Rossmoyne Senior High School

WA Exams Practice Paper D, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time for section:	one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items:	pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(4 marks)**

Determine the values of the real constants a and b if $x-4-3i$ is a factor of the polynomial $3x^3+ax^2+bx-50$.

Substitute $x=4+3i$ into $3x^3+ax^2+bx-50$:

$$7a+4b-182+(24a+3b+351)i=0$$

$$7a+4b-182=0$$
$$24a+3b+351=0$$

$$a=-26,\ b=91$$

Question 9 **(8 marks)**

Plane Π has equation $2x - y + 3z = 1$.

(a) Determine the Cartesian equation for the plane that contains the point (15, 11, -5) and is parallel to plane Π.

(2 marks)

$$2(15) - 11 + 3(-5) = c$$
$$c = 4$$

$$2x - y + 3z = 4$$

(b) Find the equation of the line L that is perpendicular to Π and passes through (-3, 6, 2), giving your answer in parametric form. (3 marks)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$x = 2\lambda - 3$$
$$y = 6 - \lambda$$
$$z = 2 + 3\lambda$$

(c) Find the point of intersection of line L and plane Π. (3 marks)

$$2(2\lambda - 3) - (6 - \lambda) + 3(2 + 3\lambda) = 1$$
$$14\lambda = 7$$
$$\lambda = \frac{1}{2}$$

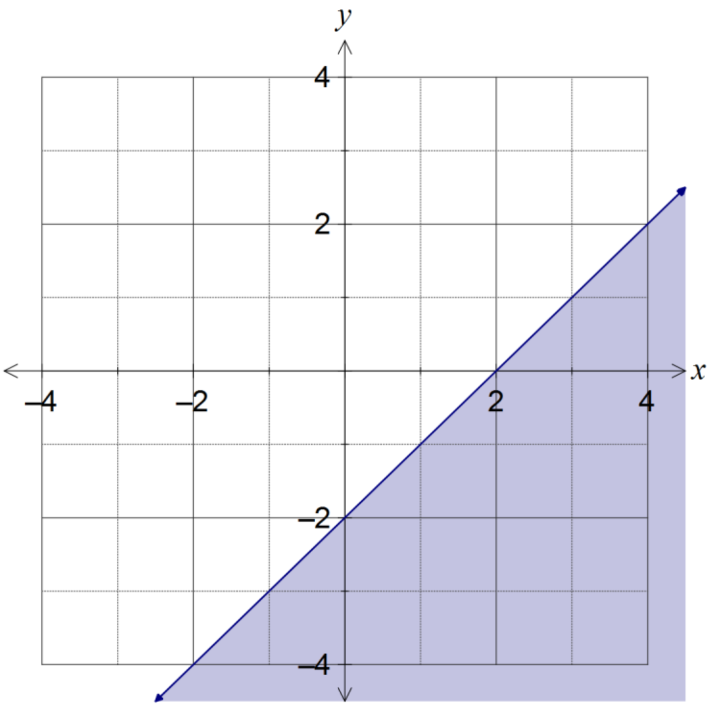
$$x = -2$$
$$y = 5.5$$
$$z = 3.5$$

Intersect at (-2, 5.5, 3.5)

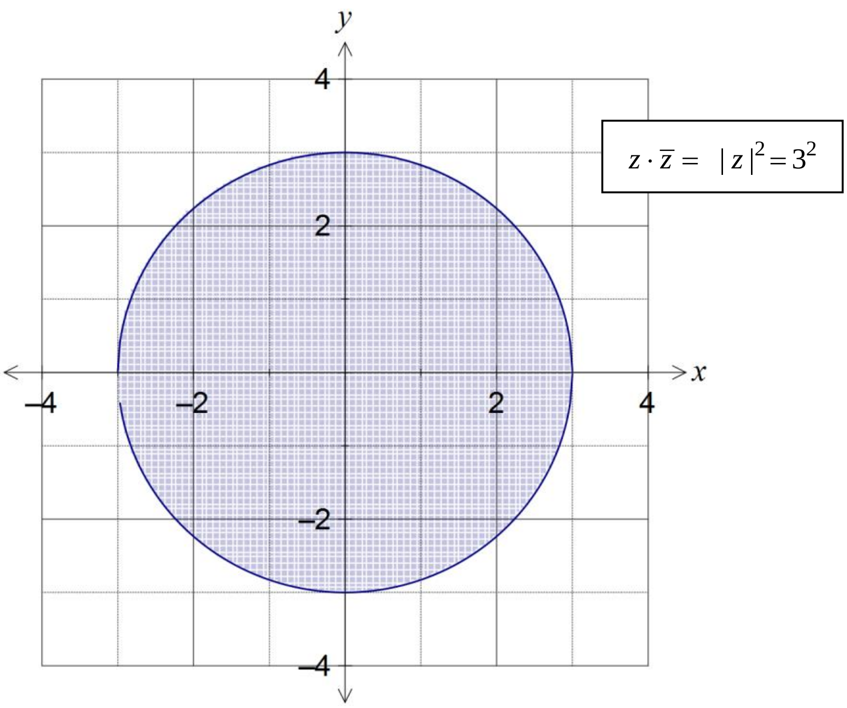
Question 10 **(6 marks)**

On the Argand diagrams below, sketch and label the regions that satisfy

(a) $|z-1-i| \geq |z-3+i|$. (3 marks)



(b) $z \cdot \overline{z} \leq 9$. (3 marks)



Question 11 **(8 marks)**

Two vectors are given by vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

(a) Determine a unit vector perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$.

(3 marks)

$$\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

(b) The plane Π contains the vectors $\mathbf{a}$, $\mathbf{b}$ and the point A(2, -3, -1). Determine the vector equation of this plane. (2 marks)

$$\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$$

$$\mathbf{r} \bullet \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \Rightarrow \mathbf{r} \bullet \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = 9$$

(c) The point B with coordinates (2b, b, -3b) lies in plane Π. Determine the Cartesian equation of the sphere with centre B that passes through the point A. (3 marks)

$$2b - 2b + 3b = 9 \ \Rightarrow \ b = 3$$

$$B(6, \ 3, \ -9)$$

$$r = \sqrt{(6-2)^2 + (3+3)^2 + (-9+1)^2} = 2\sqrt{29}$$

$$(x-6)^2 + (y-3)^2 + (z+9)^2 = 116$$

Question 12 **(6 marks)**

(a) Find all complex solutions of the equation $z^4 = 8 + 8\sqrt{3}i$. (4 marks)

$$z^4 = 16cis\left(\frac{\pi}{3}\right)$$

$$z = \sqrt[4]{16}cis\left(\frac{\pi}{3} \times \frac{1}{4} + \frac{2\pi n}{4}\right),\ n = ...,\ -1,\ 0,\ 1,\ 2,\ ...$$

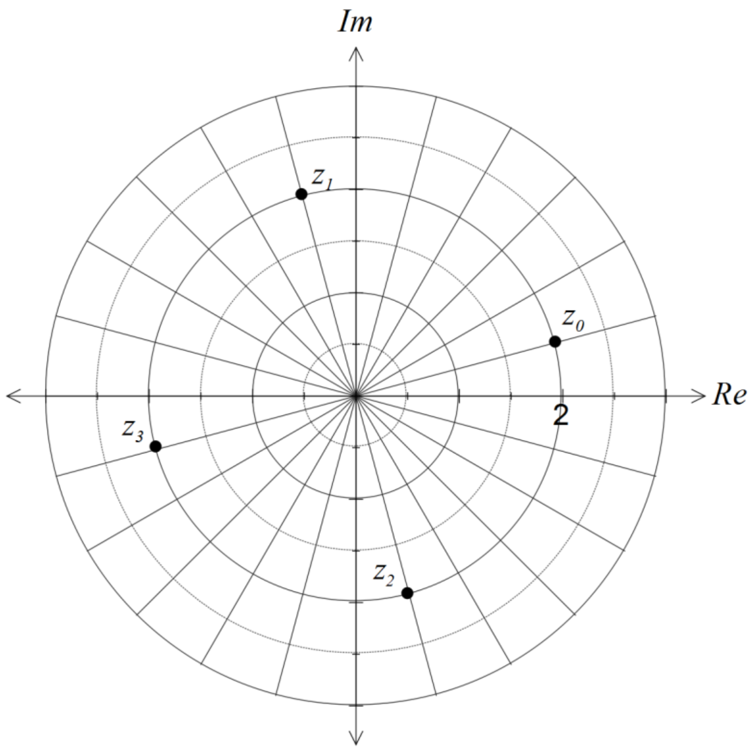
$$z_0 = 2cis\left(\frac{\pi}{12}\right)$$

$$z_1 = 2cis\left(\frac{7\pi}{12}\right)$$

$$z_2 = 2cis\left(-\frac{5\pi}{12}\right)$$

$$z_3 = 2cis\left(-\frac{11\pi}{12}\right)$$

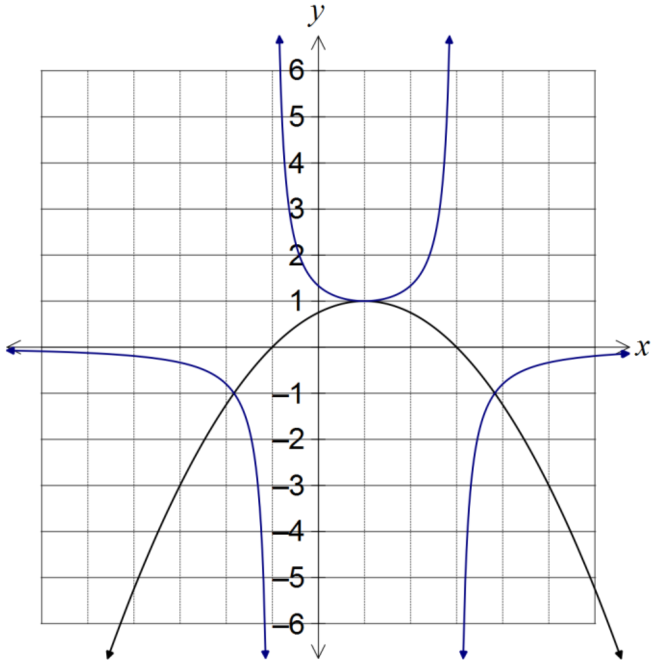
(b) Show all solutions of the equation on the Argand diagram below. (2 marks)



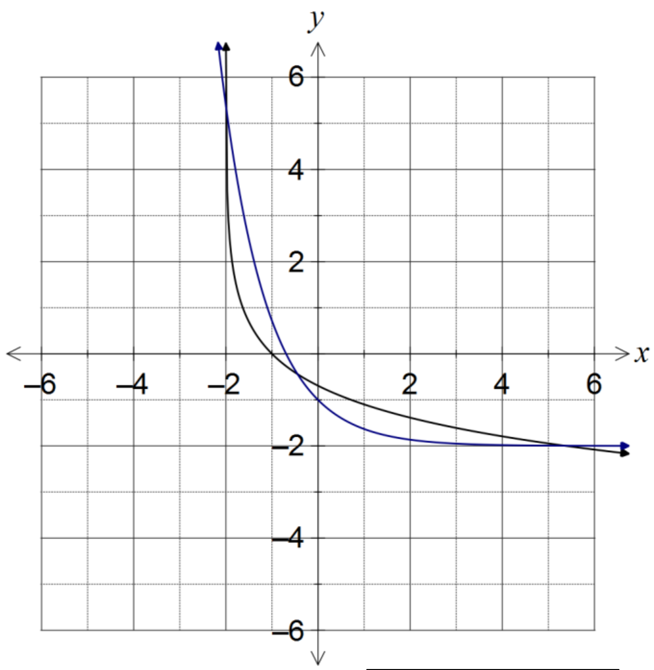
Question 13 **(8 marks)**

(a) The graph of $y = f(x)$ is shown below. On the same axes, sketch the graph of $y = \dfrac{1}{f(x)}$.

(3 marks)



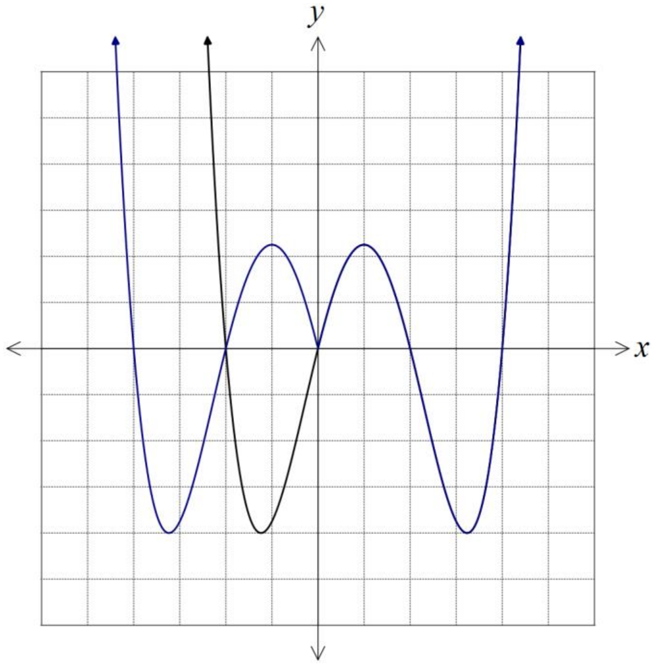
(b) The graph of $y = g(x)$ is shown below. By sketching the graph of the inverse function, $y = g^{-1}(x)$ or otherwise, determine the number of solutions to $g(x) = g^{-1}(x)$. (3 marks)



3 solutions.

(c) The graph of $y = h(x)$ is shown below.

On the same axes, sketch the graph of $y = h(|x|)$. (2 marks)



Question 14 **(5 marks)**

(a) Determine the real part of

(i) $(x+iy)^3$. (1 mark)

$x^3 - 3xy^2$

(ii) $(\cos\theta + i\cdot\sin\theta)^3$. (1 mark)

$\cos^3\theta - 3\cos\theta\cdot\sin^2\theta$

(b) Use the identity $(\cos\theta + i\cdot\sin\theta)^n = \cos n\theta + i\cdot\sin n\theta$ and the result from (a) to show that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$. (3 marks)

$\cos 3\theta + i\cdot\sin 3\theta = (\cos\theta + i\cdot\sin\theta)^3$

Equate real parts:

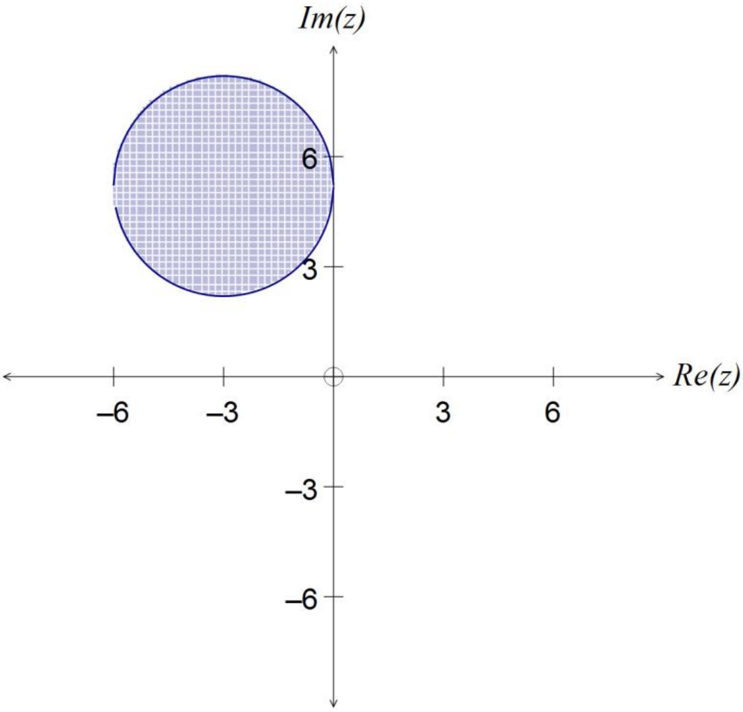
$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\cdot\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta\cdot(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta$$
$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

Question 15 **(9 marks)**

A complex inequality is given by $\left| z + 3 - 3\sqrt{3}i \right| \le 3$.

(a) Sketch the region in the complex plane defined by this inequality. (3 marks)



(b) Determine the minimum and maximum values of the modulus of z. (3 marks)

Radius of circle is 3.

Distance from (0, 0) to circle centre is $\sqrt{3^2 + (3\sqrt{3})^2} = 6$.

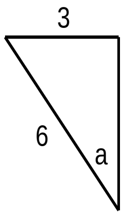
Min $|z|$ is $6 - 3 = 3$

Max $|z|$ is $6 + 3 = 9$

(c) Determine the minimum and maximum values of the argument of z. (3 marks)

Minimum value is $\frac{\pi}{2}$.

Maximum value is $\frac{\pi}{2} + 2 \times \frac{\pi}{6} = \frac{5\pi}{6}$.



sin a = 3/6

a = $\frac{\pi}{6}$

Question 16 **(9 marks)**

The function f is defined by $f(x)=\dfrac{x^2+1}{x^2-1}$. The graph of $y=f(x)$ has a local maximum and no other stationary points.

(a) State the coordinates of the local maximum. (1 mark)

(0, -1)

(b) Determine the equations of all asymptotes of the graph of $y=f(x)$. (2 marks)

$x^2-1=0 \Rightarrow x=\pm 1$

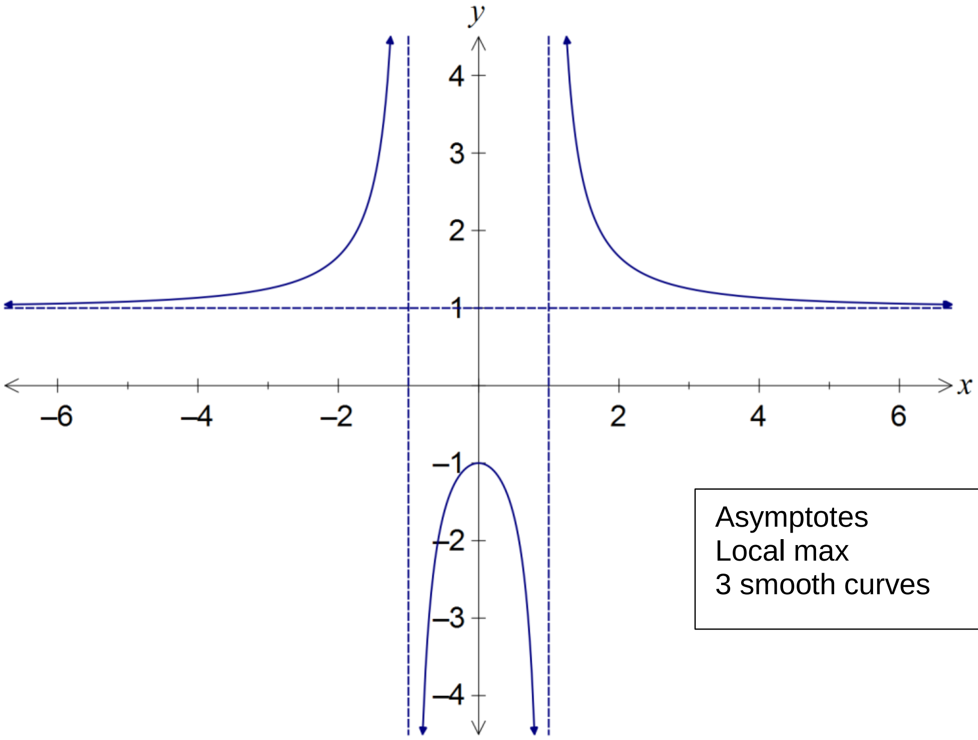
Vert asymptotes: $x=1,\ x=-1$

$x\to\infty,\ f(x)\to 1$

$x\to-\infty,\ f(x)\to 1$

Horiz asymptote: $y=1$

(c) Sketch the graph of $y=f(x)$ on the axes below. (3 marks)



Asymptotes
Local max
3 smooth curves

(d) Given the equation $\frac{x^2+1}{x^2-1}=k$, determine the value(s) of the real constant k so that this equation will have

(i) no solutions. (1 mark)

$$\boxed{-1<k\le 1}$$

(ii) one solution. (1 mark)

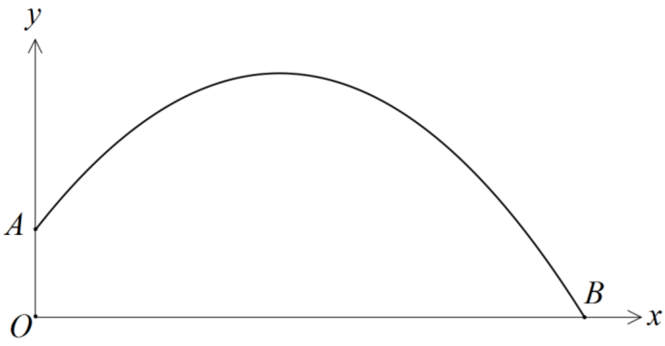
$$\boxed{k=-1}$$

(iii) more than one solution. (1 mark)

$$\boxed{k<-1 \text{ or } k>1}$$

Question 17 **(8 marks)**

A projectile is launched into the air from A, the top of a building, 15 m above the origin O. Soon after, it lands at B, at the same level as the origin, as shown below.



The velocity of the projectile at time t seconds after launch is given by $\mathbf{v}(t) = \frac{a}{2}\mathbf{i} + \left(\frac{a\sqrt{3}}{2} - 10t\right)\mathbf{j}$,

where a is a positive constant.

(a) Determine the acceleration of the projectile at any time. (1 mark)

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = -10\mathbf{j}$$

(b) Determine the position vector of the projectile in terms of a and t. (2 marks)

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ dt = \left(\frac{at}{2} + c_1\right)\mathbf{i} + \left(\frac{a\sqrt{3}t}{2} - 5t^2 + c_2\right)\mathbf{j}$$

$$\mathbf{r}(0) = 15\mathbf{j} \ \Rightarrow \ c_1 = 0, c_2 = 15$$

$$\mathbf{r}(t) = \left(\frac{at}{2}\right)\mathbf{i} + \left(\frac{a\sqrt{3}t}{2} - 5t^2 + 15\right)\mathbf{j}$$

(c) The projectile reaches maximum height when $t = 6$ seconds. Determine the value of a. (2 marks)

j-coefficient of velocity will be zero:

$$\frac{a\sqrt{3}}{2} - 10(6) = 0 \ \Rightarrow \ a = 40\sqrt{3}$$

(d) Calculate the distance from the origin O to the landing point B. (3 marks)

j-coefficient of displacement will be zero:

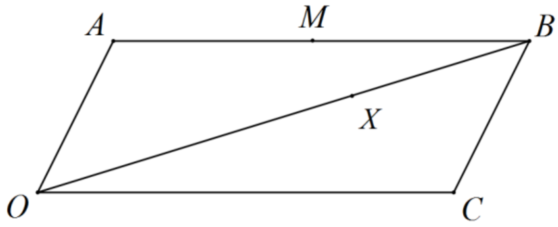
$$\frac{40\sqrt{3}\sqrt{3}t}{2} - 5t^2 + 15 = 0$$

$$5t^2 - 60t - 15 = 0 \Rightarrow t = \cancel{-0.245},\ 12.245$$

$OB = 424.18$ m

Question 18 **(10 marks)**

In the parallelogram $OABC$, where $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OC} = \mathbf{c}$, point X divides OB internally in the ratio $2:1$ and point M is the midpoint of AB.



(a) Show that $\mathbf{OX} = \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{c}$. (1 mark)

$$\begin{aligned}\mathbf{OX} &= \frac{2}{1+2}\mathbf{OB} \\ &= \frac{2}{3}(\mathbf{a}+\mathbf{c}) \\ &= \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{c}\end{aligned}$$

(b) Find $\mathbf{CX}$ in terms of **a** and **c**. (2 marks)

$$\begin{aligned}\mathbf{CX} &= \mathbf{CO} + \mathbf{OX} \\ &= -\mathbf{c} + \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{c} \\ &= \frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{c}\end{aligned}$$

(c) Use vector methods to prove that points C, X and M are collinear. (3 marks)

$$\begin{aligned}\mathbf{XM} &= \mathbf{XO} + \mathbf{OA} + \mathbf{AM} \\ &= -\frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{c} + \mathbf{a} + \frac{1}{2}\mathbf{c} \\ &= \frac{1}{3}\mathbf{a} - \frac{1}{6}\mathbf{c} \\ &= \frac{1}{2}\left(\frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{c}\right) \\ &= \frac{1}{2}\mathbf{CX}\end{aligned}$$

Hence the points C, X and M are collinear since **XM** and **CX** are parallel and both have point X in common.

(d) Let $\mathbf{a} = 3\mathbf{i} - 6\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 6\mathbf{i} - 8\mathbf{k}$. Given that the area of a parallelogram, with adjacent edges **v** and **w**, is the magnitude of the cross product $\mathbf{v} \times \mathbf{w}$, determine the area of triangle MXB.

(4 marks)

$$\mathbf{MB} = \frac{1}{2}\mathbf{c}$$

$$= \frac{1}{2}\begin{bmatrix} 6 \\ 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$\mathbf{XB} = \frac{1}{3}(\mathbf{a} + \mathbf{c}) = \frac{1}{3}\begin{bmatrix} 3+6 \\ -6+0 \\ -1-8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \times \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ -3 \\ -6 \end{bmatrix}$$

$$A = \frac{1}{2}\left(\sqrt{64+9+36}\right) = \frac{\sqrt{109}}{2} \approx 5.22 \text{ sq units}$$

Question 19 **(8 marks)**

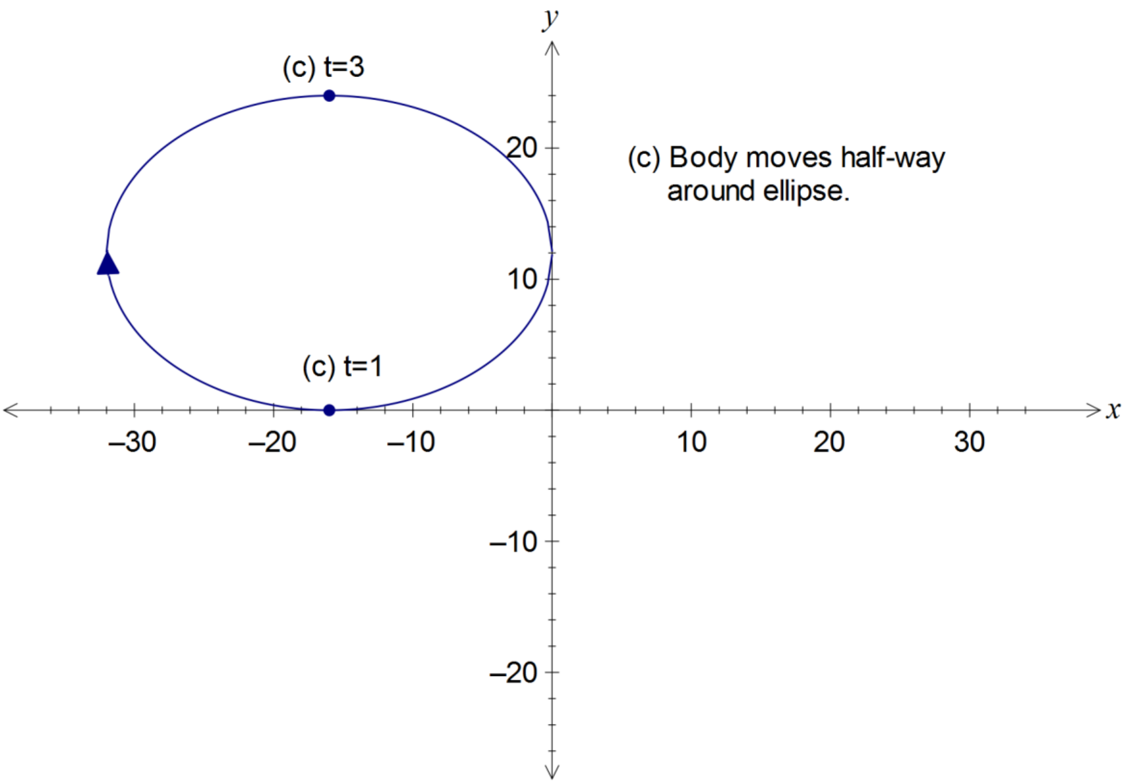
The displacement vector of a small body is given by $\mathbf{r}(t) = 10\left(\cos\left(\frac{\pi t}{2}\right) - 1\right)\mathbf{i} - 6\left(\sin\left(\frac{\pi t}{2}\right) - 1\right)\mathbf{j}$ metres at time t seconds.

(a) Determine the Cartesian equation of the path of the body. (2 marks)

$$x = 10\cos\left(\frac{\pi t}{2}\right) - 10,\ y = -6\sin\left(\frac{\pi t}{2}\right) + 6$$

$$\left(\frac{x+10}{10}\right)^2 + \left(\frac{y-6}{-6}\right)^2 = 1$$

(b) Sketch the path of the body, clearly indicating the direction of motion. (3 marks)



(c) Determine the distance travelled by the body between $t = 1$ and $t = 3$ seconds, indicating this distance on the sketch above. (3 marks)

$$\mathbf{v}(t) = -5\pi \sin\left(\frac{\pi t}{2}\right)\mathbf{i} - 3\pi \cos\left(\frac{\pi t}{2}\right)\mathbf{j}$$

$$\begin{aligned} d &= \int_1^3 \left|\mathbf{v}(t)\right| \, dt \\ &= \int_1^3 \sqrt{\left(-5\pi \sin\left(\frac{\pi t}{2}\right)\right)^2 + \left(-3\pi \cos\left(\frac{\pi t}{2}\right)\right)^2} \, dt \\ &\approx 25.527 \text{ m} \end{aligned}$$

Question 20 **(9 marks)**

A body, A, has an initial position of $-7\mathbf{i}+21\mathbf{j}+6\mathbf{k}$ metres and is moving with a constant velocity of $6\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ metres per second. A second body, B, is moving with constant velocity of $8\mathbf{i}+\mathbf{j}-\mathbf{k}$ metres per second and collides with body A after six seconds.

(a) Determine the initial distance apart of body A and body B. (4 marks)

A and B collide at

$$\begin{bmatrix}-7\\21\\6\end{bmatrix}+6\begin{bmatrix}6\\3\\-2\end{bmatrix}=\begin{bmatrix}29\\39\\-6\end{bmatrix}$$

Hence initial position of B is at

$$r_B+6\begin{bmatrix}8\\1\\-1\end{bmatrix}=\begin{bmatrix}29\\39\\-6\end{bmatrix}\Rightarrow r_B=\begin{bmatrix}-19\\33\\0\end{bmatrix}$$

Distance apart of A and B is

$$\left\|\begin{bmatrix}-19\\33\\0\end{bmatrix}-\begin{bmatrix}-7\\21\\6\end{bmatrix}\right\|=\left\|\begin{bmatrix}-12\\12\\-6\end{bmatrix}\right\|=18$$

(b) The path of a third body, C, crossed the path of body A. If C was initially at $6\mathbf{i}+32\mathbf{j}+\mathbf{k}$ metres, and was moving with a constant velocity $5\mathbf{i}+y\mathbf{j}-\mathbf{k}$, determine the value of the constant y. (5 marks)

$$R_A=\begin{bmatrix}-7\\21\\6\end{bmatrix}+\lambda\begin{bmatrix}6\\3\\-2\end{bmatrix}$$

$$R_C=\begin{bmatrix}6\\32\\1\end{bmatrix}+\mu\begin{bmatrix}5\\y\\-1\end{bmatrix}$$

$\mathbf{i}$: $-7+6\lambda=6+5\mu$

$\mathbf{k}$: $6-2\lambda=1-\mu$

$\lambda=3,\ \mu=1$

$\mathbf{j}$: $21+3(3)=32+y \Rightarrow y=-2$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________