WA Exams Practice Paper B, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
    - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
    - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

(a) The complex numbers z and w are given by $z = 8 - 6i$ and $w = 3 + ai$.

(i) Determine $z \times \bar{z}$. (1 mark)

$$\begin{aligned} z \times \bar{z} &= (8-6i)(8+6i) \\ &= 64 + 36 \\ &= 100 \end{aligned}$$

(ii) Find $\text{Im}(w^2)$ if $a = 5$. (1 mark)

$$\begin{aligned} w^2 &= (3+5i)(3+5i) \\ &= 9 - 25 + 30i \\ \text{Im}(w^2) &= 30 \end{aligned}$$

(iii) The value of a if $w^2 = z$. (2 marks)

$$\begin{aligned} z &= 8 - 6i \\ w^2 &= (3+ai)(3+ai) \\ &= 9 - a^2 + 6ai \end{aligned}$$

From equating imaginary parts

$$\begin{aligned} 6a &= -6 \\ a &= -1 \end{aligned}$$

(b) Determine the complex solutions to the equation $4x^2 + 8x + 7 = 0$. (2 marks)

$$\begin{aligned} x^2 + 2x &= -\frac{7}{4} \\ (x+1)^2 &= -\frac{3}{4} \\ x &= -1 \pm \frac{\sqrt{3}}{2} i \end{aligned}$$

Question 2 **(9 marks)**

(a) Let $P=\begin{bmatrix}5 & 4\\ -2 & -3\end{bmatrix}$ and $Q\begin{bmatrix}3 & 4\\ -2 & -5\end{bmatrix}$.

(i) Determine the matrix product PQ. (1 mark)

$$\begin{bmatrix}5 & 4\\ -2 & -3\end{bmatrix}\begin{bmatrix}3 & 4\\ -2 & -5\end{bmatrix}=\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$

(ii) State the inverse of Q. (1 mark)

$$Q^{-1}=\frac{1}{7}\begin{bmatrix}5 & 4\\ -2 & -3\end{bmatrix}$$

(b) Determine the value(s) of a if the matrix $\begin{bmatrix}a & 3\\ 2 & a-1\end{bmatrix}$ is singular. (3 marks)

$$a(a-1)-2\times 3=0$$
$$a^2-a-6=0$$
$$(a+2)(a-3)=0$$
$$a=-2,\ 3$$

(c) Find matrix A if $AB+2B=4I$, where I is the identity matrix and $B=\begin{bmatrix}2 & -1\\ -2 & 3\end{bmatrix}$. (4 marks)

$$(A+2I)B=4I$$
$$A=4B^{-1}-2I$$
$$A=4\times\frac{1}{4}\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$
$$A=\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}$$

Question 3 **(9 marks)**

(a) Solve $\tan\left(\frac{\theta}{3}\right)=1$, $-\pi \le \theta \le \pi$. (1 mark)

$\frac{\theta}{3}=\frac{\pi}{4}$ $\quad -\frac{\pi}{3} \le \frac{\theta}{3} \le \frac{\pi}{3}$

$\theta=\frac{3\pi}{4}$

(b) Determine exact solution(s) for θ, if $2\cos^2\theta - 5\sin\theta + 1 = 0$, $0 \le \theta \le 2\pi$. (3 marks)

$2(1-\sin^2\theta) - 5\sin\theta + 1 = 0$

$2\sin^2\theta + 5\sin\theta - 3 = 0$

$(2\sin\theta - 1)(\sin\theta + 3) = 0$

$\sin\theta + 3 = 0$ has no solution

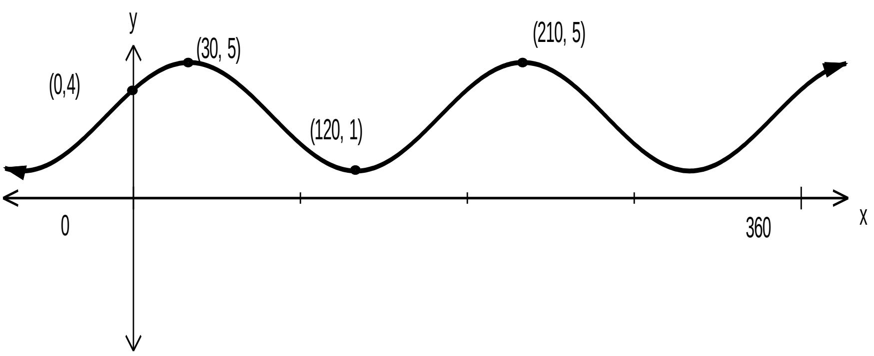
$2\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

(c) The graph of $f(x) = p\cos(q(x+r)) + s$, where $p,\ q,\ r$ and s are positive constants, has a y-intercept at (0, 4), a maximum at (210, 5), period of 180º and amplitude of 2.

(i) Sketch the graph of $y = f(x)$ for $0° \le x \le 360°$. (3 marks)



(ii) State the values of $p,\ q,\ r$ and s. (2 marks)

p = 2, q = 2, r = 150, s = 3.

Question 4 **(10 marks)**

Two vectors are given by $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 15\mathbf{j}$.

(a) Determine $|\mathbf{a} + \mathbf{b}|$. (2 marks)

$$|5\mathbf{i} + 12\mathbf{j}| = \sqrt{25 + 144} = \sqrt{169} = 13$$

(b) Determine the values of λ and μ, if $20\mathbf{i} + 9\mathbf{j} = \lambda\mathbf{a} + \mu\mathbf{b}$. (4 marks)

$$\begin{bmatrix} 20 \\ 9 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ 15 \end{bmatrix}$$

$$2\lambda + 3\mu = 20 \ (1)$$

$$-3\lambda + 15\mu = 9 \ (2)$$

$$3(1) + 2(2) \Rightarrow 39\mu = 78$$

$$\mu = 2$$

$$\lambda = 7$$

(c) Determine the vectors $\mathbf{p}$ and $\mathbf{q}$, if $\mathbf{p} + \mathbf{q} = \mathbf{a} + \mathbf{b}$ and $2\mathbf{p} - \mathbf{q} = 2\mathbf{a}$. (4 marks)

Addition of equations gives

$$3\mathbf{p} = 3\mathbf{a} + \mathbf{b}$$

$$3\mathbf{p} = 3\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 15 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

From second equation

$$\mathbf{q} = 2\mathbf{p} - 2\mathbf{a}$$

$$= 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$$

Question 5 **(5 marks)**

Use mathematical induction to prove that the sum of the first n multiples of 5 is given by

$5+10+15+\ \dots\ +5n=\dfrac{5n(n+1)}{2}$.

When $n=1$, $\left.\dfrac{5(n)(n+1)}{2}\right|_{n=1}=5$.

Assume that when $n=k$, then sum is $5+10+15+\ \dots\ +5k=\dfrac{5k(k+1)}{2}$.

When $n=k+1$, then

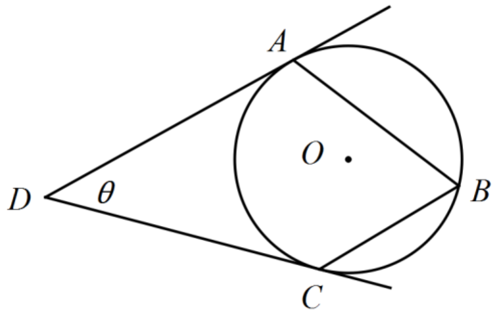
$$\begin{aligned}5+10+15+\ \dots\ +5k+(5k+5)&=\frac{5k(k+1)}{2}+5k+5\\&=\frac{5k^2+5k+10k+10}{2}\\&=\frac{5(k^2+3k+2)}{2}\\&=\frac{5(k+1)(k+2)}{2}\end{aligned}$$

Now, $\dfrac{5(k+1)(k+2)}{2}$ is the RHS of our initial equation, with $n=k+1$ and so if the initial rule is true for $n=k$, then it is also true for $n=k+1$ and since the rule is true for $n=1$, then by mathematical induction, the rule is true for all $n\geq 1$.

Question 6 **(7 marks)**

(a) In the diagram below, points A, B and C lie on a circle with centre O. The tangents to the circle at A and C meet at D.

If $\angle ADC = \theta$, determine an expression for $\angle ABC$ in terms of θ. (3 marks)



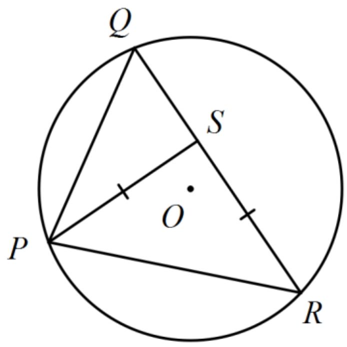
$\angle AOC = 180 - \theta$

$\angle ABC = \frac{1}{2}\angle AOC$

$= \frac{1}{2}(180 - \theta)$

$= 90 - \frac{\theta}{2}$

(b) In the diagram below, the points P, Q and R lie on the circle with centre O, and S lies on QR such that triangle PSR is isosceles. Prove that $PQSO$ is a cyclic quadrilateral.
Hint: Let $\angle QRP = x$ and determine expressions for $\angle QOP$ and $\angle QSP$. (4 marks)



$\angle QOP = 2x$ (angle at centre & circumf)

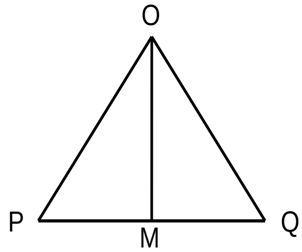
$\angle QSP = 2x$ (ext ang is sum of opp int ang)

Hence $\angle QOP = \angle QSP$

Thus S and O lie on a circle with chord QP and so $PQSO$ is a cyclic quadrilateral.

Question 7 **(6 marks)**

Consider the triangle OPQ below in which M is the midpoint of PQ.



Let $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$.

(a) Express $\overrightarrow{PM}$ and $\overrightarrow{OM}$ in terms of **p** and **q**. (2 marks)

$$\overrightarrow{PM} = \frac{1}{2}\overrightarrow{PQ}$$
$$= \frac{1}{2}(\mathbf{q} - \mathbf{p})$$
$$= \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{p}$$
$$\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$$
$$= \mathbf{p} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{p}$$
$$= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$$

(b) Given that the side PQ is perpendicular to the line OM, use this information and your expressions from (a) to prove that $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$. (4 marks)

$\overrightarrow{PM} \bullet \overrightarrow{OM} = 0$ since they are perpendicular

$$\left(\frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{p}\right) \bullet \left(\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}\right) = 0$$
$$\frac{1}{4}\left(\mathbf{q} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{q} + \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{p}\right) = 0$$
$$\frac{1}{4}\left(|\mathbf{q}| \times |\mathbf{q}| - |\mathbf{p}| \times |\mathbf{p}|\right) = 0$$
$$|\mathbf{q}|^2 = |\mathbf{p}|^2$$
$$|\overrightarrow{OQ}| = |\overrightarrow{OP}|$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________