2023 Sem 1 Willetton 12 Methods Exam



**Question 1** (8 marks)

The probability function for the random variable $X$ is $P(X=x)=\begin{cases} k^2-k+x, & x=0 \\ 5k^2x, & x=1 \\ 0, & \text{otherwise.}\end{cases}$

(a) Determine the value of the constant $k$. (4 marks)

**Solution**

$$P(X=0)+P(X=1)=1$$
$$k^2-k+5k^2=1$$
$$6k^2-k-1=0$$
$$(3k+1)(2k-1)=0$$
$$k=-\frac{1}{3}, k=\frac{1}{2}$$

$$k=-\frac{1}{3}\Rightarrow P(X=0)=\frac{4}{9}, P(X=1)=\frac{5}{9}$$
$$k=\frac{1}{2}\Rightarrow P(X=0)=-\frac{1}{4}, P(X=1)=\frac{5}{4}$$

Ignore $k=\frac{1}{2}$ as we require $0\le p\le 1$ and hence $k=-\frac{1}{3}$.

**Specific behaviours**

✓ sums probabilities to 1 and forms quadratic equation
✓ solves for both values of $k$
✓ indicates check of both values of $k$
✓ correct value of $k$

(b) Determine the mean and variance of $X$. (2 marks)

**Solution**

$$E(X)=p=\frac{5}{9},\qquad Var(X)=p(1-p)=\frac{5}{9}\times\frac{4}{9}=\frac{20}{81}$$

**Specific behaviours**

✓ mean
✓ variance

(c) The random variable $Y=3X+1$. Determine the mean and variance of $Y$. (2 marks)

**Solution**

$$E(Y)=3E(X)+1=\frac{8}{3},\qquad Var(Y)=3^2\times Var(X)=\frac{20}{9}$$

**Specific behaviours**

✓ mean
✓ variance

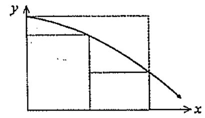




**Question 2** (7 marks)

The curve $y=17-3x-x^2$ is shown, with a bounding rectangle and two inscribed rectangles of equal width.

The shaded region is bounded by the curve, the $x$-axis, the $y$-axis and the line $x=2$.



(a) Use areas of rectangles to explain why the area of the shaded region must be between 20 and 34 square units. (4 marks)

**Solution**

Points on curve: $(0,17)$, $(1,13)$, $(2,7)$.

Area of bounding rectangle is $2\times 17=34$, which is greater than shaded area.

Area of LH rectangle is $1\times 13=13$, RH rectangle is $1\times 7=7$ and their sum is $13+7=20$, which is less than shaded area.

Hence area of the shaded region is between 20 and 34 square units.

**Specific behaviours**

✓ determines y-coordinates of points on curve
✓ derives area of bounding rectangle
✓ derives sum of inscribed rectangles
✓ explanation

(b) Determine the area of the shaded region. (3 marks)

**Solution**

$$A=\int_0^2(17-3x-x^2)\,dx$$
$$=\left[17x-\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^2$$
$$=34-\frac{12}{2}-\frac{8}{3}=25\frac{1}{3}=\frac{76}{3}\text{ u}^2$$

**Specific behaviours**

✓ writes correct integral
✓ correct antiderivative
✓ substitutes bounds to obtain correct area





**Question 3** (7 marks)

(a) Determine $f'(x)$ when $f(x)=\frac{4+\cos(x)}{4+\sin(3x)}$. There is no need to simplify the derivative. (2 marks)

**Solution**

$$f'(x)=\frac{-\sin(x)\times(4+\sin(3x))-(4+\cos(x))\times 3\cos(3x)}{(4+\sin(3x))^2}$$

**Specific behaviours**

✓ use of quotient rule
✓ correct $f'(x)$

(b) Let $y=\cos(x)$, so that when $x=150°$, $y\approx -0.8660$. Given that $1°\approx 0.017$ radians, use the increments formula to determine an approximate value for $\cos(151°)$. (5 marks)

**Solution**

When $x=150°$ and increases to $151°$ then $\delta x=1°\approx 0.017$ radians.

$$\delta y\approx\frac{dy}{dx}\delta x$$
$$\approx -\sin(x)\,\delta x$$
$$\approx -\sin(150°)\times 0.017$$
$$\approx -0.5\times 0.017$$
$$\approx -0.0085$$

Hence $\cos(151°)\approx -0.8660-0.0085\approx -0.8745$.

**Specific behaviours**

✓ correct value of $\delta x$
✓ uses increments formula to obtain expression for $\delta y$
✓ obtains correct exact value for $\sin(150°)$
✓ obtains value of $\delta y$
✓ obtains approximation





**Question 4** (9 marks)

The function $f(x)$ is defined for $x>\frac{5}{3}$, has derivative $f'(x)=\frac{6}{(3x-5)^2}$, and passes through the point $(3,1)$.

(a) Determine the rate of change of $f'(x)$ when $x=2$. (3 marks)

**Solution**

$$f'(x)=6(3x-5)^{-2}$$
$$f''(x)=6(-2)(3)(3x-5)^{-3}$$
$$=-36(3x-5)^{-3}$$

$$f''(2)=-36(1)^{-3}=-36$$

**Specific behaviours**

✓ indicates correct use of chain rule
✓ obtains correct derivative
✓ substitutes and obtains correct value

(b) Determine $f(x)$. (4 marks)

**Solution**

$$f(x)=\int 6(3x-5)^{-2}dx$$
$$=\frac{6}{(-1)(3)}(3x-5)^{-1}+c$$
$$=-2(3x-5)^{-1}+c$$

$$f(3)=1\Rightarrow -2(3\times 3-5)^{-1}+c=1\Rightarrow c=1+0.5=1.5$$

$$f(x)=-\frac{2}{3x-5}+\frac{3}{2}$$

**Specific behaviours**

✓ attempts to obtain antiderivative, with constant
✓ correct antiderivative
✓ indicates use of point to evaluate constant
✓ correct function

(c) Determine $\frac{d}{dt}\int_t^2(f'(x)-2x)\,dx$. (2 marks)

**Solution**

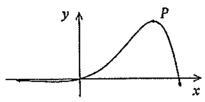
$$\frac{d}{dt}\int_t^2(f'(x)-2x)\,dx=-\frac{d}{dt}\int_2^t(f'(x)-2x)\,dx$$
$$=2t-f'(t)$$
$$=2t-\frac{6}{(3t-5)^2}$$

**Specific behaviours**

✓ adjusts integral so that variable is upper bound
✓ applies fundamental theorem to obtain correct result

**Question 5** **(8 marks)**

The graph of $y = e^{4x}\sin(4x)$ is shown.




(a) Determine the $x$-coordinate of point $P$, the first local maximum of the curve as $x$ increases from 0. (4 marks)

**Solution**

$$\frac{dy}{dx} = 4e^{4x}\times\sin(4x) + e^{4x}\times 4\cos(4x)$$

At $P$ slope is zero:

$$\begin{aligned} 4e^{4x}(\sin(4x)+\cos(4x)) &= 0 \\ \sin(4x)+\cos(4x) &= 0 \\ \sin(4x) &= -\cos(4x) \\ \tan(4x) &= -1 \\ 4x &= \frac{3\pi}{4} \\ x &= \frac{3\pi}{16} \end{aligned}$$

**Specific behaviours**

✓ indicates use of product rule
✓ correct expression for $y'$
✓ sets $y' = 0$ and simplifies to $\tan(4x) = -1$
✓ correct $x$-coordinate

(b) Determine the value of $\frac{d^2y}{dx^2}$ when $x = \frac{7\pi}{2}$ and hence describe the concavity of the curve at this point. (4 marks)

**Solution**

$$\begin{aligned} \frac{dy}{dx} &= 4e^{4x}(\sin(4x)+\cos(4x)) \\ \frac{d^2y}{dx^2} &= 16e^{4x}(\sin(4x)+\cos(4x)) + 4e^{4x}(4\cos(4x) - 4\sin(4x)) \\ &= 32e^{4x}\cos(4x) \end{aligned}$$

When $x = \frac{7\pi}{2}$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 32e^{4\times\frac{7\pi}{2}}\cos\left(4\times\frac{7\pi}{2}\right) \\ &= 32e^{14\pi}\cos(14\pi) \\ &= 32e^{14\pi} \end{aligned}$$

Since $\frac{d^2y}{dx^2} > 0$, then the curve is concave up when $x = \frac{7\pi}{2}$.

**Specific behaviours**

✓ indicates use of product rule
✓ correct expression for $y''$
✓ evaluates $y''$ at required coordinate
✓ correctly describes concavity of curve





**Question 6** **(7 marks)**

A 7 cm length of thin straight wire is bent once and laid on a level surface to form side $KL$ and diagonal $LN$ of rectangle $KLMN$. Let the length of $KL = x$.

(a) Show that the area of the rectangle is $x\sqrt{49-14x}$ cm$^2$. (3 marks)

**Solution**

$$\begin{aligned} KN^2 &= LN^2 - KL^2 \\ &= (7-x)^2 - x^2 \\ &= 49 - 14x + x^2 - x^2 \\ KN &= \sqrt{49-14x} \end{aligned}$$

$$\begin{aligned} \text{Area} &= KL \times KN \\ &= x\sqrt{49-14x} \end{aligned}$$

**Specific behaviours**

✓ indicates correct length of diagonal $LN$
✓ derives expression for length of $KN$
✓ derives expression for area

(b) Determine the maximum possible area of the rectangle. (4 marks)

**Solution**

$$A = x\sqrt{49-14x}$$

$$\frac{dA}{dx} = \sqrt{49-14x} + x\times\frac{1}{2}\frac{-14}{\sqrt{49-14x}}$$

For maximum require $\frac{dA}{dx} = 0$:

$$\begin{aligned} \sqrt{49-14x} - \frac{7x}{\sqrt{49-14x}} &= 0 \\ \frac{7x}{\sqrt{49-14x}} &= \sqrt{49-14x} \\ 7x &= 49 - 14x \\ 21x &= 49 \\ x &= \frac{7}{3} \end{aligned}$$

$$\begin{aligned} A &= \frac{7}{3}\sqrt{49 - 14\times\frac{7}{3}} \\ &= \frac{7}{3}\sqrt{\frac{3\times 49 - 2\times 49}{3}} \\ &= \frac{49}{3\sqrt{3}} = \frac{49\sqrt{3}}{9}\ \text{cm}^2 \end{aligned}$$

**Specific behaviours**

✓ indicates use of product rule
✓ correct expression for derivative
✓ equates derivative to zero and solves for $x$
✓ substitutes and simplifies to obtain maximum area

**Alt Solution**

$$A = x\sqrt{49-14x}$$

$$\frac{dA}{dx} = \sqrt{49-14x} + x\times\frac{1}{2}\frac{-14}{\sqrt{49-14x}}$$

For maximum require $\frac{dA}{dx} = 0$:

$$\begin{aligned} \frac{2(49-14x) - 14x}{2\sqrt{49-14x}} &= 0 \\ \frac{49-21x}{\sqrt{49-14x}} &= 0 \\ 21x &= 49 \\ x &= \frac{7}{3} \end{aligned}$$

$$\begin{aligned} A &= \frac{7}{3}\sqrt{49 - 14\times\frac{7}{3}} \\ &= \frac{7}{3}\sqrt{\frac{3\times 49 - 2\times 49}{3}} \\ &= \frac{49}{3\sqrt{3}} = \frac{49\sqrt{3}}{9}\ \text{cm}^2 \end{aligned}$$

**Specific behaviours**

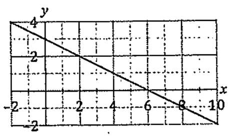
✓ indicates use of product rule
✓ correct expression for derivative
✓ equates derivative to zero and solves for $x$
✓ substitutes and simplifies to obtain maximum area





**Question 7** **(6 marks)**

The graph of the linear function $y = f(x)$ is shown.




Another function is defined as

$$A(t) = \int_2^t f(x)\,dx$$

(a) Using the graph of $y = f(x)$, or otherwise, evaluate $A(2)$ and $A(6)$. (2 marks)

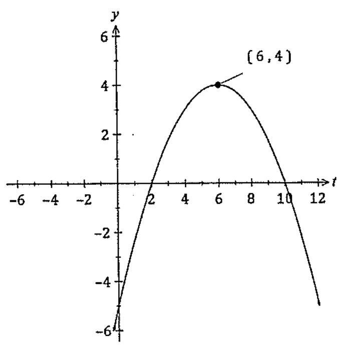
**Solution**

$A(2) = 0, \quad A(6) = \frac{1}{2}\times 4\times 2 = 4$

**Specific behaviours**

✓ one correct value
✓ second correct value

(b) Sketch the graph of $y = A(t)$ on the axes below. (4 marks)




**Solution**

Sketch is easiest using the idea that $A(t)$ is the area beneath $f(x)$ from 2 to $t$, and is a parabolic function with maximum when $t = 6$, root at $t = 2$ and vertical intercept $A(0) = -5$.

$$\begin{aligned} A(t) &= \int_2^t 3 - \frac{x}{2}\,dx \\ &= \left[3x - \frac{x^2}{4}\right]_2^t = 3t - \frac{t^2}{4} - 5 \end{aligned}$$

**Specific behaviours**

✓ maximum turning point
✓ roots
✓ vertical intercept
✓ smooth parabolic curve

End of questions





**Section Two: Calculator-assumed** **65% (98 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

**Question 8** **(6 marks)**

A bag is filled with 26 tokens numbered with the integers $1, 2, 3, \dots, 25, 26$, but otherwise identical.

Let the random variable $X$ be the number on a token drawn at random from the bag.

(a) Explain why $X$ has a uniform distribution. (1 mark)

**Solution**

Every outcome is equally likely.

**Specific behaviours**

✓ reasonable explanation indicating equally likely outcomes

(b) Determine the expected value of $X$. (1 mark)

**Solution**

Using the symmetry of a uniform distribution, $E(X) = 13.5$

**Specific behaviours**

✓ correct value

Let the random variable $Y$ take the value 0 when $X < 20$ and the value 1 otherwise.

(c) State the particular name given to two-outcome random variables such as $Y$. (1 mark)

**Solution**

Bernoulli random variable.

**Specific behaviours**

✓ correct name

(d) Determine $P(Y = 0)$. (1 mark)

**Solution**

$P(Y = 0) = \frac{19}{26}$

**Specific behaviours**

✓ correct probability

(e) Three tokens are drawn at random from the bag with each being replaced before the next is taken. Determine the probability that exactly one of the tokens is marked with a number less than 20. (2 marks)

**Solution**

$W \sim B\left(3, \frac{19}{26}\right), \quad P(W = 1) = 0.1589$

Alternative:

$p = \frac{19}{26}\times\left(\frac{7}{26}\right)^2\times 3 = \frac{2793}{17576} = 0.1589$

**Specific behaviours**

✓ indicates correct method
✓ correct probability

**Question 9** **(9 marks)**

A particle is moving in a straight line with acceleration $a = 5e^{-0.2t}$ cm/s² after $t$ seconds. When $t = 0$ it has a displacement of 2.5 m and a velocity of $-15$ cm/s.

(a) Determine the acceleration of the particle at the instant at which it comes to rest. (4 marks)

| Solution |
|---|
| $v = \int 5e^{-0.2t}\,dt$ |
| $= -25e^{-0.2t} + c$ |
| $v(0) = -25 + c = -15 \rightarrow c = 10$ |
| $v = -25e^{-0.2t} + 10$ |
| $v = 0$ |
| $-25e^{-0.2t_1} + 10 = 0$ |
| $t_1 = 4.581$ |
| $a(t_1) = 2\text{ cm/s}^2$ |
| **Specific behaviours** |
| ✓ integrates acceleration |
| ✓ expression for velocity, including constant |
| ✓ solves for root of velocity |
| ✓ substitutes to obtain acceleration |

(b) Determine an expression for the displacement of the particle in terms of $t$. (2 marks)

| Solution |
|---|
| $x = \int -25e^{-0.2t_1} + 10\,dt$ |
| $= 125e^{-0.2t} + 10t + c$ |
| $x(0) = 125 + c = 250 \rightarrow c = 125$ |
| $x = 125e^{-0.2t} + 10t + 125$ |
| **Specific behaviours** |
| ✓ integrates velocity |
| ✓ expression for displacement, including constant |

(c) Determine the velocity of the particle when it again has a displacement of 2.5 m. (3 marks)

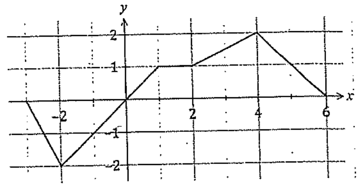
| Solution |
|---|
| $x = 250$ |
| $125e^{-0.2t_2} + 10t_2 + 125 = 250$ |
| $t_2 = 11.158$ |
| $v(t_2) = 7.32\text{ cm/s}$ |
| **Specific behaviours** |
| ✓ forms correct equation |
| ✓ solves for correct time |
| ✓ substitutes to obtain velocity |





**Question 10** **(8 marks)**

The graph of $y = f(x)$ is shown below.




Evaluate each of the following.

(a) $\int_{-3}^{4} f(x)\,dx$. (2 marks)

| Solution |
|---|
| $\int_{-3}^{4} f(x)\,dx = -3 + 4.5 = 1.5$ |
| **Specific behaviours** |
| ✓ indicates use of signed areas |
| ✓ correct value |

(b) $\int_{-2}^{-3} 4f(x)\,dx$. (2 marks)

| Solution |
|---|
| $\int_{-2}^{-3} 4f(x)\,dx = -4\int_{-3}^{-2} f(x)\,dx$ |
| $= -4(-1) = 4$ |
| **Specific behaviours** |
| ✓ adjusts integral so that LH bound < RH bound |
| ✓ correct value |

(c) $\int_{-1}^{5} (f(x) - 3)\,dx$. (2 marks)

| Solution |
|---|
| $\int_{-1}^{5} (f(x) - 3)\,dx = \int_{-1}^{5} f(x)\,dx - \int_{-1}^{5} 3\,dx$ |
| $= -0.5 + 6 - 18 = -12.5$ |
| **Specific behaviours** |
| ✓ indicates use of linearity |
| ✓ correct value |

(d) $\int_{1}^{6} f'(x)\,dx$. (2 marks)

| Solution |
|---|
| $\int_{1}^{6} f'(x)\,dx = f(6) - f(1)$ |
| $= 0 - 1 = -1$ |
| **Specific behaviours** |
| ✓ indicates use of fundamental theorem |
| ✓ correct value |





**Question 11** **(10 marks)**

A random sample of 150 households within a large town revealed that 48 households owned a cat, 60 owned a dog and 27 owned both a cat and a dog. You may assume that point estimates of probabilities derived from this sample are reliable and representative of the whole town.

(a) For households within the town, determine the probability that

(i) a randomly selected household owns neither a cat nor a dog. (2 marks)

| Solution |
|---|
| Households owning at least one cat or dog is $60 + 48 - 27 = 81$. |
| $\text{P(Neither)} = \frac{150 - 81}{150} = \frac{69}{150} = 0.46$ |
| **Specific behaviours** |
| ✓ number who own at least one cat or dog |
| ✓ correct probability |

(ii) in a random sample of 5 households, exactly 3 will not own a dog. (2 marks)

| Solution |
|---|
| $\text{P(Household does not own dog)} = (150 - 60) \div 150 = 0.6$ |
| If $X$ is number not owning dog in sample, then $X \sim B(5, 0.6)$. |
| $P(X = 3) = 0.3456$ |
| **Specific behaviours** |
| ✓ states distribution is binomial, with parameters |
| ✓ calculates probability |

(iii) in a random sample of 9 households that own a dog, at least 2 will own a cat. (3 marks)

| Solution |
|---|
| $\text{P(Household owns cat} \mid \text{owns dog)} = 27 \div 60 = 0.45$ |
| If $X$ is number owning cat in sample, then $X \sim B(9, 0.45)$. |
| $P(X \geq 2) = 0.9615$ |
| **Specific behaviours** |
| ✓ calculates conditional probability |
| ✓ states distribution is binomial, with parameters |
| ✓ calculates probability |



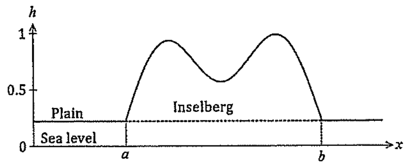


(b) If another random sample of 276 households was drawn from within the town, determine the mean and standard deviation of the probability distribution that models the number of households that own either a cat or a dog in the sample. (3 marks)

| Solution |
|---|
| $\text{P(Household owns cat or dog)} = 81 \div 150 = 0.54$ |
| If $X$ is number owning cat or dog in sample, then $X \sim B(276, 0.54)$. |
| $E(X) = 276 \times 0.54 = 149.04$ |
| $\text{sd} = \sqrt{276 \times 0.54(1 - 0.54)} = 8.28$ |
| **Specific behaviours** |
| ✓ states distribution is binomial, with parameters |
| ✓ calculates mean |
| ✓ calculates standard deviation |

**Question 12** **(11 marks)**

A vertical cross section through the highest point of an inselberg, a mountain range that rises above a surrounding level plain, is shown in the figure below.




The height of the plain and the inselberg above sea level $h$, in kilometres, is given by

$$h(x) = \begin{cases} \frac{1}{20}\left(6\cos\left(\frac{7x}{2}\right) - 6x^2 + 33x - 28\right) & a \le x \le b \\ 0.22 & \text{otherwise} \end{cases}$$

where $x$ is the horizontal displacement in kilometres from an arbitrary origin.

(a) Determine the value of $a$ and the value of $b$, the $x$ displacements where the inselberg meets the surrounding plain. (2 marks)

| Solution |
|---|
| $\frac{1}{20}\left(6\cos\left(\frac{7x}{2}\right) - 6x^2 + 33x - 28\right) = 0.22$<br>Using CAS to solve results in $a = 1.316$ and $b = 4.121$. |
| **Specific behaviours** |
| ✓ writes equation<br>✓ states both values |

(b) Use calculus to determine the cross-sectional area of the inselberg shaded in the figure above. (3 marks)

| Solution |
|---|
| $A = \int_{1.316}^{4.121} \left(\frac{1}{20}\left(6\cos\left(\frac{7x}{2}\right) - 6x^2 + 33x - 28\right) - 0.22\right) dx$<br>$= 1.435\ \text{km}^2$ |
| **Specific behaviours** |
| ✓ correct integrand<br>✓ correct bounds of integration<br>✓ correct area, with units |





(c) Use calculus to

(i) determine the maximum height of the inselberg above the surrounding plain. (4 marks)

| Solution |
|---|
| $h'(x) = \dfrac{33 - 12x - 21\sin\left(\frac{7x}{2}\right)}{20}$<br>Using CAS to solve $h'(x) = 0$ gives $x = 1.934,\ x = 2.682,\ x = 3.469$.<br>From figure shown, middle value is a minimum, so check values either side:<br>$h(1.934) = 0.934, \quad h(3.469) = 0.987$<br>Hence maximum height is 987 m above sea level, which is $987 - 220 = 767$ m above plain. |
| **Specific behaviours** |
| ✓ obtains first derivative of $h(x)$<br>✓ shows all solutions to $h'(x) = 0$<br>✓ shows reasoning for selecting root of $h'(x)$ that gives required maximum<br>✓ correct height above plain, with units |

(ii) verify that the stationary point on the curve that represents the highest part of the inselberg is a maximum. (2 marks)

| Solution |
|---|
| $h''(x) = -\frac{1}{40}\left(24 + 147\cos\left(\frac{7x}{2}\right)\right)$<br>$h''(3.469) = -3.95$<br>As the sign of the second derivative at this stationary point is negative then the curve is concave down and thus a maximum. |
| **Specific behaviours** |
| ✓ obtains second derivative as a numerical value<br>✓ uses sign of second derivative for justification |





**Question 13** **(5 marks)**

A hire company has a fleet of $n$ scooters in a city. On any given day, the probability that one of their scooters needs a repair is independent with a constant value of $p$.

The random variable $X$ is the daily number of scooters needing a repair and it has a mean of 31.68 and standard deviation 5.28.

(a) Determine the value of $n$ and the value of $p$. (3 marks)

| Solution |
|---|
| The distribution of $X \sim B(n, p)$.<br>$np = 31.68, \quad np(1-p) = 5.28^2 = 27.8784$<br>$1 - p = 27.8784 \div 31.68 = 0.88$<br>$p = 0.12, \quad n = 31.68 \div 0.12 = 264$ |
| **Specific behaviours** |
| ✓ forms equations using mean and variance of binomial distribution<br>✓ value of $p$<br>✓ value of $n$ |

(b) The daily cost to the hire company of these repairs $C$, in dollars, is also a random variable. It consists of a fixed amount of \$750 to cover workshop and labour costs plus an average of \$27.50 per scooter repaired for parts and consumables.

Determine the mean and standard deviation of the daily repair cost. (2 marks)

| Solution |
|---|
| $C = 27.5X + 750$<br>$\text{mean} = 27.5 \times 31.68 + 750 = 871.20 + 750 = \$1621.20$<br>$\text{sd} = 27.5 \times 5.28 = \$145.20$ |
| **Specific behaviours** |
| ✓ correct mean<br>✓ correct standard deviation |





**Question 14** **(8 marks)**

66 mg of a radioisotope with a half-life of 99 hours was injected into a patient before a CT scan. The mass $M$ of the radioisotope decays continuously so that $t$ hours after administration, the mass remaining is given by $M = M_0e^{-kt}$, where $M_0$ and $k$ are constants.

(a) Determine the value of the constants $M_0$ and $k$. (3 marks)

| Solution |
|---|
| $t = 0 \Rightarrow M = M_0 = 66$<br>$\frac{M}{M_0} = 0.5 = e^{-99k} \Rightarrow k = 0.007$ |
| **Specific behaviours** |
| ✓ states $M_0$<br>✓ equation for $k$<br>✓ value of $k$ |

(b) Determine the mass of the radioisotope that remains in the patient exactly 10 days after their injection. (1 mark)

| Solution |
|---|
| $t = 10 \times 24 = 240$ h, $\quad M = 66e^{-0.007 \times 240} = 12.3$ mg |
| **Specific behaviours** |
| ✓ calculates mass $M$ |

(c) Eventually, the mass of the remaining radioisotope falls to 5.5 mg.

(i) Determine how long after their injection that this occurs. (2 marks)

| Solution |
|---|
| $5.5 = 66e^{-0.007t} \Rightarrow t = 355$ h or 354.9 h |
| **Specific behaviours** |
| ✓ substitutes to form equation<br>✓ uses CAS to solve for $t$ |

(ii) Determine the rate at which the radioisotope is changing at this time. (2 marks)

| Solution |
|---|
| $\frac{dM}{dt} = -kM$<br>$= -0.007 \times 5.5 = -0.0385$ mg/h |
| **Specific behaviours** |
| ✓ uses rate of change equation<br>✓ correct rate |

**Question 15** (10 marks)

(a) Use the quotient rule to show that $\frac{d}{dx}\left(\frac{10x+5}{e^{0.2x}}\right) = \frac{9}{e^{0.2x}} - \frac{2x}{e^{0.2x}}$. (3 marks)

| Solution |
|---|
| $u = 10x+5, u' = 10, \quad v = e^{0.2x}, v' = 0.2e^{0.2x}$ <br> Using the quotient rule: <br> $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$ <br> $= \frac{10e^{0.2x} - (10x+5)0.2e^{0.2x}}{(e^{0.2x})^2}$ <br> $= \frac{10 - 0.2(10x+5)}{e^{0.2x}}$ <br> $= \frac{10 - 2x - 1}{e^{0.2x}} = \frac{9}{e^{0.2x}} - \frac{2x}{e^{0.2x}}$ |
| **Specific behaviours** |
| ✓ correct derivatives for $u, v$ <br> ✓ clearly shows use of quotient rule <br> ✓ clear simplification steps to obtain required result |

(b) Use your result from part (a) to show that $\int \frac{2x}{e^{0.2x}}dx = \frac{-10x}{e^{0.2x}} - \frac{50}{e^{0.2x}} + c$, where $c$ is a constant. (3 marks)

| Solution |
|---|
| $\frac{d}{dx}\left(\frac{10x+5}{e^{0.2x}}\right) = \frac{9}{e^{0.2x}} - \frac{2x}{e^{0.2x}}$ <br> Hence <br> $\int \frac{d}{dx}\left(\frac{10x+5}{e^{0.2x}}\right)dx = \int \frac{9}{e^{0.2x}}dx - \int \frac{2x}{e^{0.2x}}dx$ <br> $\frac{10x+5}{e^{0.2x}} = \frac{-9}{0.2e^{0.2x}} - \int \frac{2x}{e^{0.2x}}dx + c$ <br> $\int \frac{2x}{e^{0.2x}}dx = \frac{-45}{e^{0.2x}} - \frac{10x+5}{e^{0.2x}} + c$ <br> $= \frac{-10x}{e^{0.2x}} - \frac{50}{e^{0.2x}} + c$ |
| **Specific behaviours** |
| ✓ uses result from (a), wrapping integrals around terms <br> ✓ simplifies two integrals, including constant <br> ✓ rearranges for required integral and simplifies |





(c) The height $h$ of a plant, initially 35 cm, is changing at a rate given by $\frac{dh}{dt} = \frac{2t}{e^{0.2t}}$ cm/day, for $t \geq 0$.

(i) Determine an equation to model the height of the plant as a function of time and hence determine its height after 10 days. (3 marks)

| Solution |
|---|
| $h = \frac{-10t - 50}{e^{0.2t}} + c$ <br> $c = 35 - \frac{-50}{e^0} = 85$ <br> $h(t) = \frac{-10t - 50}{e^{0.2t}} + 85$ <br> $h(10) = 64.7$ cm |
| **Specific behaviours** |
| ✓ uses result from (b), changing variables <br> ✓ evaluates constant $c$ <br> ✓ correct height |

(ii) According to the model, what height will the plant never exceed? (1 mark)

| Solution |
|---|
| As $t \to \infty, h \to 85$ cm. <br> Height will not exceed 85 cm. |
| **Specific behaviours** |
| ✓ correct height |





**Question 16** (9 marks)

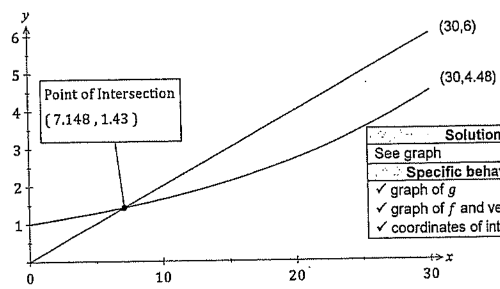
Consider the functions $f(x) = e^{0.05x}$ and $g(x) = mx$ for $x \geq 0$.

The positive constant $m$ is such that the graphs of $f$ and $g$ always intersect.

Let $R$ be the region enclosed by the $y$-axis and the graphs of $f$ and $g$.

(a) Let $m = 0.2$.

(i) Sketch the graphs of $f$ and $g$ for $0 \leq x \leq 30$, showing the coordinates of the point where they intersect on the boundary of $R$. (3 marks)




| Solution |
|---|
| See graph |
| **Specific behaviours** |
| ✓ graph of $g$ <br> ✓ graph of $f$ and vert scale <br> ✓ coordinates of intersection |

(ii) Determine the area of $R$. (2 marks)

| Solution |
|---|
| $A = \int_0^{7.148} (e^{0.05x} - 0.2x)\,dx$ <br> $= 3.483$ u$^2$ |
| **Specific behaviours** |
| ✓ correct integral <br> ✓ correct area |





(b) Determine the maximum area of $R$. (4 marks)

| Solution | Alt Solution |
|---|---|
| Maximum area when $g$ is tangential to $f$, at point $x = k$. <br> Then using $(0,0)$ and $(k, e^{0.05k})$ we get $m = \frac{e^{0.05k}}{k}$. <br> Also, $m = f'(k) \to 0.05e^{0.05k}$. <br> Hence $k = 1 \div 0.05 = 20$ and $m = 0.05e^{0.05\times 20} = 0.05e$. <br> $A_{MAX} = \int_0^{20} (e^{0.05x} - 0.05ex)\,dx$ <br> $= 10e - 20 \approx 7.183$ u$^2$ | Maximum area when $g$ is tangential to $f$, at point $x$. <br> $\begin{cases} m = 0.05e^{0.05x} \\ mx = e^{0.05x} \end{cases}$ <br> Solve simultaneously: <br> $x = 20$ and $m = 0.05e$ <br> $A_{MAX} = \int_0^{20} (e^{0.05x} - 0.05ex)\,dx$ <br> $= 10e - 20 \approx 7.183$ u$^2$ |
| **Specific behaviours** | |
| ✓ one equation relating $m$ and $k$ <br> ✓ second equation relating $m$ and $k$ <br> ✓ solves for $m$ and $k$ <br> ✓ correct maximum area | |

**Question 17** **(9 marks)**

Spinners A and B are used in a game of chance, with equally likely outcomes of $2, 3, 4, 5, 6$ for spinner A and $2, 3, 4, 5$ for spinner B after each has been spun.

A player pays \$2 for one play of the game and will win \$5 if the outcomes of spinner A and spinner B are the same, \$2 if their outcomes differ by one, and nothing otherwise.

Let $X$ be the profit (winnings minus payment) in dollars made by a player in one play of the game.

(a) Explain why $X$ is a random variable and list all possible values it can take. (2 marks)

| Solution |
|---|
| $X$ is a random variable because its value is the result of a random event or cannot be predicted. The values $X$ can take are $3, 0$ and $-2$. |
| **Specific behaviours** |
| ✓ correct values<br>✓ reasonable explanation |

(b) Determine the expected value of $X$. (4 marks)

| Solution |
|---|
| Total number of outcomes is $n_A \times n_B = 5 \times 4 = 20$. Of these, $(2,2), (3,3), (4,4), (5,5)$ are the same and $(2,3), (3,4), (4,5), (3,2), (4,3), (5,4), (6,5)$ differ by one. Hence<br>$\mathrm{P}(X = 3) = \frac{4}{20}, \quad \mathrm{P}(X = 0) = \frac{7}{20}, \quad \mathrm{P}(X = -2) = \frac{9}{20}$<br>$\mathrm{E}(X) = \frac{3 \times 4 - 0 \times 7 - 2 \times 9}{20} = -\frac{3}{10}$ |
| **Specific behaviours** |
| ✓ correct number of all possible outcomes<br>✓ one correct probability<br>✓ all correct probabilities<br>✓ correct expected value |

(c) Calculate the variance of $X$. (2 marks)

| Solution |
|---|
| $\mathrm{Var}(X) = \left(3 + \frac{3}{10}\right)^2 \times \frac{4}{20} + \left(\frac{3}{10}\right)^2 \times \frac{7}{20} + \left(-2 + \frac{3}{10}\right)^2 \times \frac{9}{20} = 3.51$ |
| **Specific behaviours** |
| ✓ indicates appropriate method<br>✓ correct variance |

(d) Determine what the cost of one play of the game should be so that in the long run, a player will break even. (1 mark)

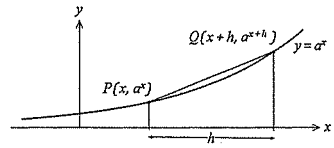
| Solution |
|---|
| Require $\mathrm{E}(X) = 0$ and so the profit per game must increase by 0.3 and hence the cost must be $2.00 - 0.30 = \$1.70$ per play. |
| **Specific behaviours** |
| ✓ correct cost per play |





**Question 18** **(6 marks)**

The graph of $y = a^x$ is shown in the diagram below, where $a$ is a positive constant.




A secant is drawn between points $P$ and $Q$ that lie on the curve with $x$-coordinates $x$ and $x + h$ respectively.

(a) Describe the property of the secant that $\frac{a^{x+h} - a^x}{h}$ represents. (1 mark)

| Solution |
|---|
| Slope/gradient/average rate of change of the secant |
| **Specific behaviours** |
| ✓ correct description |

(b) Describe the property of the curve that $\lim_{h \to 0}\left(\frac{a^{x+h} - a^x}{h}\right)$ represents. (1 mark)

| Solution |
|---|
| Slope/gradient/instantaneous rate of change of the curve at $P$. |
| **Specific behaviours** |
| ✓ correct description |

It can be shown that $\lim_{h \to 0}\left(\frac{a^{x+h} - a^x}{h}\right) = a^x \lim_{h \to 0}\left(\frac{a^h - 1}{h}\right)$.

(c) Complete the following table when $a = 4$, rounding values to 4 decimal places, and explain how the values can be used to obtain an approximation for the first derivative of $4^x$ with respect to $x$. (3 marks)

| $h$ | 0.01 | 0.001 | 0.0001 | 0.00001 |
|---|---|---|---|---|
| $\frac{a^h - 1}{h}$ | 1.3959 | 1.3873 | 1.3864 | 1.3863 |

| Solution |
|---|
| The table shows that $\lim_{h \to 0}\left(\frac{a^h - 1}{h}\right) \to 1.3863$ and so $\frac{d}{dx}(4^x) = 1.3863(4)^x$. |
| **Specific behaviours** |
| ✓ one correct value ✓ all correct values ✓ correct explanation |

(d) For what value of $a$ does $\lim_{h \to 0}\left(\frac{a^h - 1}{h}\right) = 1$? (1 mark)

| Solution |
|---|
| $a = e$<br>*(Euler's number)* |
| **Specific behaviours** |
| ✓ correct value |





**Question 19** **(7 marks)**

The values of the polynomial functions $f, g$ and $h$ and some of their derivatives are shown in the table below.

| $x$ | $f(x)$ | $g(x)$ | $h(x)$ | $f'(x)$ | $g'(x)$ | $h'(x)$ |
|---|---|---|---|---|---|---|
| 1 | 9 | −2 | 3 | −3 | −2 | 4 |
| 2 | 8 | −3 | 8 | 0 | 0 | 6 |
| 3 | 7 | −2 | 15 | −3 | 2 | 8 |

(a) Given that $g''(2) = 2$, describe the graph of $y = g(x)$ near $x = 2$. Justify your answer. (2 marks)

| Solution |
|---|
| There is a stationary point that is a minimum because $g'(2) = 0$ and $g''(2) > 0$. |
| **Specific behaviours** |
| ✓ correct description<br>✓ justification |

(b) Evaluate the derivative of $g(x) \cdot h(x)$ at $x = 3$. (2 marks)

| Solution |
|---|
| $\frac{d}{dx}(g(x) \cdot h(x))_{x=3} = g'(3) \cdot h(3) + g(3) \cdot h'(3)$<br>$= 2 \times 15 + (-2) \times 8 = 14$ |
| **Specific behaviours** |
| ✓ correct use of product rule<br>✓ correct value |

(c) Evaluate the derivative of $\frac{g(h(x))}{f(x)}$ at $x = 1$. (3 marks)

| Solution |
|---|
| $\frac{d}{dx}\left(\frac{g(h(x))}{f(x)}\right) = \frac{\frac{d}{dx}\left(g(h(x))\right) \cdot f(x) - g(h(x)) \cdot f'(x)}{[f(x)]^2}$<br>$= \frac{h'(x) \cdot g'(h(x)) \cdot f(x) - g(h(x)) \cdot f'(x)}{[f(x)]^2}$<br>$\frac{d}{dx}\left(\frac{g(h(x))}{f(x)}\right)_{x=1} = \frac{h'(1) \cdot g'(h(1)) \cdot f(1) - g(h(1)) \cdot f'(1)}{[f(1)]^2}$<br>$= \frac{4 \times g'(3) \times 9 - g(3) \times (-3)}{9^2}$<br>$= \frac{4 \times 2 \times 9 - (-2) \times (-3)}{81}$<br>$= \frac{66}{81} = \frac{22}{27} \approx 0.8148$ |
| **Specific behaviours** |
| ✓ correct use of quotient rule - second line<br>✓ correct substitution - fourth line<br>✓ obtains correct value |

**End of questions**