PHY114: QUANTUM PHYSICS (2024-25, 2nd Semester)
Homework-10

Q.1: The Bloch theorem is commonly stated in two ways (i) $\psi_k(x+a) = e^{ika}\psi_k(x)$ and (ii) $\psi_k(x) = e^{ikx}u_k(x)$. Here a is the lattice period and $u_k(x)$ is a function with periodicity same as the lattice. Prove that these two forms imply each other.

Q.2: Consider a periodic potential with total number of periodic lattices (N) as ten (10) so the total size of this 1-D crystal is $L = 10a$. **(a)** What are the possible allowed values of k according to the periodic boundary conditions? **(b)** Show that total number of discrete k-states in an interval Δk scales with length L in large L limit.

Q.3: (a) Recall Q.2 of HW07 on two ($n = 2$) negative δ-functions separated by $2a$ and at $x = \pm a$. Now generalize this to n negative δ-functions at $x_m = (2m - n - 1)a$ with $m = 1, \ldots n$. Plot these potentials together with the n lowest energy eigen state wave-functions qualitatively for to $n = 2, 3, 4$.
(b) Now recall Q.1 of HW07 on one ($n = 1$) positive δ-function at the center of the infinite well (from $-L/2$ to $+L/2$). Now generalize this to n positive δ-functions in the well at $x_m = -\frac{L}{2} + \frac{L}{n+1}$ with $m = 1, \ldots n$. Plot these potentials together with the $(n+1)$ lowest energy eigen state wave-functions qualitatively for $n = 1, 2$. Discuss how the energy separation of these states varies with δ-function strength, i.e. γ.

Q.4: Consider the the positive δ-function Kronig-Penney (K-P) model in two extreme limits: $P \to 0$ and $P \to \infty$ limits. Find and plot E vs k in these two limits. Plot also for an intermediate P value. Note that $P \to 0$ limit corresponds to a free particle while $P \to \infty$ limit corresponds to isolated infinite potential wells. Discuss how these E vs k plots change with increasing P.

Q.5: Consider the positive δ-function Kronig-Penney (K-P) model.
(a) What is the value of k for highest and lowest energy states for different bands in the K-P model?
(b) Plot the lowest and highest energy wave functions for the lowest three energy bands in the K-P model.

Q.6: The positive δ-function Kronig-Penney (K-P) model's transcendental equation works out as:

$$\cos ka = \frac{P}{\lambda a}\sin \lambda a + \cos \lambda a$$

With $E = \frac{\hbar^2\lambda^2}{2m}$ and $P = \frac{m\gamma a}{\hbar^2}$. Simplify the above equation for $k = 0$ and $k = \pi/a$ values. For each case one gets an easy solution for λa and a transcendental equation for the other. Find these and discuss the maximum and minimum energy of the bands from this in large P limit.

Q.7*: Recall that in positive δ-function K-P model transcendental equation plot, the allowed segments of this plot are almost straight lines for large enough P and between ± 1. Assuming these as exact straight lines argue that E Vs k has a parabolic nature near the top and bottom of each band and thus $\frac{dE}{dk}$ vanishes at these k-points.