Random Numbers

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Contents

Abstract

This manual provides a simple introduction to the generation of random numbers

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat.
Solution: Download the following files and execte the C program.

```
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/exrand.c
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/coeffs.h
```

And run the following commands in the terminal to execute the C program files

```
gcc −o out exrand.c coeffs.h −lm
./out
```

Then the corresponding ”uni.dat” file will be created with 10^6 samples of U.

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2 in the figs folder(Comment out or remove comments for some lines accordingly)

```
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/cdf_plot.py
```

And execute the following command in the terminal

```
python3 cdf_plot.py
```

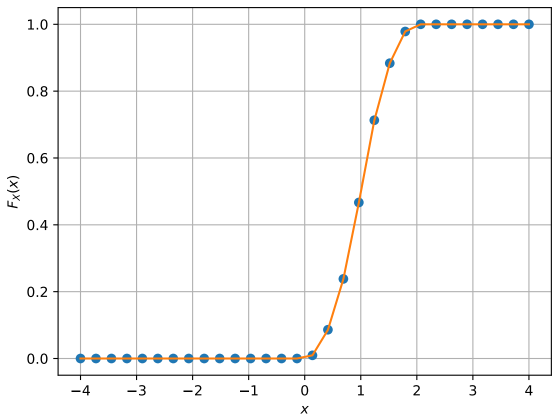

Figure 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: We know that the PDF of a uniform distribution function in a particular in-

terval (a, b) is given by,

$$f(x) = \begin{cases} \frac{1}{b-a} & , \text{for } a \leq x \leq b \\ 0 & , otherwise \end{cases} \quad (1.2)$$

Hence, the PDF of this uniform distribution is given as follows in this particular interval of (0, 1) is given by as follows,

$$f(x) = \begin{cases} 1 & , \text{for } a \leq x \leq b \\ 0 & , otherwise \end{cases} \quad (1.3)$$

Hence, the CDF of a uniform distribution function is given as follows in a particular interval (a, b) is given by,

$$F_U(x) = \int_{-\infty}^{x} f(x)\, dx \quad (1.4)$$

$$= \begin{cases} \int_{-\infty}^{x} 0\, dx & , x < 0 \\ \int_{0}^{x} dx & , 0 \leq x \leq 1 \\ \int_{0}^{1} dx & , x > 1 \end{cases} \quad (1.5)$$

$$= \begin{cases} 0 & , x < 0 \\ 1[x]_0^x & , 0 \leq x \leq 1 \\ 1[x]_0^1 + 0 & , x > 1 \end{cases} \quad (1.6)$$

$$\implies F_U(x) = \begin{cases} 0 & , \text{for } x < 0 \\ x & , \text{for } 0 \leq x \leq 1 \\ 1 & , \text{for } x > 1 \end{cases} \quad (1.7)$$

In Our case, the intervel is (0, 1). Hence the theoretical expression for $F_U(x)$ is given as follows,

$$F_U(x) = \begin{cases} 0 & , \text{for } x < 0 \\ x & , \text{for } 0 \leq x \leq 1 \\ 1 & , \text{for } x > 1 \end{cases} \quad (1.8)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \quad (1.9)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.10)$$

Write a C program to find the mean and variance of U.

Solution: Download the following files and execte the C program.

```
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/exrand.c
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/coeffs.h
```

And run the following commands in the terminal to execute the C program files

```
gcc −o out exrand.c coeffs.h −lm
./out
```

The Mean and Variance of U is written as output in the terminal.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.11)$$

Solution: If k = 1, then E[U^1] is nothing but the mean of this uniform distribution,
Theoretically, we have

$$F_U(x) = \begin{cases} 0 & , \text{for } x < 0 \\ x & , \text{for } 0 \leq x \leq 1 \\ 1 & , \text{for } x > 1 \end{cases} \quad (1.12)$$

Hence $dF_U(x)$ can be written as follows,

$$dF_U(x) = f(x) = \begin{cases} 1 & , \text{for } 0 \leq x \leq 1 \\ 0 & , otherwise \end{cases} \quad (1.13)$$

$$\implies E[U] = \int_{0}^{1} x\, dx \quad (1.14)$$

$$= 0.5 \quad (1.15)$$

Hence our theoretical mean is 0.5, where as we got our practical mean as 0.500007, which is almost same.
Hence our expression is verified for k = 1.
Similarly we can consider k = 2, then we get,

$$E[U^2] = \int_{0}^{1} x^2\, dx \quad (1.16)$$

$$= 0.333333 \quad (1.17)$$

Hence our theoretical value of $E[U^2]$ is 0.333333, where we got our practical value of $E[U^2]$ as 0.333308, which is again almost same as that of theoretical value.
Hence Verified.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execte the C program.

```
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/exrand.c
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/coeffs.h
```

And run the following commands in the terminal to execute the C program files

```
gcc -o out exrand.c coeffs.h -lm
./out
```

Then the corresponding "gau.dat" file will be created with 10^6 samples of X.

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The following code plots Fig. 2.2 in the figs folder(Comment out or remove comments for some lines accordingly)

```
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/cdf_plot.py
```

And execute the following command in the terminal

```
python3 cdf_plot.py
```

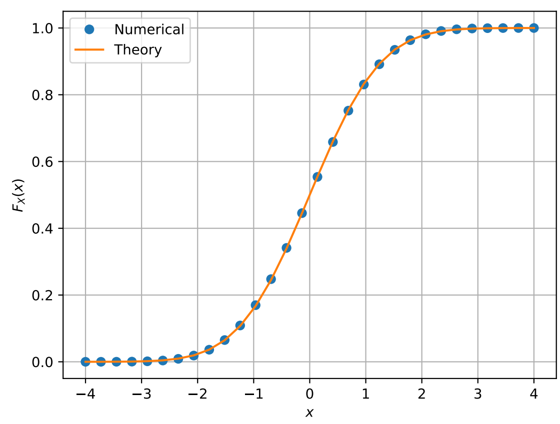

Figure 2.2: The CDF of X

To find cdf of this gaussian function, we have its pdf as,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, -\infty < x < \infty \tag{2.2}$$

$$\implies F(x) = 1 - \Pr(\mathbf{x} > x) \tag{2.3}$$

$$= 1 - Q(x) \tag{2.4}$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.5}$$

What properties does the PDF have?
Solution: The following code plots Fig. 2.3 in the figs folder(Comment out or remove comments for some lines accordingly)

```
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/pdf_plot.py
```

And execute the following command in the terminal

```
python3 pdf_plot.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execte the C program.

```
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/exrand.c
```

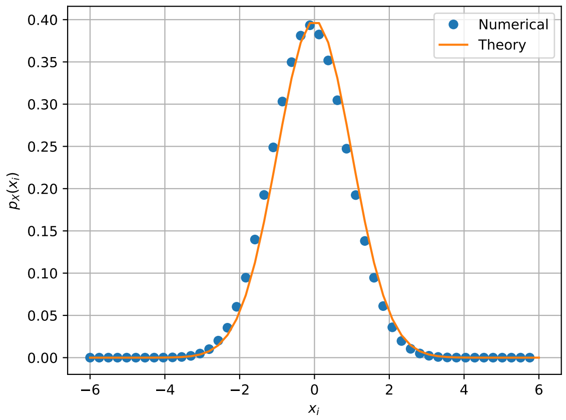

Figure 2.3: The PDF of X

```
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/coeffs.h
```

And run the following commands in the terminal to execute the C program files

```
gcc -o out exrand.c coeffs.h -lm
./out
```

The Mean and Variance of X is written as output in the terminal.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.6}$$

repeat the above exercise theoretically.
Solution: We know that,

$$Mean = E[U] = \int_{-\infty}^{\infty} x p_X(x)\,dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})\,dx \tag{2.8}$$

Since the function $x\frac{1}{\sqrt{2\pi}}exp(-\frac{x^2}{2})$ is an odd function, its integral in the interval $(-\infty, \infty)$ is zero. Hence the Theoretical Mean is 0, whereas the practical Mean we have obtained is 0.000326 which is almost as same as that of Theoretical Mean

We know that,

$$Varience = E[U^2] - E[U]^2 \tag{2.9}$$

$$\implies Variance = E[U^2] \tag{2.10}$$

$$Variance = E[U^2] \tag{2.11}$$

$$= \int_{-\infty}^{\infty} x^2 p_X(x)\,dx \tag{2.12}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \tag{2.13}$$

$$= \frac{2\sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx \tag{2.14}$$

Let $u = x$ and $v = e^{-x^2}$, then we have $\frac{dv}{dx} = -2xe^{-x^2}$. Intergration by parts for the expression

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = -\frac{1}{2}\int_{-\infty}^{\infty} x.(-2xe^{-x^2})\,dx \tag{2.15}$$

$$= -\frac{1}{2}uv|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} v\frac{du}{dx}dx \tag{2.16}$$

$$= -\frac{1}{2}x\int_{-\infty}^{\infty}(-2xe^{-x^2})dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}dx \tag{2.17}$$

since the first term of this expression is odd fuction its value is 0. And from the gaussian distribution, we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1 \tag{2.18}$$

$$\implies \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \tag{2.19}$$

$$\text{Hence, } \int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \tag{2.20}$$

Hence the Theoretical Variance is 1, whereas the practical Varaince we have obtained is 1.000907 which is almost as same as that of the Theoretical Varaince

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following files and execte the C program.

```
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/exrand.c
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/coeffs.h
```

And run the following commands in the terminal to execute the C program files

```
gcc −o out exrand.c coeffs.h −lm
./out
```

Then the corresponding "req.dat" file will be created with 10^6 samples of V. The following code plots Fig. 3.1 in the figs folder.(Comment out or remove comments for some lines accordingly)

```
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/cdf_plot.py
```

And execute the following command in the terminal

```
python3 cdf_plot.py
```

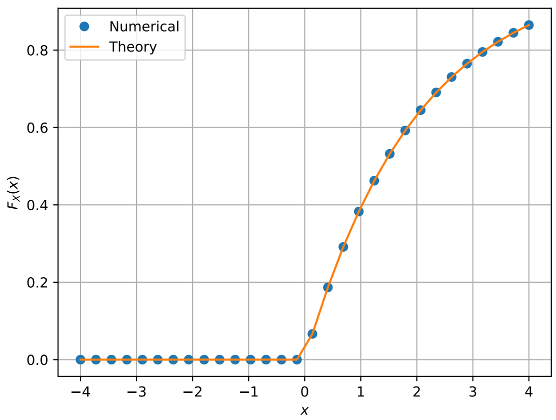

Figure 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We know that,

$$F_V(x) = P(V \le x) \tag{3.2}$$
$$F_V(x) = P(-2\ln(1-U) \le x) \tag{3.3}$$
$$F_V(x) = P(\ln(1-U) \ge -\frac{x}{2}) \tag{3.4}$$
$$F_V(x) = P(1-U \ge e^{-\frac{x}{2}}) \tag{3.5}$$
$$F_V(x) = P(U \le 1-e^{-\frac{x}{2}}) \tag{3.6}$$
$$F_V(x) = F_U(1-e^{-\frac{x}{2}}) \tag{3.7}$$

$$\text{Hence, } F_V(x) = \begin{cases} 0 & , 1-e^{-x/2} < 0 \\ 1-e^{-\frac{x}{2}} & , 0 \le 1-e^{-x/2} \le 1 \\ 1 & , 1-e^{-x/2} > 1 \end{cases} \tag{3.8}$$

From this we get,

$$F_V(x) = \begin{cases} 0 & , x < 0 \\ 1-e^{-x/2} & , x > 0 \end{cases} \tag{3.9}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execte the C program.

```
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/exrand.c
wget https://github.com/Hema−Sri−Ch/
    AI1110−Assignments/Assignment/
    codes/coeffs.h
```

And run the following commands in the terminal to execute the C program files

```
gcc −o out exrand.c coeffs.h −lm
./out
```

Then the corresponding "tri.dat" file will be created with 10^6 samples of T.

4.2 Find the CDF of T
Solution: The following code plots Fig. 4.2

in the figs folder(Comment out or remove the comments accordingly)

```
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/cdf_plot.py
```

And execute the following command in the terminal

```
python3 cdf_plot.py
```

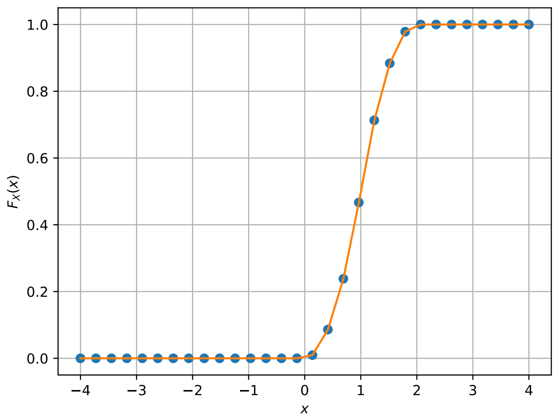

Figure 4.2: The CDF of T

4.3 Find the PDF of T
Solution: The following code plots Fig. 4.3 in the figs folder(Comment out or remove the comments accordingly)

```
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/pdf_plot.py
```

And execute the following command in the terminal

```
python3 pdf_plot.py
```

4.4 Find the theoretical expressions for the PDF and the CDF of T
Solution: We have,

$$T = U_1 + U_2 \tag{4.2}$$

Where U_1 and U_2 are independent random variables in the interval (0, 1). Hence it is clear that the random numbers of T lies between 0 and 2

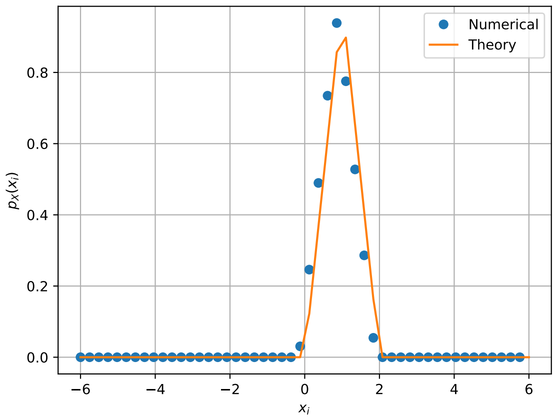

Figure 4.3: The PDF of T

Now the PDF of T is given as follows,

$$p_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\, dx \tag{4.3}$$
$$\tag{4.4}$$

We have, $p_T(t) = 0$ for $t < 0$ and for $t > 2$. Hence we need to find $p_T(t)$ in the interval (0, 2).
for $0 < t < 1$, $f_X(x)f_Y(t-x) = 1$ for some x and 0 for else. In order to have $f_Y(t-x) = 1$, $t - x \geq 0$ which imples $x \leq t$. Hence,

$$p_T(t) = \int_0^1 f_X(x) f_Y(t-x)\, dx \tag{4.5}$$
$$= \int_0^t 1\, dx \tag{4.6}$$
$$= t \tag{4.7}$$

for $1 < t < 2$, $f_X(x)f_Y(t-x) = 1$ for some x and 0 for else. In order to have $f_Y(t-x) = 1$, $t - x \leq 1$ which imples $x \geq t - 1$. Hence,

$$p_T(t) = \int_0^1 f_X(x) f_Y(t-x)\, dx \tag{4.8}$$
$$= \int_{t-1}^1 1\, dx \tag{4.9}$$
$$= 2 - t \tag{4.10}$$

Hence the PDF of T is given s follows,

$$p_T(t) = \begin{cases} 0 & ,0 < t \\ t & ,0 < t \le 1 \\ 2-t & ,1 < t \le 2 \\ 0 & ,t > 2 \end{cases} \tag{4.11}$$

The CDF of T is given as follows,

$$F_T(x) = \int_{-\infty}^{x} p_T(t)\,dt \tag{4.12}$$

$$\tag{4.13}$$

For $t < 0$;

$$F_T(x) = \int_{-\infty}^{x} p_T(t)\,dt \tag{4.14}$$

$$= \int_{-\infty}^{x} 0\,dt \tag{4.15}$$

$$= 0 \tag{4.16}$$

For $0 < t < 1$,

$$F_T(x) = \int_{-\infty}^{x} p_T(t)\,dt \tag{4.17}$$

$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} t\,dt \tag{4.18}$$

$$= 0 + \frac{t^2}{2}|_0^x \tag{4.19}$$

$$= \frac{x^2}{2} \tag{4.20}$$

For $1 < t < 2$,

$$F_T(x) = \int_{-\infty}^{x} p_T(t)\,dt \tag{4.21}$$

$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} t\,dt + \int_{1}^{x} 2-t\,dt \tag{4.22}$$

$$= 0 + \frac{t^2}{2}|_0^1 + [2t - t^2/2]_1^x \tag{4.23}$$

$$= -\frac{x^2}{2} + 2x - 1 \tag{4.24}$$

For $t > 2$

$$F_T(x) = \int_{-\infty}^{x} p_T(t)\,dt \tag{4.25}$$

$$= \int_{0}^{1} t\,dt + \int_{1}^{2} 2-t\,dt \tag{4.26}$$

$$= 0 + \frac{t^2}{2}|_0^1 + [2t - t^2/2]_1^2 \tag{4.27}$$

$$= 1 \tag{4.28}$$

Hence the CDF of T is given s follows,

$$F_T(x) = \begin{cases} 0 & ,0 < x \\ \frac{x^2}{2} & ,0 < x \le 1 \\ -\frac{x^2}{2} + 2x - 1 & ,1 < x \le 2 \\ 1 & ,x > 2 \end{cases} \tag{4.29}$$

4.5 Verify your results through a plot
Solution: The figs 4.2 and 4.3 shows the required verification

```
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/pdf_plot.py
wget https://github.com/Hema-Sri-Ch/
    AI1110-Assignments/Assignment/
    codes/pdf_plot.py
python3 pdf_plot.py
python3 cdf_plot.py
```