2021 Methods 12 Sem 1 Exam

Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine $\frac{d}{dx}(\cos^6(x))$. (2 marks)

(b) Evaluate $f'(\pi)$ when $f(x) = \frac{x + \sin 2x}{\cos x}$. (4 marks)

Question 2 **(5 marks)**

A small body is initially at the origin. It is moving along the x-axis with velocity at time t seconds given by

$$v(t) = \left(\frac{t}{3} - 2\right)^3 \text{ cm/s}.$$

(a) Determine $x(t)$, a function for the displacement of the body at time t. (3 marks)

The small body is stationary when $t = T$.

(b) Determine the displacement of the body at $T + 3$ seconds. (2 marks)

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Question 3 **(8 marks)**

(a) State two key characteristics of a chance experiment that make it suitable for modelling by a binomial random variable. (2 marks)

Research has shown that 10% of dogs between the ages of 5 and 8 have some form of heart disease. A random sample of 70 dogs is selected from a large number of dogs of this age. Let X be the number of dogs in the sample with some form of heart disease.

(b) Explain why randomly selecting one dog and recording whether it has some form of heart disease is a Bernoulli trial. (2 marks)

(c) Write a numerical expression for the probability that 8 dogs in the sample have some form of heart disease. (2 marks)

(d) State the mean and variance of X. (2 marks)

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Question 4 **(6 marks)**

Determine the area of the finite region bounded by $y = \sqrt{3x}$ and $y = \frac{x}{2}$.

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Question 5 **(5 marks)**

(a) Determine $\frac{d}{dx}(3x \cdot \sqrt[3]{e^x})$. (2 marks)

(b) Hence, or otherwise, determine $\int (3x \cdot \sqrt[3]{e^x})\, dx$. (3 marks)

Question 6 **(7 marks)**

A four-sided die has faces marked with the numbers $1, 1, 2$ and 3. All faces have an equal chance of landing face down after the die is rolled. A game, that costs \$2 to play, involves throwing the die twice and adding the two numbers that land face down. If the total score is 6, the player wins \$30, and otherwise they win nothing.

Let X be the total score obtained in one play of the game.

(a) Construct a probability distribution table for X. (3 marks)

(b) Determine $E(X)$. (1 mark)

Let Y be the net monetary **loss**, in dollars, of a player in **two** plays of the game.

(c) Determine $E(Y)$. (3 marks)

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Question 7 **(8 marks)**

The function f is defined by $f(x) = \dfrac{6}{x^2+9}$, so that $f''(x) = \dfrac{36(x^2-3)}{(x^2+9)^3}$.

(a) Describe the concavity of the graph of $y = f(x)$. (4 marks)

(b) Determine, with justification, the range of $f'(x)$. (4 marks)

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Question 8 **(7 marks)**

The following table shows the probability distribution for the random variable T.

t	0	1
$P(T=t)$	$\dfrac{11}{10}-\dfrac{2}{5k}$	$\dfrac{k}{4}+\dfrac{1}{5}$

(a) Determine the value of the positive constant k and hence state $P(T=1)$. (4 marks)

The random variable $W = 5T - 4$.

(b) Determine $E(W)$ and $\mathrm{Var}(W)$. (3 marks)

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Section Two: Calculator-assumed **65% (98 Marks)**

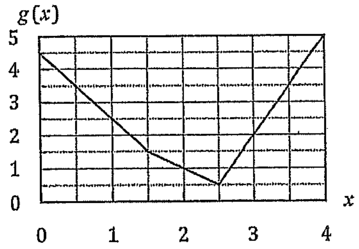
This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(8 marks)**

The graph of function g, and a table of values for function f and its derivatives are shown below.

x	1	2	3
$f(x)$	1	3	2
$f'(x)$	4	2	1
$f''(x)$	2	-1	-2



(a) Evaluate $h'(k)$ when

(i) $h(x) = f(g(x))$ and $k = 2$. (3 marks)

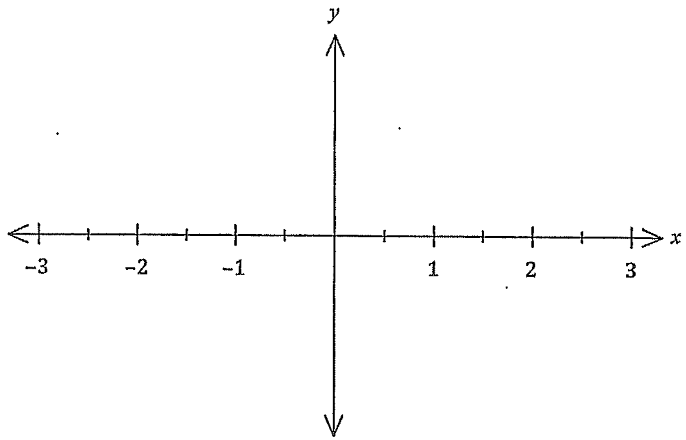
(ii) $h(x) = g(x) \div f(x)$ and $k = 3$. (3 marks)

(b) Evaluate $h''(1)$ when $h'(x) = f'(x) \times g'(x)$. (2 marks)

Question 10 **(8 marks)**

Let $f(x) = 2x^4 + ax^2 + 1$.

(a) Sketch the graph of $y = f(x)$ when $a = -16$, labelling all stationary points and intercepts. (4 marks)



(b) Show that the graph of $y = f(x)$ will always have a maximum turning point at $x = 0$ if $a < 0$. (4 marks)

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Question 11 **(7 marks)**

(a) List A contains the digits in the first 100 decimal places of π. The relative frequencies of the digits are:

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	0.08	0.08	0.12	0.11	0.10	0.08	0.09	0.08	0.12	0.14

Determine the probability that a randomly selected digit from list A

(i) is odd. (1 mark)

(ii) is a factor of 12, given that it is not odd. (2 marks)

(b) The discrete random variable X is defined by

$$P(X=x)=\begin{cases} 1/7 & x=0,1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

(i) State the name of this type of distribution. (1 mark)

(ii) Calculate the expected value and variance of X. (3 marks)

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Question 12 **(7 marks)**

A hot potato was removed from an oven and placed on a cooling rack. Its temperature T, in degrees Celsius, t minutes after being removed from the oven was modelled by

$$T=17+195e^{kt}.$$

The temperature of the potato halved between $t=0$ and $t=7.4$.

(a) Determine the value of the constant k. (3 marks)

(b) The temperature of the potato eventually reached a steady state, i.e. approaches a constant temperature. Determine the time taken for its temperature to first fall to within 3 °C of this steady state. (2 marks)

(c) Determine the time at which the potato was cooling at a rate of 3 °C per minute. (2 marks)

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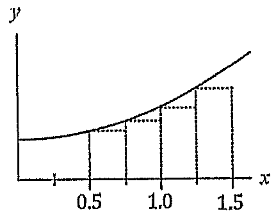
Question 13 **(8 marks)**

The graph of $y = f(x)$ is shown at right with 4 equal width inscribed rectangles. An estimate for the area under the curve between $x = 0.5$ and $x = 1.5$ is required.



The function f is defined as $f(x) = 2x^2 + 7$ and let the area sum of the 4 rectangles be S_4.

S_n, the area estimate using n inscribed rectangles can be calculated using

$$S_n = \sum_{i=1}^{i=n} f(x_i)\delta x$$

(a) State the values of x_1, x_2, x_3, x_4 and δx that should be used to determine S_4. (1 mark)

(b) Calculate the value of S_4. (3 marks)

(c) Explain, with reasons, how the value of δx and the area estimate S_n will change as the number of inscribed rectangles increase. (2 marks)

(d) Determine the limiting value of S_n as $n \to \infty$. (2 marks)

Question 14 **(8 marks)**

The area A of a regular polygon with n sides of length x is given by

$$A = \frac{n x^2 \cos\left(\frac{\pi}{n}\right)}{4 \sin\left(\frac{\pi}{n}\right)}$$

(a) Simplify the above formula when $n = 6$ to obtain a function for the area of a regular hexagon. (2 marks)

(b) Use the increments formula to estimate the change in area of a regular hexagon when its side length increases from 10 cm to 10.5 cm. (3 marks)

(c) Use the increments formula to estimate the change in area of a regular polygon with sides of length 10 cm when its number of sides increases from 29 to 31. (3 marks)

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Question 15 **(8 marks)**

(a) It is known that 17% of a large number of smoke alarms in a complex of buildings are faulty. If an electrician randomly selects 8 alarms for inspection, determine

(i) the probability that none of the alarms will be faulty. (2 marks)

(ii) the probability that more than three alarms are faulty, given that at least one is faulty. (2 marks)

(iii) the standard deviation of the distribution of the number of faulty alarms. (1 mark)

(b) In a newer complex that also has a large number of smoke alarms, only 7% are faulty. Determine, with reasoning, the minimum number of alarms that should be inspected so that the probability that at least one of them will be faulty is more than 99%. (3 marks)

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Question 16 **(8 marks)**

The volume, V litres, of fuel in a tank is reduced between $t = 0$ and $t = 48$ minutes so that

$$\frac{dV}{dt} = -175\pi \sin\left(\frac{\pi t}{48}\right)$$

(a) Determine, to the nearest litre, the amount of fuel emptied from the tank

(i) in the first minute. (3 marks)

(ii) in the last 7 minutes. (1 mark)

The tank initially held 18 600 litres of fuel.

(b) Determine the volume of fuel in the tank 5 minutes after the volume in the tank reached 12 000 litres. (4 marks)

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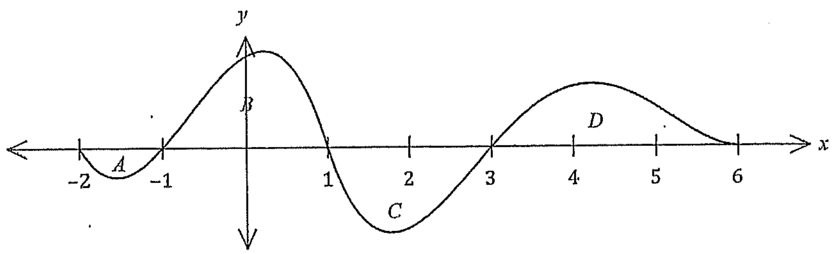
Question 17 **(7 marks)**

Regions A, B, C and D bounded by the curve $y = f(x)$ and the x-axis are shown on this graph:



The areas of A, B, C and D are $7, 25, 19$ and 17 square units respectively.

(a) Determine the value of

(i) $\displaystyle\int_{-2}^{1} f(x)\,dx$. (1 mark)

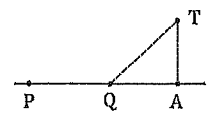
(ii) $\displaystyle\int_{1}^{6} 7f(x)\,dx$. (2 marks)

(iii) $\displaystyle\int_{-2}^{3} (3 - f(x))\,dx$. (2 marks)

(b) Explain why $\displaystyle\int_{-1}^{3} f'(x)\,dx = 0$. (2 marks)

Question 18 **(8 marks)**

An offshore wind turbine T lies 9 km away from the nearest point A on a straight coast. It must be connected to a power storage facility P that lies on the coast 40 km away from A.



Engineers will lay the cable in two straight sections, from T to Q, where Q is a point on the coast x km from A, and then from Q to P.

The cost of installing cable along the coastline is \$4000 per km and offshore is \$5000 per km.

(a) Determine, to the nearest hundred dollars, the cost of installing the cable when Q lies midway from A to P. (2 marks)

(b) Show that C, the cost in thousands of dollars, to run the cable from T to Q to P, is given by $C = 5\sqrt{x^2 + 81} - 4x + 160$. (2 marks)

(c) Use calculus techniques to determine, with justification, the minimum cost of laying the cable from T to Q to P. (4 marks)

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Question 19 **(8 marks)**

When an electronic device is run, it randomly generates one of the first four triangle numbers. The discrete random variable X is the number generated in one run of the device and the table below shows its probability distribution.

x	1	3	6	10
$P(X = x)$	a	b	0.2	0.3

The mean of X is 5.6.

(a) Determine the value of the constant a and the value of the constant b. (3 marks)

(b) The electronic device is run 3 times. Determine the probability that

(i) the number 6 will be generated exactly twice. (2 marks)

(ii) the sum of the numbers generated is at least 23. (3 marks)

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Question 20 **(5 marks)**

(a) Determine the value of the constant a and the value of the constant b that make each of the following statements true, given that $f(x)$ is a polynomial:

(i) $\int_{-4}^{-2} f(x)\,dx + \int_{-2}^{a} f(x)\,dx = \int_{b}^{1} f(x)\,dx.$ (1 mark)

(ii) $\int_{-2}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx - \int_{a}^{b} f(x)\,dx = \int_{0}^{4} f(x)\,dx.$ (2 marks)

(b) Show that $\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x))h'(x) - f(g(x))g'(x).$ (2 marks)

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Question 21 **(8 marks)**

Small body P moves in a straight line with acceleration a cm/s^2 at time t s given by

$$a = At + B$$

Initially, P has a displacement of 8 cm relative to a fixed point O and is moving with a velocity of 4 cm/s. Three seconds later, P has a displacement of 3.8 cm and a velocity of -5.9 cm/s.

(a) Determine the value of the constant A and the value of the constant B. (6 marks)

(b) Determine the minimum velocity of P. (2 marks)

End of questions

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Supplementary page

Question number: ________

Section One: Calculator-free **35% (52 Marks)**

This section has eight questions. Answer all questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

(a) Determine $\frac{d}{dx}(\cos^6(x))$. (2 marks)

Solution
$-6\sin x\cos^5 x$
Specific behaviours
✓ indicates use of chain rule ✓ correct derivative

(b) Evaluate $f'(\pi)$ when $f(x)=\frac{x+\sin 2x}{\cos x}$. (4 marks)

Solution	Solution
$=\frac{(1+2\cos 2x)(\cos x)-(x+\sin 2x)(-\sin x)}{\cos^2 x}$ $f'(\pi)=\frac{(1+2)(-1)-0}{(-1)^2}$ $=-3$	$f(x)=\frac{x}{\cos x}+\frac{\sin 2x}{\cos x}$ $f(x)=\frac{x}{\cos x}+\frac{2\sin x\cos x}{\cos x}$ $f(x)=\frac{x}{\cos x}+2\sin x$ $f'(x)=\frac{(1)(\cos x)-(x)(-\sin x)}{\cos^2 x}+2\cos x$ $f'(\pi)=\frac{(1)(-1)-(\pi)(0)}{(-1)^2}+2(-1)$ $=-3$
Specific behaviours	**Specific behaviours**
cates use of quotient rule ect u' and v' ect derivative stitutes and simplifies	✓ indicates use of quotient rule ✓ correct u' and v' ✓ correct derivative ✓ substitutes and simplifies

Question 2 **(5 marks)**

A small body is initially at the origin. It is moving along the x-axis with velocity at time t seconds given by

$$v(t)=\left(\frac{t}{3}-2\right)^3 \text{ cm/s.}$$

(a) Determine $x(t)$, a function for the displacement of the body at time t. (3 marks)

Solution
$x(t)=\int\left(\frac{t}{3}-2\right)^3 dt$ $=\frac{3}{4}\left(\frac{t}{3}-2\right)^4+c$ $t=0\Rightarrow\frac{3}{4}(-2)^4+c=0\Rightarrow c=-12$ $x(t)=\frac{3}{4}\left(\frac{t}{3}-2\right)^4-12$
Specific behaviours
✓ correct antiderivative ✓ form equation using $t=0$ and $x=-2$ ✓ solve for c and correct displacement function

The small body is stationary when $t=T$.

(b) Determine the displacement of the body at $T+3$ seconds. (2 marks)

Solution
$\frac{T}{3}-2=0\Rightarrow T=6$ s $x(9)=\frac{3}{4}(1)^4-12$ $=-11.25$ cm
Specific behaviours
✓ correct value of T ✓ correct displacement

Question 3 **(8 marks)**

(a) State two key characteristics of a chance experiment that make it suitable for modelling by a binomial random variable. (2 marks)

Solution
1. There are a fixed number of identical and independent trials. 2. There are only two possible outcomes for each trial ('success' and 'failure'). 3. The probability of 'success' is the same in every trial.
Specific behaviours
✓ identifies one characteristic ✓ identifies second characteristic

Research has shown that 10% of dogs between the ages of 5 and 8 have some form of heart disease. A random sample of 70 dogs is selected from a large number of dogs of this age. Let X be the number of dogs in the sample with some form of heart disease.

(b) Explain why randomly selecting one dog and recording whether it has some form of heart disease is a Bernoulli trial. (2 marks)

Solution
It is a chance experiment (one dog is selected at random) with two possible outcomes (dog has some form of heart disease, or it does not).
Specific behaviours
✓ mentions two possible outcomes ✓ mentions either random or one trial or one dog

(c) Write a numerical expression for the probability that 8 dogs in the sample have some form of heart disease. (2 marks)

Solution
$X\sim B(70,0.1)$ $P(X=8)=\binom{70}{8}(0.1)^8(0.9)^{62}$
Specific behaviours
✓ indicates binomial distribution ✓ correct expression

(d) State the mean and variance of X. (2 marks)

Solution
$E(X)=70\times 0.1=7$ $\text{Var}(X)=7\times 0.9=6.3$
Specific behaviours
✓ correct mean ✓ correct variance

Question 4 **(6 marks)**

Determine the area of the finite region bounded by $y=\sqrt{3x}$ and $y=\frac{x}{2}$.

Solution
Points of intersection: $\sqrt{3x}=\frac{x}{2}$ $x^2-12x=0$ $x=0,\quad x=12$ Area: $A=\int_0^{12}\sqrt{3x}-\frac{x}{2}dx$ $=\left[\frac{2(3x)^{\frac{3}{2}}}{9}-\frac{x^2}{4}\right]_0^{12}$ $=\left[\frac{2(36)^{\frac{3}{2}}}{9}-\frac{12^2}{4}\right]-0$ $=48-36$ $=12\text{ u}^2$
Specific behaviours
✓ equates two functions ✓ points of intersection ✓ writes integral for area ✓ correct antiderivative ✓ substitutes ✓ simplifies to obtain area

Question 5 (5 marks)

(a) Determine $\frac{d}{dx}(3x \cdot \sqrt[3]{e^x})$. (2 marks)

Solution
$\frac{d}{dx}\left(3x \cdot e^{\frac{x}{3}}\right) = 3e^{\frac{x}{3}} + xe^{\frac{x}{3}}$
Specific behaviours
✓ uses product rule ✓ obtains correct result

(b) Hence, or otherwise, determine $\int (3x \cdot \sqrt[3]{e^x})\, dx$. (3 marks)

Solution
$\int \frac{d}{dx}\left(3x \cdot e^{\frac{x}{3}}\right) dx = \int 3e^{\frac{x}{3}}\, dx + \int xe^{\frac{x}{3}}\, dx$ $3xe^{\frac{x}{3}} = 9e^{\frac{x}{3}} + \int xe^{\frac{x}{3}}\, dx$ $3\int xe^{\frac{x}{3}}\, dx = \int (3x \cdot \sqrt[3]{e^x})\, dx = 9xe^{\frac{x}{3}} - 27e^{\frac{x}{3}} + c$
Specific behaviours
✓ integrates all terms of result from (a) ✓ uses fundamental theorem to simplify LHS ✓ obtains required result, with constant

Question 6 (7 marks)

A four-sided die has faces marked with the numbers 1, 1, 2 and 3. All faces have an equal chance of landing face down after the die is rolled. A game, that costs \$2 to play, involves throwing the die twice and adding the two numbers that land face down. If the total score is 6, the player wins \$30, and otherwise they win nothing.

Let X be the total score obtained in one play of the game.

(a) Construct a probability distribution table for X. (3 marks)

Solution

x	2	3	4	5	6
$P(X=x)$	$^4/_{16}$	$^4/_{16}$	$^5/_{16}$	$^2/_{16}$	$^1/_{16}$

Specific behaviours

✓ table with label x and correct x values
✓ label $P(X=x)$ and at least two correct probabilities
✓ wholly correct pd table

(b) Determine $E(X)$. (1 mark)

Solution
$E(X) = \frac{8+12+20+10+6}{16} = 3.5$
Specific behaviours
✓ correct $E(X)$

Let Y be the net monetary **loss**, in dollars, of a player in **two** plays of the game.

(c) Determine $E(Y)$. (3 marks)

Solution

t	2	-28
$P(T=t)$	$^{15}/_{16}$	$^1/_{16}$

Let T be monetary loss in one game, then $E(T) = \frac{30-28}{16} = \frac{1}{8}$.

Hence $E(Y) = 2 \times E(T) = \frac{2}{8} = \$0.25$.

Specific behaviours

✓ indicates possible losses with probabilities in one game
✓ indicates expected loss in one game
✓ calculates $E(Y)$

Question 7 (8 marks)

The function f is defined by $f(x) = \frac{6}{x^2+9}$, so that $f''(x) = \frac{36(x^2-3)}{(x^2+9)^3}$.

(a) Describe the concavity of the graph of $y = f(x)$. (4 marks)

Solution
$f''(x) = 0 \Rightarrow x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$ $x < -\sqrt{3}, f''(x) > 0$ $-\sqrt{3} < x < \sqrt{3}, f''(x) < 0$ $x > \sqrt{3}, f''(x) > 0$ f is concave up when $x < -\sqrt{3}$ and $x > \sqrt{3}$. f is concave down when $-\sqrt{3} < x < \sqrt{3}$.
Specific behaviours
✓ solves $f''(x) = 0$ ✓ indicates sign of $f''(x)$ in three intervals ✓ states domains for concave up, down ✓ uses correct inequalities in domains *(penalise ambiguous language such as between $-\sqrt{3}$ and $\sqrt{3}$, etc.)*

(b) Determine, with justification, the range of $f'(x)$. (4 marks)

Solution
$f'(x) = \frac{-12x}{(x^2+9)^2}$ As $x \to \pm\infty, f'(x) \to 0$. Minimum and maximum of $f'(x)$ will be when its derivative $f''(x) = 0$, (i.e., at points of inflection) and from part (a) this is when $x = \pm\sqrt{3}$. $f'(\pm\sqrt{3}) = \pm\frac{-12\times\sqrt{3}}{12^2} = \mp\frac{\sqrt{3}}{12}$ Hence the range is: $-\frac{\sqrt{3}}{12} \le f'(x) \le \frac{\sqrt{3}}{12}$
Specific behaviours
✓ expression for $f'(x)$ ✓ states behaviour of $f'(x)$ for $x \to \pm\infty$ ✓ location of minimum and maximum values of $f'(x)$ ✓ correct range, as inequality

Question 8 (7 marks)

The following table shows the probability distribution for the random variable T.

t	0	1
$P(T=t)$	$\frac{11}{10} - \frac{2}{5k}$	$\frac{k}{4} + \frac{1}{5}$

(a) Determine the value of the positive constant k and hence state $P(T=1)$. (4 marks)

Solution
$\frac{11}{10} - \frac{2}{5k} + \frac{k}{4} + \frac{1}{5} = 1$ $22k - 8 + 5k^2 + 4k = 20k$ $5k^2 + 6k - 8 = 0$ $(5k-4)(k+2) = 0$ $k = \frac{4}{5}$ Hence $P(T=1) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$.
Specific behaviours
✓ sums probabilities to 1 ✓ forms quadratic equal to 0 ✓ solves quadratic, states unique value of k ✓ states probability

The random variable $W = 5T - 4$.

(b) Determine $E(W)$ and $\mathrm{Var}(W)$. (3 marks)

Solution
$E(W) = 5E(T) - 4 = 5\left(\frac{2}{5}\right) - 4 = -2$ $\mathrm{Var}(T) = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$ $\mathrm{Var}(W) = 5^2 \times \mathrm{Var}(T) = 5^2 \times \frac{6}{25} = 6$
Specific behaviours
✓ $E(W)$ ✓ indicates $\mathrm{Var}(T)$ ✓ $\mathrm{Var}(W)$

Section Two: Calculator-assumed 65% (98 Marks)

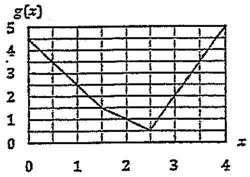
This section has thirteen questions. Answer all questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 (8 marks)

The graph of function g, and a table of values for function f and its derivatives are shown below.

x	1	2	3
$f(x)$	1	3	2
$f'(x)$	4	2	1
$f''(x)$	2	−1	−2



(a) Evaluate $h'(k)$ when

(i) $h(x) = f(g(x))$ and $k = 2$. (3 marks)

Solution
$h'(2) = f'(g(2)) \times g'(2)$ $= f'(1) \times (-1)$ $= (4)(-1) = -4$
Specific behaviours
✓ correct application of chain rule   ✓ correct values for $g(x)$ and $g'(x)$ ✓ correct value

(ii) $h(x) = g(x) \div f(x)$ and $k = 3$. (3 marks)

Solution
$h'(3) = \dfrac{g'(3)f(3) - g(3)f'(3)}{f(3)^2}$ $= \dfrac{(3)(2) - (2)(1)}{(2)^2} = 1$
Specific behaviours
✓ correct application of quotient rule   ✓ correct values for $g(x)$ and $g'(x)$ ✓ correct value

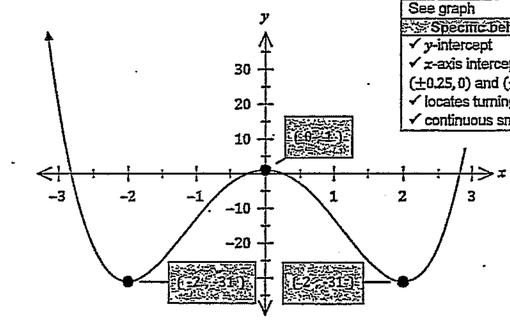
(b) Evaluate $h''(1)$ when $h'(x) = f'(x) \times g'(x)$. (2 marks)

Solution
$h''(1) = f''(1)g'(1) + f'(1)g''(1)$ $= (2)(-2) + (4)(0)$ $= -4$
Specific behaviours
✓ uses product rule with at least two correct values   ✓ correct result

Question 10 (8 marks)

Let $f(x) = 2x^4 + ax^2 + 1$.

(a) Sketch the graph of $y = f(x)$ when $a = -16$, labelling all stationary points and intercepts. (4 marks)



Solution
See graph
Specific behaviours
✓ y-intercept   ✓ x-axis intercepts   ($\pm 0.25, 0$) and ($\pm 2.82, 0$)   ✓ locates turning points   ✓ continuous smooth curve

(b) Show that the graph of $y = f(x)$ will always have a maximum turning point at $x = 0$ if $a < 0$. (4 marks)

Solution
$f'(x) = 8x^3 + 2ax$ $f'(0) = 0$ Hence curve always stationary when $x = 0$. $f''(x) = 24x^2 + 2a$ $f''(0) = 2a$ If $a < 0$ then $f''(0) < 0$ and so the curve will always be concave down. Hence a maximum at $x = 0$.
Specific behaviours
✓ determine $f'(x)$ ✓ shows $f'(0) = 0$ or states always stationary when $x = 0$ ✓ shows $f''(0) = 2a$ ✓ justifies maximum using second derivative

Question 11 (7 marks)

(a) List A contains the digits in the first 100 decimal places of π. The relative frequencies of the digits are:

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	0.08	0.08	0.12	0.11	0.10	0.08	0.09	0.08	0.12	0.14

Determine the probability that a randomly selected digit from list A

(i) is odd. (1 mark)

Solution
$P = 0.08 + 0.11 + 0.08 + 0.08 + 0.14$ $= 0.49$
Specific behaviours
✓ correct probability

(ii) is a factor of 12, given that it is not odd. (2 marks)

Solution
$P = \dfrac{0.12 + 0.1 + 0.09}{1 - 0.49} = \dfrac{31}{51} \approx 0.6078$
Specific behaviours
✓ numerator   ✓ denominator and simplifies

(b) The discrete random variable X is defined by

$$P(X = x) = \begin{cases} 1/7 & x = 0,1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

(i) State the name of this type of distribution. (1 mark)

Solution
Discrete uniform distribution
Specific behaviours
✓ states uniform distribution

(ii) Calculate the expected value and variance of X. (3 marks)

Solution
$E(X) = \dfrac{21}{7} = 3$ $\text{Var}(X) = \sum (x - \mu)^2 p(x)$ Via symmetry: $\text{Var}(X) = \dfrac{2(3^2 + 2^2 + 1^2)}{7}$ $= 4$
Specific behaviours
✓ $E(X)$ ✓ indicates calculation or use of CAS   ✓ exact variance

Question 12 (7 marks)

A hot potato was removed from an oven and placed on a cooling rack. Its temperature T, in degrees Celsius, t minutes after being removed from the oven was modelled by

$$T = 17 + 195e^{kt}.$$

The temperature of the potato halved between $t = 0$ and $t = 7.4$.

(a) Determine the value of the constant k. (3 marks)

Solution
$T_0 = 17 + 195 = 212$ $106 = 17 + 195e^{7.4k} \Rightarrow k = -0.106$
Specific behaviours
✓ indicates initial temperature   ✓ equation for temperature halving   ✓ solves for k

(b) The temperature of the potato eventually reached a steady state, i.e. approaches a constant temperature. Determine the time taken for its temperature to first fall to within 3 °C of this steady state. (2 marks)

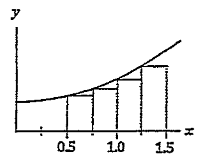
Solution
$T_\infty = 17$ $20 = 17 + 195e^{-0.106t} \Rightarrow t = 39.4$ minutes
Specific behaviours
✓ indicates steady state temperature   ✓ correct time

(c) Determine the time at which the potato was cooling at a rate of 3 °C per minute. (2 marks)

Solution
$\dfrac{dT}{dt} = -20.67e^{-0.106t}$ $-20.67e^{-0.106t} = -3 \Rightarrow t = 18.2$ minutes
Specific behaviours
✓ indicates derivative   ✓ correct time

Question 13 **(8 marks)**

The graph of $y = f(x)$ is shown at right with 4 equal width inscribed rectangles. An estimate for the area under the curve between $x = 0.5$ and $x = 1.5$ is required.



The function f is defined as $f(x) = 2x^2 + 7$ and let the area sum of the 4 rectangles be S_4.

S_n, the area estimate using n inscribed rectangles can be calculated using

$$S_n = \sum_{i=1}^{i=n} f(x_i)\delta x$$

(a) State the values of x_1, x_2, x_3, x_4 and δx that should be used to determine S_4. (1 mark)

Solution

$x_1 = 0.5, x_2 = 0.75, x_3 = 1, x_4 = 1.25, \delta x = 0.25$

Specific behaviours

✓ correctly states all values

(b) Calculate the value of S_4. (3 marks)

Solution

$$\begin{aligned} S_4 &= 0.25\left((2(0.5)^2+7)+(2(0.75)^2+7)+(2(1)^2+7)+(2(1.25)^2+7)\right) \\ &= 0.25(7.5+8.125+9+10.125) \\ &= 0.25(34.75) \\ &= \frac{139}{16} = 8.6875\ \text{u}^2 \end{aligned}$$

Specific behaviours

✓ indicates correct calculation for one rectangle
✓ correct heights of all rectangles
✓ correct value

(c) Explain, with reasons, how the value of δx and the area estimate S_n will change as the number of inscribed rectangles increase. (2 marks)

Solution

δx is the width of each rectangle and so must decrease.
S_n will increase, approaching true area under curve, as area 'lost' between curve and rectangles will decrease.

Specific behaviours

✓ indicates δx will decrease and S_n will increase
✓ reasons for both

(d) Determine the limiting value of S_n as $n \to \infty$. (2 marks)

Solution

$$S_\infty = \int_{0.5}^{1.5} f(x)\,dx = \frac{55}{6} = 9.1\dot{6}\ \text{u}^2$$

Specific behaviours

✓ correct integral
✓ correct limiting value

Question 14 **(8 marks)**

The area A of a regular polygon with n sides of length x is given by

$$A = \frac{nx^2\cos\left(\frac{\pi}{n}\right)}{4\sin\left(\frac{\pi}{n}\right)}$$

(a) Simplify the above formula when $n = 6$ to obtain a function for the area of a regular hexagon. (2 marks)

Solution

$$A = \frac{6x^2\cos\left(\frac{\pi}{6}\right)}{4\sin\left(\frac{\pi}{6}\right)} = \frac{3\sqrt{3}x^2}{2}$$

Specific behaviours

✓ correctly substitutes
✓ simplified function

(b) Use the increments formula to estimate the change in area of a regular hexagon when its side length increases from 10 cm to 10.5 cm. (3 marks)

Solution

$$\frac{dA}{dx} = 3\sqrt{3}x, \quad x = 10, \quad \delta x = 0.5$$

$$\begin{aligned} \delta A &\approx \frac{dA}{dx}\delta x \\ &\approx 3\sqrt{3}(10)(0.5) \\ &\approx 15\sqrt{3} \approx 25.98\ \text{cm}^2 \end{aligned}$$

Specific behaviours

✓ derivative of A with respect to x
✓ correct use of increments formula
✓ calculates change

(c) Use the increments formula to estimate the change in area of a regular polygon with sides of length 10 cm when its number of sides increases from 29 to 31. (3 marks)

Solution

$$\frac{dA}{dn} = -\frac{100\left(n\sin\left(\frac{2\pi}{n}\right)+2\pi\right)}{4n\cos\left(\frac{2\pi}{n}\right)-4n}, \quad n = 29, \quad \delta n = 2$$

$$= \frac{25n\sin\left(\frac{2\pi}{n}\right)+50\pi}{n-n\cos\left(\frac{2\pi}{n}\right)}$$

$$\delta A \approx \frac{dA}{dn}\delta n \approx 461.55 \times 2 \approx 923\ \text{cm}^2$$

Specific behaviours

✓ derivative of A with respect to n (CAS)
✓ indicates use of correct values of $n, \delta n, x$
✓ calculates change

Question 15 **(8 marks)**

(a) It is known that 17% of a large number of smoke alarms in a complex of buildings are faulty. If an electrician randomly selects 8 alarms for inspection, determine

(i) the probability that none of the alarms will be faulty. (2 marks)

Solution

Let X be the number of faulty alarms. Then $X \sim B(8, 0.17)$.

$$P(X = 0) = 0.2252$$

Specific behaviours

✓ defines distribution
✓ states probability

(ii) the probability that more than three alarms are faulty, given that at least one is faulty. (2 marks)

Solution

$$P(X \geq 4) = 0.0328$$

$$P(X \geq 4 | P(X \geq 1) = \frac{0.0328}{1-0.2252} = 0.0423$$

Specific behaviours

✓ indicates $P(X \geq 4)$
✓ calculates conditional probability

(iii) the standard deviation of the distribution of the number of faulty alarms. (1 mark)

Solution

$$sd = \sqrt{8 \times 0.17 \times 0.83} = 1.062$$

Specific behaviours

✓ correct value

(b) In a newer complex that also has a large number of smoke alarms, only 7% are faulty. Determine, with reasoning, the minimum number of alarms that should be inspected so that the probability that at least one of them will be faulty is more than 99%. (3 marks)

Solution

$$Y \sim B(n, 0.07)$$

$$P(Y \geq 1) \geq 0.99 \Rightarrow P(Y = 0) < 0.01$$

$$P(Y = 0) = (0.93)^n$$

$$0.93^n < 0.01$$
$$n = 64$$

Specific behaviours

✓ identifies distribution and required probability
✓ expression for no faulty alarms, in terms of n
✓ correct number

Question 16 **(8 marks)**

The volume, V litres, of fuel in a tank is reduced between $t = 0$ and $t = 48$ minutes so that

$$\frac{dV}{dt} = -175\pi\sin\left(\frac{\pi t}{48}\right)$$

(a) Determine, to the nearest litre, the amount of fuel emptied from the tank

(i) in the first minute. (3 marks)

Solution

$$\Delta V = \int_0^1 V'\,dt = -17.985$$

Hence 18 litres were emptied.

Specific behaviours

✓ writes integral for change
✓ evaluates integral
✓ answers as positive number of litres

(ii) in the last 7 minutes. (1 mark)

Solution

$$\Delta V = \int_{41}^{48} V'\,dt = -866.3$$

Hence 866 litres were emptied.

Specific behaviours

✓ correct number of litres

The tank initially held 18 600 litres of fuel.

(b) Determine the volume of fuel in the tank 5 minutes after the volume in the tank reached 12 000 litres. (4 marks)

Solution

$$\int_0^T V'\,dt = -6\,600$$
$$T = 20.70$$

$$\Delta V = \int_{20.7}^{25.7} V'\,dt = -2\,733$$

$$V(25.7) = 12\,000 - 2\,733 = 9\,267\ \text{L}$$

Specific behaviours

✓ equation for $\Delta V = -6\,600$
✓ determines T
✓ determines ΔV
✓ correct volume

Alternative Solution

$$V(t) = \int V'\,dt = 8400\cos\left(\frac{\pi t}{48}\right) + c$$

$$V(0) = 18\,600 \Rightarrow c = 10\,200$$

$$V(T) = 12\,000 \Rightarrow T = 20.70$$

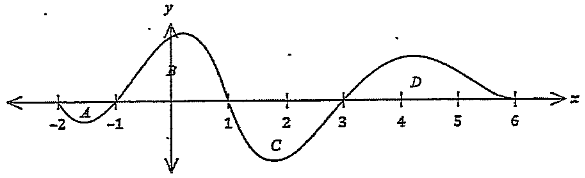
$$V(25.7) = 9\,267\ \text{L}$$

Specific behaviours

✓ antiderivative for $V(t)$
✓ determines c
✓ determines T
✓ correct volume

Question 17 **(7 marks)**

Regions A, B, C and D bounded by the curve $y = f(x)$ and the x-axis are shown on this graph:



The areas of A, B, C and D are $7, 25, 19$ and 17 square units respectively.

(a) Determine the value of

(i) $\int_{-2}^{1} f(x)\,dx.$ (1 mark)

Solution

$I = -7 + 25 = 18$

Specific behaviours

✓ correct value

(ii) $\int_{1}^{6} 7f(x)\,dx.$ (2 marks)

Solution

$I = 7(-19 + 17) = 7(-2) = -14$

Specific behaviours

✓ shows sum of signed areas
✓ uses linearity to obtain correct value

(iii) $\int_{-2}^{3} (3 - f(x))\,dx.$ (2 marks)

Solution

$I = [3(5)] - [-7 + 25 - 19]$
$I = 15 - [-1] = 16$

Specific behaviours

✓ uses linearity to obtain two integrals
✓ correct value

(b) Explain why $\int_{-1}^{3} f'(x)\,dx = 0.$ (2 marks)

Solution

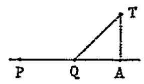
Using fundamental theorem, result is $f(3) - f(-1)$.
Since $f(-1) = f(3) = 0$, then the difference is 0.

Specific behaviours

✓ uses fundamental theorem to obtain result
✓ explains value of 0 using the two roots

Question 18 **(8 marks)**

An offshore wind turbine T lies 9 km away from the nearest point A on a straight coast. It must be connected to a power storage facility P that lies on the coast 40 km away from A.



Engineers will lay the cable in two straight sections, from T to Q, where Q is a point on the coast x km from A, and then from Q to P.

The cost of installing cable along the coastline is \$4000 per km and offshore is \$5000 per km.

(a) Determine, to the nearest hundred dollars, the cost of installing the cable when Q lies midway from A to P. (2 marks)

Solution

$C = 4000 \times 20 + 5000 \times \sqrt{20^2 + 9^2}$
$= \$189\,700$

Specific behaviours

✓ correct expression
✓ calculates cost, to the nearest hundred

(b) Show that C, the cost in thousands of dollars, to run the cable from T to Q to P, is given by $C = 5\sqrt{x^2 + 81} - 4x + 160$. (2 marks)

Solution

$C_{TQ} = 5 \times QT = 5 \times \sqrt{x^2 + 9^2}$
$C_{QP} = 4(40 - x) = 160 - 4x$

Hence

$C = C_{TQ} + C_{QP} = 5\sqrt{x^2 + 81} - 4x + 160$

Specific behaviours

✓ expression for cable from T to Q
✓ expression for cable from Q to P and shows sum

(c) Use calculus techniques to determine, with justification, the minimum cost of laying the cable from T to Q to P. (4 marks)

Solution

$C'(x) = \dfrac{5x}{\sqrt{x^2 + 81}} - 4$
$C'(x) = 0 \Rightarrow x = 12$

$C(12) = 187$

$C''(x) = \dfrac{405}{(x^2 + 81)^{\frac{3}{2}}}$
$C''(12) \approx 0.12 > 0 \Rightarrow$ minimum

Hence minimum cost is \$187 000.

Specific behaviours

✓ correct derivative
✓ solves for optimum value of x
✓ justifies minimum
✓ states minimum cost

Question 19 **(8 marks)**

When an electronic device is run, it randomly generates one of the first four triangle numbers. The discrete random variable X is the number generated in one run of the device and the table below shows its probability distribution.

x	1	3	6	10
$P(X = x)$	a	b	0.2	0.3

The mean of X is 5.6.

(a) Determine the value of the constant a and the value of the constant b. (3 marks)

Solution

Sum of probabilities:
$a + b + 0.5 = 1$
Mean:
$a + 3b + 4.2 = 5.6$
Solving simultaneously:
$a = 0.05, \quad b = 0.45$

Specific behaviours

✓ equation using sum
✓ equation using mean
✓ both correct values

(b) The electronic device is run 3 times. Determine the probability that

(i) the number 6 will be generated exactly twice. (2 marks)

Solution

$Y \sim B(3, 0.2)$
$P(Y = 2) = 0.096$

Specific behaviours

✓ indicates correct method
✓ probability

(ii) the sum of the numbers generated is at least 23. (3 marks)

Solution

Require 10, 10, 10 or 10, 10, 6 or 10, 10, 3 in any order.

$P = 0.3^3 + 3(0.3)^2(0.2) + 3(0.3)^2(0.45)$
$= \dfrac{81}{400} = 0.2025$

Specific behaviours

✓ indicates required events
✓ indicates correct probabilities for at least two events
✓ correct probability

Question 20 **(5 marks)**

(a) Determine the value of the constant a and the value of the constant b that make each of the following statements true, given that $f(x)$ is a polynomial:

(i) $\int_{-4}^{-2} f(x)\,dx + \int_{-2}^{a} f(x)\,dx = \int_{b}^{1} f(x)\,dx.$ (1 mark)

Solution

$a = 1, \quad b = -4$

Specific behaviours

✓ correct values

(ii) $\int_{-2}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx - \int_{a}^{b} f(x)\,dx = \int_{0}^{4} f(x)\,dx.$ (2 marks)

Solution

$a = -2, \quad b = 0$

Specific behaviours

✓ value of a
✓ value of b

(b) Show that $\dfrac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x))h'(x) - f(g(x))g'(x).$ (2 marks)

Solution

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = \frac{d}{dx}\left[\int_{g(x)}^{0} f(t)\,dt\right] + \frac{d}{dx}\left[\int_{0}^{h(x)} f(t)\,dt\right]$$
$$= \frac{d}{dx}\left[\int_{0}^{h(x)} f(t)\,dt\right] - \frac{d}{dx}\left[\int_{0}^{g(x)} f(t)\,dt\right]$$
$$= f(h(x))h'(x) - f(g(x))g'(x)$$

Specific behaviours

✓ uses additivity to split integral
✓ correctly uses fundamental theorem

Alternative Solution

Let $F(t)$ be an antiderivative of $f(t)$ so that $F'(t) = f(t)$.
Then

$$\int_{g(x)}^{h(x)} f(t)\,dt = [F(t)]_{g(x)}^{h(x)}$$
$$= F(h(x)) - F(g(x))$$

Hence

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = \frac{d}{dx}[F(h(x)) - F(g(x))]$$
$$= F'(h(x))h'(x) - F'(g(x))g'(x)$$
$$= f(h(x))h'(x) - f(g(x))g'(x)$$

Specific behaviours

✓ defines antiderivative and obtains definite integral
✓ correctly differentiates

Question 21 (8 marks)

Small body P moves in a straight line with acceleration a cm/s^2 at time t s given by

$$a = At + B$$

Initially, P has a displacement of 8 cm relative to a fixed point O and is moving with a velocity of 4 cm/s. Three seconds later, P has a displacement of 3.8 cm and a velocity of -5.9 cm/s.

(a) Determine the value of the constant A and the value of the constant B. (6 marks)

Solution
Velocity: $$v = \int At + B\,dt$$ $$v(t) = \frac{At^2}{2} + Bt + c$$ $$v(0) = 4 \Rightarrow c = 4$$ Displacement: $$x(t) = \int \frac{At^2}{2} + Bt + 4\,dt$$ $$x(t) = \frac{At^3}{6} + \frac{Bt^2}{2} + 4t + k$$ $$x(0) = 8 \Rightarrow k = 8$$ $$v(3) = 4.5A + 3B + 4 = -5.9$$ $$x(3) = 4.5A + 4.5B + 20 = 3.8$$ Solve: $$A = 0.6, \quad B = -4.2$$
Specific behaviours
✓ antiderivative for velocity, constant evaluated ✓ integral for displacement ✓ displacement, constant evaluated ✓ expressions for $v(3)$ and $x(3)$ ✓ value of A ✓ value of B

(b) Determine the minimum velocity of P. (2 marks)

Solution
$a = 0 \Rightarrow 0.6t - 4.2 = 0 \Rightarrow t = 7$ $v(7) = -10.7$ cm/s
Specific behaviours
✓ indicates time for minimum ✓ correct minimum velocity

End of questions

Supplementary page

Question number: ________