Neap

Trial Examination 2020

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

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Question 1 (2 marks)

Let $u = \frac{2}{x}$ and $y = \arctan(u)$.

$\frac{du}{dx} = -\frac{2}{x^2}$ and $\frac{dy}{du} = \frac{1}{1 + u^2}$

Using chain rule differentiation, $f'(x) = \left(-\frac{2}{x^2}\right)\frac{1}{\left(1 + \frac{4}{x^2}\right)}$. M1

So $f'(x) = \frac{-2}{x^2 + 4}$ $(a = -2, b = 4)$. A1

Question 2 (3 marks)

a. The conjugate root theorem cannot be applied because not all of the coefficients of the equation are real. A1

b. Let the other root be w, where $w \in C$.

$(z - (-1 + i))(z - w) = z^2 - (1 - 2i)z + 1 + 5i$

LHS $= z^2 - (-1 + i + w)z + (-1 + i)w$ M1

Comparing coefficients of z, for example, gives $-1 + i + w = 1 - 2i$.

So $w = 2 - 3i$.

The other root is $2 - 3i$. A1

Note: w can also be found by solving $1 + 5i = (-1 + i)w$.

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Question 3 (4 marks)

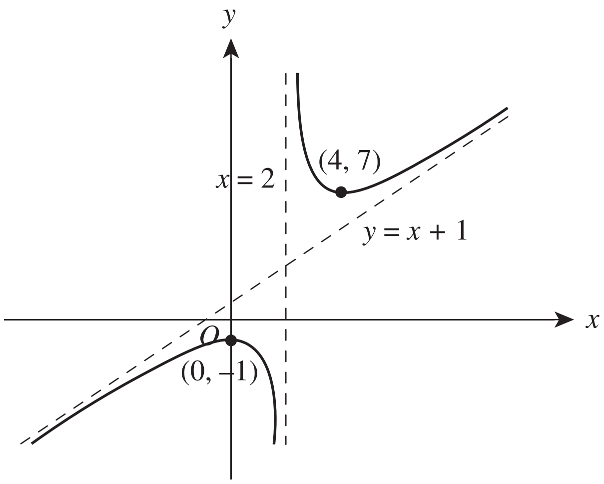
$y = x + 1 + \frac{4}{x - 2}$

The vertical asymptote is $x = 2$ and the non-vertical asymptote is $y = x + 1$.

$\frac{dy}{dx} = 1 - \frac{4}{(x-2)^2}$

$\frac{dy}{dx} = 0$ occurs for $x = 0, 4$.

The stationary points are (0, –1), which is also the y-intercept, and (4, 7).



correct shape (two branches and asymptotic behaviour) A1
correct vertical asymptote A1
correct non-vertical asymptote A1
correct stationary points A1

Question 4 (3 marks)

Method 1:

$2x + 1 = \pm x$ A1

Solving $x + 1 = 0$ and $3x + 1 = 0$ for x gives $x = -1, -\frac{1}{3}$. M1

So $-1 < x < -\frac{1}{3}$. A1

Method 2:

Either $(2x+1)^2 < x^2$ or $(2x+1)^2 = x^2$. A1

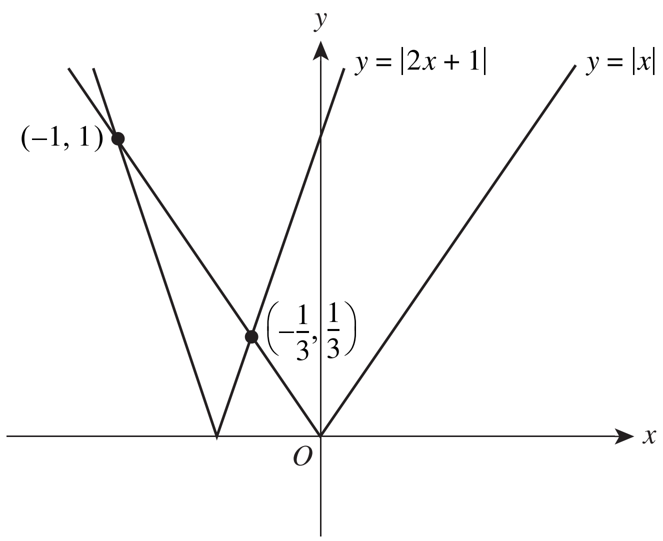
Either $3x^2 + 4x + 1 < 0$ or $3x^2 + 4x + 1 = 0$.

Either $(x+1)(3x+1) < 0$ or $(x+1)(3x+1) = 0$ gives $x = -1, -\frac{1}{3}$. M1

So $-1 < x < -\frac{1}{3}$. A1

Method 3:

Sketch the graphs of $y = |2x+1|$ and $y = |x|$. M1



The graphs intersect at $x = -1, -\frac{1}{3}$. A1

So $-1 < x < -\frac{1}{3}$. A1

Question 5 (5 marks)

a. $\overrightarrow{AB} = (t-8)\underset{\sim}{i} + (t+6)\underset{\sim}{j} + (2t-5)\underset{\sim}{k}$ M1

$$\left|\overrightarrow{AB}\right| = \sqrt{(t-8)^2 + (t+6)^2 + (2t-5)^2}$$

$$= \sqrt{(t^2 - 16t + 64) + (t^2 + 12t + 36) + (4t^2 - 20t + 25)}$$ A1

$$= \sqrt{6t^2 - 24t + 125}$$

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b. Either $\frac{d}{dt}\left|\overrightarrow{AB}\right|^2 = 12t - 24$ or $\frac{d}{dt}\left|\overrightarrow{AB}\right| = \frac{6t-12}{\sqrt{6t^2 - 24t + 125}}$. M1

Solving $12t - 24 = 0$ for t gives $t = 2$. M1

Substituting $t = 2$ into $\left|\overrightarrow{AB}\right|$ gives:

$$\left|\overrightarrow{AB}\right| = \sqrt{6(2)^2 - 24(2) + 125}$$

$$= \sqrt{101}$$ A1

The minimum distance between points A and B is $\sqrt{101}$.

Question 6 (4 marks)

Method 1:

Resolving vertically: $S\cos(\alpha) = W + T\sin(\beta)$ (1)

Resolving horizontally: $S\sin(\alpha) = T\cos(\beta)$ (2) A1

Substituting $S = \frac{T\cos(\beta)}{\sin(\alpha)}$ into (1) gives $\frac{T\cos(\beta)}{\sin(\alpha)}\cos(\alpha) = W + T\sin(\beta)$. M1

$$\frac{T\cos(\beta)}{\tan(\alpha)} = W + T\sin(\beta)$$

$$T\cos(\beta) = W\tan(\alpha) + T\sin(\beta)\tan(\alpha)$$ M1

$$T\cos(\beta) - T\sin(\beta)\tan(\alpha) = W\tan(\alpha)$$

$$T(\cos(\beta) - \sin(\beta)\tan(\alpha)) = W\tan(\alpha)$$ A1

So $T = \frac{W\tan(\alpha)}{\cos(\beta) - \sin(\beta)\tan(\alpha)}$.

Method 2:

Use of the sine rule (Lami's theorem) gives $\frac{W}{\sin(90° + \alpha + \beta)} = \frac{T}{\sin(180° - \alpha)}$. A1

$$\frac{W}{\cos(\alpha + \beta)} = \frac{T}{\sin(\alpha)}$$

$$T\cos(\alpha + \beta) = W\sin(\alpha) \Rightarrow T = \frac{W\sin(\alpha)}{\cos(\alpha + \beta)}$$ M1

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$T = \frac{\dfrac{W\sin(\alpha)}{\cos(\alpha)}}{\dfrac{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}{\cos(\alpha)}}$$ M1 A1

$$= \frac{W\tan(\alpha)}{\cos(\beta) - \sin(\beta)\tan(\alpha)}$$

Question 7 (4 marks)

Differentiating implicitly with respect to x gives $2y\frac{dy}{dx} = 2 - 2(x + y)\left(1 + \frac{dy}{dx}\right)$. M1

Tangents parallel to the x-axis satisfy the condition $\frac{dy}{dx} = 0$. That is, $0 = 2 - 2(x + y)$.

So this condition satisfies $y = 1 - x$ (or equivalent). A1

Substituting $y = 1 - x$ (or equivalent) into $y^2 = 2x - (x + y)^2$ gives $(1 - x)^2 = 2x - (x + (1 - x))^2$.

Expanding gives $1 - 2x + x^2 = 2x - 1$.

$x^2 - 4x + 2 = 0$ M1

Solving $x^2 - 4x + 2 = 0$ for x (quadratic formula or completing the square) gives $x = 2 \pm \sqrt{2}$.

For example, $x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(2)}}{2} = 2 \pm \sqrt{2}$.

Substituting $x = 2 \pm \sqrt{2}$ into $y = 1 - x$ gives $y = -1 \mp \sqrt{2}$.

So the equations of the tangents are $y = -1 + \sqrt{2}$, $y = -1 - \sqrt{2}$. A1

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Question 8 (4 marks)

Let the length be L, where $L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

$\frac{dx}{dt} = 2t - \frac{2}{t}$ and $\frac{dy}{dt} = 4$

$$L = \int_1^2 \sqrt{\left(2t - \frac{2}{t}\right)^2 + (4)^2}\, dt \qquad \text{M1}$$

$$= \int_1^2 \sqrt{\left(4t^2 + \frac{4}{t^2} + 8\right)}\, dt$$

$$= 2\int_1^2 \sqrt{\left(t + \frac{1}{t}\right)^2}\, dt$$

$$= 2\int_1^2 \left(t + \frac{1}{t}\right) dt \qquad \text{A1}$$

$$= 2\left[\frac{t^2}{2} + \log_e(t)\right]_1^2 \qquad \text{M1}$$

$$= 2\left(2 + \log_e(2) - \frac{1}{2}\right)$$

$$= 3 + 2\log_e(2) \qquad \text{A1}$$

Question 9 (4 marks)

$$\arctan(2x) + \arctan(x) = \arctan(3)$$

$$\tan(\arctan(2x) + \arctan(x)) = \tan(\arctan(3))$$

$$\frac{\tan(\arctan(2x)) + \tan(\arctan(x))}{1 - (\tan(\arctan(2x))\tan(\arctan(x)))} = 3 \qquad \text{M1}$$

$$\frac{2x + x}{1 - 2x^2} = 3 \qquad \text{A1}$$

$3(2x^2 + x - 1) = 0 \Rightarrow 3(x + 1)(2x - 1) = 0$

Solving for x gives $x = -1, \frac{1}{2}$.

When $x = -1$, $\arctan(-2) + \arctan(-1) < 0$ and $\arctan(3) > 0$. A1

Hence we reject $x = -1$.

So $x = \frac{1}{2}$. A1

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Question 10 (7 marks)

a. $\underset{\sim}{r}(t) = e^t\cos(t)\underset{\sim}{i} + e^t\sin(t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = (e^t\cos(t) - e^t\sin(t))\underset{\sim}{i} + (e^t\cos(t) + e^t\sin(t))\underset{\sim}{j}$ M1

$$|\underset{\sim}{r}(t)| = e^t\sqrt{\cos^2(t) + \sin^2(t)}$$
$$= e^t$$

$$|\dot{\underset{\sim}{r}}(t)| = e^t\sqrt{(\cos(t) - \sin(t))^2 + (\sin(t) + \cos(t))^2}$$
$$= e^t\sqrt{2\cos^2(t) + 2\sin^2(t)}$$
$$= \sqrt{2}e^t$$ A1

Attempt to find $\underset{\sim}{r}(t) \cdot \dot{\underset{\sim}{r}}(t)$. M1

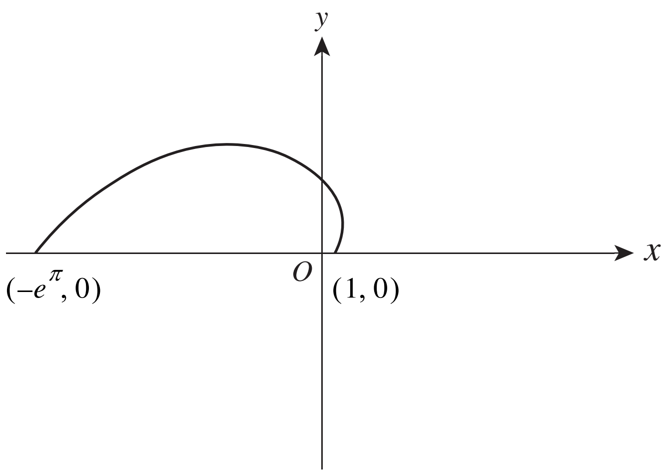
$$\underset{\sim}{r}(t) \cdot \dot{\underset{\sim}{r}}(t) = e^{2t}(\cos(t)(\cos(t) - \sin(t)) + \sin(t)(\sin(t) + \cos(t)))$$
$$= e^{2t}(\cos^2(t) + \sin^2(t))$$
$$= e^{2t}$$

Use of $\cos(\theta) = \dfrac{\underset{\sim}{r}(t) \cdot \dot{\underset{\sim}{r}}(t)}{|\underset{\sim}{r}(t)||\dot{\underset{\sim}{r}}(t)|}$ gives $\cos(\theta) = \dfrac{e^{2t}}{e^t \times \sqrt{2}e^t}$.

So $\cos(\theta) = \dfrac{1}{\sqrt{2}}$. A1

Hence, $\theta = \dfrac{\pi}{4}$ and $\underset{\sim}{r}(t)$ always makes an angle of $\dfrac{\pi}{4}$ with $\dot{\underset{\sim}{r}}(t)$.

b.



correct shape and approximately correct scale A1

correct initial position (1, 0) *and final position* $(-e^{\pi}, 0)$ A1

The particle is initially moving at an angle of $\dfrac{\pi}{4}$ above the positive x-axis. A1

Note: Students can also indicate the direction on the graph with vector and angle specified.

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