MATHEMATICS METHODS

MAWA Semester 1 (Unit 3) Examination 2017

Calculator-free

Marking Key

The release date for this exam and marking scheme is

- **the end of week 8 of term 2, 2017**

Section One: Calculator-free (50 Marks)

1(a)(i) (2 marks)

Solution	
$f(x) = \sqrt{5+x^2}$ $f'(x) = \frac{1}{2}(5+x^2)^{-1/2}.2x = \frac{2x}{2\sqrt{5+x^2}}$	
Marking key/mathematical behaviours	Marks
• correctly differentiates using chain rule • recognises $\sqrt{5+x^2}$ as $(5+x^2)^{1/2}$	1 1

Question 1(a)(ii) (2 marks)

Solution	
$f(x) = \frac{x}{e^{3x}+5}$ $f'(x) = \frac{(e^{3x}+5)1 - 3xe^{3x}}{(e^{3x}+5)^2}$	
Marking key/mathematical behaviours	Marks
• correctly differentiates using quotient rule • correctly determines derivative of denominator	1 1

Question 1(b) (3 marks)

Solution	
$y = 5\cos(3x+1)$ $\frac{dy}{dx} = -15\sin(3x+1)$ $\left(\frac{dy}{dx}\right)^2 + 9y^2 = 225\sin^2(3x+1) + 225\cos^2(3x+1) = 225$	
Marking key/mathematical behaviours	Marks
• correctly differentiates $\cos x$ • correctly differentiates using chain rule • correctly evaluates $\left(\frac{dy}{dx}\right)^2 + 9y^2$	1 1 1

Question 2 **(6 marks)**

Solution

$$\frac{dF}{d\theta}=\frac{-1200(3\cos\theta-4\sin\theta)}{(3\sin\theta+4\cos\theta)^2}$$

$\frac{dF}{d\theta}=0$ when $3\cos\theta-4\sin\theta=0$ i.e. when $\tan\theta=\frac{3}{4}$

In the interval $0\le\theta\le\frac{\pi}{2}$, $F=F(\theta)$ has just one stationary point, which occurs when $\tan\theta=\frac{3}{4}$

If $\tan\theta=\frac{3}{4}$ then $\sin\theta=\frac{3}{5}$ and $\cos\theta=\frac{4}{5}$ (3-4-5 right triangle), so $F=\frac{1200}{\frac{9}{5}+\frac{16}{5}}=240$

If $\theta=0$, $F=\frac{1200}{0+4}=300$ and if $\theta=\pi/2$, $F=\frac{1200}{3}=400$

So the minimum value of F is indeed 240

Marking key/mathematical behaviours	Marks
• differentiates correctly	1+1
• identifies the single stationary point	1
• evaluates F at the stationary point	1
• checks values of F at the end points	1
• gives correct answer	1

Question 3(a) **(2 marks)**

Solution

$v(t)=30\left(1+\cos\frac{\pi}{5}t\right)=0\Rightarrow 1+\cos\frac{\pi}{5}t=0$

$\Rightarrow\frac{\pi}{5}t=\pi\Rightarrow t=5$ (smallest positive solution)

So first at rest after 5 seconds

Marking key/mathematical behaviours	Marks
• obtains $1+\cos\frac{\pi}{5}t=0$	1
• gives correct answer	1

Question 3(b) **(2 marks)**

Solution

$a(t)=-6\pi\sin\frac{\pi}{5}t=0$ when $t=0$

So the initial acceleration is zero.

Marking key/mathematical behaviours	Marks
• differentiates correctly	1
• obtains correct answer	1

Question 3(c) **(2marks)**

Solution Since $v(t) \geq 0$ for all $t \geq 0$, the particle never moves 'backwards'. So it never returns to its starting point.	
Marking key/mathematical behaviours	Marks
• correct answer • valid reason	1 1

Question 3(d) **(2 marks)**

Solution $x(10) - x(0) = \int_0^{10} 30(1 + \cos\frac{\pi}{5}t)\, dt$ $= \left(30t + \frac{150}{\pi}\sin\frac{\pi}{5}t\right)\Big\vert_0^{10} = \left(300 + \frac{150}{\pi}\sin 2\pi\right) - \left(\frac{150}{\pi}\sin 0\right)$ $= 300$ Since the particle never moves backwards, the distance travelled is 300 m.	
Marking key/mathematical behaviours	Marks
• obtains distance travelled as the integral of $v(t)$ • evaluates integral correctly	1 1

Question 4(a) **(5 marks)**

Solution	
The shaded area = area of the square – area of the quarter circle – area of the triangle $= k^2 - \dfrac{\pi\left(\frac{k}{2}\right)^2}{4} - \dfrac{1}{2}\times\dfrac{k}{2}\times k$ $= k^2 - \dfrac{\pi k^2}{16} - \dfrac{k^2}{4}$ $= \dfrac{16k^2}{16} - \dfrac{\pi k^2}{16} - \dfrac{4k^2}{16}$ $= \left(\dfrac{12-\pi}{16}\right)\times k^2$ Hence the probability p, of a dart landing within the shaded area is, $p = \dfrac{\text{shaded area}}{\text{area of square}}$ $= \dfrac{\left(\frac{12-\pi}{16}\right)\times \cancel{k}^2}{\cancel{k}^2}$ $= \left(\dfrac{12-\pi}{16}\right)$	
Marking key/mathematical behaviours	**Marks**
• States how the shaded area may be calculated (line 1 of solution)	1
• Calculates at least one of the areas of the required regions	1
• Determines the shaded area in terms of k	1
• States the probability as a ratio of the total area	1
• Simplifies to the required result	1

Question 4(b) **(2 marks)**

Solution	
$P(\text{first and third, shaded}) = P(\text{first, shaded})\times P(\text{second, not shaded})\times P(\text{third, shaded})$ $= p\times(1-p)\times p$ $= p^2\times(1-p)$	
Marking key/mathematical behaviours	**Marks**
• Uses the result from part (a) to determine $P(\text{second, not shaded})$	1
• Applies the multiplication principle correctly	1

Question 4(c) **(2 marks)**

Solution	
Probability Jamie hits the green region only once in three throws $=P\left(S\ \bar{S}\ \bar{S}\right)+P\left(\bar{S}\ S\ \bar{S}\right)+P\left(\bar{S}\ \bar{S}\ S\right)$ $=3\times p\times(1-p)^2$	
Marking key/mathematical behaviours	Marks
• States the three ways that this can happen • Applies the addition principle and determines the correct result	1 1

Question 4(d) **(2 marks)**

Solution	
Probability Jamie hits the green region at least once in three throws $=1-P\left(\bar{S}\ \bar{S}\ \bar{S}\right)$ $=1-\left(1-p\right)^3$	
Marking key/mathematical behaviours	Marks
• Recognises the compliment • States the correct result	1 1

Question 5(a) **(2 marks)**

Solution

$$\int (e^{7x-1} + 5x^2)\ dx \ = \ \frac{e^{7x-1}}{7} + \frac{5x^3}{3} + c$$

Marking key/mathematical behaviours	Marks
• correctly integrates each term	1
• correctly adds constant of integration (1 mark penalty once only throughout the rest of question 5)	1

Question 5(b) **(2 marks)**

Solution

$$\int \frac{4x^3+3}{x^2}\ dx \ = \ \int 4x + 3x^{-2}\ dx$$

$$= \ 2x^2 - \frac{1}{x^3} + \text{c}$$

Marking key/mathematical behaviours	Marks
• correctly simplifies integral	1
• correctly integrates each term	1

Question 5(c) **(2 marks)**

Solution

$$\int 5(2x-3)^3\ dx \ = \ \frac{5(2x-3)^4}{4 \times 2} + c$$

$$= \ \frac{5}{8}\,(2x-3)^4 + c$$

Marking key/mathematical behaviours	Marks
• recognises the rule	1
• correctly integrates	1

Question 5(d) **(2 marks)**

Solution

$$\int [\sin(2x+3) + 2\cos(\pi x)]\ dx \ = \ -\frac{1}{2}\cos(2x+3) + \frac{2}{\pi}\sin(\pi x) + c$$

Marking key/mathematical behaviours	Marks
• correctly integrates first term	1
• correctly integrates second term	1

Question 6 **(4 marks)**

Solution	
$\cos 2x = \cos^2 x - \sin^2 x$ $\Rightarrow \cos 2x = 1 - \sin^2 x - \sin^2 x$ $\Rightarrow \cos 2x = 1 - 2\sin^2 x$ $\therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\int sin^2(x)\,dx = \frac{1}{2}\int(1 - \cos(2x))\,dx$ $= \frac{1}{2}\left(x - \frac{1}{2}(2x)\right) + c$	
Marking key/mathematical behaviours	Marks
• correctly manipulates the expansion to express $sin^2(x)$ in terms of cos(2x) • correctly integrates each part	2 2

Question 7(a) **(2 marks)**

Solution	
$\int_{-\pi}^{\frac{\pi}{2}} cos(\pi - x)\ dx = -\sin(\pi - x)]_{-\pi}^{\frac{\pi}{2}}$ $= -\left[\sin\left(\frac{\pi}{2}\right) - \sin(2\pi)\right]$ $= -[1 - 0]$ = -1	
Marking key/mathematical behaviours	Marks
• correctly integrates • correctly evaluates	1 1

Question 7(b) **(2 marks)**

Solution

$$\frac{d}{dx}\left[\int_x^4 \frac{4t^2-3}{\sqrt{t}}\ dt\right] = \frac{d}{dx}\left[-\int_4^x \frac{4t^2-3}{\sqrt{t}}\ dt\right]$$

$$= -\frac{4x^2-3}{\sqrt{x}}$$

Marking key/mathematical behaviours	Marks
• indicates the change of limits	1
• correctly applies fundamental theorem	1

Question 7(c) **(2 marks)**

Solution

$$\int_0^{\frac{\pi}{6}} \frac{d}{dx}[\sin(2x)]\ dx = [\sin(2x)]_0^{\frac{\pi}{6}}$$

$$= sin\left(\frac{\pi}{3}\right) - \sin(0)$$

$$= \frac{\sqrt{3}}{2} - 0$$

$$= \frac{\sqrt{3}}{2}$$

Marking key/mathematical behaviours	Marks
• correctly integrates	1
• correctly evaluates	1