PHY114: QUANTUM PHYSICS (2024-25, 2nd Semester)
Homework-08

Q.1: Use the bra-ket algebra to prove that:
(a) For any operator A, $\sum_{ij}\left|\langle u_i|A|u_j\rangle\right|^2 = \mathrm{Tr}(AA^\dagger)$.
(b) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
(c) the trace of an operator A (i.e. sum of diagonal elements of A) is same in two different orthonormal bases, i.e., the trace of a matrix operator is invariant under a change of basis.

Q.2: Given that A and B are two Hermitian operators, find if the following operators are Hermitian or not.
(a) AB **(b)** A^n (n is a positive integer) **(c)** $AB + BA$ **(d)** $-i[A, B]$

Q.3: Prove that
(a) If A is Hermitian, e^{iA} is unitary.
(b) $[A, B]^\dagger = [B^\dagger, A^\dagger]$
(c) $[A, BC] = [A, B]C + B[A, C]$
(d) Product of two unitary operators is unitary.
(e) x and p_x operators are Hermitian over the vector space of valid wave functions, i.e. the functions that are normalizable and thus vanish at infinity.

Q.4: Π is the parity operator, i.e., $\Pi\psi(x) = \langle x|\Pi|\psi\rangle = \langle -x|\psi\rangle = \psi(-x)$. Prove that
(a) Π^2 is identity operator. **(b)** Π is Hermitian **(c)** Π is Unitary
(d) What can you say about the eigen-values and eigen functions of Π?

Q.5: Given a Hermitian matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
(a) Find its eigen values and eigen vectors.
(b) Find the unitary transformation matrix S such that $SAS^\dagger$ is diagonal and verify this explicitly.

Q.6: In the basis set $\{|u_i\rangle\}$ of a 3D Hilbert space, two state-vectors are given as $|\phi\rangle = 3|u_1\rangle + i|u_2\rangle - |u_3\rangle$ and $|\psi\rangle = -i|u_1\rangle + |u_2\rangle + 2i|u_3\rangle$ and three operators are: $A = |\phi\rangle\langle\phi|$, $B = |\psi\rangle\langle\psi|$ and $C = |\phi\rangle\langle\psi|$.
(a) Write $|\phi\rangle$ and $|\psi\rangle$ and their Hermitian conjugates, i.e. $\langle\phi|$ and $\langle\psi|$ in matrix form.
(b) Find A, B and C is matrix form by multiplying appropriate matrices.
(c) Find if A, B and C are Hermitian. Do this by using both matrix form and their symbolic bra-ket form.
(d) Do A, B and C commute with each other?
(e) Show that $|\phi\rangle$ is an eigen-vector of A. What is the corresponding eigen value?
(f) Find the other two eigen-values and eigen vectors of A.
(g) Find the unitary matrix to transform to eigen-basis of A and find A, B and C in this eigen-basis.

Q.7: For a 3-dimensional vector space (Hilbert space) an orthonormal set of basis states is given by $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$. Three operators, A_1, A_2 and A_3, when operating on these basis states satisfy $A_j|u_k\rangle = i\sum_l \epsilon_{jkl}|u_l\rangle$. Here ϵ_{ijk} is ±1 for ijk as even, odd permutations of 123, respectively, and zero otherwise.
(a) Find the three operators in matrix form. Are these operators Hermitian?
(b) Do these operators commute?
(c) Find the eigen values and eigen vectors of operator $B = A_1 + A_2 + A_3$. What is the operator B in its eigen basis?

Q.8: Consider an operator $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(a) Prove that it is unitary.
(b) Find its eigen values and eigen vectors.
(c) Unitarity implies that it can be written as $A = e^{iB}$ with B as Hermitian. Find such an operator B. Is this B unique? [Hint: look at the operator in its eigen basis]

Q.9: T_a is the translation-by-a operator in 1-D, i.e. $T_a\psi(x) = \psi(x+a)$, and H is a Hamiltonian with a periodic 1-D potential with periodicity a, i.e. $H = \frac{p_x^2}{2m} + V(x)$ with $V(x+a) = V(x)$.
(a) Prove that H commutes with T_a.
(b) Is T_a Hermitian? [Hint: analyze a general matrix element T_a]
(c) Show that T_a is unitary, i.e. $T_a T_a^\dagger = 1$.
(d) What are the possible eigen values of T_a?

Q.10: Consider the TDSE, $H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$ for a time independent Hamiltonian H. By solving the eigen value equation one finds the eigen energies as $\{E_n\}$ with eigen states $\{|u_n\rangle\}$ without any degeneracy.
(a) Show that $|\psi(t)\rangle = S|\psi(0)\rangle$, where $S = e^{-\frac{iHt}{\hbar}}$ is a unitary operator.
(b) If one defines a time dependent expectation value of an operator A as $\langle A(t)\rangle = \langle\psi(t)|A|\psi(t)\rangle$, show that $\langle A(t)\rangle = \langle\psi(0)|S^\dagger AS|\psi(0)\rangle$.
(c) Finally, prove the Heisenberg equation of motion, i.e. $\frac{\partial}{\partial t}\langle A(t)\rangle = -\frac{i}{\hbar}\langle[A,H]\rangle$.

Q.11: The Hamiltonian of a system in an orthonormal basis $|u_{1,2}\rangle$ is, $H = \frac{\hbar\omega}{2}(i|u_1\rangle\langle u_2| - i|u_2\rangle\langle u_1|)$.
(a) Write H in the matrix form in basis $|u_{1,2}\rangle$ and verify that H is Hermitian.
(b) Show that $\frac{1}{\sqrt{2}}(|u_1\rangle \pm i|u_2\rangle)$ are two normalized eigen-states of this H. What are the corresponding Eigen energies?
(c) What is the unitary matrix S such that $SHS^\dagger$ is diagonal.
(d) Given that the state of the system at $t = 0$ is, $|\psi(t=0)\rangle = |u_1\rangle$; find the state $|\psi(t)\rangle$ at later time t in $|u_{1,2}\rangle$ basis.
(e) What is the probability of finding the system in the state $|u_2\rangle$ at a later time t. What is the average energy, i.e. $\langle H\rangle_\psi$, of this state?
(f) Given operators $B = (|u_1\rangle\langle u_2| + |u_2\rangle\langle u_1|)$ and $C = (|u_1\rangle\langle u_1| - |u_2\rangle\langle u_2|)$, find $\langle B(t)\rangle_\psi = \langle\psi(t)|B|\psi(t)\rangle$ and $\langle C(t)\rangle_\psi$ for the above state $|\psi(t)\rangle$.