

# Model Order Reduction Project

## Heat Diffusion

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TODO: Shall we give a small introduction to the problem where we write down the equations, boundary conditions and also add the figure?

## Problem 1

This system is non-linear and time-invariant.  
According to description, this model is isotropic.  
Therefore,

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \kappa(x, y) & 0 \\ 0 & \kappa(x, y) \end{bmatrix} \begin{bmatrix} \frac{\partial T}{\partial x}(x, y, t) \\ \frac{\partial T}{\partial y}(x, y, t) \end{bmatrix} + u(x, y, t), \quad (1)$$

i.e.,

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) = \frac{\partial}{\partial x} \left( \kappa(x, y) \frac{\partial T}{\partial x}(x, y, t) \right) + \frac{\partial}{\partial y} \left( \kappa(x, y) \frac{\partial T}{\partial y}(x, y, t) \right) + u(x, y, t). \quad (2)$$

**Remark:** Here onwards(HM: Is that correct grammar?), it is implicitly implied that  $T = T(x, y, t)$  unless specified otherwise and  $(x, y, t)$  is dropped for notational convenience.

## Non-homogeneous

### Nonlinearity

In order to check for the linearity of (2) the following solution is assumed:

$$T(x, y, t) = T_1(x, y, t) + T_2(x, y, t), \quad (3)$$

where  $T_1(x, y, t)$  &  $T_2(x, y, t)$  are solutions of the PDE (2). Substituting the solution (3) into the left hand side of the PDE (2), we get,

$$\rho(x, y)c(x, y)\frac{\partial T_1 + T_2}{\partial t}(x, y, t) = \rho \quad (4)$$

### Time-invariant

input delayed:  $u_d(t) = u(t + \delta)$

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) - \left( \frac{\partial \kappa}{\partial x}(x, y) \frac{\partial T}{\partial x}(x, y, t) + \frac{\partial \kappa}{\partial y}(x, y) \frac{\partial T}{\partial y}(x, y, t) \right) = u(x, y, t + \delta) \quad (5)$$

output delayed:  $T_d(t) = T(t + \delta)$

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t + \delta) - \left( \frac{\partial \kappa}{\partial x}(x, y) \frac{\partial T}{\partial x}(x, y, t + \delta) + \frac{\partial \kappa}{\partial y}(x, y) \frac{\partial T}{\partial y}(x, y, t + \delta) \right) = u(x, y, t + \delta) \quad (6)$$

The right side of equation(6)and (7) is equal, so it is time-invariant.

## homogeneous

### Non-linear

When  $\rho$  and  $c$  is constant, consider the right side of equation(2),

$$\frac{\partial \kappa}{\partial x}(x_1+x_2, y_1+y_2) \frac{\partial T}{\partial x}(x_1+x_2, y_1+y_2, t) + \frac{\partial \kappa}{\partial y}(x_1+x_2, y_1+y_2) \frac{\partial T}{\partial y}(x_1+x_2, y_1+y_2, t) + u(x_1+x_2, y_1+y_2, t) \quad (7)$$

since  $\kappa$  is coupled with  $T$ , the first item

$$\frac{\partial \kappa}{\partial x}(x_1+x_2) \frac{\partial T}{\partial x}(x_1+x_2) \neq \frac{\partial \kappa}{\partial x}(x_1) \frac{\partial T}{\partial x}(x_1) + \frac{\partial \kappa}{\partial x}(x_2) \frac{\partial T}{\partial x}(x_2) \quad (8)$$

### time-invariant

Because only  $u$  and  $T$  is associated with time that no matter the system is homogeneous or not, it is time-invariant. (Proof as equation(6) and (7))

## Problem 2

Homogeneous model assumption implies the following:

- $l_x = l_y = 0$ .
- $\rho(x, y) = \rho$   $\kappa(x, y) = \kappa$  and  $c(x, y) = c$ , where  $\rho, c, \kappa$  are positive constants.

Applying the above assumptions to the model gives

$$\rho c \frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + u(x, y, t), \quad (9)$$

as the final equation.

Let the source term be zero, i.e.,  $u(x, y, t) = 0$ . Consider the function

$$T(x, y, t) = a(t) \psi(x) \phi(y), \quad (10)$$

where  $a(t)$ ,  $\psi(x)$  and  $\phi(y)$  are real-valued functions on  $\mathbb{R}$ ,  $[0, L_x]$  and  $[0, L_y]$  respectively.

Substituting (10) into (9),

$$\rho c \frac{\partial (a(t) \psi(x) \phi(y))}{\partial t} = \kappa \left( \frac{\partial^2 (a(t) \psi(x) \phi(y))}{\partial x^2} + \frac{\partial^2 (a(t) \psi(x) \phi(y))}{\partial y^2} \right). \quad (11)$$

This gives,

$$\rho c \psi(x) \phi(y) \frac{da(t)}{dt} = \kappa a(t) \left( \phi(y) \frac{d^2 \psi(x)}{dx^2} + \psi(x) \frac{d^2 \phi(y)}{dy^2} \right). \quad (12)$$

Dividing throughout by (10),

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} = \kappa \left( \frac{1}{\psi(x)} \frac{d^2\psi(x)}{dx^2} + \frac{1}{\phi(y)} \frac{d^2\phi(y)}{dy^2} \right). \quad (13)$$

Rearranging, gives,

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} - \kappa \left( \frac{1}{\psi(x)} \frac{d^2\psi(x)}{dx^2} + \frac{1}{\phi(y)} \frac{d^2\phi(y)}{dy^2} \right) = 0. \quad (14)$$

The three terms in the above equation are functions of the three independent variables  $x, y, t$ . So, in order for the above equation to be satisfied, each of the term must be constant. Consider a constant  $\alpha > 0$ . The three terms are then given as

$$\frac{\rho c}{a(t)} \frac{da(t)}{dt} = \alpha, \quad \frac{1}{\psi(x)} \frac{d^2\psi(x)}{dx^2} = \frac{1}{\phi(y)} \frac{d^2\phi(y)}{dy^2} = \frac{\alpha}{2\kappa}. \quad (15)$$

Upon simplifying, the following differential equations are obtained:

$$\frac{da(t)}{dt} - \lambda a(t) = 0, \quad (16a)$$

$$\frac{d^2\psi(x)}{dx^2} - \lambda_x \psi(x) = 0, \quad (16b)$$

$$\frac{d^2\phi(y)}{dy^2} - \lambda_y \phi(y) = 0, \quad (16c)$$

where  $\lambda = \frac{\alpha}{\rho c}$ ,  $\lambda_x = \lambda_y = \frac{\alpha}{2\kappa}$ .

Thus it can be seen that for equation(10) to be a solution to the PDE (9) when  $u(x, y, t) = 0$ ,  $a(t)$ ,  $\psi(x)$  and  $\phi(y)$  need to satisfy the equations (16a), (16b) and (16c) respectively.

## Problem 3

The set of square-integrable functions is given as:

$$L_2 = \{f : \int_{\Omega} f dx < \infty\} \quad (17)$$

## Problem 4

## References