Neap

Trial Examination 2023

VCE Mathematical Methods Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	**B**	C	D	E
2	A	B	**C**	D	E
3	A	B	C	D	**E**
4	A	B	**C**	D	E
5	A	B	**C**	D	E
6	**A**	B	C	D	E
7	A	**B**	C	D	E
8	**A**	B	C	D	E
9	A	B	**C**	D	E
10	A	B	C	**D**	E

11	A	**B**	C	D	E
12	A	B	**C**	D	E
13	**A**	B	C	D	E
14	A	B	C	D	**E**
15	A	B	C	D	**E**
16	**A**	B	C	D	E
17	A	B	**C**	D	E
18	A	**B**	C	D	E
19	A	**B**	C	D	E
20	A	B	**C**	D	E

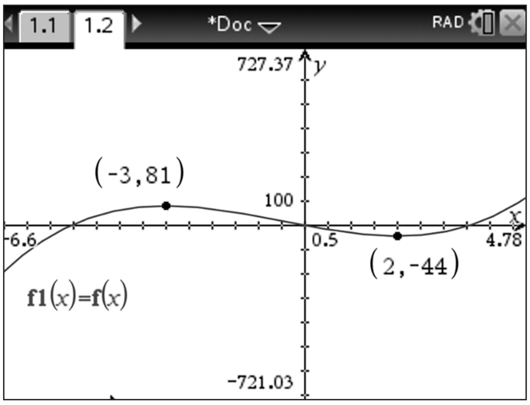
Question 1 B

$$\text{period} = \frac{2\pi}{\pi} = 2$$

$$\text{range} = [2-3, 2+3] = [-1, 5]$$

Question 2 C

Using a CAS calculator gives:



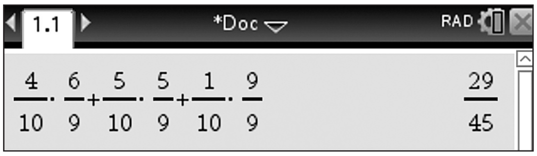
C is not a true statement and is therefore the required response. The graph does have a point of inflection.

A, **B** and **D** are true statements and are therefore not the required response.

E is a true statement and is therefore not the required response. The function does not have an inverse since it is not monotonic.

Question 3 E

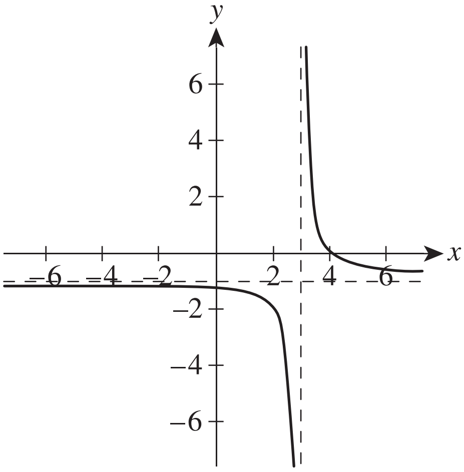
Using a CAS calculator to consider three cases where the second ball is a different colour to the first ball gives:



$$\frac{4}{10}\cdot\frac{6}{9}+\frac{5}{10}\cdot\frac{5}{9}+\frac{1}{10}\cdot\frac{9}{9} \qquad \frac{29}{45}$$

Question 4 C

C is correct. The graph of $f(x) = -1 + \dfrac{1}{x-3}$ is as follows.



A is incorrect. This option has the asymptote $x = 3$ only.

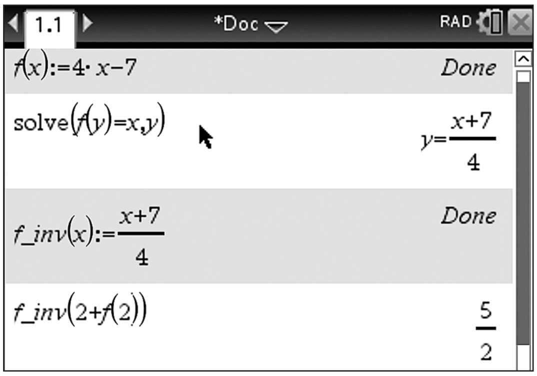
B is incorrect. This option has the asymptote $y = -1$ only.

D is incorrect. This option has the asymptotes $x = -1$ and $y = 3$.

E is incorrect. This option has the asymptotes $x = 3$ and $y = 0$.

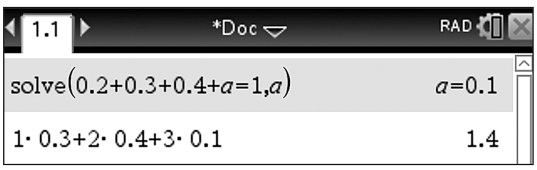
Question 5 C

Using a CAS calculator gives:



Question 6 A

Using a CAS calculator gives:



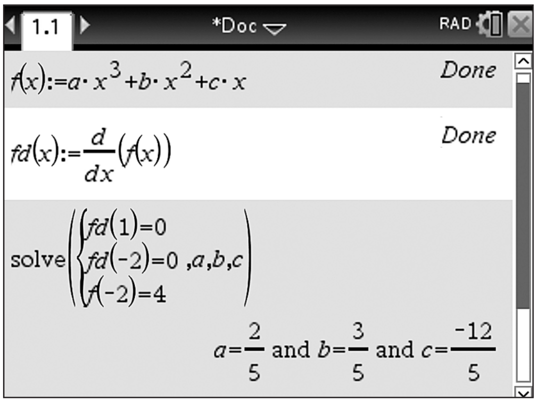
Question 7 B

Determining the average value gives:

$$\frac{1}{4-0}\int_0^4 f(x)dx = \frac{1}{4} \text{ of area under } f(x)$$
$$= \frac{1}{4}\left(\frac{3}{2}(4+1)+\frac{1}{2}(1+3)\right)$$
$$= \frac{19}{8}$$

Question 8 A

Using a CAS calculator gives:

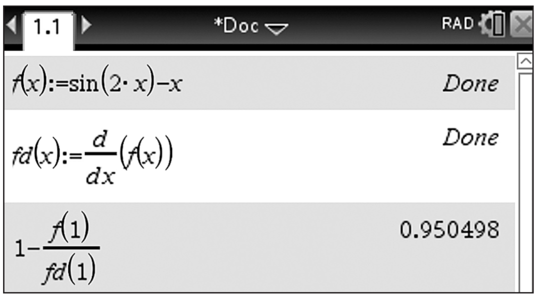


Question 9 C

The algorithm returns an angle multiplied by $\frac{180}{\pi}$ or $\frac{\pi}{180}$ depending on the initial unit. This is used to convert a given angle between degrees and radians.

Question 10 D

Using a CAS calculator gives:



Question 11 B

$$
\begin{aligned}
\Pr(17.2 \le X \le 18.4) &= \Pr\left(\frac{17.2-18}{0.4} \le Z \le \frac{18.4-18}{0.4}\right)\\
&= \Pr(-2 \le Z \le 1)\\
&= \Pr(-1 \le Z \le 2)
\end{aligned}
$$

Question 12 C

Using a CAS calculator gives:

$$\text{binomCdf}\left(12, \frac{0.7}{12}, 2, 12\right) \qquad 0.152472$$

Question 13 A

Using a CAS calculator gives:

$$f(x) := 2 \cdot x^3 + p \cdot x^2 - (p+1) \cdot x + 5 \qquad \textit{Done}$$

$$fd(x) := \frac{d}{dx}(f(x)) \qquad \textit{Done}$$

$$fd(x) \qquad 6 \cdot x^2 + 2 \cdot p \cdot x - p - 1$$

If a function has a turning point, then it is not monotonic; hence, it has no inverse.

So, looking for a positive discriminant gives:

$$
\begin{aligned}
(2p)^2 - 4 \times 6 \times (-p-1) &> 0\\
4p^2 + 24p + 24 &> 0\\
p^2 + 6p + 6 &> 0
\end{aligned}
$$

Question 14 E

Using a CAS calculator gives:

$$f(x) := 2 \cdot \sin\left(\frac{x}{4}\right) \qquad \textit{Done}$$

$$\text{solve}\left(\int_0^k f(x)\, dx = 1, k\right) \mid 0 < k < \pi \qquad k = 4 \cdot \cos^{-1}\left(\frac{7}{8}\right)$$

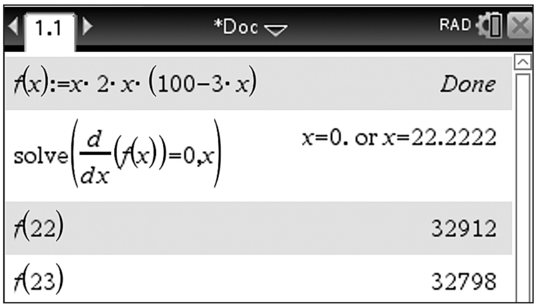
$$k := 4 \cdot \cos^{-1}\left(\frac{7}{8}\right) \qquad 4 \cdot \cos^{-1}\left(\frac{7}{8}\right)$$

$$\int_1^k f(x)\, dx \qquad 0.751299$$

Question 15 E

The three numbers can be expressed as x, $2x$ and $100 - 3x$.

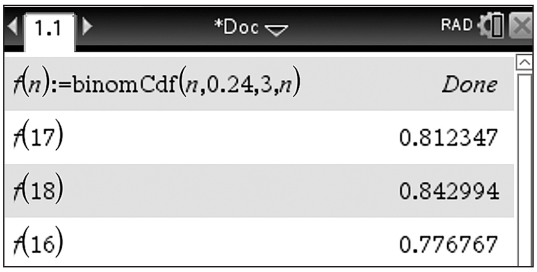
Using a CAS calculator gives:



Hence, the maximum product is 32 912.

Question 16 A

The answer can be obtained by trial and error using a CAS calculator.



Hence, the smallest possible value of n is 17.

Question 17 C

Reflection in the y-axis maps $y = \sin(2x)$ to $y = \sin(-2x)$.

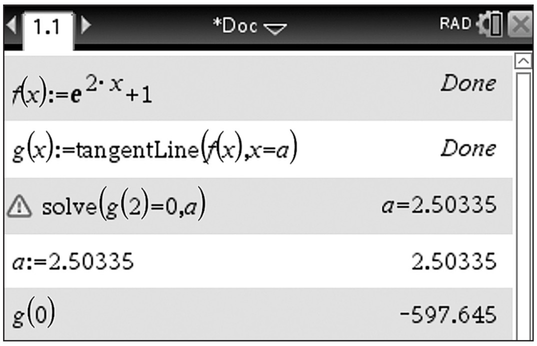
Dilation by a factor of 2 from the y-axis maps $y = \sin(-2x)$ to $y = \sin(-x)$.

Translation of $\frac{\pi}{2}$ units in the positive direction of the x-axis maps $y = \sin(-x)$ to:

$$\begin{aligned} y &= \sin\left(-\left(x - \frac{\pi}{2}\right)\right) \\ &= \sin\left(\frac{\pi}{2} - x\right) \\ &= \cos(x) \end{aligned}$$

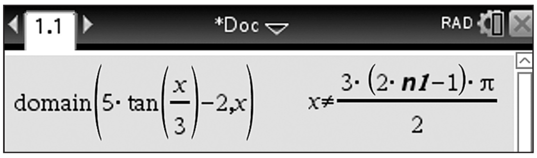
Question 18 B

Using a CAS calculator gives:



Question 19 B

Using a CAS calculator gives:



The answer is not immediately identifiable. Further manipulation gives:

$$\begin{aligned}\frac{3\pi}{2}(2k-1) &= \frac{3\pi}{2}(2k-1)+2\pi m \quad \text{(where } k \text{ and } m \text{ are any integers)}\\ &= \frac{3\pi}{2}(2k-1)+\frac{3\pi}{2}\left(\frac{2}{3\pi}\times 2\pi m\right)\\ &= \frac{3\pi}{2}\left(2k-1+\frac{4m}{3}\right)\end{aligned}$$

Choosing $m = 3$, $\frac{3\pi}{2}(2k+3)$.

Question 20 C

Stationary points exist when $\frac{d}{dx}f\left(g(x)\right)=0$.

$$\frac{d}{dx}f\left(g(x)\right)=f'\left(g(x)\right)\times g'(x)$$

$$f'\left(g(x)\right)\times g'(x)=0 \Rightarrow \left[\begin{array}{l} g'(x)=0 \\ f'\left(g(x)\right)=0 \end{array}\right.$$

Observing the graphs:

$$\left[\begin{array}{l} g'(x)=0 \\ f'\left(g(x)\right)=0 \end{array}\right. \Rightarrow \left[\begin{array}{l} 1 \text{ solution: } x=3 \\ \left[\begin{array}{l} g(x)\approx 0.5 \Rightarrow \text{ 2 solutions} \\ g(x)\approx 3.5 \Rightarrow \text{ 1 solution} \\ g(x)\approx 6.5 \Rightarrow \text{ no solution} \end{array}\right. \end{array}\right.$$

Note that the question asks for solutions for $0 \le x \le 5$. Therefore, there are 4 stationary points.

SECTION B

Question 1 (9 marks)

a. $KM = \dfrac{80 - 6x - 2x}{2} = 40 - 4x$ M1

b. $KO = \sqrt{KM^2 - MO^2}$

$= \sqrt{(40 - 4x)^2 - (2x)^2}$ M1

$= 2\sqrt{3x^2 - 80x + 400}$ A1

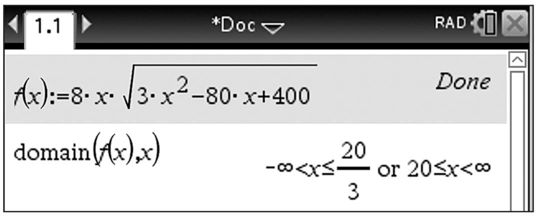
$A = \dfrac{1}{2} \times KO \times (KL + MN)$

$= \dfrac{1}{2} \times \left(2\sqrt{3x^2 - 80x + 400}\right) \times 8x$ M1

$= 8x\sqrt{3x^2 - 80x + 400}$

Note: Consequential on answer to ***Question 1a.***

c. Using a CAS calculator gives:

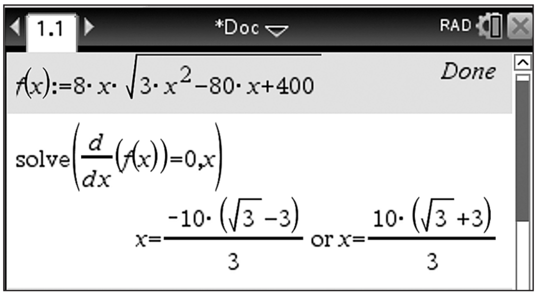


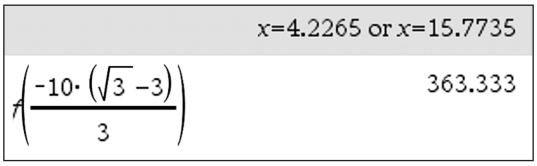
The valid interval for x such that the trapezium exists needs to be selected.

$0 < x < \dfrac{20}{3}$ A1

Note: Responses must use strict inequalities.

d. Using a CAS calculator gives:



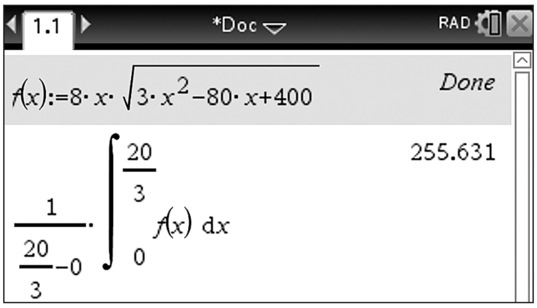


$$\frac{dA}{dx} = 0 \Rightarrow x = 4.2\ldots \text{ or } x = 15.7\ldots$$ M1

Since $15.7\ldots \notin \left(0, \frac{20}{3}\right), x = 4.2\ldots$

$A(4.2\ \ldots) = 363.3$ A1

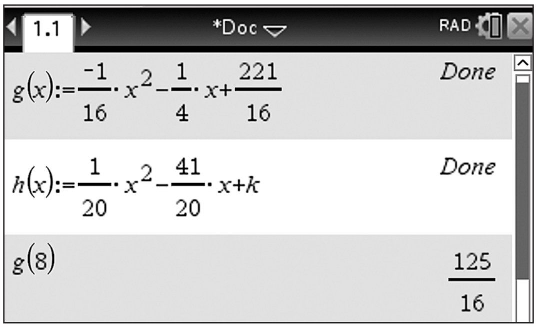
e. Using a CAS calculator gives:

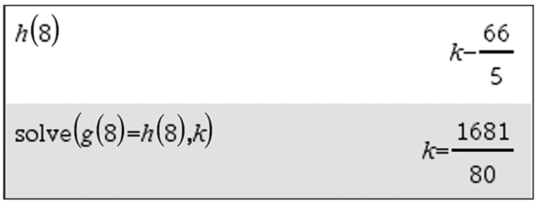


$$\frac{1}{\frac{20}{3} - 0}\int_0^{\frac{20}{3}} A(x)dx = 255.6$$ M1 A1

Question 2 (11 marks)

a. Solving for *k* using a CAS calculator gives:



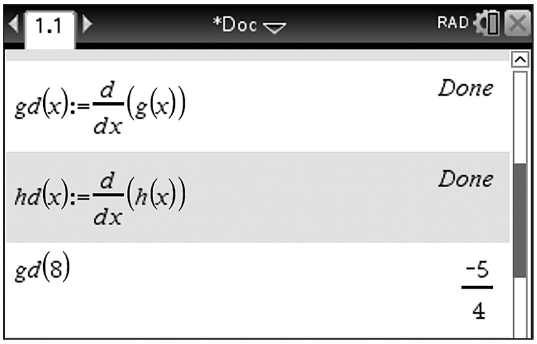


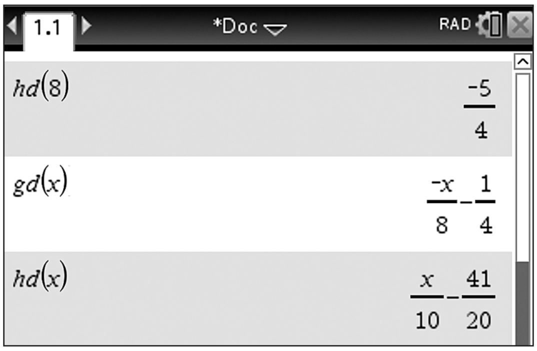
$g(8)=\dfrac{125}{16}$ M1

$h(8)=k-\dfrac{66}{5}$

$g(8)=h(8)\Rightarrow k=\dfrac{1681}{80}$ M1

b. Using a CAS calculator gives:





$$g'(x) = -\frac{1}{8}x - \frac{1}{4}$$

$$h'(x) = \frac{1}{10} - \frac{41}{20}$$

g'(x) and h'(x) M1

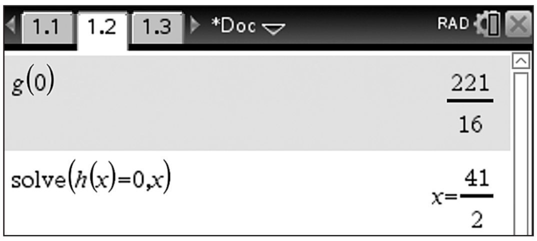
$$g'(8) = -\frac{5}{4}$$

$$h'(8) = -\frac{5}{4}$$

g'(8) and h'(8) A1

$$\begin{cases} g(8) = h(8) \\ g'(8) = h'(8) \end{cases} \Rightarrow f(x) \text{ is differentiable at } x = 8$$ M1

c. Using a CAS calculator gives:

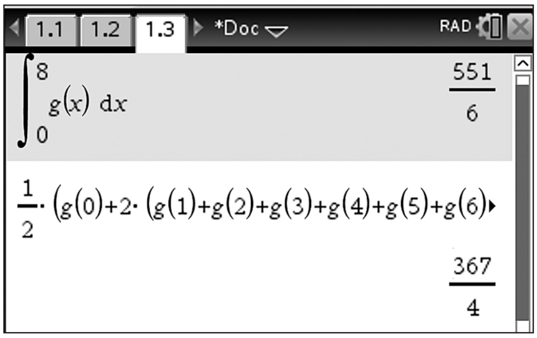


height: $g(0) = \frac{221}{16}$ m A1

length: $h(x) = 0 \Rightarrow x = \frac{41}{2}$ m A1

d. **i.** The curve is concave down for the given interval, so the trapezium will always be under the curve. Hence, the approximation will be less than the actual area. A1

ii. Using the CAS calculator gives:



actual area $= \int_0^8 g(x)dx = \frac{551}{6}$ A1

approximate area $= \frac{1}{2}\left(g(0) + 2\begin{pmatrix} g(1) + g(2) + g(3) + g(4) \\ +g(5) + g(6) + g(7) \end{pmatrix} + g(8) \right) = \frac{367}{4}$ A1

difference $= \frac{551}{6} - \frac{367}{4} = 0.083$ A1

Question 3 (14 marks)

a. $\int_8^{12} \frac{1}{40}(t-8)dt = \frac{\left[(t-8)^2\right]_8^{12}}{80} = \frac{16-0}{80} = \frac{16}{80}$ A1

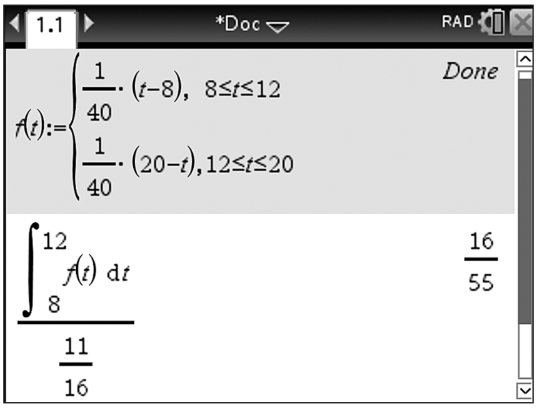
$\int_{12}^{15} \frac{1}{40}(20-t)dt = \frac{\left[(20-t)^2\right]_{12}^{15}}{-80} = \frac{25-64}{-80} = \frac{39}{80}$ A1

$\Pr(T \le 15) = \frac{16}{80} + \frac{39}{80}$ M1

$= \frac{55}{80}$

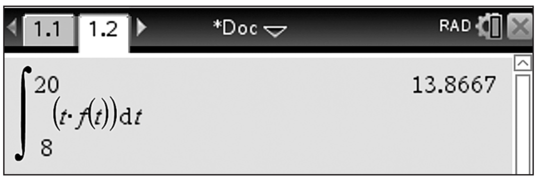
$= \frac{11}{16}$

b. Using a CAS calculator gives:



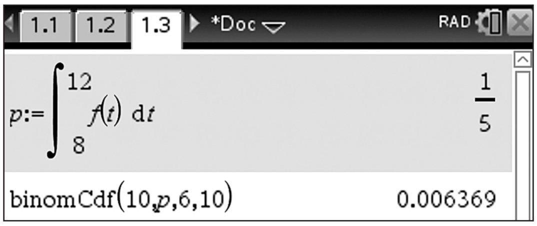
$\frac{\int_8^{12} f(t)dt}{\frac{11}{16}} = \frac{16}{55}$ M1 A1

c.



$\int_8^{20} t \times f(t)dt = 13.9$ minutes M1 A1

d.



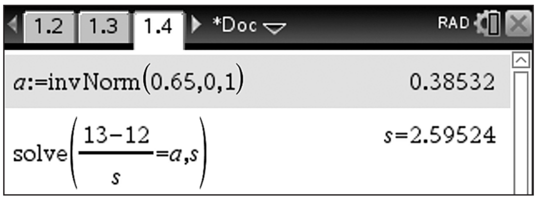
X = number of trips completed in less than 12 minutes

$X \sim \text{Bi}(10, p)$

$p = \int_8^{12} f(t)dt = \frac{1}{5}$ M1

$\Pr(X \geq 6) = 0.0064$ A1

e.



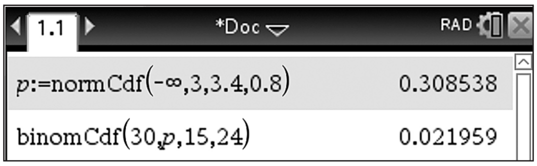
$\Pr(U < 13) = \Pr(Z < a)$

$a = 0.3853 \ldots$ M1

$\frac{13-12}{\sigma} = a$

$\sigma = 2.5952$ A1

f.



Y = number of times a trip is interrupted by red light

$Y \sim \text{N}(3.4, 0.8^2)$ M1

X = number of trips with less than three red light interruptions

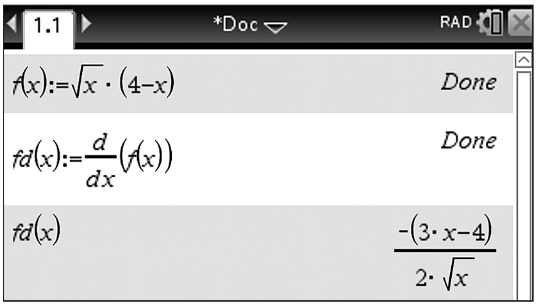
$X \sim \text{Bi}(30, p)$

$p = \Pr(Y < 3) = 0.3085 \ldots$ A1

$\hat{P} = \frac{X}{30} \Rightarrow \Pr\left(0.5 \leq \hat{P} \leq 0.8\right) = \Pr(15 \leq X \leq 24) = 0.0220$ A1

Question 4 (16 marks)

a. Using a CAS calculator gives:



$$f'(x) = \frac{4-3x}{2\sqrt{x}}$$ M1

$$f'(x) = 0 \Rightarrow x = \frac{4}{3}$$ A1

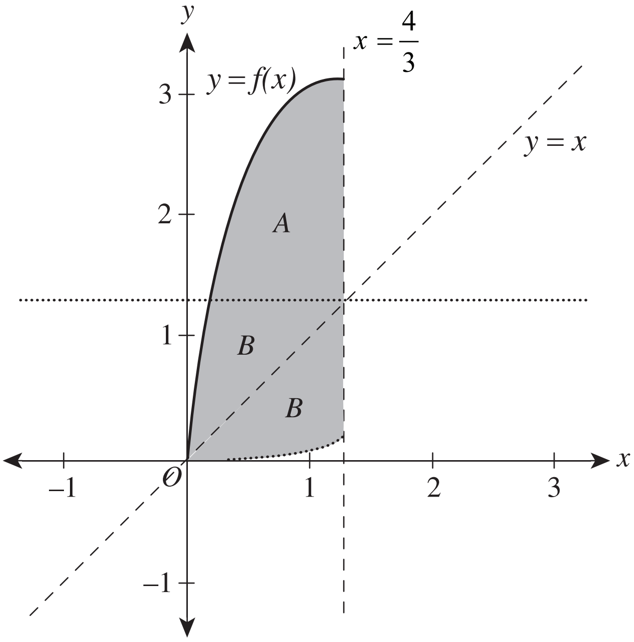
If $D_f = [0, k]$ does not contain any turning points, then f has an inverse. Therefore,

$$0 < k \le \frac{4}{3}.$$ M1

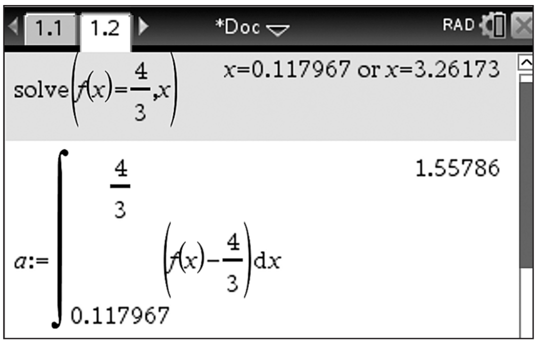
b.

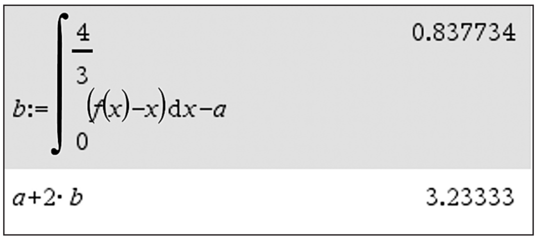
$$\int_0^{\frac{4}{3}} (f(x) - x)dx = 2.40$$ M1 A1

c. Rather than trying to find a rule for f^{-1}, the area can be found using symmetrical regions.



Using the graph above, the total area can be represented by $A + 2B$.





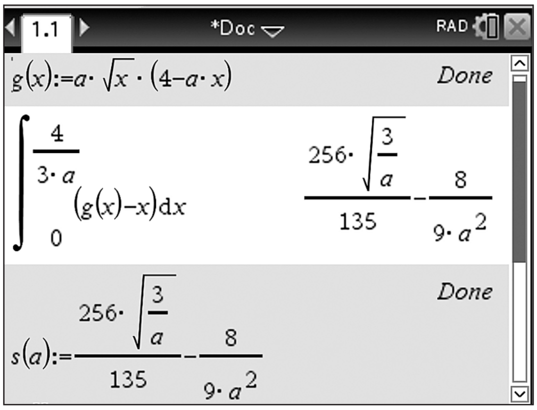
$f(x) = \frac{4}{3} \Rightarrow x = 0.1179...$ M1

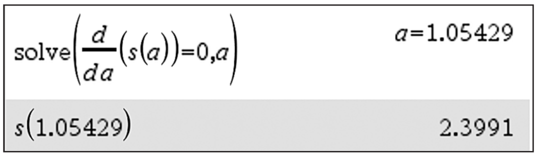
$A = \int_{0.1179...}^{\frac{4}{3}} \left(f(x) - \frac{4}{3} \right) dx = 1.5578...$ M1

$B = \int_0^{\frac{4}{3}} \left(f(x) - x \right) dx - A = 0.8377$

$A + 2B = 3.23$ A1

d.





Let the area be $S(a)$:

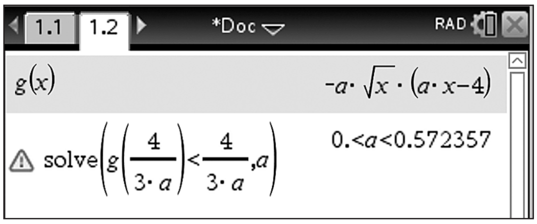
$$S(a)=\int_0^{\frac{4}{3a}}\left(g(x)-x\right)dx$$

$$=\frac{256}{135}\sqrt{\frac{3}{a}}-\frac{8}{9a^2}$$ M1

$S'(a)=0 \Rightarrow a=1.05\ldots$ A1

$S(1.05\ \ldots)=2.40$ A1

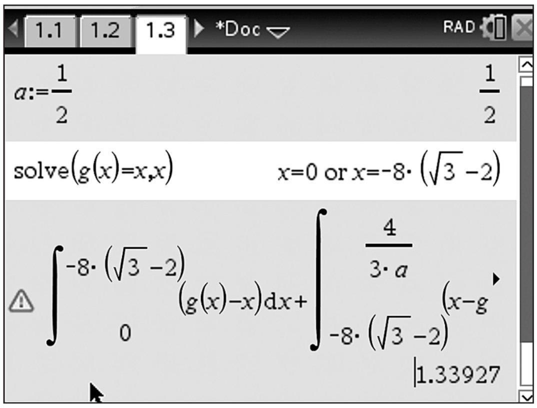
e. **i.**



$$g\left(\frac{4}{3a}\right)<\frac{4}{3a}$$ M1

$0<a<0.57$ A1

ii.



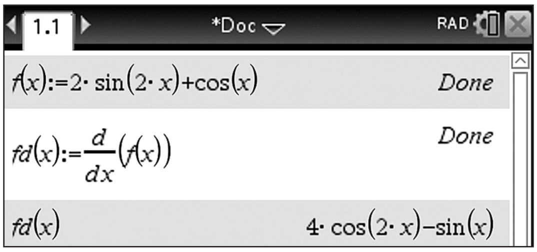
$g(x) = x \Rightarrow x = 16 - 8\sqrt{3}$ M1

$\int_{0}^{16-8\sqrt{3}} (g(x) - x)\,dx + \int_{16-8\sqrt{3}}^{\frac{8}{3}} (x - g(x))\,dx = 1.34$ M1 A1

Note: Award the second M1 for one correct definite integral.

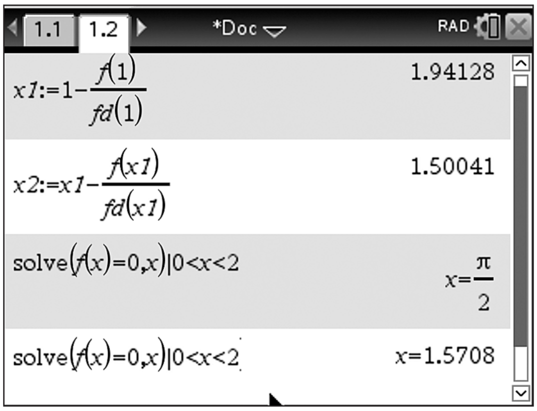
Question 5 (10 marks)

a.



Hence, $f'(x) = 4\cos(2x) - \sin(x)$. A1

b.



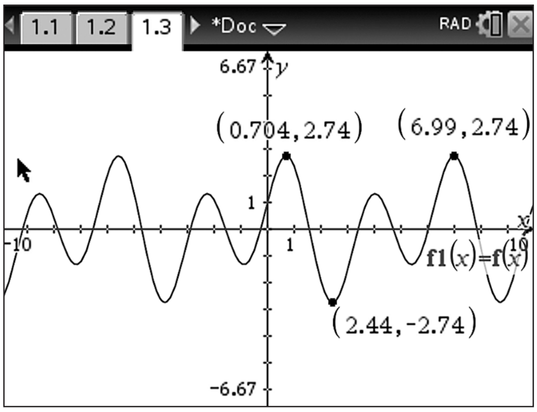
$x_1 = 1 - \frac{f(1)}{f'(1)} = 1.9412\ldots$ M1

$x_2 = 1 - \frac{f(x_1)}{f'(x_1)} = 1.5004$ A1

$f(x) = 0 \Rightarrow$ closest root is $x = \frac{\pi}{2} \approx 1.5708 > x_2$ A1

c. **i.** period $= LCM(2\pi, \pi) = 2\pi$ A1

ii.



$-2.74 \le y \le 2.74$ A1

d. **i.** Observation and trial and error give:

$a = -1$ A1

$b = -\frac{1}{2}$ A1

ii.

$$f\left(\frac{x}{2}-\pi\right) = 2\sin\left(2\left(\frac{x}{2}-\pi\right)\right)+\cos\left(\frac{x}{2}-\pi\right)$$

$$= 2\sin(x-2\pi)+\cos\left(\pi-\frac{x}{2}\right)$$ M1

$$= 2\sin(x)-\cos\left(\frac{x}{2}\right)$$

$$= -\left(-2\sin(x)+\cos\left(\frac{x}{2}\right)\right)$$

$$= -\left(2\sin(-x)+\cos\left(-\frac{x}{2}\right)\right)$$ M1

$$= -f\left(-\frac{x}{2}\right)$$