

Model Order Reduction Project

Heat Diffusion

Xinyu Zeng, 1301462
Hemaditya Malla, 1282484

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TODO: Shall we give a small introduction to the problem where we write down the equations, boundary conditions and also add the figure?

Problem 1

This system is non-linear and time-invariant.

According to description, this model is isotropic.

Therefore,

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \kappa(x, y) & 0 \\ 0 & \kappa(x, y) \end{bmatrix} \begin{bmatrix} \frac{\partial T}{\partial x}(x, y, t) \\ \frac{\partial T}{\partial y}(x, y, t) \end{bmatrix} + u(x, y, t), \quad (1)$$

i.e.,

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) = \frac{\partial}{\partial x} \left(\kappa(x, y)\frac{\partial T}{\partial x}(x, y, t) \right) + \frac{\partial}{\partial y} \left(\kappa(x, y)\frac{\partial T}{\partial y}(x, y, t) \right) + u(x, y, t). \quad (2)$$

Remark: Here onwards(HM: Is that correct grammar?), it is implicitly implied that $T = T(x, y, t)$ unless specified otherwise and (x, y, t) is dropped for notational convenience.

Non-homogeneous

Linearity

In order to check for the linearity of (2) the following solution is assumed:

$$T(x, y, t) = T_1(x, y, t) + T_2(x, y, t), \quad (3)$$

where $T_1(x, y, t)$ & $T_2(x, y, t)$ are solutions of the PDE (2). Substituting the solution (3) into the left hand side of the PDE (2), we get,

$$\rho(x, y)c(x, y)\frac{\partial T_1 + T_2}{\partial t}(x, y, t) = \rho(x, y)c(x, y) \left(\frac{\partial T_1}{\partial t}(x, y, t) + \frac{\partial T_2}{\partial t}(x, y, t) \right) \quad (4)$$

Substituting the solution (3) into the right side of the PDE (2),

$$\begin{aligned} \frac{\partial}{\partial x} \left(\kappa(x, y)\frac{\partial T_1 + T_2}{\partial x}(x, y, t) \right) + \frac{\partial}{\partial y} \left(\kappa(x, y)\frac{\partial T_1 + T_2}{\partial y}(x, y, t) \right) + u(x, y, t) = \\ \left(\frac{\partial \kappa(x, y)}{\partial x} \left(\frac{\partial T_1}{\partial x}(x, y, t) + \frac{\partial T_2}{\partial x}(x, y, t) \right) + \kappa(x, y) \left(\frac{\partial^2 T_1}{\partial x^2}(x, y, t) + \frac{\partial^2 T_2}{\partial x^2}(x, y, t) \right) \right) + \\ \left(\frac{\partial \kappa(x, y)}{\partial y} \left(\frac{\partial T_1}{\partial y}(x, y, t) + \frac{\partial T_2}{\partial y}(x, y, t) \right) + \kappa(x, y) \left(\frac{\partial^2 T_1}{\partial y^2}(x, y, t) + \frac{\partial^2 T_2}{\partial y^2}(x, y, t) \right) \right) + \\ u(x, y, t) \end{aligned} \quad (5)$$

According to equation (4) and (5), both sides are linear. Therefore this equation is linear.

Time-invariant

input delayed: $u_d(t) = u(t + \delta)$

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) - \left(\frac{\partial \kappa}{\partial x}(x, y)\frac{\partial T}{\partial x}(x, y, t) + \frac{\partial \kappa}{\partial y}(x, y)\frac{\partial T}{\partial y}(x, y, t)\right) = u(x, y, t + \delta) \quad (6)$$

output delayed: $T_d(t) = T(t + \delta)$

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t + \delta) - \left(\frac{\partial \kappa}{\partial x}(x, y)\frac{\partial T}{\partial x}(x, y, t + \delta) + \frac{\partial \kappa}{\partial y}(x, y)\frac{\partial T}{\partial y}(x, y, t + \delta)\right) = u(x, y, t + \delta) \quad (7)$$

The right side of equation (6) and (7) is equal, so it is time-invariant.

Homogeneous

Linear

When ρ and c is constant, it is not associated with substituting the solution (3) into the both sides of the PDE (2). So this system is still linear.

time-invariant

Because only u and T is associated with time that no matter the system is homogeneous or not, it is time-invariant. (Proof as equation (6) and (7))

Problem 2

Homogeneous model assumption implies the following:

- $l_x = l_y = 0$.
- $\rho(x, y) = \rho$, $\kappa(x, y) = \kappa$ and $c(x, y) = c$, where ρ, c, κ are positive constants.

Applying the above assumptions to the model gives

$$\rho c \frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + u(x, y, t), \quad (8)$$

as the final equation.

Let the source term be zero, i.e., $u(x, y, t) = 0$. Consider the function

$$T(x, y, t) = a(t)\psi(x)\phi(y), \quad (9)$$

where $a(t)$, $\psi(x)$ and $\phi(y)$ are real-valued functions on \mathbb{R} , $[0, L_x]$ and $[0, L_y]$ respectively. Substituting (9) into (8),

$$\rho c \frac{\partial (a(t)\psi(x)\phi(y))}{\partial t} = \kappa \left(\frac{\partial^2 (a(t)\psi(x)\phi(y))}{\partial x^2} + \frac{\partial^2 (a(t)\psi(x)\phi(y))}{\partial y^2} \right). \quad (10)$$

This gives,

$$\rho c \psi(x) \phi(y) \frac{da(t)}{dt} = \kappa a(t) \left(\phi(y) \frac{d^2 \psi(x)}{dx^2} + \psi(x) \frac{d^2 \phi(y)}{dy^2} \right). \quad (11)$$

Dividing throughout by (9),

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} = \kappa \left(\frac{1}{\psi(x)} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{\phi(y)} \frac{d^2 \phi(y)}{dy^2} \right). \quad (12)$$

Rearranging, gives,

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} - \kappa \left(\frac{1}{\psi(x)} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{\phi(y)} \frac{d^2 \phi(y)}{dy^2} \right) = 0. \quad (13)$$

The three terms in the above equation are functions of the three independent variables x, y, t . So, in order for the above equation to be satisfied, each of the term must be constant. Consider a constant $\alpha > 0$. The three terms are then given as

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} = \alpha, \quad \frac{1}{\psi(x)} \frac{d^2 \psi(x)}{dx^2} = \frac{1}{\phi(y)} \frac{d^2 \phi(y)}{dy^2} = \frac{\alpha}{2\kappa}. \quad (14)$$

Upon simplifying, the following differential equations are obtained:

$$\frac{da(t)}{dt} - \lambda a(t) = 0, \quad (15a)$$

$$\frac{d^2 \psi(x)}{dx^2} - \lambda_x \psi(x) = 0, \quad (15b)$$

$$\frac{d^2 \phi(y)}{dy^2} - \lambda_y \phi(y) = 0, \quad (15c)$$

where $\lambda = \frac{\alpha}{\rho c}$, $\lambda_x = \lambda_y = \frac{\alpha}{2\kappa}$.

Thus it can be seen that for equation(9) to be a solution to the PDE (8) when $u(x, y, t) = 0$, $a(t)$, $\psi(x)$ and $\phi(y)$ need to satisfy the equations (15a), (15b) and (15c) respectively.

Problem 3

The set of square-integrable functions is given as:

$$L_2 = \{f : \int_{\Omega} f dx < \infty\} \quad (16)$$

$$\phi_{k,l} = \psi_k(x) \phi_l(y) \quad (17)$$

According to question,

$$\langle \phi_{k,l}, \phi_{i,j} \rangle := \int_0^{L_x} \int_0^{L_y} \psi_k(x) \phi_l(y) \psi_i(x) \phi_j(y) dy dx \quad (18)$$

$$\mathbf{k=0,l=0;i=0,j=0}$$

$$\langle \phi_{k,l}, \phi_{i,j} \rangle = \int_0^{L_x} \int_0^{L_y} \psi_k(x) \phi_l(y) \psi_i(x) \phi_j(y) dy dx = \int_0^{L_x} \int_0^{L_y} \frac{1}{L_x} \frac{1}{L_y} dy dx = 1 \quad (19)$$

$$\mathbf{k=0,l=0;i=0,j>0}$$

$$\langle \phi_{k,l}, \phi_{i,j} \rangle = \int_0^{L_x} \int_0^{L_y} \frac{1}{L_x} \frac{\sqrt{2}}{L_y} \cos\left(\frac{j\pi y}{L_y}\right) dy dx = \frac{x}{L_x} \frac{\sqrt{2}}{j\pi} \sin\left(\frac{j\pi y}{L_y}\right) \Bigg|_0^{L_y} \Bigg|_0^{L_x} = 0 \quad (20)$$

$$\mathbf{k=0,l=0;i>0,j=0}$$

$$\langle \phi_{k,l}, \phi_{i,j} \rangle = \int_0^{L_x} \int_0^{L_y} \frac{\sqrt{2}}{L_x} \cos\left(\frac{i\pi x}{L_x}\right) \frac{1}{L_y} dy dx = \frac{\sqrt{2}}{i\pi} \sin\left(\frac{i\pi x}{L_x}\right) \frac{y}{L_y} \Bigg|_0^{L_y} \Bigg|_0^{L_x} = 0 \quad (21)$$

$$\mathbf{k=0,l>0;i=0,j=0}$$

Analogous to equation (20)

$$\mathbf{k>0,l=0;i=0,j=0}$$

Analogous to equation (21)

$$\mathbf{k=0,l=0;i>0,j>0}$$

$$\langle \phi_{k,l}, \phi_{i,j} \rangle = \int_0^{L_x} \int_0^{L_y} \frac{\sqrt{2}}{L_x} \cos\left(\frac{i\pi x}{L_x}\right) \frac{\sqrt{2}}{L_y} \cos\left(\frac{j\pi y}{L_y}\right) dy dx = \frac{\sqrt{2}}{i\pi} \sin\left(\frac{i\pi x}{L_x}\right) \frac{\sqrt{2}}{j\pi} \sin\left(\frac{j\pi y}{L_y}\right) \Bigg|_0^{L_y} \Bigg|_0^{L_x} = 0 \quad (22)$$

$$\mathbf{k>0,l>0;i=0,j=0}$$

Analogous to equation (22)

$$\mathbf{k>0,l=0;i>0,j=0}$$

$$\begin{aligned} \langle \phi_{k,l}, \phi_{i,j} \rangle &= \int_0^{L_x} \int_0^{L_y} \frac{2}{L_x} \cos\left(\frac{k\pi x}{L_x}\right) \cos\left(\frac{i\pi x}{L_x}\right) \frac{1}{L_y} dy dx \\ &= \begin{cases} \left(\frac{1}{2k\pi} \sin\left(\frac{2k\pi x}{L_x}\right) + \frac{x}{L_x} \right) \frac{y}{L_y} \Bigg|_0^{L_y} \Bigg|_0^{L_x} = 1 & k = i \\ \left(\frac{1}{(k+i)\pi} \sin\left(\frac{(k+i)\pi x}{L_x}\right) + \frac{1}{(k-i)\pi} \sin\left(\frac{(k-i)\pi x}{L_x}\right) \right) \frac{y}{L_y} \Bigg|_0^{L_y} \Bigg|_0^{L_x} = 0 & k \neq i \end{cases} \quad (23) \end{aligned}$$

$$\mathbf{k} > \mathbf{0}, \mathbf{l} = \mathbf{0}; \mathbf{i} = \mathbf{0}, \mathbf{j} > \mathbf{0}$$

Analogous to equation (22)

$$\mathbf{k} = \mathbf{0}, \mathbf{l} > \mathbf{0}; \mathbf{i} > \mathbf{0}, \mathbf{j} = \mathbf{0}$$

Analogous to equation (22)

$$\mathbf{k} = \mathbf{0}, \mathbf{l} > \mathbf{0}; \mathbf{i} = \mathbf{0}, \mathbf{j} > \mathbf{0}$$

$$\begin{aligned} \langle \phi_{k,l}, \phi_{i,j} \rangle &= \int_0^{L_x} \int_0^{L_y} \frac{1}{L_x} \frac{2}{L_y} \cos\left(\frac{l\pi y}{L_y}\right) \cos\left(\frac{j\pi y}{L_y}\right) dy dx \\ &= \begin{cases} \frac{x}{L_x} \left(\frac{1}{2l\pi} \sin\left(\frac{2l\pi y}{L_y}\right) + \frac{y}{L_y} \right) \Big|_0^{L_y} \Big|_0^{L_x} = 1 & l = j \\ \frac{x}{L_x} \left(\frac{1}{(l+j)\pi} \sin\left(\frac{(l+j)\pi y}{L_y}\right) + \frac{1}{(l-j)\pi} \sin\left(\frac{(l-j)\pi y}{L_y}\right) \right) \Big|_0^{L_y} \Big|_0^{L_x} = 0 & l \neq j \end{cases} \end{aligned} \quad (24)$$

$$\mathbf{k} = \mathbf{0}, \mathbf{l} > \mathbf{0}; \mathbf{i} > \mathbf{0}, \mathbf{j} > \mathbf{0}$$

$$\langle \phi_{k,l}, \phi_{i,j} \rangle = \int_0^{L_x} \int_0^{L_y} \frac{\sqrt{2}}{L_x} \cos\left(\frac{i\pi x}{L_x}\right) \frac{2}{L_y} \cos\left(\frac{l\pi y}{L_y}\right) \cos\left(\frac{j\pi y}{L_y}\right) dy dx = 0 \quad (25)$$

$$\mathbf{k} > \mathbf{0}, \mathbf{l} = \mathbf{0}; \mathbf{i} > \mathbf{0}, \mathbf{j} > \mathbf{0}$$

$$\langle \phi_{k,l}, \phi_{i,j} \rangle = \int_0^{L_x} \int_0^{L_y} \frac{2}{L_x} \cos\left(\frac{k\pi x}{L_x}\right) \cos\left(\frac{i\pi x}{L_x}\right) \frac{\sqrt{2}}{j\pi} \sin\left(\frac{j\pi y}{L_y}\right) dy dx = 0 \quad (26)$$

$$\mathbf{k} > \mathbf{0}, \mathbf{l} > \mathbf{0}; \mathbf{i} = \mathbf{0}, \mathbf{j} > \mathbf{0}$$

Analogous to equation (25)

$$\mathbf{k} > \mathbf{0}, \mathbf{l} > \mathbf{0}; \mathbf{i} > \mathbf{0}, \mathbf{j} = \mathbf{0}$$

Analogous to equation (26)

$$\mathbf{k} > \mathbf{0}, \mathbf{l} > \mathbf{0}; \mathbf{i} > \mathbf{0}, \mathbf{j} > \mathbf{0}$$

$$\begin{aligned} \langle \phi_{k,l}, \phi_{i,j} \rangle &= \int_0^{L_x} \int_0^{L_y} \frac{2}{L_x} \cos\left(\frac{k\pi x}{L_x}\right) \cos\left(\frac{i\pi x}{L_x}\right) \frac{2}{L_y} \cos\left(\frac{l\pi y}{L_y}\right) \cos\left(\frac{j\pi y}{L_y}\right) dy dx \\ &= \begin{cases} \left(\frac{1}{2k\pi} \sin\left(\frac{2k\pi x}{L_x}\right) + \frac{x}{L_x} \right) \left(\frac{1}{2l\pi} \sin\left(\frac{2l\pi y}{L_y}\right) + \frac{y}{L_y} \right) \Big|_0^{L_y} \Big|_0^{L_x} = 1 & k = i \wedge l = j \\ 0 & otherwise \end{cases} \end{aligned} \quad (27)$$

Problem 4

References