SEMESTER TWO

MATHEMATICS
SPECIALIST
UNITS 1 & 2

2017

SOLUTIONS

Calculator−free Solutions

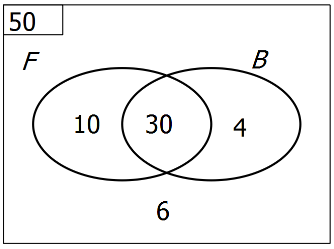
1. (a) $1000 = 310 + 650 + 440 - 170 - 150 - 180 + x$ ✔

$x = 100$ ✔

(b) $150 - 100 = 50$ ✔

(c) (i) 44 ✔

(ii)



∴ n(B) = 34 ✔✔ [6]

2. (a) (i) Substitute $z = 2\text{i}$ to get $(2\text{i})^4 - 2(2\text{i})^3 + 7(2\text{i})^2 - 8(2\text{i}) + 12$

which reduces to 0 ✔

(ii) $z = -2\text{i}$ (the conjugate) is the other root. ✔

(b) $2x^2 + 10 = 3 - 5x$ reduces to $2x^2 + 5x + 7 = 0$ ✔

∴ $x = \dfrac{-5 \pm i\sqrt{31}}{4}$ from quadratic formula ✔✔ [5]

3. (a) (i) ${}^5\mathbf{C}_2 = {}^5\mathbf{C}_3 = 10$ This statement is true. ✔✔

(ii) ${}^5\mathbf{C}_1 \neq 2 \times {}^5\mathbf{C}_0$ This statement is false ✔✔

(b) (i) ${}^5\mathbf{C}_3 = 10$ ✔✔

(ii) 2 x 4! = 48 ✔✔ [8]

4. (a) $p = 4$, $q = 0.2$ ✔✔

(b) $y = 1 - x$ becomes $y = 4[1 - 0.2x]$

i.e. $y = 4 - 0.8x$ ✔

or, if (c) is done before (b), gradient is -0.8 and intercept is 4

$\therefore \quad y = 4 - 0.8x$

(c) $A' = (5, 0)$ and $B' = (0, 4)$ ✔✔

(d) $\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$ ✔✔

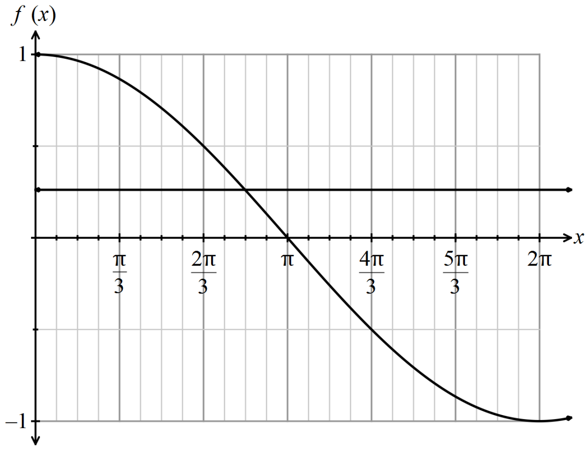
(e) $\begin{bmatrix} 0.2 & 0 \\ 0 & 0.25 \end{bmatrix}$ ✔✔

(f) Reflection across y axis

i.e. $g(x)$ becomes $-g(x)$ ✔✔

(g) $A = 0.5(ms - rn)$ ✔ [12]

5. (a)



✔✔

(b) $x \approx \frac{11\pi}{12}$ line✔

of intersection and accuracy ✔✔

(c) $\sin x$ ✔

(d) $2\cos x \cdot \sin x = \sin 2x$ ✔✔ [8]

6. (a) Let the numbers be $2k - 1$, $2k + 1$, $2k + 3$, $2k + 5$, $2k + 7$ ✔

$2k - 1 + 2k + 1 + 2k + 3 + 2k + 5 + 2k + 7$

$= 10k + 15$ ✔

Since $10k + 15 = 5(2k + 3)$ then divisible by 5. ✔

(b) Assume that $-\pi$ is rational, hence $-\pi = \frac{a}{b}$ ✔

$\therefore \quad \pi = -\frac{a}{b} = \frac{-a}{b}$ ✔

But $-a$ and b are integers, so $-\pi$ is rational. ✔

This contradicts the supposition, and

therefore by contradiction $-\pi$ must be irrational. ✔ [7]

7. For $n = 1$:

$$\frac{1 - x^1}{(1 - x)} = 1$$ $\therefore$ *True for n = 1* ✔

Assume true for $n = k$:

ie. $$1 + x + x^2 + ... x^{k-1} = \frac{1 - x^k}{(1 - x)}$$
ie ✔

Prove true for $n = k + 1$:

$$1 + x + x^2 + ... x^{(k+1)-1} = \frac{1 - x^k}{(1 - x)} + x^{(k+1)-1}$$
Proof:

$$1 + x + x^2 + ...x^k = \frac{(1 - x^k) + x^k(1 - x)}{(1 - x)}$$ ✔

$$1 + x + x^2 + ...x^k = \frac{1 - (x^{k+1})}{(1 - x)}$$ as required ✔

Therefore, True for $n = k + 1$, and since true for $n = 1$,

true for all whole numbers. ✔ [5]

Calculator−assumed Solutions

8. $wz = (2 + a\text{i})(3b + \text{i}) = 4$

$\therefore$ $6b + 2\text{i} + 3ab\text{i} - a = 4$ ✔

$\therefore$ $6b - a = 4$ and $2 + 3ab = 0$ ✔

$\therefore$ $2 + 3(6b - 4)(b) = 0$ ✔

$\therefore$ $9b^2 - 6b + 1 = 0$ ✔

$\therefore$ $b = \frac{1}{3}$ and $a = -2$ ✔ [5]

9. (a) RHS = $\dfrac{1 + \dfrac{\sin x}{\cos x}.\dfrac{\sin y}{\cos y}}{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}$ ✔

$= \dfrac{\cos x \cos y + \sin x \sin y}{\cos x \cos y} \div \dfrac{\sin x \cos y + \sin y \cos x}{\cos x \cos y}$ ✔✔

$= \dfrac{\cos(x-y)}{\sin(x+y)}$ = LHS ✔

(b) (i) $\boldsymbol{p} \cdot \boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|\cos(A - B)$ ✔

and $\boldsymbol{p} \cdot \boldsymbol{q} = \left(|\boldsymbol{p}|\cos A\, \underset{\sim}{\mathbf{i}} + |\boldsymbol{p}|\sin A\, \underset{\sim}{\mathbf{j}}\right) \cdot \left(|\boldsymbol{q}|\cos B\, \underset{\sim}{\mathbf{i}} + |\boldsymbol{q}|\sin B\, \underset{\sim}{\mathbf{j}}\right)$ ✔

$= |\boldsymbol{p}||\boldsymbol{q}|\left(\cos A\, \underset{\sim}{\mathbf{i}} + \sin A\, \underset{\sim}{\mathbf{j}}\right) \cdot \left(\cos B\, \underset{\sim}{\mathbf{i}} + \sin B\, \underset{\sim}{\mathbf{j}}\right)$ ✔

$= |\boldsymbol{p}|\,|\boldsymbol{q}|\,[\cos A \cos B + \sin A \sin B]$ ✔

$\therefore \cos(A - B) = \cos A \cos B + \sin A \sin B$

(ii) $\cos(A + B) = \cos(A - (-B)) = \cos A \cos(-B) + \sin A \sin(-B)$✔

$= \cos A \cos B - \sin A \sin B$ ✔

(iii) $\cos^2 A + [\cos(120° + A)]^2 + [\cos(120° - A)]^2$

$= \cos^2 A + [\cos 120° \cos A - \sin 120° \sin A]^2 + [\cos 120° \cos A + \sin 120° \sin A]^2$✔

$= \cos^2 A + \left[-\dfrac{\cos A}{2} - \left(\dfrac{\sqrt{3}}{2}\right)\sin A\right]^2 + \left[-\dfrac{\cos A}{2} + \left(\dfrac{\sqrt{3}}{2}\right)\sin A\right]^2$ ✔

$= 1.5\cos^2 A + 1.5\sin^2 A$ ✔

$= 1.5$ [13]

10. (a) $(3\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + 3\mathbf{b}) = 3\mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{a} + 9\mathbf{a} \cdot \mathbf{b} - 3\mathbf{b} \cdot \mathbf{b}$ ✔

$= 3|\mathbf{a}|^2 + 8\mathbf{a} \cdot \mathbf{b} - 3|\mathbf{b}|^2$ ✔

$= 3 + 8|\mathbf{a}||\mathbf{b}|\cos\theta - 3$ ✔

$= 8\cos\theta$ as required ✔

(b) $\overrightarrow{PQ} = (4,1)$ ✔

Unit vector on x axis = (1,0) ✔

Length of projection = | PQ. $\hat{x}$ | = |4| = 4 ✔ [7]

11. (a) Let the cost of a bottle of orange concentrate cost x

Let the cost of a bottle of banana concentrate cost y

$5x + 1y = 19$

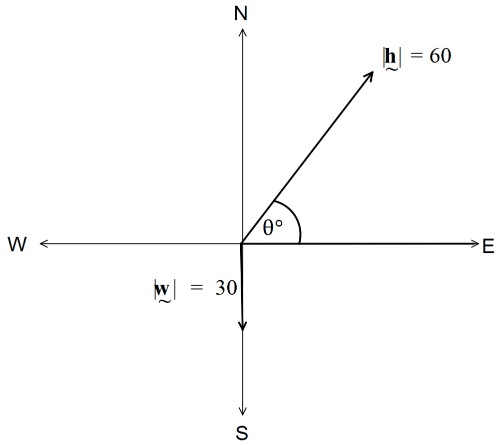
$2x + 3y = 18$ ✔

(b) $\begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 18 \end{bmatrix}$ ✔

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 19 \\ 18 \end{bmatrix}$$ ✔

$$= \frac{1}{13}\begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}\begin{pmatrix} 19 \\ 18 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$ ✔ [4]

12. (a)



✔✔

(b) $\begin{bmatrix} 0 \\ -30 \end{bmatrix} + \begin{bmatrix} 60\cos\theta \\ 60\sin\theta \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ ✔✔

$\therefore \quad 60\sin\theta = 30$ ✔

$\therefore \quad \sin\theta = \dfrac{1}{2}$

$\therefore \quad \theta = 30° \rightarrow$ Bearing is 060 °T ✔

(c) Speed in Easterly direction is $60\cos 30° = 60 \times \dfrac{\sqrt{3}}{2}$ ✔

Time taken is $\dfrac{8}{60 \times \frac{\sqrt{3}}{2}} = 9.23$ minutes ✔ [8]

13. (a) (i) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ∴ Rotation of 180° ✔✔

(ii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ∴ Rotation of 270° clockwise ✔✔

(b) $P = 6(B - 2A) \times B^{-1}$ ✔

$\therefore \quad P = \dfrac{3}{11}\begin{bmatrix} 24 & -12 \\ 8 & 18 \end{bmatrix}$ ✔✔

(c) $BA = \begin{bmatrix} -2 & 6 \\ -4 & 1 \end{bmatrix}$

$BAX = \begin{bmatrix} -2 & 6 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ -5 \end{bmatrix}$ ✔

Co−ordinates are (14, −5) ✔

(d) Det B = 22 ∴ Area = 25 x 22 = 550 ✔✔

(e) Singular matrix has det = 0 ✔

∴ Area = 0 i.e. A line ✔ [13]

14. (a) $3(2\mathbf{i} + 3\mathbf{j}) - (m\mathbf{i} - 5\mathbf{j}) = 8\mathbf{i} + 14\mathbf{j}$ ✔

$\therefore \quad (6 - m)\mathbf{i} + 14\mathbf{j} = 8\mathbf{i} + 14\mathbf{j}$

$\therefore \quad 6 - m = 8$

$\therefore \quad m = -2$ ✔

(b) $2\mathbf{i} + 3\mathbf{j} = k(m\mathbf{i} - 5\mathbf{j})$ ✔

$\therefore \quad 2 = km$ and $3 = -5k$

$\therefore \quad k = -0.6$ and by substitution, $m = -\dfrac{10}{3}$ ✔

(c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} m \\ -5 \end{pmatrix} = 0$ ✔

$\therefore \quad 2m - 15 = 0$

$\therefore \quad m = 7.5$ ✔ [6]

15. (a) $R\cos(A - \theta) = R\cos(A)\cos(\theta) + R\sin(A)\sin(\theta)$

$= -3\cos(A) + 3\sqrt{3}\sin(A)$

$\therefore R\sin(\theta) = 3\sqrt{3}$ and $R\cos(\theta) = -3$

hence, $R^2 = (-3)^2 + (3\sqrt{3})^2 = 36 \quad \therefore R = 6$ ✔✔

and $\cos(\theta) = \dfrac{-3}{6} = -\dfrac{1}{2} \quad \therefore \theta = \dfrac{2\pi}{3}$ ✔

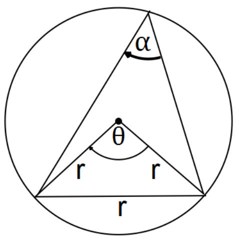
therefore, $R\cos(A - \theta) = 6\cos\left(A - \dfrac{2\pi}{3}\right)$ ✔

(b) (i) $g(x)_{min} = -6$ ✔

(ii) for $\cos\left(A - \dfrac{2\pi}{3}\right) = -1$

hence $A - \dfrac{2\pi}{3} = \pi \quad \therefore \theta = \dfrac{5\pi}{3}$ ✔✔ [7]

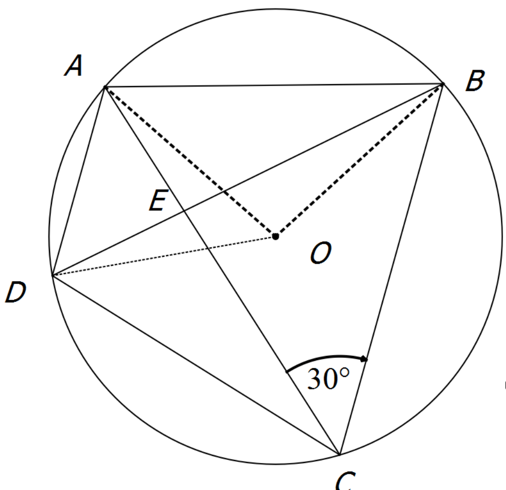
16. (a)



$\theta = 60°$ (equilateral triangle) ✔

$\therefore \alpha = 30°$ (central angle theorem)✔

(b) (i)



$AOB = 60°$ (proved in (a))

$\therefore ACB = 30°$ (theorem)

Similarly, $DBC = 30°$

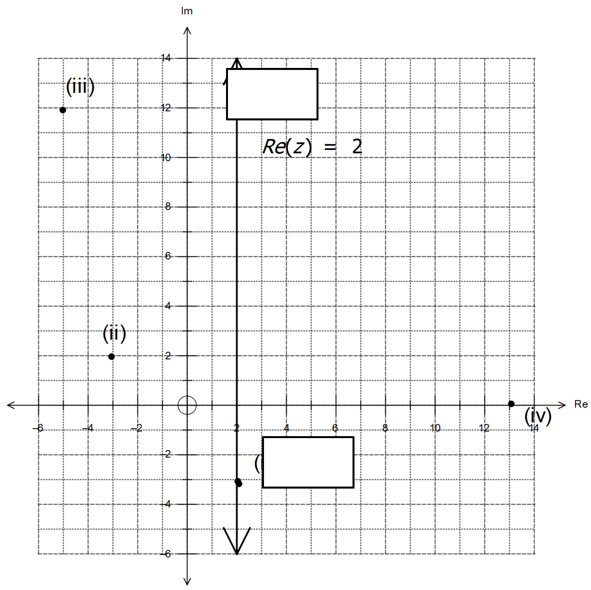
$\therefore BEC = 120°$

$\therefore AEB = 60°$ ✔✔✔

(ii) Assume E is the centre.
All angles of ΔABE = 60 $^\circ$
and all angles of ΔBEC = 60 $^\circ$
But $\angle$AEB = 2$\angle$ACB which is impossible if they are both 60 $^\circ$
$\therefore$ E is not the centre. ✔✔✔ [8]

17. $z = 2 + 3i$

(i) $2 - 3i$ ✔✔
(ii) $-3 + 2i$ ✔✔
(iii) $-5 + 12i$ ✔✔
(iv) 13✔✔



[8]

18. (a) 2m ✔

(b) $\cos\frac{\pi t}{15} = 1 \rightarrow \frac{\pi t}{15} = 2\pi$ ✔
$\therefore$ t = 30 secs ✔

(c) 138m ✔

(d) $-68\cos\frac{\pi t}{15} + 70 = 100$ ✔
$\therefore$ t = 9.68, 20.32 ✔
$\therefore$ 10.64 minutes ✔ [7]

19. (a) It is given that $(A + B)^2 = A^2 + BA + AB + B^2$ ✔
Since $AB \neq BA$, ✔
$\therefore$ $(A + B)^2 \neq A^2 + 2AB + B^2$

(b) $AB = BC$
$\therefore$ $AAB = ABC$ ✔
$\therefore$ $A^2B = BC^2$
$\therefore$ $AA^2B = ABC^2$ ✔
$\therefore$ $A^3B = BC^3$
$\therefore$ $A^3BB^{-1} = BC^3B^{-1}$ ✔

$\therefore \quad A^3 = BC^3B^{-1}$ as required ✔ [6]

20. (a) $\overrightarrow{OB} = 4i + 4j$ ✔

$\overrightarrow{CA} = 4i - 4j$ ✔

(b) $\overrightarrow{CA} . \overrightarrow{OB} = (4\mathbf{i} + 4\mathbf{j}) . (4\mathbf{i} - 4\mathbf{j}) = 0$ ✔

$\therefore \overrightarrow{CA} \perp \overrightarrow{OB}$ ✔

(c) Let k be the midpoint of $\overrightarrow{OB}$.
Then K = (2, 2)

$\therefore \overrightarrow{OK} = 2\mathbf{i} + 2\mathbf{j}$ ✔

So $\overrightarrow{CK} = \overrightarrow{KO} + \overrightarrow{OC} = -(2\mathbf{i} + 2\mathbf{j}) + 4\mathbf{j} = -2\mathbf{i} + 2\mathbf{j}$ ✔

But $\overrightarrow{CA} = -4\mathbf{i} + 4\mathbf{j} = 2\overrightarrow{CK}$

$\therefore$ K is the midpoint of $\overrightarrow{CA}$ ✔ [7]

$\therefore$ Diagonals bisect each other.