Doc | RAD

$f(x):=4\cdot x+5$	*Done*
$g(x):=x+1$	*Done*
$\text{solve}(f(x)=g(x),x)$	$x=\frac{-4}{3}$
$f\left(\frac{-4}{3}\right)$	$\frac{-1}{3}$

$\left(-\frac{4}{3}, -\frac{1}{3}\right)$ A1

ii.

$\sqrt{\left(\frac{-4}{3}\right)^2+\left(\frac{-1}{3}\right)^2}$	$\frac{\sqrt{17}}{3}$

$\frac{\sqrt{17}}{3}$ A1

iii.

$\tan^{-1}(4)-\tan^{-1}(1)$	30.9637565

$31°$ A1

b. i. There is a dilation factor of a from the y-axis and a dilation factor of $\frac{1}{a}$ from the x-axis. A1

ii. $x' = ax \rightarrow x = \dfrac{x'}{a}$

$y' = \dfrac{1}{a}y \rightarrow y = ay'$ M1

$y = x + 1$

$ay' = \dfrac{x'}{a} + 1$

$y' = \dfrac{1}{a^2}x + \dfrac{1}{a}$

$h(x) = \dfrac{1}{a^2}x + \dfrac{1}{a}$ A1

c. $f(x) = h(x)$

$4x + 5 = \dfrac{1}{a^2}x + \dfrac{1}{a}$

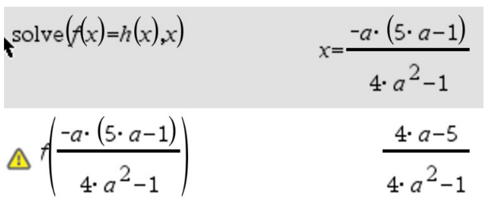
Unique solutions occur where $m_1 \neq m_2$.

$\dfrac{1}{a^2} \neq 4$

$a \neq \pm\dfrac{1}{2}$ M1

But $a \in R^+$ and therefore unique solutions occur for $a \in (0, \infty) / \left\{\dfrac{1}{2}\right\}$. A1

d.



$\left(\dfrac{-a(5a-1)}{4a^2-1}, \dfrac{4a-5}{4a^2-1}\right)$

x-coordinate A1
y-coordinate A1

e. Let distance = $d(a)$.

$$d(a) = \sqrt{\left(\frac{-a(5a-1)}{4a^2-1}\right)^2 + \left(\frac{4a-5}{4a^2-1}\right)^2}$$ M1

For min/max, let $d'(a) = 0$. M1

$$a = \frac{\pm\sqrt{21}+5}{4}$$

$a > 0$, therefore $a = \dfrac{\sqrt{21}+5}{4}$ A1

$$d(a) := \sqrt{\left(\frac{-a\cdot(5\cdot a-1)}{4\cdot a^2-1}\right)^2 + \left(\frac{4\cdot a-5}{4\cdot a^2-1}\right)^2}$$

$$\text{solve}\left(\frac{d}{da}(d(a))=0, a\right)$$

$$a=\frac{-(\sqrt{21}-5)}{4} \text{ or } a=\frac{\sqrt{21}+5}{4}$$

f. **i.** $\text{dom}_p = [-1, 1]$

Transformation T_2 represents a dilation factor of a from the y-axis and, as $a < 0$, there is also a reflection in the y-axis.

$\therefore \text{dom}_q = [a, -a]$ A1

ii. $q : [a, -a] \to R,\ q(x) = \frac{1}{a^2}x + \frac{1}{a}$

$f(x) = q(x)$

$4x + 5 = \frac{1}{a^2}x + \frac{1}{a}$

Solution is $\left(\frac{-a(5a-1)}{4a^2-1}, \frac{4a-5}{4a^2-1}\right)$ from **part d.** M1

However, the restricted domain of $\text{dom}_q = [a, -a]$.

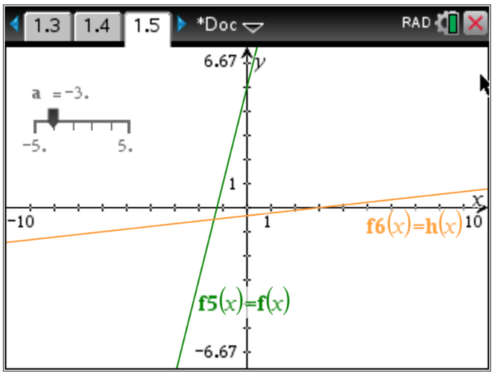
$\therefore a \le \frac{-a(5a-1)}{4a^2-1} \le -a$ for $a < 0$ and $4a^2 - 1 \ne 0$. M1

$a \in \left(-\infty, \frac{-5-\sqrt{57}}{8}\right]$ A1

$$\text{solve}\left(a \le \frac{-a\cdot(5\cdot a-1)}{4\cdot a^2-1} \le -a, a\right) \mid a<0$$

$$a \le \frac{-(\sqrt{57}+5)}{8}$$

Note: A graphical approach with sliders can be also used, as shown below.



iii. $q(x) = \frac{1}{a^2}x + \frac{1}{a}$

As $a \to -\infty$, $q(x) \to 0$.

$f(x) = q(x)$

$4x + 5 = 0$

$x = -\frac{5}{4}$

$\therefore m_1 = -\frac{5}{4}$ A1

Question 3 (13 marks)

a. **i.** average rate of change $= \dfrac{f(a)-1}{a-1}$

$$= \frac{\frac{1}{a}-1}{a-1}$$

$$= \frac{\frac{1-a}{a}}{a-1}$$

$$= -\frac{1}{a}$$ A1

ii. $f'(a) = -\dfrac{1}{x^2}$ A1

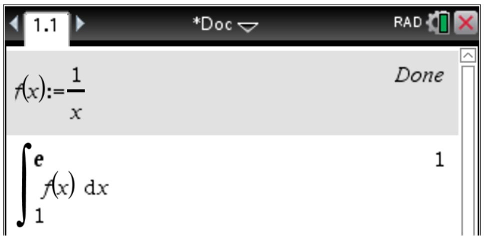
$$-\frac{1}{x^2} = -\frac{1}{a}$$

$$x^2 = a$$

As $x > 0$, $x = \sqrt{a}$.

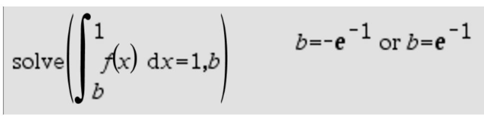
$\therefore$ point $Q = \left(\sqrt{a}, \dfrac{1}{\sqrt{a}}\right)$ A1

b. **i.**



$$\int_1^e f(x)dx = 1$$ A1

ii.



$b = \dfrac{1}{e}$ as $0 < b < 1$ A1

c. **i.** The area is a trapezium.

$$\frac{1}{2}\cdot\left(1+\frac{1}{a}\right)\cdot(a-1) \qquad \frac{(a-1)\cdot(a+1)}{2\cdot a}$$

$$A = \frac{1}{2}\left(1 + \frac{1}{a}\right) \times (a-1)$$ M1

$$= \frac{(a-1)(a+1)}{2a}$$ A1

ii.

$$\text{solve}\left(\frac{(a-1)\cdot(a+1)}{2\cdot a}=1,a\right)$$
$$a=-(\sqrt{2}-1) \text{ or } a=\sqrt{2}+1$$

$$\frac{(a-1)(a+1)}{2a} = 1$$ M1

$$a = \sqrt{2} + 1$$ A1

iii. The line PA is above the graph of f. The area below the line (the trapezium) has an area of 1 square unit so that $\int_1^a f(x)dx$ must be less than this. The integral $\int_1^e f(x)dx = 1$ so that $\int_1^a f(x)dx$ must be less than 1 and hence $a < e$. A1

d. If $\displaystyle\int_{k}^{m} f(kx)dx = \frac{1}{k}$, then $\displaystyle\int_{1}^{\frac{m}{k}} f(x)dx = 1$, as k represents a dilation factor of $\frac{1}{k}$ units from the y-axis.

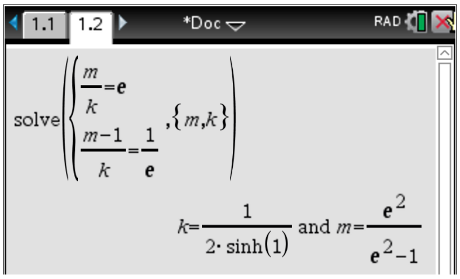
If $\displaystyle\int_{m-1}^{k} f(kx)dx = \frac{1}{k}$, then by the same logic $\displaystyle\int_{\frac{m-1}{k}}^{1} f(x)dx = 1$. M1

In **part b.** it was found that and $\displaystyle\int_{1}^{e} f(x)dx = 1$ and $\displaystyle\int_{\frac{1}{e}}^{1} f(x)dx = 1$.

Therefore $\frac{m}{k} = e$ and $\frac{m-1}{k} = \frac{1}{e}$. M1

Solve simultaneously to give $m = \dfrac{e^2}{e^2 - 1}$ and $k = \dfrac{e}{e^2 - 1}$. A1

Note: CAS gives k using hyperbolic sine, which is not permitted in the VCAA exams. Students can use the CAS but then should be able to substitute m back into the equation $\frac{m}{k} = e$ *to find the acceptable version for k.*



Question 4 (10 marks)

a. **i.** $X \sim Bi\left(5, \frac{1}{5}\right)$

$\Pr(X = 0) = \left(\frac{4}{5}\right)^5$

$= \frac{1024}{3125}$ A1

ii. $\Pr(X \geq 3 | X \geq 1) = \frac{\Pr(X \geq 3 \cap X \geq 1)}{\Pr(X \geq 1)}$

$= \frac{\Pr(X \geq 3)}{1 - \Pr(X = 0)}$ M1

$\text{binomCdf}\left(5, \frac{1}{5}, 3, 5\right) \quad 0.05792$

$\Pr(X \geq 3 | X \geq 1) = \frac{0.05792}{\left(1 - \frac{1024}{3125}\right)}$

$= 0.0861$ A1

b. $\Pr(Y = 4) + \Pr(Y = 5) = 11\Pr(Y = 5)$

$\Pr(Y = 4) = 10\Pr(Y = 5)$

${}^4C_4 p^4 (1 - p)^1 = 10\,{}^5C_4 p^5 (1 - p)^0$

$5p^4(1 - p) = 10p^5$ M1

$p^4(1 - p) = 2p^5$

$1 - p = 2p$ as $p \neq 0$

$3p = 1$

$p = \frac{1}{3}$ A1

c. $\Pr((X+Y)<2) = \Pr((X+Y)=0) + \Pr((X+Y)=1)$

$= \Pr(X=0)\times\Pr(Y=0) + \Pr(X=0)\times\Pr(Y=1) + \Pr(X=1)\times\Pr(Y=0)$ M1

$= 0.2050$ A1

2.7 3.1 3.2 *Doc RAD

$\text{binomPdf}\left(5,\frac{1}{5},0\right)\cdot\text{binomPdf}\left(5,\frac{1}{3},0\right)$ 0.043151276

$\text{binomPdf}\left(5,\frac{1}{5},0\right)\cdot\text{binomPdf}\left(5,\frac{1}{3},1\right)$ 0.107878189

$\text{binomPdf}\left(5,\frac{1}{5},1\right)\cdot\text{binomPdf}\left(5,\frac{1}{3},0\right)$ 0.053939095

0.043151275720167+0.10787818930041+0.▸ 0.20496856

d. $\Pr(Y\geq 1) = 0.8683$ M1

$\text{binomCdf}\left(5,\frac{1}{3},1,5\right)$ 0.868312757

$\Pr(T>91) = 0.8683\ldots$

$\Pr(T<91) = 1-0.8683\ldots$

$= 0.1316\ldots$

$z = \frac{x-\bar{x}}{\sigma}$ M1

invNorm(0.13168724279836,0,1) -1.11845085

$\text{solve}\left(-1.1184508519533=\frac{91-100}{s},s\right)$ $s=8.04684442$

$\sigma = 8.0468$ A1

Question 5 (12 marks)

a. **i.** $x \in (-1, \infty)$ A1

ii. $y \in [-1, 1]$ A1

b. Let $\sin(\log_e(x+1)) = 0$.

$\log_e(x+1) = n\pi, \ \ n \in Z$ M1

$n = 0 \to \log_e(x+1) = 0 \to x = 0$

$n = 1 \to \log_e(x+1) = \pi \to x + 1 = e^{\pi} \to x = e^{\pi} - 1$ A1

Therefore $(e^{\pi} - 1, 0)$ is the first positive x-intercept.

c. **i.** $g(x) = f(x-1)$

$f(x) = \sin(\log_e(x+1))$

$g(x) = \sin(\log_e(x-1+1))$ M1

$\therefore g(x) = \sin(\log_e(x))$ as required

ii. $g(x) = f(x-1)$

The graph of $y = g(x)$ is a translation of 1 unit right from the graph of $y = f(x)$.

Therefore $(1, 0)$ and $(e^{\pi}, 0)$ are the required x-intercepts of $y = g(x)$. A1

d. $\frac{d}{dx}x(\sin(\log_e(x) - \cos(\log_e(x)))$

$= 1 \times (\sin(\log_e(x)) - \cos(\log_e(x)) + x \times \left(\cos(\log_e(x)) \times \frac{1}{x} + \sin(\log_e(x)) \times \frac{1}{x}\right)$

$= 2\sin(\log_e(x))$ A1

$\to \int 2\sin(\log_e(x))\,dx = x(\sin(\log_e(x)) - \cos(\log_e(x)))$ M1

$\to \int \sin(\log_e(x))\,dx = \frac{x}{2}(\sin(\log_e(x)) - \cos(\log_e(x)))$ A1

e. $\int_0^{e^{\pi}-1} f(x)\,dx = \int_1^{e^{\pi}} g(x)\,dx$ M1

$\int_1^{e^{\pi}} g(x)\,dx = \left[\frac{x}{2}(\sin(\log_e(x)) - \cos(\log_e(x))\right]_1^{e^{\pi}}$ M1

$\int_0^{e^{\pi}-1} f(x)\,dx = \frac{1}{2}(e^{\pi} + 1)$ A1

$\int_1^{e^{\pi}} g(x)\,dx \qquad \frac{e^{\pi}}{2} + \frac{1}{2}$