Baldivis Secondary College

WA Exams Practice Paper A, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

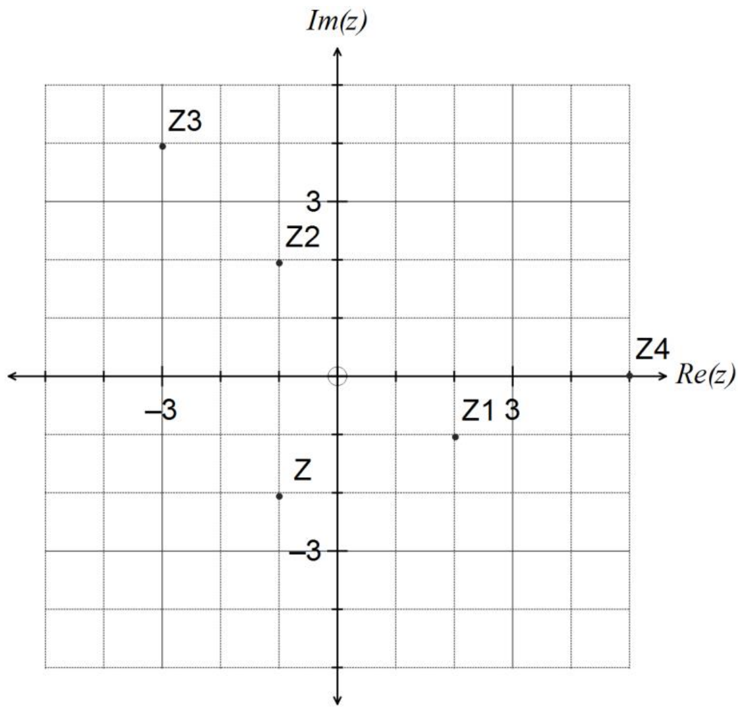
Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(5 marks)**

(a) Plot and label the complex number $z = -1 - 2i$ on the Argand diagram below. (1 mark)



(b) On the same diagram plot and label the following complex numbers: (4 marks)

(i) $z_1 = iz$

(ii) $z_2 = \overline{z}$

(iii) $z_3 = z^2$

(iv) $z_4 = z \cdot \overline{z}$

$z_1 = 2 - i$

$z_2 = -1 + 2i$

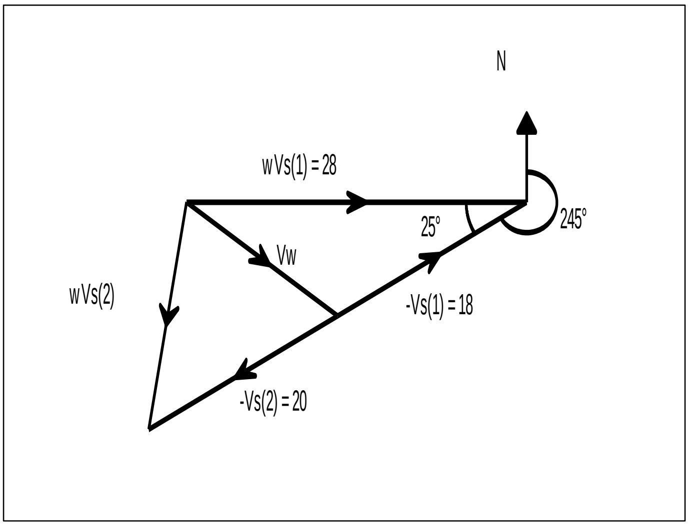
$z_3 = -3 + 4i$

$z_4 = 5$

Question 9 **(6 marks)**

A boat is motoring at 18 km/h on a bearing of 245°. To an observer on board the boat, the wind appears to be blowing at 28 km/h from due west.

If the boat turned through 180° and increased its speed to 20 km/h, find the new apparent wind speed and direction.



$$wVs(2) = \sqrt{38^2 + 28^2 - 2 \times 38 \times 28 \times \cos\ 25^\circ}$$
$$= 17.3 \text{ km/h}$$

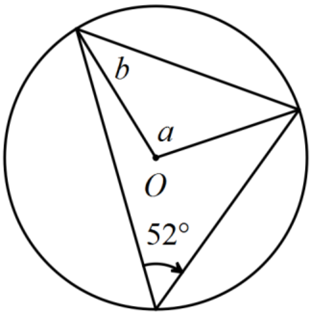
$$\frac{\sin\theta}{38} = \frac{\sin 25}{17.3}$$
$$\theta = 112^\circ$$

$$\text{Bearing} = 112 + 90$$
$$= 202^\circ$$

Question 10 **(8 marks)**

(a) In the following diagrams, O is the centre of the circle shown.

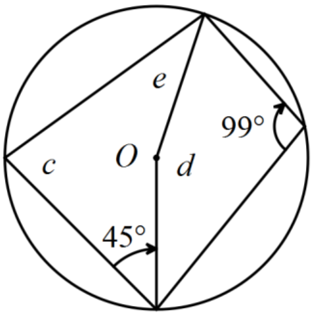
(i) Determine the values of a and b. (2 marks)



$a = 52 \times 2$
$= 104°$

$b = \dfrac{180 - 104}{2}$
$= 38°$

(ii) Determine the values of c, d and e. (3 marks)



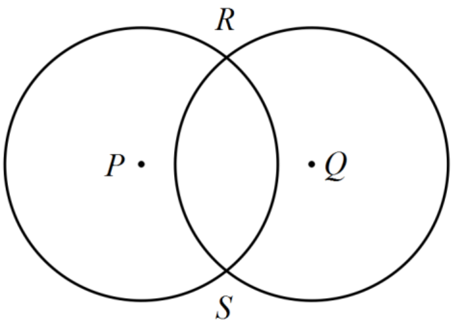
$c = 180 - 99$
$= 81°$

$d = 2 \times 81$
$= 162°$

$e = 360 - 81 - 45 - (360 - 162)$
$= 36°$

(b) Two circles, both with radius 9 cm, and centres P and Q that are 12 cm apart, intersect at R and S as shown. Determine the exact area of the quadrilateral $PRQS$.

(3 marks)



$\frac{1}{2}QP = 6$

$\frac{1}{2}RS = \sqrt{9^2 - 6^2}$
$= 3\sqrt{5}$

Area $= 2 \times 6 \times 3\sqrt{5}$
$= 36\sqrt{5}$ cm^2

Question 11 **(7 marks)**

(a) Let matrix $A=\begin{bmatrix}2 & -2\\ 7 & -6\end{bmatrix}$.

(i) Determine A^{-1}. (1 mark)

$$A^{-1}=\frac{1}{(2)(-6)-(7)(-2)}\begin{bmatrix}-6 & 2\\ -7 & 2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-6 & 2\\ -7 & 2\end{bmatrix}$$

(ii) Express the equations $7a-6b=23$ and $2a-2b=7$ as a system of matrices. (1 mark)

$$\begin{bmatrix}2 & -2\\ 7 & -6\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}7\\ 23\end{bmatrix}$$

(iii) Show use of your answer from (i) to solve the matrix equation in (ii). (2 marks)

$$\begin{bmatrix}a\\ b\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-6 & 2\\ -7 & 2\end{bmatrix}\times\begin{bmatrix}7\\ 23\end{bmatrix}$$
$$\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}2\\ -1.5\end{bmatrix}$$

(b) Solve the equation $\begin{bmatrix}3 & 0\\ 5 & 0\end{bmatrix}B=B+\begin{bmatrix}2 & 4\\ 6 & 7\end{bmatrix}$ for the 2×2 matrix B. (3 marks)

Let $X=\begin{bmatrix}3 & 0\\ 5 & 0\end{bmatrix}$ and $Y=\begin{bmatrix}2 & 4\\ 6 & 7\end{bmatrix}$ then

$$XB-B=Y$$
$$(X-I)B=Y$$
$$B=(X-I)^{-1}A$$
$$=\begin{bmatrix}2 & 0\\ 5 & -1\end{bmatrix}^{-1}\begin{bmatrix}2 & 4\\ 6 & 7\end{bmatrix}$$
$$=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}$$

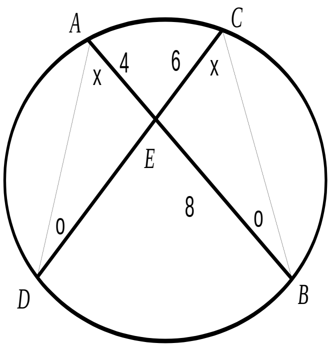
Question 12 **(7 marks)**

(a) In triangles ABC and DEF, $AC \cong DF$ and $\angle A \cong \angle D$. Is the additional fact that $BC \cong EF$ enough to prove that triangle ABC is congruent with triangle DEF? Justify your answer.
(2 marks)

No.

SAS can only be used when angle is included by the two sides.

(b) In the circle shown below, not to scale, AB and CD are chords that intersect at E. If $AE = 4$ cm, $BE = 8$ cm and $CE = 6$ cm, determine the length of DE. Justify your answer.
(3 marks)



$\angle DAB \cong \angle DCB$ (angles on common chord).

Similarly $\angle ADC \cong \angle ABC$.

Hence $\Delta AED \sim \Delta CEB$ (AA)

$\frac{4}{6} = \frac{DE}{8} \Rightarrow DE = 5\frac{1}{3}$ cm

(c) Consider the true statement 'if a quadrilateral is a square then all four sides of the quadrilateral are the same length'. Write the converse of this statement and explain whether or not the converse is also true.
(2 marks)

If all four sides of a quadrilateral are the same length then the quadrilateral is a square.

Statement is false, as a rhombus also has all sides of equal length.

Question 13 **(8 marks)**

(a) Determine the sum of all the numbers contained in the row(s) of Pascals triangle in which the number 10 appears. (3 marks)

Rows 5 and 10 contain 10.

$2^5 = 32$

$2^{10} = 1024$

$1024 + 32 = 1056$

(b) Nine students applied for four temporary positions working for a cleaning company in their holidays.

(i) How many different selections of students could the cleaning company make to fill the four positions? (1 mark)

$^9C_4 = 126$

(ii) If six of the nine applicants were male, and the company wanted to employ an equal number of males and females, in how many ways could they do this? (2 marks)

$$\begin{aligned} {}^6C_2 \times {}^3C_2 &= 15 \times 3 \\ &= 45 \end{aligned}$$

(iii) Determine the number of ways that the application forms can be sorted into order if the six male applications must be kept together. (2 marks)

$6! \times 4! = 17280$

Question 14 **(7 marks)**

(a) A triangle with vertices at $A(1,\ 1)$, $B(3,\ 1)$ and $C(3,\ 4)$ is reflected in the x-axis and then rotated $90°$ anticlockwise about the origin.

(i) Find the matrix T that will combine these two transformations in the order given. (3 marks)

$$T=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\times\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

(ii) Find the coordinates of C after transformation by T. (1 mark)

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}3\\ 4\end{bmatrix}=\begin{bmatrix}4\\ 3\end{bmatrix}$$

$C'(4,\ 3)$

(b) Another transformation matrix is given by $R=\begin{bmatrix}-0.6 & 0\\ -1.2 & -0.6\end{bmatrix}$.

Determine the area of triangle ABC after transformation by T and then by R. (3 marks)

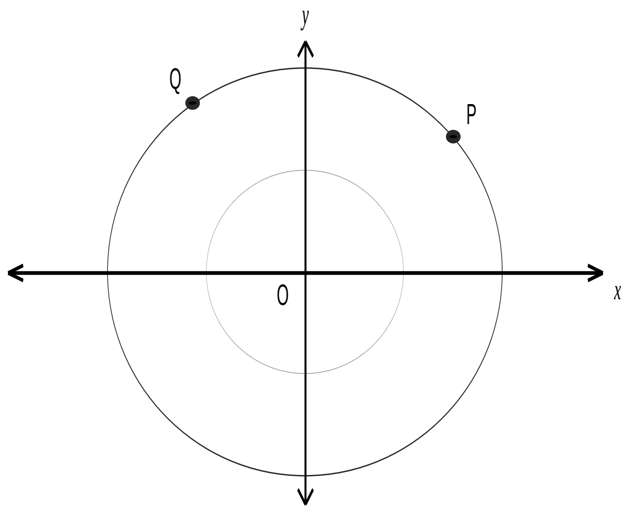
Original area of triangle ABC = 3 sq units.

Determinant of T is -1, so no change in area.

Determinant of R is 0.36, so final area $=0.36\times 3=1.08$ sq units.

Question 15 **(6 marks)**

The points P and Q lie on a circle of radius r and have polar coordinates (r,θ) and (r,ϕ) respectively, where $0<\theta<\phi<360°$.



(a) Express both of the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in the form $a\mathbf{i}+b\mathbf{j}$. (2 marks)

$$\overrightarrow{OP}=r\cos\theta\mathbf{i}+r\sin\theta\mathbf{j}$$
$$\overrightarrow{OQ}=r\cos\phi\mathbf{i}+r\sin\phi\mathbf{j}$$

(b) Use your answers from (a) to show that $\overrightarrow{OP}\bullet\overrightarrow{OQ}=r^2(\cos\theta\cdot\cos\phi+\sin\theta\cdot\sin\phi)$. (1 mark)

$$\overrightarrow{OP}\bullet\overrightarrow{OQ}=\left(r\cos\theta\mathbf{i}+r\sin\theta\mathbf{j}\right)\bullet\left(r\cos\phi\mathbf{i}+r\sin\phi\mathbf{j}\right)$$
$$=r^2(\cos\theta\cos\phi+\sin\theta\sin\phi)$$

(c) Use the diagram above to state the size of $\angle POQ$. (1 mark)

$$\angle POQ=\phi-\theta$$

(d) Use the definition $\mathbf{a}\bullet\mathbf{b}=|\mathbf{a}|\cdot|\mathbf{b}|\cos\theta$ and the result in (b) to show $\cos(\phi-\theta)=\cos\phi\cdot\cos\theta+\sin\phi\cdot\sin\theta$.

(2 marks)

$$\overrightarrow{OP}\bullet\overrightarrow{OQ}=|OP|\times|OQ|\times\cos(\phi-\theta)$$
$$=r^2\cos(\phi-\theta)$$

$$r^2\cos(\phi-\theta)=r^2(\cos\theta\cos\phi+\sin\theta\sin\phi)$$
$$\cos(\phi-\theta)=\cos\theta\cos\phi+\sin\theta\sin\phi$$

Question 16 **(6 marks)**

In a new office building, the manager's office is to have one desk, three filing cabinets, one fax, one telephone and four chairs.
The supervisors are to have one desk, two filing cabinets, no fax, one telephone and two chairs.
The clerks are to have one desk, one filing cabinet, no fax, one telephone and one chair.

(a) Express this information in matrix M. (1 mark)

$$M = \begin{bmatrix} 1 & 3 & 1 & 1 & 4 \\ 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(b) In the new building there are to be 6 managers, 12 supervisors and 40 clerks. Express this information in matrix N and then show use of a matrix operation to determine the total number of desks, filing cabinets, fax machines, telephones and chairs needed. (3 marks)

$$N = \begin{bmatrix} 6 & 12 & 40 \end{bmatrix}$$

$$MN = \begin{bmatrix} 58 & 82 & 6 & 58 & 88 \end{bmatrix}$$

ie 58 desks, 82, filing cabinets, etc.

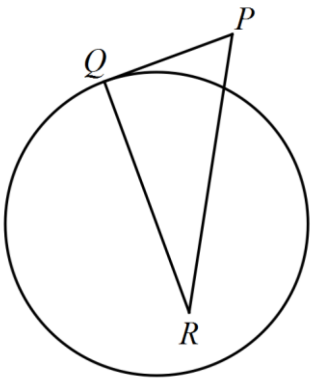
(c) The cost of a desk, a filing cabinet, a fax, a telephone and a chair are \$250, \$190, \$125, \$85 and \$160 respectively. Express these costs in matrix C and then show use of a matrix operation to determine the total cost of all new furniture and office machines. (2 marks)

$$C = \begin{bmatrix} 250 \\ 190 \\ 125 \\ 85 \\ 160 \end{bmatrix} \quad MC = \begin{bmatrix} 49840 \end{bmatrix}$$

Total cost is \$49 840.

Question 17 **(9 marks)**

(a) The circle in the diagram has a diameter of 20 cm and PQ is a tangent to the circle at Q. If $PQ = 7.5$ cm, $QR = 18$ cm and $PR = 19.5$ cm, prove that R lies on the diameter of the circle. (3 marks)



Using the converse of Pythagoras' theorem:

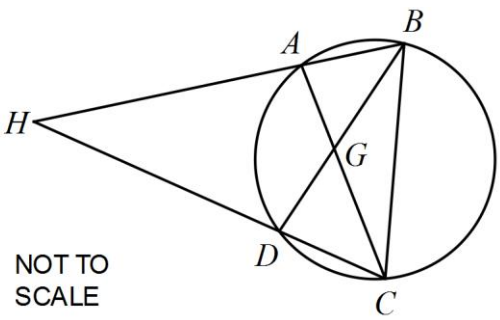
$$PQ^2 + QR^2 = 7.5^2 + 18^2$$
$$= 380.25$$
$$= 19.5^2$$
$$= PR^2$$

Hence $\angle PQR = 90°$.

Hence R must lie on the diameter of the circle since the diameter and tangent always meet at a right angle at the point of tangency.

(b) The points A, B, C and D lie on a circle of radius r. The lines AC and BD intersect at G. The lines BA and CD are produced to meet at H. $HAGD$ is a cyclic quadrilateral.

(i) Determine, with reasons, the size of $\angle BAC$. (4 marks)



$\angle BDC = \angle BAC$ (Angles on chord)

$\angle BDC = 180 - \angle HDG$ (Angles on strt line)

$\angle HAG = 180 - \angle HDG$ (Opp angles in cyclic quad)

Hence $\angle HAG = \angle BDC = \angle BAC$

But $\angle HAG + \angle BAC = 180 \Rightarrow \angle BAC = 90°$

(ii) Determine, with reasons, the length BC in terms of r. (2 marks)

$BC = 2r$

BC must be the diameter of the circle as angle in semicircle ($\angle BAC$) is 90º.

Question 18 **(9 marks)**

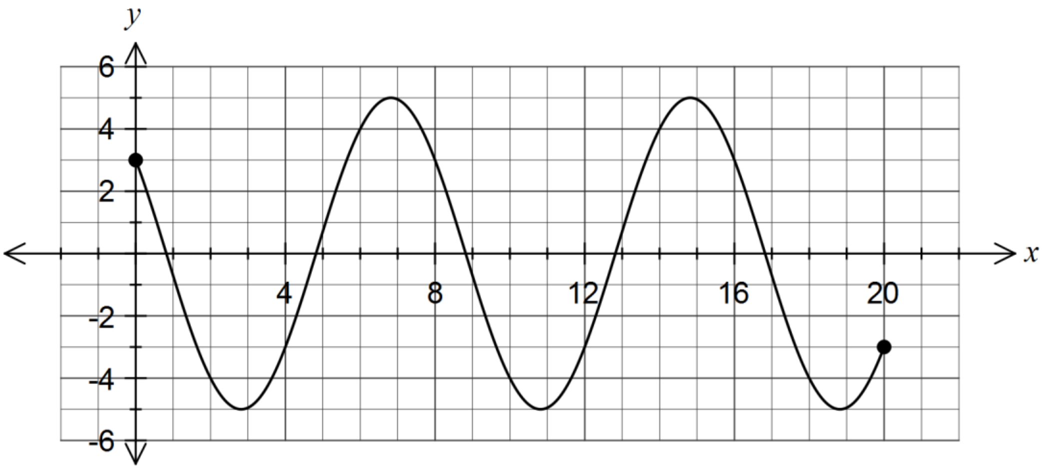
The motion of a small body moving along a straight track was recorded by a video camera for 20 seconds. An analysis of the motion showed that the distance, x cm, of the body from a fixed point O on its path t seconds after recording began was given by $x(t) = 3\cos\frac{\pi t}{4} - 4\sin\frac{\pi t}{4}$.

(a) The distance can also be given by $x(t) = a\sin\left(\frac{\pi t}{4} + b\right)$, where a and b are real constants. Determine the values of a and b. (2 marks)

$$a = -\sqrt{3^2 + 4^2}$$
$$= -5$$

$$b = \tan^{-1}\frac{3}{-4}$$
$$= -0.6435 \text{ (4 dp)}$$

(b) Graph $y = x(t)$ on the axes below for $0 \le t \le 20$. (3 marks)



(c) State the period and amplitude of the graph of $y = x(t)$. (2 marks)

$$\text{Period} = \frac{2\pi}{\pi/4} = 8 \text{ seconds}$$

$$\text{Amplitude} = 5 \text{ cm}$$

(e) Determine the percentage of the first 20 seconds that the body was at least four cm away from the point O. (2 marks)

$$7.6387 - 6 = 1.6387$$

$$\frac{1.6387 \times 5}{20} \times 100 = 40.97$$
$$\approx 41\% \text{ of the time}$$

Question 19 **(12 marks)**

(a) Consider the expression $m^2 + 7$.

(i) Write down the values of $m^2 + 7$ for $m = 1, 3, 5, 7$ and 9. (1 mark)

8, 16, 32, 56, 88

(ii) Use your values from (a) to state the largest integer, p, that $m^2 + 7$ is always divisible by, when m is a positive odd integer. (1 mark)

The largest integer is $p = 8$

(iii) Prove that $m^2 + 7$ is always divisible by p when m is a positive odd integer. (4 marks)

Let the positive odd integer $m = 2n + 1$, where n is an integer greater than or equal to 0. Then:

$$\begin{aligned}(2n+1)^2 + 7 &= 4n^2 + 4n + 8 \\ &= 4n(n+1) + 8 \\ &= 4(2k) + 8 \quad \text{(see note below)} \\ &= 8(k+1)\end{aligned}$$

Hence the expression will always be divisible by eight.

Note: Since n is an integer, then one of n and $n+1$ will be even and the other odd. Hence the product $n(n+1)$ will have a factor of 2, and so $n(n+1) = 2k$, where k is an integer greater than or equal to 0.

(b) Let $A = 2^{n+2} + 3^{2n+1},\ n \in \mathbb{N}$.

(i) Show that $A = 7 \times 25325$ when $n = 5$. (1 mark)

$$A = 2^9 + 3^{11}$$
$$= 177275$$
$$= 7 \times 25325$$

(ii) Prove by induction that A is divisible by 7. (5 marks)

When $n = 1,\ A = 2^3 + 3^3 = 35 = 7 \times 5$.

Assume that when $n = k$ then $2^{k+2} + 3^{2k+1} = 7M,\ M \in \mathbb{N}$.

When $n = k + 1$

$$\begin{aligned} 2^{(k+1)+2} + 3^{2(k+1)+1} &= 2 \cdot 2^{k+1} + 9 \cdot 3^{2k+1} \\ &= 2 \cdot 2^{k+1} + 2 \cdot 3^{2k+1} + 7 \cdot 3^{2k+1} \\ &= 2(2^{k+2} + 3^{2k+1}) + 7 \cdot 3^{2k+1} \\ &= 2(7M) + 7 \cdot 3^{2k+1} \\ &= 7\left(2M + 3^{2k+1}\right) \end{aligned}$$

Since A is divisible by 7 when $n = 1$ and as the truth for $n = k$ implies the result for $n = k + 1$ it follows that A is divisible by 7 for all positive integers.

Question 20 **(8 marks)**

A helicopter, with a maximum speed through still air of 240 km/h, leaves its base at A to fly to a destination at B.

The position vector of B relative to A is $(155\mathbf{i} + 95\mathbf{j})$ km, and a steady wind of velocity $(-17\mathbf{i} - 22\mathbf{j})$ km/h is blowing over the area.

(a) Find the velocity vector the helicopter pilot should set in order to fly directly from A to B in the shortest time. (6 marks)

Let velocity vector be $a\mathbf{i} + b\mathbf{j}$. Then

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -17 \\ -22 \end{bmatrix} = \lambda \begin{bmatrix} 155 \\ 95 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 155\lambda + 17 \\ 95\lambda + 22 \end{bmatrix}$$

But $a^2 + b^2 = 240^2$

$$(155\lambda + 17)^2 + (95\lambda + 22)^2 = 240^2$$

$\lambda = 1.1761$ (ignore $\lambda =$ -1.462 as $\lambda > 0$)

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 199.3 \\ 133.7 \end{bmatrix}$$

(b) What is the shortest journey time, to the nearest minute? (2 marks)

$$t = \frac{1}{1.1761} \times 60$$
$$= 51.02$$
$$= 51 \text{ minutes}$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________