2) a) Derivation of Motion Tracking Equation:

1.) Brightness Constancy Assumption: The brightness constancy assumption states that the intensity of a pixel remains constant over time.

$$I(x,y,t) = I(x+u,\ y+v,\ t+\Delta t) \quad \text{--- (1)}$$

where:

- $I(x,y,t)$ is the intensity of the pixel at position (x,y) in frame t.
- u & v are the horizontal and vertical components of the motion vector, representing the displacement of the pixel.
- Δt is the time difference between consecutive frames.

2) Taylor Series Expansion: We can expand $I(x+u, y+v, t+\Delta t)$ using a first-order taylor series expansion.

$$I(x+u, y+v, t+\Delta t) \simeq I(x,y,t) + \frac{dI}{dx}u + \frac{dI}{dy}v + \frac{dI}{\partial t}\Delta t \quad \text{--- (2)}$$

where:

$\frac{\partial I}{\partial x}$ and $\frac{\partial I}{\partial y}$ are the spatial gradients of the image intensity in the x and y directions.

$\frac{\partial I}{\partial t} \Rightarrow$ temporal gradient, (change of intensity over time).

3) Optical Flow Equation: By equating the two expression for $I(x,y,t) \Rightarrow$ By (1) & (2)

$$I(x,y,t) + \frac{dI}{\partial x}u + \frac{\partial I}{dy}v + \frac{\partial I}{\partial t}\Delta t = I(x,y,t)$$

Rearranging terms and dividing it by 'Δt'

$$\rightarrow \frac{\partial I}{\partial x}u + \frac{\partial I}{\partial y}v + \frac{\partial I}{\partial t}\frac{\Delta t}{\Delta t} = I(x,y,t)$$

$u \cdot I_x + v \cdot I_y + I_t = 0$.

where

I_x : $\partial I / dx$ (spatial gradient of the image intensity in the x direction)

I_y : $\partial I / \partial y$ (spatial gradient of the image intensity in the y direction)

I_z : $\partial I / \partial t$ (temporal gradient)

$\boxed{I_x u + I_y v + I_t = 0}$ → This equation represents the constraint on the motion of pixels between consecutive frames, known as the optical flow equation.

b) Procedure for performing Lucas-Kande Algorithm for motion tracking when the motion is known to be affine:

$u(x,y) = a' * x + b' * y + c1$

$v(x,y) = a'2 * x + b2 * y + c2$.

1. Affine Motion Model: This model describes motion as an affine transformation. This model includes translation, rotation, scaling and skewing. The affine motion model is represented as:

$$u(x,y) = a_1 x + b_1 y + c_1$$
$$v(x,y) = a_2 x + b_2 y + c_2$$

Here (u,v) are the horizontal and vertical components of the motion vector at position (x,y) and $(a1, b_1, c_1)$ and (a_2, b_2, c_2) are the affine motion parameters.

2. Lucas-Kanade Equations: For affine motion, we have two equations derived from the optical flow equation.

$I_x a_1 + I_y b_1 + I_t = 0$ $I_y, I_x \rightarrow$ spatial gradients of

$I_y a_2 + I_y b_2 + I_t = 0$ images intensity in the y and x direction

$I_t \rightarrow$ temporal gradient.

3. Least Squares Solution: To estimate the affine motion parameters, we solve the system of the equations using least squares. We formulate a system of linear equations for a neighborhood of pixels.

$$A \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = -b$$

A : matrix of spatial gradients (concatenated I_x and I_y gradients) for all pixels in the neighborhood.

$$A = \begin{bmatrix} I_{x1} & I_{y1} \\ I_{x2} & I_{y2} \\ \vdots & \\ I_{xM} & I_{yM} \end{bmatrix}$$

b is the vector of temporal gradients for all the pixels in the neighborhood.

We solve this overdetermined system using least squares to estimate the affine motion parameters a_1, b_1, a_2 and b_2.

4. Motion Estimation: After obtaining the motion parameters a_1, b_1, a_2, b_2 using least squares, we can compute the motion vectors $u(x,y)$ and $v(x,y)$ for each pixel using the affine motion model.

$$u(x,y) = a_1 x + b_1 y + c_1$$

$$v(x,y) = a_2 x + b_2 y + c_2.$$

Here (x,y) represents the co-ordinates of each pixel in the image and (c_1, c_2) are the translation components of the affine transformation. To estimate c_1 & c_2, we can use the centroid or the mean of the displacement vectors $u(x,y)$ and $v(x,y)$ over ROI least squares method.

$$I_x a_1 + I_y b_1 + I_t = 0$$

$$I_x a_2 + I_y b_2 + I_t = 0.$$

$Ax = b$. where $x = \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix}$; b - temporal gradients.

$$\begin{bmatrix} I_{x1} & I_{y1} \\ I_{x2} & I_{y2} \\ \vdots & \vdots \\ I_{xM} & I_{yM} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = - \begin{bmatrix} I_{t1} \\ I_{t2} \\ \vdots \\ I_{tM} \end{bmatrix}$$

Solving for the motion parameters:

To obtain the least squares solution, we minimize the error between the left & right sides of the equation.

$$\min_x ||Ax + b||^2.$$

The soln can be found using pseudo-inverse of A:

$$x = (A^T A)^{-1} A^T b.$$

Substituting A & b with respective values.

$$\begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = \left[\begin{pmatrix} I_{x1}, I_{x2} \dots I_{xM} \\ I_{y1}, I_{y2} \dots I_{yM} \end{pmatrix} \begin{bmatrix} I_{x1} \\ I_{y1} \\ \vdots \\ I_{xM} \\ I_{yM} \end{bmatrix} \right]^{-1} \begin{pmatrix} I_{x1}, I_{x2} \dots I_{xM} \\ I_{y1}, I_{y2} \dots I_{yM} \end{pmatrix} \begin{pmatrix} -I_{t1} \\ -I_{t2} \\ \vdots \\ -I_{tM} \end{pmatrix}$$

$$A^T A = \begin{bmatrix} I_{x1} & I_{y1} \\ I_{x2} & I_{y2} \\ \vdots & \vdots \\ I_{xM} & I_{yM} \end{bmatrix}^T \begin{pmatrix} I_{x1}, I_{x2} \dots I_{xM} \\ I_{y1}, I_{y2} \dots I_{yM} \end{pmatrix}$$

$$= \begin{pmatrix} I_{x_1} \\ I_{x_2} \\ \vdots \\ I_{xM} \end{pmatrix} \begin{pmatrix} I_{x_1} & I_{x_2} & \cdots & I_{xM} \end{pmatrix} + \begin{pmatrix} I_{y1} \\ I_{y2} \\ \vdots \\ I_{yM} \end{pmatrix} \begin{pmatrix} I_{y_1} & I_{y_2} & \cdots & I_{y} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_i I^2_{xi} & \sum_i I_{xi} I_{yi} \\ \sum_i I_{xi} I_{yi} & \sum_i I^2_{yi} \end{pmatrix}$$

Now compute $A^T b = \begin{bmatrix} I_{x1} & I_{y1} \\ I_{x2} & I_{y2} \\ \vdots & \vdots \\ I_{xM} & I_{yM} \end{bmatrix}^T \begin{bmatrix} -I_{t1} \\ -I_{t2} \\ \vdots \\ -I_{tM} \end{bmatrix}$

$$= \begin{bmatrix} I_{x1} \\ I_{x2} \\ \vdots \\ I_{xM} \end{bmatrix} (-I_{t1}) + \begin{bmatrix} I_{y1} \\ I_{y2} \\ \vdots \\ I_{yM} \end{bmatrix} (-I_{t2})$$

$$= \begin{pmatrix} \sum I_{xi}(-I_{ti}) \\ \sum I_{yi}(-I_{ti}) \end{pmatrix}$$

Lets consider $G = A^T A$ and $h = A^T b$.

Now that we have both. $G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$$x = \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = G^{-1} h = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$x = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} g_{22}h_1 - g_{12}h_2 \\ -g_{21}h_1 + g_{11}h_2 \end{pmatrix}$$

$$x = \frac{1}{\sum_i I^2_{xi} \sum I^2_{yi} - \left(\sum I_{xi} I_{yi}\right)^2} \begin{pmatrix} \left(\sum I^2_{yi}\right) \sum I_{xi}(-I_{ti}) - \left(\sum I_{xi} I_{yi}\right)\left(\sum_i I_{yi} (-I \right. \\ -\left(\sum I^2_{xi}\right) \sum I_{xi}(-I_{ti}) + \\ \sum I_{xi} I_{yi} \ldots \end{pmatrix}$$