Model Order Reduction Project

Heat Diffusion

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Homogeneous model assumption implies the following:

- $l_x = l_y = 0$.
- $\rho(x, y) = \rho$ $\kappa(x, y) = \kappa$ and $c(x, y) = c$, where ρ, c, κ are positive constants.

Applying the above assumptions to the model gives

$$\rho c \frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + u(x, y, t), \tag{1}$$

as the final equation.
Let the source term be zero, i.e., $u(x, y, t) = 0$. Consider the function

$$T(x, y, t) = a(t)\psi(x)\phi(y), \tag{2}$$

where $a(t)$, $\psi(x)$ and $\phi(y)$ are real-valued functions on $\mathbb{R}$, $[0, L_x]$ and $[0, L_y]$ respectively.
Substituting (2) into (1),

$$\rho c \frac{\partial\big(a(t)\psi(x)\phi(y)\big)}{\partial t} = \kappa \left(\frac{\partial^2\big(a(t)\psi(x)\phi(y)\big)}{\partial x^2} + \frac{\partial^2\big(a(t)\psi(x)\phi(y)\big)}{\partial y^2} \right). \tag{3}$$

This gives,

$$\rho c \psi(x)\phi(y) \frac{\mathrm{d}a(t)}{\mathrm{d}t} = \kappa a(t) \left(\phi(y) \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \psi(x) \frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} \right). \tag{4}$$

Dividing throughout by (2),

$$\frac{\rho c}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t} = \kappa \left(\frac{1}{\psi(x)} \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{1}{\phi(y)} \frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} \right). \tag{5}$$

Rearranging, gives,

$$\frac{\rho c}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t} - \kappa \left(\frac{1}{\psi(x)} \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{1}{\phi(y)} \frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} \right) = 0. \tag{6}$$

The three terms in the above equation are functions of the three independent variables x, y, t. So, in order for the above equation to be satisfied, each of the term must be constant. Consider a constant $\alpha > 0$. The three terms are then given as

$$\frac{\rho c}{a(t)} \frac{\mathrm{d}a(t)}{\mathrm{d}t} = \alpha, \qquad \frac{1}{\psi(x)} \frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} = \frac{1}{\phi(y)} \frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} = \frac{\alpha}{2\kappa}. \tag{7}$$

Upon simplifying, the following differential equations are obtained:

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} - \lambda a(t) = 0, \tag{8a}$$

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} - \lambda_x\psi(x) = 0, \tag{8b}$$

$$\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} - \lambda_y\phi(y) = 0, \tag{8c}$$

where $\lambda = \frac{\alpha}{\rho c}$, $\lambda_x = \lambda_y = \frac{\alpha}{2\kappa}$.
Thus it can be seen that for equation(2) to be a solution to the PDE (1) when $u(x, y, t) = 0$, $a(t)$, $\psi(x)$ and $\phi(y)$ need to satisfy the equations (8a), (8b) and (8c) respectively.

Problem 3

Problem 4

References