MATHEMATICS METHODS

MAWA Semester 2 (Units 3 and 4)
Examination 2017

Calculator-Assumed

Marking Key

The release date for this exam and marking scheme is

the end of week 1 of term 4, 2017

Section Two: Calculator-assumed (99 Marks)

Question 10(a)

Solution

Isotope A decays faster.

Reason: Its half-life is less than the half-life of isotope B, i.e. it loses half of its mass faster than isotope B loses half of its mass.

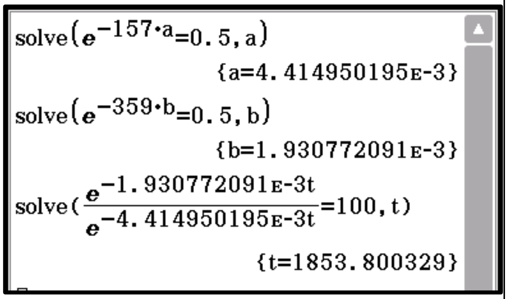
Marking key/mathematical behaviours	Marks
• answers correctly	1
• uses the concept of half-life correctly	1

Question 10(b)

Solution

May assume that $A(t) = e^{-at}$ and $B(t) = e^{-bt}$ where $A(t)$ and $B(t)$ are the amounts of isotopes A and B respectively, t years from now.

Using the half-lives: $e^{-157a} = \frac{1}{2}$ and $e^{-359b} = \frac{1}{2}$.



So $a = \frac{\ln 2}{157} \approx 4.4150 \times 10^{-3}$ and

$b = \frac{\ln 2}{359} \approx 1.9308 \times 10^{-3}$

When $\frac{B(t)}{A(t)} = 100$, $\frac{e^{-0.0019308t}}{e^{-0.0044150t}} = 100$ (#)

i.e. $e^{0.0024842t} = 100$, i.e. $t \approx 1853.8$

So it takes 1854 years before the ratio of the concentrations become 100 to 1.

Marking key/mathematical behaviours	Marks
• uses exponential models for the amounts of isotopes at time t	1
• uses half-lives to solve for the constants a and b correctly	1
• uses equation (#)	1
• solves for the time, correct to the nearest year.	1

Question 11(a)

Solution

Population would be all the people eligible to vote in the election

Sample is the 100 voters asked

Marking key/mathematical behaviours	Marks
• Identifies population correctly	1
• Identifies sample correctly	1

Question 11(b)

Solution

Use a method to randomly choose 100 people from the electoral role

Marking key/mathematical behaviours	Marks
• states a suitable method	1

Question 11(c)

Solution

For 100 estimate of proportion is 0.35

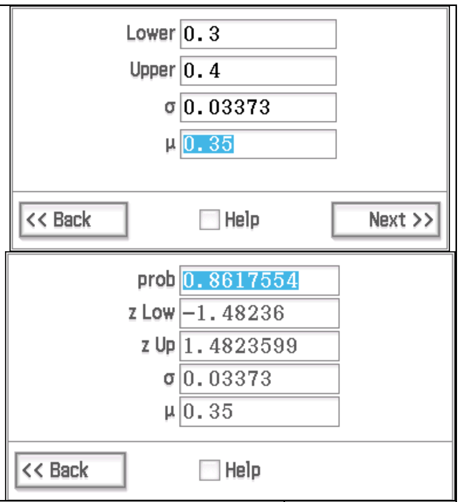
For 200 $E(\hat{p}) = 0.35$

Std Dev $(\hat{p})$ = $\sqrt{\frac{0.35(1-0.35)}{200}}$

= 0.03373

$\hat{p} \sim N(0.35.\,0.03373^2)$

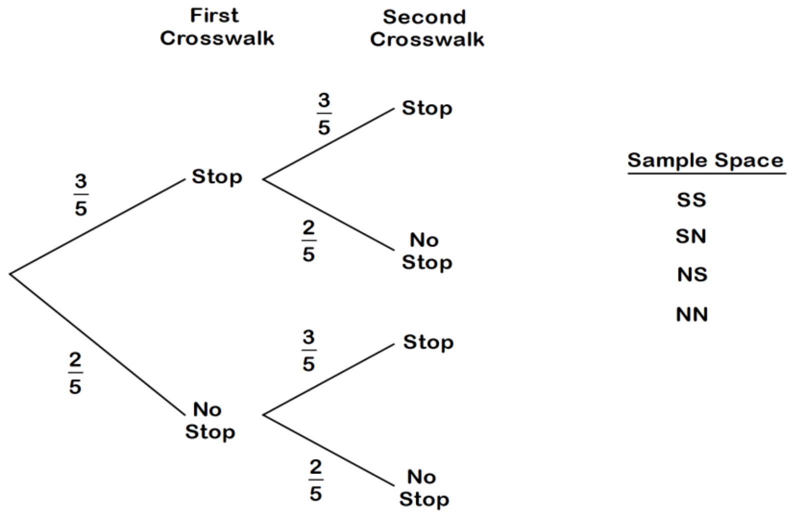
$P(0.3 < \hat{p} < 0.4) =$ 0.8618



Marking key/mathematical behaviours	Marks
• evaluates the standard deviation accurately	1
• states distribution of $\hat{p}$ correctly	1
• Evaluates correct probability	1

Question 12(a)

Solution



Marking Key/mathematical behaviours	Marks
• correctly drawn and labelled tree diagram • states the sample space	1 1

Question 12(b)

Solution

c	0	1	2
Pr(C = c)	0.16	0.48	0.36

Marking key/mathematical behaviours	Marks
• calculates correct probabilities (if only two correct, allow 1 mark)	2

Question 12(c)

Solution

$n = 5 \quad p = 0.84, \quad \mu = np$

$= 5(0.84)$

$= 4.2$

∴ The Bernesse family may expect to stop at least once, five times over the five days.

Marking key/mathematical behaviours	Marks
• recognises the binomial distribution and correctly calculates the expected value	1+1

Question 13(a)

Solution	
Pr (train is late 4 times out of 15) $= {}^{15}\mathbf{C}_4(0.7)^{11}(0.3)^4$ $= 0.219$	
Marking key/mathematical behaviours	Marks
• recognises the binomial distribution and correctly calculates the expected value	1+1

Question 13(b)

Solution	
Pr (train is late 4 times for at least 2 of the next 8 days): late 4 times per day $= 0{\cdot}219$ .... from part (a) Pr that train is not late over the 8 days $= {}^{8}\mathbf{C}_0(0{\cdot}219)^0(0{\cdot}781)^8$ $= 0{\cdot}138$ Pr train is late once over the 8 days $= {}^{8}\mathbf{C}_1(0{\cdot}219)^1(0{\cdot}781)^7$ $= 0{\cdot}311$ ∴ Pr train is late 4 times over the 8 days $= 1 - 0{\cdot}138 - 0{\cdot}311$ $= 0{\cdot}551$	
Marking key/mathematical behaviours	Marks
• calculation of probability of train not being late (using result from (a) • calculates probability for train late once • subtracts the two probabilities from one to achieve end result	1 1 1

Question 13(c)

Solution	
$(0.7)(0.7)(0.7)(0.3)$ $= 0.103$	
Marking key/mathematical behaviours	Marks
• recognizes ordered probability and uses appropriate calculation	1

Question 14(a)

Solution

Since $N \propto \log_{10}\left(\frac{P}{P_0}\right)$, where N is the noise level in decibels and P is the power and P_0 is a reference power level, and since N increases by 10 if the power increases by a factor of 10, $N = 10(\log_{10} P - \log_{10} P_0)$, (#)

So if P increases by a factor of 40, N increases by $10\log_{10} 40 \approx 16{\cdot}02\ dB$

Marking key/mathematical behaviours	Marks
• obtains equation (#) or equivalent	1
• obtains correct answer	1

Question 14(b)(i)

Solution

Since $2 \times 7^2 = 98 \approx 100 = 10^2$

it follows that $\log_{10} 2 + 2\log_{10} 7 \approx 2$ (#)

i.e. $\log_{10} 7 \approx 1 - \frac{\log_{10} 2}{2} \approx 1 - \frac{0.30}{2} = 0.85$

Marking key/mathematical behaviours	Marks
• obtains approximation (#)	1
• obtains correct answer	1

Question 14(b)(ii)

Solution

Since $2^{12} \times 3^5 = 995328$

and $995328 \approx 1000000 = 10^6$

it follows that $12\log_{10} 2 + 5\log_{10} 3 \approx 6$ (#)

and so $\log_{10} 3 \approx \frac{6 - 12\log_{10} 2}{5} \approx \frac{6 - 12 \times 0.30}{5} = 0.48$

Marking key/mathematical behaviours	Marks
• evaluates $2^{12} \times 3^5$ correctly	1
• obtains approximation (#)	1
• obtains correct answer	1

Question 15(a)

Solution	
$y_{max} = a + b = 14.5$ and $y_{min} = a - b = 9.5$ (#) and so $a = 12$ and $b = 2.5$ Since the period is 1 year, i.e. 365 days, $c = 365$	
Marking key/mathematical behaviours	Marks
• obtains equations (#) • solves for a and b correctly • obtains correct value for c	1 1 1

Question 15(b)

Solution	
When $y(t) = y_{max}$ we have $\dfrac{2\pi(t+9)}{365} = 2\pi$ (#) i.e. $t + 9 = 365$ i.e. $t = 356$ So the 356th day, (December 22nd) will be the longest day.	
Marking key/mathematical behaviours	Marks
• obtains equation (#) • obtains correct answer	1 1

Question 15(c)

Solution	
$y'(t) = -\dfrac{2\pi b}{365}\sin\dfrac{2\pi(t+9)}{365} = -\dfrac{5\pi}{365}\sin\dfrac{2\pi(t+9)}{365}$ So $y'(t) = y'_{min}$ when $\dfrac{2\pi(t+9)}{365} = \dfrac{\pi}{2}$ (#) i.e. when $t + 9 = \dfrac{365}{4}$ i.e. $t = 82.25$ So the number of daylight hours will be decreasing fastest on the 82nd day, i.e. on March 23rd.	
Marking key/mathematical behaviours	Marks
• differentiates correctly • obtains equation (#) • obtains correct answer	1 1 1

Question 15(d)

Solution

$$y'_{min} = -\frac{5\pi}{365} \approx -0.0430$$

By the increments formula $\delta y \approx y' \times \delta t$ and so if $\delta t = 1$ $\delta y \approx y' \approx -0.0430$

So the largest difference in the number of daylight hours in successive days is 0.043 hours, i.e. 2.6 minutes.

Marking key/mathematical behaviours	Marks
• correctly calculates y'_{min}	1
• uses increments formula correctly	1

Question 16(a)

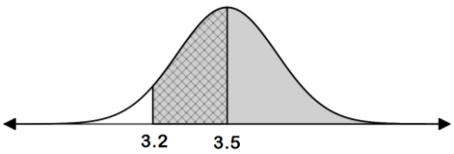
Solution

(i) $X \sim N(3.5, 0.2^2)$
$P(X = 3.5) = 0$

(ii) $X \sim N(3.5, 0.2^2)$
$P(X > 3.2) = 0.93$

(iii)

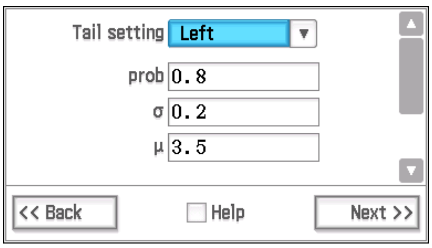
$$P(X < 3.5 | X > 3.2) = \frac{P(3.2 < X < 3.5)}{P(X > 3.2)}$$
$$= \frac{0.4332}{0.9332}$$
$$= 0.4642$$

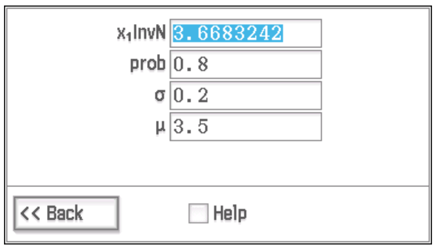


Marking key/mathematical behaviours	Marks
• recognises exact probabilities are equal to zero	1
• calculates correct probability	1
• applies the appropriate formula and associated probabilities leading to the correct answer and correct diagram	1+1+1

Question 16(b)

Solution





$P(X \le m) = 0.8$

$\Rightarrow m = 3.668$

Marking key/mathematical behaviours	Marks
• states probability condition involving m	1
• calculates the correct value for m	1

Question 16(c)

Solution	
$X \sim N(3.5, \sigma^2)$ $P(X > 3.7) = 0.1$ $P\left(Z > \dfrac{3.7-3.5}{\sigma}\right) = 0.1$ $\dfrac{3.7-3.5}{\sigma} = 1.28$ $\sigma = 0.156$ $= 16$ centimetres	
Marking key/mathematical behaviours	Marks
• uses the correct formula and substitutes values • calculation the standard score • states the correct answer	1 1 1

Question 17(a)

Solution	
$v = \int 3\sin(2t)\,dt$ $= -\frac{3}{2}\cos(2t) + c$ $t = 0 \quad \rightarrow \quad -\dfrac{3}{2} + c = 4$ $c = 5.5$ $\therefore \quad v = -\frac{3}{2}\cos(2t) + 5.5$	
Marking key/mathematical behaviours	Marks
• correctly integrates to find equation for v involving c • correctly evaluates c • writes an expression for v	1 1 1

Question 17(b)

Solution

$$x = \int \left[-\frac{3}{2}\cos(2t) + 5.5 \right] dt$$

$$= -\frac{3}{4}\sin(2t) + 5.5t + c$$

When $t = 0$ $\rightarrow$ $c = $ 2 or - 2

When $t = 2$ $\rightarrow$ $x = -\frac{3}{4}\sin(4) + 5.5(2) \mp 2$

= 13.57 m or 9.57 m

Marking key/mathematical behaviours	Marks
• determines correct integral of function plus c	1
• calculates a value for c	1
• calculates x accurately when $t = 2$ and includes both possible values	1+1

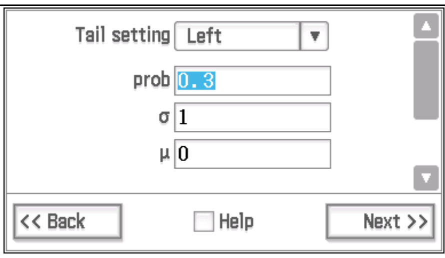
Question 18(a)

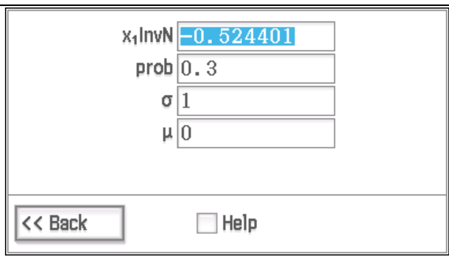
Solution

$X \sim N(0,1)$

$P(X \leq x) = 0.3$

$x = -0.5244$





Marking key/mathematical behaviours	Marks
• identifies the parameters of the standard normal and states the problem in terms of probability	1
• states the correct result	1

Question 18(b)

Solution

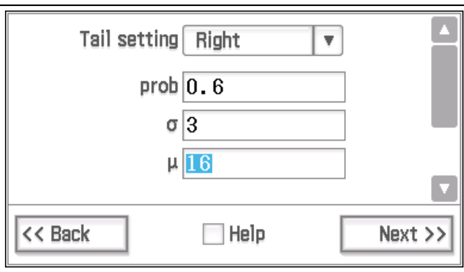
$X \sim N(16 \;\; 3^2)$

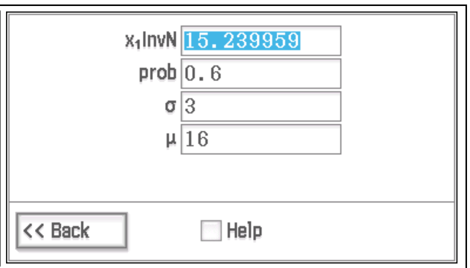
$P(X > x) = 0{\cdot}6$

$\therefore \quad x = 15{\cdot}24$

So, $3k - 1 = 15{\cdot}24$

$\therefore \quad k = 5{\cdot}41$





Marking key/mathematical behaviours	Marks
• applies the normal distribution to determine x	1
• states the correct result for k	1

Question 18(c)

Solution	
The x-value of 6 is 2.4 standard deviations away from the mean.	
Marking key/mathematical behaviours	Marks
• provided an acceptable explanation	1

Question 18(d)

Solution

$$F(x) = \int_0^x 3x^2 \ dx = \left[x^3\right]_0^x = x^3 \quad (0 < x < 1)$$

$$\therefore F(x) = \begin{cases} 0 & x \le 0 \\ x^3 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Marking key/mathematical behaviours	Marks
• evaluates the correct integral	1
• defines $F(x)$	1
• states the three domains correctly for $F(x)$	1

Question 19

Solution

Check sample size is large enough for normal approximation np >10 and $n(1 - p)$ > 10.

In this case, 1000 x 0.48 = 480 > 10
1000 x 0.52 = 520 > 10
Therefore, normal approximation can be applied.

$$CI \ = \ 0.48 \pm 1.96\sqrt{\frac{0.48 \ x \ 0.52}{1000}}$$

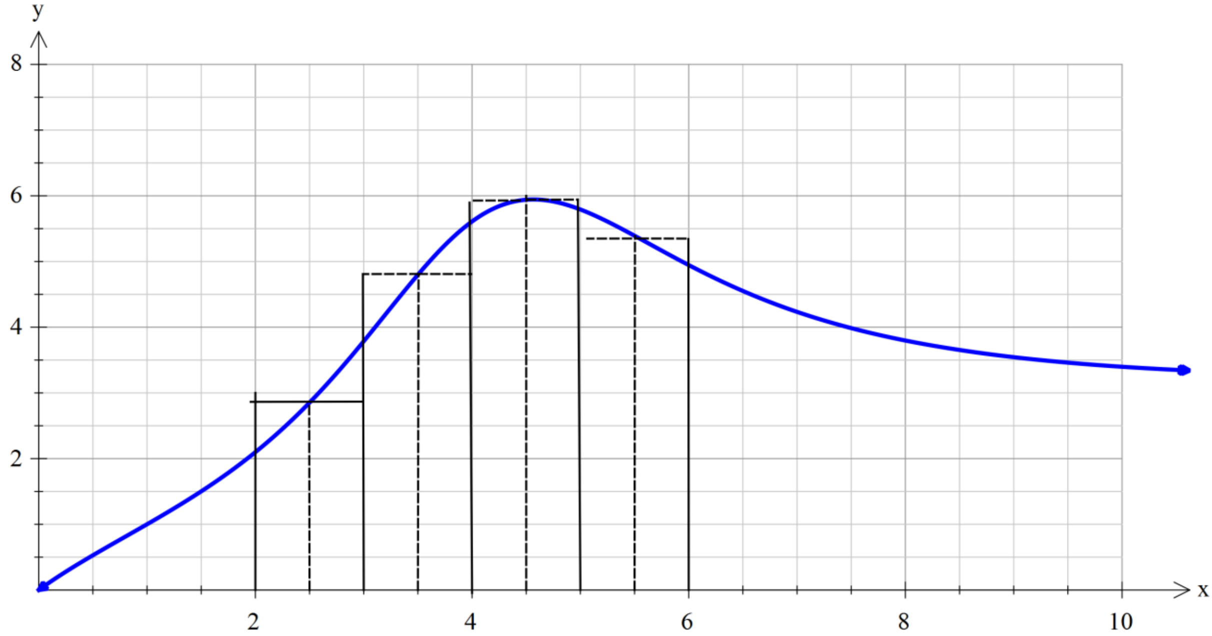
$$= \ 0.48 \ \pm 0.03097$$

= (0.45, 0.51)

(0.45, 0.51) is a 95% Confidence Interval for the true proportion of students excited by the upcoming concert.

Marking key/mathematical behaviours	Marks
• Checks the sample size for normal approximation	1+1
• Sets up CI and evaluates correctly	1+1
• correctly interprets result	1

Question 20(a)

Solution	
 $\sum_i f(x_i)\delta x_i = f(2.5)\times(1) + f(3.5)\times(1) + f(4.5)\times(1) + f(5.5)\times(1)$ (i) $= 2.8 + 5.8 + 5.9 + 5.4$ $= 19.9$ The area is approximately 20 square units. (ii) The area represents the distance travelled by the projectile between $t = 2$ and $t = 6$	
Marking key/mathematical behaviours	**Marks**
• estimates the function at the values suggested (allow ± 0.2)	2
• applies the summation correctly	1
• states the required area	1
• correctly interprets the meaning of the area as the distance travelled	1

Question 20(b)

Solution

The area of the triangle formed by $g(x)$ and the $x-\text{axis}$ (between $x=0$ and $x=2$)
= 1 square unit.

Hence,

(i) region A = $\left|\int_0^2 f(x)dx\right| - 1 = 5.1 - 1 = 4.1$

(ii)

$$\begin{aligned}\text{Region B} &= \int_2^4 f(x)dx - \int_2^4 g(x)dx \\ &= \int_0^4 f(x)dx - \int_0^2 f(x) - \int_2^4 g(x)dx \\ &= -2.18 - (-5.1) - 1 \\ &= 1.92\end{aligned}$$

Marking key/mathematical behaviours	Marks
• Calculates the area of the triangle	1
• Calculates the area of region A	1
• Defines region B in terms of integrals of $f(x)$ and $g(x)$	1
• Re-arranges the integrals using the integral properties so as to be able to use the information given	2
• Shows the required result.	1

Question 21(a)

Solution

$$\int_0^1 \frac{dx}{x+1} = [\ln(x+1)]_0^1 = \ln 2 - \ln 1 = \ln 2$$

Marking key/mathematical behaviours	Marks
• obtains $\ln(x+1)$ as the antiderivative	1
• evaluates at limits correctly	1

Question 21(b)

Solution

$$\int_0^1 \frac{x\,dx}{x^2+1} = \left[\frac{1}{2}(\ln(x^2+1))\right]_0^1 = \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 2$$

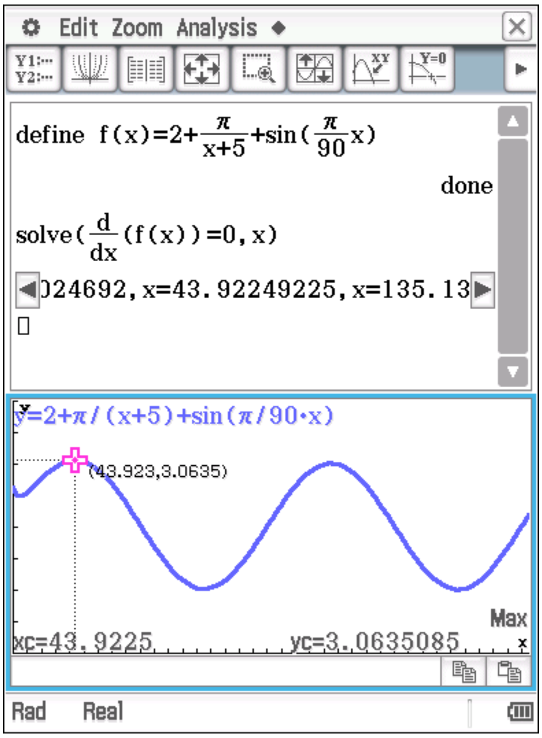
Marking key/mathematical behaviours	Marks
• obtains $\frac{1}{2}\ln(x^2+1)$ as the antiderivative	1
• evaluates at limits correctly	1

Question 21(c)(i)

Solution

Rate of heat loss is a maximum at $t \approx 44$, and $t \approx 224$ days

Maximum rate of heat loss is ~ 3 kilojoules per day.



Marking key/mathematical behaviours	Marks
• states correct values of t	1+1
• states the maximum rate of heat loss	1

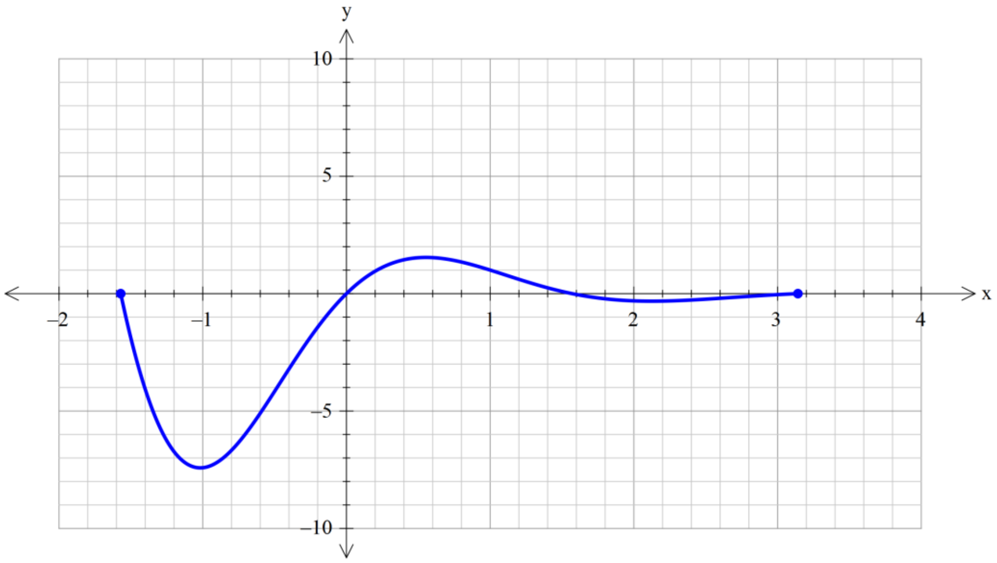
Question 21(c)(ii)

Solution:	
define $f(x)=2+\frac{\pi}{x+5}+\sin\left(\frac{\pi}{90}x\right)$ done $\int_0^{120} f(x)\,dx$ 293.0842313 The heat loss is ~293 kilojoules.	
Marking key/mathematical behaviours	**Marks**
• indicates that the heat loss in the integral from 0 to 120 of $\frac{dH}{dt}$ • states the correct result • states the correct units	1 1 1

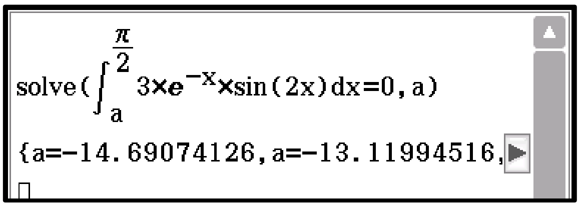
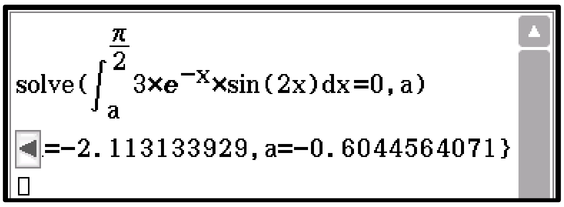
Question 21(c)(iii)

Solution	
define $f(x)=2+\frac{\pi}{x+5}+a\times\sin\left(\frac{\pi}{90}x\right)$ done solve$\left(\int_0^{120} f(x)\,dx=300, a\right)$ $\{a=1.160937246\}$ $a \approx 1.16$	
Marking key/mathematical behaviours	**Marks**
• indicates solving the integral of from 0 to 120 of $\frac{dH}{dt}$ = 300 • states the correct result	1 1

Question 22(a)

Solution	
	
Marking key/mathematical behaviours	Marks
• Clearly shows the correct intercepts	1
• Minimum and maximum points are reasonably accurate	1
• Graph is appropriately smooth	1

Question 22(b)

Solution	
Using the CAS calculator to solve for *a:*   From the graph in part (a) it is obvious that $-\frac{\pi}{2} < a < 0$ so, need to select $a \approx -0.6$	
Marking key/mathematical behaviours	Marks
• Solves correctly (if provides additional values for *a* – subtract one mark)	2