

Model Order Reduction Project

Heat Diffusion

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Introduction

TODO: Write the intro later- define the physical parameters, domain

$$\left\{ \begin{array}{l} \rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) = \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right] K(x, y) \left[\frac{\partial T(x, y, t)}{\partial x} \right] + u(x, y, t) \quad (0, T] \times \Omega, \\ \frac{\partial T(0, y, t)}{\partial x} = \frac{\partial T(L_x, y, t)}{\partial x} = 0, \quad y \in [0, L_y] \times [0, T], \\ \frac{\partial T(x, 0, t)}{\partial y} = \frac{\partial T(x, L_y, t)}{\partial y} = 0, \quad x \in [0, L_x] \times [0, T], \\ T(x, y, 0) = T_0(x, y), \quad \{t = 0\} \times \Omega, \end{array} \right. \quad (1)$$

where $\Omega := [0, L_x] \times [0, L_y]$ and $T_0(x, y)$ is a physically realistic initial temperature profile.

Problem 1

This system is non-linear and time-invariant.
According to description, this model is isotropic.
Therefore,

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) = \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right] \begin{bmatrix} \kappa(x, y) & 0 \\ 0 & \kappa(x, y) \end{bmatrix} \left[\frac{\partial T}{\partial x}(x, y, t) \right] + u(x, y, t), \quad (2)$$

i.e.,

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) = \frac{\partial}{\partial x} \left(\kappa(x, y)\frac{\partial T}{\partial x}(x, y, t) \right) + \frac{\partial}{\partial y} \left(\kappa(x, y)\frac{\partial T}{\partial y}(x, y, t) \right) + u(x, y, t). \quad (3)$$

Remark: Here onwards(HM: Is that correct grammar?), it is implicitly implied that $T = T(x, y, t)$ unless specified otherwise and (x, y, t) is dropped for notational convenience.

Non-homogeneous

Linearity

In order to check for the linearity of (3) the following solution is assumed:

$$T(x, y, t) = T_1(x, y, t) + T_2(x, y, t), \quad (4)$$

where $T_1(x, y, t)$ & $T_2(x, y, t)$ are solutions of the PDE (3). Substituting the solution (4) into the left hand side of the PDE (3), we get,

$$\rho(x, y)c(x, y)\frac{\partial T_1 + T_2}{\partial t}(x, y, t) = \rho(x, y)c(x, y) \left(\frac{\partial T_1}{\partial t}(x, y, t) + \frac{\partial T_2}{\partial t}(x, y, t) \right) \quad (5)$$

Substituting the solution (4) into the right side of the PDE (3),

$$\begin{aligned} \frac{\partial}{\partial x} \left(\kappa(x, y) \frac{\partial T_1 + T_2}{\partial x}(x, y, t) \right) + \frac{\partial}{\partial y} \left(\kappa(x, y) \frac{\partial T_1 + T_2}{\partial y}(x, y, t) \right) + u(x, y, t) = \\ \left(\frac{\partial \kappa(x, y)}{\partial x} \left(\frac{\partial T_1}{\partial x}(x, y, t) + \frac{\partial T_2}{\partial x}(x, y, t) \right) + \kappa \left(\frac{\partial^2 T_1}{\partial x^2}(x, y, t) + \frac{\partial^2 T_2}{\partial x^2}(x, y, t) \right) \right) + \\ \left(\frac{\partial \kappa(x, y)}{\partial y} \left(\frac{\partial T_1}{\partial y}(x, y, t) + \frac{\partial T_2}{\partial y}(x, y, t) \right) + \kappa \left(\frac{\partial^2 T_1}{\partial y^2}(x, y, t) + \frac{\partial^2 T_2}{\partial y^2}(x, y, t) \right) \right) + \\ u(x, y, t) \end{aligned} \quad (6)$$

According to equation (5) and (6), both sides are linear. Therefore this equation is linear.

Time-invariant

input delayed: $u_d(t) = u(t + \delta)$

$$\rho(x, y)c(x, y) \frac{\partial T}{\partial t}(x, y, t) - \left(\frac{\partial \kappa}{\partial x}(x, y) \frac{\partial T}{\partial x}(x, y, t) + \frac{\partial \kappa}{\partial y}(x, y) \frac{\partial T}{\partial y}(x, y, t) \right) = u(x, y, t + \delta) \quad (7)$$

output delayed: $T_d(t) = T(t + \delta)$

$$\rho(x, y)c(x, y) \frac{\partial T}{\partial t}(x, y, t + \delta) - \left(\frac{\partial \kappa}{\partial x}(x, y) \frac{\partial T}{\partial x}(x, y, t + \delta) + \frac{\partial \kappa}{\partial y}(x, y) \frac{\partial T}{\partial y}(x, y, t + \delta) \right) = u(x, y, t + \delta) \quad (8)$$

The right side of equation (7) and (8) is equal, so it is time-invariant.

Homogeneous

Linear

When ρ and c is constant, it is not associated with substituting the solution (4) into the both sides of the PDE (3). So this system is still linear.

time-invariant

Because only u and T is associated with time that no matter the system is homogeneous or not, it is time-invariant. (Proof as equation (7) and (8))

Problem 2

Homogeneous model assumption implies the following:

- $l_x = l_y = 0$.
- $\rho(x, y) = \rho$, $\kappa(x, y) = \kappa$ and $c(x, y) = c$, where ρ, c, κ are positive constants.

Applying the above assumptions to the model gives

$$\rho c \frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + u(x, y, t), \quad (9)$$

as the final equation.

Let the source term be zero, i.e., $u(x, y, t) = 0$. Consider the function

$$T(x, y, t) = a(t)\psi(x)\phi(y), \quad (10)$$

where $a(t)$, $\psi(x)$ and $\phi(y)$ are real-valued functions on \mathbb{R} , $[0, L_x]$ and $[0, L_y]$ respectively. Substituting (10) into (9),

$$\rho c \frac{\partial(a(t)\psi(x)\phi(y))}{\partial t} = \kappa \left(\frac{\partial^2(a(t)\psi(x)\phi(y))}{\partial x^2} + \frac{\partial^2(a(t)\psi(x)\phi(y))}{\partial y^2} \right). \quad (11)$$

This gives,

$$\rho c \psi(x)\phi(y) \frac{da(t)}{dt} = \kappa a(t) \left(\phi(y) \frac{d^2\psi(x)}{dx^2} + \psi(x) \frac{d^2\phi(y)}{dy^2} \right). \quad (12)$$

Dividing throughout by (10),

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} = \kappa \left(\frac{1}{\psi(x)} \frac{d^2\psi(x)}{dx^2} + \frac{1}{\phi(y)} \frac{d^2\phi(y)}{dy^2} \right). \quad (13)$$

Rearranging, gives,

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} - \kappa \left(\frac{1}{\psi(x)} \frac{d^2\psi(x)}{dx^2} + \frac{1}{\phi(y)} \frac{d^2\phi(y)}{dy^2} \right) = 0. \quad (14)$$

The three terms in the above equation are functions of the three independent variables x, y, t . So, in order for the above equation to be satisfied, each of the term must be constant. Consider a constant $\alpha > 0$. The three terms are then given as

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} = \alpha, \quad \frac{1}{\psi(x)} \frac{d^2\psi(x)}{dx^2} = \frac{1}{\phi(y)} \frac{d^2\phi(y)}{dy^2} = \frac{\alpha}{2\kappa}. \quad (15)$$

Upon simplifying, the following differential equations are obtained:

$$\frac{da(t)}{dt} - \lambda a(t) = 0, \quad (16a)$$

$$\frac{d^2\psi(x)}{dx^2} - \lambda_x \psi(x) = 0, \quad (16b)$$

$$\frac{d^2\phi(y)}{dy^2} - \lambda_y \phi(y) = 0, \quad (16c)$$

where $\lambda = \frac{\alpha}{\rho c}$, $\lambda_x = \lambda_y = \frac{\alpha}{2\kappa}$.

Thus it can be seen that for equation(10) to be a solution to the PDE (9) when $u(x, y, t) = 0$, $a(t)$, $\psi(x)$ and $\phi(y)$ need to satisfy the equations (16a), (16b) and (16c) respectively.

Problem 3

The space of square-integrable functions in Ω is defined as

$$L_2(\Omega) = \left\{ f(x, y) : \Omega \rightarrow \mathbb{R} \mid \int_0^{L_y} \int_0^{L_x} f(x, y)^2 dx dy < \infty \right\} \quad (17)$$

The inner product for $L_2(\Omega)$ is defined as follows:

$$\langle f, g \rangle := \int_0^{L_y} \int_0^{L_x} f(x, y) g(x, y) dx dy, \quad f, g \in L_2(\Omega). \quad (18)$$

The norm associated with $L_2(\Omega)$ is defined as follows:

$$\|f\|_{L_2} = \langle f, f \rangle^{\frac{1}{2}}, \quad f \in L_2(\Omega). \quad (19)$$

Problem 4

The spectral expansion of $T(x, y, t)$ is given as

$$T(x, y, t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}(t) \psi_k(x) \phi_l(y). \quad (20)$$

Taking $u(x, y, t) = 0$ and substituting the above solution in equation (9), the following is obtained:

$$\rho c \text{ pardert} \left(\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}(t) \psi_k(x) \phi_l(y) \right) = \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}(t) \psi_k(x) \phi_l(y). \quad (21)$$

The partial derivatives can be written as regular(HM: ?) derivatives as they are now operating on functions of a single variable.

$$\Rightarrow \rho c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{da_{kl}(t)}{dt} \psi_k(x) \phi_l(y) = \kappa \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}(t) \left(\frac{d^2 \psi_k(x)}{dx^2} \phi_l(y) + \psi_k(x) \frac{d^2 \phi_l(y)}{dy^2} \right) \quad (22)$$

Remark: It has been made clear that $\psi = \psi(x)$ and $\phi = \phi(y)$, so $(x) \& (y)$ are dropped henceforth.

The function $\psi_i(x) \phi_j(y)$ (where i, j can be any of $\{0, 1, \dots, \infty\}$) is multiplied throughout and the equation is integrated over the whole domain Ω as follows:

$$\begin{aligned} \rho c \int_0^{L_y} \int_0^{L_x} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{da_{kl}(t)}{dt} \psi_k(x) \phi_l(y) \psi_i(x) \phi_j(y) = \\ \kappa \int_0^{L_y} \int_0^{L_x} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl}(t) \left(\frac{d^2 \psi_k(x)}{dx^2} \phi_l(y) \psi_i(x) \phi_j(y) + \psi_k(x) \frac{d^2 \phi_l(y)}{dy^2} \psi_i(x) \phi_j(y) \right). \end{aligned} \quad (23)$$

Re-writing the above equation using the inner product notation (18), we get,

$$\begin{aligned} \rho c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{da_{k,l}(t)}{dt} \langle \psi_k(x) \phi_l(y), \psi_i(x) \phi_j(y) \rangle = \\ \kappa \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \left(\left\langle \frac{d^2 \psi_k(x)}{dx^2} \phi_l(y), \psi_i(x) \phi_j(y) \right\rangle + \left\langle \psi_k(x) \frac{d^2 \phi_l(y)}{dy^2}, \psi_i(x) \phi_j(y) \right\rangle \right). \end{aligned} \quad (24)$$

Since $\phi_k(x)$ and $\psi_l(y)$ are trigonometric functions, their derivatives w.r.t. x and y respectively give:

$$\frac{d^2 \psi_k(x)}{dx^2} = - \left(\frac{k\pi}{L_x} \right)^2 \psi_k(x), \quad \frac{d^2 \phi_l(y)}{dy^2} = - \left(\frac{l\pi}{L_y} \right)^2. \quad (25)$$

Using the above results in (24), the following is obtained:

$$\begin{aligned} \rho c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{da_{k,l}(t)}{dt} \langle \psi_k(x) \phi_l(y), \psi_i(x) \phi_j(y) \rangle = \\ - \kappa \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \left(\left(\frac{k\pi}{L_x} \right)^2 \langle \psi_k(x) \phi_l(y), \psi_i(x) \phi_j(y) \rangle + \left(\frac{l\pi}{L_y} \right)^2 \langle \psi_k(x) \phi_l(y), \psi_i(x) \phi_j(y) \rangle \right). \end{aligned} \quad (26)$$

Using the orthonormality result [HM: Give ref here](#) from the previous problem, the following is obtained:

$$\rho c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{da_{k,l}(t)}{dt} \delta_{ki} \delta_{lj} = - \kappa \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}(t) \left[\left(\frac{k\pi}{L_x} \right)^2 \delta_{ki} \delta_{lj} + \left(\frac{l\pi}{L_y} \right)^2 \delta_{ki} \delta_{lj} \right]. \quad (27)$$

Setting $k = i$ and $l = j$ in the equation (27) and neglecting the zero terms, we get,

$$\rho c \frac{da_{k,l}(t)}{dt} = - \kappa a_{k,l}(t) \left[\left(\frac{k\pi}{L_x} \right)^2 + \left(\frac{l\pi}{L_y} \right)^2 \right], \quad k, l = 0, 1, \dots, \infty. \quad (28)$$

It can be observed that the above set of equations are infinite, so the values of k, l are restricted to finite values.

Let $k = 0, 1, \dots, K$ and $l = 0, 1, \dots, L$. Then a set of KL equations are obtained that have the form (28).

References