WAEP Semester One Examination, 2016

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section One: Calculator-free

SOLUTIONS

Student number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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Section One: Calculator-free **35% (53 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

Consider $f(z) = z^4 + 3z^3 + 7z^2 - 21z - 26$, $z \in \mathbb{C}$. Solve $f(z) = 0$ over $\mathbb{C}$.

Solution
$f(z) = z^4 + 3z^3 + 7z^2 - 21z - 26$ $f(-1) = 1 - 3 + 7 + 21 - 26 = 0$ $f(2) = 16 + 24 + 28 - 42 - 26 = 0$ $f(z) = (z+1)(z-2)(z^2 + az + b)$ $1 \times -2 \times b = -26 \Rightarrow b = 13$ $f(z) = (z^2 - z - 2)(z^2 + az + 13)$ $z^2: \ -2 - a + 13 = 7 \Rightarrow a = 4$ $f(z) = (z^2 - 1)(z^2 + 4z + 13)$ $z^2 + 4z + 13 = 0$ $(z+2)^2 = -9$ $z + 2 = \pm 3i$ $z = -1, \ 2, \ -2 + 3i, \ -2 - 3i$
Specific behaviours
✓ uses factor theorem to find $x+1$ and $x-2$ are factors ✓ uses division or inspection to determine b ✓ uses division or inspection to determine a ✓ completes square on $z^2 + 4z + 13$ ✓ solves $z^2 + 4z + 13$ to give two complex roots ✓ clearly acknowledges all four solutions

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Question 2 **(7 marks)**

A sphere has equation $x^2 + y^2 + z^2 - 2x + 4y + 3z + 1 = 0$.

(a) Determine the coordinates of the centre and the radius of the sphere. (4 marks)

Solution
$(x-1)^2 + (y+2)^2 + (z+1.5)^2 = -1+1+4+1.5^2$ $= \frac{16+9}{4} = \left(\frac{5}{2}\right)^2$ Radius is 2.5 units Centre at (1, -2, -1.5)
Specific behaviours
✓ factorises left hand side ✓ balances right hand side ✓ states the radius ✓ states centre

(b) Determine the vector equation of the straight line that passes through the points on the sphere where $y = -2$ and $z = 0$. (3 marks)

Solution
$x^2 + 4 - 2x - 8 + 1 = 0$ $x^2 - 2x - 3 = 0$ $(x+1)(x-3) = 0 \Rightarrow x = -1,\ 3$ Point on line is (3, -2, 0) Direction of line is $\langle 1, 0, 0 \rangle$ $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \lambda\mathbf{i} = (3+\lambda)\mathbf{i} - 2\mathbf{j}$
Specific behaviours
✓ determines x-coordinates of points on sphere ✓ states direction of line ✓ states vector equation of line

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Question 3 **(8 marks)**

(a) Let $z = 2\cos\left(\frac{2\pi}{3}\right) + 2i\sin\left(\frac{2\pi}{3}\right)$.

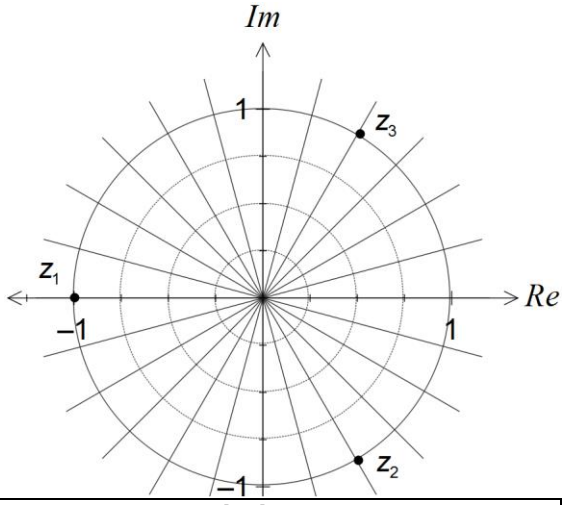
(i) Express z in Cartesian form. (2 marks)

Solution
$z = -1 + \sqrt{3}i$
Specific behaviours
✓ real part ✓ imaginary part

(ii) Determine z^5 in Cartesian form. (3 marks)

Solution
$z^5 = 2^5 cis\left(\frac{2\pi}{3} \times 5\right)$ $= 32cis\left(\frac{10\pi}{3}\right)$ $= 16 \times 2cis\left(-\frac{2\pi}{3}\right)$ $= 16 \times \overline{z}$ $= -16 - 16\sqrt{3}i$
Specific behaviours
✓ uses polar form to determine modulus ✓ uses polar form to determine argument $-\pi < \theta \leq \pi$ ✓ converts to Cartesian form

(b) If $w^3 + 1 = 0$, sketch the location of all roots of this equation on the axes below. (3 marks)



Solution
See diagram - evenly spaced points on circle
Specific behaviours
✓ Adds scale to show real root at -1 ✓ Shows second root third way around circle ✓ Shows third root as conjugate of second

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Question 4 **(7 marks)**

Consider the following system of equations, where k is a real constant.

$$\begin{aligned} x + 2y + z &= 3 \\ 2x - y - 3z &= k \\ x + 3y + kz &= 6 \end{aligned}$$

(a) Solve the system of equations when $k = 1$. (3 marks)

Solution
$x + 2y + z = 3 \quad (1)$ $2x - y - 3z = 1 \quad (2)$ $x + 3y + z = 6 \quad (3)$ $y = 3 \quad (3) - (1)$ $x + z = -3$ $2x - 3z = 4$ $5x = -5 \Rightarrow x = -1,\ z = -2$ $x = -1,\ y = 3,\ z = -2$
Specific behaviours
✓ eliminates x and z to find y ✓ eliminates and solves for another variable ✓ states values of all three variables

(b) Show that no value of k exists for the system of equations to represent three planes intersecting in a single straight line. (4 marks)

Solution
$2(1) - (2) \to (2)$ $(3) - (1) \to (3)$: $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 6-k \\ 0 & 1 & k-1 & 3 \end{bmatrix}$ $5(3) - (2) \to (3)$: $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 6-k \\ 0 & 0 & 5k-10 & k+9 \end{bmatrix}$ For infinite solns require $5k - 10 = 0 \Rightarrow k = 2$ and $k + 9 = 0 \Rightarrow k = -9$. Hence no value of k exists.
Specific behaviours
✓ reduces second and third rows in initial matrix ✓ reduces third row in second matrix ✓ indicates condition for planes to intersect in single straight line ✓ shows that no value of k exists

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Question 5 **(8 marks)**

(a) Determine the vector equation of the plane that contains the points A(1, -1, 2), B(2, 1, 0) and C(3, -1, 1). (4 marks)

Solution
$\mathbf{AB} = \langle 1, 2, -2 \rangle$
$\mathbf{AC} = \langle 2, 0, -1 \rangle$
$\mathbf{AC} \times \mathbf{AB} = \langle 2, 3, 4 \rangle$
$\mathbf{r} \cdot \langle 2, 3, 4 \rangle = \langle 2, 1, 0 \rangle \cdot \langle 2, 3, 4 \rangle$
$\mathbf{r} \cdot \langle 2, 3, 4 \rangle = 7$
Specific behaviours
✓ finds two vectors in plane ✓ calculates cross product of two vectors ✓ substitutes into vector equation of plane ✓ simplifies vector equation

(b) Plane Π has equation $x + 2y - z = 3$. Line L is perpendicular to Π and passes through the point (1, -6, 4). Determine where line L intersects plane Π. (4 marks)

Solution
$\mathbf{r}_P \cdot \langle 1, 2, -1 \rangle = 3$
$\mathbf{r}_L = \langle 1, -6, 4 \rangle + t \langle 1, 2, -1 \rangle$
$\langle 1 + t, 2t - 6, 4 - t \rangle \cdot \langle 1, 2, -1 \rangle = 3$
$1 + t + 4t - 12 - 4 + t = 3$
$6t = 18 \Rightarrow t = 3$
$\mathbf{r} = \langle 1, -6, 4 \rangle + 3\langle 1, 2, -1 \rangle$
$= \langle 4, 0, 1 \rangle \Rightarrow$ At $(4,\ 0,\ 1)$
Specific behaviours
✓ writes vector equation of plane ✓ writes vector equation of line through point ✓ substitutes line into plane and solves for t ✓ determines coordinates of point

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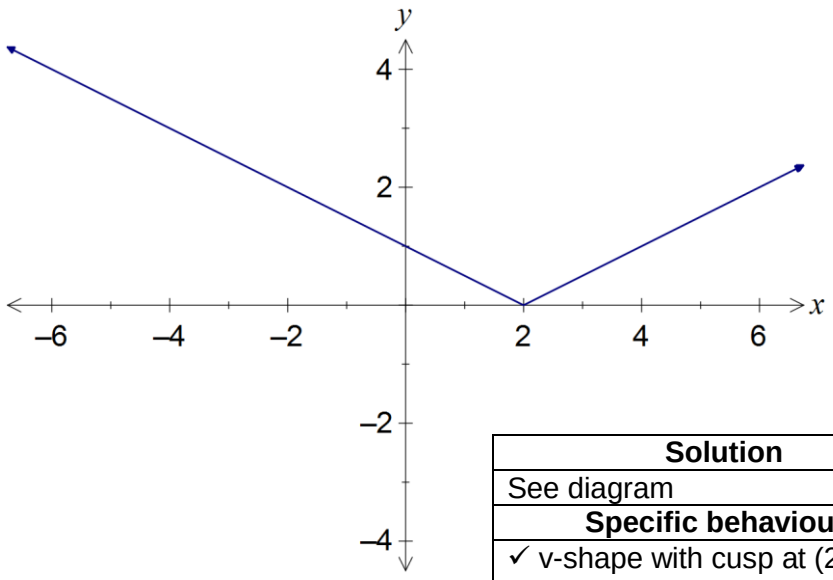
Question 6 **(7 marks)**

(a) Sketch the graph of $y = \dfrac{|x-2|}{2}$ on the axes below. (2 marks)



Solution
See diagram
Specific behaviours
✓ v-shape with cusp at (2, 0) ✓ correct y-intercept

(b) Solve the equation $4|x-8| = 38 - x$. (3 marks)

Solution
$x \geq 8 \Rightarrow 4x - 32 = 38 - x \Rightarrow 5x = 70 \Rightarrow x = 14$ $x < 8 \Rightarrow -4x + 32 = 38 - x \Rightarrow 3x = -6 \Rightarrow x = -2$ $x = -2,\ 14$
Specific behaviours
✓ separates into cases ✓ solves first case ✓ solves second case

(c) Solve the inequality $\dfrac{1}{|x+2|} \leq 1$. (2 marks)

Solution
$x > -2 \Rightarrow 1 \leq x + 2 \Rightarrow x \geq -1$ $x < -2 \Rightarrow 1 \leq -x - 2 \Rightarrow x \leq -3$ $x \leq -3,\ x \geq -1$
Specific behaviours
✓ determines correct endpoints ✓ states correct inequalities

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Question 7 **(10 marks)**

Particle A has position vector given by $\mathbf{r} = 3\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}$, where t is the time in seconds.

(a) Show that the path of the particle is circular. (2 marks)

Solution
$x = 3\cos t,\ y = 3\sin t \ \Rightarrow\ \frac{x}{3} = \cos t,\ \frac{y}{3} = \sin t$ $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \ \Rightarrow\ x^2 + y^2 = 3^2$, circle centre (0,0), radius 3.
Specific behaviours
✓ converts to Cartesian form ✓ states centre and radius

Particle B is stationary, with position vector $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

(b) Determine an expression for the distance between particles A and B in terms of t. (2 marks)

Solution
$\lvert\mathbf{BA}\rvert = \lvert\mathbf{OA} - \mathbf{OB}\rvert = \sqrt{(3\cos t - 3)^2 + (3\sin t - 4)^2 + (-5)^2}$
Specific behaviours
✓ determines vector **BA** (or **AB**) ✓ states magnitude of vector

(c) Determine the position vector of the A when it is (i) nearest and (ii) furthest from B. (6 marks)

Solution
Let S be square of distance between particles: $\frac{dS}{dt} = 2(-3\sin t)(3\cos t - 3) + 2(3\cos t)(3\sin t - 4)$ $\frac{dS}{dt} = 0 \ \Rightarrow\ -\sin t(3\cos t - 3) + \cos t(3\sin t - 4) = 0$ $3\sin t - 4\cos t = 0$ $\tan t = \frac{4}{3} \ \Rightarrow\ \sin t = \pm\frac{4}{5},\ \cos t = \pm\frac{3}{5}$ Nearest : $\mathbf{OA} = 3\left(\frac{3}{5}\right)\mathbf{i} + 3\left(\frac{4}{5}\right)\mathbf{j} = \frac{9}{5}\mathbf{i} + \frac{12}{5}\mathbf{j}$ Furthest : $\mathbf{OA} = -\frac{9}{5}\mathbf{i} - \frac{12}{5}\mathbf{j}$
Specific behaviours
✓ differentiates S ✓ simplifies and equates derivative to 0 ✓ determines solution for tan t ✓ derives possible values for sin t and cos t ✓ determines nearest position ✓ determines furthest position

End of questions

Additional working space

Question number: _____

Additional working space

Question number: _____