MATHEMATICS

3C/3D

Calculator-assumed

WACE Examination 2014

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

Section Two: Calculator assumed (100 marks)

Question 10 (5 marks)

The freshly hatched caterpillar weighs approximately 1.5 milligrams. It will grow to 1000 times its initial body weight in just three weeks. The rate of growth of the weight can be expressed by the differential equation $\frac{dw}{dt} = kw$ where $w =$ weight of the caterpillar at time t, in days, and k is a constant.

(a) Determine the value of k to three decimal places. (3 marks)

Solution
$w = 1.5e^{kt}$ When $t = 21$, $w = 1500 \Rightarrow \quad 1500 = 1.5e^{21k}$ $k = 0.329$
Specific behaviours
✓ uses $w = w_0e^{kt}$ as solution to differential equation ✓ solves for k ✓ rounds correctly to three decimal places

(b) When is the caterpillar double its initial body weight? (2 marks)

Solution
$3 = 1.5e^{0.329t}$ $t = 2.1$ After 2.1 days
Specific behaviours
✓ substitutes correct values ✓ solves correctly for t

Question 11 **(11 marks)**

A family is going camping for a week and will be taking their pet with them. They can only find room for five kilograms of food for their pet and will bring only two types of pet food, type X and type Y. Each kilogram of type X food has 10 grams of fat, 9 grams of carbohydrate and 12 grams of protein. Each kilogram of type Y has 15 grams of fat, 9 grams of carbohydrate and 6 grams of protein. The pet will need at least 30 grams of fat, 27 grams of carbohydrate and 24 grams of protein for the week.

The cost of type X food is 20 cents per kilogram and for type Y is 30 cents per kilogram.

Let x = number of kilograms of type X food and let y = number of kilograms of type Y food.

(a) The constraints above can be written as inequalities. One constraint is missing. Write the missing constraint as an inequality in terms of x and y in simplified form. (2 marks)

$9x + 9y \geq 27$

$12x + 6y \geq 24$

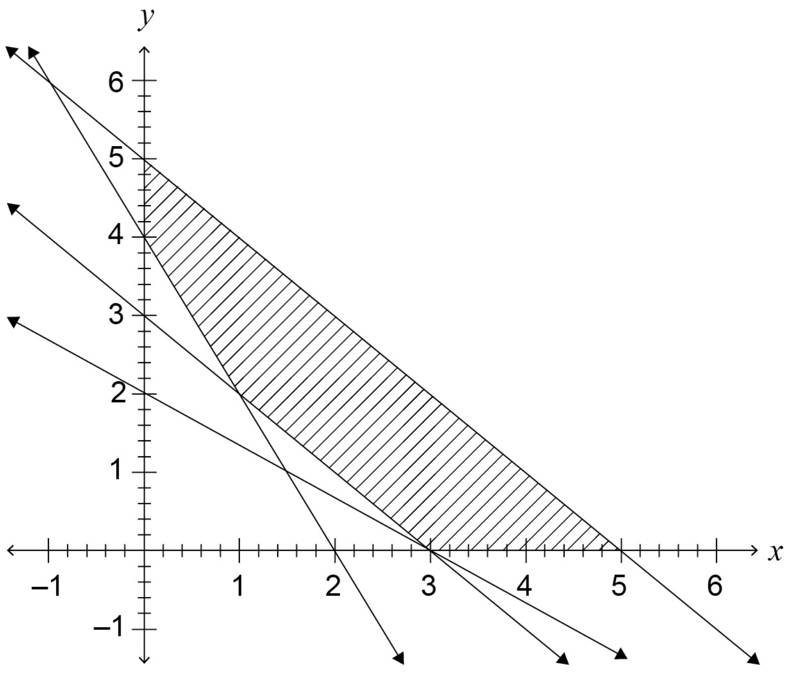
$x + y \leq 5$

$x \geq 0$

$y \geq 0$

Solution
The constraint for fat content is missing. $10x + 15y \geq 30 \Rightarrow 2x + 3y \geq 6$
Specific behaviours
✓ writes constraint for the fat content of the pet food as an inequality ✓ simplifies inequality

(b) The inequalities are shown on the graph below. Add the missing inequality and shade the feasible region. (2 marks)

Solution

Specific behaviours
✓ plots line of missing inequality ✓ shades feasible region

(c) Determine the optimal amounts of each type of pet food to minimise the cost. State this minimum cost. (4 marks)

Solution

Vertex	**Cost \$** $C = 0.20x + 0.30y$
$(0, 5)$	\$1.50
$(0, 4)$	\$1.20
$(1, 2)$	\$0.80
$(3, 0)$	\$0.60
$(5, 0)$	\$1.00

Minimum cost occurs when $x = 3$ and $y = 0$. The cost is \$0.60 (60c).

Specific behaviours

✓ examines at least three vertices
✓ states objective function
✓ determines optimal point
✓ states minimum cost

(d) To ensure a larger amount of type Y food, by how much should the price per kilogram of type Y food fall to maintain the minimum cost from part (c)? (3 marks)

Solution

Optimal changes from $(3,0)$ to $(1,2)$ for $C = 0.20x + ky$

$0.2(3) + 0 = 0.2(1) + 2k$

$0.4 = 2k$

$k = 0.2$

A drop of 10 cents per kilogram would be required.

Specific behaviours

✓ identifies that optimal point moves to $(1,2)$
✓ constructs an equation to solve for new cost per kilogram for food Y
✓ determines the price drop for food Y per kilogram

Question 12 **(5 marks)**

A bicycle is travelling at a constant speed of 20 kilometres per hour.

(a) Determine the distance, in metres, that the bicycle travels in one second. (1 mark)

Solution
$\frac{20\,\text{km}}{1\,\text{hour}} = \frac{20\,000\,\text{m}}{3600\,\text{s}} = 5.\dot{5}\,\text{m/s}$ Therefore the bicycle travels 5.5 metres. (accept 5.6)
Specific behaviours
✓ coverts speed into metres per second

When the brakes of the bicycle are applied, this results in a deceleration (negative acceleration) of 10 metres per second squared. Let t represent the time from when the brakes are initially applied, in seconds.

(b) State the velocity function of the bicycle, in metres per second, in terms of t after the brakes are applied. (2 marks)

Solution
$v = \int -10\,dt = -10t + c$ When $t = 0,\ v = 5.\dot{5};\ \Rightarrow c = 5.\dot{5}$ $\therefore v = 5.\dot{5} - 10t$ Accept 5.5 or 5.6 in function
Specific behaviours
✓ antidifferentiates a to find v ✓ substitutes values correctly to find the constant c

(c) How far will the bicycle travel while braking before it stops? (2 marks)

Solution
The bicycle stops when $5.\dot{5} - 10t = 0 \quad \Rightarrow t = 0.\dot{5}$ Distance travelled = $\int_0^{0.\dot{5}} 5.\dot{5} - 10t\ dt = 1.54$ m (Accept 1.51 to 1.56)
Specific behaviours
✓ determines the time when the bicycle stops moving ✓ determines distance travelled

Question 13 **(6 marks)**

James travels to school in one of three ways. Thirty per cent of the time he rides his bicycle (**B**), 50% of the time his mother drives him (**C**) and the rest of the time he walks (**W**). When he rides his bicycle, there is a 20% chance of his having a puncture that will make him late for school (**L**). On the days he walks, he is never late for school.
Overall, James is late to school 21% of the time.

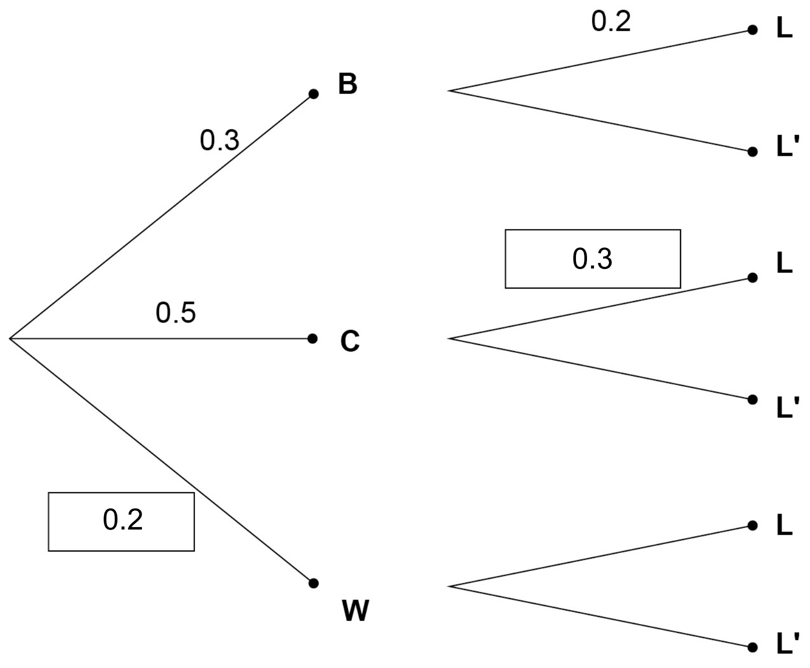
(a) Part of the tree diagram is shown above. Write the two unknown probabilities in the boxes above. (3 marks)

Solution

$P(W) = 0.2$

$P(B \cap L') + P(C \cap L') + P(W \cap L') = 0.79$
$0.24 + P(C \cap L') + 0.2 = 0.79$
$P(C \cap L') = 0.35$
$P(L'|C) = 0.70$

$\therefore$ P(late when being driven) $= 0.3$



Specific behaviours

- ✓ states the probability of walking
- ✓ calculates the probabilities associated with being on time
- ✓ determines the probability of being late when being driven

(b) On a day when he arrives late for school, what is the probability that he has ridden his bicycle? (3 marks)

Solution
$\dfrac{0.06}{0.06+0.15}=\dfrac{2}{7}$ Accept 0.286
Specific behaviours
✓ applies conditional probability to calculate reduced sample space (denominator) ✓ states correct numerator ✓ calculates probability

Question 14 **(16 marks)**

A study found that 80% of people exhibiting common influenza symptoms recovered without taking any medication. A random sample of 30 people who had developed influenza symptoms was taken.

Let X denote the number of people who recovered without taking any medication in this sample.

(a) Is X discrete or continuous? (1 mark)

Solution
Discrete as X is in integer values
Specific behaviours
✓ states X is discrete

(b) State the probability distribution of X and the mean and standard deviation of this distribution. (3 marks)

Solution
Binomial distribution $X \sim Bin(30, 0.8)$ $\mu = 30 \times 0.8 = 24$ $\sigma = \sqrt{30 \times 0.8 \times 0.2} = 2.191$
Specific behaviours
✓ states binomial distribution ✓ calculates mean ✓ calculates standard deviation

(c) What is the probability, correct to three decimal places, that

(i) exactly 25 people recovered without any medication? (1 mark)

Solution
$P(X=25)=0.172$
Specific behaviours
✓ calculates probability to three decimal places

(ii) at least 24 but no more than 28 people recover without any medication? (2 marks)

Solution
$P(24 \le X \le 28)=0.596$ Accept 0.597
Specific behaviours
✓ uses correct interval ✓ calculates probability

(d) Trial groups of 30 people from each of 15 different suburbs were then surveyed. Let $\overline{Y}$ denote the mean number of people per trial group who recover without any medication.

(i) State the probability distribution of $\overline{Y}$ and the mean and standard deviation of this distribution. (3 marks)

Solution
$\overline{Y} \sim N\left(24,\left(\frac{\sigma}{\sqrt{n}}\right)^2\right) \quad n=15$ $\overline{Y} \sim N(24, 0.566^2)$ $\therefore$ mean = 24 Standard deviation = 0.566 (Accept 0.565)
Specific behaviours
✓ states that the distribution is normal ✓ states mean ✓ calculates standard deviation

(ii) Determine $P(\overline{Y} \geq 25)$. (1 mark)

Solution
$P(\overline{Y} \geq 25) = 0.0386$ Accept 0.0385 or 0.0387
Specific behaviours
✓ determines probability

(iii) Determine a 95% confidence interval, to three decimal places, for the population mean number of people per trial who recover without medication. (3 marks)

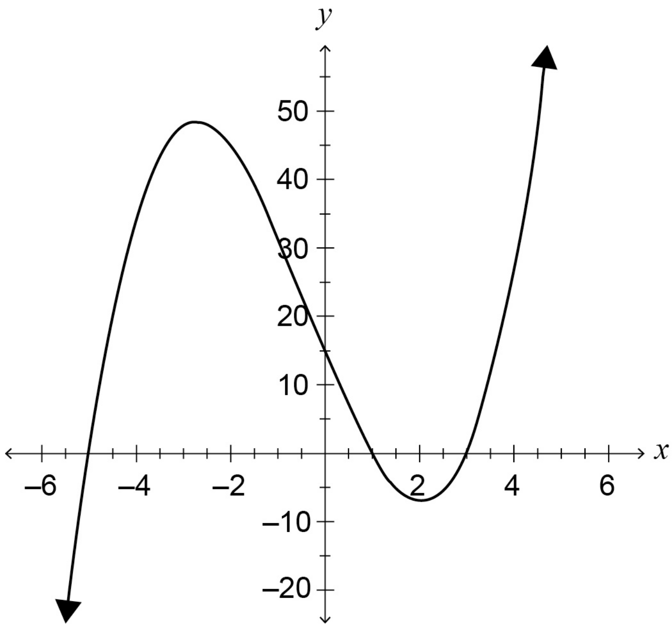
Solution
$\overline{x} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + 1.96\frac{\sigma}{\sqrt{n}}$ $24 - 1.96\frac{2.191}{\sqrt{15}} < \mu < 24 + 1.96\frac{2.191}{\sqrt{15}}$ $22.891 < \mu < 25.109$
Specific behaviours
✓ uses correct z cut-off point ✓ uses correct mean and standard deviation ✓ determines confidence interval

(iv) The researcher who conducted the trials in the 15 suburbs calculated a mean of 25 people who recovered without medication per trial. The researcher concluded that a smaller percentage of influenza sufferers take medication than has been assumed. Does this mean support her conclusion? Explain. (2 marks)

Solution
No. The mean does not support the conclusion as the mean satisfies the confidence interval.
Specific behaviours
✓ states that the conclusion is not supported by data ✓ supports answer using the confidence interval

Question 15 **(7 marks)**

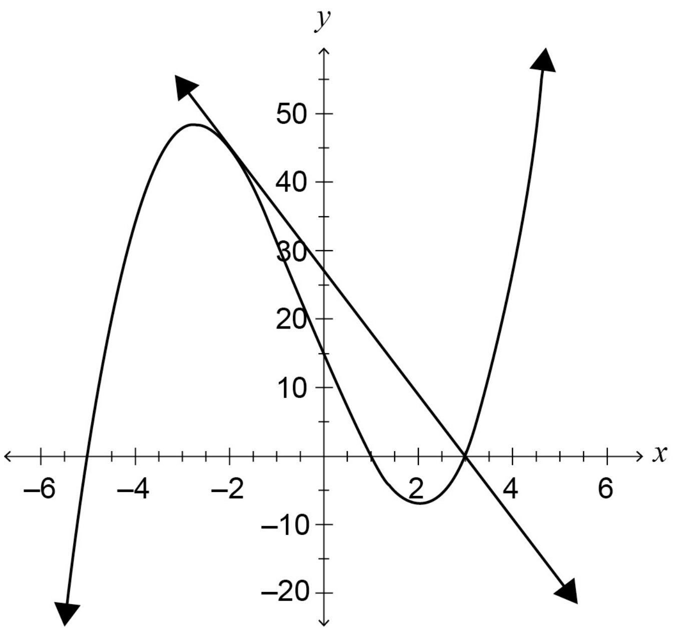
Consider the curve defined by the rule $y = x^3 + x^2 - 17x + 15$ shown below.



(a) Show that the equation of the tangent at $x = -2$ is $y = -9x + 27$. (3 marks)

Solution
$\frac{dy}{dx} = 3x^2 + 2x - 17$ $x = -2 \quad \frac{dy}{dx} = -9$ $y = -9x + c$ $x = -2 \quad y = 45$ $c = 45 - 18 = 27$ $\therefore y = -9x + 27$
Specific behaviours
✓ differentiates y ✓ calculates gradient ✓ calculates constant for tangent

(b) Determine the area enclosed between the curve and the tangent at $x = -2$. (4 marks)

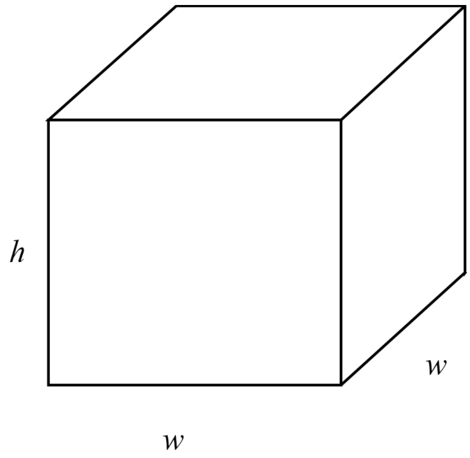
Solution

Tangent intersects curve when $x = -2, 3$
Area $= \int_{-2}^{3} (-9x+27) - \left(x^3 + x^2 - 17x + 15\right) dx$
$= \frac{625}{12} \quad (52.083)$

Specific behaviours
✓ determines where tangent meets curve ✓ expresses area as the difference between the tangent and the curve ✓ writes an integral using correct limits to determine area ✓ calculates area correctly

Question 16 **(7 marks)**

A closed box is constructed with a square base. Exactly 10 square metres of material is to be used in the construction of the box, without wastage.

Let h = height of the box, w = width of box = length of box.



(a) Show that $5 = w^2 + 2wh$. (2 marks)

Solution
Surface area $= 10 = 2w^2 + 4wh$ Simplifies to $5 = w^2 + 2wh$
Specific behaviours
✓ determines equation for total surface area in terms of h and w $(= 10\ \mathrm{m}^2)$ ✓ simplifies expression

(b) By using calculus, determine the maximum volume of the box and state the dimensions required to achieve this maximum. (5 marks)

Solution
$V = hw^2$ $h = \dfrac{5-w^2}{2w}$ So, $V = \dfrac{5-w^2}{2w}w^2 = \dfrac{5w-w^3}{2}$ $\therefore \dfrac{dV}{dw} = \dfrac{5-3w^2}{2}$ and $\dfrac{d^2V}{dw^2} = -3w$ For a maximum, $\dfrac{dV}{dw} = \dfrac{5-3w^2}{2} = 0$ $\Rightarrow w = \pm\sqrt{\dfrac{5}{3}}$ At $w = \sqrt{\dfrac{5}{3}}$, $\dfrac{d^2V}{dw^2} = -3\sqrt{\dfrac{5}{3}}$ $\therefore w = \sqrt{\dfrac{5}{3}}\,(1.2910) = h$ Volume is a maximum 2.152 cubic metres when all dimensions are 1.291 metres.
Specific behaviours
✓ expresses volume in terms of one variable only ✓ determines first derivative ✓ determines stationary points ✓ shows that positive root is a local maximum ✓ solves for the dimensions for maximum volume and states this volume

Question 17 **(7 marks)**

The label on a bottle states that it contains 330 millilitres of orange juice. The capacity of the orange juice in these bottles is normally distributed, with a mean of 365 millilitres and a standard deviation of 20 millilitres.

(a) In a batch of 100 bottles, how many bottles are expected to have less than the labelled amount? (2 marks)

Solution
$X \sim N\left(365, 20^2\right)$ $P(X < 330) = 0.04$ $\therefore$ in a batch of 100, you would expect four bottles to contained less than 330 mL
Specific behaviours
✓ determines probability ✓ determines proportion from 100 bottles

(b) Samples of 10 bottles are tested and the mean capacity for each sample is recorded.

(i) State the distribution for the sample means and their mean and standard deviation. (2 marks)

Solution
The distribution is normal. $\overline{X} \sim N\left(365, (\frac{20}{\sqrt{10}})^2\right)$ i.e. mean $= 365$ and standard deviation $= \frac{20}{\sqrt{10}}$
Specific behaviours
✓ states that the distribution is normal ✓ states mean (unchanged) and standard deviation of sample means

(ii) Calculate the probability that the sample mean is less than 360 millilitres. (1 marks)

Solution
$\overline{X} \sim N\left(365, (\frac{20}{\sqrt{10}})^2\right)$ $P(\overline{X} < 360) = 0.215$ Accept 0.214 to 0.216
Specific behaviours
✓ determines probability

(c) Determine the sample size so that there is a 99% chance that the sample mean is no more than 5 millilitres from the population mean. (2 marks)

Solution
$2.576\dfrac{\sigma}{\sqrt{n}} = 5$ $2.576\dfrac{20}{\sqrt{n}} = 5$ $n = 106.17$ $n = 107$
Specific behaviours
✓ constructs an equation to solve for sample size ✓ rounds sample size to nearest integer correctly

Question 18 **(4 marks)**

Let $I(x) = \int_{-3}^{x} g(t)dt$ with $I(5) = 208$ and $\dfrac{d^2I}{dx^2} = 6x$.

Determine the function $g(x)$.

Solution
$I(x) = \int_{-3}^{x} g(t)dt$ $I'(x) = g(x)$ $I''(x) = g'(x) = 6x$ $g(x) = 3x^2 + c$ $I(5) = \int_{-3}^{5} (3t^2 + c)dt = 208$ $\left[t^3 + ct\right]_{-3}^{5} = (125 + 5c) - (-27 - 3c) = 208$ $152 + 8c = 208$ $c = 7$ $g(x) = 3x^2 + 7$
Specific behaviours
✓ uses Fundamental Theorem of Calculus to differentiate $I(x)$ ✓ determines expression for $g(x)$ in terms of x and a constant ✓ solves for constant ✓ determines expression for $g(x)$ in terms of x only

Question 19 **(10 marks)**

A pay TV service sends a signal through the telephone lines to each of its customers every hour. The length of the signal is between two and nine seconds and follows a uniform distribution.

(a) Define the probability density function for the length of the signal. (2 marks)

Solution
$f(x)=\begin{cases}\frac{1}{7} & , 2\le x\le 9\\ 0 & \text{, elsewhere}\end{cases}$
Specific behaviours
✓ states a constant value for $2\le x\le 9$ ✓ states value of $\frac{1}{7}$ for probability density function

(b) Determine the probability that the signal is longer than four seconds. (2 marks)

Solution
Area = $(9-4)\frac{1}{7}=\frac{5}{7}$
Specific behaviours
✓ uses the area to determine probability ✓ determines probability

(c) Determine the probability that, on any given day, at least half of the signals are greater than four seconds. (3 marks)

Solution
$X\sim B\left(24,\frac{5}{7}\right)$ $P(X\ge 12)=0.9926$
Specific behaviours
✓ uses the binominal distribution ✓ uses correct values for n, p ✓ determines probability

(d) One particular day, fewer than 21 signals were longer than four seconds. Determine the probability that at least 15 were longer than four seconds. (3 marks)

Solution
$\dfrac{P(15 \le X \le 20)}{P(X \le 20)} = \dfrac{0.8246}{0.9427} = 0.8747$ Accept 0.8745 to 0.8750
Specific behaviours
✓ uses correct denominator in conditional probability calculation ✓ uses correct numerator in conditional probability calculation ✓ determines probability

Question 20 **(8 marks)**

Five members of a complex of flats form a strata committee. One of these members is an elected president. Let event A = the number of subcommittees formed with a **prime** number of people from the strata committee. Let event B = the number of subcommittees that contain the president from the strata committee. Assume that all possible subcommittees are equally likely. A subcommittee consists of one or more people from the strata committee.
(Note: the number **one** is not a prime.)

Determine

(a) the total number of subcommittees possible. (2 marks)

Solution
$\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$ $= 2^5 - 1$ $= 31$
Specific behaviours
✓ examines at least four different subcommittee sizes ✓ determines total number of subcommittees possible

(b) $P(A \cap B)$. (3 marks)

Solution
$\dfrac{\binom{1}{1}\binom{4}{1}+\binom{1}{1}\binom{4}{2}+\binom{1}{1}\binom{4}{4}}{31}$ $= \dfrac{11}{31} \approx 0.355$
Specific behaviours
✓ examines subcommittee sizes of 2, 3 and 5 people only ✓ determines total number of allowed subcommittees ✓ determines probability

(c) $P(A \cup B)$. (3 marks)

Solution
$1 - P(\overline{A \cup B})$ $= 1 - \dfrac{\binom{1}{0}\binom{4}{1} + \binom{1}{0}\binom{4}{4}}{31}$ *or* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $= \dfrac{21}{31} + \dfrac{16}{31} - \dfrac{11}{31}$ $= \dfrac{26}{31} \approx 0.839$
Specific behaviours
✓ uses an appropriate probability law ✓ determines the number of subcommittees ✓ determines probability

Question 21 **(6 marks)**

For any two unequal positive numbers a and b, the arithmetic mean is defined by $\frac{a+b}{2}$ while the geometric mean is defined by $\sqrt{ab}$.

It is conjectured that the arithmetic mean of two unequal positive numbers is always greater than the geometric mean.

(a) Provide **two** pairs of numbers to demonstrate that the conjecture is true. (2 marks)

Solution
Consider $a = 2, b = 4$ AM $= \frac{1}{2}(2+4) = 3$ GM $= \sqrt{2 \times 4} = 2.82$ $\therefore$ AM > GM Consider $a = 23, b = 47$ AM $= \frac{1}{2}(23+47) = 35$ GM $= \sqrt{23 \times 47} = 32.87$ $\therefore$ AM > GM
Specific behaviours
✓ calculates arithmetic and geometric means of two appropriate pairs of numbers ✓ shows that arithmetic means are greater than the geometric means

(b) Prove algebraically that the conjecture is true for all unequal positive numbers a and b. (4 marks)

Solution
Consider $\left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 = \frac{a-2\sqrt{ab}+b}{2}$ $= \frac{a+b}{2}-\sqrt{ab}$ $\therefore \frac{a+b}{2} = \left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 + \sqrt{ab}$ $\Rightarrow \frac{a+b}{2} > \sqrt{ab},$ since $\left(\sqrt{a}-\sqrt{b}\right)^2$ is always positive for $a \neq b$ Therefore the conjecture is true for all appropriate values of a and b.
Specific behaviours
✓ writes conjecture as an equality   ✓ rearranges inequality to facilitate a proof   ✓ derives a result that is always true for appropriate values of a and b ✓ concludes that conjecture is true for all appropriate values of a and b

OR

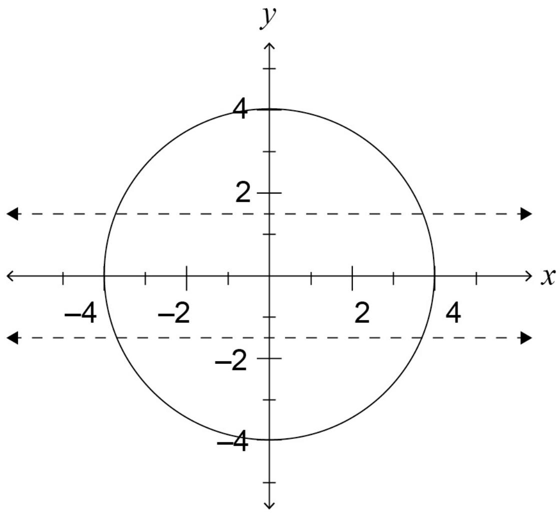
Alternative Solution
$(a-b)^2 > 0$ $a^2-2ab+b^2 > 0$ $a^2-2ab+4ab+b^2 > 0+4ab$ $a^2+2ab+b^2 > 4ab$ $(a+b)^2 > 4ab$ $(a+b) > 2\sqrt{ab}$ as both sides positive $\frac{1}{2}(a+b) > \sqrt{ab}$ Therefore the conjecture is true for all appropriate values of a and b.
Specific behaviours
✓ states a true inequality for all appropriate values of a and b ✓ rearranges inequality to facilitate a proof   ✓ derives the conjecture inequality   ✓ concludes that conjecture is true for all appropriate values of a and b

Question 22 **(8 marks)**

(a) The volume of a spherical balloon is increasing at a rate of 1200 cubic metres per minute. Determine the rate of change of the radius, in metres per minute, when the radius is 12 metres. (3 marks)

Solution
$V = \frac{4}{3}\pi R^3$ $\therefore \frac{dV}{dt} = \frac{dV}{dR}\frac{dR}{dt}$ $\Rightarrow 1200 = 4\pi(12)^2$ Substitutes $R = 12$ into $\frac{dR}{dt}$. $\therefore \frac{dR}{dt} = \frac{25}{12\pi} \approx 0.663$ m/min
Specific behaviours
✓ uses chain rule ✓ substitutes appropriate values for rates and variables ✓ determines time rate of change of radius

(b) A solid wooden sphere with a radius of four metres has a cylindrical hole drilled through the centre with a diameter of three metres, as shown in the diagram below. The cross-section of the sphere is defined by $x^2 + y^2 = 16$. Determine the volume, in cubic metres, of the remaining material in the sphere to three decimal places. (5 marks)



Solution
$x^2 + y^2 = 16$ $\Rightarrow y = \dfrac{3}{2} \quad x = \pm\dfrac{\sqrt{55}}{2} = \pm 3.7081$ $\therefore V = 2\pi \int_0^{3.7081} (y^2 - \dfrac{9}{4})dx$ $V = 2\pi \int_0^{3.7081} (16 - x^2 - \dfrac{9}{4})dx$ $V = 213.571$ Volume remaining of 213.571 cubic metres
Specific behaviours
✓ determines intersection of circle and line ✓ uses volume of revolution integral ✓ uses subtraction with volumes ✓ constructs an appropriate integral to determine volume ✓ calculates volume to three decimal places

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