Neap

Trial Examination 2021

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (4 marks)

a. $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$ A1

b. **Method 1:**

$|\overrightarrow{OA}| = \sqrt{2},\ |\overrightarrow{OB}| = 5,\ |\overrightarrow{BC}| = \sqrt{2},\ |\overrightarrow{AC}| = 5$ A1

Take the scalar product between either $\overrightarrow{OA}$ and $\overrightarrow{OB}$, $\overrightarrow{OA}$ and $\overrightarrow{AC}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ or $\overrightarrow{OB}$ and $\overrightarrow{BC}$.

For example:

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = \left(\underset{\sim}{i} + \underset{\sim}{j}\right) \cdot \left(-5\underset{\sim}{k}\right)$$

$$= 0$$ M1

As $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$, *OA* is perpendicular to *OB*. A1

Hence, *OACB* is a rectangle.

Method 2:

Find $|\overrightarrow{OC}|$ and $|\overrightarrow{AB}|$.

$$|\overrightarrow{OC}| = \sqrt{1^2 + 1^2 + (-5)^2} \qquad |\overrightarrow{AB}| = \sqrt{(-1)^2 + (-1)^2 + (-5)^2}$$ M1

$$= \sqrt{27} \qquad\qquad = \sqrt{27}$$ A1

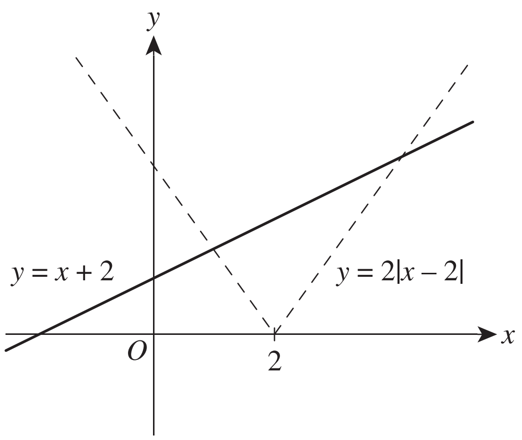
As $|\overrightarrow{OC}| = |\overrightarrow{AB}| = \sqrt{27}$, then the magnitude of the diagonals are equal. A1

Hence, *OACB* is a rectangle.

Question 2 (3 marks)

Method 1:

The values of x for which $|x+2| > 2|x-2|$ are the values of x for which the graph of $y = x + 2$ is above the graphs of $y = 2(x-2)$ and $y = -2(x-2)$.



Solve $x + 2 = 2(x-2)$ and $x + 2 = -2(x-2)$ for x.

$x + 2 = 2(x-2) \qquad x + 2 = -2(x-2)$ M1

$\Rightarrow x = 6 \qquad \Rightarrow x = \dfrac{2}{3}$ A1

$\dfrac{2}{3} < x < 6$ A1

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Method 2:

Square both sides of $|x+2|>2|x-2|$ to obtain $(x+2)^2>4(x-2)^2$. A1

Solve $(x+2)^2>4(x-2)^2$ for x.

$$(x+2)^2>4(x-2)^2$$

$$x^2+4x+4>4\left(x^2-4x+4\right)$$ M1

$$3x^2-20x+12<0$$

$$\left(3x-2\right)\left(x-6\right)<0$$

$$\frac{2}{3}<x<6$$ A1

Question 3 (3 marks)

Find $\dot{\underset{\sim}{r}}(t)$.

$\dot{\underset{\sim}{r}}\left(t\right)=-\pi\sin\left(\pi t\right)\underset{\sim}{i}+2\pi\cos\left(2\pi t\right)\underset{\sim}{j}$ M1

Find $\dot{\underset{\sim}{r}}\left(1\right)$ and $\dot{\underset{\sim}{r}}\left(\frac{1}{2}\right)$.

$$\begin{aligned}\dot{\underset{\sim}{r}}\left(1\right)&=-\pi\sin\left(\pi\right)\underset{\sim}{i}+2\pi\cos\left(2\pi\right)\underset{\sim}{j}\\&=2\pi\underset{\sim}{j}\end{aligned}\qquad\begin{aligned}\dot{\underset{\sim}{r}}\left(\frac{1}{2}\right)&=-\pi\sin\left(\frac{\pi}{2}\right)\underset{\sim}{i}+2\pi\cos\left(\pi\right)\underset{\sim}{j}\\&=-\pi\underset{\sim}{i}-2\pi\underset{\sim}{j}\end{aligned}$$ A1

Let the change in momentum be $\Delta\underset{\sim}{p}$ and $\Delta\underset{\sim}{p}=m\Delta\underset{\sim}{v}$.

$$\begin{aligned}\Delta\underset{\sim}{p}&=3\left(\dot{\underset{\sim}{r}}\left(1\right)-\dot{\underset{\sim}{r}}\left(\frac{1}{2}\right)\right)\\&=3\left(2\pi\underset{\sim}{j}-\left(-\pi\underset{\sim}{i}-2\pi\underset{\sim}{j}\right)\right)\end{aligned}$$

$\Delta\underset{\sim}{p}=3\pi\underset{\sim}{i}+12\pi\underset{\sim}{j}$ (kg ms^{-1}) A1

Question 4 (4 marks)

Using $\mathrm{E}(aX+bY)=a\mathrm{E}(X)+b\mathrm{E}(Y)$ gives $-2a-2b=2$, and
using $\mathrm{var}(aX+bY)=a^2\mathrm{var}(X)+b^2\mathrm{var}(Y)$ gives $a^2+2b^2=9$. M1

Method 1:

Substitute $a=-1-b$ into $a^2+2b^2=9$.

$$(-1-b)^2+2b^2=9$$
$$3b^2+2b-8=0$$ A1

Solve for b.

$$(b+2)(3b-4)=0$$
$$\Rightarrow b=-2,\frac{4}{3}$$ M1

$b\neq\frac{4}{3}$ as $b\in Z$ so $b=-2$.

$$\begin{aligned}a&=-1-b\\&=-1-(-2)\\&=1\end{aligned}$$

$a=1$ and $b=-2\ (a,b\in Z)$ A1

Method 2:

Substitute $b=-1-a$ into $a^2+2b^2=9$.

$$a^2+2(-1-a)^2=9$$ A1
$$3a^2+4a-7=0$$

Solve for a.

$$(3a+7)(a-1)=0$$
$$\Rightarrow a=-\frac{7}{3},1$$ M1

$a\neq-\frac{7}{3}$ as $a\in Z$ so $a=1$.

$$\begin{aligned}b&=-a-1\\&=-(1)-1\\&=-2\end{aligned}$$

$a=1$ and $b=-2\ (a,b\in Z)$ A1

Question 5 (4 marks)

a. $e^{-x}\dfrac{dy}{dx}-ye^{-x}=1+2y\dfrac{dy}{dx}$ M1

$\dfrac{dy}{dx}\left(e^{-x}-2y\right)=1+ye^{-x}$

$$\frac{dy}{dx}=\frac{1+ye^{-x}}{e^{-x}-2y}\left(=\frac{e^{x}+y}{1-2ye^{x}}\right)$$ A1

b. At $P(0,1)$:

$$\frac{dy}{dx}=\frac{1+e^{0}}{e^{0}-2}$$
$$=-2$$

Let the gradient of the normal be m_{N} and $m_{\text{N}}=\dfrac{-1}{\frac{dy}{dx}}$.

$m_N=\dfrac{1}{2}$ M1

$y-1=\dfrac{1}{2}(x-0)$

$y=\dfrac{x}{2}+1$ A1

Question 6 (4 marks)

a. Substitute $\operatorname{cosec}(x)=\dfrac{1}{\sin(x)}$ and $\cot(x)=\dfrac{\cos(x)}{\sin(x)}$ into the equation.

$$\frac{1}{\sin(x)}+\frac{\cos(x)}{\sin(x)}=2\sin(x)$$

$1+\cos(x)=2\sin^2(x)$ M1

Use $\sin^2(x)=1-\cos^2(x)$ to obtain $1+\cos(x)=2\left(1-\cos^2(x)\right)$.

$$1+\cos(x)=2\left(1-\cos^2(x)\right)$$
$$\cos(x)+1=2-2\cos^2(x)$$

$2\cos^2(x)+\cos(x)-1=0$ A1

b. **Method 1:**

Factorise and solve $2\cos^2(x)+\cos(x)-1=0$.

$(2\cos(x)-1)(\cos(x)+1)=0$ M1

$\cos(x)=\dfrac{1}{2},-1$

$\Rightarrow x=\dfrac{\pi}{3}$ as $x\neq\pi$ A1

Method 2:

Use the quadratic formula on $2\cos^2(x)+\cos(x)-1=0$.

$$\cos(x)=\frac{-1\pm\sqrt{1^2-4(2)(-1)}}{4}$$ M1

$$\cos(x)=\frac{1}{2},-1$$

$$\Rightarrow x=\frac{\pi}{3} \text{ as } x\neq\pi$$ A1

Question 7 (4 marks)

a. $$\frac{3}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}$$

$$=\frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

So, $3=A(x-2)+B(x+1)$. M1

Method 1:

When $x=2$: When $x=-1$:

$3=A(2-2)+B(2+1)$ $3=A(-1-2)+B(-1+1)$

$3=B(3)$ $3=A(-3)$

$B=1$ $A=-1$

$$\frac{3}{(x+1)(x-2)}=\frac{1}{x-2}-\frac{1}{x+1}$$ A1

Method 2:

$3=Ax-2A+Bx+B$

Equate coefficients to obtain $A+B=0$ and $-2A+B=3$.

Solving simultaneously gives $A=-1$ and $B=1$.

$$\frac{3}{(x+1)(x-2)}=\frac{1}{x-2}-\frac{1}{x+1}$$ A1

Method 3:

Use the cover-up method to obtain $\dfrac{3}{(x+1)(x-2)}=\dfrac{\frac{3}{-1-2}}{x+1}+\dfrac{\frac{3}{2+1}}{x-2}$.

$$\frac{3}{(x+1)(x-2)}=\frac{1}{x-2}-\frac{1}{x+1}$$ A1

b. Separate variables and antidifferentiate both sides.

$$\int \frac{3}{(x+1)(x-2)}dx = \int 3t^2 dt$$

$$\log_e(x-2) - \log_e(x+1) = t^3 + C$$ M1

$$\log_e\left(\frac{x-2}{x+1}\right) = t^3 + C$$

$$\frac{x-2}{x+1} = e^{t^3+C}$$

$$= e^C e^{t^3}$$

Writing $k = e^C$ gives $\frac{x-2}{x+1} = ke^{t^3}$. A1

Question 8 (3 marks)

Let the length of C be L, where $L = \int_0^{\log_e(k)} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{3x} - e^{-3x}\right)$$

Express L as a definite integral.

$$L = \int_0^{\log_e(k)} \sqrt{1+\frac{1}{4}\left(e^{3x} - e^{-3x}\right)^2}\,dx$$

$$= \int_0^{\log_e(k)} \sqrt{\frac{1}{4}\left(e^{3x} + e^{-3x}\right)^2}\,dx$$ M1

So, $L = \int_0^{\log_e(k)} \frac{1}{2}\left(e^{3x} + e^{-3x}\right)dx$. A1

$$L = \frac{1}{6}\left[e^{3x} - e^{-3x}\right]_0^{\log_e(k)}$$

$$= \frac{1}{6}\left(e^{3\log_e(k)} - e^{-3\log_e(k)}\right)$$

$$= \frac{1}{6}\left(e^{\log_e(k^3)} - e^{\log_e(k^{-3})}\right)$$

$$= \frac{1}{6}\left(k^3 - \frac{1}{k^3}\right)$$ A1

$$L = \frac{1}{6}\left(k^3 - \frac{1}{k^3}\right)$$

Question 9 (5 marks)

a. From the formula sheet, $g'(x)=\dfrac{1}{1+x^2}$.

Use chain rule differentiation on $f(x)=\arctan\left(\dfrac{x+1}{1-x}\right)$.

Let $u=\dfrac{x+1}{1-x}$ so that $y=\arctan(u)$.

Find $\dfrac{du}{dx}\left(=\dfrac{d}{dx}\left(\dfrac{x+1}{1-x}\right)\right)$.

$$\frac{du}{dx}=\frac{(1-x)(1)-(x+1)(-1)}{(1-x)^2}$$ M1

$$=\frac{2}{(1-x)^2}$$ A1

$\dfrac{du}{dx}=\dfrac{2}{(1-x)^2}$ and $\dfrac{dy}{du}=\dfrac{1}{1+u^2}$.

$$f'(x)=\frac{1}{\left(1+\left(\frac{x+1}{1-x}\right)^2\right)}\left(\frac{2}{(1-x)^2}\right)$$

$$=\left(\frac{(1-x)^2}{(1-x)^2+(x+1)^2}\right)\left(\frac{2}{(1-x)^2}\right)$$

$$=\frac{2}{2+2x^2}$$

$$=\frac{1}{1+x^2}$$

$f'(x)=\dfrac{1}{1+x^2}$ and $f'(x)=g'(x)$. A1

b. Use $\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\tan(A+B)$ with $A=\arctan(x)$ and $B=\arctan(1)$.

$$\text{LHS}=\frac{\tan(\arctan(x))+\tan(\arctan(1))}{1-\tan(\arctan(x))\tan(\arctan(1))}$$ M1

$$=\frac{x+1}{1-x}$$

$$\frac{x+1}{1-x}=\tan(\arctan(x)+\arctan(1))$$

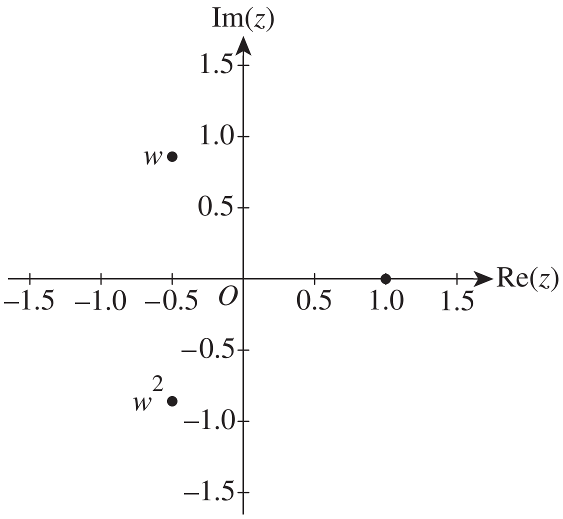
$$\Rightarrow \arctan\left(\frac{x+1}{1-x}\right)=\arctan(x)+\arctan(1)$$

So, $f(x)=\arctan(x)+\arctan(1)$ (for $x<1$) and

$\dfrac{d}{dx}(\arctan(x)+\arctan(1))=\dfrac{1}{1+x^2}=g'(x)$. A1

Question 10 (6 marks)

a.



$w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ A1

b. **Method 1:**

$w^3 = 1 \Rightarrow (w-1)\left(w^2 + w + 1\right) = 0$ A1

$\Rightarrow 1 + w + w^2 = 0$ for $w \neq 1$ A1

Method 2:

$1 + w + w^2 = 1 + \text{cis}\left(\frac{2\pi}{3}\right) + \text{cis}\left(\frac{4\pi}{3}\right)$ A1

$1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$ and so $1 + w + w^2 = 0.$ A1

Method 3:

Form the three vectors $\underset{\sim}{i}, -\frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}$ and $-\frac{1}{2}\underset{\sim}{i} - \frac{\sqrt{3}}{2}\underset{\sim}{j}.$ A1

$\underset{\sim}{i} + \left(-\frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}\right) + \left(-\frac{1}{2}\underset{\sim}{i} - \frac{\sqrt{3}}{2}\underset{\sim}{j}\right) = \underset{\sim}{0}$ (can be shown geometrically)

and so $1 + w + w^2 = 0.$ A1

Method 4:

$1 + w + w^2 = \frac{1 - w^3}{1 - w}$ A1

$= \frac{0}{1 - w} = 0 \Rightarrow 1 + w + w^2 = 0$ A1

Method 5:

Consider that, for $z^3 - 1 = 0,$ the sum of the roots is zero. A1

The sum of the roots (1, w and w^2) and so $1 + w + w^2 = 0.$ A1

c. **Method 1:**

$1+w=-w^2$ and $1+w^2=-w$ (both obtained from $1+w+w^2=0$)

Expand $(z-1)\left(z+w^2\right)(z+w)$.

$$\begin{aligned}(z-1)\left(z+w^2\right)(z+w)&=z^3-\left(1-w-w^2\right)z^2+\left(-w-w^2+w^3\right)z-w^3\\&=z^3-\left(1-\left(w+w^2\right)\right)z^2+\left(-\left(w+w^2\right)+w^3\right)z-w^3\\&=z^3-2z^2+2z-1\end{aligned}$$ M1, A1

$z^3-2z^2+2z-1=0$ A1

Note: Attempts to expand $(z-1)(z-(1+w))\left(z-(1+w)^2\right)$ *should be awarded marks as above.*

Method 2:

The required cubic equation is of the form $z^3+bz^2+cz+d=0$, where $b, c, d \in Z$.

Find the values of *b*, *c* and *d*.

Sum of roots:

$$\begin{aligned}1+(1+w)+\left(1+w^2\right)&=3+w+w^2\\&=3+(-1)\\&=2\end{aligned}$$

Product of roots:

$$\begin{aligned}(1)(1+w)\left(1+w^2\right)&=1+w+w^2+w^3\\&=0+1\\&=1\end{aligned}$$

Sum of the product of the roots taken two at a time:

$$\begin{aligned}(1)(1+w)+(1)\left(1+w^2\right)+(1+w)\left(1+w^2\right)&=1+\left(1+w+w^2\right)+\left(1+w+w^2\right)+w^3\\&=1+0+0+1\\&=2\end{aligned}$$

attempts to find sum of roots, product of roots and sum of product of roots taken two at a time M1

correct sum of roots, product of roots and sum of product of roots taken two at a time A1

$b=-2$, $c=2$ and $d=-1$

Hence, $z^3-2z^2+2z-1=0$. A1

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