WA Exams Practice Paper A, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	13	13	100	97	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed **65% (97 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(5 marks)**

Two particles, A and B, have initial positions of $2\mathbf{i}+4\mathbf{j}$ and $28\mathbf{i}-4\mathbf{j}$ respectively.

If the velocity of A is $3\mathbf{i}-2\mathbf{j}$ and the velocity of B is $-2\mathbf{i}-\mathbf{j}$, determine the point of intersection of their paths and state whether or not the particles meet at this point.

$$\mathbf{r}_A = \begin{bmatrix} 2+3\lambda \\ 4-2\lambda \end{bmatrix}$$

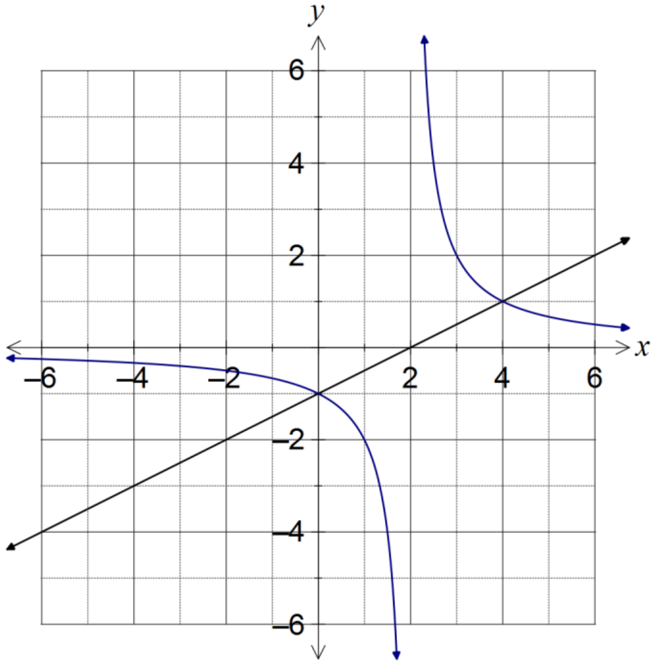
$$\mathbf{r}_B = \begin{bmatrix} 28-2\mu \\ -4-2\mu \end{bmatrix}$$

$$\begin{bmatrix} 2+3\lambda \\ 4-2\lambda \end{bmatrix} = \begin{bmatrix} 28-2\mu \\ -4-2\mu \end{bmatrix} \Rightarrow \lambda = 6,\ \mu = 4 \Rightarrow$$ don't meet, times different.

$$\begin{bmatrix} 2+3\lambda \\ 4-2\lambda \end{bmatrix}_{\lambda=6} = \begin{bmatrix} 20 \\ -8 \end{bmatrix} \Rightarrow$$ paths intersect at (20, -8)

Question 9 **(6 marks)**

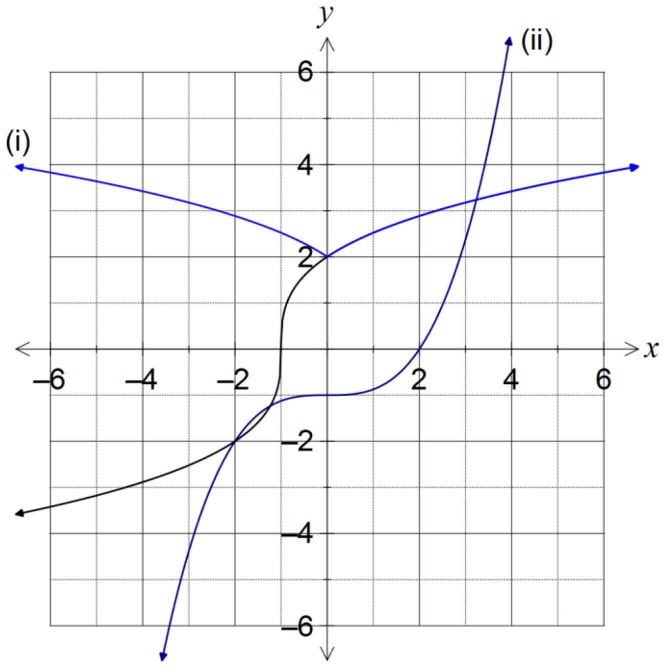
(a) The graph of $y = f(x)$ is shown below. On the same axes, sketch the graph of $y = \dfrac{1}{f(x)}$. (3 marks)



(b) The graph of $y = g(x)$ is shown below. On the same axes, sketch and label the graphs of

(i) $y = g(|x|)$. (1 mark)

(ii) the inverse function, $y = g^{-1}(x)$. (2 marks)



Question 10 **(7 marks)**

The thickness of a protective coating applied to electrical components for use in wet conditions is known to be uniformly distributed between 350 and 600 microns. The mean thickness is 475 microns and the standard deviation is 72 microns.

(a) A quality control officer needs to check that the coating is being applied properly and has access to two random samples of recent coatings, one of 20 components and the other of 50 components. Explain why the officer should choose the larger of the two samples in order to calculate an interval estimate for the mean thickness of the coating. (2 marks)

> The background distribution is not normal and so a large sample, of thirty or more, is required by the central limit theorem for the distribution of sample means to approach a normal distribution.

(b) Describe the expected distribution of sample means based on random samples of 50 components. (2 marks)

> Normally distributed with a mean of 475 microns and a standard deviation of 10.2 microns.
>
> $$\frac{72}{\sqrt{50}} = 10.18$$

(c) The officer decides to take a new random sample of 144 components. The mean thickness of these components was calculated to be 464 microns, less than the expected value of 475 microns. Calculate a 98% confidence interval for the mean thickness of the coating for all components and explain whether this suggests that the mean thickness is not 475 microns. (3 marks)

> $$464 \pm 2.326 \times \frac{72}{\sqrt{144}} = (450,\ 478)$$
>
> The stated mean of 475 microns lies within the interval, and so there is no reason to suggest that the mean is not 475 microns.

Question 11 **(6 marks)**

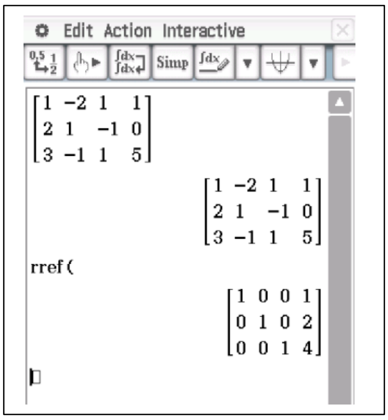
The equations of three planes are $x-2y+z=1$, $2x+y-z=0$ and $3x-y+az=b$, where a and b are constants.

Determine the number of solutions this system of equations has for the following cases, giving a brief geometric interpretation in each case.

(a) $a=1$ and $b=5$. (2 marks)

There is a unique (single) solution.

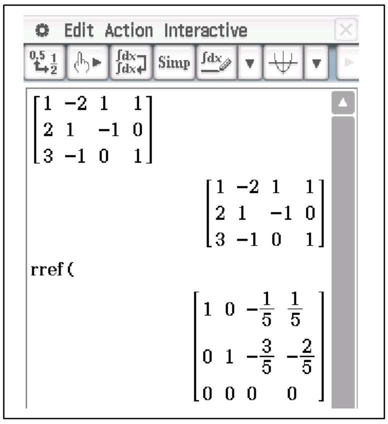
The planes intersect at the point (1, 2, 4).



(b) $a=0$ and $b=1$. (2 marks)

There are an infinite number of solutions.

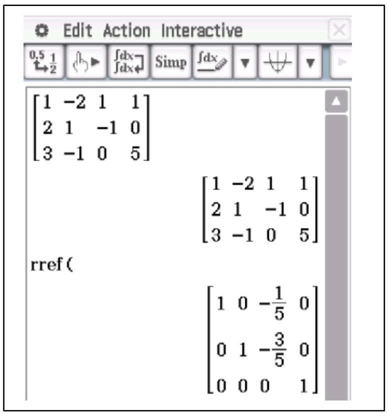
The three planes intersect in a straight line.



(c) $a=0$ and $b=5$. (2 marks)

There are no solutions.

As planes are not parallel, then they must intersect in more than one line.



Question 12 **(8 marks)**

(a) Determine all complex solutions of the equation $z^5 = 16\left(1+\sqrt{3}i\right)$, expressing each in the form $r\ \text{cis}\theta$, where $r \geq 0$ and $-\pi \leq \theta \leq \pi$. (5 marks)

$$z^5 = 16\left(1+\sqrt{3}i\right)$$

$$z^5 = 2^5\, cis\left(\frac{\pi}{3}\right)$$

$$z = \left(2^5\right)^{\frac{1}{5}} cis\left(\frac{\pi}{3}\times\frac{1}{5}+\frac{2\pi n}{5}\right),\ n = \ldots,-1,\ 0,\ 1,\ 2,\ \ldots$$

$$z_1 = 2cis\left(\frac{\pi}{15}\right)$$

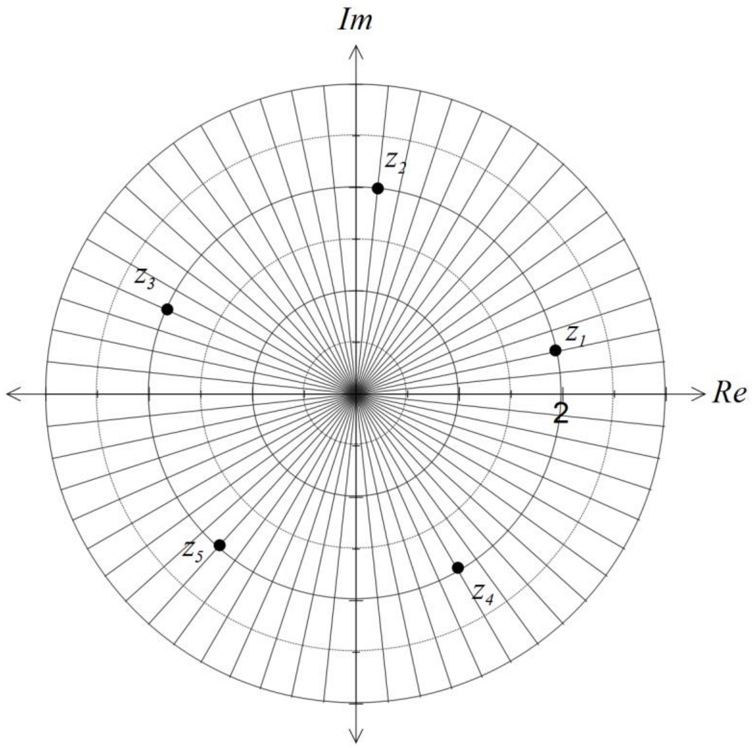
$$z_2 = 2cis\left(\frac{7\pi}{15}\right)$$

$$z_3 = 2cis\left(\frac{13\pi}{15}\right)$$

$$z_4 = 2cis\left(-\frac{5\pi}{15}\right)$$

$$z_5 = 2cis\left(-\frac{11\pi}{15}\right)$$

(b) Show all solutions of the equation on the Argand diagram below. (3 marks)



Question 13 **(7 marks)**

The vectors $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $-\mathbf{i}+\mathbf{j}+2\mathbf{k}$ lie in a plane containing the point (1, 3, 0).

(a) Determine the Cartesian equation of the plane. (3 marks)

$$\begin{bmatrix}2\\1\\-3\end{bmatrix}\times\begin{bmatrix}-1\\1\\2\end{bmatrix}=\begin{bmatrix}5\\-1\\3\end{bmatrix}$$

$$\begin{bmatrix}5\\-1\\3\end{bmatrix}\bullet\begin{bmatrix}1\\3\\0\end{bmatrix}=2$$

$$\begin{bmatrix}5\\-1\\3\end{bmatrix}\bullet\begin{bmatrix}x\\y\\z\end{bmatrix}=2 \Rightarrow 5x-y+3z=2$$

(b) Determine where the straight line that passes through the points (5, 7, -4) and (-1, 10, 5) intersects the plane. (4 marks)

$$\begin{bmatrix}5\\7\\-4\end{bmatrix}-\begin{bmatrix}-1\\10\\5\end{bmatrix}=\begin{bmatrix}6\\-3\\-9\end{bmatrix} \Rightarrow \mathbf{r}=\begin{bmatrix}5\\7\\-4\end{bmatrix}+\lambda\begin{bmatrix}2\\-1\\-3\end{bmatrix}$$

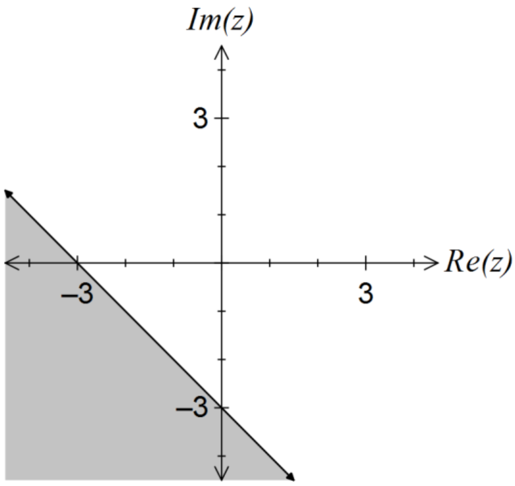
$$\begin{bmatrix}5\\-1\\3\end{bmatrix}\bullet\begin{bmatrix}5+2\lambda\\7-\lambda\\-4-3\lambda\end{bmatrix}=2 \Rightarrow \lambda=-2$$

$$\mathbf{r}=\begin{bmatrix}5\\7\\-4\end{bmatrix}-2\begin{bmatrix}2\\-1\\-3\end{bmatrix}=\begin{bmatrix}1\\9\\2\end{bmatrix}$$

Question 14 **(8 marks)**

Draw sketches of the following sets of points in the complex plane.

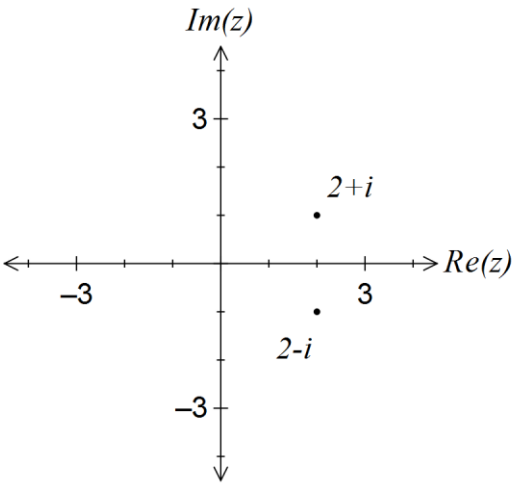
(a) $\{z: \ |z+3+3i| \le |z| \}$ (3 marks)



NB

$z+3+3i = z-(-3-3i)$

(b) $\left\{z: \ z^2 = 4z-5 \right\}$ (2 marks)

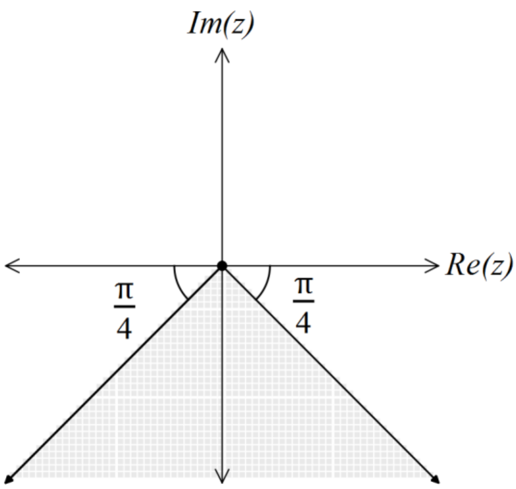


NB

$z^2 - 4z + 5 = 0$

$z = 2+i, \ z = 2-i$

(c) $\left\{z: \ -\frac{\pi}{4} \le \arg(iz) \le \frac{\pi}{4}\right\}$ (3 marks)



Question 15 **(8 marks)**

A particle, initially with position vector $\mathbf{r} = -\mathbf{j}$ cm, is moving in a horizontal plane with velocity vector at time t seconds given by $\mathbf{v} = (2\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ cms^{-1}.

(a) Calculate the initial acceleration of the particle. (2 marks)

$$\mathbf{a} = (-2\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

$$\mathbf{a}(0) = \mathbf{j} \text{ cms}^{-2}$$

(b) Determine the position vector of the particle at time t. (2 marks)

$$\mathbf{r} = (2\sin t)\mathbf{i} - (\cos t)\mathbf{j} + \mathbf{c}$$
$$r(0) = -\mathbf{j} \Rightarrow \mathbf{c} = \mathbf{0}$$

$$\mathbf{r} = (2\sin t)\mathbf{i} - (\cos t)\mathbf{j}$$

(c) Determine the Cartesian equation of the path of the particle. (2 marks)

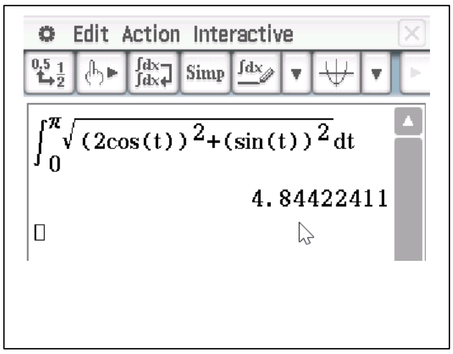
$$x = 2\sin t,\ y = -\cos t$$
$$\sin^2 t + \cos^2 t = \left(\frac{x}{2}\right)^2 + (-y)^2 = 1$$

$$x^2 + 4y^2 = 4$$

(d) Calculate the distance travelled by the particle between $t = 0$ and $t = \pi$. (2 marks)

$$s = \int |\mathbf{v}|\,dt$$
$$= \int_0^{\pi} \sqrt{(2\cos t)^2 + (\sin t)^2}\,dt$$

$$= 4.844 \text{ cm}$$



Question 16 **(8 marks)**

Lengths of climbing rope produced by a manufacturer over a long production run have breaking strengths that are normally distributed with a mean of 180.2 kg and standard deviation of 9.5 kg.

(a) Determine the probability that the mean breaking strength of a randomly chosen sample of 10 lengths will be less than 175 kg. (3 marks)

$$sd = \frac{9.5}{\sqrt{10}} \approx 3.004$$

$$X \sim N(180.2, 3.004^2)$$

$$P(X < 175) = 0.0417$$

(b) At the start of a production run, a supervisor at the factory randomly samples 20 lengths and after testing, determines that the mean breaking strength of the sample is 176.9 kg. Construct a 90% interval estimate for the population mean based on this sample. (2 marks)

$$176.9 \pm 1.645\frac{9.5}{\sqrt{20}} = (173.41, 180.39)$$

(c) If the supervisor repeated the same sampling process in (b) every day for 30 consecutive days, how many of the intervals constructed would be expected to include the known mean breaking strength of 180.2 kg? (1 mark)

$0.9 \times 30 = 27$ intervals.

(d) How large a sample should the supervisor take so that the width of a 95% confidence interval for the mean breaking strength has a width of no more than 5 kg? (2 marks)

$$n = \left(\frac{1.96 \times 9.5}{2.5}\right)^2$$
$$= 55.47$$

Hence sample 56 lengths.

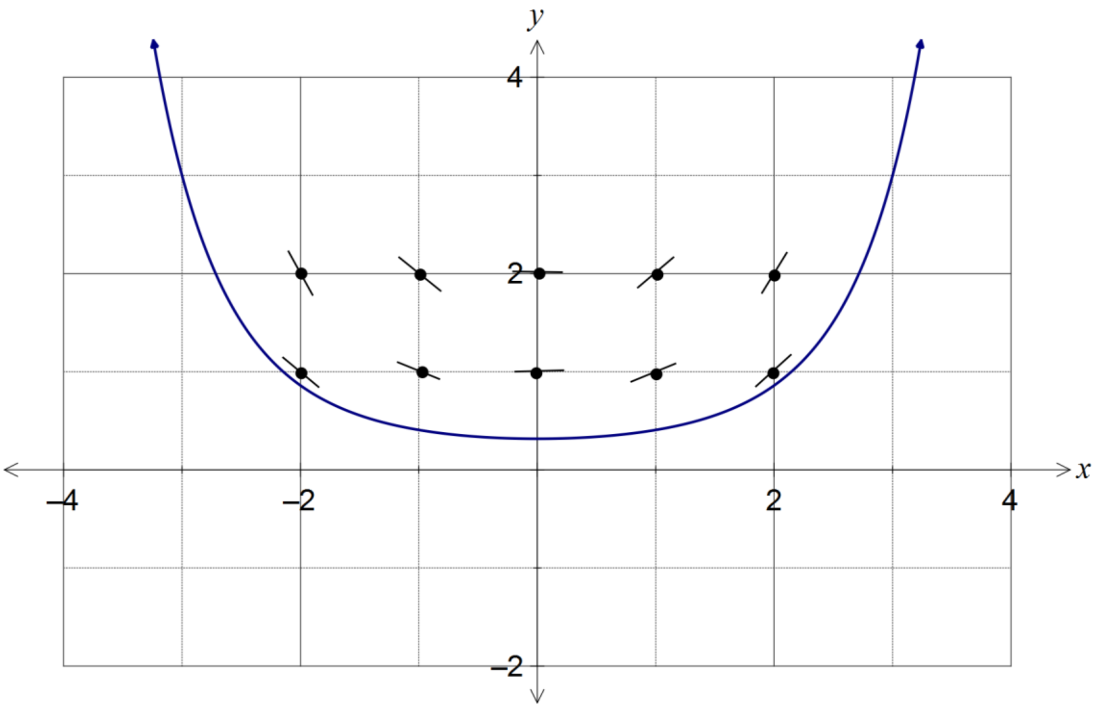
Question 17 **(12 marks)**

(a) A first-order differential equation is given by $\frac{dy}{dx} = \frac{xy}{2}$.

(i) Use the equation to complete the table below. (2 marks)

x	-2	-1	0	1	2	3
y	2	2	2	2	2	3
$\frac{dy}{dx}$	**-2**	**-1**	**0**	**1**	**2**	**4.5**

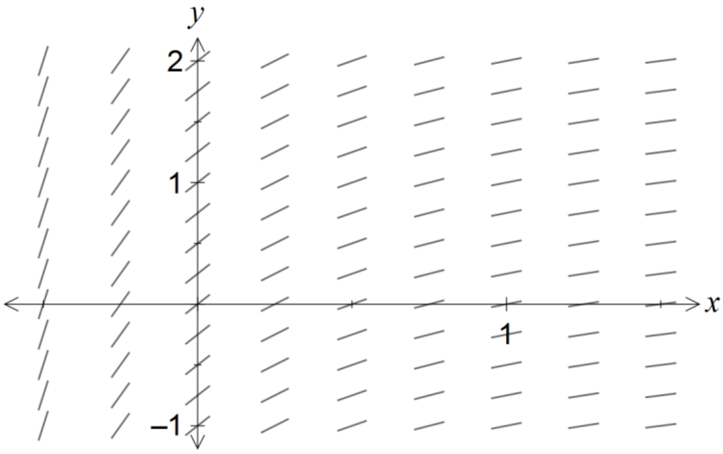
(ii) Create a slope field on the 10 points on the graph below. (2 marks)



(iii) On the graph, sketch the solution curve to the differential equation that passes through the point (3, 3). (2 marks)

Through (3, 3), u-shaped, symmetrical and close to that shown.

(b) The differential equation for a curve passing through the point (0, 1) is given by $\frac{dy}{dx} = \frac{2}{x^2 + 2x + 1}$. The slope field for the differential equation is shown below.



(i) Use the incremental formula $\delta y = \frac{dy}{dx} \times \delta x$, with $\delta x = 0.1$, to calculate an estimate for the y-coordinate of the curve when $x = 0.2$. (4 marks)

Using Euler's method with $\delta x = 0.1$:

x	y	$\frac{dy}{dx}$	$\delta y = \frac{dy}{dx} \times \delta x$
0	1	2	0.2
0.1	1.2	1.653	0.1653
0.2	1.3653		

Estimate is $y = 1.3653$, to four decimal places.

(ii) Explain whether the estimate in (a) is an over- or under-estimate for the y-coordinate. (2 marks)

Over-estimate, as the slope field shows the curve has a positive gradient but is concave down between $x = 0$ and $x = 0.2$.

Question 18 **(8 marks)**

A small body moves along the x-axis so that after t seconds it is x centimetres from the origin.

The velocity of the body for $t \geq 0$ is given by $v = \frac{x}{2} + 1$, where $x = f(t)$ and $f(0) = 0$.

(a) Show that $f(t) = 2e^{0.5t} - 2$. (3 marks)

$$f(0) = 2e^{0.5(0)} - 2 = 0$$

$$v = \frac{dx}{dt} = f\,'(t) = e^{0.5t}$$

$$x = f(t) = 2v - 2 \;\Rightarrow\; v = \frac{x}{2} + 1$$

(b) Determine the acceleration of the body when $x = 4$ cm. (3 marks)

$$x = 4 \;\Rightarrow\; v = 3$$

$$\begin{aligned}\frac{dv}{dt} &= \frac{1}{2} \cdot \frac{dx}{dt}\\ &= \frac{1}{2} \cdot v\\ &= \frac{3}{2} \text{ cm/s}^2\end{aligned}$$

(c) The body reaches a speed of 25 cm/s after $k \ln 5$ seconds. Determine the value of k. (2 marks)

$$\begin{aligned}e^{0.5t} &= 25\\ t &= 2\ln 5^2\\ &= 4\ln 5 \;\Rightarrow\; k = 4\end{aligned}$$

Question 19 **(7 marks)**

The complex number $w = \sqrt{2} - i\sqrt{2}$.

(a) Use de Moivre's theorem to show that for any integer n, $w^n = 2^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$. (3 marks)

$$w = 2cis\left(-\frac{\pi}{4}\right)$$

$$w^n = 2^n cis\left(-\frac{n\pi}{4}\right)$$

$$= 2^n\left(\cos\left(-\frac{n\pi}{4}\right) + i\sin\left(-\frac{n\pi}{4}\right)\right)$$

$$= 2^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

(b) Given that w is a root of the equation $z^6 - 8(1-i)z^4 + a + ib = 0$, find the values of the real constants a and b. (4 marks)

$$w^6 = 2^6\left(\cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2}\right)$$

$$= 64i$$

$$w^4 = 2^4\left(\cos\pi - i\sin\pi\right)$$

$$= -16$$

$$64i - 8(1-i)(-16) = 64i + 128 - 128i$$

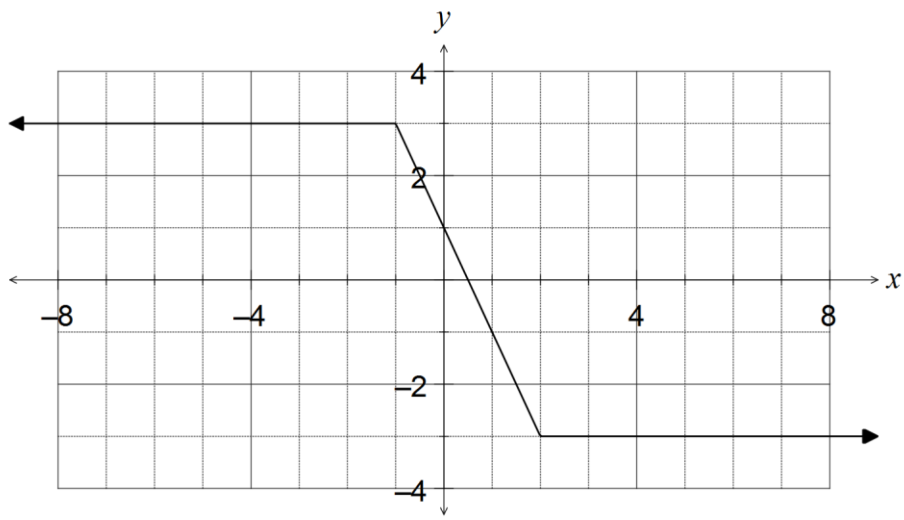
$$= 128 - 64i$$

$$a = -128,\ b = 64$$

Question 20 **(7 marks)**

Consider the function $f(x) = |x-2|-|x+1|$.

(a) Draw the graph of $y = f(x)$ on the axes below. (2 marks)



Let $g(x) = |x+a|-|x+b|$, where a and b are positive constants such that $b > a$.

(b) For which values of x is $g(x)$ constant? (2 marks)

$x \geq -a$ or $x \leq -b$

(c) Write a piecewise definition of $g(x)$. (3 marks)

$$x \leq -b: \ y = -(x+a) - -(x+b)$$
$$-b < x < -a: \ y = -(x+a) - (x+b)$$
$$x \geq -a: \ y = (x+a) - (x+b)$$

$$g(x) = \begin{cases} b-a & x \leq -b \\ -2x-a-b & -b < x < -a \\ a-b & x \geq -a \end{cases}$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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