Neap

Trial Examination 2021

VCE Specialist Mathematics Units 3&4

Written Examination 2

Question and Answer Booklet

Reading time: 15 minutes
Writing time: 2 hours

Student's Name: ______________________________

Teacher's Name: ______________________________

Structure of booklet

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.

Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

Question and answer booklet of 25 pages

Formula sheet

Answer sheet for multiple-choice questions

Instructions

Write your **name** and your **teacher's name** in the space provided above on this page, and on the answer sheet for multiple-choice questions.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

All written responses must be in English.

At the end of the examination

Place the answer sheet for multiple-choice questions inside the front cover of this booklet.

You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – MULTIPLE-CHOICE QUESTIONS

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this booklet are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1

The graph with equation $y = \dfrac{x^2 - 4k^2}{x - k}$, where $x \neq k$ and k is a non-zero constant, has asymptotes given by

A. $x = k$ and $y = x - k$.

B. $x = k$ and $y = x + k$.

C. $x = k$ only.

D. $y = x + k$ only.

E. $x = k$ and $y = -3k^2$.

Question 2

Consider the graph of $y = \arccos(a - 2x) - \dfrac{\pi}{4}$, where $\dfrac{a-1}{2} \leq x \leq \dfrac{a+1}{2}$ and $a \in R$.

The graph has a

A. maximum gradient of 2 at the point $\left(\dfrac{a}{2}, -\dfrac{\pi}{4}\right)$.

B. minimum gradient of $\dfrac{1}{2}$ at the point $\left(\dfrac{a}{2}, \dfrac{\pi}{4}\right)$.

C. minimum gradient of 2 at the point $\left(\dfrac{a}{2}, -\dfrac{\pi}{4}\right)$.

D. minimum gradient of 2 at the point $\left(\dfrac{a}{2}, \dfrac{\pi}{4}\right)$.

E. maximum gradient of 2 at the point $\left(\dfrac{a}{2}, \dfrac{\pi}{4}\right)$.

Question 3

The expression $\cot(ax)+\tan(bx)$, where $a, b \in Z$, is equal to

A. $\dfrac{\sin((a-b)x)}{\sin(ax)\cos(bx)}$

B. $\dfrac{\sin((a+b)x)}{\sin(ax)\cos(bx)}$

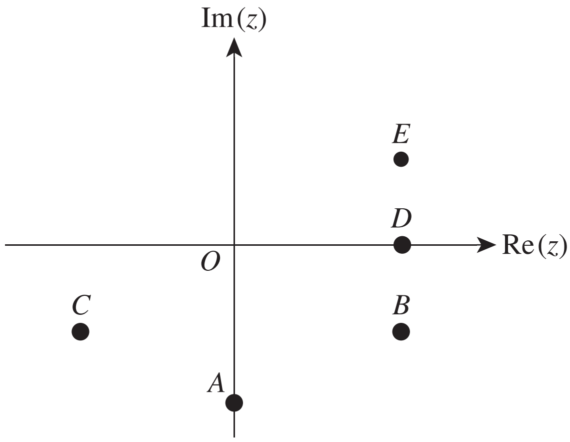
C. $\dfrac{\cos((a+b)x)}{\sin(ax)\cos(bx)}$

D. $\dfrac{\cos(ax)+\sin(bx)}{\sin(ax)\cos(bx)}$

E. $\dfrac{\cos((a-b)x)}{\sin(ax)\cos(bx)}$

Question 4

Consider the following points on the complex plane.



Given that $z = x + iy$, two points in the complex plane that could represent the square roots of z are

A. C and E.

B. A and D.

C. B and E.

D. B and C.

E. D and E.

Question 5

Consider the complex number $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

The smallest integer value of n such that $\left|z^n\right| > 100$ and z^n is real is

A. 6

B. 7

C. 12

D. 13

E. 18

Question 6

Let u, v and $\overline{v}$, where $\overline{v}$ is the complex conjugate of v, be the roots of the equation $z^3 - 7z^2 + 17z - 15 = 0$.

It is **not** true that

A. $v + \overline{v} = 4$

B. $\left|v - \overline{v}\right| = 2$

C. $v\overline{v} = 3$

D. $u + v + \overline{v} = 7$

E. $\left|v\right| + \left|\overline{v}\right| = 2\sqrt{5}$

Question 7

Consider the curve $y = -x^3 + 2x^2 + 1$ for $x \in R$.

This curve

A. does not change concavity for $x \in R$.

B. is concave down on the interval $\left(-\infty, -\frac{2}{3}\right)$ and concave up on the interval $\left(-\frac{2}{3}, \infty\right)$.

C. is concave up on the interval $\left(-\infty, -\frac{2}{3}\right)$ and concave down on the interval $\left(-\frac{2}{3}, \infty\right)$.

D. is concave down on the interval $\left(-\infty, \frac{2}{3}\right)$ and concave up on the interval $\left(\frac{2}{3}, \infty\right)$.

E. is concave up on the interval $\left(-\infty, \frac{2}{3}\right)$ and concave down on the interval $\left(\frac{2}{3}, \infty\right)$.

Question 8

A solid is formed by rotating the curve $y = \sqrt{x}\sin(x)$ through 2π radians about the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

The volume of this solid is given by

A. $\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(x - x\cos(2x)\right)dx$

B. $\pi\int_0^{\frac{\pi}{2}}\left(x - x\cos(2x)\right)dx$

C. $\pi\int_0^{\frac{\pi}{2}}\left(\sqrt{x}\sin(x)\right)dx$

D. $\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(x\cos(2x) - x\right)dx$

E. $\pi\int_0^{\frac{\pi}{2}}\left(x\cos(2x) - x\right)dx$

Question 9

Water containing 7 grams of sugar per litre is flowing at 6 L per minute into a large tank that initially contained 150 L of pure water. The concentration of sugar in the tank is kept uniform by stirring, and the mixture flows out of the tank at 8 L per minute.

If S grams is the amount of sugar in the tank t minutes after the water begins to flow, then $\frac{dS}{dt}$ is equal to

A. $42 - \frac{S}{150 - 2t}$

B. $42 - \frac{8S}{150}$

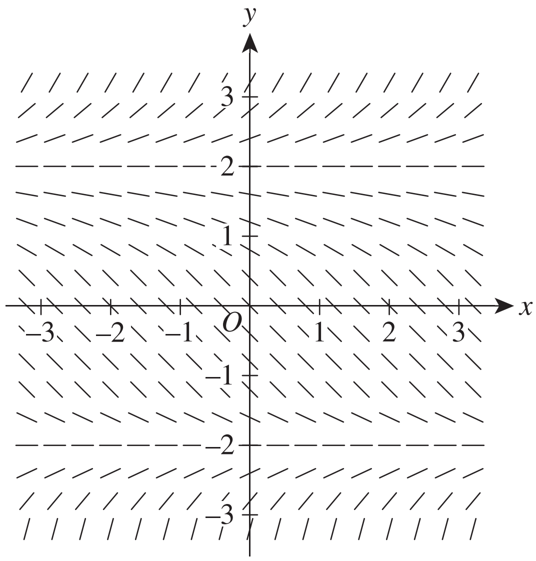
C. $42 - \frac{8S}{150 - 2t}$

D. $56 - \frac{8S}{150 + 2t}$

E. $56 - \frac{8S}{150 - 2t}$

Question 10

Consider the following direction field.



The differential equation that is best represented by this direction field is

A. $\dfrac{dy}{dx} = \dfrac{y-2}{2}$

B. $\dfrac{dy}{dx} = \dfrac{y^2-4}{4}$

C. $\dfrac{dy}{dx} = \dfrac{x-2}{2}$

D. $\dfrac{dy}{dx} = \dfrac{x^2-4}{4}$

E. $\dfrac{dy}{dx} = \dfrac{(y-2)^2}{4}$

Question 11

A unit vector makes an angle of 60° with the positive direction of the x-axis, an angle of 45° with the positive direction of the y-axis and an obtuse angle with the positive direction of the z-axis.

This unit vector is represented by

A. $\dfrac{1}{2}\left(\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j} + \underset{\sim}{k}\right)$

B. $\dfrac{1}{\sqrt{6}}\left(\sqrt{3}\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j} - \underset{\sim}{k}\right)$

C. $\dfrac{1}{\sqrt{6}}\left(\sqrt{3}\underset{\sim}{i} - \sqrt{2}\underset{\sim}{j} + \underset{\sim}{k}\right)$

D. $\dfrac{1}{2}\left(\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j} - \underset{\sim}{k}\right)$

E. $\dfrac{1}{\sqrt{2}}\left(\sqrt{3}\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$

Question 12

Given two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, a geometrical interpretation of the expression $\frac{1}{|\underset{\sim}{b}|}|\underset{\sim}{b}\cdot\underset{\sim}{a}|$ is the

A. vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.

B. scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$.

C. scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.

D. magnitude of the scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$.

E. magnitude of the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.

Question 13

The position vector of a particle at time *t* is given by $\underset{\sim}{r}(t)=(2t-1)\underset{\sim}{i}+t^2\underset{\sim}{j}$, where $t \geq 0$.

The cartesian equation of the particle's path is given by

A. $y=\left(\frac{x-1}{2}\right)^2, x \geq -1$

B. $y=\left(\frac{x+1}{2}\right)^2, x \geq 1$

C. $y=\left(\frac{x+1}{2}\right)^2, x \geq 0$

D. $y=\left(\frac{x+1}{2}\right)^2, x \geq -1$

E. $y^2=\frac{x+1}{2}, x \geq 0$

Question 14

The position vector of a particle at time *t* is given by $\underset{\sim}{r}(t)=3\cos(t)\underset{\sim}{i}+\sqrt{3}\sin(t)\underset{\sim}{j}$. $t \geq 0$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the directions east and north respectively.

At $t=\frac{\pi}{6}$, the particle is moving in a

A. south-easterly direction.

B. south-westerly direction.

C. north-easterly direction.

D. westerly direction.

E. north-westerly direction.

Question 15

The velocity, v, of a particle moving in a straight line at position x from the origin is given by $v^2 = 6 + 4x - 2x^2$, where $-1 \le x \le 3$.

The particle does **not**

A. change direction at $x = -1$ and $x = 3$.

B. have acceleration given by $a = -2(x-1)$.

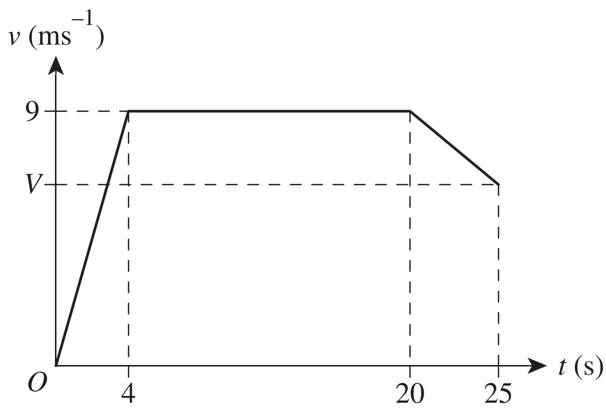
C. have an acceleration of zero at $x = 1$.

D. remain stationary at $x = 1$.

E. travel at maximum velocity at $x = 1$.

Question 16

An athlete completes a 200 m race in 25 s. The athlete starts from rest and accelerates uniformly to a velocity of 9 ms^{-1} in 4 s. They maintain this velocity for 16 s before decelerating uniformly to a speed of V ms^{-1} at the end of the race. The athlete's motion is represented by the velocity–time graph below.



The athlete finishes the race running at a velocity of

A. 0.56 ms^{-1}

B. 6.2 ms^{-1}

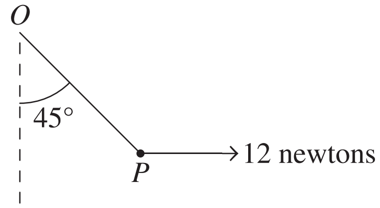
C. 7 ms^{-1}

D. 8 ms^{-1}

E. 9 ms^{-1}

Question 17

A particle P of mass m kg is attached to one end of a light inextensible string. The other end of the string is attached to a fixed point O. A horizontal force of magnitude 12 newtons is applied to P. The particle P is in equilibrium with the string taut and OP makes an angle of 45° with the downward vertical, as shown in the diagram below.



The mass of particle P, in kilograms, is

A. $\dfrac{g}{12}$

B. $12\sqrt{2}g$

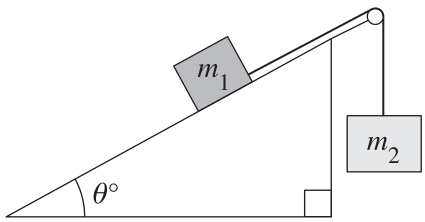
C. $\dfrac{12}{g}$

D. 12g

E. 12

Question 18

A particle of mass m_1 kg is held at rest on a smooth plane inclined at θ degrees to the horizontal. It is connected to a particle of mass m_2 kg by a light inextensible string parallel to the plane that passes over a smooth pulley at the end of the plane, as shown in the diagram below. The tension in the string is T newtons.



The angle θ is equal to

A. $\arcsin\left(\dfrac{m_2}{m_1}\right)$

B. $\arccos\left(\dfrac{m_2}{m_1}\right)$

C. $\arcsin\left(\dfrac{m_1}{m_2}\right)$

D. $\arccos\left(\dfrac{m_1}{m_2}\right)$

E. $\arctan\left(\dfrac{m_2}{m_1}\right)$

Question 19

Let $x_1, x_2, ..., x_{50}$ be 50 independent observations from a distribution of a random variable X that is not normal. It is known that the mean of this distribution is 24 and the standard deviation is 3.

Regarding the distribution of the sample mean $\bar{X}$, it is correct to say that

A. the distribution of $\bar{X}$ is approximately normal with mean 24 and standard deviation $\frac{3}{\sqrt{50}}$.

B. the distribution of $\bar{X}$ is approximately normal with mean 24 and standard deviation 3.

C. the distribution of $\bar{X}$ is approximately normal with mean 24 and standard deviation $\frac{9}{50}$.

D. the distribution of $\bar{X}$ cannot be approximated with the normal distribution since X is not normal.

E. $\bar{X} = 24$.

Question 20

A battery manufacturer wishes to find a confidence interval for μ, the mean voltage of a certain type of battery. The manufacturer randomly selects n batteries, measures their voltages in volts and obtains an approximate 90% confidence interval for μ of $(4.884, 4.916)$.

Given that the voltages of batteries are known to be normally distributed with a standard deviation of 0.1 volts, the value of n is closest to

A. 27

B. 38

C. 65

D. 106

E. 151

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (10 marks)

Let $I_1 = \int_0^{\frac{\pi}{4}} \tan(x)\, dx$.

a. Show that $I_1 = \frac{1}{2}\log_e(2)$. 2 marks

Consider $I_n = \int_0^{\frac{\pi}{4}} \tan^n(x)\, dx$ for $n \in Z, n \geq 2$.

b. Show that $I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\left(\sec^2(x) - 1\right) dx$ for $n \in Z, n \geq 2$. 2 marks

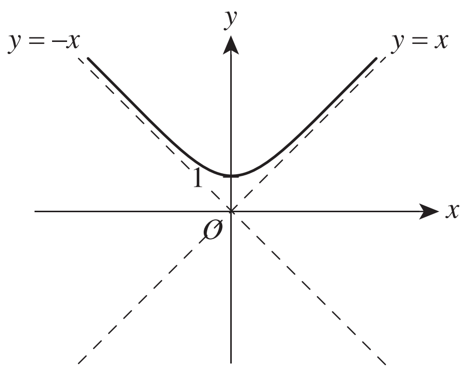
c. Show that $I_n = \frac{1}{n-1} - I_{n-2}$ for $n \in Z, n \geq 2$. 3 marks

d. Using the results from **part a.** and **part c.**, show that $I_5 = \frac{1}{2}\log_e(2) - \frac{1}{4}$. 3 marks

Question 2 (9 marks)

A curve C is defined parametrically by $x = \tan(s)$, $y = \sec(s)$, where $-\frac{\pi}{2} < s < \frac{\pi}{2}$.

Curve C has oblique asymptotes $y = x$ and $y = -x$, as shown in the graph below.



a. Show that the cartesian equation of curve C is given by $y^2 - x^2 = 1$. 1 mark

b. Show that the normal to curve C at the point $P(\tan(s), \sec(s))$ for $0 < s < \frac{\pi}{2}$ is given by $y = -x\operatorname{cosec}(s) + 2\sec(s)$. 3 marks

The normal to curve *C* at point *P* intersects the *x*-axis at point *N*. Let *A* be the area of triangle *OPN*, where *O* is the origin.

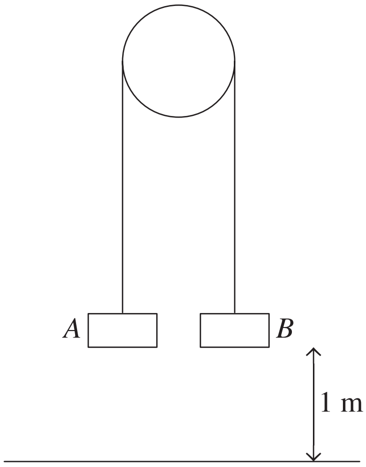
c. Show that $A = \tan(s)\sec(s)$. 2 marks

d. Point *P* moves along curve *C* such that the rate of change of *s* with respect to time, *t*, is given by $\frac{ds}{dt} = \cos(s)$.

Find $\frac{dA}{dt}$ when $s = \frac{\pi}{6}$. 3 marks

Question 3 (9 marks)

Two particles, *A* and *B*, of mass 0.6 kg and 0.4 kg respectively are attached to the ends of a light inextensible string that passes over a smooth pulley. Both particles are held, with the string taut, at a height of 1 m above a horizontal floor as shown in the diagram below.



a. On the diagram above, show all the forces acting on both particles. 1 mark

The two particles are released from rest and, in the subsequent motion, particle *B* does not reach the pulley.

b. Find the tension in the string immediately after the particles are released. 3 marks

c. Find the acceleration of particle *A* immediately after the particles are released. 1 mark

d. After the particles have been moving for 0.5 seconds, the string breaks.

How long does it take for particle B to hit the floor after the string breaks? Give your answer correct to two decimal places. 4 marks

Question 4 (12 marks)

A particle of mass m kg falls from rest from a point O. While in free fall, the forces acting on the particle are its weight mg newtons and air resistance mkv newtons, where k is a positive constant and v ms^{-1} is the velocity of the particle at time t seconds.

a. Draw a diagram showing the forces acting on the particle. 1 mark

b. Show that $a = g - kv$, where a ms^{-2} is the particle's acceleration. 1 mark

c. Given that the particle's limiting (terminal) velocity is V, show that $V = \dfrac{g}{k}$. 1 mark

d. By solving the differential equation from **part b.** and using the result from **part c.**, show that $v = V\left(1 - e^{-kt}\right)$. 3 marks

At the same time the free-falling particle starts its motion, a second particle of mass m kg is projected upwards from O with initial velocity U ms^{-1}. The forces acting on this particle are its weight mg newtons and air resistance mkv newtons, where k is a positive constant and v ms^{-1} is the velocity of the particle at time t seconds.

e. Find the time taken for the second particle to reach its maximum height. 3 marks

f. Show that, when the second particle reaches its maximum height, the velocity of the first particle is $\dfrac{UV}{U+V}$ ms^{-1}. 3 marks

Question 5 (13 marks)

A relation in the complex plane is given by $\text{Arg}(u-8i)=-\dfrac{\pi}{6}$, where $u \in C$.

a. Show that the cartesian equation of this relation is given by $y=-\dfrac{1}{\sqrt{3}}x+8,\ x>0$. 3 marks

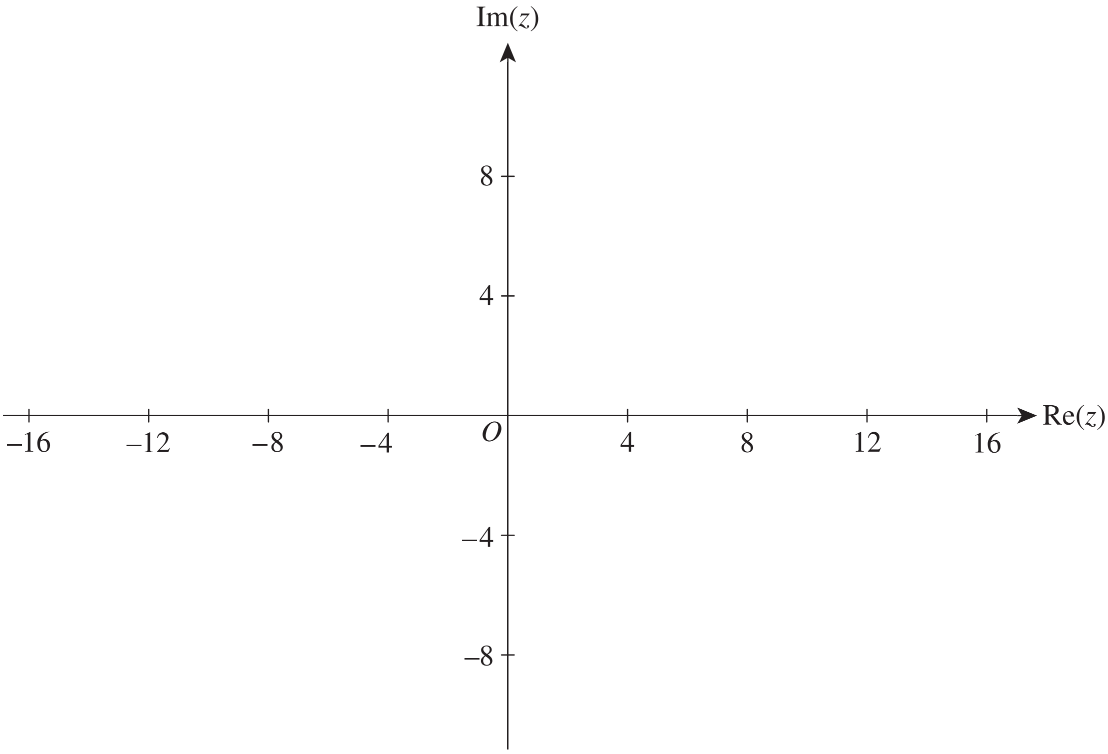
b. State a geometrical interpretation of this relation. 1 mark

A relation in the complex plane is given by $(v-2-2i)(\overline{v}-2+2i)=8,$ where $v \in C$.

c. Show that the cartesian equation of this relation is given by $(x-2)^2+(y-2)^2=8$. 2 marks

d. State a geometrical interpretation of this relation. 1 mark

e. Sketch the two relations $\text{Arg}(u-8i)=-\frac{\pi}{6}$ and $(v-2-2i)(\overline{v}-2+2i)=8$ on the Argand diagram below. 2 marks



f. Find the least exact value of $|v - u|$. 4 marks

Question 6 (7 marks)

The weights of apples, in grams, are known to be normally distributed with a standard deviation of 7 grams. Tom wants to test whether apples grown at his farm have a mean weight greater than 83 grams.

a. Write down suitable hypotheses, H_0 and H_1, for this test. 1 mark

Tom weighs eight apples and determines that their mean weight is 86 grams.

b. Testing at the 5% significance level, would Tom's apples have a mean weight greater than 83 grams? Justify your answer. 2 marks

c. Write down the probability of rejecting H_0 when it is true. 1 mark

d. Find the minimum mean weight of the eight apples that would lead Tom to reject H_0. Give your answer correct to one decimal place. 2 marks

e. Given that the mean weight of Tom's apples is actually 81.8 grams, find the probability of not rejecting H_0. Give your answer correct to three decimal places. 1 mark

END OF QUESTION AND ANSWER BOOKLET