MATHEMATICS

SECTION A

February 6, 2024

1 Vectors

1. Show that the points **A, B, C** with position vectors $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle.
2. Find the coordinates of the point where the line through the points $(3, -4, -5)$ and $(2, -3, 1)$, crosses the plane determined by the points $(1, 2, 3), (4, 2, -3)$ and $(0, 4, 3)$.

2 Linear Forms

3. The x-coordinate of a point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4 . Find its z-coordinate.
4. Find the value of x such that the points $A(3, 2, 1), B(4, x, 5), C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar.
5. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

3 Probability

6. A die, whose faces are marked $1, 2, 3$ in red and $4, 5, 6$ in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events.
7. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X.
8. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer.

4 Optimization

9. Two tailors, A and B, earn ₹300 and ₹400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP.
10. Solve the following linear programming problem graphically :
    Maximise $Z = 34x + 45y$
    under the following constraints

$$\begin{aligned} x + y &\leq 300 \\ 2x + 3y &\leq 70 \\ x \geq 0, y &\geq 0 \end{aligned}$$

5 Geometry

11. AB is the diameter of a circle and C is any point on the circle. Show that the area of triangle ABC is maximum, when it is an isosceles triangle.

12. A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$.

6 Differentiation

13. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R.

14. The length x, of a rectangle is decreasing at the rate of 5 cm minute and the width y, is increasing at the rate of 4 $cm/$ minute. When $x = 8$ cm and $y = 6$ cm, find the rate of change of the area of the rectangle.

15. Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = (x + 2y)$, given that $y = 0$ when $x = 1$.

16. Find the general solution of the differential equation

$$ydx - \left(x + 2y^2\right) dy = 0.$$

17. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

18. If $e^y (x + 1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

7 Integration

19. Find :

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$$

20. Find :

$$\int \frac{dx}{5 - 8x - x^2}$$

21. Evaluate :

$$\int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx$$

22. Evaluate :

$$\int_1^4 \{|x - 1| + |x - 2| + |x - 4|\} dx$$

23. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$.

24. Find the area enclosed between the parabola $4y = 3x^2$ and line $3x - 2y + 12 = 0$.

25. Find :

$$\int \frac{\sin\theta \, d\theta}{(4 + \cos^2\theta)\left(2 - \sin^2\theta\right)}$$

8 Functions

26. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $\left[-\sqrt{3}, 0\right]$.

27. Consider $f : R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ given by $f(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}(0)$ and x such that $\mathrm{f}^{-1}(x) = 2$.

28. Let $A = Q \times Q$ and let $*$ be a binary operation on A defined by $(a,b) * (c,d) = (ac, b + ad)$ for $(a,b), (c,d) \in A$. Determine, whether $*$ is commutative and associative. Then, with respect to $*$ on A.
    (a) Find the identity element in A.
    (b) Find the invertible elements of A.

9 Matrices

29. If for any 2×2 square matrix A, $A\,(Adj A) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$,then write the value of det A.

30. If A is a skew-symmetric matrix of order 3 , then prove that $\left|A\right| = 0$.

31. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

32. Find matrix A such that

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$$

33. If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find A^{-1}. Hence using A^{-1} solve the system of equations

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x + 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$

10 Trigonometry

34. If $\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \frac{\pi}{4}$, then find the value of x.

11 Limits and Continuity

35. Determine the value of 'k' for which the following function is continuous at $x = 3$:

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3} & , x \neq 3 \\ k & , x = 3 \end{cases}$$