Government of **Western Australia**
School Curriculum and Standards Authority

WACE
Western Australian Certificate of Education

MATHEMATICS

3C/3D

Calculator-assumed

WACE Examination 2012

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

When examiners design an examination, they develop provisional marking keys that can be reviewed at a marking key ratification meeting and modified as necessary in the light of candidate responses.

Section Two: Calculator-assumed (100 Marks)

Question 9 (4 marks)

Let $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{x+5}$.

(a) Determine an expression for $f(g(x))$. (1 mark)

Solution
$f(g(x))=f\left(\dfrac{1}{x+5}\right)$ $=\sqrt{\dfrac{1}{x+5}}$
Specific behaviours
✓ states correct expression

(b) Determine the domain of $f(g(x))$. (1 mark)

Solution
Domain of $f(g(x)): x>-5,\ x$ real
Specific behaviours
✓ states correct domain

(c) For what value(s) of x will $f(f(x)) = x$? (2 marks)

Solution
$f(x)=\sqrt{x}=x^{\frac{1}{2}}$ $\therefore f(f(x))=\sqrt{x^{\frac{1}{2}}}=(x^{\frac{1}{2}})^{\frac{1}{2}}$ If $x^{\frac{1}{4}}=x$, then $x=0$ or $x=1$
Specific behaviours
✓ evaluates $f(f(x))$ in terms of x ✓ correctly solves equation for x

Question 10 **(11 marks)**

A company makes two models of aircraft, the Airglide and the Skymaster.

The Airglide requires 200 hours of labour and costs $100 000 to make. The Skymaster requires 100 hours of labour and costs $200 000 to make.

Each month, the company can spend at most $1 200 000, and can use up to 1200 hours of labour. It needs to make a total of at least seven aircraft each month, but no more than ten.

Let x = the number of Airglide aircraft produced each month, and
y = the number of Skymaster aircraft produced each month.

Some of the constraints relating to the information above can be represented by the following inequalities:

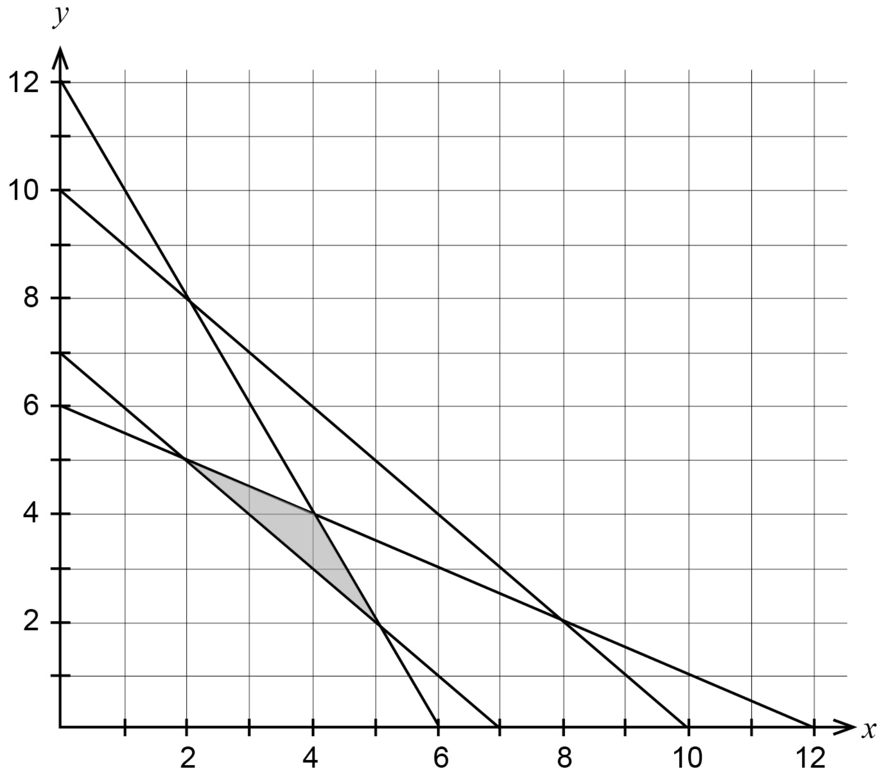
$$x \geq 0 \qquad y \geq 0 \qquad x + y \geq 7 \qquad x + y \leq 10 \qquad x + 2y \leq 12.$$

(a) State one more inequality which, along with those shown above, is sufficient to determine the feasible region.

(1 mark)

Solution
$2x + y \leq 12$ or $200x + 100y \leq 1200$
Specific behaviours
✓ states correct inequality

(b) Draw this inequality on the axes below and shade the feasible region. (2 marks)



Solution
As drawn on axes above
Specific behaviours
✓ draws constraint correctly ✓ shades correct feasible region

(c) Each Airglide aircraft produced makes a profit of \$500 000 for the company and each Skymaster aircraft makes a profit of \$300 000.

Determine the number of each model that should be made each month in order to maximise the total profit, and state this maximum profit. (4 marks)

Solution

Profit = $5x+3y$ (× \$100 000)

x	y	$5x+3y$
2	5	25
5	2	31
4	4	32

The company should produce four of each model, for a total profit of \$3 200 000

Specific behaviours

✓ states correct profit function
✓ determines correct vertices of feasible region
✓ identifies optimum number of aircraft to produce
✓ states maximum profit in correct units

(d) By how much can the profit on each Airglide aircraft be reduced before the optimal number of aircraft in Part (c) above is changed? (4 marks)

Solution

The new optimal point would be (2, 5)
Let the new profit on each Airglide aircraft be p (× \$100 000).
Total profit = $px+3y$

The optimal point will change just as

$$p(2)+3(5)=p(4)+3(4)$$
$$2p+15=4p+12$$
$$p=1.5$$

Can reduce the profit on each Airglide aircraft by up to \$350 000

Specific behaviours

✓ identifies new optimal point
✓ states condition for change of optimal point
✓ solves for profit per unit of x
✓ states answer as a reduction from original profit

Question 11 **(5 marks)**

Iodine-131 is present in radioactive waste from the nuclear power industry.

It has a half-life of eight days. This means that every eight days, one half of the iodine-131 decays to a form that is not radioactive.

This decay can be represented by the equation $N = N_0 e^{kt}$,

where N = amount of iodine-131 present after t days, and
N_0 = amount of iodine-131 present initially.

(a) Determine the value of k correct to **three (3)** decimal places. (3 marks)

Solution
$0.5N_0 = N_0 e^{8k}$ $0.5 = e^{8k}$ $k = -0.087$ solve(0.5a=ae^8k,k) {k=-0.08664339757}
Specific behaviours
✓ identifies condition for half-life ✓ solves for k ✓ rounds correctly to three decimal places

(b) If 125 milligrams of iodine-131 are considered to be safe, how many days will it take for 88 grams of iodine-131 to decay to a safe amount? (2 marks)

Solution
$0.125 = 88e^{-0.087t}$ $t = 75.4 \approx 75$ days solve(0.125=88e^-0.08664t,t) {t=75.67842055} Accept answers between 75 and 76 days
Specific behaviours
✓ states equation with correct unit conversion ✓ solves for t

Question 12 **(8 marks)**

The standard deviation of the lifetimes of Xact computer chips is 550 hours.

Quality control experts plan to estimate μ, the mean lifetime of Xact chips, using the mean lifetime of a random sample of Xact chips.

(a) The experts would like to be 95% confident that the mean lifetime of chips in the sample is within 10 hours of μ. How large a sample should they take? (3 marks)

Solution

$$10 = 1.96\left(\frac{550}{\sqrt{n}}\right)$$

$$n = 11620.84$$

solve$\left(10=1.96\left(\frac{550}{\sqrt{n}}\right), n\right)$

{n=11620.84}

They should take a sample of at least 11 621.

Specific behaviours

✓ identifies that a 95% confidence interval is within 1.96 standard deviations of the mean

✓ uses $\frac{550}{\sqrt{n}}$ as the standard deviation

✓ calculates the correct sample size, rounded up to 11 621

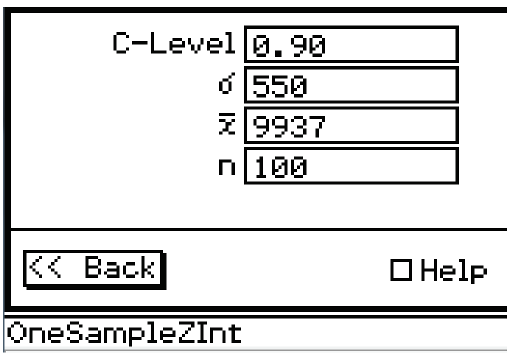
(b) Suppose that a random sample of 100 Xact chips is taken, and the mean lifetime of these chips is 9937 hours.

Based on this sample, determine a 90% confidence interval for μ. (3 marks)

Solution

The 90% interval is $9937 \pm 1.645\left(\frac{550}{\sqrt{100}}\right)$

$= (9846.5, 10\,027.5)$



```
Stat Calculation
One-Sample ZInterval
Data=Variable
Lower  =9846.5331
Upper  =10027.467
x̄      =9937
n      =100
```

Accept lower endpoint from 9846–9847, upper endpoint from 10 027–10 028

Specific behaviours
✓ identifies that a 90% CI is within 1.645 standard deviations of the mean. ✓ uses $\dfrac{550}{\sqrt{100}}$ as the standard deviation ✓ calculates the correct interval

(c) The manufacturer claims that the mean lifetime of Xact chips is at least 10 000 hours. Does the sample in Part (b) provide a strong reason to doubt this claim? Justify your answer. (2 marks)

Solution
No. 10 000 lies within the 90% confidence interval. We would generally doubt the claim only if 10 000 lay outside the confidence interval – and a 90% interval is in any event a narrower interval than some we might use (say 95%, 99%) to test the claim. 10 000 would lie well within these intervals.
Specific behaviours
✓ states correct conclusion ✓ provides a reason referring to the role of confidence intervals in hypothesis testing

Question 13 **(8 marks)**

(a) Ten per cent of a large population is left-handed.

If six people are selected randomly from the population, what is the probability that two or more of these six people are left-handed? (3 marks)

Solution
Let X be the number of left-handed people in the sample of six Then X ~ Bin(6, 0.1) and $P\left(X \geq 2\right) = 0.1143$ binomialCDf(2,6,6,0.1) 0.114265
Specific behaviours
✓ identifies that the binomial distribution applies ✓ uses correct parameters of the binomial distribution ✓ determines correct probability

(b) In a group of 30 people, three are left-handed.

Six people are selected randomly from this group. What is the probability that

(i) two or more of the six people are left-handed? (3 marks)

Solution
P(2 or more are left-handed) = 1 – P(0 or 1 are left-handed) $= 1 - \dfrac{\binom{3}{0}\binom{27}{6}}{\binom{30}{6}} - \dfrac{\binom{3}{1}\binom{27}{5}}{\binom{30}{6}}$ $= 0.0936$ Alternative solution: P(2 or more are left-handed) = P(2 or 3 are left-handed) $= \dfrac{\binom{3}{2}\binom{27}{4}}{\binom{30}{6}} + \dfrac{\binom{3}{3}\binom{27}{3}}{\binom{30}{6}} = 0.0936$

```
(nCr(3,2)nCr(27,4)+nCr(3,3)nCr(27,3))/nCr(30,6)
                                  0.09359605911
```

Specific behaviours
✓ uses $\begin{pmatrix} n \\ r \end{pmatrix}$ correctly for at least one term of the solution ✓ uses correct denominator ✓ determines correct probability

(ii) three of the six people are left-handed, given that two or more are left-handed? (2 marks)

Solution

$P(X=3)=\dfrac{\begin{pmatrix} 3 \\ 3 \end{pmatrix}\begin{pmatrix} 27 \\ 3 \end{pmatrix}}{\begin{pmatrix} 30 \\ 6 \end{pmatrix}}=0.00493$
$P(X=3 \mid X \geq 2)=\dfrac{0.00493}{0.0936}=0.053$
`(nCr(3,3)nCr(27,3))/(nCr(3,3)nCr(27,3)+nCr(3,2)nCr(27,4))`
`0.05263157895`
Accept 0.05 as a response.

Specific behaviours
✓ determines a correct expression for $P(X=3)$ ✓ determines $P(X=3 \mid X \geq 2)$

Question 14 **(10 marks)**

The velocity of a robotic engine moving on a monorail is given by

$v = 3t^2 - 12t + 9$ metres per second, where t = time in seconds.

Determine

(a) the acceleration after 4 seconds. (2 marks)

Solution
$a = 6t - 12.$ When $t = 4,\ a = 12\ \text{ms}^{-2}$
Specific behaviours
✓ differentiates to find a ✓ correctly evaluates a at $t = 4$

(b) how far the engine is from its starting point after 10 seconds. (3 marks)

Solution
$x = t^3 - 6t^2 + 9t + c,$ with $c = 0.$ When $t = 10,\ x = 490\ \text{m}$
Specific behaviours
✓ integrates to find x ✓ correctly reasons that $c = 0$ ✓ correctly evaluates x at $t = 10$

(c) the total distance travelled by the engine in the first 10 seconds. (3 marks)

Solution
Engine is stationary when $0 = 3t^2 - 12t + 9,\ \text{so } t = 1 \text{ or } 3.$ $x(1) = 4,\ x(3) = 0$ Total distance = $4 + 4 + 490 = 498\ \text{m}$ Alternatively, $\int_0^{10} \lvert 3t^2 - 12t + 9 \rvert\, dt = 498$ $\int_0^{10} \lvert 3t^2-12t+9 \rvert\, dt$ 498
Specific behaviours
✓ finds times when engine is stationary ✓ determines displacements at these times ✓ calculates total distance

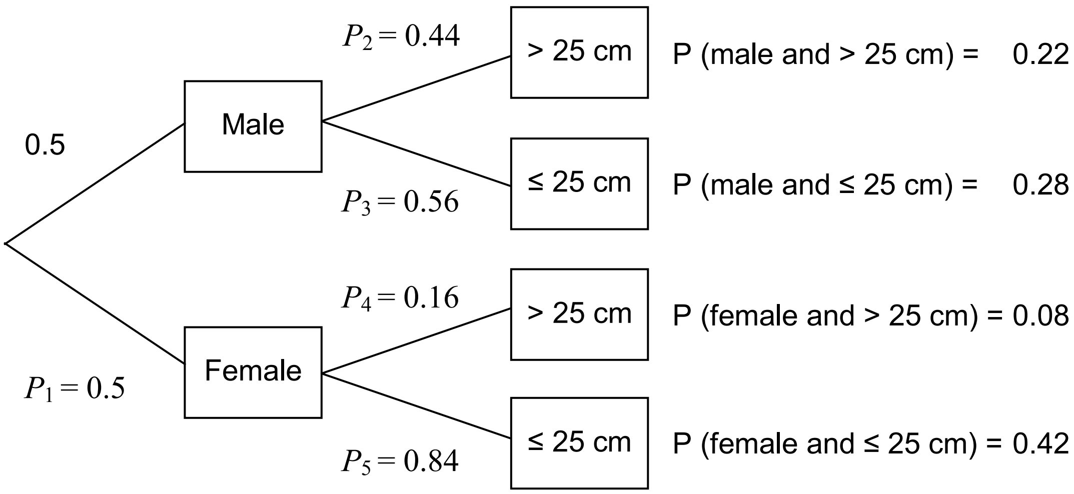
(d) the average velocity of the engine during the first 10 seconds. (2 marks)

Solution
Average velocity = $\dfrac{490\ \text{m}}{10\ \text{s}} = 49\ \text{ms}^{-1}$
Specific behaviours
✓ interprets average velocity as a total distance or displacement, divided by time ✓ uses displacement rather than distance

Question 15 **(9 marks)**

In a population of fish, 50% are male and 50% are female. Overall, 30% of the fish are over 25 cm in length. Furthermore 42% of the fish are female and 25 cm or under in length.

(a) Use this information to complete the tree diagram below by determining the probabilities P_1 to P_5 on the branches of the diagram, and the probabilities to the right of the tree diagram. (5 marks)



Solution
On diagram above
Specific behaviours
✓ calculates probabilities of complementary events ($P_1 = 0.5,\ P_2 + P_3 = 1,\ P_4 + P_5 = 1$) ✓ determines that $P_5 = 0.84$ ✓ determines a correct representation that 30% of fish are over 25 cm in length ✓ determines that $P_2 + P_4 = 0.6$ ✓ calculates other probabilities consistently

(b) What is the probability that a randomly caught fish will be:

(i) 25 cm or under in length? (1 mark)

Solution
0.7
Specific behaviours
✓ states correct answer (from diagram)

(ii) either female or over 25 cm in length? (1 mark)

Solution
0.72
Specific behaviours
✓ states correct answer (from diagram)

(iii) female, given that it is over 25 cm in length? (2 marks)

Solution
$\frac{0.08}{0.3} = \frac{4}{15} = 0.27$
Specific behaviours
✓ states correct numerator ✓ states correct denominator

Question 16 **(8 marks)**

A spherical balloon has volume $V = \dfrac{4\pi r^3}{3}$, where r is its radius.

(a) Determine an expression for $\dfrac{dV}{dr}$. (1 mark)

Solution
$\dfrac{dV}{dr} = 4\pi r^2$
Specific behaviours
✓ differentiates correctly

(b) The balloon is being inflated at a rate of 100 cm^3 per second. At what rate is the balloon's radius increasing at the time when the radius is 5 cm? (3 marks)

Solution
$\dfrac{dV}{dr} = \dfrac{dV}{dt} \times \dfrac{dt}{dr} = \dfrac{\left(\dfrac{dV}{dt}\right)}{\left(\dfrac{dr}{dt}\right)}$ $4\pi(5)^2 = \dfrac{100}{\left(\dfrac{dr}{dt}\right)}$ $\dfrac{dr}{dt} = \dfrac{1}{\pi} = 0.32$ cm per second   Alternative solution: $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$ $100 = 4\pi(5^2)\dfrac{dr}{dt}$ $\dfrac{dr}{dt} = \dfrac{1}{\pi} = 0.32$ cm per second
Specific behaviours
✓ correctly uses chain rule in Leibniz form for related rates   ✓ substitutes correct values into equation   ✓ solves to find correct rate with appropriate units

(c) Using the formula $\delta V \approx \frac{dV}{dr}\delta r$, find the approximate percentage increase in the balloon's volume when its radius increases by 1%. (4 marks)

Solution
Dividing both sides by V, $\frac{\delta V}{V} \approx 4\pi r^2 \times \frac{\delta r}{\left(\frac{4\pi r^3}{3}\right)}$ $\frac{\delta V}{V} \approx 3\frac{\delta r}{r}$ So the volume increases by 3% when the radius increases by 1%
Specific behaviours
✓ represents $\frac{\delta r}{r}$ as 0.01 ✓ represents proportional change in change in volume as $\frac{\delta V}{V}$ ✓ substitutes $\frac{4\pi r^3}{3}$ for V ✓ manipulates formula to find 3% increase

Question 17 **(6 marks)**

A coffee machine is intended to produce cups of coffee with a mean temperature between 74 °C and 78 °C.

However, the temperature of coffee produced is in fact uniformly distributed between 70 °C and 80 °C, with a mean of 75 °C and a standard deviation of 2.89 °C.

(a) What is the probability that a cup of coffee produced by the machine will have a temperature between 74 °C and 78 °C? (1 mark)

Solution
0.4
Specific behaviours
✓ states correct probability

(b) If two cups of coffee are produced by the machine, what is the probability that exactly one of the two cups has a temperature between 74 °C and 78 °C? (2 marks)

Solution
P(exactly one cup in range) = P(Cup 1 in range and Cup 2 out) + P(Cup 2 in range and Cup 1 out) $= 0.4\,(0.6) + 0.4\,(0.6)$ $= 0.48$
Specific behaviours
✓ determines probability that first cup is within the range while the second is not ✓ determines probability that exactly one cup is within the range

(c) Use the Central Limit Theorem to estimate the probability that the mean temperature of the next 50 cups of coffee produced by the machine will lie between 74 °C and 78 °C. Give the answer correct to **two (2)** decimal places. (3 marks)

Solution
Let X be the mean temperature of the 50 cups. Then $X \sim N\left(75, \left(\frac{2.89}{\sqrt{50}}\right)^2\right)$ $P(74 < X < 78) = 0.99$ $\text{normCDf}\left(74, 78, \frac{2.89}{\sqrt{50}}, 75\right)$ 0.992792186
Specific behaviours
✓ identifies that a normal distribution is required ✓ uses correct parameters for the distribution ✓ calculates probability, correct to two decimal places

Question 18 **(9 marks)**

A new treatment for back pain is being tested.

A trial group consists of 100 randomly chosen patients with back pain. There is a 25% chance that any one of these patients will report an improvement after one month if no treatment is given.

Let X denote the number of patients who will report an improvement after one month, assuming no treatment is given.

(a) Is the random variable X discrete or continuous? (1 mark)

Solution
discrete
Specific behaviours
✓ states that X is discrete.

(b) State the probability distribution of X. (2 marks)

Solution
$X \sim \text{Bin}(100, 0.25)$
Specific behaviours
✓ states that X is a binomial distribution ✓ states correct parameters of the distribution

(c) Calculate the mean and standard deviation of X. (2 marks)

Solution
Mean = 25 Standard deviation = $\sqrt{100(0.25)(0.75)} = 4.33$
Specific behaviours
✓ calculates mean ✓ calculates standard deviation

(d) What is the probability that 35 or more of the patients in the trial group will report an improvement after one month, assuming that no treatment is given? (2 marks)

Solution
$P(X \geq 35) = 0.016$ binomialCDf(35,100,100,0.25) 0.01642674067
Specific behaviours
✓ uses correct inclusive inequality ✓ calculates correct probability

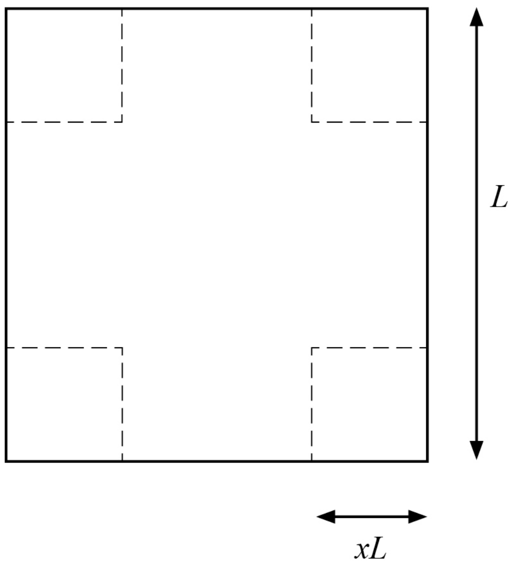
(e) Now suppose that each patient in the trial group is given the new treatment and that 35 of them report an improvement after one month. Is this strong evidence that the treatment is effective? Justify your answer. (2 marks)

Solution
Yes. This trial provides evidence that the treatment is effective. 35% improved with treatment, while only 25% improved without treatment. There is only a 1.6% chance that this improvement would have occurred without treatment.
Specific behaviours
✓ concludes that the trial is effective ✓ provides reasoning about the likelihood of this improvement.

Question 19 **(7 marks)**

A square sheet of metal has sides of fixed length L cm.

A tray is constructed by cutting smaller square pieces out of the corners of the metal sheet and folding up the sides. Each of the pieces has side length xL cm.



(a) Show that the volume of the tray is given by $V = L^3\left(x - 4x^2 + 4x^3\right)$ cm^3. (3 marks)

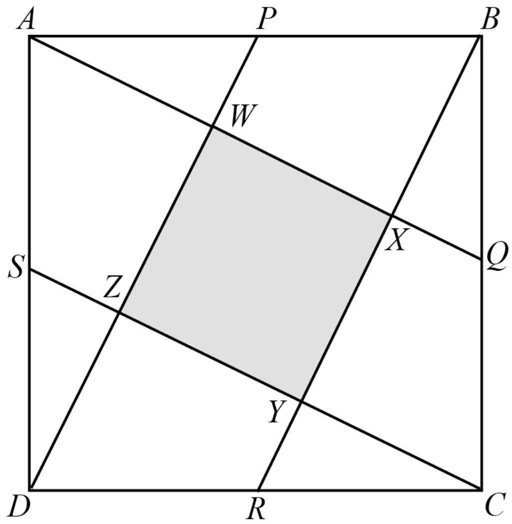
Solution
$V = xL\left(L - 2xL\right)^2$ $= xL\left[L^2\left(1-2x\right)^2\right]$ $= L^3\left[x\left(1-2x\right)^2\right]$ $= L^3\left(x - 4x^2 + 4x^3\right)$ expand$(x \cdot L \cdot (L-2 \cdot x \cdot L)^2)$ $4 \cdot L^3 \cdot x^3 - 4 \cdot L^3 \cdot x^2 + L^3 \cdot x$ factorOut$(4 \cdot L^3 \cdot x^3 - 4 \cdot L^3 \cdot x^2 + L^3 \cdot x, L^3)$ $L^3 \cdot (4 \cdot x^3 - 4 \cdot x^2 + x)$
Specific behaviours
✓ states correct expression for side length of base of tray ✓ states correct expression for volume ✓ shows correct simplification

(b) What is the maximum possible volume of the tray, in terms of L? (4 marks)

Solution
Varying x changes the volume of the tray. So we need: $\frac{dV}{dx}=0$ $L^3\left(1-8x+12x^2\right)=0$ $x=\frac{1}{6}$ or $\frac{1}{2}$ $\text{solve}\left(\frac{d}{dx}\left(L^3\cdot\left(4\cdot x^3-4\cdot x^2+x\right)\right)=0,x\right)$ $\left\{x=\frac{1}{2},x=\frac{1}{6}\right\}$ $x=\frac{1}{6}$ gives the maximum volume. This can be justified by a sign test, a second-derivative test or by noting the shape of the cubic in x. Then $V=L^3\left(\frac{1}{6}\right)\left(\frac{2}{3}\right)^2=\frac{2L^3}{27}$ cm^3
Specific behaviours
✓ differentiates correctly with respect to x ✓ solves for x ✓ justifies which value of x gives the maximum volume ✓ calculates the maximum volume in terms of L

Question 20 **(8 marks)**

In the diagram below, P, Q, R and S bisect the sides of the square $ABCD$.



(a) Prove that ΔAWP is similar to ΔABQ. (3 marks)

Solution	
$\angle PAW \cong \angle QAB$	(common)
$\Delta APD \cong \Delta BQA$	SAS
$\therefore \angle APW \cong \angle BQA$	corresponding angles of $\Delta APD, \Delta BQA$
So $\Delta AWP \cong \Delta ABQ$	AA
Specific behaviours	
✓ identifies the common angle and labels it correctly ✓ reasons via other triangles to the congruence of $\angle APW$ and $\angle BQA$ ✓ uses the congruence of two angles to demonstrate similarity	

(b) If $ABCD$ has side length 10 cm,

(i) d

etermine the ratio $\dfrac{AQ}{AP}$. (2 marks)

Solution
$\dfrac{AQ}{AP} = \dfrac{\sqrt{10^2+5^2}}{5} = \sqrt{5} = 2.236$
Specific behaviours

✓ determines an expression for AQ ✓ calculates the correct ratio

(ii) d
etermine the area of the shaded region $WXYZ$. (3 marks)

Solution
By similarity $\dfrac{AQ}{AP}=\dfrac{AB}{AW}=\dfrac{QB}{PW}=\sqrt{5}$ Area $\Delta ABQ = 25 \text{ cm}^2$ Area $\Delta AWP = \dfrac{1}{2}(AW \times PW) = \dfrac{1}{2}\left(\dfrac{AB}{\sqrt{5}} \times \dfrac{QB}{\sqrt{5}}\right) = 5 \text{ cm}^2$ Area $WXYZ$ = Area $ABCD - 4(25-5) = 20 \text{ cm}^2$
Specific behaviours
✓ determines the area of ΔABQ or another conguent triangle ✓ determines the area of ΔAWP or another conguent triangle ✓ determines the shaded area

Question 21 **(7 marks)**

Let $f(n) = n^2 + n + 11$.

Here are two possible conjectures:

Conjecture P: For each positive integer n, $f(n)$ is prime.

Conjecture Q: For each positive integer n, $f(n)$ is not a multiple of 3.

One of these conjectures is false, and the other is true.

(a) Which conjecture is false? Justify your answer. (3 marks)

Solution
P is false For example, $f(10) = 121$ $121 = 11 \times 11$, so $f(n)$ is not prime when $n = 10$. Note: n must be a multiple of 11, or one less than a multiple of 11, to serve as a counter example.
Specific behaviours
✓ states that P is false ✓ provides a value of n which serves as a counter example ✓ demonstrates falsity of P by showing factors of $f(n)$

(b) Prove the true conjecture. (4 marks)

Solution
Any positive integer n will be of the form $3k$, $3k+1$ or $3k+2$, where k is a non-negative integer. So there are 3 cases. If $n=3k$, then $f(n)=9k^2+3k+11$, which has remainder 2 when divided by 3. If $n=3k+1$, then $f(n)=(3k+1)^2+(3k+1)+11=9k^2+9k+13$, which has remainder 1 when divided by 3. If $n=3k+2$, then $f(n)=(3k+2)^2+(3k+2)+11=9k^2+15k+17$, which has remainder 2 when divided by 3. This exhausts all possibilities, implying that $f(n)$ is not a multiple of 3 Define $f(x)=x^2+x+11$ done f(3k) $9\cdot k^2+3\cdot k+11$ f(3k+1) $(3\cdot k+1)^2+3\cdot k+12$ expand$\left((3\cdot k+1)^2+3\cdot k+12\right)$ $9\cdot k^2+9\cdot k+13$ f(3k+2) $(3\cdot k+2)^2+3\cdot k+13$ expand$\left((3\cdot k+2)^2+3\cdot k+13\right)$ $9\cdot k^2+15\cdot k+17$
Specific behaviours
✓ divides into cases ✓ justifies argument correctly, if not rigorously, for one or more cases ✓ justifies argument correctly, if not rigorously, for all cases ✓ completes argument rigorously