Model Order Reduction Project

Heat Diffusion

Xinyu Zeng, 1301462
Hemaditya Malla, 1282484

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Introduction

TODO: Write the intro later- define the physical parameters, domain

$$\begin{cases} \rho(x,y)c(x,y)\frac{\partial T}{\partial t}(x,y,t) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} K(x,y) \begin{bmatrix} \frac{\partial T(x,y,t)}{\partial x} \\ \frac{\partial T(x,y,t)}{\partial y} \end{bmatrix} + u(x,y,t) & (0,T] \times \Omega, \\ \frac{\partial T(0,y,t)}{\partial x} = \frac{\partial L_x,y,t}{\partial x} = 0, \quad y \in [0,L_y] \times [0,T], \\ \frac{\partial T(x,0,t)}{\partial y} = \frac{\partial x,L_y,t}{\partial y} = 0, \quad x \in [0,L_x] \times [0,T], \\ T(x,y,0) = T_0(x,y), \quad \{t=0\} \times \Omega, \end{cases} \tag{1}$$

where $\Omega := [0,L_x] \times [0,L_y]$ and $T_0(x,y)$ is a physically realistic initial temperature profile.

Problem 1

This system is non-linear and time-invariant.
According to description, this model is isotropic.
Therefore,

$$\rho(x,y)c(x,y)\frac{\partial T}{\partial t}(x,y,t) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \kappa(x,y) & 0 \\ 0 & \kappa(x,y) \end{bmatrix} \begin{bmatrix} \frac{\partial T}{\partial x}(x,y,t) \\ \frac{\partial T}{\partial y}(x,y,t) \end{bmatrix} + u(x,y,t), \tag{2}$$

i.e.,

$$\rho(x,y)c(x,y)\frac{\partial T}{\partial t}(x,y,t) = \frac{\partial}{\partial x}\left(\kappa(x,y)\frac{\partial T}{\partial x}(x,y,t)\right) + \frac{\partial}{\partial y}\left(\kappa(x,y)\frac{\partial T}{\partial y}(x,y,t)\right) + u(x,y,t). \tag{3}$$

Remark: Here onwards(HM: Is that correct grammar?), it is implicitly implied that $T = T(x,y,t)$ unless specified otherwise and (x,y,t) is dropped for notational convenience.

Non-homogeneous

Linearity

In order to check for the linearity of (3) the following solution is assumed:

$$T(x,y,t) = T_1(x,y,t) + T_2(x,y,t), \tag{4}$$

where $T_1(x,y,t)$ & $T_2(x,y,t)$ are solutions of the PDE (3). Substituting the solution (4) into the left hand side of the PDE (3), we get,

$$\rho(x,y)c(x,y)\frac{\partial T_1 + T_2}{\partial t}(x,y,t) = \rho(x,y)c(x,y)\left(\frac{\partial T_1}{\partial t}(x,y,t) + \frac{\partial T_2}{\partial t}(x,y,t)\right) \tag{5}$$

Substituting the solution (4) into the right side of the PDE (3),

$$\frac{\partial}{\partial x}\left(\kappa(x,y)\frac{\partial T_1+T_2}{\partial x}(x,y,t)\right)+\frac{\partial}{\partial y}\left(\kappa(x,y)\frac{\partial T_1+T_2}{\partial y}(x,y,t)\right)+u(x,y,t)=$$
$$\left(\frac{\partial\kappa(x,y)}{\partial x}\Big(\frac{\partial T_1}{\partial x}(x,y,t)+\frac{\partial T_2}{\partial x}(x,y,t)\Big)+\kappa(\frac{\partial^2 T_1}{\partial x^2}(x,y,t)+\frac{\partial^2}{\partial x^2}T_2(x,y,t))\right)+$$
$$\left(\frac{\partial\kappa(x,y)}{\partial y}(\frac{\partial T_1}{\partial y}(x,y,t)+\frac{\partial T_2}{\partial y}(x,y,t))+\kappa(\frac{\partial^2}{\partial y^2}T_1(x,y,t)+\frac{\partial^2}{\partial y^2}T_2(x,y,t))\right)+$$
$$u(x,y,t) \tag{6}$$

According to equation (5) and (6), both sides are linear. Therefore this equation is linear.

Time-invariant

input delayed: $u_d(t)=u(t+\delta)$

$$\rho(x,y)c(x,y)\frac{\partial T}{\partial t}(x,y,t)-(\frac{\partial\kappa}{\partial x}(x,y)\frac{\partial T}{\partial x}(x,y,t)+\frac{\partial\kappa}{\partial y}(x,y)\frac{\partial T}{\partial y}(x,y,t))=u(x,y,t+\delta) \tag{7}$$

output delayed:$T_d(t)=T(t+\delta)$

$$\rho(x,y)c(x,y)\frac{\partial T}{\partial t}(x,y,t+\delta)-(\frac{\partial\kappa}{\partial x}(x,y)\frac{\partial T}{\partial x}(x,y,t+\delta)+\frac{\partial\kappa}{\partial y}(x,y)\frac{\partial T}{\partial y}(x,y,t+\delta))=u(x,y,t+\delta) \tag{8}$$

The right side of equation(7) and (8) is equal, so it is time-invariant.

Homogeneous

Linear

When ρ and c is constant, it is not associated with substituting the solution (4) into the both sides of the PDE (3). So this system is still linear.

time-invariant

Because only u and T is associated with time that no matter the system is homogeneous or not, it is time-invariant. (Proof as equation (7) and (8))

Problem 2

Homogeneous model assumption implies the following:

- $l_x=l_y=0$.
- $\rho(x,y)=\rho\ \kappa(x,y)=\kappa$ and $c(x,y)=c$, where ρ,c,κ are positive constants.

Applying the above assumptions to the model gives

$$\rho c \frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + u(x, y, t), \tag{9}$$

as the final equation.
Let the source term be zero, i.e., $u(x, y, t) = 0$. Consider the function

$$T(x, y, t) = a(t)\psi(x)\phi(y), \tag{10}$$

where $a(t)$, $\psi(x)$ and $\phi(y)$ are real-valued functions on $\mathbb{R}$, $[0, L_x]$ and $[0, L_y]$ respectively.
Substituting (10) into (9),

$$\rho c \frac{\partial\big(a(t)\psi(x)\phi(y)\big)}{\partial t} = \kappa \left(\frac{\partial^2 \big(a(t)\psi(x)\phi(y)\big)}{\partial x^2} + \frac{\partial^2 \big(a(t)\psi(x)\phi(y)\big)}{\partial y^2} \right). \tag{11}$$

This gives,

$$\rho c \psi(x)\phi(y) \frac{\mathrm{d}a(t)}{\mathrm{d}t} = \kappa a(t) \left(\phi(y) \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \psi(x) \frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} \right). \tag{12}$$

Dividing throughout by (10),

$$\frac{\rho c}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t} = \kappa \left(\frac{1}{\psi(x)} \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{1}{\phi(y)} \frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} \right). \tag{13}$$

Rearranging, gives,

$$\frac{\rho c}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t} - \kappa \left(\frac{1}{\psi(x)} \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{1}{\phi(y)} \frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} \right) = 0. \tag{14}$$

The three terms in the above equation are functions of the three independent variables x, y, t. So, in order for the above equation to be satisfied, each of the term must be constant. Consider a constant $\alpha > 0$. The three terms are then given as

$$\frac{\rho c}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t} = \alpha, \qquad \frac{1}{\psi(x)} \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} = \frac{1}{\phi(y)} \frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} = \frac{\alpha}{2\kappa}. \tag{15}$$

Upon simplifying, the following differential equations are obtained:

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} - \lambda a(t) = 0, \tag{16a}$$

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} - \lambda_x \psi(x) = 0, \tag{16b}$$

$$\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} - \lambda_y \phi(y) = 0, \tag{16c}$$

where $\lambda = \frac{\alpha}{\rho c}$, $\lambda_x = \lambda_y = \frac{\alpha}{2\kappa}$.
Thus it can be seen that for equation(10) to be a solution to the PDE (9) when $u(x, y, t) = 0$, $a(t)$, $\psi(x)$ and $\phi(y)$ need to satisfy the equations (16a), (16b) and (16c) respectively.

Problem 3

The space of square-integrable functions in Ω is defined as

$$L_2(\Omega) = \left\{ f(x,y) : \Omega \to \mathbb{R} \;\middle|\; \int_0^{L_y} \int_0^{L_x} f(x,y)^2 \mathrm{d}x\mathrm{d}y < \infty \right\} \tag{17}$$

The inner product for $L_2(\Omega)$ is defined as follows:

$$\langle f, g \rangle := \int_0^{L_y} \int_0^{L_x} f(x,y) g(x,y) \mathrm{d}x\mathrm{d}y, \qquad f, g \in L_2(\Omega). \tag{18}$$

The norm associated with $L_2(\Omega)$ is defined as follows:

$$\|f\|_{L_2} = \langle f, f \rangle^{\frac{1}{2}}, \qquad f \in L_2(\Omega). \tag{19}$$

Problem 4

The spectral expansion of $T(x,y,t)$ is given as

$$T(x,y,t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}(t) \psi_k(x) \phi_l(y). \tag{20}$$

Taking $u(x,y,t) = 0$ and substituting the above solution in equation (9), the following is obtained:

$$\rho c \; pardert \left(\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \psi_k(x) \phi_l(y) \right) = \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}(t) \psi_k(x) \phi_l(y). \tag{21}$$

The partial derivatives can be written as regular(HM: ?) derivatives as they are now operating on functions of a single variable.

$$\Rightarrow \rho c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\mathrm{d}a_{k,l}(t)}{\mathrm{d}t} \psi_k(x) \phi_l(y) = \kappa \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \left(\frac{\mathrm{d}^2 \psi_k(x)}{\mathrm{d}x^2} \phi_l(y) + \psi_k(x) \frac{\mathrm{d}^2 \phi_l(y)}{\mathrm{d}y^2} \right) \tag{22}$$

Remark: It has been made clear that $\psi = \psi(x)$ and $\phi = \phi(y)$, so $(x)\&(y)$ are dropped henceforth.
The function $\psi_i(x)\phi_j(y)$ (where i, j can be any of $\{0, 1, \ldots, \infty\}$) is multiplied throughout and the equation is integrated over the whole domain Ω as follows:

$$\begin{aligned} &\rho c \int_0^{L_y} \int_0^{L_x} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\mathrm{d}a_{k,l}(t)}{\mathrm{d}t} \psi_k(x) \phi_l(y) \psi_i(x) \phi_j(y) = \\ &\quad \kappa \int_0^{L_y} \int_0^{L_x} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \left(\frac{\mathrm{d}^2 \psi_k(x)}{\mathrm{d}x^2} \phi_l(y) \psi_i(x) \phi_j(y) + \psi_k(x) \frac{\mathrm{d}^2 \phi_l(y)}{\mathrm{d}y^2} \psi_i(x) \phi_j(y) \right). \end{aligned} \tag{23}$$

Re-writing the above equation using the inner product notation (18), we get,

$$\rho c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\mathrm{d}a_{k,l}(t)}{\mathrm{d}t} \langle \psi_k(x)\phi_l(y), \psi_i(x)\phi_j(y) \rangle =$$
$$\kappa \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \left(\left\langle \frac{\mathrm{d}^2\psi_k(x)}{\mathrm{d}x^2}\phi_l(y), \psi_i(x)\phi_j(y) \right\rangle + \left\langle \psi_k(x)\frac{\mathrm{d}^2\phi_l(y)}{\mathrm{d}y^2}, \psi_i(x)\phi_j(y) \right\rangle \right). \quad (24)$$

Since $\phi_k(x)$ and $\psi_l(y)$ are trigonometric functions, their derivatives w.r.t. x and y respectively give:

$$\frac{\mathrm{d}^2\psi_k(x)}{\mathrm{d}x^2} = -\left(\frac{k\pi}{L_x}\right)^2 \psi_k(x), \qquad \frac{\mathrm{d}^2\phi_l(y)}{\mathrm{d}y^2} = -\left(\frac{l\pi}{L_y}\right)^2. \quad (25)$$

Using the above results in (24), the following is obtained:

$$\rho c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\mathrm{d}a_{k,l}(t)}{\mathrm{d}t} \langle \psi_k(x)\phi_l(y), \psi_i(x)\phi_j(y) \rangle =$$
$$-\kappa \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \left(\left(\frac{k\pi}{L_x}\right)^2 \langle \psi_k(x)\phi_l(y), \psi_i(x)\phi_j(y) \rangle + \left(\frac{l\pi}{L_y}\right)^2 \langle \psi_k(x)\phi_l(y), \psi_i(x)\phi_j(y) \rangle \right). \quad (26)$$

Using the orthonormality result HM: Give ref here from the previous problem, the following is obtained:

$$\rho c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\mathrm{d}a_{k,l}(t)}{\mathrm{d}t} \delta_{ki}\delta_{lj} = -\kappa \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \left[\left(\frac{k\pi}{L_x}\right)^2 \delta_{ki}\delta_{lj} + \left(\frac{l\pi}{L_y}\right)^2 \delta_{ki}\delta_{lj} \right]. \quad (27)$$

Setting $k = i$ and $l = j$ in the equation (27) and neglecting the zero terms, we get,

$$\rho c \frac{\mathrm{d}a_{k,l}(t)}{\mathrm{d}t} = -\kappa a_{k,l}(t) \left[\left(\frac{k\pi}{L_x}\right)^2 + \left(\frac{l\pi}{L_y}\right)^2 \right], \qquad k, l = 0, 1, \ldots, \infty. \quad (28)$$

It can be observed that the above set of equations are infinite, so the values of k, l are restricted to finite values.
Let $k = 0, 1, \ldots, K$ and $l = 0, 1, \ldots, L$. Then a set of KL equations are obtained that have the form (28).

References