Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS

3C/3D

Calculator-assumed

WACE Examination 2015

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

2015/101883

Section Two: Calculator assumed **66⅔% (100 marks)**

Question 9 **(6 marks)**

For any two numbers $a > b > 0$, it is conjectured that $\sqrt{a} - \sqrt{b} < \sqrt{a-b}$.

(a) Provide two pairs of numbers to demonstrate that the conjecture appears to be true. (2 marks)

Solution
Consider $a = 2$ and $b = 1$, then $\sqrt{2} - \sqrt{1} = \sqrt{2} - 1 \approx 0.4142 < \sqrt{2-1} = \sqrt{1} = 1$ Consider $a = 3$ and $b = 1$, then $\sqrt{3} - \sqrt{1} = \sqrt{3} - 1 \approx 0.7321 < \sqrt{3-1} = \sqrt{2} = 1.4142$
Specific behaviours
✓ calculates both sides of the inequality for two appropriate pairs of numbers ✓ shows the inequality holds

(b) If $a - b = c$ where $c > 0$, show that the conjecture is equivalent to $\sqrt{b+c} < \sqrt{b} + \sqrt{c}$. (1 mark)

Solution
If $a - b = c \Rightarrow a = b + c$. Then we have $\sqrt{a} - \sqrt{b} < \sqrt{a-b} \Rightarrow \sqrt{b+c} - \sqrt{b} < \sqrt{c}$ $\Rightarrow \sqrt{b+c} < \sqrt{b} + \sqrt{c}$
Specific behaviours
✓ deduces new conjecture

(c) Prove algebraically that the conjecture in part (b) is true for all positive numbers b and c. (3 marks)

Solution
$\left(\sqrt{b} + \sqrt{c}\right)^2 = b + c + 2\sqrt{bc} > \left(\sqrt{b+c}\right)^2$, as $2\sqrt{bc} > 0$ If $\left(\sqrt{b} + \sqrt{c}\right)^2 > \left(\sqrt{b+c}\right)^2$ then $\sqrt{b} + \sqrt{c} > \sqrt{b+c}$ Since b and c are positive, this statement is true Hence the conjecture is true for all positive numbers b and c
Specific behaviours
✓ rearranges inequality to facilitate a proof ✓ derives statement $\sqrt{b+c} > 0$ ✓ concludes conjecture is true for $b,\ c$

Question 10 **(10 marks)**

The events A and B have probabilities $P(A)=0.3$, $P(\overline{B}\mid\overline{A})=0.2$ and $P(B\mid A)=0.4$.

(a) Show that $P(A\cap B)=0.12$. (1 mark)

Solution
$P(A\cap B)=P(A)\times P(B\mid A)=0.3\times 0.4=0.12$
Specific behaviours
✓ uses conditional probability formula correctly

(b) Show that $P(A\cup B)=0.86$. (3 marks)

Solution
$P(\overline{B}\mid\overline{A})=0.2=\dfrac{P(\overline{B}\cap\overline{A})}{P(\overline{A})}=\dfrac{P(\overline{A\cup B})}{0.7}$ Hence $1-P(A\cup B)=0.2\times 0.7$ $P(A\cup B)=0.86$
Specific behaviours
✓ uses correct probability formula ✓ recognises that $P(\overline{B}\cap\overline{A})=P(\overline{A\cup B})$ ✓ show that $P(A\cup B)=0.86$

(c) Determine $P(B)$. (2 marks)

Solution
$P(A\cup B)=P(A)+P(B)-P(A\cap B)$ $0.86=0.3+P(B)-0.12$ $P(B)=0.68$
Specific behaviours
✓ uses correct probability law ✓ solves for $P(B)$

(d) Determine $P(A \mid B)$. (2 marks)

Solution
$P(A \mid B)=\dfrac{P(A\cap B)}{P(B)}$ $=\dfrac{0.12}{0.68}$ $=0.1765$
Specific behaviours
✓ uses correct probability formula   ✓ solves for $P(A \mid B)$

(e) Are events A and B independent? Justify your answer. (2 marks)

Solution
$P(A\cap B)\neq P(A)\times P(B)$ $0.12\neq 0.3\times 0.68=0.204$ Hence events are not independent
Specific behaviours
✓ states that events are not independent   ✓ justifies using numerical values

Question 11 **(4 marks)**

The points $P(-2,1)$ and $Q(6,9)$ lie on the parabola $y=\frac{x^2}{4}$.

(a) Find the equations of the tangents to the parabola at P and Q. (2 marks)

Solution
$y=\frac{1}{4}x^2 \Rightarrow \frac{dy}{dx}=\frac{1}{2}x$ $\left.\frac{dy}{dx}\right\|_{x=-2}=-1,\quad \left.\frac{dy}{dx}\right\|_{x=6}=3$ Tangent at P: $y-1=-1(x+2)\Rightarrow y=-x-1$ Tangent at Q: $y-9=3(x-6)\Rightarrow y=3x-9$
Specific behaviours
✓ determines derivative to calculate gradients ✓ determines equations of tangent lines

(b) The tangents to the parabola at P and Q meet at point R. Find the coordinates of R. (2 marks)

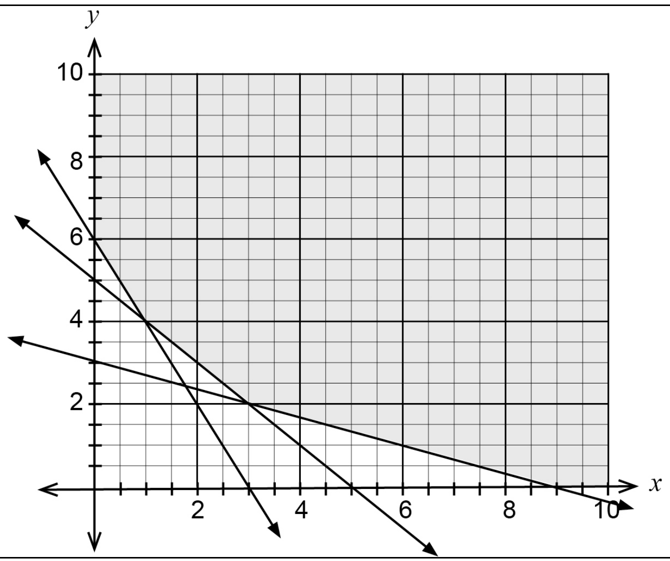
Solution
$-x-1=3x-9$ $\Rightarrow 4x=8$ $\Rightarrow x=2\Rightarrow y=-3$ Hence $R(2,-3)$
Specific behaviours
✓ finds the value of x where the tangent lines intersect ✓ determines coordinates of R

Question 12 **(12 marks)**

(a) Two of the three above-mentioned constraints have been drawn on the axes below. Write down the missing constraint in terms of x and y. (Note : $x \geq 0, y \geq 0$.) (2 marks)

Solution
Poison $12x + 6y \geq 36$ $2x + y \geq 6$
Specific behaviours
✓ identifies the poison constraint ✓ states a correct inequality for poison

(b) Draw in the missing constraint on the axes above, and then shade the feasible region that satisfies all of the constraints. (3 marks)

Solution

Specific behaviours
✓ draws in the line $2x + y = 6$ ✓ shades for $x \geq 0, y \geq 0$ ✓ shades correct region that satisfies all constraints

(c) Given that each kilogram of Type A fertiliser costs \$12 and each kilogram of Type B fertiliser costs \$15, determine the number of kilograms that the farmer must buy so as to minimise the cost and still satisfy the constraints. State this minimum cost. (4 marks)

Solution

Objective function Cost C

$C = 12x + 15y$

Extreme points	Cost \$
(0,6)	\$90
(1,4)	\$72
(3,2)	\$66
(9,0)	\$108

(3,2) is optimal point with a minimum cost of \$66

Specific behaviours

✓ states objective function
✓ examines at least three extreme points
✓ determines optimal point (3,2)
✓ states minimal cost

(d) By how much can the price of Type B fertiliser change so as to increase the amount of the fertiliser while still maintaining the minimum cost found in part (c)? (3 marks)

Solution

Optimal point (3,2) changing to (1,4)

$C = 12x + by$

$12(3) + b(2) = 12(1) + b(4)$

$24 = 2b$

$b = 12$

Cost of Type B fertilizer must drop by \$3 from \$15 to \$12

Specific behaviours

✓ identifies (1,4) as new optimal point
✓ solves for new cost
✓ states drop in price

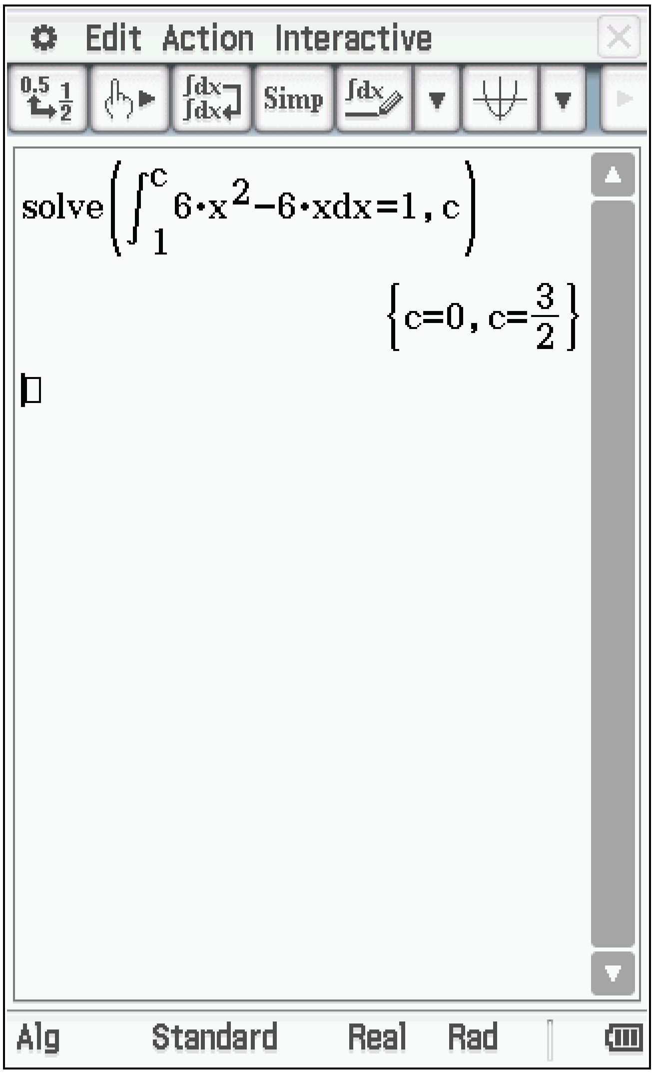
Question 13 **(4 marks)**

The area bound by the parabola $y = 6x^2 - 6x,$ the $x-$axis and the lines $x = 1$ and $x = c$, $(c > 1)$, is equal to 1 unit^2. Find the value of the constant.

Solution
$\int_1^c 6x^2 - 6xdx = \left[2x^3 - 3x^2\right]_1^c$ $= \left(2c^3 - 3c^2\right) - \left(2 - 3\right)$ $= 2c^3 - 3c^2 + 1$ $\therefore 2c^3 - 3c^2 + 1 = 1$ $\Rightarrow 2c^3 - 3c^2 = 0$ $\Rightarrow c^2\left(2c - 3\right) = 0$ $\Rightarrow c = 0, \frac{3}{2}$ But $c > 1 \Rightarrow c = \frac{3}{2}$
Specific behaviours
✓ evaluates integral correctly ✓ substitutes correct values for the two limits for x ✓ deduces equation to solve for c ✓ solves for c

Alternative solution:

Solution



But $c > 1 \Rightarrow c = \frac{3}{2}$

Specific behaviours

✓ sets up integral correctly
✓ substitutes correct values for the two limits for x
✓ solves for c
✓ discards $c = 0$

Question 14 **(8 marks)**

(a) What is the probability that the property is sold to the second person viewing it? (2 marks)

Solution
$0.9\times 0.1=0.09$
Specific behaviours
✓ uses 0.9 for first buyer ✓ determines probability

(b) What is the probability that more than two people view the property before it is sold? (3 marks)

Solution
Let X = then number of people to view the property before being bought $P(X>2)=1-P(X\le 2)=1-(0.1+0.9\times 0.1)$ $=0.81$
Specific behaviours
✓ uses complement ✓ determines $P(X=1,2)$ ✓ determines probability

(c) Four people are scheduled to view the property. What is the probability that one of them buys it? (3 marks)

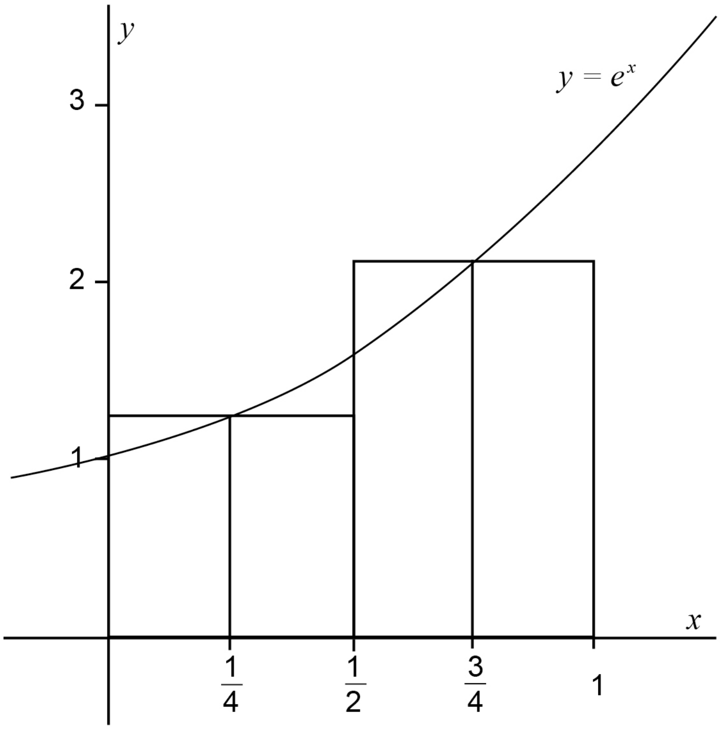
Solution
$0.1+0.9\times 0.1+0.9^2\times 0.1+0.9^3\times 0.1=0.3439$
Specific behaviours
✓ considers that each viewer may buy the property ✓ uses independent probabilities ✓ determines probability

OR

Solution
$1-0.9^4=0.3439$
Specific behaviours
✓ considers complement that no one buys the property ✓ uses independent probabilities ✓ determines probability

Question 15 **(4 marks)**

A plot of the exponential function $y = e^x$ is shown below.



The integral $\int_0^1 e^x \, dx$ may be approximated by the areas of the rectangles as shown above.

(a) Show that the value of the integral $\int_0^1 e^x \, dx$ is approximately given by

$\int_0^1 e^x \, dx \approx \frac{1}{2}\left(e^{\frac{1}{4}} + e^{\frac{3}{4}}\right)$. (2 marks)

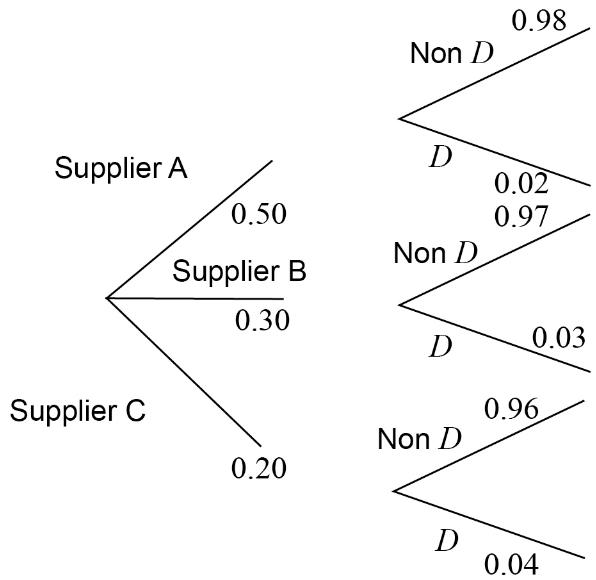
Solution
Each rectangle has a width of $\frac{1}{2}$ The first rectangle has a height of $e^{\frac{1}{4}}$ and the second has a height of $e^{\frac{3}{4}}$ Hence the area of the two rectangles is $\frac{1}{2}\left(e^{\frac{1}{4}} + e^{\frac{3}{4}}\right)$
Specific behaviours
✓ treats the integral as the sum of the area of rectangles ✓ determines area of the rectangles

(b) Determine upper and lower limits for the integral $\int_0^1 e^x\,dx$ using the areas of the rectangles. (2 marks)

Solution
Upper = $\frac{1}{2}\left(e^{\frac{1}{2}}+e\right)$ Lower = $\frac{1}{2}\left(e^0+e^{\frac{1}{2}}\right)=\frac{1}{2}\left(1+e^{\frac{1}{2}}\right)$
Specific behaviours
✓ determines lower limit ✓ determines upper limit

Question 16 **(7 marks)**

(a) Construct a probability tree diagram for the above information. (4 marks)

Solution

Specific behaviours
✓ shows all suppliers ✓ labels defective and non-defective to each supplier ✓ shows probabilities for most branches ✓ uses idea that probabilities at each node add to one

(b) Given that a robotic arm is defective, determine the probability that the arm did not come from Supplier A. (3 marks)

Solution
$\dfrac{0.30\times0.03+0.20\times0.04}{0.50\times0.02+0.30\times0.03+0.20\times0.04}$ $=\dfrac{0.017}{0.027}$ $=0.6296$ $=\dfrac{17}{27}$
Specific behaviours
✓ uses conditional probabilities ✓ determines numerator ✓ determines denominator

Question 17 **(6 marks)**

(a) When ℓ =12 metres, the length of rope is changing at a rate of $\frac{d\ell}{dt} = 0.1$ metres per second. Determine $\frac{dT}{dt}$. (3 marks)

Solution
$\frac{dT}{dt} = \frac{dT}{d\ell}\frac{d\ell}{dt} = \frac{\pi}{\sqrt{10}} \times l^{-\frac{1}{2}} \times 0.1$ $= \frac{\pi}{\sqrt{10}} \times 12^{-\frac{1}{2}} \times 0.1$ $= 0.0287$sec/sec
Specific behaviours
✓ uses chain rule ✓ differentiates T ✓ determines $\frac{dT}{dt}$ when ℓ =12

(b) Use the increments formula $\delta T \approx \frac{dT}{d\ell}\delta\ell$ to determine the approximate percentage change in T if ℓ changes by 2% (that is, $\frac{\delta\ell}{\ell} = 0.02$). (3 marks)

Solution
$\frac{\delta T}{T} \approx \frac{\frac{dT}{d\ell}\delta\ell}{2\pi\sqrt{\frac{\ell}{10}}}$ $= \frac{\frac{\pi}{\sqrt{10}}\ell^{-\frac{1}{2}}\delta\ell}{2\pi\sqrt{\frac{\ell}{10}}}$ $= \frac{\delta\ell}{2\ell}$ $= \frac{2\%}{2}$ $= 1\%$
Specific behaviours
✓ obtains expression for $\frac{\delta T}{T}$ using increments formula ✓ simplifies expression ✓ determines approximate percentage

Question 18 **(6 marks)**

Consider two circles, the first having a radius R_1 and the other radius R_2, with the sum of the two radii being constant, $R_1 + R_2 = C$.

Use calculus to prove that if the sum of the radii of two circles is constant, then the sum of the areas of the two circles is at a minimum when the circles have equal radii.

Solution
The sum of the areas is $A = \pi R_1^2 + \pi R_2^2$ Hence $A = \pi R_1^2 + \pi\left(C - R_1\right)^2 = 2\pi R_1^2 - 2\pi C R_1 + \pi C^2$, and $\dfrac{dA}{dR_1} = 4\pi R_1 - 2\pi C = 0 \Rightarrow R_1 = \dfrac{C}{2}$ Since $\dfrac{d^2 A}{dR_1^2} = 4\pi > 0$, this is a minimum Therefore the sum of the area of the two circles is minimised when $R_2 = C - R_1 = C - \dfrac{C}{2} = \dfrac{C}{2}$; that is, the circles have equal radii
Specific behaviours
✓ derives a formula for the sum of the areas ✓ expresses the area in terms of one variable only ✓ finds the first derivative ✓ equates first derivative to zero and solves for critical point ✓ uses either the first or second derivative test to check the critical point is a minimum ✓ deduces that the radii are equal

Question 19 **(10 marks)**

(a) Determine the displacement function x from the depot, in terms of t. (2 marks)

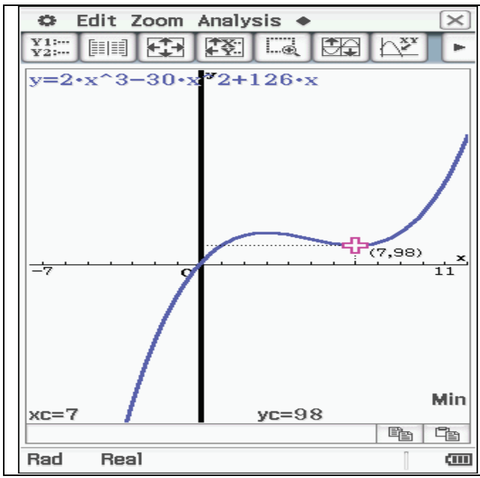
Solution
$x = 2t^3 - 30t^2 + 126t + c$ $x = 0, t = 0 \Rightarrow c = 0$ $x = 2t^3 - 30t^2 + 126t$
Specific behaviours
✓ anti-differentiates velocity ✓ states zero for constant

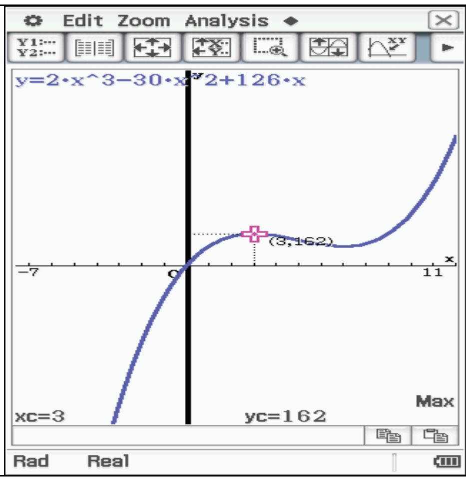
(b) Determine the times that the monorail will stop at Towns A and B. (3 marks)

Solution
$v = 0 = 6t^2 - 60t + 126$ $t = 3, 7$ Times 3 and 7 hours at towns B and A respectively.
Specific behaviours
✓ equates velocity to zero ✓ solves for times ✓ matches times with towns

(c) What is the distance between the two towns? (2 marks)

Solution





Turning points for displacement occur at (3,162) and (7, 98)

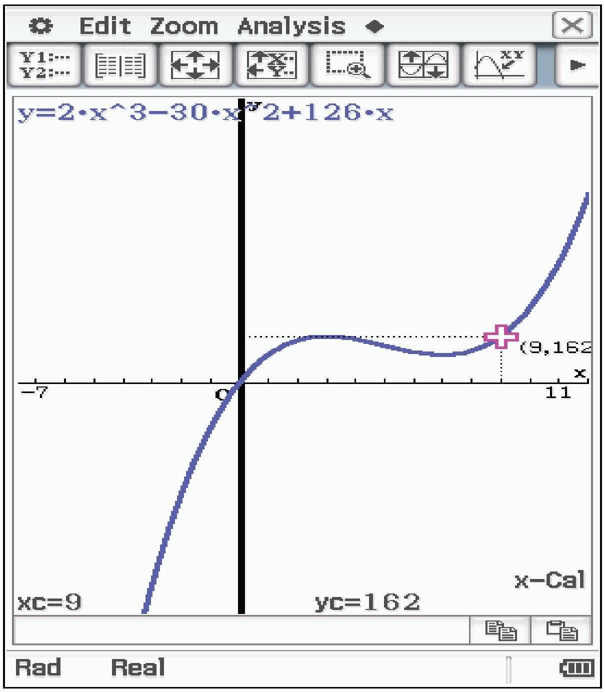
$$x = 2t^3 - 30t^2 + 126t$$
$$x(3) = 162$$
$$x(7) = 98$$

Distance between towns = 162-98 = 64 kilometres

Specific behaviours

✓ solves for turning points of displacement function
✓ determines distance

(d) Determine the distance travelled and the time taken when the monorail enters Town B for the second time. (3 marks)

Solution
 $x = 2t^3 - 30t^2 + 126t$ $x(t) = 162$ $t = 9$ Distance travelled = 162 + 64 + 64 = 290 km At nine hours, monorail enters town B for second time, that is at (9, 162)
Specific behaviours
✓ solves for point (y = 162) on displacement function ✓ states distance travelled ✓ states time taken

Question 20 **(15 marks)**

The strength of steel cables produced by a manufacturer are normally distributed with a specified mean of 1000 tonnes and a standard deviation of 100 tonnes.

(a) Of 50 steel cables, how many would be expected to have a strength of less than 990 tonnes? (3 marks)

Solution
$X \sim N\left(1000, 100^2\right)$ $P\left(X < 990\right) = 0.4602$ $50 \times 0.4602 = 23$ Approximately 23 cables
Specific behaviours
✓ uses Normal probabilities ✓ determines probability ✓ determines proportion of 50 cables

(b) What is the probability that out of 10 cables selected at random, at least nine have a strength of less than 990 tonnes? (3 marks)

Solution
$Y \sim Bin\left(10, 0.4602\right)$ $P\left(Y \geq 9\right) = 0.0054$
Specific behaviours
✓ states Binomial distribution ✓ uses correct parameters ✓ determines probability

(c) (i) Obtain a 99% confidence interval for the population mean strength of the cables, correct to two decimal places. (4 marks)

Solution
$995 - 2.576 \times \frac{100}{\sqrt{200}} < \mu < 995 + 2.576 \times \frac{100}{\sqrt{200}}$ $976.785 < \mu < 1013.215$ $976.79 < \mu < 1013.22$ (rounded to two decimal places)
Specific behaviours
✓ uses correct parameter for 99 % confidence interval ✓ uses $\frac{\sigma}{\sqrt{200}}$ ✓ determines lower limit ✓ determines upper limit to two decimal places

(ii) What would you advise the engineer regarding the suitability of the cables for the crane? Justify your answer. (2 marks)

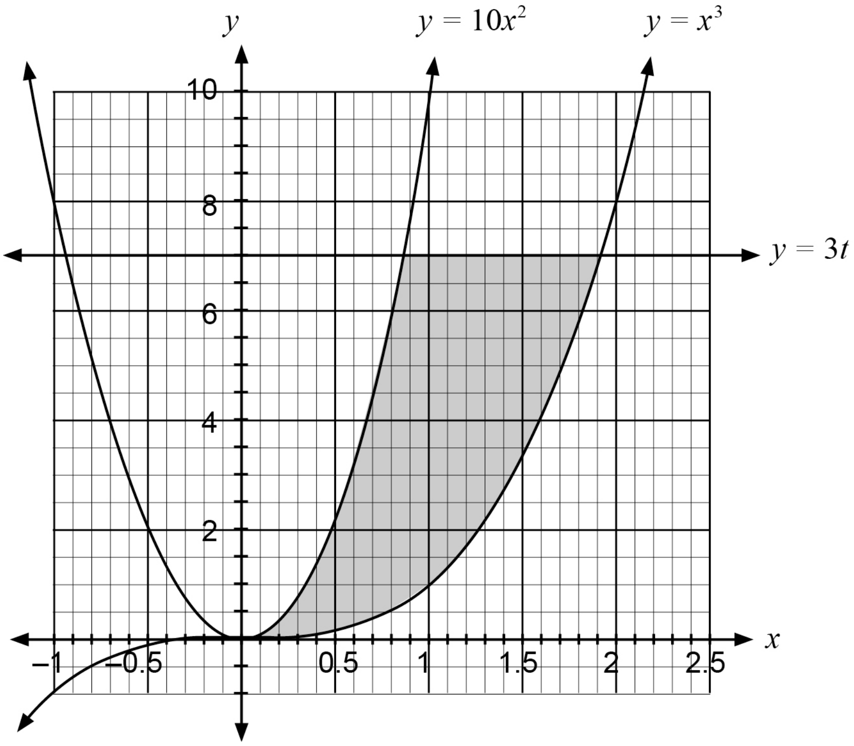
Solution
Would advise the engineer **not** to use the cables, as 1014 tonnes is above the confidence interval.
Specific behaviours
✓ states that strength is not suitable ✓ justifies using confidence interval

(iii) The engineer wants to obtain a 99% confidence interval no wider than 20 tonnes for the population mean strength of the cables. What sample size should she take? (3 marks)

Solution
$2.576 \times \frac{100}{\sqrt{n}} = 10$ $n \approx 663.578$ $n = 664$
Specific behaviours
✓ uses 10 with formula ✓ solves for sample size ✓ rounds sample size up

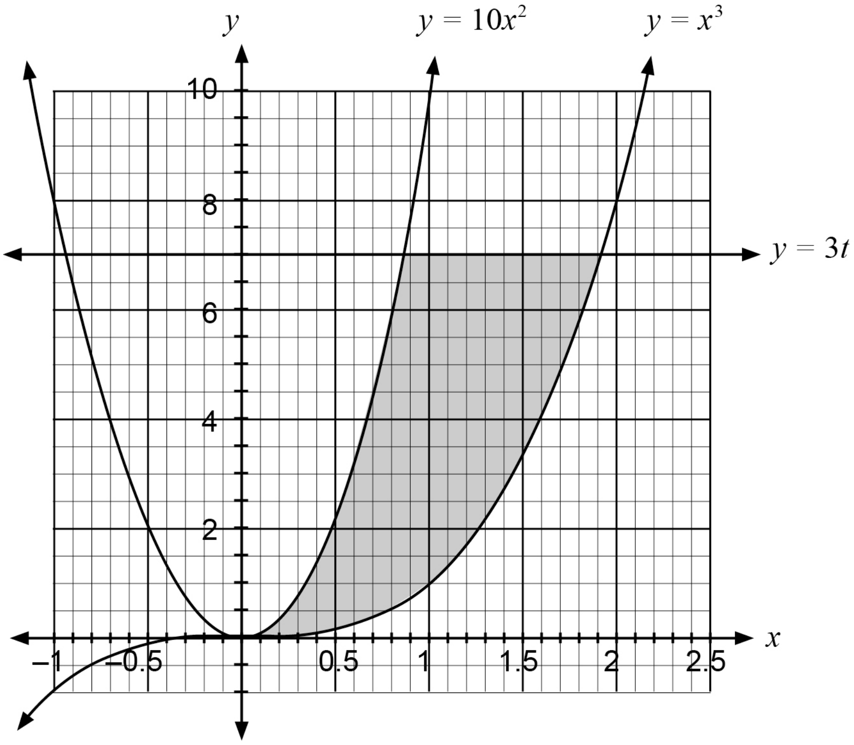
Question 21 **(8 marks)**

(a) Determine the volume of the solid generated when the shaded area enclosed by the curves and lines $y = 10x^2$, $y = x^3$, and $y = 5$ (see below) is revolved around the y axis. (4 marks)



Solution
$V = \pi \int_0^5 \left(y^{\frac{2}{3}} - \frac{y}{10} \right) dy$ $= 23.631$ Volume is 23.631 cubic units
Specific behaviours
✓ uses $\int \pi x^2 dy$ ✓ rearranges x^2 for each curve ✓ sets up correct integral with y limits ✓ determines volume

The line $y = 5$ is replaced with the line $y = 3t$ where $0 \le t \le 3,$ as can be seen in the diagram below for a particular value of t. The area enclosed is revolved around the y axis, forming a solid of revolution.



(b) Derive an expression for the volume, $V,$ of the solid of revolution as a function of t (may be left as an integral). (2 marks)

Solution
$V = \pi \int_0^{3t} \left(y^{\frac{2}{3}} - \dfrac{y}{10} \right) dy$
Specific bahaviours
✓ uses integral from part (a) ✓ uses $3t$ for upper limit

(c) Determine $\frac{dV}{dt}$ when $t = 2$. (2 marks)

Solution
$\frac{dV}{dt} = \frac{d}{dt}\pi\int_0^{3t}\left(y^{\frac{2}{3}} - \frac{y}{10}\right)dy$ $= 3 \times \pi \times \left((3t)^{\frac{2}{3}} - \frac{3t}{10}\right)\Big\|_{t=2}$ $= 25.465$
Specific behaviours
✓ uses fundamental theorem of calculus ✓ uses chain rule and substitutes for $t = 2$

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