Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section One:
Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	51	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	149	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Booklet.

Section One: Calculator-free **35% (51 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

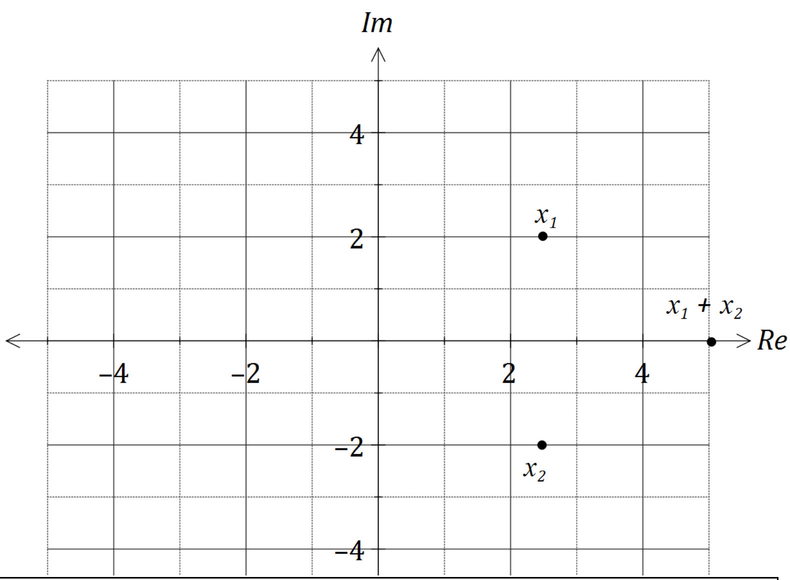
(a) Determine the number of real solutions to the equation $x^2 + x + 1 = 0$. (1 mark)

Solution
$b^2 - 4ac = -3 \Rightarrow$ no real solutions
Specific behaviours
✓ uses discriminant, or otherwise, to determine no solutions

(b) Determine all complex solutions to the equation $x^2 + 2x + 10 = 0$. (2 marks)

Solution
$(x+1)^2 = -9 = 9i^2$ $x = -1 \pm 3i$
Specific behaviours
✓ completes square or uses quadratic formula ✓ states both complex solutions

(c) x_1 and x_2 are the complex solutions to the equation $4x^2 = 20x - 41$. If $x_1 = 2.5 + 2i$, plot x_1, x_2 and $x_1 + x_2$ in the complex plane below. (3 marks)



Solution
x_2 must be conjugate, so $x_2 = 2.5 - 2i$ and $x_1 + x_2 = 5$
Specific behaviours
✓ determines x_2 ✓ plots x_1 and x_2 ✓ determines sum and plots

Question 2 **(7 marks)**

Three vectors are given by $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j}$.

Determine

(a) a unit vector $\mathbf{d}$, parallel to $\mathbf{a} + 2\mathbf{b}$. (3 marks)

Solution

$\mathbf{d} = \mathbf{a} + 2\mathbf{b} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$ and $|\mathbf{d}| = \sqrt{80} = 4\sqrt{5}$

$$\hat{\mathbf{d}} = \frac{1}{4\sqrt{5}}\begin{bmatrix} 4 \\ -8 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$$

Specific behaviours

✓ calculates $\mathbf{a} + 2\mathbf{b}$
✓ calculates magnitude
✓ states unit vector in simplified form

(b) the value(s) of k so that the magnitude of the vector $\mathbf{a} + k\mathbf{b}$ is 4. (4 marks)

Solution

$\mathbf{a} + k\mathbf{b} = \begin{bmatrix} 2 + k \\ -2 - 3k \end{bmatrix}$

Require $(2 + k)^2 + (-2 - 3k)^2 = 4^2$

$$\begin{aligned} 4 + 4k + k^2 + 4 + 12k + 9k^2 - 16 &= 0 \\ 10k^2 + 16k - 8 &= 0 \\ 5k^2 + 8k - 4 &= 0 \\ (5k - 2)(k + 2) &= 0 \end{aligned}$$

$k = \frac{2}{5}$ or $k = -2$

Specific behaviours

✓ writes magnitude equation
✓ expands and simplifies equation
✓ factorises equation
✓ states both solutions

Question 3 **(9 marks)**

Consider the matrices $A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & -5 \end{bmatrix}$.

(a) It is possible to form the product of all four matrices. State the dimensions of the resulting product. (2 marks)

Solution
$ABDC$ or $BDAC$ are possible, both resulting in a 2×3 matrix.
Specific behaviours
✓ lists possible product ✓ states dimensions of product

(b) Determine the matrix $\frac{1}{2}DC$. (2 marks)

Solution
$\frac{1}{2} \times \begin{bmatrix} 4 & -5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 4 & -10 & 6 \end{bmatrix}$ $= \begin{bmatrix} 2 & -5 & 3 \end{bmatrix}$
Specific behaviours
✓ calculates DC ✓ calculates required result

(c) Determine the inverse of matrix A. (2 marks)

Solution
$A^{-1} = \dfrac{1}{8-(-6)} \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$ $= \begin{bmatrix} 2 & 1.5 \\ 1 & 1 \end{bmatrix}$
Specific behaviours
✓ uses determinant ✓ determines inverse

(d) Clearly show use of matrix algebra to solve the system of equations $2x - 3y + 3 = 0$ and $4y = 2x + 2$. (3 marks)

Solution
$\begin{matrix} 2x - 3y = -3 \\ -2x + 4y = 2 \end{matrix} \Rightarrow AX = B$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $X = A^{-1}B = \begin{bmatrix} 2 & 1.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ $x = -3, y = -1$
Specific behaviours
✓ shows system can be written as matrix equation ✓ shows pre-multiplication of equation by inverse from (c) ✓ states solution of system

Question 4 **(7 marks)**

Let $z_1 = 2 - 2i$ and $z_2 = 3 + i$.

(a) Simplify

(i) $2z_1 - z_2$. (1 mark)

Solution
$4 - 4i - 3 - i = 1 - 5i$
Specific behaviours
✓ simplifies result

(ii) z_1^3. (2 marks)

Solution
$(2-2i)(2-2i)(2-2i) = (-8i)(2-2i)$ $= -16 - 16i$
Specific behaviours
✓ simplifies z_1^2 ✓ simplifies z_1^3

(iii) $\frac{z_1}{z_2}$. (2 marks)

Solution
$\frac{(2-2i)(3-i)}{(3+i)(3-i)} = \frac{4-8i}{9+1}$ $= \frac{2}{5} - \frac{4}{5}i$
Specific behaviours
✓ multiplies by conjugate   ✓ simplifies

(b) Show that $\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2}$. (2 marks)

Solution
$LHS = (2+2i)(3-i) = 8 + 4i$ $RHS = \overline{(2-2i)(3+i)} = \overline{8-4i} = 8 + 4i$
Specific behaviours
✓ evaluates RHS   ✓ evaluates LHS

Question 5 **(7 marks)**

(a) Solve the equation $\tan\left(\frac{x+25°}{2}\right) = \sqrt{3}$ for $0° \leq x \leq 540°$. (3 marks)

Solution
$0° \leq x \leq 540° \Rightarrow 0° \leq \dfrac{x}{2} \leq 270°$ $\dfrac{x+25°}{2} = 60°, 240°$ $x = 95°, x = 455°$
Specific behaviours
✓ uses $\tan 60° = \sqrt{3}$ ✓ determines first solution   ✓ determines second solution

(b) Prove that $(1 - \cos x)(1 + \sec x) = \sin x \tan x$. (4 marks)

Solution
$LHS = 1 + \sec x - \cos x - \cos x \sec x$ $= \sec x - \cos x$ $= \dfrac{1 - \cos^2 x}{\cos x}$ $= \dfrac{\sin^2 x}{\cos x}$ $= \sin x \tan x$ $= RHS$
Specific behaviours
✓ expands and simplifies LHS   ✓ combines into single fraction   ✓ uses Pythagorean identity   ✓ simplifies to RHS

Question 6 **(7 marks)**

(a) Determine the value(s) of a for which the matrix $\begin{bmatrix} a & a \\ 3 & 2a \end{bmatrix}$ is singular. (2 marks)

Solution
Singular ⇒ determinant is zero, so require $2a^2 - 3a = 0$. $a(2a-3) = 0 \Rightarrow a = 0$ or $a = \frac{3}{2}$
Specific behaviours
✓ writes determinant in terms of a and equates to 0 ✓ solves equation for a

(b) The non-singular matrix B is such that $[-3 \quad 2] \times B = [8 \quad 3]$ and $[2 \quad 6] \times B = [10 \quad 4]$.

(i) Use these results to show that $[-1 \quad 8] \times B = [18 \quad 7]$. (2 marks)

Solution
$[-3 \quad 2] \times B + [2 \quad 6] \times B = [8 \quad 3] + [10 \quad 4]$ $([-3 \quad 2] + [2 \quad 6]) \times B = [18 \quad 7]$ $[-1 \quad 8] \times B = [18 \quad 7]$
Specific behaviours
✓ uses sum of equations ✓ illustrates distributive law

(ii) Determine $[2 \quad 1] \times B^{-1}$. (3 marks)

Solution
$([2 \quad 6] - [-3 \quad 2]) \times B = [10 \quad 4] - [8 \quad 3]$ $[5 \quad 4] \times B = [2 \quad 1]$ $[5 \quad 4] \times B \times B^{-1} = [2 \quad 1] \times B^{-1}$ $[5 \quad 4] = [2 \quad 1] \times B^{-1}$
Specific behaviours
✓ uses difference of equations ✓ shows post-multiplication by inverse ✓ clearly shows result

Question 7 **(8 marks)**

(a) Prove that the sum of any three consecutive terms of an arithmetic sequence with first term a and common difference d is always a multiple of three, for $a, d \in \mathbb{N}$. (3 marks)

Solution
Let $T_n = a + (n-1)d$ so that $T_n + T_{n+1} + T_{(n+2)} = (a + nd - d) + (a + nd) + (a + nd + d)$ $= 3a + 3nd = 3(a + nd) \Rightarrow$ always a multiple of 3
Specific behaviours
✓ writes expression for three consecutive terms of arithmetic sequence ✓ simplifies expression ✓ factors 3 out and states conclusion

(b) Use mathematical induction to prove that $7^{2n-1} + 5$ is always divisible by 12, for $n \in \mathbb{N}$. (5 marks)

Solution
Let $f(n) = 7^{2n-1} + 5$, so clearly true when $n = 1$ as $f(1) = 12$. Assume that $f(k)$ is always true, so that $f(k) = 7^{2k-1} + 5 = 12I$, where I is an integer. $f(k+1) = 7^{2(k+1)-1} + 5$ $= 7^{2+2k-1} + 5$ $= 7^2 \times 7^{2k-1} + 5$ $= 49 \times 7^{2k-1} + 5$ $= 48 \times 7^{2k-1} + 7^{2k-1} + 5$ $= 48 \times 7^{2k-1} + 12I$ $= 12(4 \times 7^{2k-1} + I)$ Since $f(1)$ is divisible by 12, and it has been shown that if $f(k)$ is, so is $f(k+1)$, then $7^{2n-1} + 5$ is divisible by 12 for all $n \geq 1$.
Specific behaviours
✓ shows true for initial case ✓ assumes true for $n = k$ and equates result to multiple of 12 ✓ uses index laws to achieve $49 \times 7^{2k-1} + 5$ ✓ factors 12 out of expression ✓ makes summary statement

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________