Carine Senior High School

Semester Two Examination, 2023

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3&4

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	48	35
Section Two: Calculator-assumed	12	12	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (90 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(6 marks)**

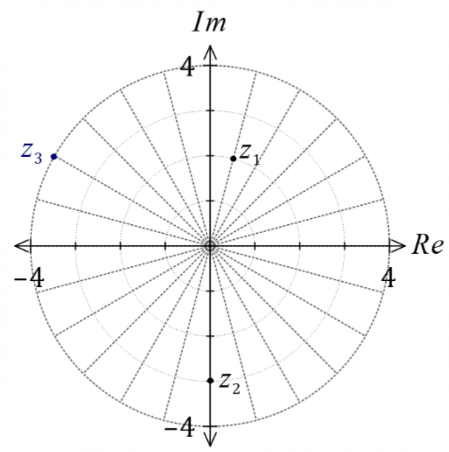
A small helium balloon is released and rises vertically so that its height h metres above its launch site after t seconds is given by $h = 4.5\left(\sqrt[3]{t^4}\right)$. A video camera is located 36 metres horizontally from the launch site of the balloon and automatically rotates so that it is always pointing directly at the balloon.

Determine the rate at which the camera is rotating 8 seconds after the balloon is released.

Solution
Since $h = 4.5\left(\sqrt[3]{t^4}\right) = 4.5t^{1.\overline{3}}$ then given rate is $\dfrac{dh}{dt} = 6t^{0.\overline{3}} = 6\sqrt[3]{t}$. Required rate is $\dfrac{d\theta}{dt}$ and the relation between variables is $h = 36\tan\theta$. $$\frac{dh}{d\theta} = 36\sec^2\theta$$ $$\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt} = \frac{6\sqrt[3]{t}}{36\sec^2\theta}$$ When $t = 8, h = 4.5(8)^{1.\overline{3}} = 72$ and so $\tan\theta = 72 \div 36 = 2$ and then $\sec^2\theta = 1 + 2^2 = 5$. (NB $\theta \approx 63.4°$ or 1.107^{r}). Hence: $$\frac{d\theta}{dt} = \frac{6 \times \sqrt[3]{8}}{36 \times 5} = \frac{1}{15} = 0.0\overline{6}$$ The camera is rotating at $0.0\overline{6}$ radians per second.
Specific behaviours
✓ differentiates to obtain given rate ✓ obtains formula for h in terms of θ ✓ obtains formula for required rate in terms of t and θ ✓ obtains value for θ (or $\tan\theta$) at required time ✓ substitutes values into required rate formula ✓ states correct rate, with units.

Question 9 **(9 marks)**

The complex numbers z_1 and z_2 are shown in the complex plane at right.



Solution (c)
See diagram for z_3
Specific behaviours
✓ correct magnitude
✓ correct argument

(a) Express z_1 in polar form. (1 mark)

Solution
$z_1 = 2\,\text{cis}\,\dfrac{5\pi}{12}$
Specific behaviours
✓ correct expression

(b) Express z_2 in Cartesian form. (1 mark)

Solution
$z_2 = -3i$
Specific behaviours
✓ correct expression

(c) Plot z_3 in the complex plane above, given that $z_3 = z_1^2$. (2 marks)

(d) Determine the argument of z_4 when $z_4 = (3 + z_2)(-\sqrt{3} + i)$. (2 marks)

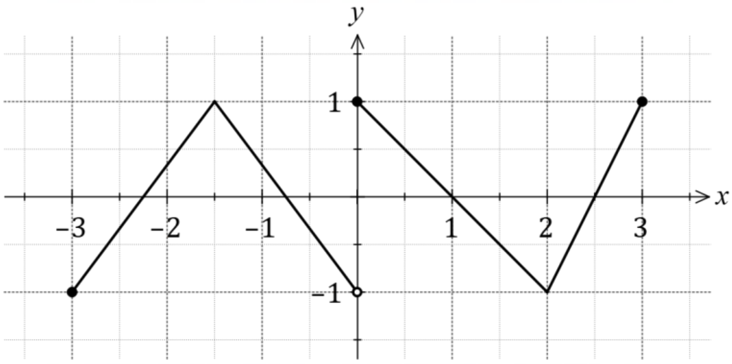
Solution
$\arg(3 - 3i) = -\dfrac{\pi}{4}, \qquad \arg(-\sqrt{3} + i) = \dfrac{5\pi}{6}$ $\arg(z_4) = -\dfrac{\pi}{4} + \dfrac{5\pi}{6} = \dfrac{7\pi}{12}$
Specific behaviours
✓ indicates one correct argument of factors
✓ correct argument

(e) Let $w = a\,\text{cis}\,\phi$. Express $\dfrac{z_1}{w^2(1-i)}$ in polar form in terms of the real constants a and ϕ. (3 marks)

Solution
$\dfrac{z_1}{w^2(1-i)} = 2\,\text{cis}\left(\dfrac{5\pi}{12}\right) \div \left(a^2\,\text{cis}(2\phi) \times \sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)\right)$ $= \dfrac{2}{a^2\sqrt{2}}\,\text{cis}\left(\dfrac{5\pi}{12} - 2\phi + \dfrac{\pi}{4}\right)$ $= \dfrac{\sqrt{2}}{a^2}\,\text{cis}\left(\dfrac{2\pi}{3} - 2\phi\right)$
Specific behaviours
✓ expresses all terms in polar form
✓ correct magnitude in terms of a
✓ correct argument in terms of ϕ

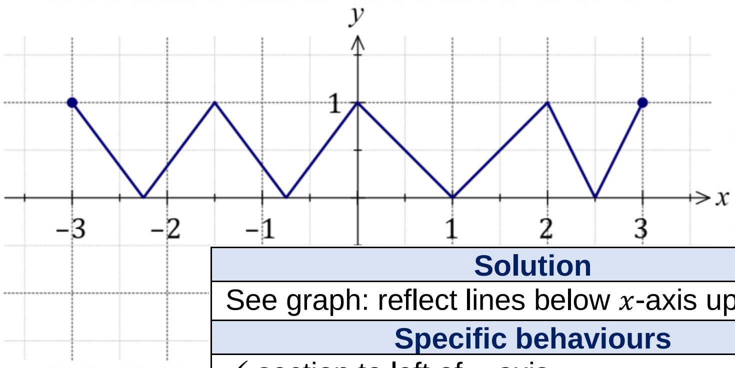
Question 10 **(6 marks)**

The graph of $y = f(x)$ is shown.



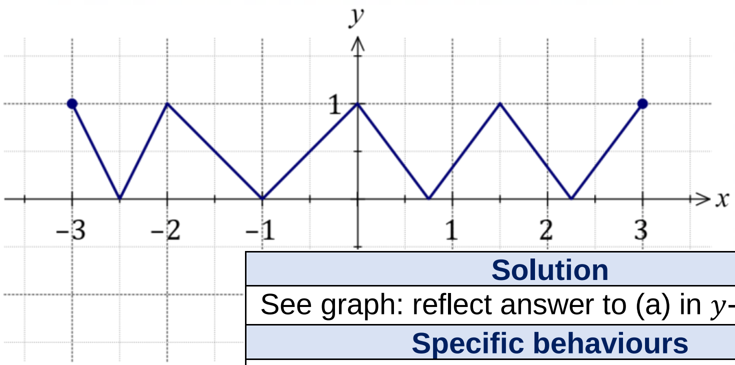
Using the set of axes provided, draw the graph of

(a) $y = |f(x)|$. (2 marks)



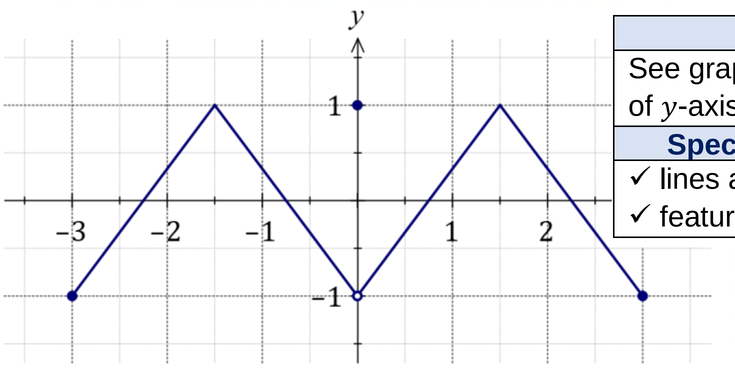
Solution
See graph: reflect lines below x-axis upwards
Specific behaviours
✓ section to left of y-axis
✓ section to right of y-axis

(b) $y = |f(-x)|$. (2 marks)



Solution
See graph: reflect answer to (a) in y-axis
Specific behaviours
✓ section to left of y-axis
✓ section to right of y-axis

(c) $y = f(-|x|)$. (2 marks)



Solution
See graph: reflect lines left of y-axis to right
Specific behaviours
✓ lines as shown
✓ features when $x = 0$

Question 11 **(7 marks)**

A factory advertises that its tea light candles burn for an average of 97.5 minutes. The standard deviation of the burn times is known to be 7.5 minutes.

(a) Quality control took a random sample of 80 candles from the factory production line and recorded their burn times. These times were used to calculate the P percent confidence interval for the population mean burn time as $93.34 \leq \mu \leq 97.66$ minutes. Determine the value of P. (3 marks)

Solution
$se = \dfrac{7.5}{\sqrt{80}} = 0.8385, \qquad E = \dfrac{97.66 - 93.34}{2} = 2.16$ $z = 2.16 \div 0.8385 = 2.576 \Rightarrow P = 99$
Specific behaviours
✓ indicates standard error or margin of error   ✓ forms equation for z-score   ✓ correct value of P

A consumer watchdog tested a random sample of 32 candles made by the factory and their mean burn time was 95.4 minutes.

(b) Describe and construct a suitable interval estimate based on this sample that can be used to advise the watchdog on the reasonableness of the factory's advertising and use the interval estimate to provide that advice. (4 marks)

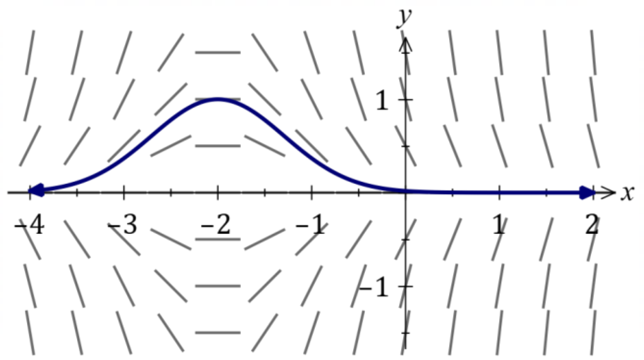
Solution
A suitable interval estimate is a 95% confidence interval for the population mean burn time of a candle that can be constructed using the sample mean and known population standard deviation: $se = \dfrac{7.5}{\sqrt{32}} = 1.3258, \qquad z_{0.95} = 1.96, \qquad E = 1.3258 \times 1.96 = 2.5986$ Interval estimate for population mean: $95.4 \pm 2.5986 \rightarrow 92.80 < \mu < 98.00$ Advice to watchdog is that because the interval estimate for the mean burn time of all candles contains the advertised time of 97.5 minutes, then the factory's advertising is reasonable.   *Other intervals:* $90\% \rightarrow 93.22 < \mu < 97.58$ *and* $99\% \rightarrow 91.98 < \mu < 98.82$.
Specific behaviours
✓ correctly describes interval for population mean using $90\% \leq$ confidence level $\leq 99\%$ ✓ calculates variance of sampling distribution   ✓ constructs an interval estimate for population mean   ✓ advises watchdog, with reasoning, that advertised average is reasonable

Question 12 **(7 marks)**

The slope field for the differential equation

$$\frac{dy}{dx} + y(2x + k) = 0$$

where k is a constant, is shown at right.



Solution (c)
See graph
Specific behaviours
✓ 'normal' curve thru' $(-2, 1)$

(a) Use a feature of the slope field to explain why $k = 4$ and hence determine the slope at the point $A(-3, -1)$. (2 marks)

Solution
$y' = -y(2x + k)$ When $x = -2$ and $y \neq 0$ it can be seen that $y' = 0$ and so $2(-2) + k = 0 \Rightarrow k = 4$. $A(-3, -1) \rightarrow y' = -(-1)(2(-3) + 4) = -2$. Slope at $A(-3, -1)$ is -2.
Specific behaviours
✓ explains using $y' = 0$ at $x = -2$ ✓ correct slope at A

(b) Determine the solution of the differential equation that contains the point $B(-2, 1)$ in the form $y = f(x)$. (4 marks)

Solution
$\frac{dy}{dx} = -y(2x + 4)$ $\int \frac{1}{y} dy = -\int 2x + 4\, dx$ $\ln\lvert y\rvert = -x^2 - 4x + c$ At $B(-2, 1)$, $y > 0$ and so require $\ln(y) = -x^2 - 4x + c$ $-(-2)^2 - 4(-2) + c = 0 \Rightarrow c = -4$ $y = e^{-x^2-4x-4} = e^{-(x+2)^2}$ $(\approx 0.0183e^{-x^2-4x})$
Specific behaviours
✓ separates variables and antidifferentiates   ✓ recognises that $y > 0$ to replace $\lvert y\rvert \rightarrow (y)$ ✓ evaluates constant   ✓ correctly expresses y as a function of x

(c) Sketch the solution curve that contains the point $B(-2, 1)$ on the slope field. (1 mark)

Question 13 **(9 marks)**

The Cartesian equation of sphere S is $(x+4)^2+(y-2)^2+(z-5)^2=14$.

(a) State the vector equation of sphere S. (1 mark)

Solution
$\left\| \underset{\sim}{r} - \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix} \right\| = \sqrt{14}$
Specific behaviours
✓ correct equation

The position vector of particle A at time t seconds is given by $\underset{\sim}{r}(t) = \begin{pmatrix} -13 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$.

(b) Show that the path of A is tangential to sphere S. (3 marks)

Solution
Path of A in parametric form: $x=-13+3t, \quad y=1+t, \quad z=-1+4t$ Substituting for x, y, z into equation of sphere: $(-13+3t+4)^2+(1+t-2)^2+(-1+4t-5)^2=14$ Expanding and solving this equation: $26t^2-104t+104=0$ $(t-2)^2=0 \Rightarrow t=2$ Since there is exactly one solution to this quadratic equation, then the path of particle must be tangential to sphere.
Specific behaviours
✓ substitutes for x, y, z in sphere equation   ✓ simplifies equation   ✓ explains meaning of exactly one solution

Particle B is moving with a constant velocity and has position vector $\begin{pmatrix} -7 \\ 11 \\ 3 \end{pmatrix}$ when $t = 0$. Three seconds later, its position vector is $\begin{pmatrix} -4 \\ 8 \\ 9 \end{pmatrix}$.

(c) Show that the paths of A and B cross but that they do not collide. (5 marks)

Solution

Velocity of B is $\frac{1}{3}\left(\begin{pmatrix} -4 \\ 8 \\ 9 \end{pmatrix} - \begin{pmatrix} -7 \\ 11 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Position vector of B, s seconds after $t = 0$ is:

$$\underset{\sim}{r_B}(s) = \begin{pmatrix} -7 \\ 11 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

For collision $\begin{pmatrix} -7 \\ 11 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -13 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$.

Equating $\underset{\sim}{i}$ and $\underset{\sim}{j}$ coefficients:

$$-7 + s = -13 + 3t$$
$$11 - s = 1 + t$$

Solving these equations simultaneously we get $s = 6, t = 4$.

$$\underset{\sim}{r_B}(6) = \begin{pmatrix} -7 \\ 11 \\ 3 \end{pmatrix} + 6\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 15 \end{pmatrix}$$

$$\underset{\sim}{r_A}(4) = \begin{pmatrix} -13 \\ 1 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 15 \end{pmatrix}$$

Since particles A and B pass through $(-1, 5, 15)$ but at different times then their paths cross but they do not collide.

Specific behaviours

✓ obtains velocity vector for B
✓ obtains position vector for B
✓ equates two pairs of coefficients
✓ solves simultaneously
✓ shows that particles have same position vectors at different times

Question 14 **(7 marks)**

(a) Show that $\frac{4}{x-3}+\frac{6}{x^2+9}=\frac{4x^2+6x+18}{x^3-3x^2+9x-27}$. (1 mark)

Solution
$$\begin{aligned}\text{LHS} &= \frac{4}{x-3}+\frac{6}{x^2+9}\\ &= \frac{4x^2+36+6x-18}{(x-3)(x^2+9)}\\ &= \frac{4x^2+6x+18}{x^3-3x^2+9x-27}=\text{RHS}\end{aligned}$$
Specific behaviours
✓ combines into single fraction with expanded numerator

(b) Hence show that $\int_{-3}^{0}\left(\frac{4x^2+6x+18}{x^3-3x^2+9x-27}\right)dx=\frac{\pi}{2}-4\ln 2$, using the substitution $x=3\tan\theta$ where appropriate. (6 marks)

Solution
$$I=\int_{-3}^{0}\left(\frac{4x^2+6x+18}{x^3-3x^2+9x-27}\right)dx=\int_{-3}^{0}\frac{4}{x-3}dx+\int_{-3}^{0}\frac{6}{x^2+9}dx$$ $$\begin{aligned}\int_{-3}^{0}\frac{4}{x-3}dx &= [4\ln\lvert x-3\rvert]_{-3}^{0}\\ &= 4\ln 3-4\ln 6=4\ln\frac{1}{2}=-4\ln 2\end{aligned}$$ When $x=3\tan\theta$: $dx=3\sec^2\theta\,d\theta;\quad x^2+9=9\tan^2\theta+9=9\sec^2\theta;\quad x=0,\theta=0;\quad x=-3,\theta=-\frac{\pi}{4}$ $$\begin{aligned}\int_{-3}^{0}\frac{6}{x^2+9}dx &= \int_{-\frac{\pi}{4}}^{0}\frac{6\times 3\sec^2\theta}{9\sec^2\theta}d\theta\\ &= \int_{-\frac{\pi}{4}}^{0}2\,d\theta\\ &= [2\theta]_{-\frac{\pi}{4}}^{0}=\frac{\pi}{2}\end{aligned}$$ Hence $I=\frac{\pi}{2}-4\ln 2$.
Specific behaviours
✓ antidifferentiates first integral, using absolute value brackets ✓ substitutes and simplifies to obtain $-4\ln 2$ ✓ uses substitution to relate dx and $d\theta$ ✓ adjusts bounds of integration ✓ writes and simplifies second integral in terms of θ ✓ antidifferentiates and substitutes to obtain $\frac{\pi}{2}$ and hence I

Question 15 **(7 marks)**

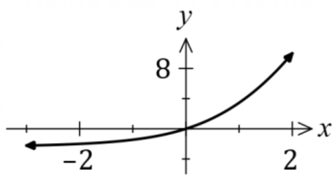
(a) Use the substitution $u^2 = 2y + 5$ to show that $\displaystyle\int \frac{y}{\sqrt{2y+5}}\,dy = \frac{(y-5)\sqrt{2y+5}}{3} + c$, where c is a constant of integration. (4 marks)

Solution
$u^2 = 2y + 5 \Rightarrow dy = u\,du, \qquad y = \dfrac{u^2 - 5}{2}$ $\begin{aligned}\int \frac{y}{\sqrt{2y+5}}\,dy &= \int \frac{u^2-5}{2u} \times u\,du \\ &= \frac{1}{2}\int u^2 - 5\,du \\ &= \frac{1}{2}\left(\frac{1}{3}u^3 - 5u\right) + c \\ &= \frac{u}{6}(u^2 - 15) + c \\ &= \frac{\sqrt{2y+5}(2y+5-15)}{6} + c \\ &= \frac{(y-5)\sqrt{2y+5}}{3} + c\end{aligned}$
Specific behaviours
✓ obtains y and dy in terms of u and du ✓ obtains simplified integral in terms of u ✓ obtains correct antiderivative ✓ shows step(s) that clearly lead to required result

(b) The equation of the curve shown is

$y = x\sqrt{2y+5}$.

Determine the area enclosed by the curve and the line $3x - y + 4 = 0$. (3 marks)



Solution
$x = \dfrac{y}{\sqrt{2y+5}}, \qquad x = \dfrac{y-4}{3}$ Lines intersect when $y = -2, y = 10$. $\begin{aligned}A &= \int_{-2}^{10} \left(\frac{y}{\sqrt{2y+5}}\right) - \left(\frac{y-4}{3}\right) dy \\ &= \frac{32}{3} = 10.\overline{6}\end{aligned}$
Specific behaviours
✓ obtains bounds of integral ✓ writes correct integral for area ✓ correct area

Question 16 **(8 marks)**

A machine fills bags with salt. The mean and standard deviation of the weight of salt it delivers into a bag is 125 and 11.5 grams respectively. An inspector routinely takes a random sample of 58 bags filled by the machine.

(a) For repeated random sampling of 58 bags of salt filled by this machine, state the approximate distribution of the sample mean that the inspector should expect. (3 marks)

Solution
Let $\bar{X}$ be the sample mean. Since the sample size is large then the distribution of $\bar{X}$ will be approximately normal with mean 125 g. The standard deviation of $\bar{X}$ is $\dfrac{11.5}{\sqrt{58}} = 1.510$ grams (variance≈ 2.28) Hence $\bar{X} \sim N(125, 1.51^2)$.
Specific behaviours
✓ states that sample mean will be normally distributed ✓ states the mean of the distribution ✓ states the variance or standard deviation of the distribution

(b) Determine the probability that the mean weight of a random sample of 58 bags of salt is less than 128 grams, given that the sample mean is greater than 125 grams. (2 marks)

Solution
$P(\bar{X} < 128 \mid \bar{X} > 125) = \dfrac{P(125 < \bar{X} < 128)}{P(\bar{X} > 125)} = \dfrac{0.4765}{0.5} = 0.953$
Specific behaviours
✓ forms correct probability statement ✓ correct probability

(c) Occasionally, the inspector only has enough time to take a random sample of 46 bags. In the long run, 75% of sample means derived from samples with this smaller size will lie in the range $125 \pm k$ grams. Determine the value of k. (3 marks)

Solution
The new standard deviation of $\bar{X}$ is $\dfrac{11.5}{\sqrt{46}} = 1.696$ grams (variance$= 2.875$). $\bar{X} \sim N(125, 2.875)$ $P(125 - k < \bar{X} < 125 + k) = 0.75$ $k = 1.95$ g
Specific behaviours
✓ states new parameters of distribution of sample mean ✓ writes correct probability statement ✓ correct value of k

Question 17 **(6 marks)**

Consider the function $f(x) = \dfrac{ax^2 - ax - b}{x^2 - c}$, where a, b and c are positive constants.

The graph of $y = f(x)$ cuts the x-axis at $x = -2$, has a horizontal asymptote with equation $y = 4$ and has a vertical asymptote with equation $x = -1$.

(a) Determine $f(0)$. (3 marks)

Solution
Horizontal asymptote $y = 4 \Rightarrow a = 4$. $f(-2) = 0 \Rightarrow 4(-2)^2 - 4(-2) - b = 0 \Rightarrow b = 24$ Vertical asymptote $x = -1 \Rightarrow (-1)^2 - c = 0 \Rightarrow c = 1$ $f(0) = \dfrac{-24}{-1} = 24$
Specific behaviours
✓ obtains value of one constant ✓ obtains value of second constant ✓ correct value of $f(0)$

(b) Now consider the graph of $y = \dfrac{1}{f(x)}$. State the

(i) equation of its horizontal asymptote. (1 mark)

Solution
$y = \dfrac{1}{4}$
Specific behaviours
✓ correct equation

(ii) x-axis intercepts. (1 mark)

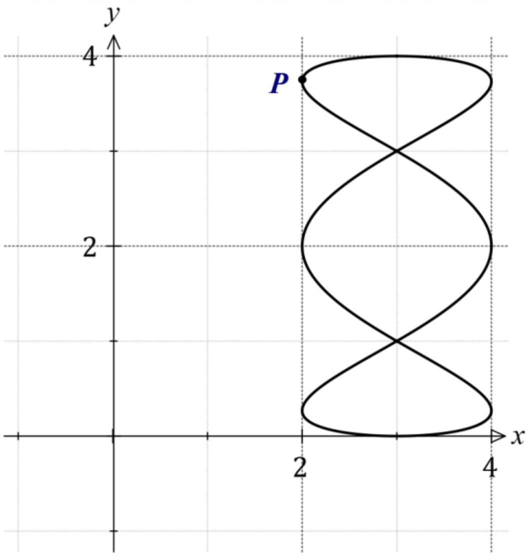
Solution
Vertical asymptotes → roots: $x = \pm 1$.
Specific behaviours
✓ correct intercepts

(iii) equations of its vertical asymptotes. (1 mark)

Solution
Roots → vertical asymptotes: $4x^2 - 4x - 24 = 0 \Rightarrow x = -2$ and $x = 3$.
Specific behaviours
✓ correct equations

Question 18 **(9 marks)**

A particle is moving and has position vector $\underset{\sim}{r}(t) = \begin{pmatrix} 3 - \sin(3t) \\ 2\cos(t) + 2 \end{pmatrix}$ metres, where t is the time in seconds since motion began. Its path is shown in the diagram below.



(a) Mark point P on the diagram above to show the position of the particle when $t = \frac{\pi}{6}$, and state the time taken for the particle to next return to this position. (2 marks)

Solution
Locate point at $P\left(2, \sqrt{3} + 2\right) \approx (2, 3.73)$. Using period of $\cos t$, the particle will return here after 2π seconds.
Specific behaviours
✓ correctly marks point ✓ states period of motion

(b) Determine the velocity of the particle when $t = \frac{\pi}{6}$. (2 marks)

Solution
$\underset{\sim}{v}(t) = \frac{d}{dt}\underset{\sim}{r}(t)$ $= \begin{pmatrix} -3\cos(3t) \\ -2\sin(t) \end{pmatrix}$ $\underset{\sim}{v}\left(\frac{\pi}{6}\right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
Specific behaviours
✓ obtains correct velocity vector ✓ correct velocity

(c) Determine the distance moved by the particle during its third second of motion. (2 marks)

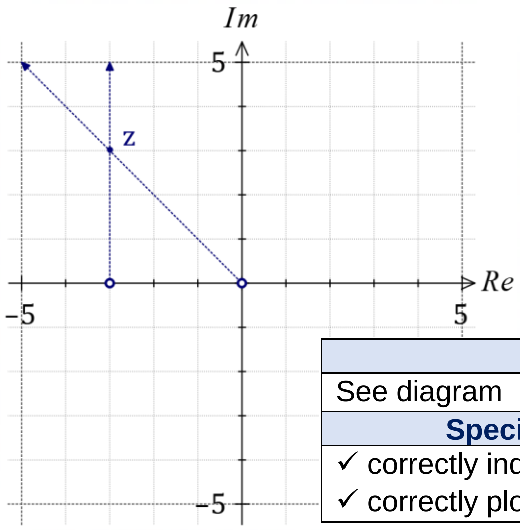
Solution
Let $s = \lvert \underset{\sim}{v}(t) \rvert = \sqrt{9\cos^2 3t + 4\sin^2 t}$ Then distance is: $$d = \int_2^3 s\,dt \approx 2.27\text{ m}$$
Specific behaviours
✓ indicates correct integral for distance ✓ correct distance

(d) Using the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, or otherwise, determine the Cartesian equation of the path of the particle. (3 marks)

Solution
$$y = 2\cos t + 2 \Rightarrow \cos t = \frac{y-2}{2}$$ $$\begin{aligned} x &= 3 - \sin 3t \\ 3 - x &= 3\sin t - 4\sin^3 t \\ &= \sin t\,(3 - 4\sin^2 t) \\ &= \sin t\,\big(3 - 4(1-\cos^2 t)\big) \\ &= -\sin t\,(1 - 4\cos^2 t) \\ \sin t &= \frac{x-3}{1 - 4\left(\frac{y-2}{2}\right)^2} \\ &= \frac{x-3}{1-(y-2)^2} \end{aligned}$$ Hence $$\left(\frac{y-2}{2}\right)^2 + \left(\frac{x-3}{1-(y-2)^2}\right)^2 = 1$$
Specific behaviours
✓ obtains Cartesian expression for $\cos t$ ✓ obtains Cartesian expression for $\sin t$ ✓ combines to obtain correct Cartesian equation

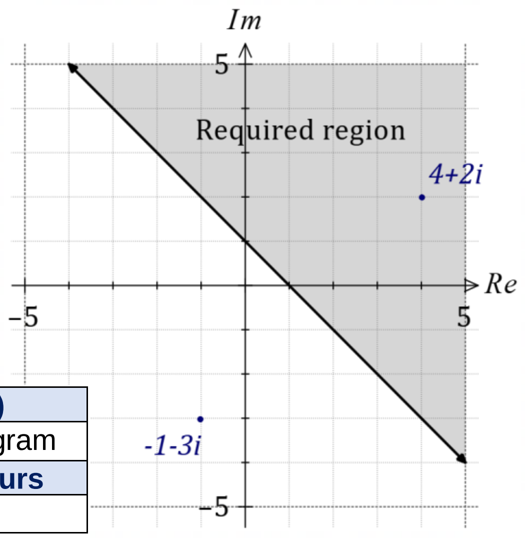
Question 19 **(9 marks)**

(a) Plot the complex number that satisfies the conditions $\arg(z) = \dfrac{3\pi}{4}$ and $\arg(z+3) = \dfrac{\pi}{2}$ on the Argand diagram below. (2 marks)



Solution
See diagram
Specific behaviours
✓ correctly indicates at least one ray
✓ correctly plots complex number

(b) Let $z_1 = 4 + 2i$ and z_2 be another complex number. The locus of a complex number z satisfies the condition $|z - z_1| = |z - z_2|$ and is shown in the diagram below.



Solution (b)(ii)
See shading on diagram
Specific behaviours
✓ correct shading

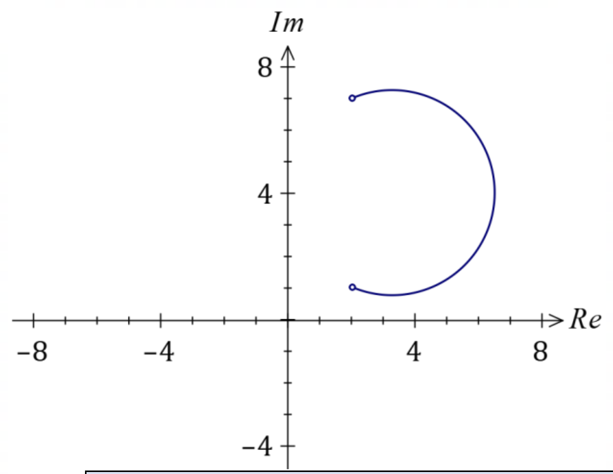
(i) Determine the complex number z_2. (2 marks)

Solution
$z_2 = -1 - 3i$
Specific behaviours
✓ indicates point lies on perpendicular to locus through z_1
✓ correct complex number

(ii) On the same diagram, indicate the locus of a complex number z that satisfies the condition $|z - z_1| \le |z - z_2|$. (1 mark)

(c) The locus of points that satisfy $\arg\left(\dfrac{z-2-i}{z-2-7i}\right)=\dfrac{\pi}{3}$ is an arc of a circle.

(i) Sketch the locus of z in the complex plane. (2 marks)



Solution
$\arg\big(z-(2+i)\big)=\arg\big(z-(2+7i)\big)+\dfrac{\pi}{3}$ Anticlockwise major arc from $2+i$ to $2+7i$.   *NB Marks for location of major arc rather than neatness/curvature*
Specific behaviours
✓ major arc of a circle drawn anywhere   ✓ correctly locates endpoints and major arc drawn to their right

(ii) Determine, with justification, the exact location of the centre of the circle. (2 marks)

Solution
When $\arg\big(z-(2+7i)\big)=0$, then $\arg\big(z-(2+i)\big)=\frac{\pi}{3}$ and a right-triangle is formed in the circle. The midpoint of the hypotenuse of this triangle must be the centre of the circle. Hence the centre is at $\left(2+\sqrt{3}\right)+4i$
Specific behaviours
✓ indicates adoption of suitable method   ✓ correct centre, fully justified

End of questions

Supplementary page

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Supplementary page

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