Government of **Western Australia**
School Curriculum and Standards Authority

WACE
Western Australian Certificate of Education

MATHEMATICS

3C/3D

Calculator-assumed

WACE Examination 2013

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

Section Two: Calculator assumed **(100 marks)**

Question 8 **(7 marks)**

Alex finishes work between 5 pm and 6 pm every weekday. His finishing time T, in minutes after 5 pm, is a uniformly distributed random variable:

$$T \sim U(0,60).$$

(a) What is the probability that Alex will finish work after 5.15 pm? (1 mark)

Solution
$\frac{45}{60}=\frac{3}{4}=0.75$
Specific behaviours
✓ states correct probability

(b) Determine

(i) the mean of T. (1 mark)

Solution
30
Specific behaviours
✓ states correct mean (5.30 pm not acceptable)

(ii) $P(T=55)$. (1 mark)

Solution
Since T is continuous, $P(T=55)=0$
Specific behaviours
✓ states correct probability

(iii) $P(T>55 \mid T>40)$. (2 marks)

Solution
$\frac{5}{20}=\frac{1}{4}=0.25$
Specific behaviours
✓ determines correct denominator, reflecting conditional probability ✓ determines correct numerator

(iv) the value of t for which $P(T>t)=P(T<2t)$. (2 marks)

Solution
$\frac{60-t}{60}=\frac{2t}{60}$ $t=20$
Specific behaviours
✓ determines a correct equation in t ✓ correctly solves for t

Question 9 **(12 marks)**

A mining company has two sources of gold ore: mine A and mine B.

- 2 grams of gold can be extracted from each tonne of ore from mine A.
- 3 grams of gold can be extracted from each tonne of ore from mine B.

Ore from both sources is processed at a single processing plant.

- In order to keep the plant running, a total of at least 3 tonnes of ore must be processed each hour.
- Staffing constraints at the mines determine that the amount of ore processed from mine B cannot exceed twice the amount of ore processed from mine A.
- Ore from mine A costs $20 per tonne to process, and ore from mine B costs $10 per tonne to process. Processing costs must be kept to no more than $80 per hour.

Let $x=$ the number of tonnes of ore per hour processed from mine A,

and $y=$ the number of tonnes of ore per hour processed from mine B.

The following four constraints can be obtained from the information above.

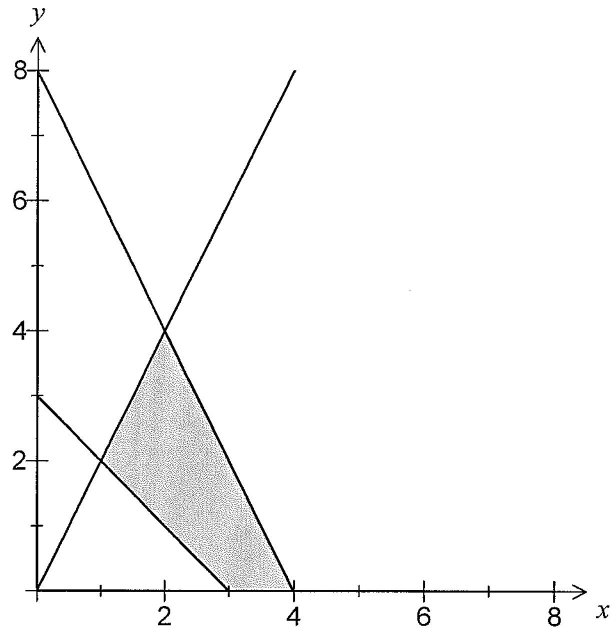
$$x \geq 0 \qquad y \geq 0 \qquad x+y \geq 3 \qquad 20x+10y \leq 80$$

(a) State the final constraint that applies to this situation. (2 marks)

Solution
$y \leq 2x$
Specific behaviours
✓ identifies correct relationship between y and x ✓ uses correct inequality symbol

(b) Draw all the constraints on the axes below and indicate the feasible region. (4 marks)

Solution



Specific behaviours	
✓	correctly draws line $x + y = 3$
✓	correctly draws line $y = 2x$
✓	correctly draws line $20x + 10y = 80$
✓	correctly shades region satisfying all inequalities

(c) The company wants to maximise the total weight of gold extracted. How many tonnes of ore from each mine should be processed each hour? (4 marks)

Solution

Point	$2x + 3y$ grams of gold
(1,2)	8
(2,4)	16
(3,0)	6
(4,0)	8

Hence the maximum amount of gold is extracted by processing 2 tonnes from mine A and 4 tonnes from mine B each hour.

Specific behaviours	
✓	states objective function
✓	identifies extreme points – at least (2, 4) and (4, 0)
✓	examines extreme points – at least (2, 4) and (4, 0)
✓	states optimal solution

(d) A new technique enables k grams of gold to be extracted from each tonne of ore from mine A, where $k > 2$.

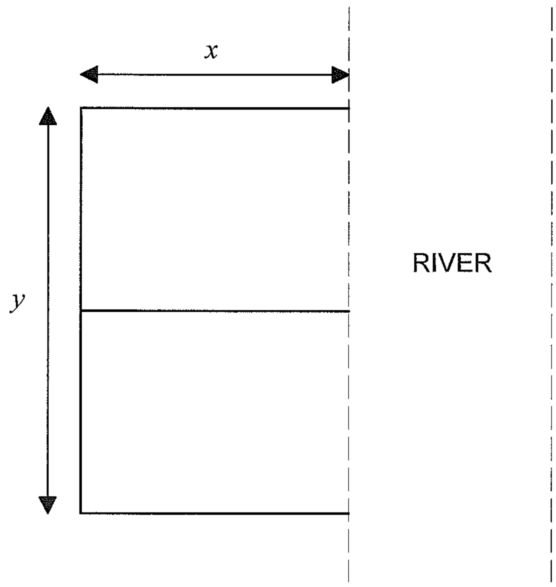
What is the smallest value of k that would enable the maximum weight of gold to be extracted without processing any ore from mine B? (2 marks)

Solution
Need to change optimal point from (2, 4) to (4, 0) $2k + 3(4) = 4k + 3(0)$ $2k = 12$ $k = 6$ The smallest value of k is 6
Specific behaviours
✓ uses k in a modified objective function for mine A ✓ solves for k correctly

Question 10 **(7 marks)**

A farmer has $1500 available to build an E-shaped fence along a straight river so as to create two identical rectangular pastures.

The materials for the side parallel to the river cost $6 per metre and the materials for the three sides perpendicular to the river cost $5 per metre.



Each of the sides perpendicular to the river is x metres long, and the side parallel to the river is y metres long.

(a) Assuming that the farmer spends the entire \$1500, show that the total area $A(x)$ of the two pastures, in square metres, is $A(x)=\frac{5}{2}\left(100x-x^2\right)$. (3 marks)

Solution
Total cost = Cost of side parallel to river + Cost of sides perpendicular to river
$1500=6y+5(3x)$
$1500=6y+15x$
$y=\frac{1500-15x}{6}=\frac{5}{2}(100-x)$
Area $=xy=\frac{5}{2}\left(100x-x^2\right)$
Specific behaviours
✓ determines an equation representing cost
✓ rearranges to make y the subject
✓ substitutes into the equation for area

(b) Use calculus methods to determine the dimensions of the fence which maximise the total area, and state this area. (4 marks)

Solution
$\frac{dA}{dx}=\frac{5}{2}(100-2x)$
$\frac{dA}{dx}=0 \Rightarrow x=50$
$y=\frac{5}{2}(100-50)=125$
∴ The fence will be 50 m wide and 125 m long.
$A=6250$ m^2
Specific behaviours
✓ differentiates correctly
✓ solves for x
✓ substitutes to find y
✓ determines maximum area

Question 11 **(13 marks)**

The size of a population of birds is changing according to the rule $\frac{dP}{dt} = -0.08P$, where P is the number of birds in the population, and t is the time in years from the initial population measurement.

There are initially 1000 birds in the population.

(a) Describe the type of relationship between P and t. (2 marks)

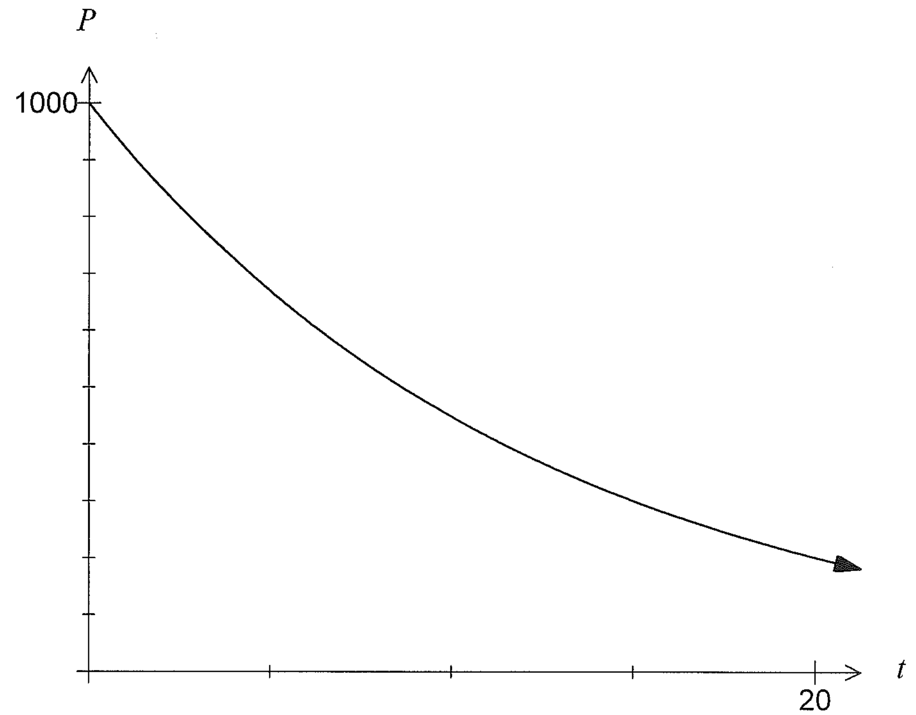
Solution	
Exponential decay	
Specific behaviours	
✓	identifies an exponential relationship
✓	states that P is decreasing over time

(b) State an equation for P in terms of t. (1 mark)

Solution	
$P = 1000e^{-0.08t}$	
Specific behaviours	
✓	states the correct equation

(c) Sketch the graph of P against t on the axes below. (3 marks)

Solution



Specific behaviours	
✓	draws graph with correct shape
✓	labels initial value
✓	shows accurate rate of decay

(d) Determine

(i) the number of birds in the population after 10 years. (1 mark)

Solution
$P(10) \approx 449$ Accept 449 – 450
Specific behaviours
✓ calculates correct population

(ii) the number of years, to one decimal place, after which there are 800 birds in the population. (3 marks)

Solution
$800 = 1000e^{-0.08t}$ $t = 2.8$
Specific behaviours
✓ establishes correct equation in t ✓ solves correctly for t ✓ rounds correctly to one decimal place

(e) What is the value of $\frac{dP}{dt}$ when $t = 10$?

Interpret this answer in terms of the bird population. (3 marks)

Solution
$\left.\frac{dP}{dt}\right\|_{t=10} = -0.08(449) = -35.9$. The population is decreasing at a rate of approximately 36 birds per year.
Specific behaviours
✓ calculates correct value of $\frac{dP}{dt}$ ✓ states that the bird population is decreasing ✓ quantifies the decrease as a rate with correct units

Question 12 **(13 marks)**

The length of barramundi is approximately normally distributed with a mean of 650 mm and a standard deviation of 100 mm. For game fishing, a barramundi must be between 550 mm and 800 mm long to be considered of legal size.

(a) What is the probability that a randomly caught barramundi is of legal size? (1 mark)

Solution	
0.7745	
Specific behaviours	
✓	calculates correct probability

(b) A fisherman catches 100 barramundi in a week. What is the expected number of legal-sized fish in his catch? (1 mark)

Solution	
77 (accept 77–78)	
Specific behaviours	
✓	calculates correct expected value

(c) What is the probability that a legal-sized barramundi is over 750 mm in length? (2 marks)

Solution	
Let X be the length of barramundi. $P(X > 750 \mid 550 < X < 800) = \frac{P(750 < X < 800)}{P(550 < X < 800)} = \frac{0.0918}{0.7745} \approx 0.12$	
Specific behaviours	
✓	calculates correct denominator
✓	calculates correct numerator

(d) Calculate the interquartile range of the barramundi population. (3 marks)

Solution
$P(X < Q3) = 0.75 \Rightarrow Q3 = 717.45$ $P(X < Q1) = 0.25 \Rightarrow Q1 = 582.55$ $IQR = Q3 - Q1 \approx 135$ Accept 134–135
Specific behaviours
✓ interprets quartiles as 25th and 75th percentiles   ✓ calculates quartiles correctly   ✓ states interquartile range correctly

(e) A fisheries researcher suspects that the length of the barramundi population may have changed over time. She intends to investigate this by taking random samples of barramundi and calculating the mean length. Assume that the standard deviation of the fish population is still 100 mm.

(i) Her first sample of 50 barramundi had a mean length of 668 mm. Use this to calculate a 90% confidence interval for the mean length of the population, and explain whether this provides strong evidence that the population mean had changed from 650 mm. (4 marks)

Solution
The mean $\overline{X}$ will have standard deviation $\frac{100}{\sqrt{50}} = 14.14$ The 90% confidence interval will be $668 \pm 1.645\left(\frac{100}{\sqrt{50}}\right) = (644.7, 691.3)$ Accept 644–645, 691–692.   This does not provide strong evidence.   650 lies within the confidence interval; and would also lie within other common confidence intervals such as 95% or 99%.
Specific behaviours
✓ uses correct z-value for a 90% confidence interval   ✓ calculates correct interval   ✓ states that there is not strong evidence to conclude that the mean had changed   ✓ explains by stating that 650 lies within the confidence interval

(ii) With her second sample, she wants to obtain a 95% confidence interval for the mean length of the barramundi population which has a width of no more than 20 mm. What sample size should she select? (2 marks)

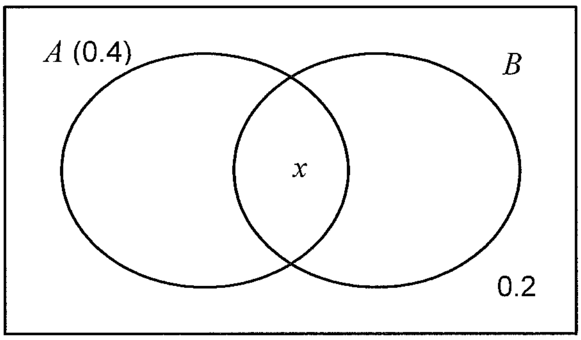
Solution
$10 = 1.96\left(\frac{100}{\sqrt{n}}\right)$ $n \approx 384$ A sample size of >384 should be selected Accept 380–400 to allow for approximate z-values from 1.95 to 2
Specific behaviours
✓ establishes correct equation ✓ correctly solves for sample size

Question 13 **(9 marks)**

For two events A and B, $P(A) = 0.4$ and $P(\overline{A \cup B}) = 0.2$.

Let $P(A \cap B) = x$.

(a) Determine an expression for $P(B)$ in terms of x. (2 marks)

Solution
 $P(B \cap \overline{A}) = 1 - 0.4 - 0.2 = 0.4$ $P(B) = 0.4 + x$
Specific behaviours
✓ represents given information correctly in diagram or equation ✓ determines correct expression for $P(B)$

(b) Determine the value of x under each of the following conditions.

(i) A and B are mutually exclusive events. (1 mark)

Solution
$x = 0$
Specific behaviours
✓ states correct probability

(ii) $P(B \mid A) = 0.25$. (2 marks)

Solution
$P(A \cap B) = P(B \mid A)P(A)$ $x = 0.25(0.4) = 0.1$
Specific behaviours
✓ uses correct equation for conditional probability ✓ calculates correct probability

(iii) $P(A \mid B) = \frac{1}{3}$. (2 marks)

Solution
$\frac{x}{0.4 + x} = \frac{1}{3}$ $x = 0.2$
Specific behaviours
✓ establishes correct equation ✓ calculates correct probability

(iv) A and B are independent events. (2 marks)

Solution
$P(A \cap B) = P(A)P(B)$ $x = 0.4(0.4 + x)$ $x = \frac{4}{15} \approx 0.267$
Specific behaviours
✓ establishes correct equation for independence ✓ calculates correct probability

Question 14 **(7 marks)**

A computer store room contains 25 computers. Of these, 20 are working and five need to be repaired. An order for five computers is received. The computer technician selects five computers at random from the store room.

(a) What is the probability that all five are working? (2 marks)

Solution
$\dfrac{\binom{20}{5}\binom{5}{0}}{\binom{25}{5}} = \dfrac{15\ 504}{53\ 130} \approx 0.29$
Specific behaviours
✓ calculates correct numerator ✓ calculates correct denominator

(b) What is the probability that more than one needs to be repaired? (3 marks)

Solution
$1 - \dfrac{\binom{20}{5}\binom{5}{0}}{\binom{25}{5}} - \dfrac{\binom{20}{4}\binom{5}{1}}{\binom{25}{5}} = \dfrac{13\ 401}{53\ 130} \approx 0.25$
Specific behaviours
✓ correctly identifies cases which enumerate the probability ✓ determines correct calculation for exactly one needing to be repaired ✓ calculates correct probability

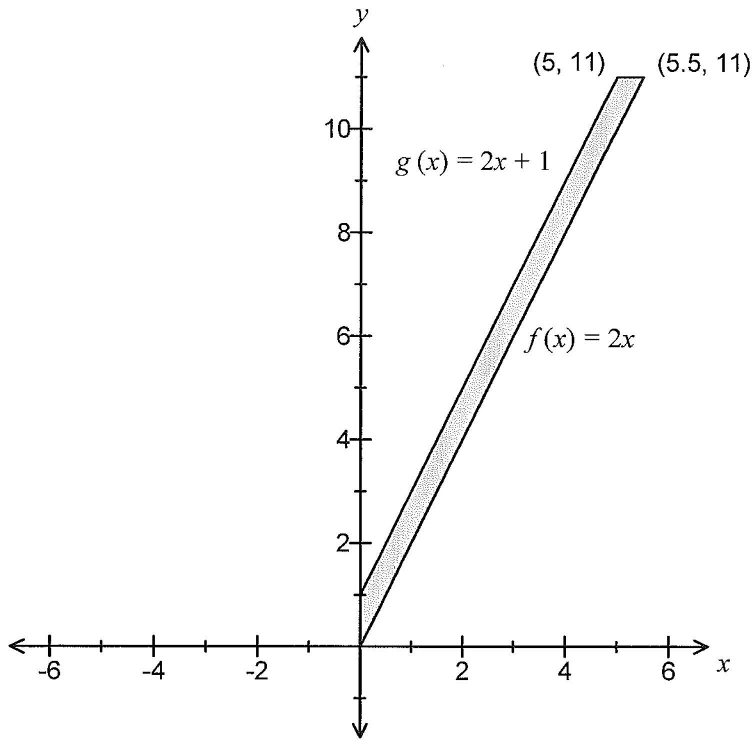
(c) The technician checks the first three of the five computers and finds that they are all working. What is the probability that all five are working? (2 marks)

Solution
If the first three are working, the task is equivalent to selecting two more working computers from 22 in total, 17 of which are working. $P = \dfrac{\binom{17}{2}\binom{5}{0}}{\binom{22}{2}} = \dfrac{136}{231} \approx 0.59$ Alternatively, $P = \dfrac{17}{22}\left(\dfrac{16}{21}\right) = \dfrac{272}{462} = \dfrac{136}{231} \approx 0.59$
Specific behaviours
✓ establishes a calculation showing correct interpretation of requirements ✓ calculates correct probability

Question 15 **(5 marks)**

A conical container is made by rotating the shaded area shown in the graph around the y-axis.

The area is bounded by the lines $x=0$, $y=11$, $f(x)=2x$ and $g(x)=2x+1$.



Use calculus methods to determine the volume of material required to construct the container.

Solution
$V=\pi\int_1^{11}\left(\frac{y}{2}\right)^2-\left(\frac{y-1}{2}\right)^2 dy+\pi\int_0^1\left(\frac{y}{2}\right)^2 dy$ $=27.5\pi+\frac{\pi}{12}$ $=86.66$ cubic units
Specific behaviours
✓ selects correct technique for rotation around the y-axis
✓ rearranges equations to obtain correct expressions for x^2
✓ correctly establishes difference between the rotations of $f(x)$ and $g(x)$
✓ calculates a volume of revolution from $y=1$ to $y=11$
✓ calculates a volume of revolution from $y=0$ to $y=1$

Question 16 **(11 marks)**

A factory makes low-cost batteries. Unfortunately, 20% of the batteries are faulty.

Batteries are sold in packets of 20.

(a) Let X be the number of faulty batteries in a packet.

(i) State the distribution of X and determine its mean and standard deviation. (3 marks)

Solution
$X \sim$ Bin (20, 0.2) Mean = 4 Standard deviation = $\sqrt{20(0.2)(0.8)} = \sqrt{3.2} \approx 1.79$
Specific behaviours
✓ states that the distribution is binomial ✓ calculates the mean ✓ calculates the standard deviation

(ii) Calculate $P(X \geq 5)$. (1 mark)

Solution
$P(X \geq 5) = 0.370$
Specific behaviours
✓ calculates correct probability

(b) A customer buys 20 packets of batteries.
Let Y be the mean number of faulty batteries per packet.
According to the Central Limit Theorem, Y will be approximately normally distributed.

(i) Determine the mean and standard deviation of Y. (2 marks)

Solution
Mean = 4 Standard deviation = $\dfrac{\sqrt{3.2}}{\sqrt{20}} = 0.4$
Specific behaviours
✓ calculates correct mean ✓ calculates correct standard deviation

(ii) Calculate $P(Y \geq 4.2)$. (1 mark)

Solution
$P(Y \geq 4.2) = 0.309$
Specific behaviours
✓ calculates correct probability

(c) In a sample of 400 batteries, what is the probability that at least 84 will be faulty? (2 marks)

Solution
Let F be the number of faulty batteries. $F \sim$ Bin (400, 0.2) $P(F \geq 84) = 0.327$
Specific behaviours
✓ selects correct distribution ✓ calculates correct probability

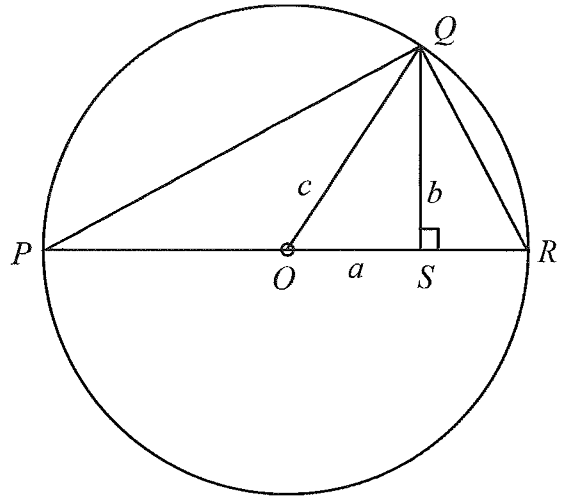
(d) Explain why the answer in part (c) should be close to the answer in part (b) (ii). (1 mark)

Solution
84 faulty batteries out of 400 is analogous to a mean of 4.2 per packet in 20 packets of 20 batteries.
Specific behaviours
✓ explains why the two calculations are analogous

(e) Explain why the answer in part (c) should **not** be exactly equal to the answer in part (b) (ii). (1 mark)

Solution
The Central Limit Theorem enables an approximate answer, not an exact answer. The sampling distribution of the mean is close to normal, but with a sample size of only 20 the approximation will be some way from exact. Alternatively, the approximation would be better if it calculated $P(Y \geq 4.175)$. This is equivalent to $P(F > 83.5)$, which accounts for the fact that the binomial distribution is discrete.
Specific behaviours
✓ identifies a reason why the calculations are only approximately equal

Question 17 **(7 marks)**



The circle above has centre O. PR is a diameter of the circle. QS is drawn perpendicular to PR.

In triangle OSQ, let $OQ = c$, $OS = a$ and $SQ = b$.

(a) Prove that triangle PSQ is similar to triangle QSR. (4 marks)

Solution	
$\angle PSQ = 90^\circ = \angle QSR$	given
$\angle SQR + \angle PQS = 90^\circ$	angle in a semi circle
$\angle QPS + \angle PQS = 90^\circ$	complementary angles in right-angled triangle PSQ
Hence $\angle SQR = \angle QPS$	
Therefore triangle PSQ is similar to triangle QSR due to AA test	
Specific behaviours	
✓ identifies common right angle	
✓ uses angle in a semi-circle to show that $\angle SQR + \angle PQS = 90^\circ$	
✓ proves that $\angle SQR = \angle QPS$	
✓ states correct condition for similarity	

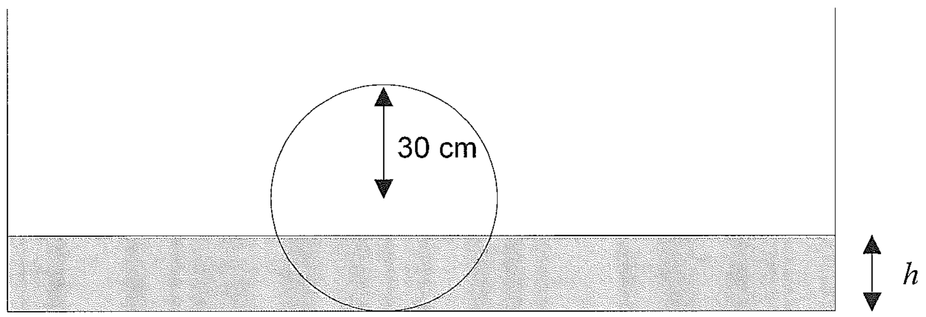
(b) Use the result from part (a) to prove the Pythagorean theorem $c^2 = a^2 + b^2$. (3 marks)

Solution
Based on similar triangles:
$\frac{PS}{QS} = \frac{QS}{SR}$
$\frac{c+a}{b} = \frac{b}{c-a}$
$c^2 - a^2 = b^2$
$c^2 = a^2 + b^2$
Specific behaviours
✓ uses common side length ratios from both triangles
✓ expresses relevant side lengths in terms of a, b and c
✓ derives the Pythagorean result $c^2 = a^2 + b^2$

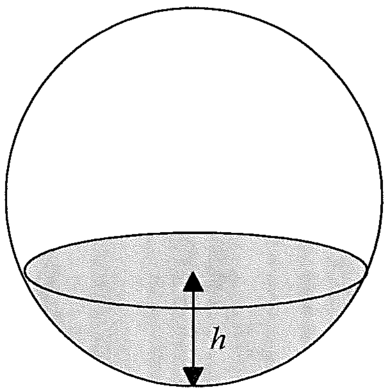
Question 18 **(9 marks)**

A sphere of radius 30 cm is sitting on the base of a container of water. The water level is rising at a rate of 2 cm per second, so that the sphere is gradually becoming submerged.

Let h be the height of the water.



The submerged portion of the sphere is as shown below.



The volume of the submerged portion is given by

$V = \frac{\pi h^2}{3}(3r - h)$, $0 \le h \le 2r$, where r is the radius of the sphere.

(a) Use this formula to calculate the volume of the entire sphere, to the nearest cubic centimetre. (1 mark)

Solution
$V = \frac{\pi\left(60^2\right)}{3}(90-60) = 36\ 000\pi = 113\ 097\text{cm}^3$
Specific behaviours
✓ calculates correct volume

(b) Find the values of h for which $\frac{dV}{dh} = 0$. (3 marks)

Solution
$V = \frac{\pi h^2}{3}(90 - h)$ $V = \pi\left(30h^2 - \frac{h^3}{3}\right)$ $\frac{dV}{dh} = \pi\left(60h - h^2\right)$ $\frac{dV}{dh} = 0 \Rightarrow h = 0 \text{ or } 60$
Specific behaviours
✓ differentiates correctly ✓ determines first value ✓ determines second value

(c) Calculate the rate at which the submerged volume of the sphere is changing at the time when the sphere is half submerged. (3 marks)

Solution
$\frac{dV}{dt} = \frac{dV}{dh}\left(\frac{dh}{dt}\right)$ $\frac{dV}{dt} = \pi\left(60(30) - 30^2\right)(2)$ $\frac{dV}{dt} = 1800\pi \approx 5655 \text{ cm}^3\text{s}^{-1}$
Specific behaviours
✓ uses chain rule correctly ✓ substitutes for $\frac{dV}{dh}$ and $\frac{dh}{dt}$ ✓ calculates correct rate, with units

(d) Consider the following conjecture:

Every 1% increase in h leads to approximately a 3% increase in V.

Using the formula $\delta V \approx \frac{dV}{dh}\delta h$, explain whether the conjecture is true or false. (2 marks)

Solution	
The conjecture is false. Counter examples include $h = 0 \ or \ 60$, for which $\frac{dV}{dh} = 0$. For these values, a 1% increase in height will therefore lead to approximately a 0% increase in volume.	
Specific behaviours	
✓	chooses a height value which will serve as a counter example
✓	demonstrates the falsity of the conjecture

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