PHY114: QUANTUM PHYSICS (2024-25, 2nd Semester)
Homework-12

Q.1: Consider the three angular momentum operators $L_x = yp_z - zp_y$ and $L_y = zp_x - xp_z$ and $L_z = xp_y - yp_x$ in terms of the position operators x, y and z and the momentum operators p_x, p_y and p_z.
(a) Argue that L_x, L_y and L_z are Hermitian and show that $\left[L_x, L_y\right] = i\hbar L_z$.
(b) Using L_x, L_y and L_z commutators show that the operator $L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with L_x.
(c) Show that $Y_{lm}(\theta, \phi)$ is an eigen state of the operator $L_x^2 + L_y^2$. Find the corresponding eigen value? Argue that in state $Y_{lm}(\theta, \phi)$ the vector $\vec{L}$ makes an angle α with the xy-plane, given by $\sin\alpha = \frac{m}{\sqrt{l(l+1)}}$. What are different α values for $l = 1$?
(d) Given an eigen-state of L_z, i.e. $L_z|m\rangle = m\hbar|m\rangle$, and relation $\left[L_y, L_z\right] = i\hbar L_x$, prove that the expectation value of L_x over state $|m\rangle$ is zero.

Q.2: (a) The spin angular momentum of a spin-1/2 particle is given by $\vec{S} = (\hbar/2)\,\vec{\sigma}$ with the Pauli matrices given by: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
(a) Show that the Pauli matrices satisfy the commutation relation $\left[\sigma_x, \sigma_y\right] = 2i\sigma_z$ and its cyclic permutations. What can you conclude about the commutator $[S_x, S_y]$ from this?
(b) Find σ_x^2, σ_y^2 and σ_z^2 and thus the operator S^2 in matrix form.
(c) Show that the expectation values of S_x and S_y are zero for both the eigen-states of S_z. What are the expectation values of S_x^2 and S_y^2 for these eigen states?

Q.3: A particle is known to be in a state given by the wave-function $\psi(\vec{r}) = N(r + x + y + z)e^{-r^2/a^2}$, where N is a normalization constant.
(a) Write this wave-function in terms of spherical harmonics and an r dependent function. Suppose L^2 is measured for this state, what are the possible results of the measurements and with what probabilities?
(b) What are the possible results when L_z is measured for above state and with what probabilities?

Q.4: (a) What is the Bohr Magneton value in SI units for the electron and the proton?
(b) What is the magnetic potential energy of an electron with its spin magnetic moment aligned anti-parallel to a magnetic field of 100 G (i.e. 0.01 Tesla)?
(c) What is the Larmour precession frequency of the electron's spin in this magnetic field?

Q.5: Consider an angular momentum l system with gyromagnetic ratio γ (i.e. $\vec{\mu} = \gamma\vec{L}$) in a magnetic field $\vec{B} = B\hat{z}$ leading to the Zeeman Hamiltonian $H = -\vec{\mu}_l.\vec{B} = -\gamma B L_z$.
(a) Use the Heisenberg equation of motion to find how the expectation values of the three components of $\vec{\mu}$ evolve with time.
(b) Argue that these agree with the classical equations of motion for these components and amount to precession. What is the precession frequency?

Q.6: Two orthonormal states $|\pm n\rangle$ of a spin-1/2 particle in $|\pm z\rangle$ (or $|\uparrow\rangle$ and $|\downarrow\rangle$) basis are given as $|+n\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$ and $|-n\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix}$ together with an operator $S_n = \frac{\hbar}{2}\hat{n}.\vec{\sigma}$. Here $\hat{n} =$

$\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$ and $\vec{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$ with σ_x, σ_y and σ_z as the three Pauli matrices.

(a) Find the expectation values of $S_x = \frac{\hbar}{2}\sigma_x$, S_y and S_z for the state $|+n\rangle$.

(b) Use the explicit matrix form of Pauli matrices to find S_n in matrix form and simplify it.

(c) Prove that $|\pm n\rangle$ states are eigen states of S_n operator. What are the corresponding eigen values?

(d) Let us construct an operator U using the $|\pm n\rangle$ states as $U = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2}e^{-i\phi} \end{pmatrix}$. Prove that U is unitary, i.e. $UU^\dagger$ is identity operator.

(e) U represents the transformation from the $|\pm z\rangle$ basis to the new $|\pm n\rangle$ basis. Find the $|\pm n\rangle$ states and the operator S_n in the new basis, i.e. $U|\pm n\rangle$ and $US_nU^\dagger$ in matrix form.

Q.7: The Hamiltonian of a proton with spin-1/2 having magnetic moment $\vec{\mu} = g_p\mu_N\vec{S}/\hbar$ in a magnetic field $B\hat{k}$ is given by $H = -\frac{g_p\mu_N}{\hbar}\vec{S}.\vec{B} = -\frac{g_p\mu_N}{2}B\sigma_z$.

(a) Show that the time dependent spin state given by $|\chi(t)\rangle = \begin{pmatrix} e^{i\omega_0 t/2}\cos\frac{\theta}{2} \\ e^{i(\phi_0-\omega_0 t/2)}\sin\frac{\theta}{2} \end{pmatrix}$ satisfies the time dependent Schrodinger equation, i.e. $H|\chi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\chi(t)\rangle$ with a suitable ω_0.

(b) Interpret the time evolution of state $|\chi(t)\rangle$ physically. For this you may find the expectation values of the three magnetic moment (or spin) components for this time dependent state. Also, the previous problem provides some hints for this.