Assignment 8

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Outline

Question

The random variables x and y are independent with,

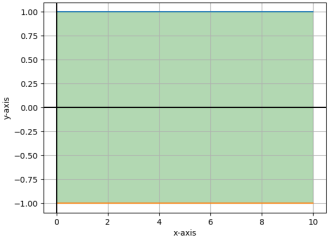
$$f_x = \frac{x}{\alpha^2} e^{\frac{-x^2}{2\alpha^2}} U(x) \tag{1.0.1}$$

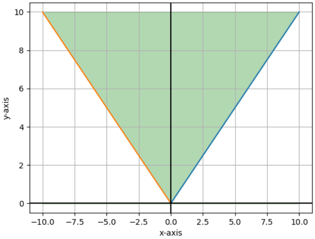
$$f_y = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}} & , |y| < 1 \\ 0 & , |y| > 1 \end{cases} \tag{1.0.2}$$

Show that the random variable z=xy is $N(0, \alpha^2)$

Solution

we have z=xy and if we assume x = w, then y = z/w
Now their respective regions are as follows





$x > 0$ and $|y| < 1$ and x = w and y = z/w

Solution

since y = z/w,

$$f_{wz}(w,z) = \frac{1}{|x|} f_{xy}(x,y) \tag{2.0.1}$$

$$f_{wz}(w,z) = \frac{1}{x}\frac{x}{\alpha^2} e^{\frac{-x^2}{2\alpha^2}} \frac{1}{\pi\sqrt{1-y^2}} \tag{2.0.2}$$

$$\implies f_{wz}(w,z) = \frac{1}{\pi\alpha^2} \frac{e^{\frac{-w^2}{2\alpha^2}}}{\pi\sqrt{1-\frac{z^2}{w^2}}} \tag{2.0.3}$$

Solution

$$f_z(z) = \frac{1}{\pi\alpha^2}\int_{|w|}^{\infty}\frac{e^{\frac{-w^2}{2\alpha^2}}}{\sqrt{w^2 - z^2}}w\,dw \tag{2.0.4}$$

Assume,

$$\sqrt{w^2 - z^2} = t \tag{2.0.5}$$

$$\frac{2w\,dw}{2\sqrt{w^2 - z^2}} = dt \tag{2.0.6}$$

$$\implies w\,dw = t\,dt \tag{2.0.7}$$

Solution

$$\implies f_z(z) = \frac{1}{\pi\alpha^2}\int_0^\infty \frac{e^{\frac{-(t^2+z^2)}{2\alpha^2}}}{t} t\,dt \tag{2.0.8}$$

$$= \frac{1}{\pi\alpha^2} e^{\frac{-z^2}{2\alpha^2}} \int_0^\infty e^{-(\frac{t}{\sqrt{2\alpha^2}})^2}\,dt \tag{2.0.9}$$

$$\text{Let } \frac{t}{\sqrt{2\alpha^2}} = k \tag{2.0.10}$$

$$\implies dt = \sqrt{2\alpha^2}\,dk \tag{2.0.11}$$

$$\text{Hence } f_z(z) = \frac{1}{\pi\alpha^2} e^{\frac{-z^2}{2\alpha^2}} \sqrt{2\alpha^2} \int_0^\infty e^{-k^2}\,dk \tag{2.0.12}$$

Solution

From the known result of

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \tag{2.0.13}$$

$$\implies f_z(z) = \frac{1}{\pi\alpha^2} e^{\frac{-z^2}{2\alpha^2}} \sqrt{2\alpha^2}\frac{\sqrt{\pi}}{2} \tag{2.0.14}$$

$$f_z(z) = \frac{1}{\sqrt{2\pi\alpha^2}} e^{\frac{-z^2}{2\alpha^2}} \tag{2.0.15}$$

Hence the random variable $z = xy$ is $N(0, \alpha^2)$