Neap

Trial Examination 2021

# VCE Mathematical Methods Units 3&4

Written Examination 2

## Suggested Solutions

### SECTION A – MULTIPLE-CHOICE QUESTIONS

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **A** | B | C | D | E | 11 | A | B | **C** | D | E |
| 2 | A | B | C | **D** | E | 12 | A | **B** | C | D | E |
| 3 | **A** | B | C | D | E | 13 | A | **B** | C | D | E |
| 4 | A | **B** | C | D | E | 14 | A | B | C | **D** | E |
| 5 | A | B | **C** | D | E | 15 | A | B | C | D | **E** |
| 6 | A | B | **C** | D | E | 16 | A | **B** | C | D | E |
| 7 | A | B | C | **D** | E | 17 | A | B | **C** | D | E |
| 8 | A | B | C | D | **E** | 18 | A | B | C | **D** | E |
| 9 | A | B | C | D | **E** | 19 | A | **B** | C | D | E |
| 10 | A | **B** | C | D | E | 20 | **A** | B | C | D | E |

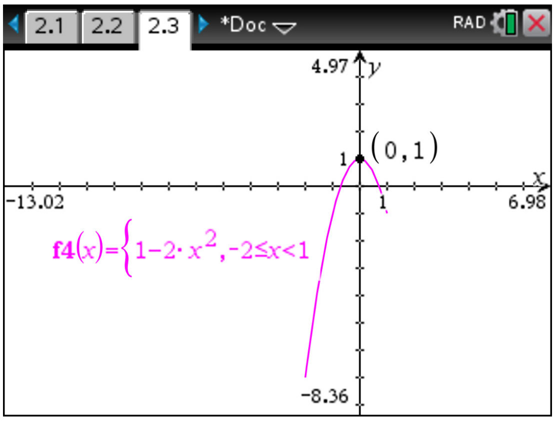
**Question 1 A**

tangentLine$(x^2+2\cdot x-1,x,1)$ $\quad 4\cdot x-2$

**Question 2 D**

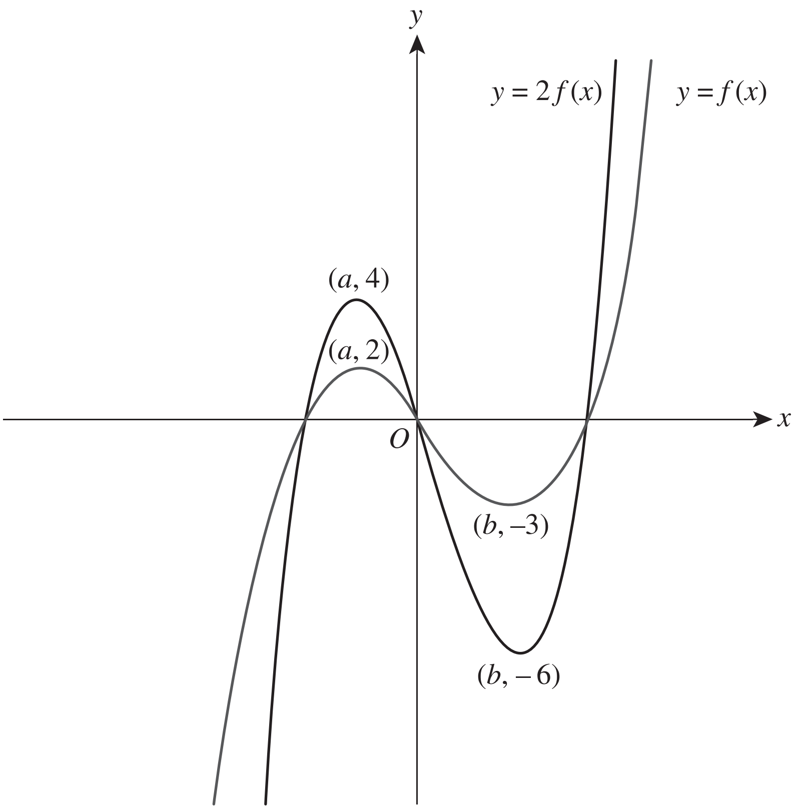
$f(-2)=-7$

$\therefore$ range $=[-7,1]$



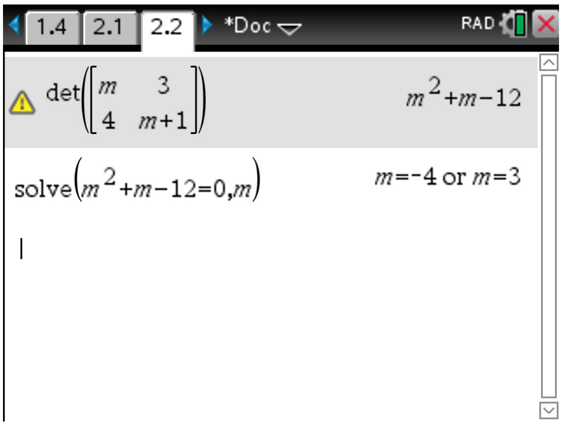

**Question 3 A**

Graphing $y=2f(x)$ gives turning points that provide the restrictions for $c$. The graph of $y=c$ is a horizontal line. The solution of $2f(x)=c$ is the intersection point of $y=2f(x)$ and $y=c$. Therefore, $c<-6$ or $c>4$ gives only one point of intersection and one solution.

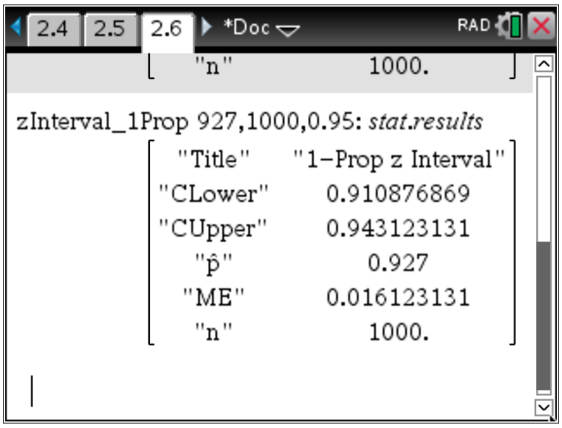


**Question 4 B**

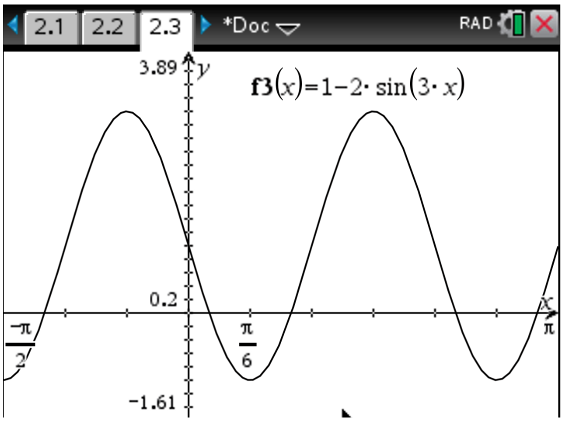
$$\begin{bmatrix} m & 3 \\ 4 & m+1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ m \end{bmatrix}$$




**Question 5 C**




**Question 6 C**




The local minimum is $\left(\frac{\pi}{6}, -1\right)$ and the local maximum is $\left(\frac{\pi}{2}, 3\right)$. Therefore, the range is $[-1, 3]$.

$$\begin{aligned} \text{period} &= \frac{2\pi}{n} \\ &= \frac{2\pi}{3} \end{aligned}$$

**Question 7 D**

$g(x) = 2f(x-5)+1$

Transformations:

- dilation factor of 2 from $x$-axis $(-1, 2) \to (-1, 4)$
- translation of 5 units right $(-1, 4) \to (4, 4)$
- translation of 1 unit up $(4, 4) \to (4, 5)$

$\to g(4) = 5$

**Question 8 E**

$$\begin{aligned}\int_5^3 1-2f(x)\,dx &= \int_3^5 2f(x)-1\,dx\\ &= 2\int_3^5 f(x)\,dx - \int_3^5 1\,dx\\ &= 2\times 10-2\\ &= 18\end{aligned}$$

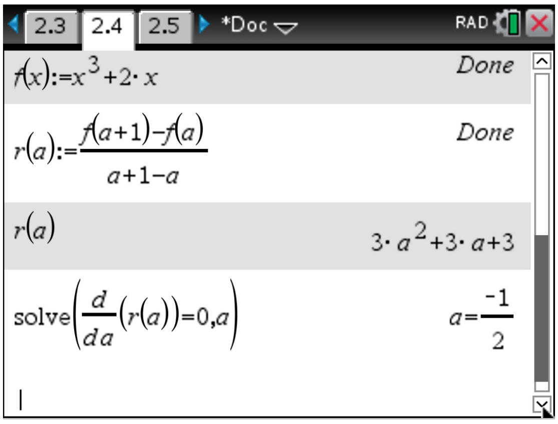
**Question 9 E**

As $0 \le \Pr(x=2) \le 1$, $0 \le k-\frac{1}{4} \le 1$, therefore the minimum value of $k$ is equal to $\frac{1}{4}$.

Maximum value of $m$:

$$\begin{aligned}m &= 1-\left(\frac{1}{4}+0+\left(\frac{1}{4}\right)^2\right)\\ &= \frac{11}{16}\end{aligned}$$

**Question 10 B**




**Question 11 C**

Degree must be even as range $\neq R$.

There is a point of inflection at $x = -a$ and a local maximum at $x = a$.

Therefore, $f$ is at minimum a degree-4 polynomial.

**Question 12 B**

Let S = success, M = miss, A = Any (success or miss)

Pr(exactly 3) = Pr(SSSM) + Pr(MSSS)

$$\Pr(\text{exactly } 3) = \left(\frac{4}{5}\right)^3 \times \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^3 \times \left(\frac{1}{5}\right)$$

$$= \frac{27}{128}$$

**Question 13 B**

$$\text{solve}\left(\int_{\pi}^{a} \sin(2\cdot x)\ dx = \frac{1}{4}, a\right) | \pi < a < \frac{3\cdot\pi}{2}$$

$$a = \frac{7\cdot\pi}{6}$$

**Question 14 D**

$$\text{solve}(1 - m\cdot x > 0, x) | m > 0 \qquad x < \frac{1}{m} \text{ and } m > 0$$

**Question 15 E**

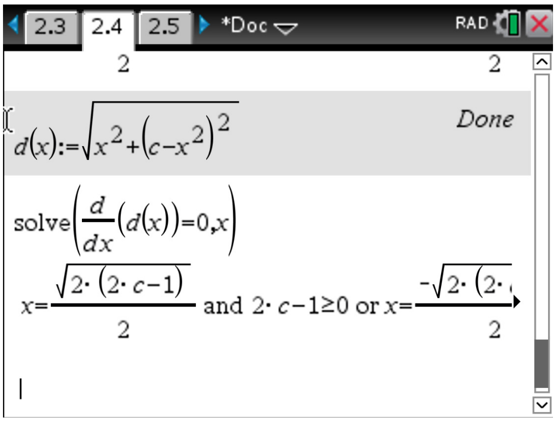
**E** is correct. This graph has two $x$-intercepts and has matching gradients with the derivative function $f'$.

**Question 16 B**

Differentiating the distance formula from the origin to a point $P\left(x, c-x^2\right)$ reveals that the minimum (or maximum) distance occurs at an $x$-value of $x = \dfrac{\pm\sqrt{2(2c-1)}}{2}$ if $2c-1 \geq 0$ or $c \geq \dfrac{1}{2}$.

If $c \leq \dfrac{1}{2}$, however, the minimum distance will always be found at the $y$-intercept, which has a value of $c$.

*Note: A sketch graph could also be used.*




*Note:* $x = 0$ *gives the minimum distance when* $c \geq \dfrac{1}{2}$, *but the derivative indicates a local maximum when* $c > \dfrac{1}{2}$.

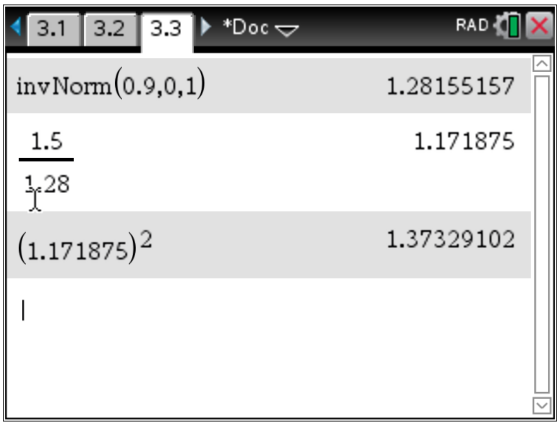
**Question 17 C**

$z = \dfrac{x-\mu}{\sigma}$

$\mu = 0$

$\sigma = \dfrac{x}{z}$

$Var(X) = \sigma^2$

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**Question 18 D**

$$y = \sqrt{\frac{1}{f(x)}}$$

$$= [f(x)]^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2} \times [f(x)]^{-\frac{3}{2}} \times f'(x)$$

$$\frac{dy}{dx} = \frac{-f'(x)}{2\left[\sqrt{f(x)}\right]^3}$$

**Question 19 B**

$\sin(\theta)$ is a one-to-one function for $0 \le \theta \le \frac{\pi}{2}$.

$$0 \le a^x \le \frac{\pi}{2}$$

$$-\infty < x \le \log_a\left(\frac{\pi}{2}\right)$$

**Question 20 A**

Maximum area occurs when the parabola is tangential to the line at $x = a$.

$$g(x) = kx(x-a)$$

$$g'(x) = k(2x-a)$$

When $x = a$, $g'(a) = ak$.

$$m_T = -\frac{b}{a}$$

Let $g'(a) = m_T$.

$$ak = -\frac{b}{a}$$

$$k = -\frac{b}{a^2}$$

$$g(x) = -\frac{b}{a^2}x(x-a)$$

$$\text{maximum area} = \int_0^a -\frac{b}{a^2}x(x-a)dx$$

$$= \frac{ab}{6}$$

## SECTION B

**Question 1** (10 marks)

**a.** $y = 3x - 2$ A1

| | |
|---|---|
| $f(x):=x^3$ | *Done* |
| tangentLine$(f(x),x,1)$ | $3\cdot x-2$ |

**b.** $A = \int_0^{\frac{2}{3}} x^3\,dx + \int_{\frac{2}{3}}^1 x^3 - (3x-2)\,dx$ M2

$= \frac{1}{12}$ A1

| | |
|---|---|
| $\int_0^{\frac{2}{3}} f(x)\,dx + \int_{\frac{2}{3}}^1 (f(x)-(3\cdot x-2))\,dx$ | $\frac{1}{12}$ |

**c.** $y = 3x - 2 + k$ M1

Let $y = 0$.

$3x - 2 + k = 0$

$x = \frac{2}{3} - \frac{k}{3}$

$\left(\frac{2}{3} - \frac{k}{3}, 0\right)$ A1

**d.** $$\text{area } 1 = \int_0^{\frac{2}{3}-\frac{k}{3}} x^3 + k\,dx + \int_{\frac{2}{3}-\frac{k}{3}}^{1} x^3 + k - (3x - 2 + k)dx$$ M1

$$= -\frac{k^2}{6} + \frac{2k}{3} + \frac{1}{12}$$ M1

$$\text{area } 2 = -\frac{1}{2} \times (k-2) \times \left(\frac{2}{3} - \frac{k}{3}\right)$$

$$= \frac{k^2}{6} - \frac{2k}{3} + \frac{2}{3}$$

$$\text{area } 1 = \text{area } 2$$

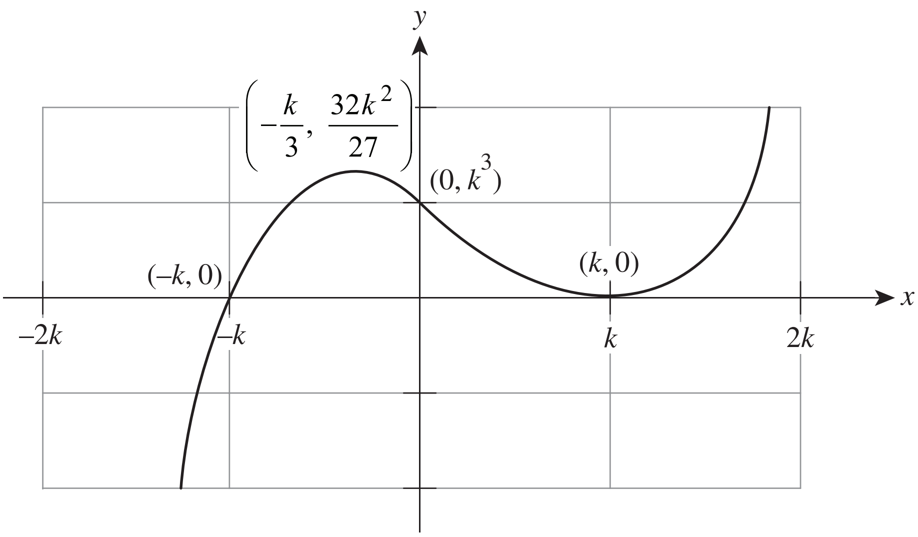
$$-\frac{k^2}{6} + \frac{2k}{3} + \frac{1}{12} = \frac{k^2}{6} - \frac{2k}{3} + \frac{2}{3}$$ M1

$$k = \frac{1}{2} \text{ or } k = \frac{7}{2}$$

Since $0 < k < 2$, $k = \frac{1}{2}$ A1

**Question 2** (6 marks)

**a.**




*correct intercepts* A1
*correct turning point* A1
*correct shape* A1

**b. i.** $$x^2 - k^2 = \frac{1}{x-k}$$

$$(x-k)(x+k) = \frac{1}{x-k}$$

$$(x-k)^2(x+k) = 1$$ A1

**ii.** Let $f(x) = (x-k)^2(x+k)$.

Since $(x-k)^2(x+k) = 1$, $f(x) = 1$.

Two solutions occur when $\frac{32k^3}{27} = 1$, as seen on the graph in **part a.** M1

$k = \frac{3 \times 2^{\frac{1}{3}}}{4}$ A1

$\text{solve}\left(\frac{32 \cdot k^3}{27} = 1, k\right) \qquad k = \frac{3 \cdot 2^{\frac{1}{3}}}{4}$

**Question 3** (17 marks)

**a.** Let $B \sim N(280, 24^2)$

$\Pr(250 < B < 300) = 0.692$ A1

1.1 1.2 1.3 *Doc RAD

normCdf(250,300,280,24) 0.692021836

**b.** $\Pr(\text{elite}) = \Pr(B < 240)$

$= 0.04779...$ M1

$\Pr(\text{purebred} \cap \text{elite}) = 0.35 \times 0.04779...$

$= 0.017$ A1

**c.** **i.** $X \sim Bi(20, 0.35)$ A1

$\Pr(X = 6) = 0.171$ A1

binomPdf(20,0.35,6) 0.171229673

**ii.** $\Pr\left(\hat{P} > 33\%\right) = \Pr(X \geq 7)$ M1

$= 0.583$ A1

binomCdf(20,0.35,7,20) 0.583374582

**iii.** $\Pr\left(\hat{P} \geq \frac{2}{n}\right) = 1 - \Pr(X = 0) - \Pr(X = 1)$ M1

$0.65^n + n \times 0.35^1 \times 0.65^{n-1} < 0.04$

Let $0.65^n + n \times 0.35^1 \times 0.65^{n-1} = 0.04$ to solve on CAS.

$\text{solve}\left((0.65)^n + n \cdot 0.35 \cdot (0.65)^{n-1} = 0.04, n\right)$

$n = -1.82327443 \text{ or } n = 12.1650004$

As $n > 12.165\ldots$, $n = 13$. A1

**d.** **i.** $\hat{p} = \dfrac{0.1184 + 0.3047}{2}$

$= 0.21155$ A1

**ii.** Confidence interval formula: $\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

$0.21155 - z\sqrt{\dfrac{0.21155(1-0.21155)}{52}} = 0.1184$

$z = 1.6438...$ A1

$\Pr(-1.64 < Z < 1.64) = 0.9$

$= 90\%$ confidence A1

**e.** **i.** Let $y = \left(t^2 + 2t + 2\right)e^{-t}$.

$\dfrac{dy}{dt} = -t^2 e^{-t}$ A1

$\dfrac{d}{dx}\left(\left(x^2+2\cdot x+2\right)\cdot \mathbf{e}^{-x}\right) \qquad -x^2\cdot \mathbf{e}^{-x}$

$\int -t^2 e^{-t}\, dt = \left(t^2 + 2t + 2\right)e^{-t}$

$\int \dfrac{mt^2}{e^t} dt = -m\left(t^2 + 2t + 2\right)e^{-t}$

$\int_0^{\infty} \dfrac{mt^2}{e^t}\, dt = 1$ for a probability density function.

$\left[\left(t^2 + 2t + 2\right)e^{-t}\right]_0^{\infty} = \lim\limits_{x\to\infty}\left(\left(t^2 + 2t + 2\right)e^{-t}\right) - \left(\dfrac{0^2 + 2(0) + 2}{e^0}\right)$ M1

$= 0 - 2$

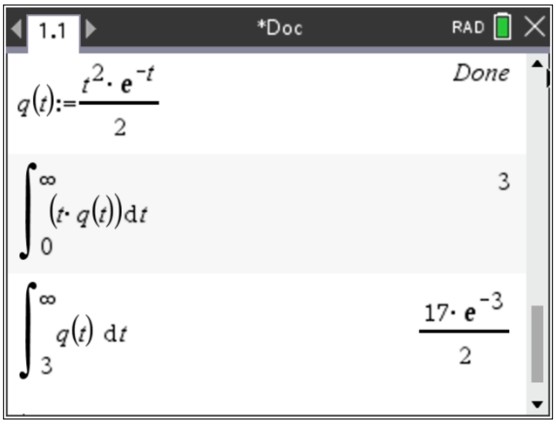
$= -2$

$m \times -2 = -1$

$= \dfrac{1}{2}$ A1

**ii.** mean age = 3 years A1

$\Pr(Q > 3) = \dfrac{17}{2e^3}$ A1




**Question 4** (7 marks)

**a.** $f(x) = e^{2x} - 2e^x$

$f'(x) = 2e^{2x} - 2e^x$

$= 2e^x\left(e^x - 1\right)$

Let $f'(x) = 0$

$\therefore e^x - 1 = 0$ M1

$x = 0$

$f(0) = -1$

Turning point: $(0, -1)$ A1

**b.** $a = 0$, as the turning point is at $x = 0$ and $g$ must be a one-to-one function. A1

**c.** domain $g^{-1}$ = range $g$

domain $g^{-1} = [-1, \infty)$ A1

Let $y = e^{2x} - 2e^x$.

For inverse, swap $x$ and $y$.

$x = e^{2y} - 2e^y$

$= (e^y - 1)^2 - 1$ M1

$e^y = \sqrt{x+1} + 1$

$f^{-1}(x) = \log_e\left(\sqrt{x+1} + 1\right)$ A1

**d.** $(0.86, 0.86)$ A1

solve$\left(\ln\left(\sqrt{x+1}+1\right)=x,x\right)$ $x=0.86032924$

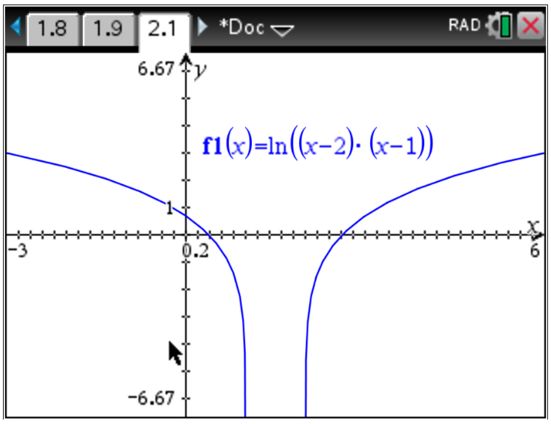
solve$\left(e^{2\cdot x}-2\cdot e^{x}=x,x\right)$
$x=-0.731054944$ or $x=0.86032924$

solve$\left(\ln\left(\sqrt{x+1}+1\right)=e^{2\cdot x}-2\cdot e^{x},x\right)$
$x=0.86032924$

**Question 5** (20 marks)

**a.** **i.** domain: $R\,/\,[1, 2]$ A1

range: $R$ A1




**ii.** $\dfrac{d[g(f(x))]}{dx}=\dfrac{2x-3}{(x-2)(x-1)}$ A1

Let $\dfrac{d[g(f(x))]}{dx}=0.$

$$\frac{2x-3}{(x-2)(x-1)}=0$$

$$x=\frac{3}{2}$$

However, domain is equal to $R\,/\,[1, 2]$ and therefore there are no valid solutions and no stationary points. A1

**b.** **i.**

$$\begin{aligned} h(x) &= f(g(x)) \\ &= \left(\log_e(x)-1\right)\left(\log_e(x)-2\right) \end{aligned}$$ M1

$$\begin{aligned} &= \left(\log_e(x)\right)^2 - 3\log_e(x) + 2 \\ &= \left(\log_e(x) - \frac{3}{2}\right)^2 - \frac{9}{4} + 2 \\ &= \left(\log_e(x) - \frac{3}{2}\right)^2 - \frac{1}{4} \end{aligned}$$ A1

**ii.** $\left(\log_e(x) - \frac{3}{2}\right)^2 \geq 0$

range: $\left[-\frac{1}{4}, \infty\right)$ A1

**c.** **i.** $h'(x) = \dfrac{2\log_e(x) - 3}{x}$ A1

$$\frac{d}{dx}\left(h(x)\right) \qquad \frac{2\cdot \ln(x)-3}{x}$$

**ii.** $2\log_e(x) - 3 = 0$

$$x_M = e^{\frac{3}{2}}$$ A1

**d.** $x_P$ is the $x$-coordinate of the intersection of the tangent lines.

$y_1 = 1 - \frac{x}{e}$ and $y_2 = \frac{x}{e^2} - 1$. M1

$$\begin{aligned} 1 - \frac{x}{e} &= \frac{x}{e^2} - 1 \\ 2 &= \frac{x}{e^2} + \frac{x}{e} \\ 2 &= x\left(\frac{1}{e^2} + \frac{1}{e}\right) \\ 2 &= x\left(\frac{1+e}{e^2}\right) \\ x_P &= \frac{2e^2}{e+1} \end{aligned}$$ A1

**e.** If $x_P < x_M$, then $\frac{2e^2}{e+1} < e^{\frac{3}{2}}$.

$$\frac{2e^{\frac{1}{2}}}{e+1} \leq 1$$

$$2e^{\frac{1}{2}} < e+1 \qquad \text{M1}$$

$$4e < (e+1)^2$$

$$4e < e^2 + 2e + 1$$

$$0 < e^2 - 2e + 1$$

$$0 < (e-1)^2$$

As $e > 1$ and $(e-1)^2 > 0$, then $x_P < x_M$ must be true. A1

**f.** **i.** $h'(e) = -\frac{1}{e}$ and $h'(e^2) = \frac{1}{e^2}$. M1

$$\angle APB = \pi + \tan^{-1}\left(-\frac{1}{e}\right) - \tan^{-1}\left(\frac{1}{e^2}\right)$$

$$= \tan^{-1}\left(e^2\right) + \tan^{-1}\left(e\right) \qquad \text{A1}$$

$\pi + \tan^{-1}\left(-\mathbf{e}^{-1}\right) - \tan^{-1}\left(\mathbf{e}^{-2}\right)$ $\qquad$ $\tan^{-1}\left(\mathbf{e}^{2}\right) + \tan^{-1}\left(\mathbf{e}\right)$

**ii.** $\tan^{-1}(1) = \frac{\pi}{4}$ and $e > 1$.

$\therefore \tan^{-1}(e) > \frac{\pi}{4}$ and $\tan^{-1}\left(e^2\right) > \frac{\pi}{4}$.

$$\angle APB = \tan^{-1}\left(e^2\right) + \tan^{-1}\left(e\right)$$

$$\angle APB > \frac{\pi}{4} + \frac{\pi}{4}$$

$$> \frac{\pi}{2} \qquad \text{A1}$$

$\therefore \angle APB$ is obtuse

**g.** area of triangle $ABP = \frac{1}{2} \times \left(e^2 - e\right) \times \left(-y_P\right)$

$$x_P = \frac{2e^2}{e+1}$$

$$y_1 = 1 - \frac{x}{e}$$

$$y_P = \frac{2}{e+1} - 1$$

$$A_1 = \frac{1}{2} \times \left(e^2 - e\right) \times \left(1 - \frac{2}{e+1}\right)$$ M1

Area bound by $y = h(x)$ and $x$-axis:

$$A_2 = -\int_e^{e^2} h(x)\,dx$$ M1

$$= 3e - e^2$$

$$-\int_{e}^{e^2} h(x)\,dx \qquad 3 \cdot e - e^2$$

total area $= A_1 - A_2$

$$= \frac{1}{2} \times \left(e^2 - e\right) \times \left(1 - \frac{2}{e+1}\right) - \left(3e - e^2\right)$$ M1

$$= \frac{3e^3 - 6e^2 - 5e}{2e+2}$$ A1

1.1 1.2 1.3 *Doc RAD

$$\frac{-1}{2} \cdot \left(\frac{2}{e+1} - 1\right) \cdot \left(e^2 - e\right) - \left(3 \cdot e - e^2\right)$$

$$\frac{3 \cdot e^2}{2} - \frac{2}{e+1} - \frac{9 \cdot e}{2} + 2$$

$$\text{comDenom}\left(\frac{3 \cdot e^2}{2} - \frac{2}{e+1} - \frac{9 \cdot e}{2} + 2\right)$$

$$\frac{3 \cdot e^3 - 6 \cdot e^2 - 5 \cdot e}{2 \cdot e + 2}$$