



### Problem 1: Multiples of 3 and 5

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23.  
Find the sum of all the multiples of 3 or 5 below 1000.

### Problem 2: Even Fibonacci numbers

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.

### Problem 3: Largest prime factor

The prime factors of 13195 are 5, 7, 13 and 29.

What is the largest prime factor of the number 600851475143 ?

### Problem 4: Largest palindrome product

A palindromic number reads the same both ways. The largest palindrome made from the product of two 2-digit numbers is  $9009 = 91 \times 99$ .

Find the largest palindrome made from the product of two 3-digit numbers.

### Problem 5: Smallest multiple

2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder.

What is the smallest positive number that is evenly divisible by all of the numbers from 1 to 20?

### Problem 6: Sum square difference

The sum of the squares of the first ten natural numbers is,

$$1^2 + 2^2 + \dots + 10^2 = 385$$

The square of the sum of the first ten natural numbers is,

$$(1 + 2 + \dots + 10)^2 = 55^2 = 3025$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is  $3025 - 385 = 2640$ .

Find the difference between the sum of the squares of the first one hundred natural numbers and the square of the sum.

### Problem 7: 10001st prime

By listing the first six prime numbers: 2, 3, 5, 7, 11, and 13, we can see that the 6th prime is 13.

What is the 10 001st prime number?

### Problem 8: Largest product in a series

The four adjacent digits in the 1000-digit number that have the greatest product are  $9 \times 9 \times 8 \times 9 = 5832$ .

```
73167176531330624919225119674426574742355349194934
96983520312774506326239578318016984801869478851843
85861560789112949495459501737958331952853208805511
12540698747158523863050715693290963295227443043557
66896648950445244523161731856403098711121722383113
62229893423380308135336276614282806444486645238749
30358907296290491560440772390713810515859307960866
70172427121883998797908792274921901699720888093776
65727333001053367881220235421809751254540594752243
52584907711670556013604839586446706324415722155397
53697817977846174064955149290862569321978468622482
83972241375657056057490261407972968652414535100474
82166370484403199890008895243450658541227588666881
16427171479924442928230863465674813919123162824586
17866458359124566529476545682848912883142607690042
24219022671055626321111109370544217506941658960408
07198403850962455444362981230987879927244284909188
84580156166097919133875499200524063689912560717606
05886116467109405077541002256983155200055935729725
71636269561882670428252483600823257530420752963450
```

Find the thirteen adjacent digits in the 1000-digit number that have the greatest product. What is the value of this product?

### Problem 9: Special Pythagorean triplet

A Pythagorean triplet is a set of three natural numbers,  $a < b < c$ , for which,

$$a^2 + b^2 = c^2$$

For example,  $3^2 + 4^2 = 9 + 16 = 25 = 5^2$ .

There exists exactly one Pythagorean triplet for which  $a + b + c = 1000$ .

Find the product  $abc$ .

### Problem 10: Summation of primes

The sum of the primes below 10 is  $2 + 3 + 5 + 7 = 17$ .

Find the sum of all the primes below two million.

Problem 11: Largest product in a grid

In the 20×20 grid below, four numbers along a diagonal line have been marked in red.

08	02	22	97	38	15	00	40	00	75	04	05	07	78	52	12	50	77	91	08
49	49	99	40	17	81	18	57	60	87	17	40	98	43	69	48	04	56	62	00
81	49	31	73	55	79	14	29	93	71	40	67	53	88	30	03	49	13	36	65
52	70	95	23	04	60	11	42	69	24	68	56	01	32	56	71	37	02	36	91
22	31	16	71	51	67	63	89	41	92	36	54	22	40	40	28	66	33	13	80
24	47	32	60	99	03	45	02	44	75	33	53	78	36	84	20	35	17	12	50
32	98	81	28	64	23	67	10	26	38	40	67	59	54	70	66	18	38	64	70
67	26	20	68	02	62	12	20	95	63	94	39	63	08	40	91	66	49	94	21
24	55	58	05	66	73	99	26	97	17	78	78	96	83	14	88	34	89	63	72
21	36	23	09	75	00	76	44	20	45	35	00	61	33	97	34	31	33	95	
78	17	53	28	22	75	31	67	15	94	03	80	04	62	16	14	09	53	56	92
16	39	05	42	96	35	31	47	55	58	88	24	00	17	54	24	36	29	85	57
86	56	00	48	35	71	89	07	05	44	44	37	44	60	21	58	51	54	17	58
19	80	81	68	05	94	47	69	28	73	92	13	86	52	17	77	04	89	55	40
04	52	08	83	97	35	99	16	07	97	57	32	16	26	26	79	33	27	98	66
88	36	68	87	57	62	20	72	03	46	33	67	46	55	12	32	63	93	53	69
04	42	16	73	38	25	39	11	24	94	72	18	08	46	29	32	40	62	76	36
20	69	36	41	72	30	23	88	34	62	99	69	82	67	59	85	74	04	36	16
20	73	35	29	78	31	90	01	74	31	49	71	48	86	81	16	23	57	05	54
01	70	54	71	83	51	54	69	16	92	33	48	61	43	52	01	89	19	67	48

The product of these numbers is  $26 \times 63 \times 78 \times 14 = 1788696$ .

What is the greatest product of four adjacent numbers in the same direction (up, down, left, right, or diagonally) in the 20×20 grid?

Problem 12: Highly divisible triangular number

The sequence of triangle numbers is generated by adding the natural numbers. So the  $n^{\text{th}}$  triangle number would be  $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ . The first ten terms would be:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

Let us list the factors of the first seven triangle numbers:

- 1: 1
- 3: 1,3
- 6: 1,2,3,6
- 10: 1,2,5,10
- 15: 1,3,5,15
- 21: 1,3,7,21
- 28: 1,2,4,7,14,28

We can see that 28 is the first triangle number to have over five divisors.

What is the value of the first triangle number to have over five hundred divisors?

Problem 13: Large sum

Work out the first ten digits of the sum of the following one-hundred 50-digit numbers.

37107287533902102798797998220837590246510135740250
46376937677490009712648124896970078050417018260538
74324986199524741059474233309513058123726617309629
91942213363574161572522430563301811072406154908250
23067588207539346171171980310421047513778063246676
89261670696623633820136378418383684178734361726757
28112879812849979408065481931592621691275889832738
44274228917432520321923589422876796487670272189318
47451445736001306439091167216856844588711603153276
70386486105843025439939619828917593665686757934951
62176457141856560629502157223196586755079324193331
64906352462741904929101432445813822663347944758178
92575867718337217661963751590579239728245598838407
58203565325359399008402633568948830189458628227828
80181199384826282014278194139940567587151170094390
35398664372827112653829987240784473053190104293586
86515506006295864861532075273371959191420517255829
71693888707715466499115593487603532921714970056938
54370070576826684624621495650076471787294438377604
53282654108756828443191190634694037855217779295145
36123272525000296071075082563815656710885258350721
45876576172410976447339110607218265236877223636045
17423706905851860660448207621209813287860733969412

81142660418086830619328460811191061556940512689692  
51934325451728388641918047049293215058642563049483  
62467221648435076201727918039944693004732956340691  
15732444386908125794514089057706229429197107928209  
55037687525678773091862540744969844508330393682126  
18336384825330154686196124348767681297534375946515  
80386287592878490201521685554828717201219257766954  
78182833757993103614740356856449095527097864797581  
16726320100436897842553539920931837441497806860984  
48403098129077791799088218795327364475675590848030  
87086987551392711854517078544161852424320693150332  
59959406895756536782107074926966537676326235447210  
69793950679652694742597709739166693763042633987085  
41052684708299085211399427365734116182760315001271  
65378607361501080857009149939512557028198746004375  
35829035317434717326932123578154982629742552737307  
94953759765105305946966067683156574377167401875275  
88902802571733229619176668713819931811048770190271  
25267680276078003013678680992525463401061632866526  
36270218540497705585629946580636237993140746255962  
24074486908231174977792365466257246923322810917141  
9143028819710328859780666976089293863828502533403  
34413065578016127815921815005561868836468420090470  
23053081172816430487623791969842487255036638784583  
11487696932154902810424020138335124462181441773470  
63783299490636259666498587618221225225512486764533  
67720186971698544312419572409913959008952310058822  
95548255300263520781532296796249481641953868218774  
76085327132285723110424803456124867697064507995236  
37774242535411291684276865538926205024910326572967  
23701913275725675285653248258265463092207058596522  
29798860272258331913126375147341994889534765745501  
18495701454879288984856827726077713721403798879715  
38298203783031473527721580348144513491373226651381  
34829543829199918180278916522431027392251122869539  
40957953066405232632538044100059654939159879593635  
29746152185502371307642255121183693803580388584903  
41698116222072977186158236678424689157993532961922  
62467957194401269043877107275048102390895523597457  
23189706772547915061505504953922979530901129967519  
86188088225875314529584099251203829009407770775672  
11306739708304724483816533873502340845647058077308  
82959174767140363198008187129011875491310547126581  
97623331044818386269515456334926366572897563400500  
42846280183517070527831839425882145521227251250327  
55121603546981200581762165212827652751691296897789  
32238195734329339946437501907836945765883352399886  
75506164965184775180738168837861091527357929701337  
62177842752192623401942399639168044983993173312731  
32924185707147349566916674687634660915035914677504  
99518671430235219628894890102423325116913619626622  
73267460800591547471830798392868535206946944540724  
76841822524674417161514036427982273348055556214818  
97142617910342598647204516893989422179826088076852  
87783646182799346313767754307809363333018982642090  
10848802521674670883215120185883543223812876952786  
71329612474782464538636993009049310363619763878039  
62184073572399794223406235393808339651327408011116  
66627891981488087797941876876144230030984490851411  
60661826293682836764744779239180335110989069790714  
85786944089552990653640447425576083659976645795096  
66024396409905389607120198219976047599490197230297  
64913982680032973156037120041377903785566085089252  
16730939319872750275468906903707539413042652315011  
94809377245048795150954100921645863754710598436791  
78639167021187492431995700641917969777599028300699  
15368713711936614952811305876380278410754449733078  
40789923115535562561142322423255033685442488917353  
44889911501440648020369068063960672322193204149535  
41503128880339536053299340368006977710650566631954  
81234880673210146739058568557934581403627822703280  
82616570773948327592232845941706525094512325230608  
22918802058777319719839450180888072429661980811197  
77158542502016545090413245809786882778948721859617  
72107838435069186155435662884062257473692284509516  
20849603980134001723930671666823555245252804609722  
53503534226472524250874054075591789781264330331690

#### Problem 14: Longest Collatz sequence

The following iterative sequence is defined for the set of positive integers:

$$n \rightarrow n/2 \text{ (} n \text{ is even)}$$

$$n \rightarrow 3n + 1 \text{ (} n \text{ is odd)}$$

Using the rule above and starting with 13, we generate the following sequence:

$$13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

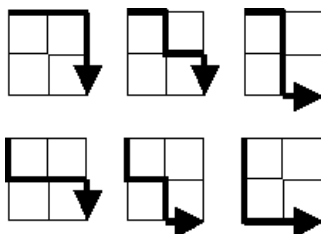
It can be seen that this sequence (starting at 13 and finishing at 1) contains 10 terms. Although it has not been proved yet (Collatz Problem), it is thought that all starting numbers finish at 1.

Which starting number, under one million, produces the longest chain?

**NOTE:** Once the chain starts the terms are allowed to go above one million.

#### Problem 15: Lattice paths

Starting in the top left corner of a  $2 \times 2$  grid, and only being able to move to the right and down, there are exactly 6 routes to the bottom right corner.



How many such routes are there through a  $20 \times 20$  grid?

#### Problem 16: Power digit sum

$2^{15} = 32768$  and the sum of its digits is  $3 + 2 + 7 + 6 + 8 = 26$ .

What is the sum of the digits of the number  $2^{1000}$ ?

#### Problem 17: Number letter counts

If the numbers 1 to 5 are written out in words: one, two, three, four, five, then there are  $3 + 3 + 5 + 4 + 4 = 19$  letters used in total.

If all the numbers from 1 to 1000 (one thousand) inclusive were written out in words, how many letters would be used?

**NOTE:** Do not count spaces or hyphens. For example, 342 (three hundred and forty-two) contains 23 letters and 115 (one hundred and fifteen) contains 20 letters. The use of "and" when writing out numbers is in compliance with British usage.

### Problem 18: Maximum path sum I

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.

```
      3
     7 4
    2 4 6
   8 5 9 3
```

That is,  $3 + 7 + 4 + 9 = 23$ .

Find the maximum total from top to bottom of the triangle below:

```
          75
        95 64
       17 47 82
      18 35 87 10
     20 04 82 47 65
    19 01 23 75 03 34
   88 02 77 73 07 63 67
  99 65 04 28 06 16 70 92
 41 41 26 56 83 40 80 70 33
41 48 72 33 47 32 37 16 94 29
53 71 44 65 25 43 91 52 97 51 14
70 11 33 28 77 73 17 78 39 68 17 57
91 71 52 38 17 14 91 43 58 50 27 29 48
63 66 04 68 89 53 67 30 73 16 69 87 40 31
04 62 98 27 23 09 70 98 73 93 38 53 60 04 23
```

**NOTE:** As there are only 16384 routes, it is possible to solve this problem by trying every route. However, [Problem 67](#), is the same challenge with a triangle containing one-hundred rows; it cannot be solved by brute force, and requires a clever method! ;o)

### Problem 19: Counting Sundays

You are given the following information, but you may prefer to do some research for yourself.

- 1 Jan 1900 was a Monday.
- Thirty days has September,  
April, June and November.  
All the rest have thirty-one,  
Saving February alone,  
Which has twenty-eight, rain or shine.  
And on leap years, twenty-nine.
- A leap year occurs on any year evenly divisible by 4, but not on a century unless it is divisible by 400.

How many Sundays fell on the first of the month during the twentieth century (1 Jan 1901 to 31 Dec 2000)?

### Problem 20: Factorial digit sum

$n!$  means  $n \times (n - 1) \times \dots \times 3 \times 2 \times 1$

For example,  $10! = 10 \times 9 \times \dots \times 3 \times 2 \times 1 = 3628800$ ,  
and the sum of the digits in the number  $10!$  is  $3 + 6 + 2 + 8 + 8 + 0 + 0 = 27$ .

Find the sum of the digits in the number  $100!$

### Problem 21: Amicable numbers

Let  $d(n)$  be defined as the sum of proper divisors of  $n$  (numbers less than  $n$  which divide evenly into  $n$ ).  
If  $d(a) = b$  and  $d(b) = a$ , where  $a \neq b$ , then  $a$  and  $b$  are an amicable pair and each of  $a$  and  $b$  are called amicable numbers.

For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore  $d(220) = 284$ . The proper divisors of 284 are 1, 2, 4, 71 and 142; so  $d(284) = 220$ .

Evaluate the sum of all the amicable numbers under 10000.

### Problem 22: Names scores

Using `names.txt` (right click and 'Save Link/Target As...'), a 46K text file containing over five-thousand first names, begin by sorting it into alphabetical order. Then working out the alphabetical value for each name, multiply this value by its alphabetical position in the list to obtain a name score.

For example, when the list is sorted into alphabetical order, COLIN, which is worth  $3 + 15 + 12 + 9 + 14 = 53$ , is the 938th name in the list. So, COLIN would obtain a score of  $938 \times 53 = 49714$ .

What is the total of all the name scores in the file?

### Problem 23: Non-abundant sums

A perfect number is a number for which the sum of its proper divisors is exactly equal to the number. For example, the sum of the proper divisors of 28 would be  $1 + 2 + 4 + 7 + 14 = 28$ , which means that 28 is a perfect number.

A number  $n$  is called deficient if the sum of its proper divisors is less than  $n$  and it is called abundant if this sum exceeds  $n$ .

As 12 is the smallest abundant number,  $1 + 2 + 3 + 4 + 6 = 16$ , the smallest number that can be written as the sum of two abundant numbers is 24. By mathematical analysis, it can be shown that all integers greater than 28123 can be written as the sum of two abundant numbers. However, this upper limit cannot be reduced any further by analysis even though it is known that the greatest number that cannot be expressed as the sum of two abundant numbers is less than this limit.

Find the sum of all the positive integers which cannot be written as the sum of two abundant numbers.

### Problem 24: Lexicographic permutations

A permutation is an ordered arrangement of objects. For example, 3124 is one possible permutation of the digits 1, 2, 3 and 4. If all of the permutations are listed numerically or alphabetically, we call it lexicographic order. The lexicographic permutations of 0, 1 and 2 are:

012 021 102 120 201 210

What is the millionth lexicographic permutation of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9?

### Problem 25: 1000-digit Fibonacci number

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_1 = 1 \text{ and } F_2 = 1.$$

Hence the first 12 terms will be:

$F_1 = 1$   
 $F_2 = 1$   
 $F_3 = 2$   
 $F_4 = 3$   
 $F_5 = 5$   
 $F_6 = 8$   
 $F_7 = 13$   
 $F_8 = 21$   
 $F_9 = 34$   
 $F_{10} = 55$   
 $F_{11} = 89$   
 $F_{12} = 144$

The 12th term,  $F_{12}$ , is the first term to contain three digits.

What is the index of the first term in the Fibonacci sequence to contain 1000 digits?

### Problem 26: Reciprocal cycles

A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given:

$$\frac{1}{2} = 0.5$$

$$\frac{1}{3} = 0.(3)$$

$$\frac{1}{4} = 0.25$$

$$\frac{1}{5} = 0.2$$

$$\frac{1}{6} = 0.1(6)$$

$$\frac{1}{7} = 0.(142857)$$

$$\frac{1}{8} = 0.125$$

$$\frac{1}{9} = 0.(1)$$

$$\frac{1}{10} = 0.1$$

Where 0.1(6) means 0.166666..., and has a 1-digit recurring cycle. It can be seen that  $\frac{1}{7}$  has a 6-digit recurring cycle.

Find the value of  $d < 1000$  for which  $\frac{1}{d}$  contains the longest recurring cycle in its decimal fraction part.

### Problem 27: Quadratic primes

Euler discovered the remarkable quadratic formula:

$$n^2 + n + 41$$

It turns out that the formula will produce 40 primes for the consecutive integer values  $0 \leq n \leq 39$ . However, when  $n = 40$ ,  $40^2 + 40 + 41 = 40(40 + 1) + 41$  is divisible by 41, and certainly when  $n = 41$ ,  $41^2 + 41 + 41$  is clearly divisible by 41.

The incredible formula  $n^2 - 79n + 1601$  was discovered, which produces 80 primes for the consecutive values  $0 \leq n \leq 79$ . The product of the coefficients,  $-79$  and  $1601$ , is  $-126479$ .

Considering quadratics of the form:

$$n^2 + an + b, \text{ where } |a| < 1000 \text{ and } |b| \leq 1000$$

where  $|n|$  is the modulus/absolute value of  $n$

e.g.  $|11| = 11$  and  $|-4| = 4$

Find the product of the coefficients,  $a$  and  $b$ , for the quadratic expression that produces the maximum number of primes for consecutive values of  $n$ , starting with  $n = 0$ .

### Problem 28: Number spiral diagonals

Starting with the number 1 and moving to the right in a clockwise direction a 5 by 5 spiral is formed as follows:

```
21 22 23 24 25
20 7 8 9 10
19 6 1 2 11
18 5 4 3 12
17 16 15 14 13
```

It can be verified that the sum of the numbers on the diagonals is 101.

What is the sum of the numbers on the diagonals in a 1001 by 1001 spiral formed in the same way?



### Problem 29: Distinct powers

Consider all integer combinations of  $a^b$  for  $2 \leq a \leq 5$  and  $2 \leq b \leq 5$ :

$$2^2=4, 2^3=8, 2^4=16, 2^5=32$$

$$3^2=9, 3^3=27, 3^4=81, 3^5=243$$

$$4^2=16, 4^3=64, 4^4=256, 4^5=1024$$

$$5^2=25, 5^3=125, 5^4=625, 5^5=3125$$

If they are then placed in numerical order, with any repeats removed, we get the following sequence of 15 distinct terms:

$$4, 8, 9, 16, 25, 27, 32, 64, 81, 125, 243, 256, 625, 1024, 3125$$

How many distinct terms are in the sequence generated by  $a^b$  for  $2 \leq a \leq 100$  and  $2 \leq b \leq 100$ ?

### Problem 30: Digit fifth powers

Surprisingly there are only three numbers that can be written as the sum of fourth powers of their digits:

$$1634 = 1^4 + 6^4 + 3^4 + 4^4$$

$$8208 = 8^4 + 2^4 + 0^4 + 8^4$$

$$9474 = 9^4 + 4^4 + 7^4 + 4^4$$

As  $1 = 1^4$  is not a sum it is not included.

The sum of these numbers is  $1634 + 8208 + 9474 = 19316$ .

Find the sum of all the numbers that can be written as the sum of fifth powers of their digits.

### Problem 31: Coin sums

In England the currency is made up of pound, £, and pence, p, and there are eight coins in general circulation:

1p, 2p, 5p, 10p, 20p, 50p, £1 (100p) and £2 (200p).

It is possible to make £2 in the following way:

$$1 \times £1 + 1 \times 50p + 2 \times 20p + 1 \times 5p + 1 \times 2p + 3 \times 1p$$

How many different ways can £2 be made using any number of coins?

### Problem 32: Pandigital products

We shall say that an  $n$ -digit number is pandigital if it makes use of all the digits 1 to  $n$  exactly once; for example, the 5-digit number, 15234, is 1 through 5 pandigital.

The product 7254 is unusual, as the identity,  $39 \times 186 = 7254$ , containing multiplicand, multiplier, and product is 1 through 9 pandigital.

Find the sum of all products whose multiplicand/multiplier/product identity can be written as a 1 through 9 pandigital.

HINT: Some products can be obtained in more than one way so be sure to only include it once in your sum.

### Problem 33: Digit cancelling fractions

The fraction  $\frac{49}{98}$  is a curious fraction, as an inexperienced mathematician in attempting to simplify it may incorrectly believe that  $\frac{49}{98} = \frac{4}{8}$ , which is correct, is obtained by cancelling the 9s.

We shall consider fractions like,  $\frac{30}{50} = \frac{3}{5}$ , to be trivial examples.

There are exactly four non-trivial examples of this type of fraction, less than one in value, and containing two digits in the numerator and denominator.

If the product of these four fractions is given in its lowest common terms, find the value of the denominator.

### Problem 34: Digit factorials

145 is a curious number, as  $1! + 4! + 5! = 1 + 24 + 120 = 145$ .

Find the sum of all numbers which are equal to the sum of the factorial of their digits.

Note: as  $1! = 1$  and  $2! = 2$  are not sums they are not included.

### Problem 35: Circular primes

The number, 197, is called a circular prime because all rotations of the digits: 197, 971, and 719, are themselves prime.

There are thirteen such primes below 100: 2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, and 97.

How many circular primes are there below one million?

### Problem 36: Double-base palindromes

The decimal number,  $585 = 1001001001_2$  (binary), is palindromic in both bases.

Find the sum of all numbers, less than one million, which are palindromic in base 10 and base 2.

(Please note that the palindromic number, in either base, may not include leading zeros.)

### Problem 37: Truncatable primes

The number 3797 has an interesting property. Being prime itself, it is possible to continuously remove digits from left to right, and remain prime at each stage: 3797, 797, 97, and 7. Similarly we can work from right to left: 3797, 379, 37, and 3.

Find the sum of the only eleven primes that are both truncatable from left to right and right to left.

NOTE: 2, 3, 5, and 7 are not considered to be truncatable primes.

### Problem 38: Pandigital multiples

Take the number 192 and multiply it by each of 1, 2, and 3:

$$\begin{aligned}192 \times 1 &= 192 \\192 \times 2 &= 384 \\192 \times 3 &= 576\end{aligned}$$

By concatenating each product we get the 1 to 9 pandigital, 192384576. We will call 192384576 the concatenated product of 192 and (1,2,3)

The same can be achieved by starting with 9 and multiplying by 1, 2, 3, 4, and 5, giving the pandigital, 918273645, which is the concatenated product of 9 and (1,2,3,4,5).

What is the largest 1 to 9 pandigital 9-digit number that can be formed as the concatenated product of an integer with (1,2, ...,  $n$ ) where  $n > 1$ ?

### Problem 39: Integer right triangles

If  $p$  is the perimeter of a right angle triangle with integral length sides,  $\{a,b,c\}$ , there are exactly three solutions for  $p = 120$ .

$\{20,48,52\}$ ,  $\{24,45,51\}$ ,  $\{30,40,50\}$

For which value of  $p \leq 1000$ , is the number of solutions maximised?

### Problem 40: Champernowne's constant

An irrational decimal fraction is created by concatenating the positive integers:

0.12345678910**1**112131415161718192021...

It can be seen that the 12<sup>th</sup> digit of the fractional part is 1.

If  $d_n$  represents the  $n^{\text{th}}$  digit of the fractional part, find the value of the following expression.

$$d_1 \times d_{10} \times d_{100} \times d_{1000} \times d_{10000} \times d_{100000} \times d_{1000000}$$

### Problem 41: Pandigital prime

We shall say that an  $n$ -digit number is pandigital if it makes use of all the digits 1 to  $n$  exactly once. For example, 2143 is a 4-digit pandigital and is also prime.

What is the largest  $n$ -digit pandigital prime that exists?

### Problem 42: Coded triangle numbers

The  $n^{\text{th}}$  term of the sequence of triangle numbers is given by  $t_n = \frac{1}{2}n(n+1)$ ; so the first ten triangle numbers are:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

By converting each letter in a word to a number corresponding to its alphabetical position and adding these values we form a word value. For example, the word value for SKY is  $19 + 11 + 25 = 55 = t_{10}$ . If the word value is a triangle number then we shall call the word a triangle word.

Using words.txt (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, how many are triangle words?

### Problem 43: Sub-string divisibility

The number, 1406357289, is a 0 to 9 pandigital number because it is made up of each of the digits 0 to 9 in some order, but it also has a rather interesting sub-string divisibility property.

Let  $d_1$  be the 1<sup>st</sup> digit,  $d_2$  be the 2<sup>nd</sup> digit, and so on. In this way, we note the following:

- $d_2d_3d_4=406$  is divisible by 2
- $d_3d_4d_5=063$  is divisible by 3
- $d_4d_5d_6=635$  is divisible by 5
- $d_5d_6d_7=357$  is divisible by 7
- $d_6d_7d_8=572$  is divisible by 11
- $d_7d_8d_9=728$  is divisible by 13
- $d_8d_9d_{10}=289$  is divisible by 17

Find the sum of all 0 to 9 pandigital numbers with this property.

### Problem 44: Pentagon numbers

Pentagonal numbers are generated by the formula,  $P_n=n(3n-1)/2$ . The first ten pentagonal numbers are:

1, 5, 12, 22, 35, 51, 70, 92, 117, 145, ...

It can be seen that  $P_4 + P_7 = 22 + 70 = 92 = P_8$ . However, their difference,  $70 - 22 = 48$ , is not pentagonal.

Find the pair of pentagonal numbers,  $P_j$  and  $P_k$ , for which their sum and difference are pentagonal and  $D = |P_k - P_j|$  is minimised; what is the value of  $D$ ?

### Problem 45: Triangular, pentagonal, and hexagonal

Triangle, pentagonal, and hexagonal numbers are generated by the following formulae:

Triangle  $T_n=n(n+1)/2$  1, 3, 6, 10, 15, ...

Pentagonal  $P_n=n(3n-1)/2$  1, 5, 12, 22, 35,  
...

Hexagonal  $H_n=n(2n-1)$  1, 6, 15, 28, 45,  
...

It can be verified that  $T_{285} = P_{165} = H_{143} = 40755$ .

Find the next triangle number that is also pentagonal and hexagonal.

### Problem 46: Goldbach's other conjecture

It was proposed by Christian Goldbach that every odd composite number can be written as the sum of a prime and twice a square.

$$9 = 7 + 2 \times 1^2$$

$$15 = 7 + 2 \times 2^2$$

$$21 = 3 + 2 \times 3^2$$

$$25 = 7 + 2 \times 3^2$$

$$27 = 19 + 2 \times 2^2$$

$$33 = 31 + 2 \times 1^2$$

It turns out that the conjecture was false.

What is the smallest odd composite that cannot be written as the sum of a prime and twice a square?

#### Problem 47: Distinct primes factors

The first two consecutive numbers to have two distinct prime factors are:

$$\begin{aligned}14 &= 2 \times 7 \\15 &= 3 \times 5\end{aligned}$$

The first three consecutive numbers to have three distinct prime factors are:

$$\begin{aligned}644 &= 2^2 \times 7 \times 23 \\645 &= 3 \times 5 \times 43 \\646 &= 2 \times 17 \times 19.\end{aligned}$$

Find the first four consecutive integers to have four distinct prime factors each. What is the first of these numbers?

#### Problem 48: Self powers

The series,  $1^1 + 2^2 + 3^3 + \dots + 10^{10} = 10405071317$ .

Find the last ten digits of the series,  $1^1 + 2^2 + 3^3 + \dots + 1000^{1000}$ .

#### Problem 49: Prime permutations

The arithmetic sequence, 1487, 4817, 8147, in which each of the terms increases by 3330, is unusual in two ways: (i) each of the three terms are prime, and, (ii) each of the 4-digit numbers are permutations of one another.

There are no arithmetic sequences made up of three 1-, 2-, or 3-digit primes, exhibiting this property, but there is one other 4-digit increasing sequence.

What 12-digit number do you form by concatenating the three terms in this sequence?

#### Problem 50: Consecutive prime sum

The prime 41, can be written as the sum of six consecutive primes:

$$41 = 2 + 3 + 5 + 7 + 11 + 13$$

This is the longest sum of consecutive primes that adds to a prime below one-hundred.

The longest sum of consecutive primes below one-thousand that adds to a prime, contains 21 terms, and is equal to 953.

Which prime, below one-million, can be written as the sum of the most consecutive primes?

#### Problem 51: Prime digit replacements

By replacing the 1<sup>st</sup> digit of the 2-digit number \*3, it turns out that six of the nine possible values: 13, 23, 43, 53, 73, and 83, are all prime.

By replacing the 3<sup>rd</sup> and 4<sup>th</sup> digits of 56\*\*3 with the same digit, this 5-digit number is the first example having seven primes among the ten generated numbers, yielding the family: 56003, 56113, 56333, 56443, 56663, 56773, and 56993. Consequently 56003, being the first member of this family, is the smallest prime with this property.

Find the smallest prime which, by replacing part of the number (not necessarily adjacent digits) with the same digit, is part of an eight prime value family.

#### Problem 52: Permuted multiples

It can be seen that the number, 125874, and its double, 251748, contain exactly the same digits, but in a different order.

Find the smallest positive integer,  $x$ , such that  $2x$ ,  $3x$ ,  $4x$ ,  $5x$ , and  $6x$ , contain the same digits.

### Problem 53: Combinatoric selections

There are exactly ten ways of selecting three from five, 12345:

123, 124, 125, 134, 135, 145, 234, 235, 245, and 345

In combinatorics, we use the notation,  ${}^5C_3 = 10$ .

In general,

$${}^nC_r = \frac{n!}{r!(n-r)!}, \text{ where } r \leq n, n! = n \times (n-1) \times \dots \times 3 \times 2 \times 1, \text{ and } 0! = 1.$$

It is not until  $n = 23$ , that a value exceeds one-million:  ${}^{23}C_{10} = 1144066$ .

How many, not necessarily distinct, values of  ${}^nC_r$ , for  $1 \leq n \leq 100$ , are greater than one-million?

### Problem 54: Poker hands

In the card game poker, a hand consists of five cards and are ranked, from lowest to highest, in the following way:

- **High Card:** Highest value card.
- **One Pair:** Two cards of the same value.
- **Two Pairs:** Two different pairs.
- **Three of a Kind:** Three cards of the same value.
- **Straight:** All cards are consecutive values.
- **Flush:** All cards of the same suit.
- **Full House:** Three of a kind and a pair.
- **Four of a Kind:** Four cards of the same value.
- **Straight Flush:** All cards are consecutive values of same suit.
- **Royal Flush:** Ten, Jack, Queen, King, Ace, in same suit.

The cards are valued in the order:

2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace.

If two players have the same ranked hands then the rank made up of the highest value wins; for example, a pair of eights beats a pair of fives (see example 1 below). But if two ranks tie, for example, both players have a pair of queens, then highest cards in each hand are compared (see example 4 below); if the highest cards tie then the next highest cards are compared, and so on.

Consider the following five hands dealt to two players:

Hand	Player 1	Player 2	Winner
1	5H 5C 6S 7S KD Pair of Fives	2C 3S 8S 8D TD Pair of Eights	Player 2
2	5D 8C 9S JS AC Highest card Ace	2C 5C 7D 8S QH Highest card Queen	Player 1
3	2D 9C AS AH AC Three Aces	3D 6D 7D TD QD Flush with Diamonds	Player 2
4	4D 6S 9H QH QC Pair of Queens Highest card Nine	3D 6D 7H QD QS Pair of Queens Highest card Seven	Player 1
5	2H 2D 4C 4D 4S Full House With Three Fours	3C 3D 3S 9S 9D Full House with Three Threes	Player 1

The file, `poker.txt`, contains one-thousand random hands dealt to two players. Each line of the file contains ten cards (separated by a single space): the first five are Player 1's cards and the last five are Player 2's cards. You can assume that all hands are valid (no invalid characters or repeated cards), each player's hand is in no specific order, and in each hand there is a clear winner.

How many hands does Player 1 win?

### Problem 55: Lychrel numbers

If we take 47, reverse and add,  $47 + 74 = 121$ , which is palindromic.

Not all numbers produce palindromes so quickly. For example,

$$\begin{aligned}349 + 943 &= 1292, \\1292 + 2921 &= 4213 \\4213 + 3124 &= 7337\end{aligned}$$

That is, 349 took three iterations to arrive at a palindrome.

Although no one has proved it yet, it is thought that some numbers, like 196, never produce a palindrome. A number that never forms a palindrome through the reverse and add process is called a Lychrel number. Due to the theoretical nature of these numbers, and for the purpose of this problem, we shall assume that a number is Lychrel until proven otherwise. In addition you are given that for every number below ten-thousand, it will either (i) become a palindrome in less than fifty iterations, or, (ii) no one, with all the computing power that exists, has managed so far to map it to a palindrome. In fact, 10677 is the first number to be shown to require over fifty iterations before producing a palindrome: 4668731596684224866951378664 (53 iterations, 28-digits).

Surprisingly, there are palindromic numbers that are themselves Lychrel numbers; the first example is 4994.

How many Lychrel numbers are there below ten-thousand?

NOTE: Wording was modified slightly on 24 April 2007 to emphasise the theoretical nature of Lychrel numbers.

### Problem 56: Powerful digit sum

A googol ( $10^{100}$ ) is a massive number: one followed by one-hundred zeros;  $100^{100}$  is almost unimaginably large: one followed by two-hundred zeros. Despite their size, the sum of the digits in each number is only 1.

Considering natural numbers of the form,  $a^b$ , where  $a, b < 100$ , what is the maximum digital sum?

### Problem 57: Square root convergents

It is possible to show that the square root of two can be expressed as an infinite continued fraction.

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}} = 1.414213\dots$$

By expanding this for the first four iterations, we get:

$$\begin{aligned}1 + \frac{1}{2} &= \frac{3}{2} = 1.5 \\1 + \frac{1}{2 + \frac{1}{2}} &= \frac{7}{5} = 1.4 \\1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} &= \frac{17}{12} = 1.41666\dots \\1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} &= \frac{41}{29} = 1.41379\dots\end{aligned}$$

The next three expansions are 99/70, 239/169, and 577/408, but the eighth expansion, 1393/985, is the first example where the number of digits in the numerator exceeds the number of digits in the denominator.

In the first one-thousand expansions, how many fractions contain a numerator with more digits than denominator?

### Problem 58: Spiral primes

Starting with 1 and spiralling anticlockwise in the following way, a square spiral with side length 7 is formed.

```
 37 36 35 34 33 32 31
38 17 16 15 14 13 30
39 18 5 4 3 12 29
40 19 6 1 2 11 28
41 20 7 8 9 10 27
42 21 22 23 24 25 26
43 44 45 46 47 48 49
```

It is interesting to note that the odd squares lie along the bottom right diagonal, but what is more interesting is that 8 out of the 13 numbers lying along both diagonals are prime; that is, a ratio of  $8/13 \approx 62\%$ .

If one complete new layer is wrapped around the spiral above, a square spiral with side length 9 will be formed. If this process is continued, what is the side length of the square spiral for which the ratio of primes along both diagonals first falls below 10%?

### Problem 59: XOR decryption

Each character on a computer is assigned a unique code and the preferred standard is ASCII (American Standard Code for Information Interchange). For example, uppercase A = 65, asterisk (\*) = 42, and lowercase k = 107.

A modern encryption method is to take a text file, convert the bytes to ASCII, then XOR each byte with a given value, taken from a secret key. The advantage with the XOR function is that using the same encryption key on the cipher text, restores the plain text; for example,  $65 \text{ XOR } 42 = 107$ , then  $107 \text{ XOR } 42 = 65$ .

For unbreakable encryption, the key is the same length as the plain text message, and the key is made up of random bytes. The user would keep the encrypted message and the encryption key in different locations, and without both "halves", it is impossible to decrypt the message.

Unfortunately, this method is impractical for most users, so the modified method is to use a password as a key. If the password is shorter than the message, which is likely, the key is repeated cyclically throughout the message. The balance for this method is using a sufficiently long password key for security, but short enough to be memorable.

Your task has been made easy, as the encryption key consists of three lower case characters. Using `cipher.txt` (right click and 'Save Link/Target As...'), a file containing the encrypted ASCII codes, and the knowledge that the plain text must contain common English words, decrypt the message and find the sum of the ASCII values in the original text.

### Problem 60: Prime pair sets

The primes 3, 7, 109, and 673, are quite remarkable. By taking any two primes and concatenating them in any order the result will always be prime. For example, taking 7 and 109, both 7109 and 1097 are prime. The sum of these four primes, 792, represents the lowest sum for a set of four primes with this property.

Find the lowest sum for a set of five primes for which any two primes concatenate to produce another prime.



### Problem 61: Cyclical figurate numbers

Triangle, square, pentagonal, hexagonal, heptagonal, and octagonal numbers are all figurate (polygonal) numbers and are generated by the following formulae:

Triangle	$P_{3,n}=n(n+1)/2$	1, 3, 6, 10, 15, ...
Square	$P_{4,n}=n^2$	1, 4, 9, 16, 25, ...
Pentagonal	$P_{5,n}=n(3n-1)/2$	1, 5, 12, 22, 35, ...
Hexagonal	$P_{6,n}=n(2n-1)$	1, 6, 15, 28, 45, ...
Heptagonal	$P_{7,n}=n(5n-3)/2$	1, 7, 18, 34, 55, ...
Octagonal	$P_{8,n}=n(3n-2)$	1, 8, 21, 40, 65, ...

The ordered set of three 4-digit numbers: 8128, 2882, 8281, has three interesting properties.

1. The set is cyclic, in that the last two digits of each number is the first two digits of the next number (including the last number with the first).
2. Each polygonal type: triangle ( $P_{3,127}=8128$ ), square ( $P_{4,91}=8281$ ), and pentagonal ( $P_{5,44}=2882$ ), is represented by a different number in the set.
3. This is the only set of 4-digit numbers with this property.

Find the sum of the only ordered set of six cyclic 4-digit numbers for which each polygonal type: triangle, square, pentagonal, hexagonal, heptagonal, and octagonal, is represented by a different number in the set.

### Problem 62: Cubic permutations

The cube, 41063625 ( $345^3$ ), can be permuted to produce two other cubes: 56623104 ( $384^3$ ) and 66430125 ( $405^3$ ). In fact, 41063625 is the smallest cube which has exactly three permutations of its digits which are also cube.

Find the smallest cube for which exactly five permutations of its digits are cube.

### Problem 63: Powerful digit counts

The 5-digit number,  $16807=7^5$ , is also a fifth power. Similarly, the 9-digit number,  $134217728=8^9$ , is a ninth power.

How many  $n$ -digit positive integers exist which are also  $n$ th power?

## Problem 64: Odd period square roots

All square roots are periodic when written as continued fractions and can be written in the form:

$$\sqrt{N} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

For example, let us consider  $\sqrt{23}$ :

$$\sqrt{23} = 4 + \sqrt{23} - 4 = 4 + \cfrac{1}{\cfrac{1}{\sqrt{23}-4}} = 4 + \cfrac{1}{1 + \cfrac{\sqrt{23}-3}{7}}$$

If we continue we would get the following expansion:

$$\sqrt{23} = 4 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{8 + \dots}}}}$$

The process can be summarised as follows:

$$\begin{aligned} a_0 &= \cfrac{1}{\sqrt{23}-4} = \cfrac{\sqrt{23}+4}{7} = 1 + \cfrac{\sqrt{23}-3}{7} \\ 4, & \\ a_1 &= \cfrac{7}{\sqrt{23}-3} = \cfrac{7(\sqrt{23}+3)}{14} = 3 + \cfrac{\sqrt{23}-3}{2} \\ 1, & \\ a_2 &= \cfrac{2}{\sqrt{23}-3} = \cfrac{2(\sqrt{23}+3)}{14} = 1 + \cfrac{\sqrt{23}-4}{7} \\ 3, & \\ a_3 &= \cfrac{7}{\sqrt{23}-4} = \cfrac{7(\sqrt{23}+4)}{7} = 8 + \sqrt{23}-4 \\ 1, & \\ a_4 &= \cfrac{1}{\sqrt{23}-4} = \cfrac{\sqrt{23}+4}{7} = 1 + \cfrac{\sqrt{23}-3}{7} \\ 8, & \\ a_5 &= \cfrac{7}{\sqrt{23}-3} = \cfrac{7(\sqrt{23}+3)}{14} = 3 + \cfrac{\sqrt{23}-3}{2} \\ 1, & \\ a_6 &= \cfrac{2}{\sqrt{23}-3} = \cfrac{2(\sqrt{23}+3)}{14} = 1 + \cfrac{\sqrt{23}-4}{7} \\ 3, & \\ a_7 &= \cfrac{7}{\sqrt{23}-4} = \cfrac{7(\sqrt{23}+4)}{7} = 8 + \sqrt{23}-4 \\ 1, & \end{aligned}$$

It can be seen that the sequence is repeating. For conciseness, we use the notation  $\sqrt{23} = [4;(1,3,1,8)]$ , to indicate that the block (1,3,1,8) repeats indefinitely.

The first ten continued fraction representations of (irrational) square roots are:

$$\begin{aligned} \sqrt{2} &= [1;(2)], \text{ period}=1 \\ \sqrt{3} &= [1;(1,2)], \text{ period}=2 \\ \sqrt{5} &= [2;(4)], \text{ period}=1 \\ \sqrt{6} &= [2;(2,4)], \text{ period}=2 \\ \sqrt{7} &= [2;(1,1,1,4)], \text{ period}=4 \\ \sqrt{8} &= [2;(1,4)], \text{ period}=2 \\ \sqrt{10} &= [3;(6)], \text{ period}=1 \\ \sqrt{11} &= [3;(3,6)], \text{ period}=2 \\ \sqrt{12} &= [3;(2,6)], \text{ period}=2 \\ \sqrt{13} &= [3;(1,1,1,1,6)], \text{ period}=5 \end{aligned}$$

Exactly four continued fractions, for  $N \leq 13$ , have an odd period.

How many continued fractions for  $N \leq 10000$  have an odd period?

## Problem 65: Convergents of e

The square root of 2 can be written as an infinite continued fraction.

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

The infinite continued fraction can be written,  $\sqrt{2} = [1;(2)]$ , (2) indicates that 2 repeats *ad infinitum*. In a similar way,  $\sqrt{23} = [4;(1,3,1,8)]$ .

It turns out that the sequence of partial values of continued fractions for square roots provide the best rational approximations. Let us consider the convergents for  $\sqrt{2}$ .

$$\begin{aligned} \frac{1}{1} &= 1 \\ \frac{1}{1} + \frac{1}{2} &= \frac{3}{2} \\ \frac{1}{1} + \frac{1}{2 + \frac{1}{2}} &= \frac{7}{5} \\ \frac{1}{1} + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} &= \frac{17}{12} \\ \frac{1}{1} + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} &= \frac{41}{29} \end{aligned}$$

Hence the sequence of the first ten convergents for  $\sqrt{2}$  are:

1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169, 577/408, 1393/985, 3363/2378, ...

What is most surprising is that the important mathematical constant,  $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots, 1, \text{2}, 1, \dots]$ .

The first ten terms in the sequence of convergents for e are:

2, 3, 8/3, 11/4, 19/7, 87/32, 106/39, 193/71, 1264/465, 1457/536, ...

The sum of digits in the numerator of the 10<sup>th</sup> convergent is 1+4+5+7=17.

Find the sum of digits in the numerator of the 100<sup>th</sup> convergent of the continued fraction for e.

Problem 66: Diophantine equation

Consider quadratic Diophantine equations of the form:

$$x^2 - Dy^2 = 1$$

For example, when D=13, the minimal solution in x is  $649^2 - 13 \times 180^2 = 1$ .

It can be assumed that there are no solutions in positive integers when D is square.

By finding minimal solutions in x for D = {2, 3, 5, 6, 7}, we obtain the following:

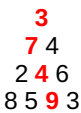
$3^2 - 2 \times 2^2 = 1$   
 $2^2 - 3 \times 1^2 = 1$   
 $9^2 - 5 \times 4^2 = 1$   
 $5^2 - 6 \times 2^2 = 1$   
 $8^2 - 7 \times 3^2 = 1$

Hence, by considering minimal solutions in x for D ≤ 7, the largest x is obtained when D=5.

Find the value of D ≤ 1000 in minimal solutions of x for which the largest value of x is obtained.

Problem 67: Maximum path sum II

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.



That is, 3 + 7 + 4 + 9 = 23.

Find the maximum total from top to bottom in triangle.txt (right click and 'Save Link/Target As...'), a 15K text file containing a triangle with one-hundred rows.

**NOTE:** This is a much more difficult version of Problem 18. It is not possible to try every route to solve this problem, as there are 2<sup>99</sup> altogether! If you could check one trillion (10<sup>12</sup>) routes every second it would take over twenty billion years to check them all. There is an efficient algorithm to solve it. ;o)

Problem 68: Magic 5-gon ring

Consider the following "magic" 3-gon ring, filled with the numbers 1 to 6, and each line adding to nine.



Working **clockwise**, and starting from the group of three with the numerically lowest external node (4,3,2 in this example), each solution can be described uniquely. For example, the above solution can be described by the set: 4,3,2; 6,2,1; 5,1,3.

It is possible to complete the ring with four different totals: 9, 10, 11, and 12. There are eight solutions in total.

Total	Solution Set
9	4,2,3; 5,3,1; 6,1,2
9	4,3,2; 6,2,1; 5,1,3
10	2,3,5; 4,5,1; 6,1,3
10	2,5,3; 6,3,1; 4,1,5
11	1,4,6; 3,6,2; 5,2,4
11	1,6,4; 5,4,2; 3,2,6
12	1,5,6; 2,6,4; 3,4,5
12	1,6,5; 3,5,4; 2,4,6

By concatenating each group it is possible to form 9-digit strings; the maximum string for a 3-gon ring is 432621513.

Using the numbers 1 to 10, and depending on arrangements, it is possible to form 16- and 17-digit strings. What is the maximum **16-digit** string for a "magic" 5-gon ring?



### Problem 69: Totient maximum

Euler's Totient function,  $\phi(n)$  [sometimes called the phi function], is used to determine the number of numbers less than  $n$  which are relatively prime to  $n$ . For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine,  $\phi(9)=6$ .

$n$	Relatively Prime	$\phi(n)$	$n/\phi(n)$
2	1	1	2
3	1,2	2	1.5
4	1,3	2	2
5	1,2,3,4	4	1.25
6	1,5	2	3
7	1,2,3,4,5,6	6	1.1666...
8	1,3,5,7	4	2
9	1,2,4,5,7,8	6	1.5
10	1,3,7,9	4	2.5

It can be seen that  $n=6$  produces a maximum  $n/\phi(n)$  for  $n \leq 10$ .

Find the value of  $n \leq 1,000,000$  for which  $n/\phi(n)$  is a maximum.

### Problem 70: Totient permutation

Euler's Totient function,  $\phi(n)$  [sometimes called the phi function], is used to determine the number of positive numbers less than or equal to  $n$  which are relatively prime to  $n$ . For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine,  $\phi(9)=6$ .

The number 1 is considered to be relatively prime to every positive number, so  $\phi(1)=1$ .

Interestingly,  $\phi(87109)=79180$ , and it can be seen that 87109 is a permutation of 79180.

Find the value of  $n$ ,  $1 < n < 10^7$ , for which  $\phi(n)$  is a permutation of  $n$  and the ratio  $n/\phi(n)$  produces a minimum.

### Problem 71: Ordered fractions

Consider the fraction,  $n/d$ , where  $n$  and  $d$  are positive integers. If  $n < d$  and  $\text{HCF}(n,d)=1$ , it is called a reduced proper fraction.

If we list the set of reduced proper fractions for  $d \leq 8$  in ascending order of size, we get:

1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8

It can be seen that 2/5 is the fraction immediately to the left of 3/7.

By listing the set of reduced proper fractions for  $d \leq 1,000,000$  in ascending order of size, find the numerator of the fraction immediately to the left of 3/7.

### Problem 72: Counting fractions

Consider the fraction,  $n/d$ , where  $n$  and  $d$  are positive integers. If  $n < d$  and  $\text{HCF}(n,d)=1$ , it is called a reduced proper fraction.

If we list the set of reduced proper fractions for  $d \leq 8$  in ascending order of size, we get:

1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8

It can be seen that there are 21 elements in this set.

How many elements would be contained in the set of reduced proper fractions for  $d \leq 1,000,000$ ?

### Problem 73: Counting fractions in a range

Consider the fraction,  $n/d$ , where  $n$  and  $d$  are positive integers. If  $n < d$  and  $\text{HCF}(n,d)=1$ , it is called a reduced proper fraction.

If we list the set of reduced proper fractions for  $d \leq 8$  in ascending order of size, we get:

$1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8$

It can be seen that there are 3 fractions between  $1/3$  and  $1/2$ .

How many fractions lie between  $1/3$  and  $1/2$  in the sorted set of reduced proper fractions for  $d \leq 12,000$ ?

### Problem 74: Digit factorial chains

The number 145 is well known for the property that the sum of the factorial of its digits is equal to 145:

$$1! + 4! + 5! = 1 + 24 + 120 = 145$$

Perhaps less well known is 169, in that it produces the longest chain of numbers that link back to 169; it turns out that there are only three such loops that exist:

$169 \rightarrow 363601 \rightarrow 1454 \rightarrow 169$   
 $871 \rightarrow 45361 \rightarrow 871$   
 $872 \rightarrow 45362 \rightarrow 872$

It is not difficult to prove that EVERY starting number will eventually get stuck in a loop. For example,

$69 \rightarrow 363600 \rightarrow 1454 \rightarrow 169 \rightarrow 363601 (\rightarrow 1454)$   
 $78 \rightarrow 45360 \rightarrow 871 \rightarrow 45361 (\rightarrow 871)$   
 $540 \rightarrow 145 (\rightarrow 145)$

Starting with 69 produces a chain of five non-repeating terms, but the longest non-repeating chain with a starting number below one million is sixty terms.

How many chains, with a starting number below one million, contain exactly sixty non-repeating terms?

### Problem 75: Singular integer right triangles

It turns out that 12 cm is the smallest length of wire that can be bent to form an integer sided right angle triangle in exactly one way, but there are many more examples.

**12 cm:** (3,4,5)  
**24 cm:** (6,8,10)  
**30 cm:** (5,12,13)  
**36 cm:** (9,12,15)  
**40 cm:** (8,15,17)  
**48 cm:** (12,16,20)

In contrast, some lengths of wire, like 20 cm, cannot be bent to form an integer sided right angle triangle, and other lengths allow more than one solution to be found; for example, using 120 cm it is possible to form exactly three different integer sided right angle triangles.

**120 cm:** (30,40,50), (20,48,52), (24,45,51)

Given that  $L$  is the length of the wire, for how many values of  $L \leq 1,500,000$  can exactly one integer sided right angle triangle be formed?

### Problem 76: Counting summations

It is possible to write five as a sum in exactly six different ways:

4 + 1  
3 + 2  
3 + 1 + 1  
2 + 2 + 1  
2 + 1 + 1 + 1  
1 + 1 + 1 + 1 + 1

How many different ways can one hundred be written as a sum of at least two positive integers?

### Problem 77: Prime summations

It is possible to write ten as the sum of primes in exactly five different ways:

7 + 3  
5 + 5  
5 + 3 + 2  
3 + 3 + 2 + 2  
2 + 2 + 2 + 2 + 2

What is the first value which can be written as the sum of primes in over five thousand different ways?

### Problem 78: Coin partitions

Let  $p(n)$  represent the number of different ways in which  $n$  coins can be separated into piles. For example, five coins can be separated into piles in exactly seven different ways, so  $p(5)=7$ .

OOOOO  
OOOO O  
OOO OO  
OOO O O  
OO OO O  
OO O O O  
O O O O  
O

Find the least value of  $n$  for which  $p(n)$  is divisible by one million.

### Problem 79: Passcode derivation

A common security method used for online banking is to ask the user for three random characters from a passcode. For example, if the passcode was 531278, they may ask for the 2nd, 3rd, and 5th characters; the expected reply would be: 317.

The text file, `keylog.txt`, contains fifty successful login attempts.

Given that the three characters are always asked for in order, analyse the file so as to determine the shortest possible secret passcode of unknown length.

### Problem 80: Square root digital expansion

It is well known that if the square root of a natural number is not an integer, then it is irrational. The decimal expansion of such square roots is infinite without any repeating pattern at all.

The square root of two is 1.41421356237309504880..., and the digital sum of the first one hundred decimal digits is 475.

For the first one hundred natural numbers, find the total of the digital sums of the first one hundred decimal digits for all the irrational square roots.

### Problem 81: Path sum: two ways

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by **only moving to the right and down**, is indicated in bold red and is equal to 2427.

131	673	234	103	18
201	96	342	965	150
630	803	746	422	111
537	699	497	121	956
805	732	524	37	331

Find the minimal path sum, in `matrix.txt` (right click and "Save Link/Target As..."), a 31K text file containing a 80 by 80 matrix, from the top left to the bottom right by only moving right and down.

### Problem 82: Path sum: three ways

NOTE: This problem is a more challenging version of Problem 81.

The minimal path sum in the 5 by 5 matrix below, by starting in any cell in the left column and finishing in any cell in the right column, and only moving up, down, and right, is indicated in red and bold; the sum is equal to 994.

131	673	234	103	18
201	96	342	965	150
630	803	746	422	111
537	699	497	121	956
805	732	524	37	331

Find the minimal path sum, in `matrix.txt` (right click and "Save Link/Target As..."), a 31K text file containing a 80 by 80 matrix, from the left column to the right column.

### Problem 83: Path sum: four ways

NOTE: This problem is a significantly more challenging version of Problem 81.

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by moving left, right, up, and down, is indicated in bold red and is equal to 2297.

131	673	234	103	18
201	96	342	965	150
630	803	746	422	111
537	699	497	121	956
805	732	524	37	331

Find the minimal path sum, in `matrix.txt` (right click and "Save Link/Target As..."), a 31K text file containing a 80 by 80 matrix, from the top left to the bottom right by moving left, right, up, and down.



## Problem 84: Monopoly odds

In the game, *Monopoly*, the standard board is set up in the following way:

GO	A1	CC1	A2	T1	R1	B1	CH1	B2	B3	JAIL
H2										C1
T2										U1
H1										C2
CH3										C3
R4										R2
G3										D1
CC3										CC2
G2										D2
G1										D3
G2J	F3	U2	F2	F1	R3	E3	E2	CH2	E1	FP

A player starts on the GO square and adds the scores on two 6-sided dice to determine the number of squares they advance in a clockwise direction. Without any further rules we would expect to visit each square with equal probability: 2.5%. However, landing on G2J (Go To Jail), CC (community chest), and CH (chance) changes this distribution.

In addition to G2J, and one card from each of CC and CH, that orders the player to go directly to jail, if a player rolls three consecutive doubles, they do not advance the result of their 3rd roll. Instead they proceed directly to jail.

At the beginning of the game, the CC and CH cards are shuffled. When a player lands on CC or CH they take a card from the top of the respective pile and, after following the instructions, it is returned to the bottom of the pile. There are sixteen cards in each pile, but for the purpose of this problem we are only concerned with cards that order a movement; any instruction not concerned with movement will be ignored and the player will remain on the CC/CH square.

- Community Chest (2/16 cards):
  1. Advance to GO
  2. Go to JAIL
- Chance (10/16 cards):
  1. Advance to GO
  2. Go to JAIL
  3. Go to C1
  4. Go to E3
  5. Go to H2
  6. Go to R1
  7. Go to next R (railway company)
  8. Go to next R
  9. Go to next U (utility company)
  10. Go back 3 squares.

The heart of this problem concerns the likelihood of visiting a particular square. That is, the probability of finishing at that square after a roll. For this reason it should be clear that, with the exception of G2J for which the probability of finishing on it is zero, the CH squares will have the lowest probabilities, as 5/8 request a movement to another square, and it is the final square that the player finishes at on each roll that we are interested in. We shall make no distinction between "Just Visiting" and being sent to JAIL, and we shall also ignore the rule about requiring a double to "get out of jail", assuming that they pay to get out on their next turn.

By starting at GO and numbering the squares sequentially from 00 to 39 we can concatenate these two-digit numbers to produce strings that correspond with sets of squares.

Statistically it can be shown that the three most popular squares, in order, are JAIL (6.24%) = Square 10, E3 (3.18%) = Square 24, and GO (3.09%) = Square 00. So these three most popular squares can be listed with the six-digit modal string: 102400.

If, instead of using two 6-sided dice, two 4-sided dice are used, find the six-digit modal string.

### Problem 85: Counting rectangles

By counting carefully it can be seen that a rectangular grid measuring 3 by 2 contains eighteen rectangles:



Although there exists no rectangular grid that contains exactly two million rectangles, find the area of the grid with the nearest solution.

### Problem 86: Cuboid route

A spider, S, sits in one corner of a cuboid room, measuring 6 by 5 by 3, and a fly, F, sits in the opposite corner. By travelling on the surfaces of the room the shortest "straight line" distance from S to F is 10 and the path is shown on the diagram.



However, there are up to three "shortest" path candidates for any given cuboid and the shortest route doesn't always have integer length.

It can be shown that there are exactly 2060 distinct cuboids, ignoring rotations, with integer dimensions, up to a maximum size of M by M by M, for which the shortest route has integer length when M = 100. This is the least value of M for which the number of solutions first exceeds two thousand; the number of solutions when M = 99 is 1975.

Find the least value of M such that the number of solutions first exceeds one million.

### Problem 87: Prime power triples

The smallest number expressible as the sum of a prime square, prime cube, and prime fourth power is 28. In fact, there are exactly four numbers below fifty that can be expressed in such a way:

$$28 = 2^2 + 2^3 + 2^4$$

$$33 = 3^2 + 2^3 + 2^4$$

$$49 = 5^2 + 2^3 + 2^4$$

$$47 = 2^2 + 3^3 + 2^4$$

How many numbers below fifty million can be expressed as the sum of a prime square, prime cube, and prime fourth power?

### Problem 88: Product-sum numbers

A natural number, N, that can be written as the sum and product of a given set of at least two natural numbers,  $\{a_1, a_2, \dots, a_k\}$  is called a product-sum number:  $N = a_1 + a_2 + \dots + a_k = a_1 \times a_2 \times \dots \times a_k$ .

For example,  $6 = 1 + 2 + 3 = 1 \times 2 \times 3$ .

For a given set of size, k, we shall call the smallest N with this property a minimal product-sum number. The minimal product-sum numbers for sets of size,  $k = 2, 3, 4, 5$ , and 6 are as follows.

$$k=2: 4 = 2 \times 2 = 2 + 2$$

$$k=3: 6 = 1 \times 2 \times 3 = 1 + 2 + 3$$

$$k=4: 8 = 1 \times 1 \times 2 \times 4 = 1 + 1 + 2 + 4$$

$$k=5: 8 = 1 \times 1 \times 2 \times 2 \times 2 = 1 + 1 + 2 + 2 + 2$$

$$k=6: 12 = 1 \times 1 \times 1 \times 1 \times 2 \times 6 = 1 + 1 + 1 + 1 + 2 + 6$$

Hence for  $2 \leq k \leq 6$ , the sum of all the minimal product-sum numbers is  $4+6+8+12 = 30$ ; note that 8 is only counted once in the sum.

In fact, as the complete set of minimal product-sum numbers for  $2 \leq k \leq 12$  is  $\{4, 6, 8, 12, 15, 16\}$ , the sum is 61.

What is the sum of all the minimal product-sum numbers for  $2 \leq k \leq 12000$ ?

### Problem 89: Roman numerals

For a number written in Roman numerals to be considered valid there are basic rules which must be followed. Even though the rules allow some numbers to be expressed in more than one way there is always a "best" way of writing a particular number.

For example, it would appear that there are at least six ways of writing the number sixteen:

IIIIIIIIIIII  
VIIIIIIIIII  
VVIIIIIII  
XIIIIIIII  
VVVI  
XVI

However, according to the rules only XIIIIII and XVI are valid, and the last example is considered to be the most efficient, as it uses the least number of numerals.

The 11K text file, [roman.txt](#) (right click and 'Save Link/Target As...'), contains one thousand numbers written in valid, but not necessarily minimal, Roman numerals; see [About... Roman Numerals](#) for the definitive rules for this problem.

Find the number of characters saved by writing each of these in their minimal form.

Note: You can assume that all the Roman numerals in the file contain no more than four consecutive identical units.

### Problem 90: Cube digit pairs

Each of the six faces on a cube has a different digit (0 to 9) written on it; the same is done to a second cube. By placing the two cubes side-by-side in different positions we can form a variety of 2-digit numbers.

For example, the square number 64 could be formed:

□

In fact, by carefully choosing the digits on both cubes it is possible to display all of the square numbers below one-hundred: 01, 04, 09, 16, 25, 36, 49, 64, and 81.

For example, one way this can be achieved is by placing {0, 5, 6, 7, 8, 9} on one cube and {1, 2, 3, 4, 8, 9} on the other cube.

However, for this problem we shall allow the 6 or 9 to be turned upside-down so that an arrangement like {0, 5, 6, 7, 8, 9} and {1, 2, 3, 4, 6, 7} allows for all nine square numbers to be displayed; otherwise it would be impossible to obtain 09.

In determining a distinct arrangement we are interested in the digits on each cube, not the order.

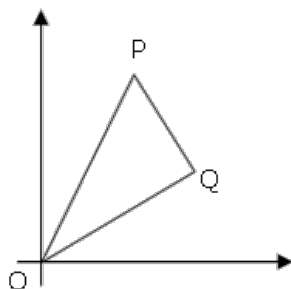
{1, 2, 3, 4, 5, 6} is equivalent to {3, 6, 4, 1, 2, 5}  
{1, 2, 3, 4, 5, 6} is distinct from {1, 2, 3, 4, 5, 9}

But because we are allowing 6 and 9 to be reversed, the two distinct sets in the last example both represent the extended set {1, 2, 3, 4, 5, 6, 9} for the purpose of forming 2-digit numbers.

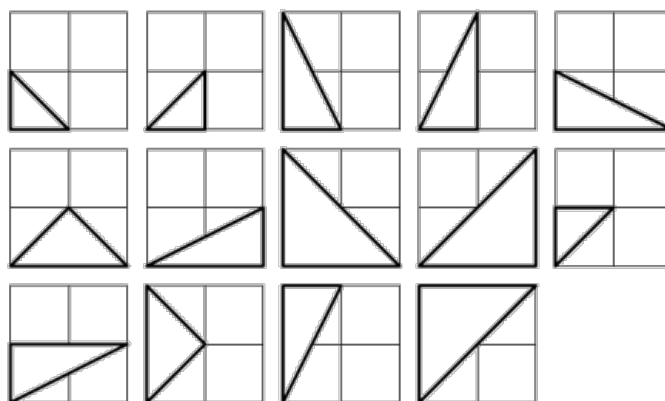
How many distinct arrangements of the two cubes allow for all of the square numbers to be displayed?

### Problem 91: Right triangles with integer coordinates

The points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  are plotted at integer co-ordinates and are joined to the origin,  $O(0,0)$ , to form  $\triangle OPQ$ .



There are exactly fourteen triangles containing a right angle that can be formed when each co-ordinate lies between 0 and 2 inclusive; that is,  
 $0 \leq x_1, y_1, x_2, y_2 \leq 2$ .



Given that  $0 \leq x_1, y_1, x_2, y_2 \leq 50$ , how many right triangles can be formed?

### Problem 92: Square digit chains

A number chain is created by continuously adding the square of the digits in a number to form a new number until it has been seen before.

For example,

$44 \rightarrow 32 \rightarrow 13 \rightarrow 10 \rightarrow \mathbf{1} \rightarrow \mathbf{1}$

$85 \rightarrow \mathbf{89} \rightarrow 145 \rightarrow 42 \rightarrow 20 \rightarrow 4 \rightarrow 16 \rightarrow 37 \rightarrow 58 \rightarrow \mathbf{89}$

Therefore any chain that arrives at 1 or 89 will become stuck in an endless loop. What is most amazing is that EVERY starting number will eventually arrive at 1 or 89.

How many starting numbers below ten million will arrive at 89?

### Problem 93: Arithmetic expressions

By using each of the digits from the set, {1, 2, 3, 4}, exactly once, and making use of the four arithmetic operations (+, −, \*, /) and brackets/parentheses, it is possible to form different positive integer targets.

For example,

$$8 = (4 * (1 + 3)) / 2$$

$$14 = 4 * (3 + 1 / 2)$$

$$19 = 4 * (2 + 3) - 1$$

$$36 = 3 * 4 * (2 + 1)$$

Note that concatenations of the digits, like 12 + 34, are not allowed.

Using the set, {1, 2, 3, 4}, it is possible to obtain thirty-one different target numbers of which 36 is the maximum, and each of the numbers 1 to 28 can be obtained before encountering the first non-expressible number.

Find the set of four distinct digits,  $a < b < c < d$ , for which the longest set of consecutive positive integers, 1 to  $n$ , can be obtained, giving your answer as a string:  $abcd$ .

### Problem 94: Almost equilateral triangles

It is easily proved that no equilateral triangle exists with integral length sides and integral area. However, the *almost equilateral triangle* 5-5-6 has an area of 12 square units.

We shall define an *almost equilateral triangle* to be a triangle for which two sides are equal and the third differs by no more than one unit.

Find the sum of the perimeters of all *almost equilateral triangles* with integral side lengths and area and whose perimeters do not exceed one billion (1,000,000,000).

### Problem 95: Amicable chains

The proper divisors of a number are all the divisors excluding the number itself. For example, the proper divisors of 28 are 1, 2, 4, 7, and 14. As the sum of these divisors is equal to 28, we call it a perfect number.

Interestingly the sum of the proper divisors of 220 is 284 and the sum of the proper divisors of 284 is 220, forming a chain of two numbers. For this reason, 220 and 284 are called an amicable pair.

Perhaps less well known are longer chains. For example, starting with 12496, we form a chain of five numbers:

$$12496 \rightarrow 14288 \rightarrow 15472 \rightarrow 14536 \rightarrow 14264 (\rightarrow 12496 \rightarrow \dots)$$

Since this chain returns to its starting point, it is called an amicable chain.

Find the smallest member of the longest amicable chain with no element exceeding one million.

### Problem 96: Su Doku

Su Doku (Japanese meaning *number place*) is the name given to a popular puzzle concept. Its origin is unclear, but credit must be attributed to Leonhard Euler who invented a similar, and much more difficult, puzzle idea called Latin Squares. The objective of Su Doku puzzles, however, is to replace the blanks (or zeros) in a 9 by 9 grid in such that each row, column, and 3 by 3 box contains each of the digits 1 to 9. Below is an example of a typical starting puzzle grid and its solution grid.

0 0 3	0 2 0	6 0 0	4 8 3	9 2 1	6 5 7
9 0 0	3 0 5	0 0 1	9 6 7	3 4 5	8 2 1
0 0 1	8 0 6	4 0 0	2 5 1	8 7 6	4 9 3
0 0 8	1 0 2	9 0 0	5 4 8	1 3 2	9 7 6
7 0 0	0 0 0	0 0 8	7 2 9	5 6 4	1 3 8
0 0 6	7 0 8	2 0 0	1 3 6	7 9 8	2 4 5
0 0 2	6 0 9	5 0 0	3 7 2	6 8 9	5 1 4
8 0 0	2 0 3	0 0 9	8 1 4	2 5 3	7 6 9
0 0 5	0 1 0	3 0 0	6 9 5	4 1 7	3 8 2

A well constructed Su Doku puzzle has a unique solution and can be solved by logic, although it may be necessary to employ "guess and test" methods in order to eliminate options (there is much contested opinion over this). The complexity of the search determines the difficulty of the puzzle; the example above is considered easy because it can be solved by straight forward direct deduction.

The 6K text file, [sudoku.txt](#) (right click and 'Save Link/Target As...'), contains fifty different Su Doku puzzles ranging in difficulty, but all with unique solutions (the first puzzle in the file is the example above).

By solving all fifty puzzles find the sum of the 3-digit numbers found in the top left corner of each solution grid; for example, 483 is the 3-digit number found in the top left corner of the solution grid above.

### Problem 97: Large non-Mersenne prime

The first known prime found to exceed one million digits was discovered in 1999, and is a Mersenne prime of the form  $2^{6972593}-1$ ; it contains exactly 2,098,960 digits. Subsequently other Mersenne primes, of the form  $2^n-1$ , have been found which contain more digits.

However, in 2004 there was found a massive non-Mersenne prime which contains 2,357,207 digits:  $28433 \times 2^{830457} + 1$ .

Find the last ten digits of this prime number.

### Problem 98: Anagramic squares

By replacing each of the letters in the word CARE with 1, 2, 9, and 6 respectively, we form a square number:  $1296 = 36^2$ . What is remarkable is that, by using the same digital substitutions, the anagram, RACE, also forms a square number:  $9216 = 96^2$ . We shall call CARE (and RACE) a square anagram word pair and specify further that leading zeroes are not permitted, neither may a different letter have the same digital value as another letter.

Using [words.txt](#) (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, find all the square anagram word pairs (a palindromic word is NOT considered to be an anagram of itself).

What is the largest square number formed by any member of such a pair?

NOTE: All anagrams formed must be contained in the given text file.

### Problem 99: Largest exponential

Comparing two numbers written in index form like  $2^{11}$  and  $3^7$  is not difficult, as any calculator would confirm that  $2^{11} = 2048 < 3^7 = 2187$ .

However, confirming that  $632382^{518061} > 519432^{525806}$  would be much more difficult, as both numbers contain over three million digits.

Using `base_exp.txt` (right click and 'Save Link/Target As...'), a 22K text file containing one thousand lines with a base/exponent pair on each line, determine which line number has the greatest numerical value.

NOTE: The first two lines in the file represent the numbers in the example given above.

### Problem 100: Arranged probability

If a box contains twenty-one coloured discs, composed of fifteen blue discs and six red discs, and two discs were taken at random, it can be seen that the probability of taking two blue discs,  $P(BB) = (15/21) \times (14/20) = 1/2$ .

The next such arrangement, for which there is exactly 50% chance of taking two blue discs at random, is a box containing eighty-five blue discs and thirty-five red discs.

By finding the first arrangement to contain over  $10^{12} = 1,000,000,000,000$  discs in total, determine the number of blue discs that the box would contain.

### Problem 101: Optimum polynomial

If we are presented with the first  $k$  terms of a sequence it is impossible to say with certainty the value of the next term, as there are infinitely many polynomial functions that can model the sequence.

As an example, let us consider the sequence of cube numbers. This is defined by the generating function,  $u_n = n^3$ : 1, 8, 27, 64, 125, 216, ...

Suppose we were only given the first two terms of this sequence. Working on the principle that "simple is best" we should assume a linear relationship and predict the next term to be 15 (common difference 7). Even if we were presented with the first three terms, by the same principle of simplicity, a quadratic relationship should be assumed.

We shall define  $OP(k, n)$  to be the  $n^{\text{th}}$  term of the optimum polynomial generating function for the first  $k$  terms of a sequence. It should be clear that  $OP(k, n)$  will accurately generate the terms of the sequence for  $n \leq k$ , and potentially the *first incorrect term* (FIT) will be  $OP(k, k+1)$ ; in which case we shall call it *abad OP* (BOP).

As a basis, if we were only given the first term of sequence, it would be most sensible to assume constancy; that is, for  $n \geq 2$ ,  $OP(1, n) = u_1$ .

Hence we obtain the following OPs for the cubic sequence:

$OP(1, n) = 1$	1, <b>1</b> , 1, 1, ...
$OP(2, n) = 7n - 6$	1, 8, <b>15</b> , ...
$OP(3, n) = 6n^2 - 11n + 6$	1, 8, 27, <b>58</b> , ...
$OP(4, n) = n^3$	1, 8, 27, 64, 125, ...

Clearly no BOPs exist for  $k \geq 4$ .

By considering the sum of FITs generated by the BOPs (indicated in **red** above), we obtain  $1 + 15 + 58 = 74$ .

Consider the following tenth degree polynomial generating function:

$$u_n = 1 - n + n^2 - n^3 + n^4 - n^5 + n^6 - n^7 + n^8 - n^9 + n^{10}$$

Find the sum of FITs for the BOPs.

### Problem 102: Triangle containment

Three distinct points are plotted at random on a Cartesian plane, for which  $-1000 \leq x, y \leq 1000$ , such that a triangle is formed.

Consider the following two triangles:

A(-340,495), B(-153,-910), C(835,-947)

X(-175,41), Y(-421,-714), Z(574,-645)

It can be verified that triangle ABC contains the origin, whereas triangle XYZ does not.

Using `triangles.txt` (right click and 'Save Link/Target As...'), a 27K text file containing the co-ordinates of one thousand "random" triangles, find the number of triangles for which the interior contains the origin.

NOTE: The first two examples in the file represent the triangles in the example given above.

### Problem 103: Special subset sums: optimum

Let  $S(A)$  represent the sum of elements in set  $A$  of size  $n$ . We shall call it a special sum set if for any two non-empty disjoint subsets,  $B$  and  $C$ , the following properties are true:

- $S(B) \neq S(C)$ ; that is, sums of subsets cannot be equal.
- If  $B$  contains more elements than  $C$  then  $S(B) > S(C)$ .

If  $S(A)$  is minimised for a given  $n$ , we shall call it an optimum special sum set. The first five optimum special sum sets are given below.

$n = 1: \{1\}$

$n = 2: \{1, 2\}$

$n = 3: \{2, 3, 4\}$

$n = 4: \{3, 5, 6, 7\}$

$n = 5: \{6, 9, 11, 12, 13\}$

It seems that for a given optimum set,  $A = \{a_1, a_2, \dots, a_n\}$ , the next optimum set is of the form  $B = \{b, a_1+b, a_2+b, \dots, a_n+b\}$ , where  $b$  is the "middle" element on the previous row.

By applying this "rule" we would expect the optimum set for  $n = 6$  to be  $A = \{11, 17, 20, 22, 23, 24\}$ , with  $S(A) = 117$ . However, this is not the optimum set, as we have merely applied an algorithm to provide a near optimum set. The optimum set for  $n = 6$  is  $A = \{11, 18, 19, 20, 22, 25\}$ , with  $S(A) = 115$  and corresponding set string: 111819202225.

Given that  $A$  is an optimum special sum set for  $n = 7$ , find its set string.

NOTE: This problem is related to [Problem 105](#) and [Problem 106](#).

### Problem 104: Pandigital Fibonacci ends

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_1 = 1 \text{ and } F_2 = 1.$$

It turns out that  $F_{541}$ , which contains 113 digits, is the first Fibonacci number for which the last nine digits are 1-9 pandigital (contain all the digits 1 to 9, but not necessarily in order). And  $F_{2749}$ , which contains 575 digits, is the first Fibonacci number for which the first nine digits are 1-9 pandigital.

Given that  $F_k$  is the first Fibonacci number for which the first nine digits AND the last nine digits are 1-9 pandigital, find  $k$ .



### Problem 105: Special subset sums: testing

Let  $S(A)$  represent the sum of elements in set  $A$  of size  $n$ . We shall call it a special sum set if for any two non-empty disjoint subsets,  $B$  and  $C$ , the following properties are true:

- i.  $S(B) \neq S(C)$ ; that is, sums of subsets cannot be equal.
- ii. If  $B$  contains more elements than  $C$  then  $S(B) > S(C)$ .

For example,  $\{81, 88, 75, 42, 87, 84, 86, 65\}$  is not a special sum set because  $65 + 87 + 88 = 75 + 81 + 84$ , whereas  $\{157, 150, 164, 119, 79, 159, 161, 139, 158\}$  satisfies both rules for all possible subset pair combinations and  $S(A) = 1286$ .

Using `sets.txt` (right click and "Save Link/Target As..."), a 4K text file with one-hundred sets containing seven to twelve elements (the two examples given above are the first two sets in the file), identify all the special sum sets,  $A_1, A_2, \dots, A_k$ , and find the value of  $S(A_1) + S(A_2) + \dots + S(A_k)$ .

NOTE: This problem is related to [Problem 103](#) and [Problem 106](#).

### Problem 106: Special subset sums: meta-testing

Let  $S(A)$  represent the sum of elements in set  $A$  of size  $n$ . We shall call it a special sum set if for any two non-empty disjoint subsets,  $B$  and  $C$ , the following properties are true:

- i.  $S(B) \neq S(C)$ ; that is, sums of subsets cannot be equal.
- ii. If  $B$  contains more elements than  $C$  then  $S(B) > S(C)$ .

For this problem we shall assume that a given set contains  $n$  strictly increasing elements and it already satisfies the second rule.

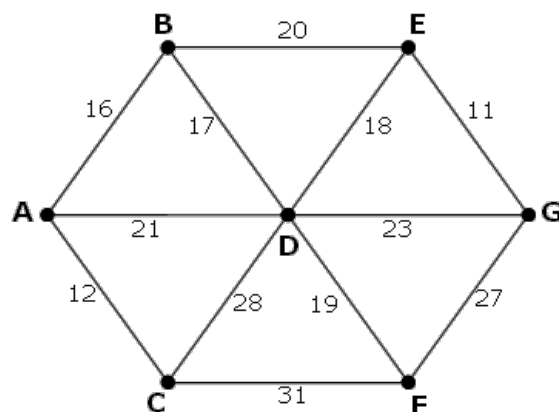
Surprisingly, out of the 25 possible subset pairs that can be obtained from a set for which  $n = 4$ , only 1 of these pairs need to be tested for equality (first rule). Similarly, when  $n = 7$ , only 70 out of the 966 subset pairs need to be tested.

For  $n = 12$ , how many of the 261625 subset pairs that can be obtained need to be tested for equality?

NOTE: This problem is related to [Problem 103](#) and [Problem 105](#).

### Problem 107: Minimal network

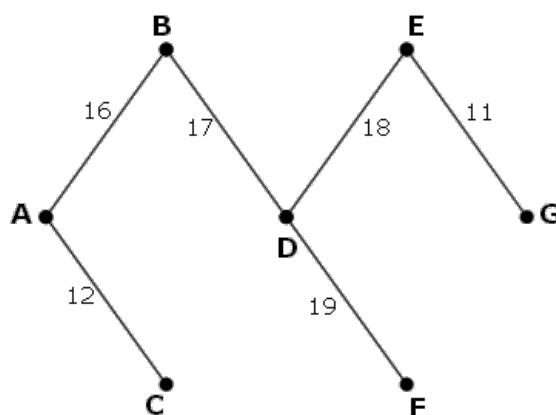
The following undirected network consists of seven vertices and twelve edges with a total weight of 243.



The same network can be represented by the matrix below.

	A	B	C	D	E	F	G
A	-	16	12	21	-	-	-
B	16	-	-	17	20	-	-
C	12	-	-	28	-	31	-
D	21	17	28	-	18	19	23
E	-	20	-	18	-	-	11
F	-	-	31	19	-	-	27
G	-	-	-	23	11	27	-

However, it is possible to optimise the network by removing some edges and still ensure that all points on the network remain connected. The network which achieves the maximum saving is shown below. It has a weight of 93, representing a saving of  $243 - 93 = 150$  from the original network.



Using [network.txt](#) (right click and 'Save Link/Target As...'), a 6K text file containing a network with forty vertices, and given in matrix form, find the maximum saving which can be achieved by removing redundant edges whilst ensuring that the network remains connected.

### Problem 108: Diophantine reciprocals I

In the following equation  $x$ ,  $y$ , and  $n$  are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

For  $n = 4$  there are exactly three distinct solutions:

$$\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$$

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$

$$\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

What is the least value of  $n$  for which the number of distinct solutions exceeds one-thousand?

NOTE: This problem is an easier version of [Problem 110](#); it is strongly advised that you solve this one first.

Problem 109: Darts

In the game of darts a player throws three darts at a target board which is split into twenty equal sized sections numbered one to twenty.



The score of a dart is determined by the number of the region that the dart lands in. A dart landing outside the red/green outer ring scores zero. The black and cream regions inside this ring represent single scores. However, the red/green outer ring and middle ring score double and treble scores respectively.

At the centre of the board are two concentric circles called the bull region, or bulls-eye. The outer bull is worth 25 points and the inner bull is a double, worth 50 points.

There are many variations of rules but in the most popular game the players will begin with a score 301 or 501 and the first player to reduce their running total to zero is a winner. However, it is normal to play a "doubles out" system, which means that the player must land a double (including the double bulls-eye at the centre of the board) on their final dart to win; any other dart that would reduce their running total to one or lower means the score for that set of three darts is "bust".

When a player is able to finish on their current score it is called a "checkout" and the highest checkout is 170: T20 T20 D25 (two treble 20s and double bull).

There are exactly eleven distinct ways to checkout on a score of 6:

D3		
D1	D2	
S2	D2	
D2	D1	
S4	D1	
S1	S1	D2
S1	T1	D1
S1	S3	D1
D1	D1	D1
D1	S2	D1
S2	S2	D1

Note that D1 D2 is considered **different** to D2 D1 as they finish on different doubles. However, the combination S1 T1 D1 is considered the **same** as T1 S1 D1.

In addition we shall not include misses in considering combinations; for example, D3 is the **same** as 0 D3 and 0 0 D3.

Incredibly there are 42336 distinct ways of checking out in total.

How many distinct ways can a player checkout with a score less than 100?

Problem 110: Diophantine reciprocals II

In the following equation  $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ ,  $x$ ,  $y$ , and  $n$  are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

It can be verified that when  $n = 1260$  there are 113 distinct solutions and this is the least value of  $n$  for which the total number of distinct solutions exceeds one hundred.

What is the least value of  $n$  for which the number of distinct solutions exceeds four million?

NOTE: This problem is a much more difficult version of Problem 108 and as it is well beyond the limitations of a brute force approach it requires a clever implementation.

Problem 111: Primes with runs

Considering 4-digit primes containing repeated digits it is clear that they cannot all be the same: 1111 is divisible by 11, 2222 is divisible by 22, and so on. But there are nine 4-digit primes containing three ones:

1117, 1151, 1171, 1181, 1511, 1811, 2111, 4111, 8111

We shall say that  $M(n, d)$  represents the maximum number of repeated digits for an  $n$ -digit prime where  $d$  is the repeated digit,  $N(n, d)$  represents the number of such primes, and  $S(n, d)$  represents the sum of these primes.

So  $M(4, 1) = 3$  is the maximum number of repeated digits for a 4-digit prime where one is the repeated digit, there are  $N(4, 1) = 9$  such primes, and the sum of these primes is  $S(4, 1) = 22275$ . It turns out that for  $d = 0$ , it is only possible to have  $M(4, 0) = 2$  repeated digits, but there are  $N(4, 0) = 13$  such cases.

In the same way we obtain the following results for 4-digit primes.

Digit, $d$	$M(4, d)$	$N(4, d)$	$S(4, d)$
0	2	13	67061
1	3	9	22275
2	3	1	2221
3	3	12	46214
4	3	2	8888
5	3	1	5557
6	3	1	6661
7	3	9	57863
8	3	1	8887
9	3	7	48073

For  $d = 0$  to 9, the sum of all  $S(4, d)$  is 273700.

Find the sum of all  $S(10, d)$ .

Problem 112: Bouncy numbers

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

Clearly there cannot be any bouncy numbers below one-hundred, but just over half of the numbers below one-thousand (525) are bouncy. In fact, the least number for which the proportion of bouncy numbers first reaches 50% is 538.

Surprisingly, bouncy numbers become more and more common and by the time we reach 21780 the proportion of bouncy numbers is equal to 90%.

Find the least number for which the proportion of bouncy numbers is exactly 99%.

### Problem 113: Non-bouncy numbers

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

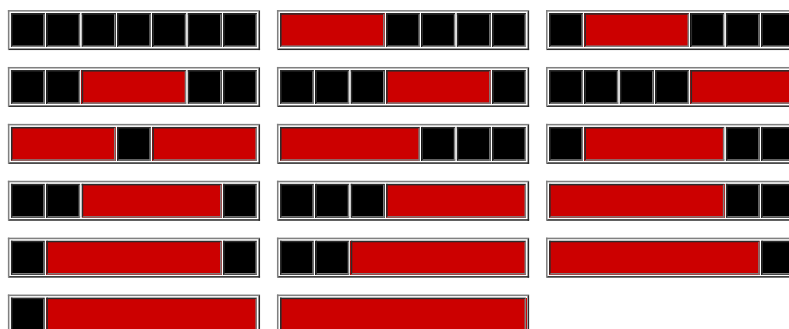
We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

As  $n$  increases, the proportion of bouncy numbers below  $n$  increases such that there are only 12951 numbers below one-million that are not bouncy and only 277032 non-bouncy numbers below  $10^{10}$ .

How many numbers below a googol ( $10^{100}$ ) are not bouncy?

### Problem 114: Counting block combinations I

A row measuring seven units in length has red blocks with a minimum length of three units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square. There are exactly seventeen ways of doing this.



How many ways can a row measuring fifty units in length be filled?

NOTE: Although the example above does not lend itself to the possibility, in general it is permitted to mix block sizes. For example, on a row measuring eight units in length you could use red (3), black (1), and red (4).

### Problem 115: Counting block combinations II

NOTE: This is a more difficult version of Problem 114.

A row measuring  $n$  units in length has red blocks with a minimum length of  $m$  units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square.

Let the fill-count function,  $F(m, n)$ , represent the number of ways that a row can be filled.

For example,  $F(3, 29) = 673135$  and  $F(3, 30) = 1089155$ .

That is, for  $m = 3$ , it can be seen that  $n = 30$  is the smallest value for which the fill-count function first exceeds one million.

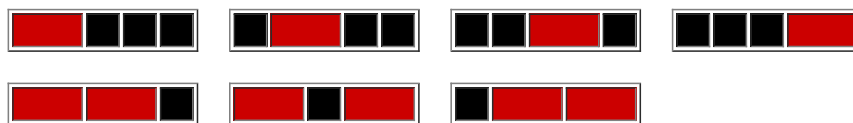
In the same way, for  $m = 10$ , it can be verified that  $F(10, 56) = 880711$  and  $F(10, 57) = 1148904$ , so  $n = 57$  is the least value for which the fill-count function first exceeds one million.

For  $m = 50$ , find the least value of  $n$  for which the fill-count function first exceeds one million.

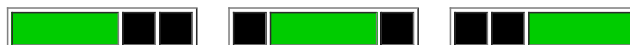
### Problem 116: Red, green or blue tiles

A row of five black square tiles is to have a number of its tiles replaced with coloured oblong tiles chosen from red (length two), green (length three), or blue (length four).

If red tiles are chosen there are exactly seven ways this can be done.



If green tiles are chosen there are three ways.



And if blue tiles are chosen there are two ways.



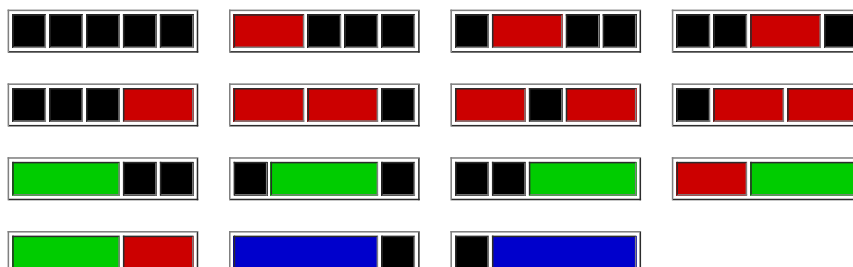
Assuming that colours cannot be mixed there are  $7 + 3 + 2 = 12$  ways of replacing the black tiles in a row measuring five units in length.

How many different ways can the black tiles in a row measuring fifty units in length be replaced if colours cannot be mixed and at least one coloured tile must be used?

NOTE: This is related to Problem 117.

### Problem 117: Red, green, and blue tiles

Using a combination of black square tiles and oblong tiles chosen from: red tiles measuring two units, green tiles measuring three units, and blue tiles measuring four units, it is possible to tile a row measuring five units in length in exactly fifteen different ways.



How many ways can a row measuring fifty units in length be tiled?

NOTE: This is related to Problem 116.

### Problem 118: Pandigital prime sets

Using all of the digits 1 through 9 and concatenating them freely to form decimal integers, different sets can be formed. Interestingly with the set {2,5,47,89,631}, all of the elements belonging to it are prime.

How many distinct sets containing each of the digits one through nine exactly once contain only prime elements?

### Problem 119: Digit power sum

The number 512 is interesting because it is equal to the sum of its digits raised to some power:  $5 + 1 + 2 = 8$ , and  $8^3 = 512$ . Another example of a number with this property is  $614656 = 28^4$ .

We shall define  $a_n$  to be the  $n$ th term of this sequence and insist that a number must contain at least two digits to have a sum.

You are given that  $a_2 = 512$  and  $a_{10} = 614656$ .

Find  $a_{30}$ .

### Problem 120: Square remainders

Let  $r$  be the remainder when  $(a-1)^n + (a+1)^n$  is divided by  $a^2$ .

For example, if  $a = 7$  and  $n = 3$ , then  $r = 42$ :  $6^3 + 8^3 = 728 \equiv 42 \pmod{49}$ . And as  $n$  varies, so too will  $r$ , but for  $a = 7$  it turns out that  $r_{\max} = 42$ .

For  $3 \leq a \leq 1000$ , find  $\sum r_{\max}$ .

### Problem 121: Disc game prize fund

A bag contains one red disc and one blue disc. In a game of chance a player takes a disc at random and its colour is noted. After each turn the disc is returned to the bag, an extra red disc is added, and another disc is taken at random.

The player pays £1 to play and wins if they have taken more blue discs than red discs at the end of the game.

If the game is played for four turns, the probability of a player winning is exactly  $11/120$ , and so the maximum prize fund the banker should allocate for winning in this game would be £10 before they would expect to incur a loss. Note that any payout will be a whole number of pounds and also includes the original £1 paid to play the game, so in the example given the player actually wins £9.

Find the maximum prize fund that should be allocated to a single game in which fifteen turns are played.

### Problem 122: Efficient exponentiation

The most naive way of computing  $n^{15}$  requires fourteen multiplications:

$$n \times n \times \dots \times n = n^{15}$$

But using a "binary" method you can compute it in six multiplications:

$$n \times n = n^2$$

$$n^2 \times n^2 = n^4$$

$$n^4 \times n^4 = n^8$$

$$n^8 \times n^4 = n^{12}$$

$$n^{12} \times n^2 = n^{14}$$

$$n^{14} \times n = n^{15}$$

However it is yet possible to compute it in only five multiplications:

$$n \times n = n^2$$

$$n^2 \times n = n^3$$

$$n^3 \times n^3 = n^6$$

$$n^6 \times n^6 = n^{12}$$

$$n^{12} \times n^3 = n^{15}$$

We shall define  $m(k)$  to be the minimum number of multiplications to compute  $n^k$ ; for example  $m(15) = 5$ .

For  $1 \leq k \leq 200$ , find  $\sum m(k)$ .



Problem 123: Prime square remainders

Let  $p_n$  be the  $n$ th prime: 2, 3, 5, 7, 11, ..., and let  $r$  be the remainder when  $(p_{n-1})^n + (p_{n+1})^n$  is divided by  $p_n^2$ .

For example, when  $n = 3, p_3 = 5$ , and  $4^3 + 6^3 = 280 \equiv 5 \pmod{25}$ .

The least value of  $n$  for which the remainder first exceeds  $10^9$  is 7037.

Find the least value of  $n$  for which the remainder first exceeds  $10^{10}$ .

Problem 124: Ordered radicals

The radical of  $n$ ,  $\text{rad}(n)$ , is the product of the distinct prime factors of  $n$ . For example,  $504 = 2^3 \times 3^2 \times 7$ , so  $\text{rad}(504) = 2 \times 3 \times 7 = 42$ .

If we calculate  $\text{rad}(n)$  for  $1 \leq n \leq 10$ , then sort them on  $\text{rad}(n)$ , and sorting on  $n$  if the radical values are equal, we get:

Unsorted		Sorted		
$n$	$\text{rad}(n)$	$n$	$\text{rad}(n)$	$k$
1	1	1	1	1
2	2	2	2	2
3	3	4	2	3
4	2	8	2	4
5	5	3	3	5
6	6	9	3	6
7	7	5	5	7
8	2	6	6	8
9	3	7	7	9
10	10	10	10	10

Let  $E(k)$  be the  $k$ th element in the sorted  $n$  column; for example,  $E(4) = 8$  and  $E(6) = 9$ .

If  $\text{rad}(n)$  is sorted for  $1 \leq n \leq 100000$ , find  $E(10000)$ .

Problem 125: Palindromic sums

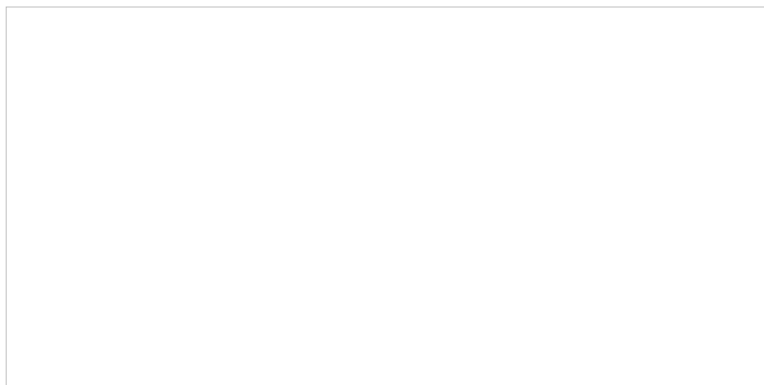
The palindromic number 595 is interesting because it can be written as the sum of consecutive squares:  $6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2$ .

There are exactly eleven palindromes below one-thousand that can be written as consecutive square sums, and the sum of these palindromes is 4164. Note that  $1 = 0^2 + 1^2$  has not been included as this problem is concerned with the squares of positive integers.

Find the sum of all the numbers less than  $10^8$  that are both palindromic and can be written as the sum of consecutive squares.

### Problem 126: Cuboid layers

The minimum number of cubes to cover every visible face on a cuboid measuring  $3 \times 2 \times 1$  is twenty-two.



If we then add a second layer to this solid it would require forty-six cubes to cover every visible face, the third layer would require seventy-eight cubes, and the fourth layer would require one-hundred and eighteen cubes to cover every visible face.

However, the first layer on a cuboid measuring  $5 \times 1 \times 1$  also requires twenty-two cubes; similarly the first layer on cuboids measuring  $5 \times 3 \times 1$ ,  $7 \times 2 \times 1$ , and  $11 \times 1 \times 1$  all contain forty-six cubes.

We shall define  $C(n)$  to represent the number of cuboids that contain  $n$  cubes in one of its layers. So  $C(22) = 2$ ,  $C(46) = 4$ ,  $C(78) = 5$ , and  $C(118) = 8$ .

It turns out that 154 is the least value of  $n$  for which  $C(n) = 10$ .

Find the least value of  $n$  for which  $C(n) = 1000$ .

### Problem 127: abc-hits

The radical of  $n$ ,  $\text{rad}(n)$ , is the product of distinct prime factors of  $n$ . For example,  $504 = 2^3 \times 3^2 \times 7$ , so  $\text{rad}(504) = 2 \times 3 \times 7 = 42$ .

We shall define the triplet of positive integers  $(a, b, c)$  to be an abc-hit if:

1.  $\text{GCD}(a, b) = \text{GCD}(a, c) = \text{GCD}(b, c) = 1$
2.  $a < b$
3.  $a + b = c$
4.  $\text{rad}(abc) < c$

For example,  $(5, 27, 32)$  is an abc-hit, because:

1.  $\text{GCD}(5, 27) = \text{GCD}(5, 32) = \text{GCD}(27, 32) = 1$
2.  $5 < 27$
3.  $5 + 27 = 32$
4.  $\text{rad}(4320) = 30 < 32$

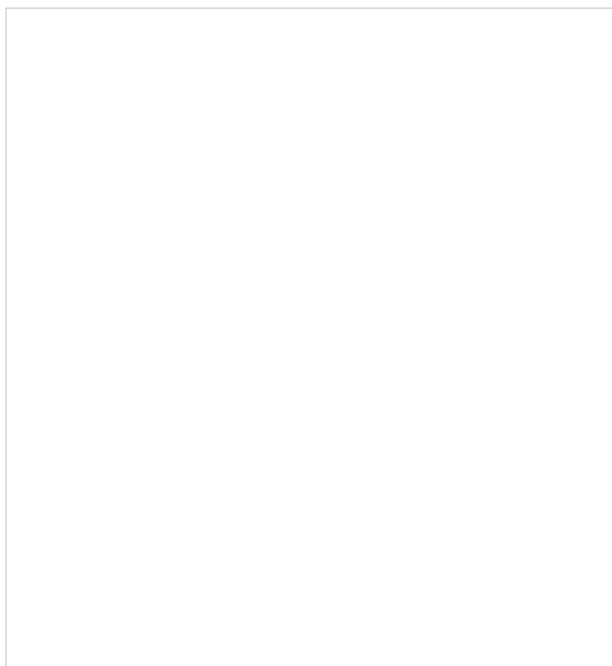
It turns out that abc-hits are quite rare and there are only thirty-one abc-hits for  $c < 1000$ , with  $\sum c = 12523$ .

Find  $\sum c$  for  $c < 120000$ .

### Problem 128: Hexagonal tile differences

A hexagonal tile with number 1 is surrounded by a ring of six hexagonal tiles, starting at "12 o'clock" and numbering the tiles 2 to 7 in an anti-clockwise direction.

New rings are added in the same fashion, with the next rings being numbered 8 to 19, 20 to 37, 38 to 61, and so on. The diagram below shows the first three rings.



By finding the difference between tile  $n$  and each of its six neighbours we shall define  $PD(n)$  to be the number of those differences which are prime.

For example, working clockwise around tile 8 the differences are 12, 29, 11, 6, 1, and 13. So  $PD(8) = 3$ .

In the same way, the differences around tile 17 are 1, 17, 16, 1, 11, and 10, hence  $PD(17) = 2$ .

It can be shown that the maximum value of  $PD(n)$  is 3.

If all of the tiles for which  $PD(n) = 3$  are listed in ascending order to form a sequence, the 10th tile would be 271.

Find the 2000th tile in this sequence.

### Problem 129: Repunit divisibility

A number consisting entirely of ones is called a repunit. We shall define  $R(k)$  to be a repunit of length  $k$ ; for example,  $R(6) = 111111$ .

Given that  $n$  is a positive integer and  $\text{GCD}(n, 10) = 1$ , it can be shown that there always exists a value  $k$ , for which  $R(k)$  is divisible by  $n$ , and let  $A(n)$  be the least such value of  $k$ ; for example,  $A(7) = 6$  and  $A(41) = 5$ .

The least value of  $n$  for which  $A(n)$  first exceeds ten is 17.

Find the least value of  $n$  for which  $A(n)$  first exceeds one-million.

### Problem 130: Composites with prime repunit property

A number consisting entirely of ones is called a repunit. We shall define  $R(k)$  to be a repunit of length  $k$ ; for example,  $R(6) = 111111$ .

Given that  $n$  is a positive integer and  $\text{GCD}(n, 10) = 1$ , it can be shown that there always exists a value  $k$ , for which  $R(k)$  is divisible by  $n$ , and let  $A(n)$  be the least such value of  $k$ ; for example,  $A(7) = 6$  and  $A(41) = 5$ .

You are given that for all primes,  $p > 5$ , that  $p - 1$  is divisible by  $A(p)$ . For example, when  $p = 41$ ,  $A(41) = 5$ , and 40 is divisible by 5.

However, there are rare composite values for which this is also true; the first five examples being 91, 259, 451, 481, and 703.

Find the sum of the first twenty-five composite values of  $n$  for which  $\text{GCD}(n, 10) = 1$  and  $n - 1$  is divisible by  $A(n)$ .

### Problem 131: Prime cube partnership

There are some prime values,  $p$ , for which there exists a positive integer,  $n$ , such that the expression  $n^3 + n^2p$  is a perfect cube.

For example, when  $p = 19$ ,  $8^3 + 8^2 \times 19 = 12^3$ .

What is perhaps most surprising is that for each prime with this property the value of  $n$  is unique, and there are only four such primes below one-hundred.

How many primes below one million have this remarkable property?

### Problem 132: Large repunit factors

A number consisting entirely of ones is called a repunit. We shall define  $R(k)$  to be a repunit of length  $k$ .

For example,  $R(10) = 1111111111 = 11 \times 41 \times 271 \times 9091$ , and the sum of these prime factors is 9414.

Find the sum of the first forty prime factors of  $R(10^9)$ .

### Problem 133: Repunit nonfactors

A number consisting entirely of ones is called a repunit. We shall define  $R(k)$  to be a repunit of length  $k$ ; for example,  $R(6) = 111111$ .

Let us consider repunits of the form  $R(10^n)$ .

Although  $R(10)$ ,  $R(100)$ , or  $R(1000)$  are not divisible by 17,  $R(10000)$  is divisible by 17. Yet there is no value of  $n$  for which  $R(10^n)$  will divide by 19. In fact, it is remarkable that 11, 17, 41, and 73 are the only four primes below one-hundred that can be a factor of  $R(10^n)$ .

Find the sum of all the primes below one-hundred thousand that will never be a factor of  $R(10^n)$ .

### Problem 134: Prime pair connection

Consider the consecutive primes  $p_1 = 19$  and  $p_2 = 23$ . It can be verified that 1219 is the smallest number such that the last digits are formed by  $p_1$  whilst also being divisible by  $p_2$ .

In fact, with the exception of  $p_1 = 3$  and  $p_2 = 5$ , for every pair of consecutive primes,  $p_2 > p_1$ , there exist values of  $n$  for which the last digits are formed by  $p_1$  and  $n$  is divisible by  $p_2$ . Let  $S$  be the smallest of these values of  $n$ .

Find  $\sum S$  for every pair of consecutive primes with  $5 \leq p_1 \leq 1000000$ .

### Problem 135: Same differences

Given the positive integers,  $x$ ,  $y$ , and  $z$ , are consecutive terms of an arithmetic progression, the least value of the positive integer,  $n$ , for which the equation,  $x^2 - y^2 - z^2 = n$ , has exactly two solutions is  $n = 27$ :

$$34^2 - 27^2 - 20^2 = 12^2 - 9^2 - 6^2 = 27$$

It turns out that  $n = 1155$  is the least value which has exactly ten solutions.

How many values of  $n$  less than one million have exactly ten distinct solutions?

### Problem 136: Singleton difference

The positive integers,  $x$ ,  $y$ , and  $z$ , are consecutive terms of an arithmetic progression. Given that  $n$  is a positive integer, the equation,  $x^2 - y^2 - z^2 = n$ , has exactly one solution when  $n = 20$ :

$$13^2 - 10^2 - 7^2 = 20$$

In fact there are twenty-five values of  $n$  below one hundred for which the equation has a unique solution.

How many values of  $n$  less than fifty million have exactly one solution?

### Problem 137: Fibonacci golden nuggets

Consider the infinite polynomial series  $A_F(x) = xF_1 + x^2F_2 + x^3F_3 + \dots$ , where  $F_k$  is the  $k$ th term in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, ...; that is,  $F_k = F_{k-1} + F_{k-2}$ ,  $F_1 = 1$  and  $F_2 = 1$ .

For this problem we shall be interested in values of  $x$  for which  $A_F(x)$  is a positive integer.

$$\begin{aligned} \text{Surprisingly } A_F(1/2) &= (1/2) \cdot 1 + (1/2)^2 \cdot 1 + (1/2)^3 \cdot 2 + (1/2)^4 \cdot 3 + (1/2)^5 \cdot 5 + \\ &\quad \dots \\ &= 1/2 + 1/4 + 2/8 + 3/16 + 5/32 + \dots \\ &= 2 \end{aligned}$$

The corresponding values of  $x$  for the first five natural numbers are shown below.

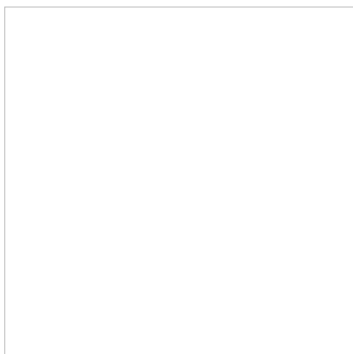
$x$	$A_F(x)$
$\sqrt{2}-1$	1
$1/2$	2
$(\sqrt{13}-2)/3$	3
$(\sqrt{89}-5)/8$	4
$(\sqrt{34}-3)/5$	5

We shall call  $A_F(x)$  a golden nugget if  $x$  is rational, because they become increasingly rarer; for example, the 10th golden nugget is 74049690.

Find the 15th golden nugget.

### Problem 138: Special isosceles triangles

Consider the isosceles triangle with base length,  $b = 16$ , and legs,  $L = 17$ .



By using the Pythagorean theorem it can be seen that the height of the triangle,  $h = \sqrt{17^2 - 8^2} = 15$ , which is one less than the base length.

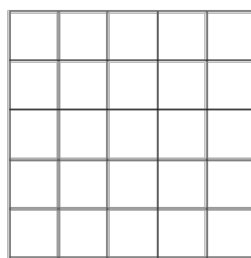
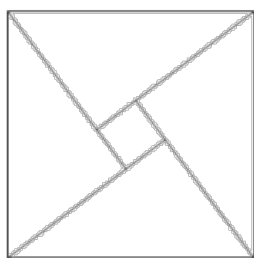
With  $b = 272$  and  $L = 305$ , we get  $h = 273$ , which is one more than the base length, and this is the second smallest isosceles triangle with the property that  $h = b \pm 1$ .

Find  $\sum L$  for the twelve smallest isosceles triangles for which  $h = b \pm 1$  and  $b, L$  are positive integers.

### Problem 139: Pythagorean tiles

Let  $(a, b, c)$  represent the three sides of a right angle triangle with integral length sides. It is possible to place four such triangles together to form a square with length  $c$ .

For example,  $(3, 4, 5)$  triangles can be placed together to form a 5 by 5 square with a 1 by 1 hole in the middle and it can be seen that the 5 by 5 square can be tiled with twenty-five 1 by 1 squares.



However, if  $(5, 12, 13)$  triangles were used then the hole would measure 7 by 7 and these could not be used to tile the 13 by 13 square.

Given that the perimeter of the right triangle is less than one-hundred million, how many Pythagorean triangles would allow such a tiling to take place?

**Problem 140: Modified Fibonacci golden nuggets**

Consider the infinite polynomial series  $A_G(x) = xG_1 + x^2G_2 + x^3G_3 + \dots$ , where  $G_k$  is the  $k$ th term of the second order recurrence relation  $G_k = G_{k-1} + G_{k-2}$ ,  $G_1 = 1$  and  $G_2 = 4$ ; that is, 1, 4, 5, 9, 14, 23, ... .

For this problem we shall be concerned with values of  $x$  for which  $A_G(x)$  is a positive integer.

The corresponding values of  $x$  for the first five natural numbers are shown below.

$x$	$A_G(x)$
$(\sqrt{5}-1)/4$	1
$2/5$	2
$(\sqrt{22}-2)/6$	3
$(\sqrt{137}-5)/14$	4
$1/2$	5

We shall call  $A_G(x)$  a golden nugget if  $x$  is rational, because they become increasingly rarer; for example, the 20th golden nugget is 211345365.

Find the sum of the first thirty golden nuggets.

**Problem 141: Investigating progressive numbers,  $n$ , which are also square**

A positive integer,  $n$ , is divided by  $d$  and the quotient and remainder are  $q$  and  $r$  respectively. In addition  $d$ ,  $q$ , and  $r$  are consecutive positive integer terms in a geometric sequence, but not necessarily in that order.

For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio  $3/2$ ).

We will call such numbers,  $n$ , progressive.

Some progressive numbers, such as 9 and  $10404 = 102^2$ , happen to also be perfect squares.

The sum of all progressive perfect squares below one hundred thousand is 124657.

Find the sum of all progressive perfect squares below one trillion ( $10^{12}$ ).

**Problem 142: Perfect Square Collection**

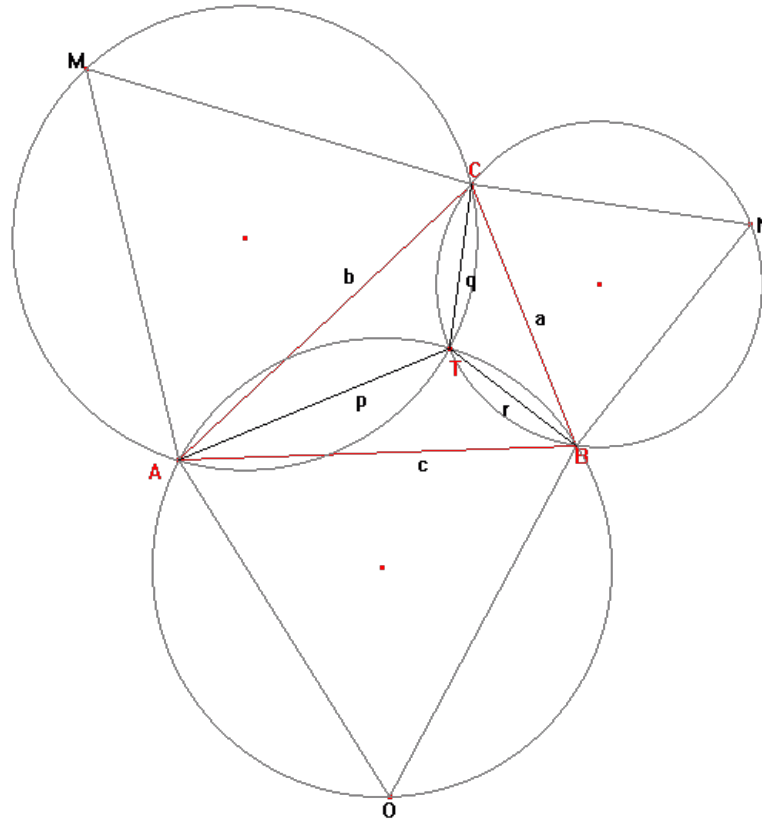
Find the smallest  $x + y + z$  with integers  $x > y > z > 0$  such that  $x + y$ ,  $x - y$ ,  $x + z$ ,  $x - z$ ,  $y + z$ ,  $y - z$  are all perfect squares.

### Problem 143: Investigating the Torricelli point of a triangle

Let  $ABC$  be a triangle with all interior angles being less than 120 degrees. Let  $X$  be any point inside the triangle and let  $XA = p$ ,  $XC = q$ , and  $XB = r$ .

Fermat challenged Torricelli to find the position of  $X$  such that  $p + q + r$  was minimised.

Toricelli was able to prove that if equilateral triangles  $AOB$ ,  $BNC$  and  $AMC$  are constructed on each side of triangle  $ABC$ , the circumscribed circles of  $AOB$ ,  $BNC$ , and  $AMC$  will intersect at a single point,  $T$ , inside the triangle. Moreover he proved that  $T$ , called the Torricelli/Fermat point, minimises  $p + q + r$ . Even more remarkable, it can be shown that when the sum is minimised,  $AN = BM = CO = p + q + r$  and that  $AN$ ,  $BM$  and  $CO$  also intersect at  $T$ .



If the sum is minimised and  $a$ ,  $b$ ,  $c$ ,  $p$ ,  $q$  and  $r$  are all positive integers we shall call triangle  $ABC$  a Torricelli triangle. For example,  $a = 399$ ,  $b = 455$ ,  $c = 511$  is an example of a Torricelli triangle, with  $p + q + r = 784$ .

Find the sum of all distinct values of  $p + q + r \leq 120000$  for Torricelli triangles.



#### Problem 144: Investigating multiple reflections of a laser beam

In laser physics, a "white cell" is a mirror system that acts as a delay line for the laser beam. The beam enters the cell, bounces around on the mirrors, and eventually works its way back out.

The specific white cell we will be considering is an ellipse with the equation  $4x^2 + y^2 = 100$

The section corresponding to  $-0.01 \leq x \leq +0.01$  at the top is missing, allowing the light to enter and exit through the hole.



The light beam in this problem starts at the point  $(0.0, 10.1)$  just outside the white cell, and the beam first impacts the mirror at  $(1.4, -9.6)$ .

Each time the laser beam hits the surface of the ellipse, it follows the usual law of reflection "angle of incidence equals angle of reflection." That is, both the incident and reflected beams make the same angle with the normal line at the point of incidence.

In the figure on the left, the red line shows the first two points of contact between the laser beam and the wall of the white cell; the blue line shows the line tangent to the ellipse at the point of incidence of the first bounce.

The slope  $m$  of the tangent line at any point  $(x, y)$  of the given ellipse is:  $m = -4x/y$

The normal line is perpendicular to this tangent line at the point of incidence.

The animation on the right shows the first 10 reflections of the beam.

How many times does the beam hit the internal surface of the white cell before exiting?

#### Problem 145: How many reversible numbers are there below one-billion?

Some positive integers  $n$  have the property that the sum  $[n + \text{reverse}(n)]$  consists entirely of odd (decimal) digits. For instance,  $36 + 63 = 99$  and  $409 + 904 = 1313$ . We will call such numbers *reversible*; so 36, 63, 409, and 904 are reversible. Leading zeroes are not allowed in either  $n$  or  $\text{reverse}(n)$ .

There are 120 reversible numbers below one-thousand.

How many reversible numbers are there below one-billion ( $10^9$ )?

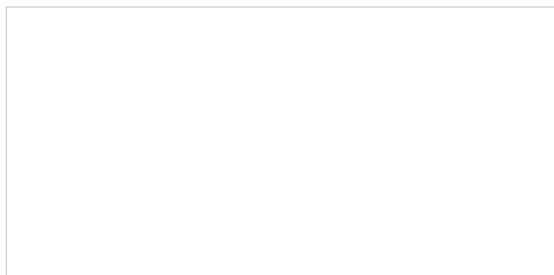
#### Problem 146: Investigating a Prime Pattern

The smallest positive integer  $n$  for which the numbers  $n^2+1$ ,  $n^2+3$ ,  $n^2+7$ ,  $n^2+9$ ,  $n^2+13$ , and  $n^2+27$  are consecutive primes is 10. The sum of all such integers  $n$  below one-million is 1242490.

What is the sum of all such integers  $n$  below 150 million?

### Problem 147: Rectangles in cross-hatched grids

In a 3x2 cross-hatched grid, a total of 37 different rectangles could be situated within that grid as indicated in the sketch.



There are 5 grids smaller than 3x2, vertical and horizontal dimensions being important, i.e. 1x1, 2x1, 3x1, 1x2 and 2x2. If each of them is cross-hatched, the following number of different rectangles could be situated within those smaller grids:

1x1: 1

2x1: 4

3x1: 8

1x2: 4

2x2: 18

Adding those to the 37 of the 3x2 grid, a total of 72 different rectangles could be situated within 3x2 and smaller grids.

How many different rectangles could be situated within 47x43 and smaller grids?

### Problem 148: Exploring Pascal's triangle

We can easily verify that none of the entries in the first seven rows of Pascal's triangle are divisible by 7:

				1				
			1		1			
		1		2		1		
	1		3		3		1	
	1	4		6		4		1
	1	5	10		10	5		1
	1	6	15	20		15	6	1

However, if we check the first one hundred rows, we will find that only 2361 of the 5050 entries are *not* divisible by 7.

Find the number of entries which are *not* divisible by 7 in the first one billion ( $10^9$ ) rows of Pascal's triangle.

### Problem 149: Searching for a maximum-sum subsequence

Looking at the table below, it is easy to verify that the maximum possible sum of adjacent numbers in any direction (horizontal, vertical, diagonal or anti-diagonal) is 16 (= 8 + 7 + 1).

-2	5	3	2
9	-6	5	1
3	2	7	3
-1	8	-4	8

Now, let us repeat the search, but on a much larger scale:

First, generate four million pseudo-random numbers using a specific form of what is known as a "Lagged Fibonacci Generator":

For  $1 \leq k \leq 55$ ,  $s_k = [100003 - 200003k + 300007k^3] \text{ (modulo } 1000000) - 500000$ .

For  $56 \leq k \leq 4000000$ ,  $s_k = [s_{k-24} + s_{k-55} + 1000000] \text{ (modulo } 1000000) - 500000$ .

Thus,  $s_{10} = -393027$  and  $s_{100} = 86613$ .

The terms of  $s$  are then arranged in a  $2000 \times 2000$  table, using the first 2000 numbers to fill the first row (sequentially), the next 2000 numbers to fill the second row, and so on.

Finally, find the greatest sum of (any number of) adjacent entries in any direction (horizontal, vertical, diagonal or anti-diagonal).

### Problem 150: Searching a triangular array for a sub-triangle having minimum-sum

In a triangular array of positive and negative integers, we wish to find a sub-triangle such that the sum of the numbers it contains is the smallest possible.

In the example below, it can be easily verified that the marked triangle satisfies this condition having a sum of -42.

□

We wish to make such a triangular array with one thousand rows, so we generate 500500 pseudo-random numbers  $s_k$  in the range  $\pm 2^{19}$ , using a type of random number generator (known as a Linear Congruential Generator) as follows:

```
t := 0
for k = 1 up to k = 500500:
  t := (615949*t + 797807) modulo 220
  sk := t - 219
```

Thus:  $s_1 = 273519$ ,  $s_2 = -153582$ ,  $s_3 = 450905$  etc

Our triangular array is then formed using the pseudo-random numbers thus:

```
      s1
     s2 s3
    s4 s5 s6
   s7 s8 s9 s10
  ...
```

Sub-triangles can start at any element of the array and extend down as far as we like (taking-in the two elements directly below it from the next row, the three elements directly below from the row after that, and so on).

The "sum of a sub-triangle" is defined as the sum of all the elements it contains.

Find the smallest possible sub-triangle sum.

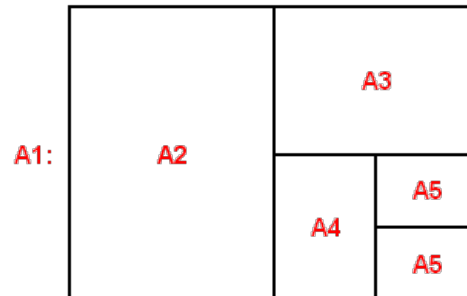
### Problem 151: Paper sheets of standard sizes: an expected-value problem

A printing shop runs 16 batches (jobs) every week and each batch requires a sheet of special colour-proofing paper of size A5.

Every Monday morning, the foreman opens a new envelope, containing a large sheet of the special paper with size A1.

He proceeds to cut it in half, thus getting two sheets of size A2. Then he cuts one of them in half to get two sheets of size A3 and so on until he obtains the A5-size sheet needed for the first batch of the week.

All the unused sheets are placed back in the envelope.



At the beginning of each subsequent batch, he takes from the envelope one sheet of paper at random. If it is of size A5, he uses it. If it is larger, he repeats the 'cut-in-half' procedure until he has what he needs and any remaining sheets are always placed back in the envelope.

Excluding the first and last batch of the week, find the expected number of times (during each week) that the foreman finds a single sheet of paper in the envelope.

Give your answer rounded to six decimal places using the format x.xxxxxx .

### Problem 152: Writing $1/2$ as a sum of inverse squares

There are several ways to write the number  $1/2$  as a sum of inverse squares using *distinct* integers.

For instance, the numbers {2,3,4,5,7,12,15,20,28,35} can be used:

$$\frac{1}{2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2} + \frac{1}{28^2} + \frac{1}{35^2}$$

In fact, only using integers between 2 and 45 inclusive, there are exactly three ways to do it, the remaining two being: {2,3,4,6,7,9,10,20,28,35,36,45} and {2,3,4,6,7,9,12,15,28,30,35,36,45}.

How many ways are there to write the number  $1/2$  as a sum of inverse squares using distinct integers between 2 and 80 inclusive?

### Problem 153: Investigating Gaussian Integers

As we all know the equation  $x^2 = -1$  has no solutions for real  $x$ .

If we however introduce the imaginary number  $i$  this equation has two solutions:  $x=i$  and  $x=-i$ .

If we go a step further the equation  $(x-3)^2 = -4$  has two complex solutions:  $x=3+2i$  and  $x=3-2i$ .

$x=3+2i$  and  $x=3-2i$  are called each others' complex conjugate.

Numbers of the form  $a+bi$  are called complex numbers.

In general  $a+bi$  and  $a-bi$  are each other's complex conjugate.

A Gaussian Integer is a complex number  $a+bi$  such that both  $a$  and  $b$  are integers.

The regular integers are also Gaussian integers (with  $b=0$ ).

To distinguish them from Gaussian integers with  $b \neq 0$  we call such integers "rational integers."

A Gaussian integer is called a divisor of a rational integer  $n$  if the result is also a Gaussian integer.

If for example we divide 5 by  $1+2i$  we can simplify  $\frac{5}{1+2i}$  in the following manner:

Multiply numerator and denominator by the complex conjugate of  $1+2i$ :  $1-2i$ .

The result is  $\frac{5}{1+2i} = \frac{5}{1+2i} \frac{1-2i}{1-2i} = \frac{5(1-2i)}{1-(2i)^2} = \frac{5(1-2i)}{1-(-4)} = \frac{5(1-2i)}{5} = 1-2i$ .

So  $1+2i$  is a divisor of 5.

Note that  $1+i$  is not a divisor of 5 because:

Note also that if the Gaussian Integer  $(a+bi)$  is a divisor of a rational integer  $n$ , then its complex conjugate  $(a-bi)$  is also a divisor of  $n$ .

In fact, 5 has six divisors such that the real part is positive:  $\{1, 1+i, 1-2i, 2+i, 2-i, 5\}$ .

The following is a table of all of the divisors for the first five positive rational integers:

$n$	Gaussian integer divisors with positive real part	Sum $s(n)$ of these divisors
1	1	1
2	1, $1+i$ , $1-i$ , 2	5
3	1, 3	4
4	1, $1+i$ , $1-i$ , 2, $2+2i$ , $2-2i$ , 4	13
5	1, $1+2i$ , $1-2i$ , $2+i$ , $2-i$ , 5	12

For divisors with positive real parts, then, we have:

For  $1 \leq n \leq 10^5$ ,  $\sum s(n) = 17924657155$ .

What is  $\sum s(n)$  for  $1 \leq n \leq 10^8$ ?

### Problem 154: Exploring Pascal's pyramid

A triangular pyramid is constructed using spherical balls so that each ball rests on exactly three balls of the next lower level.



Then, we calculate the number of paths leading from the apex to each position:

A path starts at the apex and progresses downwards to any of the three spheres directly below the current position.

Consequently, the number of paths to reach a certain position is the sum of the numbers immediately above it (depending on the position, there are up to three numbers above it).

The result is *Pascal's pyramid* and the numbers at each level  $n$  are the coefficients of the trinomial expansion  $(x + y + z)^n$ .

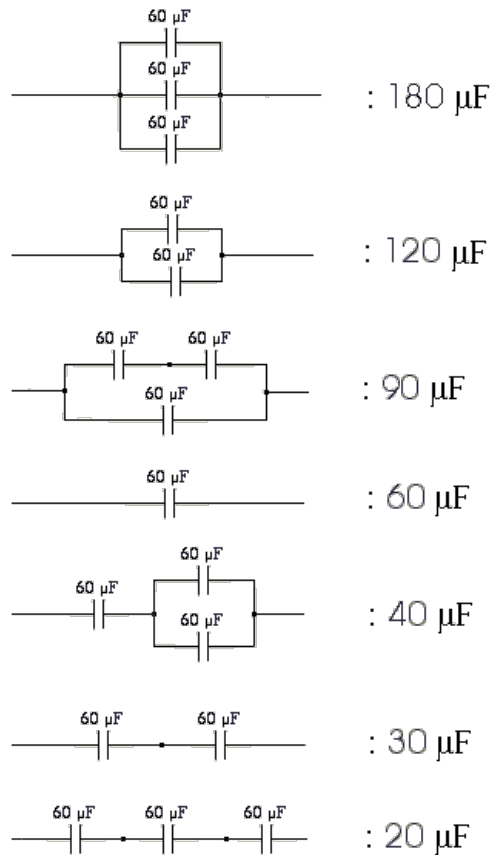
How many coefficients in the expansion of  $(x + y + z)^{200000}$  are multiples of  $10^{12}$ ?

### Problem 155: Counting Capacitor Circuits

An electric circuit uses exclusively identical capacitors of the same value  $C$ .

The capacitors can be connected in series or in parallel to form sub-units, which can then be connected in series or in parallel with other capacitors or other sub-units to form larger sub-units, and so on up to a final circuit.

Using this simple procedure and up to  $n$  identical capacitors, we can make circuits having a range of different total capacitances. For example, using up to  $n=3$  capacitors of  $60\ \mu\text{F}$  each, we can obtain the following 7 distinct total capacitance values:



If we denote by  $D(n)$  the number of distinct total capacitance values we can obtain when using up to  $n$  equal-valued capacitors and the simple procedure described above, we have:  $D(1)=1$ ,  $D(2)=3$ ,  $D(3)=7$  ...

Find  $D(18)$ .

*Reminder* : When connecting capacitors  $C_1$ ,  $C_2$  etc in parallel, the total capacitance is  $C_T = C_1 + C_2 + \dots$ ,

whereas when connecting them in series, the overall capacitance is given by:

### Problem 156: Counting Digits

Starting from zero the natural numbers are written down in base 10 like this:  
0 1 2 3 4 5 6 7 8 9 10 11 12....

Consider the digit  $d=1$ . After we write down each number  $n$ , we will update the number of ones that have occurred and call this number  $f(n,1)$ . The first values for  $f(n,1)$ , then, are as follows:

$n$	$f(n,1)$
0	0
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	2
11	4
12	5

Note that  $f(n,1)$  never equals 3.

So the first two solutions of the equation  $f(n,1)=n$  are  $n=0$  and  $n=1$ . The next solution is  $n=199981$ .

In the same manner the function  $f(n,d)$  gives the total number of digits  $d$  that have been written down after the number  $n$  has been written.

In fact, for every digit  $d \neq 0$ , 0 is the first solution of the equation  $f(n,d)=n$ .

Let  $S(d)$  be the sum of all the solutions for which  $f(n,d)=n$ .

You are given that  $S(1)=22786974071$ .

Find  $\sum S(d)$  for  $1 \leq d \leq 9$ .

Note: if, for some  $n$ ,  $f(n,d)=n$  for more than one value of  $d$  this value of  $n$  is counted again for every value of  $d$  for which  $f(n,d)=n$ .

### Problem 157: Solving the diophantine equation $\frac{1}{a} + \frac{1}{b} = \frac{p}{10^n}$

Consider the diophantine equation  $\frac{1}{a} + \frac{1}{b} = \frac{p}{10^n}$  with  $a, b, p, n$  positive integers and  $a \leq b$ .

For  $n=1$  this equation has 20 solutions that are listed below:

$$\begin{array}{ccccc}
 \frac{1}{1} + \frac{1}{1} = \frac{20}{10} & \frac{1}{1} + \frac{1}{2} = \frac{15}{10} & \frac{1}{1} + \frac{1}{5} = \frac{12}{10} & \frac{1}{1} + \frac{1}{10} = \frac{11}{10} & \frac{1}{2} + \frac{1}{2} = \frac{10}{10} \\
 \frac{1}{2} + \frac{1}{5} = \frac{7}{10} & \frac{1}{2} + \frac{1}{10} = \frac{6}{10} & \frac{1}{3} + \frac{1}{6} = \frac{5}{10} & \frac{1}{3} + \frac{1}{15} = \frac{4}{10} & \frac{1}{4} + \frac{1}{4} = \frac{5}{10} \\
 \frac{1}{4} + \frac{1}{20} = \frac{3}{10} & \frac{1}{5} + \frac{1}{5} = \frac{4}{10} & \frac{1}{5} + \frac{1}{10} = \frac{3}{10} & \frac{1}{6} + \frac{1}{30} = \frac{2}{10} & \frac{1}{10} + \frac{1}{10} = \frac{2}{10} \\
 \frac{1}{11} + \frac{1}{110} = \frac{1}{10} & \frac{1}{12} + \frac{1}{60} = \frac{1}{10} & \frac{1}{14} + \frac{1}{35} = \frac{1}{10} & \frac{1}{15} + \frac{1}{30} = \frac{1}{10} & \frac{1}{20} + \frac{1}{20} = \frac{1}{10}
 \end{array}$$

How many solutions has this equation for  $1 \leq n \leq 9$ ?



**Problem 158: Exploring strings for which only one character comes lexicographically after its neighbour to the left**

Taking three different letters from the 26 letters of the alphabet, character strings of length three can be formed.

Examples are 'abc', 'hat' and 'zyx'.

When we study these three examples we see that for 'abc' two characters come lexicographically after its neighbour to the left.

For 'hat' there is exactly one character that comes lexicographically after its neighbour to the left. For 'zyx' there are zero characters that come lexicographically after its neighbour to the left.

In all there are 10400 strings of length 3 for which exactly one character comes lexicographically after its neighbour to the left.

We now consider strings of  $n \leq 26$  different characters from the alphabet.

For every  $n$ ,  $p(n)$  is the number of strings of length  $n$  for which exactly one character comes lexicographically after its neighbour to the left.

What is the maximum value of  $p(n)$ ?

**Problem 159: Digital root sums of factorisations**

A composite number can be factored many different ways. For instance, not including multiplication by one, 24 can be factored in 7 distinct ways:

$$24 = 2 \times 2 \times 2 \times 3$$

$$24 = 2 \times 3 \times 4$$

$$24 = 2 \times 2 \times 6$$

$$24 = 4 \times 6$$

$$24 = 3 \times 8$$

$$24 = 2 \times 12$$

$$24 = 24$$

Recall that the digital root of a number, in base 10, is found by adding together the digits of that number, and repeating that process until a number is arrived at that is less than 10. Thus the digital root of 467 is 8.

We shall call a Digital Root Sum (DRS) the sum of the digital roots of the individual factors of our number.

The chart below demonstrates all of the DRS values for 24.

Factorisation	Digital Root Sum
$2 \times 2 \times 2 \times 3$	9
$2 \times 3 \times 4$	9
$2 \times 2 \times 6$	10
$4 \times 6$	10
$3 \times 8$	11
$2 \times 12$	5
24	6

The maximum Digital Root Sum of 24 is 11.

The function  $\text{mdrs}(n)$  gives the maximum Digital Root Sum of  $n$ . So  $\text{mdrs}(24)=11$ .

Find  $\sum \text{mdrs}(n)$  for  $1 < n < 1,000,000$ .

**Problem 160: Factorial trailing digits**

For any  $N$ , let  $f(N)$  be the last five digits before the trailing zeroes in  $N!$ .

For example,

$$9! = 362880 \text{ so } f(9)=36288$$

$$10! = 3628800 \text{ so } f(10)=36288$$

$$20! = 2432902008176640000 \text{ so } f(20)=17664$$

Find  $f(1,000,000,000,000)$

### Problem 161: Triominoes

A triomino is a shape consisting of three squares joined via the edges. There are two basic forms:

□

If all possible orientations are taken into account there are six:

□

Any  $n$  by  $m$  grid for which  $n \times m$  is divisible by 3 can be tiled with triominoes.

If we consider tilings that can be obtained by reflection or rotation from another tiling as different there are 41 ways a 2 by 9 grid can be tiled with triominoes:

1



In how many ways can a 9 by 12 grid be tiled in this way by triominoes?

### Problem 162: Hexadecimal numbers

In the hexadecimal number system numbers are represented using 16 different digits:

0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

The hexadecimal number AF when written in the decimal number system equals  $10 \times 16 + 15 = 175$ .

In the 3-digit hexadecimal numbers 10A, 1A0, A10, and A01 the digits 0,1 and A are all present. Like numbers written in base ten we write hexadecimal numbers without leading zeroes.

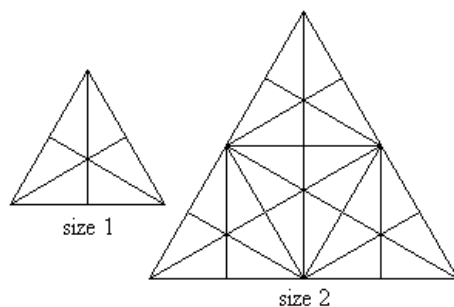
How many hexadecimal numbers containing at most sixteen hexadecimal digits exist with all of the digits 0,1, and A present at least once?

Give your answer as a hexadecimal number.

(A,B,C,D,E and F in upper case, without any leading or trailing code that marks the number as hexadecimal and without leading zeroes, e.g. 1A3F and not: 1a3f and not 0x1a3f and not \$1A3F and not #1A3F and not 0000001A3F)

### Problem 163: Cross-hatched triangles

Consider an equilateral triangle in which straight lines are drawn from each vertex to the middle of the opposite side, such as in the *size 1* triangle in the sketch below.



Sixteen triangles of either different shape or size or orientation or location can now be observed in that triangle. Using *size 1* triangles as building blocks, larger triangles can be formed, such as the *size 2* triangle in the above sketch. One-hundred and four triangles of either different shape or size or orientation or location can now be observed in that *size 2* triangle.

It can be observed that the *size 2* triangle contains 4 *size 1* triangle building blocks. A *size 3* triangle would contain 9 *size 1* triangle building blocks and a *size n* triangle would thus contain  $n^2$  *size 1* triangle building blocks.

If we denote  $T(n)$  as the number of triangles present in a triangle of *size n*, then

$$T(1) = 16$$

$$T(2) = 104$$

Find  $T(36)$ .

### Problem 164: Numbers for which no three consecutive digits have a sum greater than a given value

How many 20 digit numbers  $n$  (without any leading zero) exist such that no three consecutive digits of  $n$  have a sum greater than 9?

### Problem 165: Intersections

A segment is uniquely defined by its two endpoints.

By considering two line segments in plane geometry there are three possibilities: the segments have zero points, one point, or infinitely many points in common.

Moreover when two segments have exactly one point in common it might be the case that that common point is an endpoint of either one of the segments or of both. If a common point of two segments is not an endpoint of either of the segments it is an interior point of both segments.

We will call a common point  $T$  of two segments  $L_1$  and  $L_2$  a true intersection point of  $L_1$  and  $L_2$  if  $T$  is the only common point of  $L_1$  and  $L_2$  and  $T$  is an interior point of both segments.

Consider the three segments  $L_1$ ,  $L_2$ , and  $L_3$ :

$L_1$ : (27, 44) to (12, 32)

$L_2$ : (46, 53) to (17, 62)

$L_3$ : (46, 70) to (22, 40)

It can be verified that line segments  $L_2$  and  $L_3$  have a true intersection point. We note that as the one of the end points of  $L_3$ : (22,40) lies on  $L_1$  this is not considered to be a true point of intersection.  $L_1$  and  $L_2$  have no common point. So among the three line segments, we find one true intersection point.

Now let us do the same for 5000 line segments. To this end, we generate 20000 numbers using the so-called "Blum Blum Shub" pseudo-random number generator.

$$s_0 = 290797$$

$$s_{n+1} = s_n \times s_n \text{ (modulo } 50515093\text{)}$$

$$t_n = s_n \text{ (modulo } 500\text{)}$$

To create each line segment, we use four consecutive numbers  $t_i$ . That is, the first line segment is given by:

$(t_1, t_2)$  to  $(t_3, t_4)$

The first four numbers computed according to the above generator should be: 27, 144, 12 and 232. The first segment would thus be (27,144) to (12,232).

How many distinct true intersection points are found among the 5000 line segments?

### Problem 166: Criss Cross

A 4x4 grid is filled with digits  $d$ ,  $0 \leq d \leq 9$ .

It can be seen that in the grid

6	3	3	0
5	0	4	3
0	7	1	4
1	2	4	5

the sum of each row and each column has the value 12. Moreover the sum of each diagonal is also 12.

In how many ways can you fill a 4x4 grid with the digits  $d$ ,  $0 \leq d \leq 9$  so that each row, each column, and both diagonals have the same sum?

### Problem 167: Investigating Ulam sequences

For two positive integers  $a$  and  $b$ , the Ulam sequence  $U(a,b)$  is defined by  $U(a,b)_1 = a$ ,  $U(a,b)_2 = b$  and for  $k > 2$ ,  $U(a,b)_k$  is the smallest integer greater than  $U(a,b)_{(k-1)}$  which can be written in exactly one way as the sum of two distinct previous members of  $U(a,b)$ .

For example, the sequence  $U(1,2)$  begins with

1, 2,  $3 = 1 + 2$ ,  $4 = 1 + 3$ ,  $6 = 2 + 4$ ,  $8 = 2 + 6$ ,  $11 = 3 + 8$ ;

5 does not belong to it because  $5 = 1 + 4 = 2 + 3$  has two representations as the sum of two previous members, likewise  $7 = 1 + 6 = 3 + 4$ .

Find  $\sum U(2,2n+1)_k$  for  $2 \leq n \leq 10$ , where  $k = 10^{11}$ .

### Problem 168: Number Rotations

Consider the number 142857. We can right-rotate this number by moving the last digit (7) to the front of it, giving us 714285. It can be verified that  $714285 = 5 \times 142857$ .

This demonstrates an unusual property of 142857: it is a divisor of its right-rotation.

Find the last 5 digits of the sum of all integers  $n$ ,  $10 < n < 10^{100}$ , that have this property.

### Problem 169: Exploring the number of different ways a number can be expressed as a sum of powers of 2

Define  $f(0)=1$  and  $f(n)$  to be the number of different ways  $n$  can be expressed as a sum of integer powers of 2 using each power no more than twice.

For example,  $f(10)=5$  since there are five different ways to express 10:

1 + 1 + 8  
1 + 1 + 4 + 4  
1 + 1 + 2 + 2 + 4  
2 + 4 + 4  
2 + 8

What is  $f(10^{25})$ ?

### Problem 170: Find the largest 0 to 9 pandigital that can be formed by concatenating products

Take the number 6 and multiply it by each of 1273 and 9854:

$6 \times 1273 = 7638$   
 $6 \times 9854 = 59124$

By concatenating these products we get the 1 to 9 pandigital 763859124. We will call 763859124 the "concatenated product of 6 and (1273,9854)". Notice too, that the concatenation of the input numbers, 612739854, is also 1 to 9 pandigital.

The same can be done for 0 to 9 pandigital numbers.

What is the largest 0 to 9 pandigital 10-digit concatenated product of an integer with two or more other integers, such that the concatenation of the input numbers is also a 0 to 9 pandigital 10-digit number?

### Problem 171: Finding numbers for which the sum of the squares of the digits is a square

For a positive integer  $n$ , let  $f(n)$  be the sum of the squares of the digits (in base 10) of  $n$ , e.g.

$f(3) = 3^2 = 9$ ,  
 $f(25) = 2^2 + 5^2 = 4 + 25 = 29$ ,  
 $f(442) = 4^2 + 4^2 + 2^2 = 16 + 16 + 4 = 36$

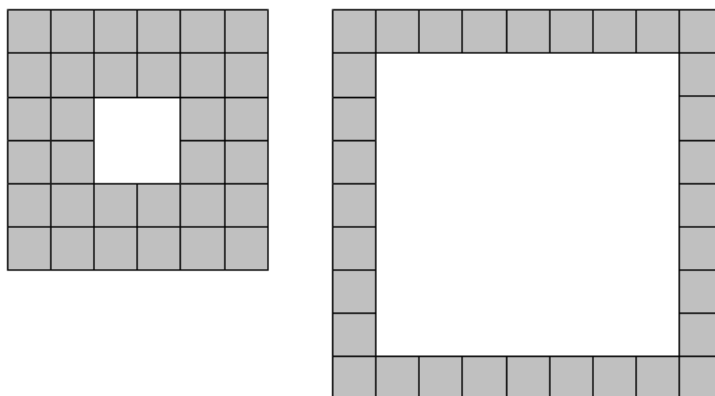
Find the last nine digits of the sum of all  $n$ ,  $0 < n < 10^{20}$ , such that  $f(n)$  is a perfect square.

### Problem 172: Investigating numbers with few repeated digits

How many 18-digit numbers  $n$  (without leading zeros) are there such that no digit occurs more than three times in  $n$ ?

### Problem 173: Using up to one million tiles how many different "hollow" square laminae can be formed?

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry. For example, using exactly thirty-two square tiles we can form two different square laminae:



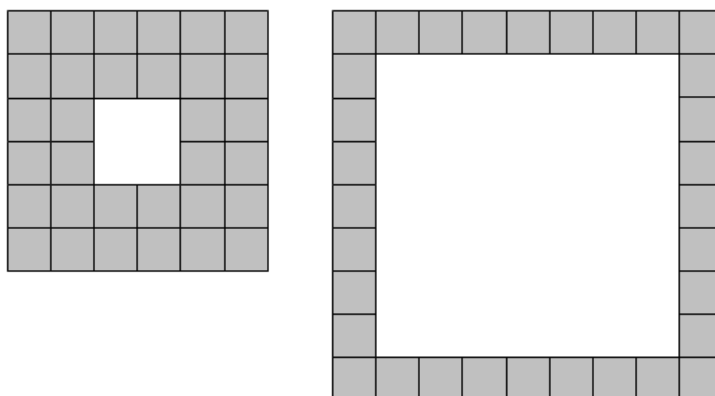
With one-hundred tiles, and not necessarily using all of the tiles at one time, it is possible to form forty-one different square laminae.

Using up to one million tiles how many different square laminae can be formed?

### Problem 174: Counting the number of "hollow" square laminae that can form one, two, three, ... distinct arrangements

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry.

Given eight tiles it is possible to form a lamina in only one way: 3x3 square with a 1x1 hole in the middle. However, using thirty-two tiles it is possible to form two distinct laminae.



If  $t$  represents the number of tiles used, we shall say that  $t = 8$  is type L(1) and  $t = 32$  is type L(2).

Let  $N(n)$  be the number of  $t \leq 1000000$  such that  $t$  is type L( $n$ ); for example,  $N(15) = 832$ .

What is  $\sum N(n)$  for  $1 \leq n \leq 10$ ?

### Problem 175: Fractions involving the number of different ways a number can be expressed as a sum of powers of 2

Define  $f(0)=1$  and  $f(n)$  to be the number of ways to write  $n$  as a sum of powers of 2 where no power occurs more than twice.

For example,  $f(10)=5$  since there are five different ways to express 10:

$$10 = 8+2 = 8+1+1 = 4+4+2 = 4+2+2+1+1 = 4+4+1+1$$

It can be shown that for every fraction  $p/q$  ( $p>0$ ,  $q>0$ ) there exists at least one integer  $n$  such that  $f(n)/f(n-1)=p/q$ .

For instance, the smallest  $n$  for which  $f(n)/f(n-1)=13/17$  is 241.

The binary expansion of 241 is 11110001.

Reading this binary number from the most significant bit to the least significant bit there are 4 one's, 3 zeroes and 1 one. We shall call the string 4,3,1 the *Shortened Binary Expansion* of 241.

Find the Shortened Binary Expansion of the smallest  $n$  for which

$$f(n)/f(n-1)=123456789/987654321.$$

Give your answer as comma separated integers, without any whitespaces.

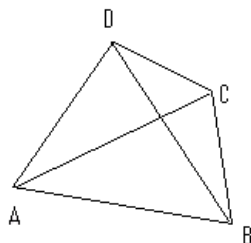
### Problem 176: Right-angled triangles that share a cathetus

The four right-angled triangles with sides (9,12,15), (12,16,20), (5,12,13) and (12,35,37) all have one of the shorter sides (catheti) equal to 12. It can be shown that no other integer sided right-angled triangle exists with one of the catheti equal to 12.

Find the smallest integer that can be the length of a cathetus of exactly 47547 different integer sided right-angled triangles.

### Problem 177: Integer angled Quadrilaterals

Let ABCD be a convex quadrilateral, with diagonals AC and BD. At each vertex the diagonal makes an angle with each of the two sides, creating eight corner angles.



For example, at vertex A, the two angles are CAD, CAB.

We call such a quadrilateral for which all eight corner angles have integer values when measured in degrees an "integer angled quadrilateral". An example of an integer angled quadrilateral is a square, where all eight corner angles are  $45^\circ$ . Another example is given by  $DAC = 20^\circ$ ,  $BAC = 60^\circ$ ,  $ABD = 50^\circ$ ,  $CBD = 30^\circ$ ,  $BCA = 40^\circ$ ,  $DCA = 30^\circ$ ,  $CDB = 80^\circ$ ,  $ADB = 50^\circ$ .

What is the total number of non-similar integer angled quadrilaterals?

Note: In your calculations you may assume that a calculated angle is integral if it is within a tolerance of  $10^9$  of an integer value.

### Problem 178: Step Numbers

Consider the number 45656.

It can be seen that each pair of consecutive digits of 45656 has a difference of one.

A number for which every pair of consecutive digits has a difference of one is called a step number.

A pandigital number contains every decimal digit from 0 to 9 at least once.

How many pandigital step numbers less than  $10^{40}$  are there?

**Problem 179: Consecutive positive divisors**

Find the number of integers  $1 < n < 10^7$ , for which  $n$  and  $n + 1$  have the same number of positive divisors. For example, 14 has the positive divisors 1, 2, 7, 14 while 15 has 1, 3, 5, 15.

**Problem 180: Rational zeros of a function of three variables**

For any integer  $n$ , consider the three functions

$$f_{1,n}(X,Y,Z) = X^{n+1} + Y^{n+1} - Z^{n+1}$$

$$f_{2,n}(X,Y,Z) = (XY + YZ + ZX)(X^{n-1} + Y^{n-1} - Z^{n-1})$$

$$f_{3,n}(X,Y,Z) = XYZ(X^{n-2} + Y^{n-2} - Z^{n-2})$$

and their combination

$$f_n(X,Y,Z) = f_{1,n}(X,Y,Z) + f_{2,n}(X,Y,Z) - f_{3,n}(X,Y,Z)$$

We call  $(X,Y,Z)$  a golden triple of order  $k$  if  $X$ ,  $Y$ , and  $Z$  are all rational numbers of the form  $a/b$  with  $0 < a < b \leq k$  and there is (at least) one integer  $n$ , so that  $f_n(X,Y,Z) = 0$ .

Let  $S(X,Y,Z) = X + Y + Z$ .

Let  $t = u/v$  be the sum of all distinct  $S(X,Y,Z)$  for all golden triples  $(X,Y,Z)$  of order 35.

All the  $S(X,Y,Z)$  and  $t$  must be in reduced form.

Find  $u + v$ .

**Problem 181: Investigating in how many ways objects of two different colours can be grouped**

Having three black objects B and one white object W they can be grouped in 7 ways like this:

(BBBW) (B,BBW) (B,B,BW) (B,B,B,W) (B,BB,W) (BBB,W) (BB,BW)

In how many ways can sixty black objects B and forty white objects W be thus grouped?



### Problem 182: RSA encryption

The RSA encryption is based on the following procedure:

Generate two distinct primes  $p$  and  $q$ .

Compute  $n=pq$  and  $\phi=(p-1)(q-1)$ .

Find an integer  $e$ ,  $1 < e < \phi$ , such that  $\gcd(e, \phi) = 1$ .

A message in this system is a number in the interval  $[0, n-1]$ .

A text to be encrypted is then somehow converted to messages (numbers in the interval  $[0, n-1]$ ).

To encrypt the text, for each message,  $m$ ,  $c = m^e \bmod n$  is calculated.

To decrypt the text, the following procedure is needed: calculate  $d$  such that  $ed = 1 \bmod \phi$ , then for each encrypted message,  $c$ , calculate  $m = c^d \bmod n$ .

There exist values of  $e$  and  $m$  such that  $m^e \bmod n = m$ .

We call messages  $m$  for which  $m^e \bmod n = m$  unconcealed messages.

An issue when choosing  $e$  is that there should not be too many unconcealed messages.

For instance, let  $p=19$  and  $q=37$ .

Then  $n=19 \cdot 37=703$  and  $\phi=18 \cdot 36=648$ .

If we choose  $e=181$ , then, although  $\gcd(181, 648)=1$  it turns out that all possible messages

$m$  ( $0 \leq m \leq n-1$ ) are unconcealed when calculating  $m^e \bmod n$ .

For any valid choice of  $e$  there exist some unconcealed messages.

It's important that the number of unconcealed messages is at a minimum.

Choose  $p=1009$  and  $q=3643$ .

Find the sum of all values of  $e$ ,  $1 < e < \phi(1009, 3643)$  and  $\gcd(e, \phi)=1$ , so that the number of unconcealed messages for this value of  $e$  is at a minimum.

### Problem 183: Maximum product of parts

Let  $N$  be a positive integer and let  $N$  be split into  $k$  equal parts,  $r = N/k$ , so that  $N = r + r + \dots + r$ .

Let  $P$  be the product of these parts,  $P = r \times r \times \dots \times r = r^k$ .

For example, if 11 is split into five equal parts,  $11 = 2.2 + 2.2 + 2.2 + 2.2 + 2.2$ , then  $P = 2.2^5 = 51.53632$ .

Let  $M(N) = P_{\max}$  for a given value of  $N$ .

It turns out that the maximum for  $N = 11$  is found by splitting eleven into four equal parts which leads to  $P_{\max} = (11/4)^4$ ; that is,  $M(11) = 14641/256 = 57.19140625$ , which is a terminating decimal.

However, for  $N = 8$  the maximum is achieved by splitting it into three equal parts, so  $M(8) = 512/27$ , which is a non-terminating decimal.

Let  $D(N) = N$  if  $M(N)$  is a non-terminating decimal and  $D(N) = -N$  if  $M(N)$  is a terminating decimal.

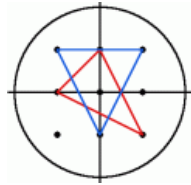
For example,  $\sum D(N)$  for  $5 \leq N \leq 100$  is 2438.

Find  $\sum D(N)$  for  $5 \leq N \leq 10000$ .

### Problem 184: Triangles containing the origin

Consider the set  $I_r$  of points  $(x,y)$  with integer co-ordinates in the interior of the circle with radius  $r$ , centered at the origin, i.e.  $x^2 + y^2 < r^2$ .

For a radius of 2,  $I_2$  contains the nine points  $(0,0)$ ,  $(1,0)$ ,  $(1,1)$ ,  $(0,1)$ ,  $(-1,1)$ ,  $(-1,0)$ ,  $(-1,-1)$ ,  $(0,-1)$  and  $(1,-1)$ . There are eight triangles having all three vertices in  $I_2$  which contain the origin in the interior. Two of them are shown below, the others are obtained from these by rotation.



For a radius of 3, there are 360 triangles containing the origin in the interior and having all vertices in  $I_3$  and for  $I_5$  the number is 10600.

How many triangles are there containing the origin in the interior and having all three vertices in  $I_{105}$ ?

### Problem 185: Number Mind

The game Number Mind is a variant of the well known game Master Mind.

Instead of coloured pegs, you have to guess a secret sequence of digits. After each guess you're only told in how many places you've guessed the correct digit. So, if the sequence was 1234 and you guessed 2036, you'd be told that you have one correct digit; however, you would NOT be told that you also have another digit in the wrong place.

For instance, given the following guesses for a 5-digit secret sequence,

90342 ;2 correct  
70794 ;0 correct  
39458 ;2 correct  
34109 ;1 correct  
51545 ;2 correct  
12531 ;1 correct

The correct sequence 39542 is unique.

Based on the following guesses,

5616185650518293 ;2 correct  
3847439647293047 ;1 correct  
5855462940810587 ;3 correct  
9742855507068353 ;3 correct  
4296849643607543 ;3 correct  
3174248439465858 ;1 correct  
4513559094146117 ;2 correct  
7890971548908067 ;3 correct  
8157356344118483 ;1 correct  
2615250744386899 ;2 correct  
8690095851526254 ;3 correct  
6375711915077050 ;1 correct  
6913859173121360 ;1 correct  
6442889055042768 ;2 correct  
2321386104303845 ;0 correct  
2326509471271448 ;2 correct  
5251583379644322 ;2 correct  
1748270476758276 ;3 correct  
4895722652190306 ;1 correct  
3041631117224635 ;3 correct  
1841236454324589 ;3 correct  
2659862637316867 ;2 correct

Find the unique 16-digit secret sequence.

### Problem 186: Connectedness of a network

Here are the records from a busy telephone system with one million users:

RecNr	Caller	Called
1	200007	100053
2	600183	500439
3	600863	701497
...	...	...

The telephone number of the caller and the called number in record  $n$  are  $\text{Caller}(n) = S_{n-1}$  and  $\text{Called}(n) = S_{2n}$  where  $S_{1,2,3,\dots}$  come from the "Lagged Fibonacci Generator":

For  $1 \leq k \leq 55$ ,  $S_k = [100003 - 200003k + 300007k^2] \pmod{1000000}$

For  $56 \leq k$ ,  $S_k = [S_{k-24} + S_{k-55}] \pmod{1000000}$

If  $\text{Caller}(n) = \text{Called}(n)$  then the user is assumed to have misdialled and the call fails; otherwise the call is successful.

From the start of the records, we say that any pair of users  $X$  and  $Y$  are friends if  $X$  calls  $Y$  or vice-versa. Similarly,  $X$  is a friend of a friend of  $Z$  if  $X$  is a friend of  $Y$  and  $Y$  is a friend of  $Z$ ; and so on for longer chains.

The Prime Minister's phone number is 524287. After how many successful calls, not counting misdials, will 99% of the users (including the PM) be a friend, or a friend of a friend etc., of the Prime Minister?

### Problem 187: Semiprimes

A composite is a number containing at least two prime factors. For example,  $15 = 3 \times 5$ ;  $9 = 3 \times 3$ ;  $12 = 2 \times 2 \times 3$ .

There are ten composites below thirty containing precisely two, not necessarily distinct, prime factors: 4, 6, 9, 10, 14, 15, 21, 22, 25, 26.

How many composite integers,  $n < 10^8$ , have precisely two, not necessarily distinct, prime factors?

### Problem 188: The hyperexponentiation of a number

The *hyperexponentiation* or *tetration* of a number  $a$  by a positive integer  $b$ , denoted by  $a \uparrow\uparrow b$  or  ${}^b a$ , is recursively defined by:

$$a \uparrow\uparrow 1 = a,$$

$$a \uparrow\uparrow (k+1) = a^{(a \uparrow\uparrow k)}.$$

Thus we have e.g.  $3 \uparrow\uparrow 2 = 3^3 = 27$ , hence  $3 \uparrow\uparrow 3 = 3^{27} = 7625597484987$  and  $3 \uparrow\uparrow 4$  is roughly  $10^{3.6383346400240996 \times 10^{12}}$ .

Find the last 8 digits of  $1777 \uparrow\uparrow 1855$ .

### Problem 189: Tri-colouring a triangular grid

Consider the following configuration of 64 triangles:



We wish to colour the interior of each triangle with one of three colours: red, green or blue, so that no two neighbouring triangles have the same colour. Such a colouring shall be called valid. Here, two triangles are said to be neighbouring if they share an edge.

Note: if they only share a vertex, then they are not neighbours.

For example, here is a valid colouring of the above grid:



A colouring  $C'$  which is obtained from a colouring  $C$  by rotation or reflection is considered *distinct* from  $C$  unless the two are identical.

How many distinct valid colourings are there for the above configuration?

### Problem 190: Maximising a weighted product

Let  $S_m = (x_1, x_2, \dots, x_m)$  be the  $m$ -tuple of positive real numbers with  $x_1 + x_2 + \dots + x_m = m$  for which  $P_m = x_1 * x_2^2 * \dots * x_m^m$  is maximised.

For example, it can be verified that  $[P_{10}] = 4112$  ( $[ ]$  is the integer part function).

Find  $\sum [P_m]$  for  $2 \leq m \leq 15$ .

### Problem 191: Prize Strings

A particular school offers cash rewards to children with good attendance and punctuality. If they are absent for three consecutive days or late on more than one occasion then they forfeit their prize.

During an  $n$ -day period a trinary string is formed for each child consisting of L's (late), O's (on time), and A's (absent).

Although there are eighty-one trinary strings for a 4-day period that can be formed, exactly forty-three strings would lead to a prize:

OOOO OOOA OOO L OOA OOA OOA OOLA OAO OAO  
OAO OAO OAO OALO OALO OLO OLO OLA OLA AOO  
AOO AOOL AOAO AOAA AOAL AOLO AOLA AAO AOA AOL  
AALO AALA ALOO ALOA ALAO ALAA LOOO LOOA LOAO LOA  
LAOO LAOA LAOO

How many "prize" strings exist over a 30-day period?

### Problem 192: Best Approximations

Let  $X$  be a real number.

A *best approximation* to  $X$  for the *denominator bound*  $d$  is a rational number  $r/s$  in reduced form, with  $s \leq d$ , such that any rational number which is closer to  $X$  than  $r/s$  has a denominator larger than  $d$ :

$$|p/q - X| < |r/s - X| \Rightarrow q > d$$

For example, the best approximation to  $\sqrt{13}$  for the denominator bound 20 is  $18/5$  and the best approximation to  $\sqrt{13}$  for the denominator bound 30 is  $101/28$ .



Find the sum of all denominators of the best approximations to  $\sqrt{n}$  for the denominator bound  $10^{12}$ , where  $n$  is not a perfect square and  $1 < n \leq 100000$ .

### Problem 193: Squarefree Numbers

A positive integer  $n$  is called squarefree, if no square of a prime divides  $n$ , thus 1, 2, 3, 5, 6, 7, 10, 11 are squarefree, but not 4, 8, 9, 12.

How many squarefree numbers are there below  $2^{60}$ ?

### Problem 194: Coloured Configurations

Consider graphs built with the units A:  and B: , where the units are glued along the vertical edges as in the graph.

A configuration of type  $(a,b,c)$  is a graph thus built of  $a$  units A and  $b$  units B, where the graph's vertices are coloured using up to  $c$  colours, so that no two adjacent vertices have the same colour.

The compound graph above is an example of a configuration of type  $(2,2,6)$ , in fact of type  $(2,2,c)$  for all  $c \geq 4$ .

Let  $N(a,b,c)$  be the number of configurations of type  $(a,b,c)$ .

For example,  $N(1,0,3) = 24$ ,  $N(0,2,4) = 92928$  and  $N(2,2,3) = 20736$ .

Find the last 8 digits of  $N(25,75,1984)$ .

### Problem 195: Inscribed circles of triangles with one angle of 60 degrees

Let's call an integer sided triangle with exactly one angle of 60 degrees a 60-degree triangle.

Let  $r$  be the radius of the inscribed circle of such a 60-degree triangle.

There are 1234 60-degree triangles for which  $r \leq 100$ .

Let  $T(n)$  be the number of 60-degree triangles for which  $r \leq n$ , so

$T(100) = 1234$ ,  $T(1000) = 22767$ , and  $T(10000) = 359912$ .

Find  $T(1053779)$ .

### Problem 196: Prime triplets

Build a triangle from all positive integers in the following way:

```

1
2 3
4 5 6
7 8 9 10
11 12 13 14 15
16 17 18 19 20 21
22 23 24 25 26 27 28
29 30 31 32 33 34 35 36
37 38 39 40 41 42 43 44 45
46 47 48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63 64 65 66
...
```

Each positive integer has up to eight neighbours in the triangle.

A set of three primes is called a *prime triplet* if one of the three primes has the other two as neighbours in the triangle.

For example, in the second row, the prime numbers 2 and 3 are elements of some prime triplet.

If row 8 is considered, it contains two primes which are elements of some prime triplet, i.e. 29 and 31.

If row 9 is considered, it contains only one prime which is an element of some prime triplet: 37.

Define  $S(n)$  as the sum of the primes in row  $n$  which are elements of any prime triplet.

Then  $S(8)=60$  and  $S(9)=37$ .

You are given that  $S(10000)=950007619$ .

Find  $S(5678027) + S(7208785)$ .

### Problem 197: Investigating the behaviour of a recursively defined sequence

Given is the function  $f(x) = \lfloor 2^{30.403243784-x^2} \rfloor \times 10^{-9}$  ( $\lfloor \cdot \rfloor$  is the floor-function), the sequence  $u_n$  is defined by  $u_0 = -1$  and  $u_{n+1} = f(u_n)$ .

Find  $u_n + u_{n+1}$  for  $n = 10^{12}$ .

Give your answer with 9 digits after the decimal point.

### Problem 198: Ambiguous Numbers

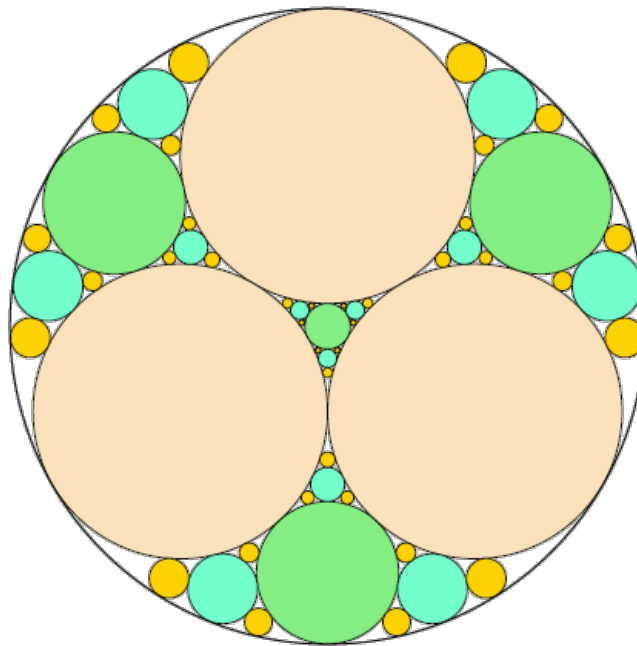
A best approximation to a real number  $x$  for the denominator bound  $d$  is a rational number  $r/s$  (in reduced form) with  $s \leq d$ , so that any rational number  $p/q$  which is closer to  $x$  than  $r/s$  has  $q > d$ .

Usually the best approximation to a real number is uniquely determined for all denominator bounds. However, there are some exceptions, e.g.  $9/40$  has the two best approximations  $1/4$  and  $1/5$  for the denominator bound 6. We shall call a real number  $x$  *ambiguous*, if there is at least one denominator bound for which  $x$  possesses two best approximations. Clearly, an ambiguous number is necessarily rational.

How many ambiguous numbers  $x = p/q$ ,  $0 < x < 1/100$ , are there whose denominator  $q$  does not exceed  $10^8$ ?

### Problem 199: Iterative Circle Packing

Three circles of equal radius are placed inside a larger circle such that each pair of circles is tangent to one another and the inner circles do not overlap. There are four uncovered "gaps" which are to be filled iteratively with more tangent circles.



At each iteration, a maximally sized circle is placed in each gap, which creates more gaps for the next iteration. After 3 iterations (pictured), there are 108 gaps and the fraction of the area which is not covered by circles is 0.06790342, rounded to eight decimal places.

What fraction of the area is not covered by circles after 10 iterations?

Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

### Problem 200: Find the 200th prime-proof sqube containing the contiguous sub-string "200"

We shall define a sqube to be a number of the form  $p^2q^3$ , where  $p$  and  $q$  are distinct primes.

For example,  $200 = 5^2 2^3$  or  $120072949 = 23^2 61^3$ .

The first five squbes are 72, 108, 200, 392, and 500.

Interestingly, 200 is also the first number for which you cannot change any single digit to make a prime; we shall call such numbers, prime-proof. The next prime-proof sqube which contains the contiguous sub-string "200" is 1992008.

Find the 200th prime-proof sqube containing the contiguous sub-string "200".

### Problem 201: Subsets with a unique sum

For any set  $A$  of numbers, let  $\text{sum}(A)$  be the sum of the elements of  $A$ .

Consider the set  $B = \{1, 3, 6, 8, 10, 11\}$ .

There are 20 subsets of  $B$  containing three elements, and their sums are:

$\text{sum}(\{1, 3, 6\}) = 10,$   
 $\text{sum}(\{1, 3, 8\}) = 12,$   
 $\text{sum}(\{1, 3, 10\}) = 14,$   
 $\text{sum}(\{1, 3, 11\}) = 15,$   
 $\text{sum}(\{1, 6, 8\}) = 15,$   
 $\text{sum}(\{1, 6, 10\}) = 17,$   
 $\text{sum}(\{1, 6, 11\}) = 18,$   
 $\text{sum}(\{1, 8, 10\}) = 19,$   
 $\text{sum}(\{1, 8, 11\}) = 20,$   
 $\text{sum}(\{1, 10, 11\}) = 22,$   
 $\text{sum}(\{3, 6, 8\}) = 17,$   
 $\text{sum}(\{3, 6, 10\}) = 19,$   
 $\text{sum}(\{3, 6, 11\}) = 20,$   
 $\text{sum}(\{3, 8, 10\}) = 21,$   
 $\text{sum}(\{3, 8, 11\}) = 22,$   
 $\text{sum}(\{3, 10, 11\}) = 24,$   
 $\text{sum}(\{6, 8, 10\}) = 24,$   
 $\text{sum}(\{6, 8, 11\}) = 25,$   
 $\text{sum}(\{6, 10, 11\}) = 27,$   
 $\text{sum}(\{8, 10, 11\}) = 29.$

Some of these sums occur more than once, others are unique.

For a set  $A$ , let  $U(A, k)$  be the set of unique sums of  $k$ -element subsets of  $A$ , in our example we find  $U(B, 3) = \{10, 12, 14, 18, 21, 25, 27, 29\}$  and  $\text{sum}(U(B, 3)) = 156$ .

Now consider the 100-element set  $S = \{1^2, 2^2, \dots, 100^2\}$ .

$S$  has 100891344545564193334812497256 50-element subsets.

Determine the sum of all integers which are the sum of exactly one of the 50-element subsets of  $S$ , i.e. find  $\text{sum}(U(S, 50))$ .

### Problem 202: Laserbeam

Three mirrors are arranged in the shape of an equilateral triangle, with their reflective surfaces pointing inwards. There is an infinitesimal gap at each vertex of the triangle through which a laser beam may pass.

Label the vertices  $A$ ,  $B$  and  $C$ . There are 2 ways in which a laser beam may enter vertex  $C$ , bounce off 11 surfaces, then exit through the same vertex: one way is shown below; the other is the reverse of that.

□

There are 80840 ways in which a laser beam may enter vertex  $C$ , bounce off 1000001 surfaces, then exit through the same vertex.

In how many ways can a laser beam enter at vertex  $C$ , bounce off 12017639147 surfaces, then exit through the same vertex?

The binomial coefficients  ${}^nC_k$  can be arranged in triangular form, Pascal's triangle, like this:

$$\begin{array}{ccccccccccc} & & & & 1 & & & & & & \\ & & & 1 & & 1 & & & & & \\ & & 1 & & 2 & & 1 & & & & \\ & 1 & & 3 & & 3 & & 1 & & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\ & & & & & & & \dots & & & & & & & & \end{array}$$

It can be seen that the first eight rows of Pascal's triangle contain twelve distinct numbers: 1, 2, 3, 4, 5, 6, 7, 10, 15, 20, 21 and 35.

A positive integer  $n$  is called squarefree if no square of a prime divides  $n$ . Of the twelve distinct numbers in the first eight rows of Pascal's triangle, all except 4 and 20 are squarefree. The sum of the distinct squarefree numbers in the first eight rows is 105.

Find the sum of the distinct squarefree numbers in the first 51 rows of Pascal's triangle.

A Hamming number is a positive number which has no prime factor larger than 5.  
So the first few Hamming numbers are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15.  
There are 1105 Hamming numbers not exceeding  $10^8$ .

We will call a positive number a generalised Hamming number of type  $n$ , if it has no prime factor larger than  $n$ .  
Hence the Hamming numbers are the generalised Hamming numbers of type 5.

How many generalised Hamming numbers of type 100 are there which don't exceed  $10^8$ ?

Peter has nine four-sided (pyramidal) dice, each with faces numbered 1, 2, 3, 4.  
Colin has six six-sided (cubic) dice, each with faces numbered 1, 2, 3, 4, 5, 6.

Peter and Colin roll their dice and compare totals: the highest total wins. The result is a draw if the totals are equal.

What is the probability that Pyramidal Pete beats Cubic Colin? Give your answer rounded to seven decimal places in the form 0.abcdefg

Find the unique positive integer whose square has the form  $1\_2\_3\_4\_5\_6\_7\_8\_9\_0$ , where each “ $\_$ ” is a single digit.



### Problem 207: Integer partition equations

For some positive integers  $k$ , there exists an integer partition of the form  $4 = 2^t + k$ , where  $4^t$ ,  $2^t$ , and  $k$  are all positive integers and  $t$  is a real number.

The first two such partitions are  $4^1 = 2^1 + 2$  and  $4^{1.5849625\dots} = 2^{1.5849625\dots} + 6$ .

Partitions where  $t$  is also an integer are called *perfect*.

For any  $m \geq 1$  let  $P(m)$  be the proportion of such partitions that are perfect with  $k \leq m$ .

Thus  $P(6) = 1/2$ .

In the following table are listed some values of  $P(m)$

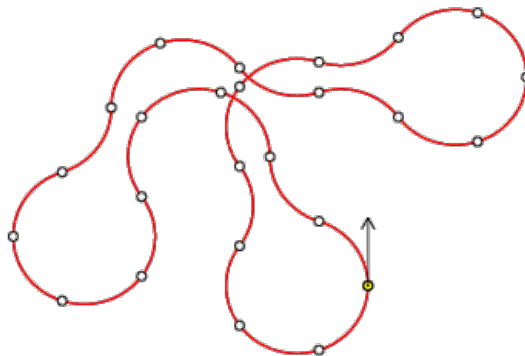
$P(5) = 1/1$   
 $P(10) = 1/2$   
 $P(15) = 2/3$   
 $P(20) = 1/2$   
 $P(25) = 1/2$   
 $P(30) = 2/5$   
...  
 $P(180) = 1/4$   
 $P(185) = 3/13$

Find the smallest  $m$  for which  $P(m) < 1/12345$

### Problem 208: Robot Walks

A robot moves in a series of one-fifth circular arcs ( $72^\circ$ ), with a free choice of a clockwise or an anticlockwise arc for each step, but no turning on the spot.

One of 70932 possible closed paths of 25 arcs starting northward is



Given that the robot starts facing North, how many journeys of 70 arcs in length can it take that return it, after the final arc, to its starting position?

(Any arc may be traversed multiple times.)

### Problem 209: Circular Logic

A  $k$ -input *binary truth table* is a map from  $k$  input bits (binary digits, 0 [false] or 1 [true]) to 1 output bit. For example, the 2-input binary truth tables for the logical AND and XOR functions are:

$x$	$y$	$x \text{ AND } y$
0	0	0
0	1	0
1	0	0
1	1	1

$x$	$y$	$x \text{ XOR } y$
0	0	0
0	1	1
1	0	1
1	1	0

How many 6-input binary truth tables,  $\tau$ , satisfy the formula

$$\tau(a, b, c, d, e, f) \text{ AND } \tau(b, c, d, e, f, a \text{ XOR } (b \text{ AND } c)) = 0$$

for all 6-bit inputs  $(a, b, c, d, e, f)$ ?

### Problem 210: Obtuse Angled Triangles

Consider the set  $S(r)$  of points  $(x, y)$  with integer coordinates satisfying  $|x| + |y| \leq r$ .

Let  $O$  be the point  $(0, 0)$  and  $C$  the point  $(r/4, r/4)$ .

Let  $N(r)$  be the number of points  $B$  in  $S(r)$ , so that the triangle  $OBC$  has an obtuse angle, i.e. the largest angle  $\alpha$  satisfies  $90^\circ < \alpha < 180^\circ$ .

So, for example,  $N(4)=24$  and  $N(8)=100$ .

What is  $N(1,000,000,000)$ ?

### Problem 211: Divisor Square Sum

For a positive integer  $n$ , let  $\sigma_2(n)$  be the sum of the squares of its divisors. For example,

$$\sigma_2(10) = 1 + 4 + 25 + 100 = 130.$$

Find the sum of all  $n$ ,  $0 < n < 64,000,000$  such that  $\sigma_2(n)$  is a perfect square.

### Problem 212: Combined Volume of Cuboids

An *axis-aligned cuboid*, specified by parameters  $\{ (x_0, y_0, z_0), (dx, dy, dz) \}$ , consists of all points  $(X, Y, Z)$  such that  $x_0 \leq X \leq x_0 + dx$ ,  $y_0 \leq Y \leq y_0 + dy$  and  $z_0 \leq Z \leq z_0 + dz$ . The volume of the cuboid is the product,  $dx \times dy \times dz$ . The *combined volume* of a collection of cuboids is the volume of their union and will be less than the sum of the individual volumes if any cuboids overlap.

Let  $C_1, \dots, C_{50000}$  be a collection of 50000 axis-aligned cuboids such that  $C_n$  has parameters

$$\begin{aligned}x_0 &= S_{6n-5} \text{ modulo } 10000 \\y_0 &= S_{6n-4} \text{ modulo } 10000 \\z_0 &= S_{6n-3} \text{ modulo } 10000 \\dx &= 1 + (S_{6n-2} \text{ modulo } 399) \\dy &= 1 + (S_{6n-1} \text{ modulo } 399) \\dz &= 1 + (S_{6n} \text{ modulo } 399)\end{aligned}$$

where  $S_1, \dots, S_{300000}$  come from the "Lagged Fibonacci Generator":

$$\begin{aligned}\text{For } 1 \leq k \leq 55, S_k &= [100003 - 200003k + 300007k^3] \text{ (modulo } 1000000) \\ \text{For } 56 \leq k, S_k &= [S_{k-24} + S_{k-55}] \text{ (modulo } 1000000)\end{aligned}$$

Thus,  $C_1$  has parameters  $\{(7, 53, 183), (94, 369, 56)\}$ ,  $C_2$  has parameters  $\{(2383, 3563, 5079), (42, 212, 344)\}$ , and so on.

The combined volume of the first 100 cuboids,  $C_1, \dots, C_{100}$ , is 723581599.

What is the combined volume of all 50000 cuboids,  $C_1, \dots, C_{50000}$  ?

### Problem 213: Flea Circus

A  $30 \times 30$  grid of squares contains 900 fleas, initially one flea per square.

When a bell is rung, each flea jumps to an adjacent square at random (usually 4 possibilities, except for fleas on the edge of the grid or at the corners).

What is the expected number of unoccupied squares after 50 rings of the bell? Give your answer rounded to six decimal places.

### Problem 214: Totient Chains

Let  $\phi$  be Euler's totient function, i.e. for a natural number  $n$ ,  $\phi(n)$  is the number of  $k$ ,  $1 \leq k \leq n$ , for which  $\gcd(k, n) = 1$ .

By iterating  $\phi$ , each positive integer generates a decreasing chain of numbers ending in 1.

E.g. if we start with 5 the sequence 5, 4, 2, 1 is generated.

Here is a listing of all chains with length 4:

5, 4, 2, 1  
7, 6, 2, 1  
8, 4, 2, 1  
9, 6, 2, 1  
10, 4, 2, 1  
12, 4, 2, 1  
14, 6, 2, 1  
18, 6, 2, 1

Only two of these chains start with a prime, their sum is 12.

What is the sum of all primes less than 40000000 which generate a chain of length 25?

### Problem 215: Crack-free Walls

Consider the problem of building a wall out of  $2 \times 1$  and  $3 \times 1$  bricks (horizontal  $\times$  vertical dimensions) such that, for extra strength, the gaps between horizontally-adjacent bricks never line up in consecutive layers, i.e. never form a "running crack".

For example, the following  $9 \times 3$  wall is not acceptable due to the running crack shown in red:

□

There are eight ways of forming a crack-free  $9 \times 3$  wall, written  $W(9,3) = 8$ .

Calculate  $W(32,10)$ .

### Problem 216: Investigating the primality of numbers of the form $2n^2-1$

Consider numbers  $t(n)$  of the form  $t(n) = 2n^2-1$  with  $n > 1$ .  
The first such numbers are 7, 17, 31, 49, 71, 97, 127 and 161.  
It turns out that only  $49 = 7 \cdot 7$  and  $161 = 7 \cdot 23$  are not prime.  
For  $n \leq 10000$  there are 2202 numbers  $t(n)$  that are prime.

How many numbers  $t(n)$  are prime for  $n \leq 50,000,000$  ?

### Problem 217: Balanced Numbers

A positive integer with  $k$  (decimal) digits is called balanced if its first  $\lceil k/2 \rceil$  digits sum to the same value as its last  $\lfloor k/2 \rfloor$  digits, where  $\lceil x \rceil$ , pronounced *ceiling* of  $x$ , is the smallest integer  $\geq x$ , thus  $\lceil \pi \rceil = 4$  and  $\lfloor 5 \rfloor = 5$ .

So, for example, all palindromes are balanced, as is 13722.

Let  $T(n)$  be the sum of all balanced numbers less than  $10^n$ .  
Thus:  $T(1) = 45$ ,  $T(2) = 540$  and  $T(5) = 334795890$ .

Find  $T(47) \bmod 3^{15}$

### Problem 218: Perfect right-angled triangles

Consider the right angled triangle with sides  $a=7$ ,  $b=24$  and  $c=25$ . The area of this triangle is 84, which is divisible by the perfect numbers 6 and 28.  
Moreover it is a primitive right angled triangle as  $\gcd(a,b)=1$  and  $\gcd(b,c)=1$ .  
Also  $c$  is a perfect square.

We will call a right angled triangle perfect if  
-it is a primitive right angled triangle  
-its hypotenuse is a perfect square

We will call a right angled triangle super-perfect if  
-it is a perfect right angled triangle and  
-its area is a multiple of the perfect numbers 6 and 28.

How many perfect right-angled triangles with  $c \leq 10^{16}$  exist that are not super-perfect?

### Problem 219: Skew-cost coding

Let **A** and **B** be bit strings (sequences of 0's and 1's).

If **A** is equal to the leftmost length(**A**) bits of **B**, then **A** is said to be a *prefix* of **B**.

For example, 00110 is a prefix of 001101001, but not of 00111 or 100110.

A *prefix-free code* of size  $n$  is a collection of  $n$  distinct bit strings such that no string is a prefix of any other. For example, this is a prefix-free code of size 6:

0000, 0001, 001, 01, 10, 11

Now suppose that it costs one penny to transmit a '0' bit, but four pence to transmit a '1'.

Then the total cost of the prefix-free code shown above is 35 pence, which happens to be the cheapest possible for the skewed pricing scheme in question.

In short, we write  $\text{Cost}(6) = 35$ .

What is  $\text{Cost}(10^9)$  ?

### Problem 220: Highway Dragon

Let  $D_0$  be the two-letter string "Fa". For  $n \geq 1$ , derive  $D_n$  from  $D_{n-1}$  by the string-rewriting rules:

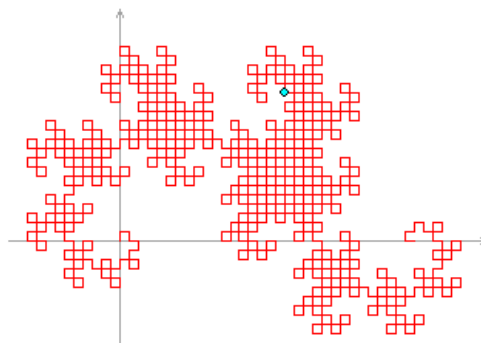
"a"  $\rightarrow$  "aRbFR"

"b"  $\rightarrow$  "LFaLb"

Thus,  $D_0 = \text{"Fa"}$ ,  $D_1 = \text{"FaRbFR"}$ ,  $D_2 = \text{"FaRbFRRLFaLbFR"}$ , and so on.

These strings can be interpreted as instructions to a computer graphics program, with "F" meaning "draw forward one unit", "L" meaning "turn left 90 degrees", "R" meaning "turn right 90 degrees", and "a" and "b" being ignored. The initial position of the computer cursor is (0,0), pointing up towards (0,1).

Then  $D_n$  is an exotic drawing known as the *Highway Dragon* of order  $n$ . For example,  $D_{10}$  is shown below; counting each "F" as one step, the highlighted spot at (18,16) is the position reached after 500 steps.



What is the position of the cursor after  $10^{12}$  steps in  $D_{50}$  ?

Give your answer in the form  $x,y$  with no spaces.

### Problem 221: Alexandrian Integers

We shall call a positive integer  $A$  an "Alexandrian integer", if there exist integers  $p, q, r$  such that:

$$A = p \cdot q \cdot r \quad \text{and} \quad \frac{1}{A} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

For example, 630 is an Alexandrian integer ( $p = 5, q = -7, r = -18$ ). In fact, 630 is the 6<sup>th</sup> Alexandrian integer, the first 6 Alexandrian integers being: 6, 42, 120, 156, 420 and 630.

Find the 150000<sup>th</sup> Alexandrian integer.

### Problem 222: Sphere Packing

What is the length of the shortest pipe, of internal radius 50mm, that can fully contain 21 balls of radii 30mm, 31mm, ..., 50mm?

Give your answer in micrometres ( $10^6$  m) rounded to the nearest integer.

### Problem 223: Almost right-angled triangles I

Let us call an integer sided triangle with sides  $a \leq b \leq c$  *barely acute* if the sides satisfy  $a^2 + b^2 = c^2 + 1$ .

How many barely acute triangles are there with perimeter  $\leq 25,000,000$ ?

### Problem 224: Almost right-angled triangles II

Let us call an integer sided triangle with sides  $a \leq b \leq c$  *barely obtuse* if the sides satisfy  $a^2 + b^2 = c^2 - 1$ .

How many barely obtuse triangles are there with perimeter  $\leq 75,000,000$ ?

### Problem 225: Tribonacci non-divisors

The sequence 1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201 ... is defined by  $T_1 = T_2 = T_3 = 1$  and  $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ .

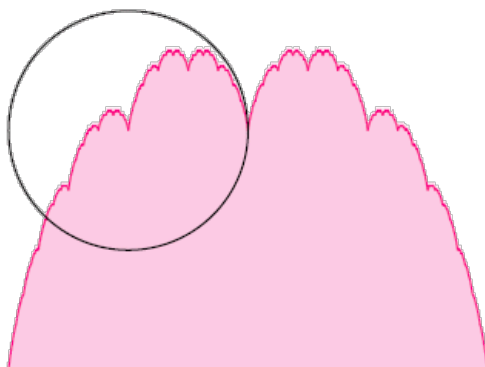
It can be shown that 27 does not divide any terms of this sequence. In fact, 27 is the first odd number with this property.

Find the 124<sup>th</sup> odd number that does not divide any terms of the above sequence.

### Problem 226: A Scoop of Blancmange

The *blancmange curve* is the set of points  $(x,y)$  such that  $0 \leq x \leq 1$  and  $y = S(x)$ , where  $S(x)$  = the distance from  $x$  to the nearest integer.

The area under the blancmange curve is equal to  $\frac{1}{2}$ , shown in pink in the diagram below.



Let  $C$  be the circle with centre  $(\frac{1}{4}, \frac{1}{2})$  and radius  $\frac{1}{4}$ , shown in black in the diagram.

What area under the blancmange curve is enclosed by  $C$ ?

Give your answer rounded to eight decimal places in the form 0.abcdefgh

### Problem 227: The Chase

"The Chase" is a game played with two dice and an even number of players.

The players sit around a table; the game begins with two opposite players having one die each. On each turn, the two players with a die roll it.

If a player rolls a 1, he passes the die to his neighbour on the left; if he rolls a 6, he passes the die to his neighbour on the right; otherwise, he keeps the die for the next turn.

The game ends when one player has both dice after they have been rolled and passed; that player has then lost.

In a game with 100 players, what is the expected number of turns the game lasts?

Give your answer rounded to ten significant digits.

### Problem 228: Minkowski Sums

Let  $S_n$  be the regular  $n$ -sided polygon – or *shape* – whose vertices  $v_k$  ( $k = 1, 2, \dots, n$ ) have coordinates:

$$x_k = \cos(2^{k-1}/n \times 180^\circ)$$

$$y_k = \sin(2^{k-1}/n \times 180^\circ)$$

Each  $S_n$  is to be interpreted as a filled shape consisting of all points on the perimeter and in the interior.

The *Minkowski sum*,  $S+T$ , of two shapes  $S$  and  $T$  is the result of adding every point in  $S$  to every point in  $T$ , where point addition is performed coordinate-wise:  $(u, v) + (x, y) = (u+x, v+y)$ .

For example, the sum of  $S_3$  and  $S_4$  is the six-sided shape shown in pink below:



How many sides does  $S_{1864} + S_{1865} + \dots + S_{1909}$  have?

### Problem 229: Four Representations using Squares

Consider the number 3600. It is very special, because

$$3600 = 48^2 + 36^2$$

$$3600 = 20^2 + 2 \times 40^2$$

$$3600 = 30^2 + 3 \times 30^2$$

$$3600 = 45^2 + 7 \times 15^2$$

Similarly, we find that  $88201 = 99^2 + 280^2 = 287^2 + 2 \times 54^2 = 283^2 + 3 \times 52^2 = 197^2 + 7 \times 84^2$ .

In 1747, Euler proved which numbers are representable as a sum of two squares. We are interested in the numbers  $n$  which admit representations of all of the following four types:

$$n = a_1^2 + b_1^2$$

$$n = a_2^2 + 2b_2^2$$

$$n = a_3^2 + 3b_3^2$$

$$n = a_7^2 + 7b_7^2,$$

where the  $a_k$  and  $b_k$  are positive integers.

There are 75373 such numbers that do not exceed  $10^7$ .

How many such numbers are there that do not exceed  $2 \times 10^9$ ?

### Problem 230: Fibonacci Words

For any two strings of digits, A and B, we define  $F_{A,B}$  to be the sequence (A,B,AB,BAB,ABBAB,...) in which each term is the concatenation of the previous two.

Further, we define  $D_{A,B}(n)$  to be the  $n^{\text{th}}$  digit in the first term of  $F_{A,B}$  that contains at least  $n$  digits.

Example:

Let A=1415926535, B=8979323846. We wish to find  $D_{A,B}(35)$ , say.

The first few terms of  $F_{A,B}$  are:

1415926535

8979323846

14159265358979323846

897932384614159265358979323846

14159265358979323846897932384614159265358979323846

Then  $D_{A,B}(35)$  is the 35<sup>th</sup> digit in the fifth term, which is 9.

Now we use for A the first 100 digits of  $\pi$  behind the decimal point:

14159265358979323846264338327950288419716939937510

58209749445923078164062862089986280348253421170679

and for B the next hundred digits:

82148086513282306647093844609550582231725359408128

48111745028410270193852110555964462294895493038196 .

Find  $\sum_{n=0,1,\dots,17} 10^n \times D_{A,B}((127+19n) \times 7^n)$  .



### Problem 231: The prime factorisation of binomial coefficients

The binomial coefficient  $^{10}C_3 = 120$ .

$120 = 2^3 \times 3 \times 5 = 2 \times 2 \times 2 \times 3 \times 5$ , and  $2 + 2 + 2 + 3 + 5 = 14$ .

So the sum of the terms in the prime factorisation of  $^{10}C_3$  is 14.

Find the sum of the terms in the prime factorisation of  $^{20000000}C_{15000000}$ .

### Problem 232: The Race

Two players share an unbiased coin and take it in turns to play "The Race". On Player 1's turn, he tosses the coin once: if it comes up Heads, he scores one point; if it comes up Tails, he scores nothing. On Player 2's turn, she chooses a positive integer  $T$  and tosses the coin  $T$  times: if it comes up all Heads, she scores  $2^{T-1}$  points; otherwise, she scores nothing. Player 1 goes first. The winner is the first to 100 or more points.

On each turn Player 2 selects the number,  $T$ , of coin tosses that maximises the probability of her winning.

What is the probability that Player 2 wins?

Give your answer rounded to eight decimal places in the form  $0.abcdefgh$ .

### Problem 233: Lattice points on a circle

Let  $f(N)$  be the number of points with integer coordinates that are on a circle passing through  $(0,0)$ ,  $(N,0)$ ,  $(0,N)$ , and  $(N,N)$ .

It can be shown that  $f(10000) = 36$ .

What is the sum of all positive integers  $N \leq 10^{11}$  such that  $f(N) = 420$ ?

### Problem 234: Semidivisible numbers

For an integer  $n \geq 4$ , we define the *lower prime square root* of  $n$ , denoted by  $\text{lps}(n)$ , as the largest prime  $\leq \sqrt{n}$  and the *upper prime square root* of  $n$ ,  $\text{ups}(n)$ , as the smallest prime  $\geq \sqrt{n}$ .

So, for example,  $\text{lps}(4) = 2 = \text{ups}(4)$ ,  $\text{lps}(1000) = 31$ ,  $\text{ups}(1000) = 37$ .

Let us call an integer  $n \geq 4$  *semidivisible*, if one of  $\text{lps}(n)$  and  $\text{ups}(n)$  divides  $n$ , but not both.

The sum of the semidivisible numbers not exceeding 15 is 30, the numbers are 8, 10 and 12.

15 is not semidivisible because it is a multiple of both  $\text{lps}(15) = 3$  and  $\text{ups}(15) = 5$ .

As a further example, the sum of the 92 semidivisible numbers up to 1000 is 34825.

What is the sum of all semidivisible numbers not exceeding 999966663333?

### Problem 235: An Arithmetic Geometric sequence

Given is the arithmetic-geometric sequence  $u(k) = (900 - 3k)r^{k-1}$ .

Let  $s(n) = \sum_{k=1}^n u(k)$ .

Find the value of  $r$  for which  $s(5000) = -600,000,000,000$ .

Give your answer rounded to 12 places behind the decimal point.

### Problem 236: Luxury Hampers

Suppliers 'A' and 'B' provided the following numbers of products for the luxury hamper market:

Product	'A'	'B'
Beluga Caviar	5248	640
Christmas Cake	1312	1888
Gammon Joint	2624	3776
Vintage Port	5760	3776
Champagne	3936	5664
Truffles		

Although the suppliers try very hard to ship their goods in perfect condition, there is inevitably some spoilage *i.e.* products gone bad.

The suppliers compare their performance using two types of statistic:

- The five *per-product spoilage rates* for each supplier are equal to the number of products gone bad divided by the number of products supplied, for each of the five products in turn.
- The *overall spoilage rate* for each supplier is equal to the total number of products gone bad divided by the total number of products provided by that supplier.

To their surprise, the suppliers found that each of the five per-product spoilage rates was worse (higher) for 'B' than for 'A' by the same factor (ratio of spoilage rates),  $m > 1$ ; and yet, paradoxically, the overall spoilage rate was worse for 'A' than for 'B', also by a factor of  $m$ .

There are thirty-five  $m > 1$  for which this surprising result could have occurred, the smallest of which is  $1476/1475$ .

What's the largest possible value of  $m$ ?

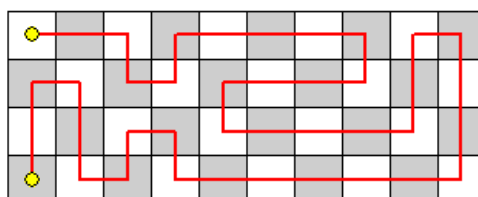
Give your answer as a fraction reduced to its lowest terms, in the form  $U/V$ .

### Problem 237: Tours on a $4 \times n$ playing board

Let  $T(n)$  be the number of tours over a  $4 \times n$  playing board such that:

- The tour starts in the top left corner.
- The tour consists of moves that are up, down, left, or right one square.
- The tour visits each square exactly once.
- The tour ends in the bottom left corner.

The diagram shows one tour over a  $4 \times 10$  board:



$T(10)$  is 2329. What is  $T(10^{12})$  modulo  $10^8$ ?

### Problem 238: Infinite string tour

Create a sequence of numbers using the "Blum Blum Shub" pseudo-random number generator:

$$S_0 = 14025256$$
$$S_{n+1} = S_n^2 \bmod 20300713$$

Concatenate these numbers  $S_0S_1S_2\dots$  to create a string  $W$  of infinite length.

Then,  $W = 14025256741014958470038053646\dots$

For a positive integer  $k$ , if no substring of  $W$  exists with a sum of digits equal to  $k$ ,  $p(k)$  is defined to be zero. If at least one substring of  $W$  exists with a sum of digits equal to  $k$ , we define  $p(k) = Z$ , where  $Z$  is the starting position of the earliest such substring.

For instance:

The substrings **1**, **14**, **1402**, ...  
with respective sums of digits equal to 1, 5, 7, ...  
start at position **1**, hence  $p(1) = p(5) = p(7) = \dots = \mathbf{1}$ .

The substrings **4**, **402**, **4025**, ...  
with respective sums of digits equal to 4, 6, 11, ...  
start at position **2**, hence  $p(4) = p(6) = p(11) = \dots = \mathbf{2}$ .

The substrings **02**, **0252**, ...  
with respective sums of digits equal to 2, 9, ...  
start at position **3**, hence  $p(2) = p(9) = \dots = \mathbf{3}$ .

Note that substring **025** starting at position **3**, has a sum of digits equal to 7, but there was an earlier substring (starting at position **1**) with a sum of digits equal to 7, so  $p(7) = 1$ , not 3.

We can verify that, for  $0 < k \leq 10^3$ ,  $\sum p(k) = 4742$ .

Find  $\sum p(k)$ , for  $0 < k \leq 2 \cdot 10^{15}$ .

### Problem 239: Twenty-two Foolish Primes

A set of disks numbered 1 through 100 are placed in a line in random order.

What is the probability that we have a partial derangement such that exactly 22 prime number discs are found away from their natural positions?

(Any number of non-prime disks may also be found in or out of their natural positions.)

Give your answer rounded to 12 places behind the decimal point in the form 0.abcdefghijkl.

### Problem 240: Top Dice

There are 1111 ways in which five 6-sided dice (sides numbered 1 to 6) can be rolled so that the top three sum to 15. Some examples are:

$$D_1, D_2, D_3, D_4, D_5 = 4, 3, 6, 3, 5$$

$$D_1, D_2, D_3, D_4, D_5 = 4, 3, 3, 5, 6$$

$$D_1, D_2, D_3, D_4, D_5 = 3, 3, 3, 6, 6$$

$$D_1, D_2, D_3, D_4, D_5 = 6, 6, 3, 3, 3$$

In how many ways can twenty 12-sided dice (sides numbered 1 to 12) be rolled so that the top ten sum to 70?

### Problem 241: Perfection Quotients

For a positive integer  $n$ , let  $\sigma(n)$  be the sum of all divisors of  $n$ , so e.g.  $\sigma(6) = 1 + 2 + 3 + 6 = 12$ .

A perfect number, as you probably know, is a number with  $\sigma(n) = 2n$ .

Let us define the **perfection quotient** of a positive integer  $n$  as  $p(n) = \frac{\sigma(n)}{n}$ .

Find the sum of all positive integers  $n \leq 10^{18}$  for which  $p(n)$  has the form  $k + \frac{1}{2}$ , where  $k$  is an integer.

### Problem 242: Odd Triplets

Given the set  $\{1, 2, \dots, n\}$ , we define  $f(n, k)$  as the number of its  $k$ -element subsets with an odd sum of elements. For example,  $f(5, 3) = 4$ , since the set  $\{1, 2, 3, 4, 5\}$  has four 3-element subsets having an odd sum of elements, i.e.:  $\{1, 2, 4\}$ ,  $\{1, 3, 5\}$ ,  $\{2, 3, 4\}$  and  $\{2, 4, 5\}$ .

When all three values  $n$ ,  $k$  and  $f(n, k)$  are odd, we say that they make an *odd-triplet*  $[n, k, f(n, k)]$ .

There are exactly five odd-triplets with  $n \leq 10$ , namely:

$[1, 1, f(1, 1) = 1]$ ,  $[5, 1, f(5, 1) = 3]$ ,  $[5, 5, f(5, 5) = 1]$ ,  $[9, 1, f(9, 1) = 5]$  and  $[9, 9, f(9, 9) = 1]$ .

How many odd-triplets are there with  $n \leq 10^{12}$ ?

### Problem 243: Resilience

A positive fraction whose numerator is less than its denominator is called a proper fraction.

For any denominator,  $d$ , there will be  $d-1$  proper fractions; for example, with  $d = 12$ :

$\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}$ .

We shall call a fraction that cannot be cancelled down a *resilient fraction*.

Furthermore we shall define the *resilience* of a denominator,  $R(d)$ , to be the ratio of its proper fractions that are resilient; for example,  $R(12) = \frac{4}{11}$ .

In fact,  $d = 12$  is the smallest denominator having a resilience  $R(d) < \frac{4}{10}$ .

Find the smallest denominator  $d$ , having a resilience  $R(d) < \frac{15499}{94744}$ .

Problem 244: Sliders

You probably know the game *Fifteen Puzzle*. Here, instead of numbered tiles, we have seven red tiles and eight blue tiles. A move is denoted by the uppercase initial of the direction (Left, Right, Up, Down) in which the tile is slid, e.g. starting from configuration (S), by the sequence **LULUR** we reach the configuration (E):

(S) 


 , (E)

For each path, its checksum is calculated by (pseudocode):

```
checksum = 0
checksum = (checksum × 243 + m1) mod 100 000 007
checksum = (checksum × 243 + m2) mod 100 000 007
...
checksum = (checksum × 243 + mn) mod 100 000 007
```

where  $m_k$  is the ASCII value of the  $k^{th}$  letter in the move sequence and the ASCII values for the moves are:

L	76
R	82
U	85
D	68

For the sequence **LULUR** given above, the checksum would be 19761398. Now, starting from configuration (S), find all shortest ways to reach configuration (T).

(S) 


 , (T)

What is the sum of all checksums for the paths having the minimal length?

Problem 245: Coresilience

We shall call a fraction that cannot be cancelled down a resilient fraction. Furthermore we shall define the resilience of a denominator,  $R(d)$ , to be the ratio of its proper fractions that are resilient; for example,  $R(12) = \frac{4}{11}$ .

The resilience of a number  $d > 1$  is  $\frac{\varphi(d)}{d - 1}$ , where  $\varphi$  is Euler's totient function.

We further define the **coresilience** of a number  $n > 1$  as  $C(n) = \frac{n - \varphi(n)}{n - 1}$ .

The coresilience of a prime  $p$  is  $C(p) = \frac{1}{p - 1}$ .

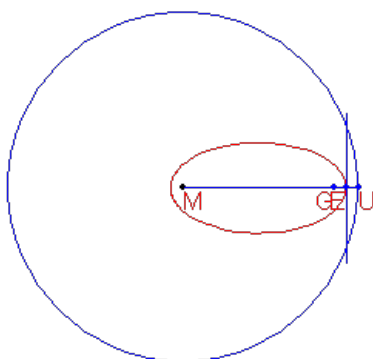
Find the sum of all **composite** integers  $1 < n \leq 2 \times 10^{11}$ , for which  $C(n)$  is a unit fraction.

# Problem 246: Tangents to an ellipse

A definition for an ellipse is:

Given a circle  $c$  with centre  $M$  and radius  $r$  and a point  $G$  such that  $d(G,M) < r$ , the locus of the points that are equidistant from  $c$  and  $G$  form an ellipse.

The construction of the points of the ellipse is shown below.



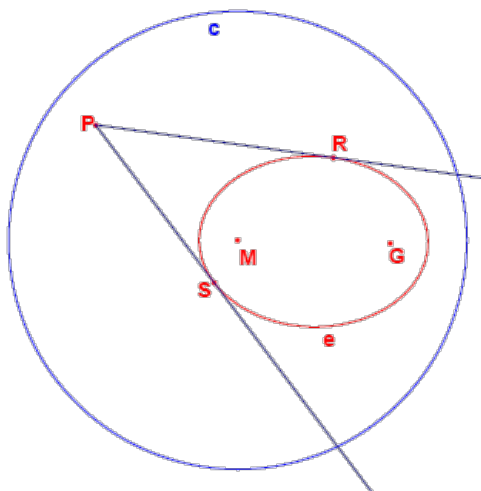
Given are the points  $M(-2000,1500)$  and  $G(8000,1500)$ .

Given is also the circle  $c$  with centre  $M$  and radius 15000.

The locus of the points that are equidistant from  $G$  and  $c$  form an ellipse  $e$ .

From a point  $P$  outside  $e$  the two tangents  $t_1$  and  $t_2$  to the ellipse are drawn.

Let the points where  $t_1$  and  $t_2$  touch the ellipse be  $R$  and  $S$ .



For how many lattice points  $P$  is angle  $RPS$  greater than 45 degrees?

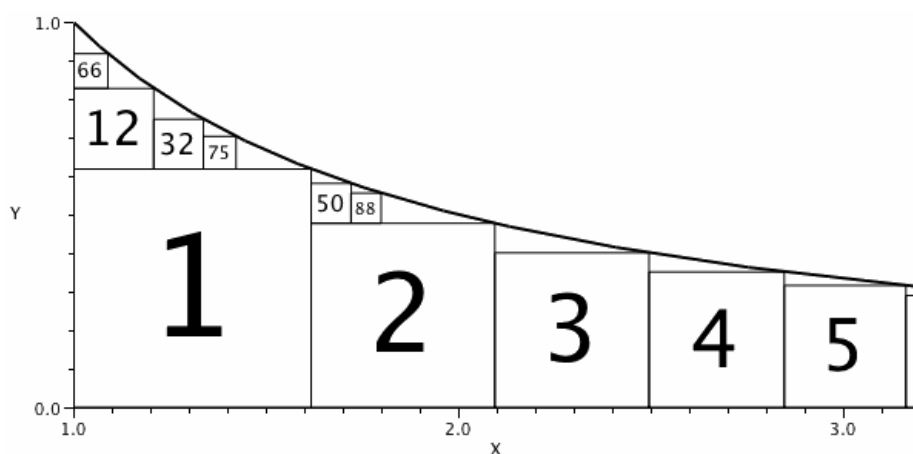
### Problem 247: Squares under a hyperbola

Consider the region constrained by  $1 \leq x$  and  $0 \leq y \leq 1/x$ .

Let  $S_1$  be the largest square that can fit under the curve.

Let  $S_2$  be the largest square that fits in the remaining area, and so on.

Let the *index* of  $S_n$  be the pair (left, below) indicating the number of squares to the left of  $S_n$  and the number of squares below  $S_n$ .



The diagram shows some such squares labelled by number.

$S_2$  has one square to its left and none below, so the index of  $S_2$  is (1,0).

It can be seen that the index of  $S_{32}$  is (1,1) as is the index of  $S_{50}$ .

50 is the largest  $n$  for which the index of  $S_n$  is (1,1).

What is the largest  $n$  for which the index of  $S_n$  is (3,3)?

### Problem 248: Numbers for which Euler's totient function equals 13!

The first number  $n$  for which  $\phi(n)=13!$  is 6227180929.

Find the 150,000<sup>th</sup> such number.

### Problem 249: Prime Subset Sums

Let  $S = \{2, 3, 5, \dots, 4999\}$  be the set of prime numbers less than 5000.

Find the number of subsets of  $S$ , the sum of whose elements is a prime number.

Enter the rightmost 16 digits as your answer.

### Problem 250: 250250

Find the number of non-empty subsets of  $\{1^1, 2^2, 3^3, \dots, 250250^{250250}\}$ , the sum of whose elements is divisible by 250. Enter the rightmost 16 digits as your answer.

### Problem 251: Cardano Triplets

A triplet of positive integers  $(a,b,c)$  is called a Cardano Triplet if it satisfies the condition:

$$\sqrt[3]{a+b\sqrt{c}} + \sqrt[3]{a-b\sqrt{c}} = 1$$

For example,  $(2,1,5)$  is a Cardano Triplet.

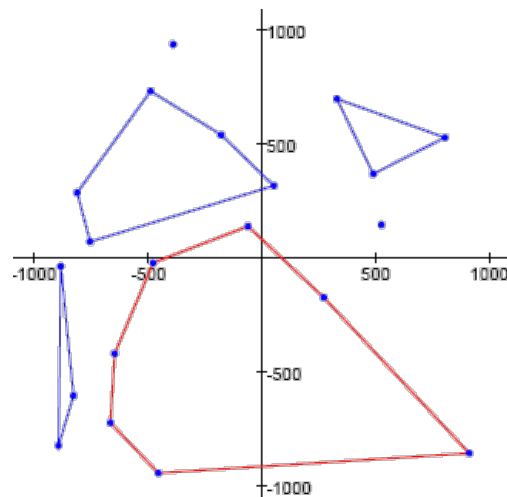
There exist 149 Cardano Triplets for which  $a+b+c \leq 1000$ .

Find how many Cardano Triplets exist such that  $a+b+c \leq 110,000,000$ .

### Problem 252: Convex Holes

Given a set of points on a plane, we define a convex hole to be a convex polygon having as vertices any of the given points and not containing any of the given points in its interior (in addition to the vertices, other given points may lie on the perimeter of the polygon).

As an example, the image below shows a set of twenty points and a few such convex holes. The convex hole shown as a red heptagon has an area equal to 1049694.5 square units, which is the highest possible area for a convex hole on the given set of points.



For our example, we used the first 20 points  $(T_{2k-1}, T_{2k})$ , for  $k = 1, 2, \dots, 20$ , produced with the pseudo-random number generator:

$$\begin{aligned} S_0 &= 290797 \\ S_{n+1} &= S_n^2 \bmod 50515093 \\ T_n &= (S_n \bmod 2000) - 1000 \end{aligned}$$

i.e.  $(527, 144), (-488, 732), (-454, -947), \dots$

What is the maximum area for a convex hole on the set containing the first 500 points in the pseudo-random sequence? Specify your answer including one digit after the decimal point.



Problem 253: Tidying up

A small child has a “number caterpillar” consisting of forty jigsaw pieces, each with one number on it, which, when connected together in a line, reveal the numbers 1 to 40 in order.

Every night, the child's father has to pick up the pieces of the caterpillar that have been scattered across the play room. He picks up the pieces at random and places them in the correct order.

As the caterpillar is built up in this way, it forms distinct segments that gradually merge together.

The number of segments starts at zero (no pieces placed), generally increases up to about eleven or twelve, then tends to drop again before finishing at a single segment (all pieces placed).

For example:

Piece Placed	Segments So Far
12	1
4	2
29	3
6	4
34	5
5	4
35	4
...	...

Let  $M$  be the maximum number of segments encountered during a random tidy-up of the caterpillar.

For a caterpillar of ten pieces, the number of possibilities for each  $M$  is

$M$	Possibilities
1	512
2	250912
3	1815264
4	1418112
5	144000

so the most likely value of  $M$  is 3 and the average value is  $\frac{385643}{113400} = 3.400732$ , rounded to six decimal places.

The most likely value of  $M$  for a forty-piece caterpillar is 11; but what is the average value of  $M$ ?

Give your answer rounded to six decimal places.

Problem 254: Sums of Digit Factorials

Define  $f(n)$  as the sum of the factorials of the digits of  $n$ . For example,  $f(342) = 3! + 4! + 2! = 32$ .

Define  $sf(n)$  as the sum of the digits of  $f(n)$ . So  $sf(342) = 3 + 2 = 5$ .

Define  $g(i)$  to be the smallest positive integer  $n$  such that  $sf(n) = i$ . Though  $sf(342)$  is 5,  $sf(25)$  is also 5, and it can be verified that  $g(5)$  is 25.

Define  $sg(i)$  as the sum of the digits of  $g(i)$ . So  $sg(5) = 2 + 5 = 7$ .

Further, it can be verified that  $g(20)$  is 267 and  $\sum sg(i)$  for  $1 \leq i \leq 20$  is 156.

What is  $\sum sg(i)$  for  $1 \leq i \leq 150$ ?

### Problem 255: Rounded Square Roots

We define the *rounded-square-root* of a positive integer  $n$  as the square root of  $n$  rounded to the nearest integer.

The following procedure (essentially Heron's method adapted to integer arithmetic) finds the rounded-square-root of  $n$ :

Let  $d$  be the number of digits of the number  $n$ .

If  $d$  is odd, set  $x_0 = 2 \times 10^{(d-1)/2}$ .

If  $d$  is even, set  $x_0 = 7 \times 10^{(d-2)/2}$ .

Repeat:

$$x_{k+1} = \left\lfloor \frac{x_k + \lceil n/x_k \rceil}{2} \right\rfloor$$

until  $x_{k+1} = x_k$ .

As an example, let us find the rounded-square-root of  $n = 4321$ .

$n$  has 4 digits, so  $x_0 = 7 \times 10^{(4-2)/2} = 70$ .

$$x_1 = \left\lfloor \frac{70 + \lceil 4321/70 \rceil}{2} \right\rfloor = 66, x_2 = \left\lfloor \frac{66 + \lceil 4321/66 \rceil}{2} \right\rfloor = 66$$

Since  $x_2 = x_1$ , we stop here.

So, after just two iterations, we have found that the rounded-square-root of 4321 is 66 (the actual square root is 65.7343137...).

The number of iterations required when using this method is surprisingly low.

For example, we can find the rounded-square-root of a 5-digit integer ( $10,000 \leq n \leq 99,999$ ) with an average of 3.210288889 iterations (the average value was rounded to 10 decimal places).

Using the procedure described above, what is the average number of iterations required to find the rounded-square-root of a 14-digit number ( $10^{13} \leq n < 10^{14}$ )?

Give your answer rounded to 10 decimal places.

Note: The symbols  $\lfloor x \rfloor$  and  $\lceil x \rceil$  represent the floor function and ceiling function respectively.

### Problem 256: Tatami-Free Rooms

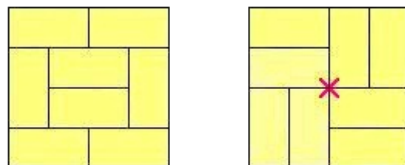
Tatami are rectangular mats, used to completely cover the floor of a room, without overlap.

Assuming that the only type of available tatami has dimensions  $1 \times 2$ , there are obviously some limitations for the shape and size of the rooms that can be covered.

For this problem, we consider only rectangular rooms with integer dimensions  $a, b$  and even size  $S = a \cdot b$ .

We use the term 'size' to denote the floor surface area of the room, and — without loss of generality — we add the condition  $a \leq b$ .

There is one rule to follow when laying out tatami: there must be no points where corners of four different mats meet. For example, consider the two arrangements below for a  $4 \times 4$  room:



The arrangement on the left is acceptable, whereas the one on the right is not: a red 'X' in the middle, marks the point where four tatami meet.

Because of this rule, certain even-sized rooms cannot be covered with tatami: we call them tatami-free rooms.

Further, we define  $T(S)$  as the number of tatami-free rooms of size  $S$ .

The smallest tatami-free room has size  $S = 70$  and dimensions  $7 \times 10$ .

All the other rooms of size  $S = 70$  can be covered with tatami; they are:  $1 \times 70$ ,  $2 \times 35$  and  $5 \times 14$ .

Hence,  $T(70) = 1$ .

Similarly, we can verify that  $T(1320) = 5$  because there are exactly 5 tatami-free rooms of size  $S = 1320$ :

$20 \times 66$ ,  $22 \times 60$ ,  $24 \times 55$ ,  $30 \times 44$  and  $33 \times 40$ .

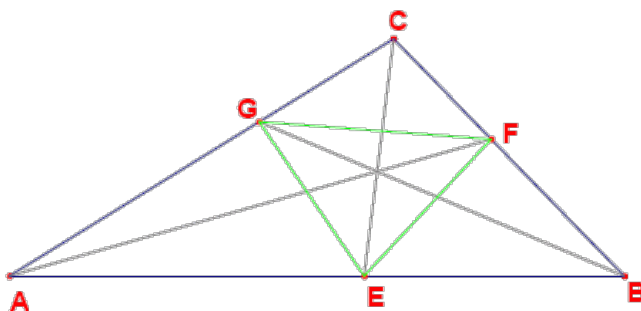
In fact,  $S = 1320$  is the smallest room-size  $S$  for which  $T(S) = 5$ .

Find the smallest room-size  $S$  for which  $T(S) = 200$ .

### Problem 257: Angular Bisectors

Given is an integer sided triangle  $ABC$  with sides  $a \leq b \leq c$ . ( $AB = c$ ,  $BC = a$  and  $AC = b$ ).

The angular bisectors of the triangle intersect the sides at points  $E, F$  and  $G$  (see picture below).



The segments  $EF, EG$  and  $FG$  partition the triangle  $ABC$  into four smaller triangles:  $AEG, BFE, CGF$  and  $EFG$ .

It can be proven that for each of these four triangles the ratio  $\text{area}(ABC)/\text{area}(\text{subtriangle})$  is rational.

However, there exist triangles for which some or all of these ratios are integral.

How many triangles  $ABC$  with perimeter  $\leq 100,000,000$  exist so that the ratio  $\text{area}(ABC)/\text{area}(AEG)$  is integral?

### Problem 258: A lagged Fibonacci sequence

A sequence is defined as:

- $g_k = 1$ , for  $0 \leq k \leq 1999$
- $g_k = g_{k-2000} + g_{k-1999}$ , for  $k \geq 2000$ .

Find  $g_k \bmod 20092010$  for  $k = 10^{18}$ .

### Problem 259: Reachable Numbers

A positive integer will be called *reachable* if it can result from an arithmetic expression obeying the following rules:

- Uses the digits 1 through 9, in that order and exactly once each.
- Any successive digits can be concatenated (for example, using the digits 2, 3 and 4 we obtain the number 234).
- Only the four usual binary arithmetic operations (addition, subtraction, multiplication and division) are allowed.
- Each operation can be used any number of times, or not at all.
- Unary minus is not allowed.
- Any number of (possibly nested) parentheses may be used to define the order of operations.

For example, 42 is reachable, since  $(1/23) * ((4*5)-6) * (78-9) = 42$ .

What is the sum of all positive reachable integers?

### Problem 260: Stone Game

A game is played with three piles of stones and two players.

At her turn, a player removes one or more stones from the piles. However, if she takes stones from more than one pile, she must remove the same number of stones from each of the selected piles.

In other words, the player chooses some  $N > 0$  and removes:

- $N$  stones from any single pile; or
- $N$  stones from each of any two piles ( $2N$  total); or
- $N$  stones from each of the three piles ( $3N$  total).

The player taking the last stone(s) wins the game.

A *winning configuration* is one where the first player can force a win.

For example,  $(0,0,13)$ ,  $(0,11,11)$  and  $(5,5,5)$  are winning configurations because the first player can immediately remove all stones.

A *losing configuration* is one where the second player can force a win, no matter what the first player does.

For example,  $(0,1,2)$  and  $(1,3,3)$  are losing configurations: any legal move leaves a winning configuration for the second player.

Consider all losing configurations  $(x_i, y_i, z_i)$  where  $x_i \leq y_i \leq z_i \leq 100$ .

We can verify that  $\Sigma(x_i + y_i + z_i) = 173895$  for these.

Find  $\Sigma(x_i + y_i + z_i)$  where  $(x_i, y_i, z_i)$  ranges over the losing configurations with  $x_i \leq y_i \leq z_i \leq 1000$ .

### Problem 261: Pivotal Square Sums

Let us call a positive integer  $k$  a *square-pivot*, if there is a pair of integers  $m > 0$  and  $n \geq k$ , such that the sum of the  $(n+1)$  consecutive squares up to  $k$  equals the sum of the  $m$  consecutive squares from  $(n+1)$  on:

$$(k-m)^2 + \dots + k^2 = (n+1)^2 + \dots + (n+m)^2.$$

Some small square-pivots are

- **4**:  $3^2 + 4^2 = 5^2$
- **21**:  $20^2 + 21^2 = 29^2$
- **24**:  $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$
- **110**:  $108^2 + 109^2 + 110^2 = 133^2 + 134^2$

Find the sum of all **distinct** square-pivots  $\leq 10^{10}$ .

### Problem 262: Mountain Range

The following equation represents the *continuous* topography of a mountainous region, giving the elevation  $h$  at any point  $(x,y)$ :



A mosquito intends to fly from A(200,200) to B(1400,1400), without leaving the area given by  $0 \leq x, y \leq 1600$ .

Because of the intervening mountains, it first rises straight up to a point A', having elevation  $f$ . Then, while remaining at the same elevation  $f$ , it flies around any obstacles until it arrives at a point B' directly above B.

First, determine  $f_{min}$  which is the minimum constant elevation allowing such a trip from A to B, while remaining in the specified area.

Then, find the length of the shortest path between A' and B', while flying at that constant elevation  $f_{min}$ .

Give that length as your answer, rounded to three decimal places.

**Note:** For convenience, the elevation function shown above is repeated below, in a form suitable for most programming languages:

$h = (5000 - 0.005 \cdot (x \cdot x + y \cdot y + x \cdot y) + 12.5 \cdot (x + y)) \cdot \exp(-\text{abs}(0.000001 \cdot (x \cdot x + y \cdot y) - 0.0015 \cdot (x + y) + 0.7))$

### Problem 263: An engineers' dream come true

Consider the number 6. The divisors of 6 are: 1,2,3 and 6.

Every number from 1 up to and including 6 can be written as a sum of distinct divisors of 6:

1=1, 2=2, 3=1+2, 4=1+3, 5=2+3, 6=6.

A number  $n$  is called a *practical number* if every number from 1 up to and including  $n$  can be expressed as a sum of distinct divisors of  $n$ .

A pair of consecutive prime numbers with a difference of six is called a *sexy pair* (since "sex" is the Latin word for "six"). The first sexy pair is (23, 29).

We may occasionally find a *triple-pair*, which means three consecutive sexy prime pairs, such that the second member of each pair is the first member of the next pair.

We shall call a number  $n$  such that :

- $(n-9, n-3)$ ,  $(n-3, n+3)$ ,  $(n+3, n+9)$  form a triple-pair, and
- the numbers  $n-8$ ,  $n-4$ ,  $n$ ,  $n+4$  and  $n+8$  are all practical,

an engineers' paradise.

Find the sum of the first four engineers' paradises.

### Problem 264: Triangle Centres

Consider all the triangles having:

- All their vertices on lattice points.
- Circumcentre at the origin O.
- Orthocentre at the point H(5, 0).

There are nine such triangles having a perimeter  $\leq 50$ .

Listed and shown in ascending order of their perimeter, they are:

A(-4, 3), B(5, 0), C(4, -3)

A(4, 3), B(5, 0), C(-4, -3)

A(-3, 4), B(5, 0), C(3, -4)

A(3, 4), B(5, 0), C(-3, -4)

A(0, 5), B(5, 0), C(0, -5)

A(1, 8), B(8, -1), C(-4, -  
7)

A(8, 1), B(1, -8), C(-4, 7)

A(2, 9), B(9, -2), C(-6, -  
7)

A(9, 2), B(2, -9), C(-6, 7)

The sum of their perimeters, rounded to four decimal places, is 291.0089.

Find all such triangles with a perimeter  $\leq 10^5$ .

Enter as your answer the sum of their perimeters rounded to four decimal places.

### Problem 265: Binary Circles

$2^N$  binary digits can be placed in a circle so that all the N-digit clockwise subsequences are distinct.

For N=3, two such circular arrangements are possible, ignoring rotations:

For the first arrangement, the 3-digit subsequences, in clockwise order, are:  
000, 001, 010, 101, 011, 111, 110 and 100.

Each circular arrangement can be encoded as a number by concatenating the binary digits starting with the subsequence of all zeros as the most significant bits and proceeding clockwise. The two arrangements for N=3 are thus represented as 23 and 29:

$$00010111_2 = 23$$

$$00011101_2 = 29$$

Calling S(N) the sum of the unique numeric representations, we can see that  $S(3) = 23 + 29 = 52$ .

Find S(5).

### Problem 266: Pseudo Square Root

The divisors of 12 are: 1,2,3,4,6 and 12.

The largest divisor of 12 that does not exceed the square root of 12 is 3.

We shall call the largest divisor of an integer  $n$  that does not exceed the square root of  $n$  the pseudo square root (PSR) of  $n$ . It can be seen that  $\text{PSR}(3102)=47$ .

Let  $p$  be the product of the primes below 190.

Find  $\text{PSR}(p) \bmod 10^{16}$ .

### Problem 267: Billionaire

You are given a unique investment opportunity.

Starting with £1 of capital, you can choose a fixed proportion,  $f$ , of your capital to bet on a fair coin toss repeatedly for 1000 tosses.

Your return is double your bet for heads and you lose your bet for tails.

For example, if  $f = 1/4$ , for the first toss you bet £0.25, and if heads comes up you win £0.5 and so then have £1.5. You then bet £0.375 and if the second toss is tails, you have £1.125.

Choosing  $f$  to maximize your chances of having at least £1,000,000,000 after 1,000 flips, what is the chance that you become a billionaire?

All computations are assumed to be exact (no rounding), but give your answer rounded to 12 digits behind the decimal point in the form 0.abcdefghijkl.

### Problem 268: Counting numbers with at least four distinct prime factors less than 100

It can be verified that there are 23 positive integers less than 1000 that are divisible by at least four distinct primes less than 100.

Find how many positive integers less than  $10^{16}$  are divisible by at least four distinct primes less than 100.

### Problem 269: Polynomials with at least one integer root

A root or zero of a polynomial  $P(x)$  is a solution to the equation  $P(x) = 0$ .

Define  $P_n$  as the polynomial whose coefficients are the digits of  $n$ .

For example,  $P_{5703}(x) = 5x^3 + 7x^2 + 3$ .

We can see that:

- $P_n(0)$  is the last digit of  $n$ ,
- $P_n(1)$  is the sum of the digits of  $n$ ,
- $P_n(10)$  is  $n$  itself.

Define  $Z(k)$  as the number of positive integers,  $n$ , not exceeding  $k$  for which the polynomial  $P_n$  has at least one integer root.

It can be verified that  $Z(100\,000)$  is 14696.

What is  $Z(10^{16})$ ?

### Problem 270: Cutting Squares

A square piece of paper with integer dimensions  $N \times N$  is placed with a corner at the origin and two of its sides along the  $x$ - and  $y$ -axes. Then, we cut it up respecting the following rules:

- We only make straight cuts between two points lying on different sides of the square, and having integer coordinates.
- Two cuts cannot cross, but several cuts can meet at the same border point.
- Proceed until no more legal cuts can be made.

Counting any reflections or rotations as distinct, we call  $C(N)$  the number of ways to cut an  $N \times N$  square. For example,  $C(1) = 2$  and  $C(2) = 30$  (shown below).



What is  $C(30) \bmod 10^8$ ?

### Problem 271: Modular Cubes, part 1

For a positive number  $n$ , define  $S(n)$  as the sum of the integers  $x$ , for which  $1 < x < n$  and  $x^3 \equiv 1 \pmod{n}$ .

When  $n=91$ , there are 8 possible values for  $x$ , namely : 9, 16, 22, 29, 53, 74, 79, 81.  
Thus,  $S(91)=9+16+22+29+53+74+79+81=363$ .

Find  $S(13082761331670030)$ .

### Problem 272: Modular Cubes, part 2

For a positive number  $n$ , define  $C(n)$  as the number of the integers  $x$ , for which  $1 < x < n$  and  $x^3 \equiv 1 \pmod{n}$ .

When  $n=91$ , there are 8 possible values for  $x$ , namely : 9, 16, 22, 29, 53, 74, 79, 81.  
Thus,  $C(91)=8$ .

Find the sum of the positive numbers  $n \leq 10^{11}$  for which  $C(n)=242$ .

### Problem 273: Sum of Squares

Consider equations of the form:  $a^2 + b^2 = N$ ,  $0 \leq a \leq b$ ,  $a, b$  and  $N$  integer.

For  $N=65$  there are two solutions:

$a=1, b=8$  and  $a=4, b=7$ .

We call  $S(N)$  the sum of the values of  $a$  of all solutions of  $a^2 + b^2 = N$ ,  $0 \leq a \leq b$ ,  $a, b$  and  $N$  integer.

Thus  $S(65) = 1 + 4 = 5$ .

Find  $\sum S(N)$ , for all squarefree  $N$  only divisible by primes of the form  $4k+1$  with  $4k+1 < 150$ .

### Problem 274: Divisibility Multipliers

For each integer  $p > 1$  coprime to 10 there is a positive *divisibility multiplier*  $m < p$  which preserves divisibility by  $p$  for the following function on any positive integer,  $n$ :

$f(n) = (\text{all but the last digit of } n) + (\text{the last digit of } n) * m$

That is, if  $m$  is the divisibility multiplier for  $p$ , then  $f(n)$  is divisible by  $p$  if and only if  $n$  is divisible by  $p$ .

(When  $n$  is much larger than  $p$ ,  $f(n)$  will be less than  $n$  and repeated application of  $f$  provides a multiplicative divisibility test for  $p$ .)

For example, the divisibility multiplier for 113 is 34.

$f(76275) = 7627 + 5 * 34 = 7797$  : 76275 and 7797 are both divisible by 113

$f(12345) = 1234 + 5 * 34 = 1404$  : 12345 and 1404 are both not divisible by 113

The sum of the divisibility multipliers for the primes that are coprime to 10 and less than 1000 is 39517. What is the sum of the divisibility multipliers for the primes that are coprime to 10 and less than  $10^7$ ?



### Problem 275: Balanced Sculptures

Let us define a *balanced sculpture* of order  $n$  as follows:

- A polyomino made up of  $n+1$  tiles known as the *blocks* ( $n$  tiles) and the *plinth* (remaining tile);
- the plinth has its centre at position  $(x = 0, y = 0)$ ;
- the blocks have  $y$ -coordinates greater than zero (so the plinth is the unique lowest tile);
- the centre of mass of all the blocks, combined, has  $x$ -coordinate equal to zero.

When counting the sculptures, any arrangements which are simply reflections about the  $y$ -axis, are not counted as distinct. For example, the 18 balanced sculptures of order 6 are shown below; note that each pair of mirror images (about the  $y$ -axis) is counted as one sculpture:



There are 964 balanced sculptures of order 10 and 360505 of order 15.  
How many balanced sculptures are there of order 18?

### Problem 276: Primitive Triangles

Consider the triangles with integer sides  $a$ ,  $b$  and  $c$  with  $a \leq b \leq c$ .  
An integer sided triangle  $(a,b,c)$  is called primitive if  $\gcd(a,b,c)=1$ .  
How many primitive integer sided triangles exist with a perimeter not exceeding 10 000 000?

### Problem 277: A Modified Collatz sequence

A modified Collatz sequence of integers is obtained from a starting value  $a_1$  in the following way:

$a_{n+1} = a_n/3$  if  $a_n$  is divisible by 3. We shall denote this as a large downward step, "D".

$a_{n+1} = (4a_n + 2)/3$  if  $a_n$  divided by 3 gives a remainder of 1. We shall denote this as an upward step, "U".

$a_{n+1} = (2a_n - 1)/3$  if  $a_n$  divided by 3 gives a remainder of 2. We shall denote this as a small downward step, "d".

The sequence terminates when some  $a_n = 1$ .

Given any integer, we can list out the sequence of steps.

For instance if  $a_1=231$ , then the sequence  $\{a_n\}=\{231,77,51,17,11,7,10,14,9,3,1\}$  corresponds to the steps "DdDddUUdDD".

Of course, there are other sequences that begin with that same sequence "DdDddUUdDD....".

For instance, if  $a_1=1004064$ , then the sequence is DdDddUUdDDdUDUUUdDdUUDDDUdDD.

In fact, 1004064 is the smallest possible  $a_1 > 10^6$  that begins with the sequence DdDddUUdDD.

What is the smallest  $a_1 > 10^{15}$  that begins with the sequence "UDDDUdddDDUDDddDdDddDDUDDdUUdD"?

### Problem 278: Linear Combinations of Semiprimes

Given the values of integers  $1 < a_1 < a_2 < \dots < a_n$ , consider the linear combination

$q_1 a_1 + q_2 a_2 + \dots + q_n a_n = b$ , using only integer values  $q_k \geq 0$ .

Note that for a given set of  $a_k$ , it may be that not all values of  $b$  are possible.

For instance, if  $a_1 = 5$  and  $a_2 = 7$ , there are no  $q_1 \geq 0$  and  $q_2 \geq 0$  such that  $b$  could be 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18 or 23.

In fact, 23 is the largest impossible value of  $b$  for  $a_1 = 5$  and  $a_2 = 7$ .

We therefore call  $f(5, 7) = 23$ .

Similarly, it can be shown that  $f(6, 10)=29$  and  $f(14, 22, 77) = 195$ .

Find  $\sum f(p^*q, p^*r, q^*r)$ , where  $p$ ,  $q$  and  $r$  are prime numbers and  $p < q < r < 5000$ .

### Problem 279: Triangles with integral sides and an integral angle

How many triangles are there with integral sides, at least one integral angle (measured in degrees), and a perimeter that does not exceed  $10^8$ ?

### Problem 280: Ant and seeds

A laborious ant walks randomly on a  $5 \times 5$  grid. The walk starts from the central square. At each step, the ant moves to an adjacent square at random, without leaving the grid; thus there are 2, 3 or 4 possible moves at each step depending on the ant's position.

At the start of the walk, a seed is placed on each square of the lower row. When the ant isn't carrying a seed and reaches a square of the lower row containing a seed, it will start to carry the seed. The ant will drop the seed on the first empty square of the upper row it eventually reaches.

What's the expected number of steps until all seeds have been dropped in the top row?  
Give your answer rounded to 6 decimal places.

### Problem 281: Pizza Toppings

You are given a pizza (perfect circle) that has been cut into  $m \cdot n$  equal pieces and you want to have exactly one topping on each slice.

Let  $f(m, n)$  denote the number of ways you can have toppings on the pizza with  $m$  different toppings ( $m \geq 2$ ), using each topping on exactly  $n$  slices ( $n \geq 1$ ).

Reflections are considered distinct, rotations are not.

Thus, for instance,  $f(2, 1) = 1$ ,  $f(2, 2) = f(3, 1) = 2$  and  $f(3, 2) = 16$ .

$f(3, 2)$  is shown below:



Find the sum of all  $f(m, n)$  such that  $f(m, n) \leq 10^{15}$ .

### Problem 282: The Ackermann function

For non-negative integers  $m, n$ , the Ackermann function  $A(m, n)$  is defined as follows:  $A(m, n) = \begin{cases} n+1 & \text{if } m=0 \\ A(m-1, 1) & \text{if } m>0 \text{ and } n=0 \\ A(m-1, A(m, n-1)) & \text{if } m>0 \text{ and } n>0 \end{cases}$

For example  $A(1, 0) = 2$ ,  $A(2, 2) = 7$  and  $A(3, 4) = 125$ .

Find  $\sum_{n=0}^6 A(n, n)$  and give your answer mod  $14^8$ .

### Problem 283: Integer sided triangles for which the area/perimeter ratio is integral

Consider the triangle with sides 6, 8 and 10. It can be seen that the perimeter and the area are both equal to 24. So the area/perimeter ratio is equal to 1.

Consider also the triangle with sides 13, 14 and 15. The perimeter equals 42 while the area is equal to 84. So for this triangle the area/perimeter ratio is equal to 2.

Find the sum of the perimeters of all integer sided triangles for which the area/perimeter ratios are equal to positive integers not exceeding 1000.

### Problem 284: Steady Squares

The 3-digit number 376 in the decimal numbering system is an example of numbers with the special property that its square ends with the same digits:  $376^2 = 141376$ . Let's call a number with this property a steady square.

Steady squares can also be observed in other numbering systems. In the base 14 numbering system, the 3-digit number c37 is also a steady square:  $c37^2 = aa0c37$ , and the sum of its digits is  $c+3+7=18$  in the same numbering system. The letters a, b, c and d are used for the 10, 11, 12 and 13 digits respectively, in a manner similar to the hexadecimal numbering system.

For  $1 \leq n \leq 9$ , the sum of the digits of all the  $n$ -digit steady squares in the base 14 numbering system is 2d8 (582 decimal). Steady squares with leading 0's are not allowed.

Find the sum of the digits of all the  $n$ -digit steady squares in the base 14 numbering system for  $1 \leq n \leq 10000$  (decimal) and give your answer in the base 14 system using lower case letters where necessary.

### Problem 285: Pythagorean odds

Albert chooses a positive integer  $k$ , then two real numbers  $a, b$  are randomly chosen in the interval  $[0,1]$  with uniform distribution.

The square root of the sum  $(k \cdot a + 1)^2 + (k \cdot b + 1)^2$  is then computed and rounded to the nearest integer. If the result is equal to  $k$ , he scores  $k$  points; otherwise he scores nothing.

For example, if  $k = 6$ ,  $a = 0.2$  and  $b = 0.85$ , then  $(k \cdot a + 1)^2 + (k \cdot b + 1)^2 = 42.05$ .

The square root of 42.05 is 6.484... and when rounded to the nearest integer, it becomes 6.

This is equal to  $k$ , so he scores 6 points.

It can be shown that if he plays 10 turns with  $k = 1, k = 2, \dots, k = 10$ , the expected value of his total score, rounded to five decimal places, is 10.20914.

If he plays  $10^5$  turns with  $k = 1, k = 2, k = 3, \dots, k = 10^5$ , what is the expected value of his total score, rounded to five decimal places?

### Problem 286: Scoring probabilities

Barbara is a mathematician and a basketball player. She has found that the probability of scoring a point when shooting from a distance  $X$  is exactly  $(1 - X/q)$ , where  $q$  is a real constant greater than 50.

During each practice run, she takes shots from distances  $X = 1, X = 2, \dots, X = 50$  and, according to her records, she has precisely a 2 % chance to score a total of exactly 20 points.

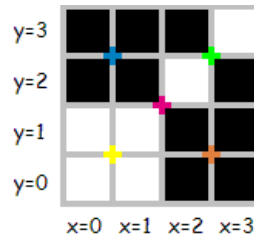
Find  $q$  and give your answer rounded to 10 decimal places.

### Problem 287: Quadtree encoding (a simple compression algorithm)

The quadtree encoding allows us to describe a  $2^N \times 2^N$  black and white image as a sequence of bits (0 and 1). Those sequences are to be read from left to right like this:

- the first bit deals with the complete  $2^N \times 2^N$  region;
- "0" denotes a split:  
the current  $2^n \times 2^n$  region is divided into 4 sub-regions of dimension  $2^{n-1} \times 2^{n-1}$ ,  
the next bits contains the description of the top left, top right, bottom left and bottom right sub-regions - in that order;
- "10" indicates that the current region contains only black pixels;
- "11" indicates that the current region contains only white pixels.

Consider the following  $4 \times 4$  image (colored marks denote places where a split can occur):



This image can be described by several sequences, for example : "010101010101011110110101010", of length 30, or "0100101111101110", of length 16, which is the minimal sequence for this image.

For a positive integer  $N$ , define  $D_N$  as the  $2^N \times 2^N$  image with the following coloring scheme:

- the pixel with coordinates  $X = 0, Y = 0$  corresponds to the bottom left pixel,
- if  $(X - 2^{N-1})^2 + (Y - 2^{N-1})^2 \leq 2^{2N-2}$  then the pixel is black,
- otherwise the pixel is white.

What is the length of the minimal sequence describing  $D_{24}$  ?

### Problem 288: An enormous factorial

For any prime  $p$  the number  $N(p, q)$  is defined by  $N(p, q) = \sum_{n=0}^q T_n \cdot p^n$   
with  $T_n$  generated by the following random number generator:

$$S_0 = 290797$$

$$S_{n+1} = S_n^2 \bmod 50515093$$

$$T_n = S_n \bmod p$$

Let  $Nfac(p, q)$  be the factorial of  $N(p, q)$ .

Let  $NF(p, q)$  be the number of factors  $p$  in  $Nfac(p, q)$ .

You are given that  $NF(3, 10000) \bmod 3^{20} = 624955285$ .

Find  $NF(61, 10^7) \bmod 61^{10}$

### Problem 289: Eulerian Cycles

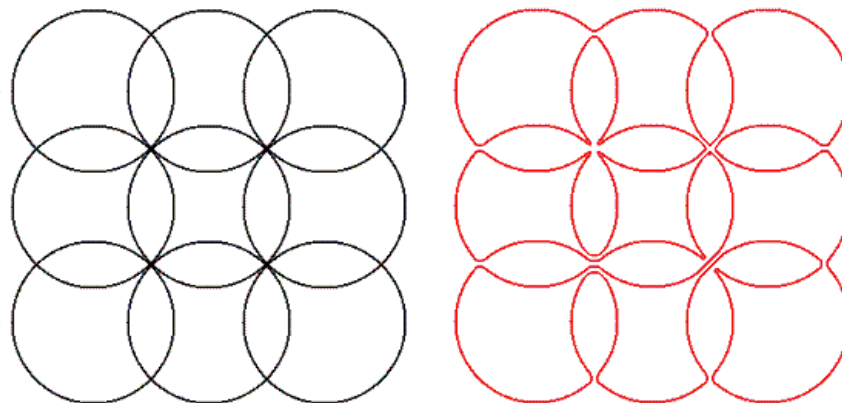
Let  $C(x,y)$  be a circle passing through the points  $(x,y)$ ,  $(x,y+1)$ ,  $(x+1,y)$  and  $(x+1,y+1)$ .

For positive integers  $m$  and  $n$ , let  $E(m,n)$  be a configuration which consists of the  $m \cdot n$  circles:  
 $\{ C(x,y): 0 \leq x < m, 0 \leq y < n, x \text{ and } y \text{ are integers} \}$

An Eulerian cycle on  $E(m,n)$  is a closed path that passes through each arc exactly once.

Many such paths are possible on  $E(m,n)$ , but we are only interested in those which are not self-crossing: A non-crossing path just touches itself at lattice points, but it never crosses itself.

The image below shows  $E(3,3)$  and an example of an Eulerian non-crossing path.



Let  $L(m,n)$  be the number of Eulerian non-crossing paths on  $E(m,n)$ .

For example,  $L(1,2) = 2$ ,  $L(2,2) = 37$  and  $L(3,3) = 104290$ .

Find  $L(6,10) \bmod 10^{10}$ .

### Problem 290: Digital Signature

How many integers  $0 \leq n < 10^{18}$  have the property that the sum of the digits of  $n$  equals the sum of digits of  $137n$ ?

### Problem 291: Panaitopol Primes

A prime number  $p$  is called a Panaitopol prime if  $p = \frac{x^4 - y^4}{x^3 + y^3}$  for some positive integers

$x$  and  $y$ .

Find how many Panaitopol primes are less than  $5 \times 10^{15}$ .

### Problem 292: Pythagorean Polygons

We shall define a *pythagorean polygon* to be a **convex polygon** with the following properties:

- there are at least three vertices,
- no three vertices are aligned,
- each vertex has **integer coordinates**,
- each edge has **integer length**.

For a given integer  $n$ , define  $P(n)$  as the number of distinct pythagorean polygons for which the perimeter is  $\leq n$ . Pythagorean polygons should be considered distinct as long as none is a translation of another.

You are given that  $P(4) = 1$ ,  $P(30) = 3655$  and  $P(60) = 891045$ .  
 Find  $P(120)$ .

### Problem 293: Pseudo-Fortunate Numbers

An even positive integer  $N$  will be called admissible, if it is a power of 2 or its distinct prime factors are consecutive primes. The first twelve admissible numbers are 2,4,6,8,12,16,18,24,30,32,36,48.

If  $N$  is admissible, the smallest integer  $M > 1$  such that  $N+M$  is prime, will be called the pseudo-Fortunate number for  $N$ .

For example,  $N=630$  is admissible since it is even and its distinct prime factors are the consecutive primes 2,3,5 and 7. The next prime number after 631 is 641; hence, the pseudo-Fortunate number for 630 is  $M=11$ . It can also be seen that the pseudo-Fortunate number for 16 is 3.

Find the sum of all distinct pseudo-Fortunate numbers for admissible numbers  $N$  less than  $10^9$ .

### Problem 294: Sum of digits - experience #23

For a positive integer  $k$ , define  $d(k)$  as the sum of the digits of  $k$  in its usual decimal representation. Thus  $d(42) = 4+2 = 6$ .

For a positive integer  $n$ , define  $S(n)$  as the number of positive integers  $k < 10^n$  with the following properties :

- $k$  is divisible by 23 and
- $d(k) = 23$ .

You are given that  $S(9) = 263626$  and  $S(42) = 6377168878570056$ .

Find  $S(11^{12})$  and give your answer mod  $10^9$ .

### Problem 295: Lenticular holes

We call the convex area enclosed by two circles a *lenticular hole* if:

- The centres of both circles are on lattice points.
- The two circles intersect at two distinct lattice points.
- The interior of the convex area enclosed by both circles does not contain any lattice points.

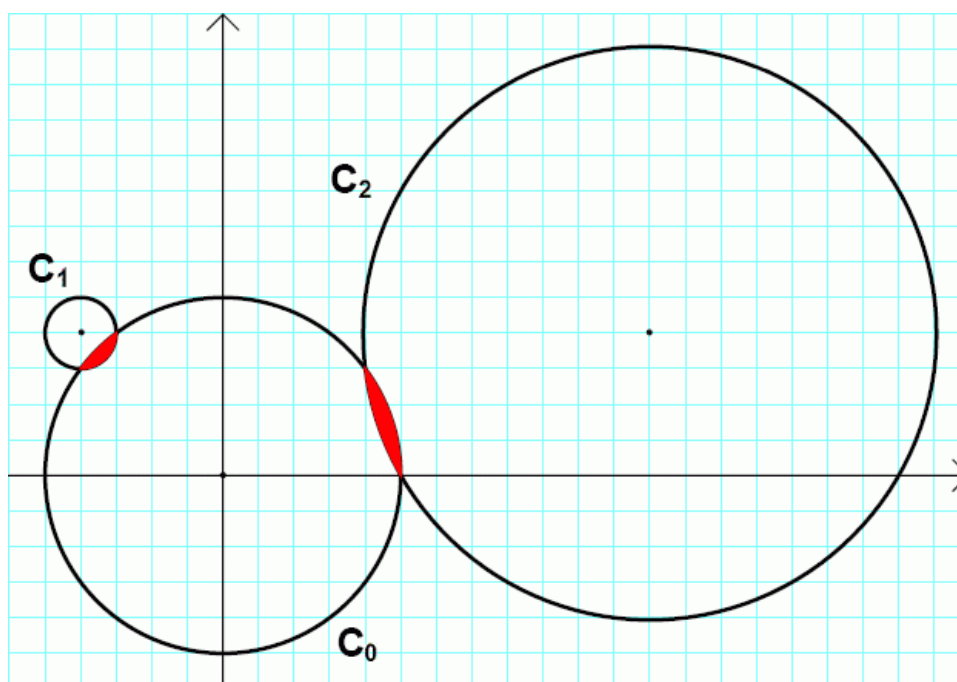
Consider the circles:

$$C_0: x^2 + y^2 = 25$$

$$C_1: (x+4)^2 + (y-4)^2 = 1$$

$$C_2: (x-12)^2 + (y-4)^2 = 65$$

The circles  $C_0$ ,  $C_1$  and  $C_2$  are drawn in the picture below.



$C_0$  and  $C_1$  form a lenticular hole, as well as  $C_0$  and  $C_2$ .

We call an ordered pair of positive real numbers  $(r_1, r_2)$  a *lenticular pair* if there exist two circles with radii  $r_1$  and  $r_2$  that form a lenticular hole. We can verify that  $(1, 5)$  and  $(5, \sqrt{65})$  are the lenticular pairs of the example above.

Let  $L(N)$  be the number of **distinct** lenticular pairs  $(r_1, r_2)$  for which  $0 < r_1 \leq r_2 \leq N$ .

We can verify that  $L(10) = 30$  and  $L(100) = 3442$ .

Find  $L(100\,000)$ .

### Problem 296: Angular Bisector and Tangent

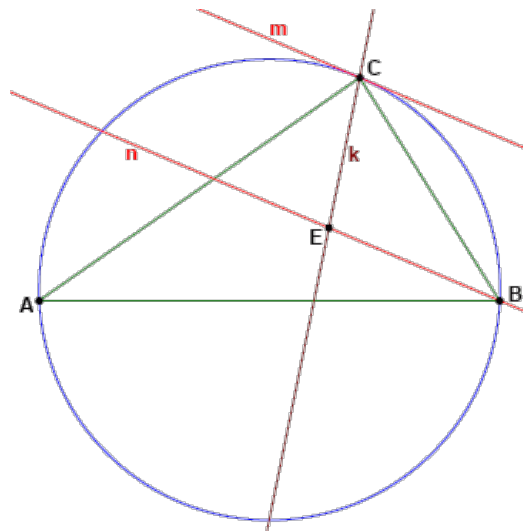
Given is an integer sided triangle  $ABC$  with  $BC \leq AC \leq AB$ .

$k$  is the angular bisector of angle  $ACB$ .

$m$  is the tangent at  $C$  to the circumscribed circle of  $ABC$ .

$n$  is a line parallel to  $m$  through  $B$ .

The intersection of  $n$  and  $k$  is called  $E$ .



How many triangles  $ABC$  with a perimeter not exceeding 100 000 exist such that  $BE$  has integral length?

### Problem 297: Zeckendorf Representation

Each new term in the Fibonacci sequence is generated by adding the previous two terms.

Starting with 1 and 2, the first 10 terms will be: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.

Every positive integer can be uniquely written as a sum of nonconsecutive terms of the Fibonacci sequence. For example,  $100 = 3 + 8 + 89$ .

Such a sum is called the **Zeckendorf representation** of the number.

For any integer  $n > 0$ , let  $Z(n)$  be the number of terms in the Zeckendorf representation of  $n$ .

Thus,  $Z(5) = 1$ ,  $Z(14) = 2$ ,  $Z(100) = 3$  etc.

Also, for  $0 < n < 10^6$ ,  $\sum Z(n) = 7894453$ .

Find  $\sum Z(n)$  for  $0 < n < 10^{17}$ .



### Problem 298: Selective Amnesia

Larry and Robin play a memory game involving of a sequence of random numbers between 1 and 10, inclusive, that are called out one at a time. Each player can remember up to 5 previous numbers. When the called number is in a player's memory, that player is awarded a point. If it's not, the player adds the called number to his memory, removing another number if his memory is full.

Both players start with empty memories. Both players always add new missed numbers to their memory but use a different strategy in deciding which number to remove:

Larry's strategy is to remove the number that hasn't been called in the longest time.

Robin's strategy is to remove the number that's been in the memory the longest time.

Example game:

Turn	Called number	Larry's memory	Larry's score	Robin's memory	Robin's score
1	1		1		0
2	2		1,2		0
3	4		1,2,4		0
4	6		1,2,4,6		0
5	1		1,2,4,6,1		1
6	8		1,2,4,6,8		1
7	10		1,4,6,8,10		1
8	2		1,2,6,8,10		2
9	4		1,2,4,8,10		3
10	1		1,2,4,8,10		2

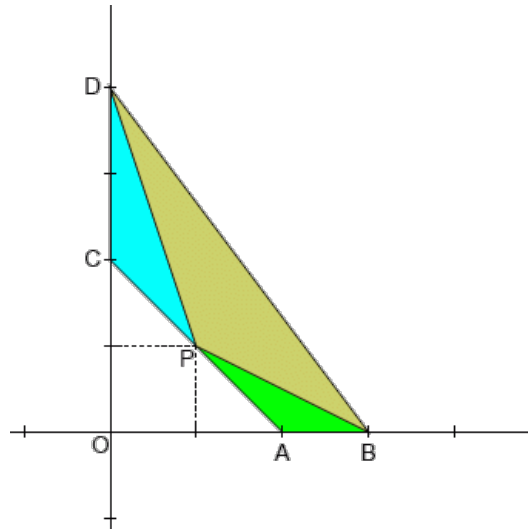
Denoting Larry's score by  $L$  and Robin's score by  $R$ , what is the expected value of  $|L - R|$  after 50 turns? Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

### Problem 299: Three similar triangles

Four points with integer coordinates are selected:

$A(a, 0)$ ,  $B(b, 0)$ ,  $C(0, c)$  and  $D(0, d)$ , with  $0 < a < b$  and  $0 < c < d$ .

Point  $P$ , also with integer coordinates, is chosen on the line  $AC$  so that the three triangles  $ABP$ ,  $CDP$  and  $BDP$  are all similar.



It is easy to prove that the three triangles can be similar, only if  $a=c$ .

So, given that  $a=c$ , we are looking for triplets  $(a, b, d)$  such that at least one point  $P$  (with integer coordinates) exists on  $AC$ , making the three triangles  $ABP$ ,  $CDP$  and  $BDP$  all similar.

For example, if  $(a, b, d) = (2, 3, 4)$ , it can be easily verified that point  $P(1, 1)$  satisfies the above condition. Note that the triplets  $(2, 3, 4)$  and  $(2, 4, 3)$  are considered as distinct, although point  $P(1, 1)$  is common for both.

If  $b+d < 100$ , there are 92 distinct triplets  $(a, b, d)$  such that point  $P$  exists.

If  $b+d < 100\,000$ , there are 320471 distinct triplets  $(a, b, d)$  such that point  $P$  exists.

If  $b+d < 100\,000\,000$ , how many distinct triplets  $(a, b, d)$  are there such that point  $P$  exists?

### Problem 300: Protein folding

In a very simplified form, we can consider proteins as strings consisting of hydrophobic (H) and polar (P) elements, e.g. HHPHHPHHPH.

For this problem, the orientation of a protein is important; e.g. HPP is considered distinct from PPH. Thus, there are  $2^n$  distinct proteins consisting of  $n$  elements.

When one encounters these strings in nature, they are always folded in such a way that the number of H-H contact points is as large as possible, since this is energetically advantageous.

As a result, the H-elements tend to accumulate in the inner part, with the P-elements on the outside.

Natural proteins are folded in three dimensions of course, but we will only consider protein folding in two dimensions.

The figure below shows two possible ways that our example protein could be folded (H-H contact points are shown with red dots).



The folding on the left has only six H-H contact points, thus it would never occur naturally.

On the other hand, the folding on the right has nine H-H contact points, which is optimal for this string.

Assuming that H and P elements are equally likely to occur in any position along the string, the average number of H-H contact points in an optimal folding of a random protein string of length 8 turns out to be  $850 / 2^8 = 3.3203125$ .

What is the average number of H-H contact points in an optimal folding of a random protein string of length 15?  
Give your answer using as many decimal places as necessary for an exact result.

### Problem 301: Nim

*Nim* is a game played with heaps of stones, where two players take it in turn to remove any number of stones from any heap until no stones remain.

We'll consider the three-heap normal-play version of Nim, which works as follows:

- At the start of the game there are three heaps of stones.
- On his turn the player removes any positive number of stones from any single heap.
- The first player unable to move (because no stones remain) loses.

If  $(n_1, n_2, n_3)$  indicates a Nim position consisting of heaps of size  $n_1$ ,  $n_2$  and  $n_3$  then there is a simple function  $X(n_1, n_2, n_3)$  — that you may look up or attempt to deduce for yourself — that returns:

- zero if, with perfect strategy, the player about to move will eventually lose; or
- non-zero if, with perfect strategy, the player about to move will eventually win.

For example  $X(1, 2, 3) = 0$  because, no matter what the current player does, his opponent can respond with a move that leaves two heaps of equal size, at which point every move by the current player can be mirrored by his opponent until no stones remain; so the current player loses. To illustrate:

- current player moves to  $(1, 2, 1)$
- opponent moves to  $(1, 0, 1)$
- current player moves to  $(0, 0, 1)$
- opponent moves to  $(0, 0, 0)$ , and so wins.

For how many positive integers  $n \leq 2^{30}$  does  $X(n, 2n, 3n) = 0$  ?

### Problem 302: Strong Achilles Numbers

A positive integer  $n$  is **powerful** if  $p^2$  is a divisor of  $n$  for every prime factor  $p$  in  $n$ .

A positive integer  $n$  is a **perfect power** if  $n$  can be expressed as a power of another positive integer.

A positive integer  $n$  is an **Achilles number** if  $n$  is powerful but not a perfect power. For example, 864 and 1800 are Achilles numbers:  $864 = 2^5 \cdot 3^3$  and  $1800 = 2^3 \cdot 3^2 \cdot 5^2$ .

We shall call a positive integer  $S$  a *Strong Achilles number* if both  $S$  and  $\varphi(S)$  are Achilles numbers.<sup>1</sup>

For example, 864 is a Strong Achilles number:  $\varphi(864) = 288 = 2^5 \cdot 3^2$ . However, 1800 isn't a Strong Achilles number because:  $\varphi(1800) = 480 = 2^5 \cdot 3^1 \cdot 5^1$ .

There are 7 Strong Achilles numbers below  $10^4$  and 656 below  $10^8$ .

How many Strong Achilles numbers are there below  $10^{18}$ ?

<sup>1</sup>  $\varphi$  denotes **Euler's totient function**.

### Problem 303: Multiples with small digits

For a positive integer  $n$ , define  $f(n)$  as the least positive multiple of  $n$  that, written in base 10, uses only digits  $\leq 2$ .

Thus  $f(2)=2$ ,  $f(3)=12$ ,  $f(7)=21$ ,  $f(42)=210$ ,  $f(89)=1121222$ .

Also, .

Find .

#### Problem 304: Primonacci

For any positive integer  $n$  the function  $\text{next\_prime}(n)$  returns the smallest prime  $p$  such that  $p > n$ .

The sequence  $a(n)$  is defined by:

$a(1) = \text{next\_prime}(10^{14})$  and  $a(n) = \text{next\_prime}(a(n-1))$  for  $n > 1$ .

The fibonacci sequence  $f(n)$  is defined by:  $f(0) = 0$ ,  $f(1) = 1$  and  $f(n) = f(n-1) + f(n-2)$  for  $n > 1$ .

The sequence  $b(n)$  is defined as  $f(a(n))$ .

Find  $\sum b(n)$  for  $1 \leq n \leq 100\,000$ . Give your answer mod 1234567891011.

#### Problem 305: Reflexive Position

Let's call  $S$  the (infinite) string that is made by concatenating the consecutive positive integers (starting from 1) written down in base 10.

Thus,  $S = 1234567891011121314151617181920212223242\dots$

It's easy to see that any number will show up an infinite number of times in  $S$ .

Let's call  $f(n)$  the starting position of the  $n^{\text{th}}$  occurrence of  $n$  in  $S$ .

For example,  $f(1) = 1$ ,  $f(5) = 81$ ,  $f(12) = 271$  and  $f(7780) = 111111365$ .

Find  $\sum f(3^k)$  for  $1 \leq k \leq 13$ .

#### Problem 306: Paper-strip Game

The following game is a classic example of Combinatorial Game Theory:

Two players start with a strip of  $n$  white squares and they take alternate turns.  
On each turn, a player picks two contiguous white squares and paints them black.  
The first player who cannot make a move loses.

- If  $n = 1$ , there are no valid moves, so the first player loses automatically.
- If  $n = 2$ , there is only one valid move, after which the second player loses.
- If  $n = 3$ , there are two valid moves, but both leave a situation where the second player loses.
- If  $n = 4$ , there are three valid moves for the first player; she can win the game by painting the two middle squares.
- If  $n = 5$ , there are four valid moves for the first player (shown below in red); but no matter what she does, the second player (blue) wins.



So, for  $1 \leq n \leq 5$ , there are 3 values of  $n$  for which the first player can force a win.

Similarly, for  $1 \leq n \leq 50$ , there are 40 values of  $n$  for which the first player can force a win.

For  $1 \leq n \leq 1\,000\,000$ , how many values of  $n$  are there for which the first player can force a win?

#### Problem 307: Chip Defects

$k$  defects are randomly distributed amongst  $n$  integrated-circuit chips produced by a factory (any number of defects may be found on a chip and each defect is independent of the other defects).

Let  $p(k, n)$  represent the probability that there is a chip with at least 3 defects.

For instance  $p(3, 7) \approx 0.0204081633$ .

Find  $p(20\,000, 1\,000\,000)$  and give your answer rounded to 10 decimal places in the form 0.abcdefghij

### Problem 308: An amazing Prime-generating Automaton

A program written in the programming language Fractran consists of a list of fractions.

The internal state of the Fractran Virtual Machine is a positive integer, which is initially set to a seed value. Each iteration of a Fractran program multiplies the state integer by the first fraction in the list which will leave it an integer.

For example, one of the Fractran programs that John Horton Conway wrote for prime-generation consists of the following 14 fractions:

$$\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1}.$$

Starting with the seed integer 2, successive iterations of the program produce the sequence:

15, 825, 725, 1925, 2275, 425, ..., 68, **4**, 30, ..., 136, **8**, 60, ..., 544, **32**, 240, ...

The powers of 2 that appear in this sequence are  $2^2, 2^3, 2^5, \dots$

It can be shown that *all* the powers of 2 in this sequence have prime exponents and that *all* the primes appear as exponents of powers of 2, in proper order!

If someone uses the above Fractran program to solve Project Euler Problem 7 (find the 1000<sup>th</sup> prime), how many iterations would be needed until the program produces  $2^{10001\text{st prime}}$ ?

### Problem 309: Integer Ladders

In the classic "Crossing Ladders" problem, we are given the lengths  $x$  and  $y$  of two ladders resting on the opposite walls of a narrow, level street. We are also given the height  $h$  above the street where the two ladders cross and we are asked to find the width of the street ( $w$ ).



Here, we are only concerned with instances where all four variables are positive integers.

For example, if  $x = 70, y = 119$  and  $h = 30$ , we can calculate that  $w = 56$ .

In fact, for integer values  $x, y, h$  and  $0 < x < y < 200$ , there are only five triplets  $(x, y, h)$  producing integer solutions for  $w$ : (70, 119, 30), (74, 182, 21), (87, 105, 35), (100, 116, 35) and (119, 175, 40).

For integer values  $x, y, h$  and  $0 < x < y < 1\,000\,000$ , how many triplets  $(x, y, h)$  produce integer solutions for  $w$ ?

### Problem 310: Nim Square

Alice and Bob play the game Nim Square.

Nim Square is just like ordinary three-heap normal play Nim, but the players may only remove a square number of stones from a heap.

The number of stones in the three heaps is represented by the ordered triple  $(a, b, c)$ .

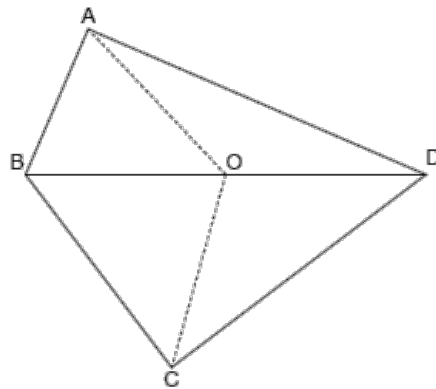
If  $0 \leq a \leq b \leq c \leq 29$  then the number of losing positions for the next player is 1160.

Find the number of losing positions for the next player if  $0 \leq a \leq b \leq c \leq 100\,000$ .

### Problem 311: Biclinic Integral Quadrilaterals

ABCD is a convex, integer sided quadrilateral with  $1 \leq AB < BC < CD < AD$ .  
BD has integer length. O is the midpoint of BD. AO has integer length.  
We'll call ABCD a *biclinic integral quadrilateral* if  $AO = CO \leq BO = DO$ .

For example, the following quadrilateral is a biclinic integral quadrilateral:  
 $AB = 19$ ,  $BC = 29$ ,  $CD = 37$ ,  $AD = 43$ ,  $BD = 48$  and  $AO = CO = 23$ .

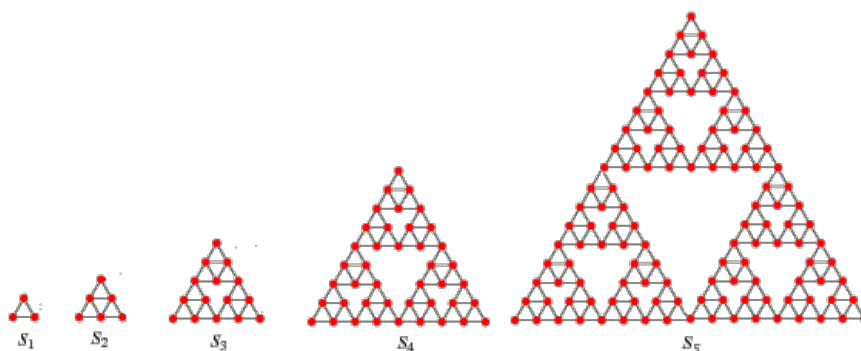


Let  $B(N)$  be the number of distinct biclinic integral quadrilaterals ABCD that satisfy  $AB^2 + BC^2 + CD^2 + AD^2 \leq N$ .  
We can verify that  $B(10\,000) = 49$  and  $B(1\,000\,000) = 38239$ .

Find  $B(10\,000\,000\,000)$ .

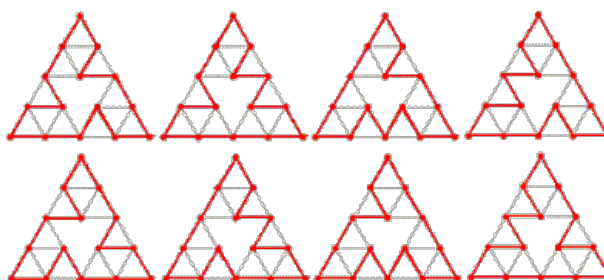
### Problem 312: Cyclic paths on Sierpiński graphs

- A **Sierpiński graph** of order-1 ( $S_1$ ) is an equilateral triangle.
- $S_{n+1}$  is obtained from  $S_n$  by positioning three copies of  $S_n$  so that every pair of copies has one common corner.



Let  $C(n)$  be the number of cycles that pass exactly once through all the vertices of  $S_n$ .

For example,  $C(3) = 8$  because eight such cycles can be drawn on  $S_3$ , as shown below:



It can also be verified that :

$$C(1) = C(2) = 1$$

$$C(5) = 71328803586048$$

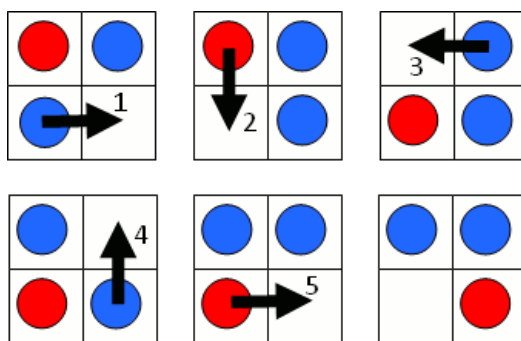
$$C(10\,000) \bmod 10^8 = 37652224$$

$$C(10\,000) \bmod 13^8 = 617720485$$

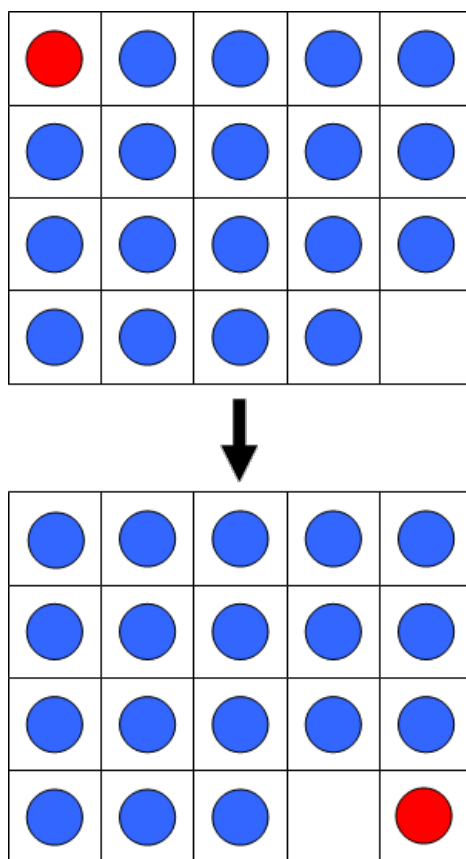
Find  $C(C(C(10\,000))) \bmod 13^8$ .

### Problem 313: Sliding game

In a sliding game a counter may slide horizontally or vertically into an empty space. The objective of the game is to move the red counter from the top left corner of a grid to the bottom right corner; the space always starts in the bottom right corner. For example, the following sequence of pictures show how the game can be completed in five moves on a 2 by 2 grid.



Let  $S(m,n)$  represent the minimum number of moves to complete the game on an  $m$  by  $n$  grid. For example, it can be verified that  $S(5,4) = 25$ .



There are exactly 5482 grids for which  $S(m,n) = p^2$ , where  $p < 100$  is prime.

How many grids does  $S(m,n) = p^2$ , where  $p < 10^6$  is prime?



### Problem 314: The Mouse on the Moon

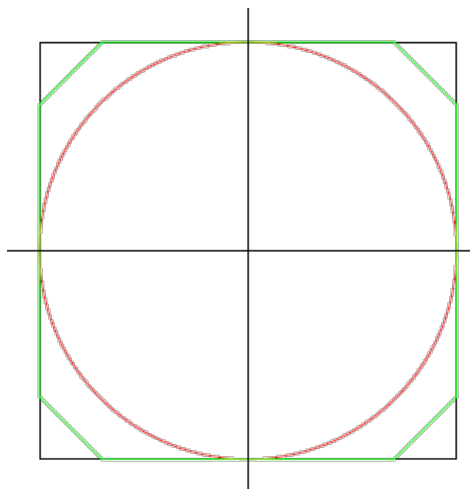
The moon has been opened up, and land can be obtained for free, but there is a catch. You have to build a wall around the land that you stake out, and building a wall on the moon is expensive. Every country has been allotted a 500 m by 500 m square area, but they will possess only that area which they wall in. 251001 posts have been placed in a rectangular grid with 1 meter spacing. The wall must be a closed series of straight lines, each line running from post to post.

The bigger countries of course have built a 2000 m wall enclosing the entire 250 000 m<sup>2</sup> area. The Duchy of Grand Fenwick, has a tighter budget, and has asked you (their Royal Programmer) to compute what shape would get best maximum enclosed-area/wall-length ratio.

You have done some preliminary calculations on a sheet of paper. For a 2000 meter wall enclosing the 250 000 m<sup>2</sup> area the enclosed-area/wall-length ratio is 125.

Although not allowed, but to get an idea if this is anything better: if you place a circle inside the square area touching the four sides the area will be equal to  $\pi \cdot 250^2$  m<sup>2</sup> and the perimeter will be  $\pi \cdot 500$  m, so the enclosed-area/wall-length ratio will also be 125.

However, if you cut off from the square four triangles with sides 75 m, 75 m and  $75\sqrt{2}$  m the total area becomes 238750 m<sup>2</sup> and the perimeter becomes  $1400 + 300\sqrt{2}$  m. So this gives an enclosed-area/wall-length ratio of 130.87, which is significantly better.



Find the maximum enclosed-area/wall-length ratio.

Give your answer rounded to 8 places behind the decimal point in the form abc.defghijk.

### Problem 315: Digital root clocks



Sam and Max are asked to transform two digital clocks into two "digital root" clocks.  
A digital root clock is a digital clock that calculates digital roots step by step.

When a clock is fed a number, it will show it and then it will start the calculation, showing all the intermediate values until it gets to the result.

For example, if the clock is fed the number 137, it will show: "**137**" → "**11**" → "**2**" and then it will go black, waiting for the next number.

Every digital number consists of some light segments: three horizontal (top, middle, bottom) and four vertical (top-left, top-right, bottom-left, bottom-right).

Number "**1**" is made of vertical top-right and bottom-right, number "**4**" is made by middle horizontal and vertical top-left, top-right and bottom-right. Number "**8**" lights them all.

The clocks consume energy only when segments are turned on/off.  
To turn on a "**2**" will cost 5 transitions, while a "**7**" will cost only 4 transitions.

Sam and Max built two different clocks.

Sam's clock is fed e.g. number 137: the clock shows **137**", then the panel is turned off, then the next number (**11**") is turned on, then the panel is turned off again and finally the last number ("**2**") is turned on and, after some time, off.  
For the example, with number 137, Sam's clock requires:

"**137**" :  $(2 + 5 + 4) \times 2 = 22$  transitions ("**137**" on/off).

"**11**" :  $(2 + 2) \times 2 = 8$  transitions ("**11**" on/off).

"**2**" :  $(5) \times 2 = 10$  transitions ("**2**" on/off).

For a grand total of 40 transitions.

Max's clock works differently. Instead of turning off the whole panel, it is smart enough to turn off only those segments that won't be needed for the next number.

For number 137, Max's clock requires:

"**137**" :  $2 + 5 + 4 = 11$  transitions ("**137**" on)  
7 transitions (to turn off the segments that are not needed for number "**11**").

"**11**" : 0 transitions (number "**11**" is already turned on correctly)  
3 transitions (to turn off the first "**1**" and the bottom part of the second "**1**"; the top part is common with number "**2**").

"**2**" : 4 transitions (to turn on the remaining segments in order to get a **2**)  
5 transitions (to turn off number "**2**").

For a grand total of 30 transitions.

Of course, Max's clock consumes less power than Sam's one.

The two clocks are fed all the prime numbers between  $A = 10^7$  and  $B = 2 \times 10^7$ .

Find the difference between the total number of transitions needed by Sam's clock and that needed by Max's one.

### Problem 316: Numbers in decimal expansions

Let  $p = p_1 p_2 p_3 \dots$  be an infinite sequence of random digits, selected from  $\{0,1,2,3,4,5,6,7,8,9\}$  with equal probability.

It can be seen that  $p$  corresponds to the real number  $0p_1 p_2 p_3 \dots$ .

It can also be seen that choosing a random real number from the interval  $[0,1)$  is equivalent to choosing an infinite sequence of random digits selected from  $\{0,1,2,3,4,5,6,7,8,9\}$  with equal probability.

For any positive integer  $n$  with  $d$  decimal digits, let  $k$  be the smallest index such that

$p_k, p_{k+1}, \dots, p_{k+d-1}$  are the decimal digits of  $n$ , in the same order.

Also, let  $g(n)$  be the expected value of  $k$ ; it can be proven that  $g(n)$  is always finite and, interestingly, always an integer number.

For example, if  $n = 535$ , then

for  $p = 314159265\mathbf{535}897\dots$ , we get  $k = 9$

for  $p = 35528714365004956000049084876408468\mathbf{535}4\dots$ , we get  $k = 36$

etc and we find that  $g(535) = 1008$ .

Given that  $\sum_{n=2}^{999} g\left(\left\lfloor \frac{10^6}{n} \right\rfloor\right) = 27280188$ , find

Note:  represents the floor function.

### Problem 317: Firecracker

A firecracker explodes at a height of 100 m above level ground. It breaks into a large number of very small fragments, which move in every direction; all of them have the same initial velocity of 20 m/s.

We assume that the fragments move without air resistance, in a uniform gravitational field with  $g=9.81 \text{ m/s}^2$ .

Find the volume (in  $\text{m}^3$ ) of the region through which the fragments move before reaching the ground. Give your answer rounded to four decimal places.

### Problem 318: 2011 nines

Consider the real number  $\sqrt{2}+\sqrt{3}$ .

When we calculate the even powers of  $\sqrt{2}+\sqrt{3}$  we get:

$$(\sqrt{2}+\sqrt{3})^2 = 9.898979485566356\dots$$

$$(\sqrt{2}+\sqrt{3})^4 = 97.98979485566356\dots$$

$$(\sqrt{2}+\sqrt{3})^6 = 969.998969071069263\dots$$

$$(\sqrt{2}+\sqrt{3})^8 = 9601.99989585502907\dots$$

$$(\sqrt{2}+\sqrt{3})^{10} = 95049.999989479221\dots$$

$$(\sqrt{2}+\sqrt{3})^{12} = 940897.9999989371855\dots$$

$$(\sqrt{2}+\sqrt{3})^{14} = 9313929.99999989263\dots$$

$$(\sqrt{2}+\sqrt{3})^{16} = 92198401.99999998915\dots$$

It looks like that the number of consecutive nines at the beginning of the fractional part of these powers is non-decreasing.

In fact it can be proven that the fractional part of  $(\sqrt{2}+\sqrt{3})^{2n}$  approaches 1 for large  $n$ .

Consider all real numbers of the form  $\sqrt{p}+\sqrt{q}$  with  $p$  and  $q$  positive integers and  $p < q$ , such that the fractional part of  $(\sqrt{p}+\sqrt{q})^{2n}$  approaches 1 for large  $n$ .

Let  $C(p,q,n)$  be the number of consecutive nines at the beginning of the fractional part of  $(\sqrt{p}+\sqrt{q})^{2n}$ .

Let  $N(p,q)$  be the minimal value of  $n$  such that  $C(p,q,n) \geq 2011$ .

Find  $\sum N(p,q)$  for  $p+q \leq 2011$ .

### Problem 319: Bounded Sequences

Let  $X_1, X_2, \dots, X_n$  be a sequence of length  $n$  such that:

- $X_1 = 2$
- for all  $1 < i \leq n : X_{i-1} < X_i$
- for all  $i$  and  $j$  with  $1 \leq i, j \leq n : (X_i)^j < (X_j + 1)^i$

There are only five such sequences of length 2, namely:  $\{2,4\}$ ,  $\{2,5\}$ ,  $\{2,6\}$ ,  $\{2,7\}$  and  $\{2,8\}$ .

There are 293 such sequences of length 5; three examples are given below:

$\{2,5,11,25,55\}$ ,  $\{2,6,14,36,88\}$ ,  $\{2,8,22,64,181\}$ .

Let  $t(n)$  denote the number of such sequences of length  $n$ .

You are given that  $t(10) = 86195$  and  $t(20) = 5227991891$ .

Find  $t(10^{10})$  and give your answer modulo  $10^9$ .

### Problem 320: Factorials divisible by a huge integer

Let  $N(i)$  be the smallest integer  $n$  such that  $n!$  is divisible by  $(i!)^{1234567890}$

Let  $S(u) = \sum N(i)$  for  $10 \leq i \leq u$ .

$S(1000) = 614538266565663$ .

Find  $S(1\,000\,000) \bmod 10^{18}$ .

### Problem 321: Swapping Counters

A horizontal row comprising of  $2n + 1$  squares has  $n$  red counters placed at one end and  $n$  blue counters at the other end, being separated by a single empty square in the centre. For example, when  $n = 3$ .



A counter can move from one square to the next (slide) or can jump over another counter (hop) as long as the square next to that counter is unoccupied.



Let  $M(n)$  represent the minimum number of moves/actions to completely reverse the positions of the coloured counters; that is, move all the red counters to the right and all the blue counters to the left.

It can be verified  $M(3) = 15$ , which also happens to be a triangle number.

If we create a sequence based on the values of  $n$  for which  $M(n)$  is a triangle number then the first five terms would be: 1, 3, 10, 22, and 63, and their sum would be 99.

Find the sum of the first forty terms of this sequence.

### Problem 322: Binomial coefficients divisible by 10

Let  $T(m, n)$  be the number of the binomial coefficients  ${}^nC_n$  that are divisible by 10 for  $n \leq i < m$  ( $i, m$  and  $n$  are positive integers).

You are given that  $T(10^9, 10^7 - 10) = 989697000$ .

Find  $T(10^{18}, 10^{12} - 10)$ .

### Problem 323: Bitwise-OR operations on random integers

Let  $y_0, y_1, y_2, \dots$  be a sequence of random unsigned 32 bit integers (i.e.  $0 \leq y_i < 2^{32}$ , every value equally likely).

For the sequence  $X_i$  the following recursion is given:

- $X_0 = 0$  and
- $X_i = X_{i-1} | y_{i-1}$ , for  $i > 0$ . ( $|$  is the bitwise-OR operator)

It can be seen that eventually there will be an index  $N$  such that  $X_i = 2^{32} - 1$  (a bit-pattern of all ones) for all  $i \geq N$ .

Find the expected value of  $N$ .

Give your answer rounded to 10 digits after the decimal point.

### Problem 324: Building a tower

Let  $f(n)$  represent the number of ways one can fill a  $3 \times 3 \times n$  tower with blocks of  $2 \times 1 \times 1$ .

You're allowed to rotate the blocks in any way you like; however, rotations, reflections etc of the tower itself are counted as distinct.

For example (with  $q = 100000007$ ) :

$$f(2) = 229,$$

$$f(4) = 117805,$$

$$f(10) \bmod q = 96149360,$$

$$f(10^3) \bmod q = 24806056,$$

$$f(10^6) \bmod q = 30808124.$$

Find  $f(10^{10000}) \bmod 100000007$ .

### Problem 325: Stone Game II

A game is played with two piles of stones and two players. At her turn, a player removes a number of stones from the larger pile. The number of stones she removes must be a positive multiple of the number of stones in the smaller pile.

E.g., let the ordered pair (6,14) describe a configuration with 6 stones in the smaller pile and 14 stones in the larger pile, then the first player can remove 6 or 12 stones from the larger pile.

The player taking all the stones from a pile wins the game.

A *winning configuration* is one where the first player can force a win. For example, (1,5), (2,6) and (3,12) are winning configurations because the first player can immediately remove all stones in the second pile.

A *losing configuration* is one where the second player can force a win, no matter what the first player does. For example, (2,3) and (3,4) are losing configurations: any legal move leaves a winning configuration for the second player.

Define  $S(N)$  as the sum of  $(x_i + y_i)$  for all losing configurations  $(x_i, y_i)$ ,  $0 < x_i < y_i \leq N$ . We can verify that  $S(10) = 211$  and  $S(10^4) = 230312207313$ .

Find  $S(10^{16}) \bmod 7^{10}$ .

### Problem 326: Modulo Summations

Let  $a_n$  be a sequence recursively defined by  $a_1=1$ ,  $a_n=\biggl(\sum_{k=1}^{n-1}k\cdot a_k\biggr)\bmod n$ .

So the first 10 elements of  $a_n$  are: 1,1,0,3,0,3,5,4,1,9.

Let  $f(N,M)$  represent the number of pairs  $(p,q)$  such that:

$$1\leq p\leq q\leq N$$

$$\text{and}\biggl(\sum_{i=p}^qa_i\biggr)\bmod M=0$$

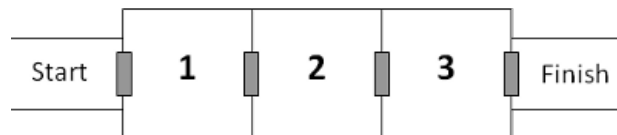
It can be seen that  $f(10,10)=4$  with the pairs (3,3), (5,5), (7,9) and (9,10).

You are also given that  $f(10^4,10^3)=97158$ .

Find  $f(10^{12},10^6)$ .

### Problem 327: Rooms of Doom

A series of three rooms are connected to each other by automatic doors.



Each door is operated by a security card. Once you enter a room the door automatically closes and that security card cannot be used again. A machine at the start will dispense an unlimited number of cards, but each room (including the starting room) contains scanners and if they detect that you are holding more than three security cards or if they detect an unattended security card on the floor, then all the doors will become permanently locked. However, each room contains a box where you may safely store any number of security cards for use at a later stage.

If you simply tried to travel through the rooms one at a time then as you entered room 3 you would have used all three cards and would be trapped in that room forever!

However, if you make use of the storage boxes, then escape is possible. For example, you could enter room 1 using your first card, place one card in the storage box, and use your third card to exit the room back to the start. Then after collecting three more cards from the dispensing machine you could use one to enter room 1 and collect the card you placed in the box a moment ago. You now have three cards again and will be able to travel through the remaining three doors. This method allows you to travel through all three rooms using six security cards in total.

It is possible to travel through six rooms using a total of 123 security cards while carrying a maximum of 3 cards.

Let  $C$  be the maximum number of cards which can be carried at any time.

Let  $R$  be the number of rooms to travel through.

Let  $M(C,R)$  be the minimum number of cards required from the dispensing machine to travel through  $R$  rooms carrying up to a maximum of  $C$  cards at any time.

For example,  $M(3,6)=123$  and  $M(4,6)=23$ .

And,  $\sum M(C,6)=146$  for  $3\leq C\leq 4$ .

You are given that  $\sum M(C,10)=10382$  for  $3\leq C\leq 10$ .

Find  $\sum M(C,30)$  for  $3\leq C\leq 40$ .

### Problem 328: Lowest-cost Search

We are trying to find a hidden number selected from the set of integers  $\{1, 2, \dots, n\}$  by asking questions. Each number (question) we ask, has a cost equal to the number asked and we get one of three possible answers:

- "Your guess is lower than the hidden number", or
- "Yes, that's it!", or
- "Your guess is higher than the hidden number".

Given the value of  $n$ , an *optimal strategy* minimizes the total cost (i.e. the sum of all the questions asked for the worst possible case). E.g.

If  $n=3$ , the best we can do is obviously to ask the number **2**". The answer will immediately lead us to find the hidden number (at a total cost = 2).

If  $n=8$ , we might decide to use a "binary search" type of strategy: Our first question would be **4**" and if the hidden number is higher than 4 we will need one or two additional questions.

Let our second question be **6**". If the hidden number is still higher than 6, we will need a third question in order to discriminate between 7 and 8.

Thus, our third question will be **7**" and the total cost for this worst-case scenario will be  $4+6+7=17$ .

We can improve considerably the worst-case cost for  $n=8$ , by asking **5**" as our first question.

If we are told that the hidden number is higher than 5, our second question will be **7**", then we'll know for certain what the hidden number is (for a total cost of  $5+7=12$ ).

If we are told that the hidden number is lower than 5, our second question will be **3**" and if the hidden number is lower than 3 our third question will be **1**", giving a total cost of  $5+3+1=9$ .

Since  $12 > 9$ , the worst-case cost for this strategy is **12**. That's better than what we achieved previously with the "binary search" strategy; it is also better than or equal to any other strategy.

So, in fact, we have just described an optimal strategy for  $n=8$ .

Let  $C(n)$  be the worst-case cost achieved by an optimal strategy for  $n$ , as described above.

Thus  $C(1) = 0$ ,  $C(2) = 1$ ,  $C(3) = 2$  and  $C(8) = 12$ .

Similarly,  $C(100) = 400$  and  $\sum_{n=1}^{100} C(n) = 17575$ .

Find  $\sum_{n=1}^{2000000} C(n)$ .

### Problem 329: Prime Frog

Susan has a prime frog.

Her frog is jumping around over 500 squares numbered 1 to 500. He can only jump one square to the left or to the right, with equal probability, and he cannot jump outside the range  $[1;500]$ .

(if it lands at either end, it automatically jumps to the only available square on the next move.)

When he is on a square with a prime number on it, he croaks 'P' (PRIME) with probability  $2/3$  or 'N' (NOT PRIME) with probability  $1/3$  just before jumping to the next square.

When he is on a square with a number on it that is not a prime he croaks 'P' with probability  $1/3$  or 'N' with probability  $2/3$  just before jumping to the next square.

Given that the frog's starting position is random with the same probability for every square, and given that she listens to his first 15 croaks, what is the probability that she hears the sequence PPPPNPPPNPPNPN?

Give your answer as a fraction  $p/q$  in reduced form.

### Problem 330: Euler's Number

An infinite sequence of real numbers  $a(n)$  is defined for all integers  $n$  as follows:

$$a(n) = \begin{cases} 1 & n < 0 \\ \sum_{i=1}^{\infty} \frac{a(n-i)}{i!} & n \geq 0 \end{cases}$$

For example,

$$a(0) = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e - 1$$

$$a(1) = \frac{e-1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = 2e - \frac{3}{2}$$

$$a(2) = \frac{2e-3}{1!} + \frac{e-1}{2!} + \frac{1}{3!} + \dots = \frac{7}{2}e - \frac{11}{6}$$

with  $e = 2.7182818\dots$  being Euler's constant.

It can be shown that  $a(n)$  is of the form  $\frac{A(n)e + B(n)}{n!}$  for integers  $A(n)$  and  $B(n)$ .

$$\text{For example } a(10) = \frac{328161643e - 652694486}{10!}.$$

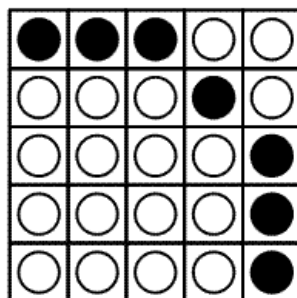
Find  $A(10^9) + B(10^9)$  and give your answer mod 77 777 777.

### Problem 331: Cross flips

$N \times N$  disks are placed on a square game board. Each disk has a black side and white side.

At each turn, you may choose a disk and flip all the disks in the same row and the same column as this disk: thus  $2N-1$  disks are flipped. The game ends when all disks show their white side. The following example shows a game on a  $5 \times 5$  board.

**Start**



It can be proven that 3 is the minimal number of turns to finish this game.

The bottom left disk on the  $N \times N$  board has coordinates  $(0,0)$ ; the bottom right disk has coordinates  $(N-1,0)$  and the top left disk has coordinates  $(0,N-1)$ .

Let  $C_N$  be the following configuration of a board with  $N \times N$  disks:

A disk at  $(X,Y)$  satisfying  $N-1 \leq \sqrt{X^2 + Y^2} < N$ , shows its black side; otherwise, it shows its white side.  $C_5$  is shown above.

Let  $T(N)$  be the minimal number of turns to finish a game starting from configuration  $C_N$  or 0 if configuration  $C_N$  is unsolvable. We have shown that  $T(5)=3$ . You are also given that  $T(10)=29$  and  $T(1\,000)=395253$ .

Find .



### Problem 332: Spherical triangles

A **spherical triangle** is a figure formed on the surface of a sphere by three **great circular arcs** intersecting pairwise in three vertices.

Let  $C(r)$  be the sphere with the centre  $(0,0,0)$  and radius  $r$ .

Let  $Z(r)$  be the set of points on the surface of  $C(r)$  with integer coordinates.

Let  $T(r)$  be the set of spherical triangles with vertices in  $Z(r)$ . Degenerate spherical triangles, formed by three points on the same great arc, are not included in  $T(r)$ .

Let  $A(r)$  be the area of the smallest spherical triangle in  $T(r)$ .

For example  $A(14)$  is 3.294040 rounded to six decimal places.

Find  $\sum_{r=1}^{50} A(r)$ . Give your answer rounded to six decimal places.

### Problem 333: Special partitions

All positive integers can be partitioned in such a way that each and every term of the partition can be expressed as  $2^i 3^j$ , where  $i, j \geq 0$ .

Let's consider only those such partitions where none of the terms can divide any of the other terms.

For example, the partition of  $17 = 2 + 6 + 9 = (2^1 3^0 + 2^1 3^1 + 2^0 3^2)$  would not be valid since 2 can divide 6. Neither would the partition  $17 = 16 + 1 = (2^4 3^0 + 2^0 3^0)$  since 1 can divide 16. The only valid partition of 17 would be  $8 + 9 = (2^3 3^0 + 2^0 3^2)$ .

Many integers have more than one valid partition, the first being 11 having the following two partitions.

$$11 = 2 + 9 = (2^1 3^0 + 2^0 3^2)$$

$$11 = 8 + 3 = (2^3 3^0 + 2^0 3^1)$$

Let's define  $P(n)$  as the number of valid partitions of  $n$ . For example,  $P(11) = 2$ .

Let's consider only the prime integers  $q$  which would have a single valid partition such as  $P(17)$ .

The sum of the primes  $q < 100$  such that  $P(q) = 1$  equals 233.

Find the sum of the primes  $q < 1000000$  such that  $P(q) = 1$ .

### Problem 334: Spilling the beans

In Plato's heaven, there exist an infinite number of bowls in a straight line.

Each bowl either contains some or none of a finite number of beans.

A child plays a game, which allows only one kind of move: removing two beans from any bowl, and putting one in each of the two adjacent bowls.

The game ends when each bowl contains either one or no beans.

For example, consider two adjacent bowls containing 2 and 3 beans respectively, all other bowls being empty. The following eight moves will finish the game:

You are given the following sequences:

```
\def\text#1{\style{font-family:inherit;}\text{#1}} \begin{align} \quad t_0 &= 123456, \cr \quad t_i &= \cases{ \lfloor \frac{t_{i-1}}{2} \rfloor, & \text{if } t_{i-1} \text{ is even} \cr \lfloor \frac{t_{i-1}}{2} \rfloor + 926252, & \text{if } t_{i-1} \text{ is odd} \cr } \cr \quad \text{where } \lfloor x \rfloor &\text{ is the floor function } \cr \quad \text{and } \oplus &\text{ is the bitwise XOR operator.} \cr \quad b_i &= (t_i \bmod 2^{11}) + 1. \cr \end{align}
```

The first two terms of the last sequence are  $b_1 = 289$  and  $b_2 = 145$ .

If we start with  $b_1$  and  $b_2$  beans in two adjacent bowls, 3419100 moves would be required to finish the game.

Consider now 1500 adjacent bowls containing  $b_1, b_2, \dots, b_{1500}$  beans respectively, all other bowls being empty. Find how many moves it takes before the game ends.

### Problem 335: Gathering the beans

Whenever Peter feels bored, he places some bowls, containing one bean each, in a circle. After this, he takes all the beans out of a certain bowl and drops them one by one in the bowls going clockwise. He repeats this, starting from the bowl he dropped the last bean in, until the initial situation appears again. For example with 5 bowls he acts as follows:



So with 5 bowls it takes Peter 15 moves to return to the initial situation.

Let  $M(x)$  represent the number of moves required to return to the initial situation, starting with  $x$  bowls. Thus,  $M(5) = 15$ . It can also be verified that  $M(100) = 10920$ .

Find   $M(2^{k+1})$ . Give your answer modulo  $7^9$ .

### Problem 336: Maximix Arrangements

A train is used to transport four carriages in the order: ABCD. However, sometimes when the train arrives to collect the carriages they are not in the correct order.

To rearrange the carriages they are all shunted on to a large rotating turntable. After the carriages are uncoupled at a specific point the train moves off the turntable pulling the carriages still attached with it. The remaining carriages are rotated 180 degrees. All of the carriages are then rejoined and this process is repeated as often as necessary in order to obtain the least number of uses of the turntable.

Some arrangements, such as ADCB, can be solved easily: the carriages are separated between A and D, and after DCB are rotated the correct order has been achieved.

However, Simple Simon, the train driver, is not known for his efficiency, so he always solves the problem by initially getting carriage A in the correct place, then carriage B, and so on.

Using four carriages, the worst possible arrangements for Simon, which we shall call *maximix arrangements*, are DACB and DBAC; each requiring him five rotations (although, using the most efficient approach, they could be solved using just three rotations). The process he uses for DACB is shown below.



It can be verified that there are 24 maximix arrangements for six carriages, of which the tenth lexicographic maximix arrangement is DFAECB.

Find the 2011<sup>th</sup> lexicographic maximix arrangement for eleven carriages.

### Problem 337: Totient Stairstep Sequences

Let  $\{a_1, a_2, \dots, a_n\}$  be an integer sequence of length  $n$  such that:

- $a_1 = 6$
- for all  $1 \leq i < n$  :  $\varphi(a_i) < \varphi(a_{i+1}) < a_i < a_{i+1}$ <sup>1</sup>

Let  $S(N)$  be the number of such sequences with  $a_n \leq N$ .

For example,  $S(10) = 4$ :  $\{6\}$ ,  $\{6, 8\}$ ,  $\{6, 8, 9\}$  and  $\{6, 10\}$ .

We can verify that  $S(100) = 482073668$  and  $S(10\,000) \bmod 10^8 = 73808307$ .

Find  $S(20\,000\,000) \bmod 10^8$ .

<sup>1</sup>  $\varphi$  denotes **Euler's totient function**.

### Problem 338: Cutting Rectangular Grid Paper

A rectangular sheet of grid paper with integer dimensions  $w \times h$  is given. Its grid spacing is 1.

When we cut the sheet along the grid lines into two pieces and rearrange those pieces without overlap, we can make new rectangles with different dimensions.

For example, from a sheet with dimensions  $9 \times 4$ , we can make rectangles with dimensions  $18 \times 2$ ,  $12 \times 3$  and  $6 \times 6$  by cutting and rearranging as below:



Similarly, from a sheet with dimensions  $9 \times 8$ , we can make rectangles with dimensions  $18 \times 4$  and  $12 \times 6$ .

For a pair  $w$  and  $h$ , let  $F(w, h)$  be the number of distinct rectangles that can be made from a sheet with dimensions  $w \times h$ .

For example,  $F(2, 1) = 0$ ,  $F(2, 2) = 1$ ,  $F(9, 4) = 3$  and  $F(9, 8) = 2$ .

Note that rectangles congruent to the initial one are not counted in  $F(w, h)$ .

Note also that rectangles with dimensions  $w \times h$  and dimensions  $h \times w$  are not considered distinct.

For an integer  $N$ , let  $G(N)$  be the sum of  $F(w, h)$  for all pairs  $w$  and  $h$  which satisfy  $0 < h \leq w \leq N$ .

We can verify that  $G(10) = 55$ ,  $G(10^3) = 971745$  and  $G(10^5) = 9992617687$ .

Find  $G(10^{12})$ . Give your answer modulo  $10^8$ .

### Problem 339: Peredur fab Efwarg

*"And he came towards a valley, through which ran a river; and the borders of the valley were wooded, and on each side of the river were level meadows. And on one side of the river he saw a flock of white sheep, and on the other a flock of black sheep. And whenever one of the white sheep bleated, one of the black sheep would cross over and become white; and when one of the black sheep bleated, one of the white sheep would cross over and become black."*

[en.wikisource.org](http://en.wikisource.org)

Initially each flock consists of  $n$  sheep. Each sheep (regardless of colour) is equally likely to be the next sheep to bleat. After a sheep has bleated and a sheep from the other flock has crossed over, Peredur may remove a number of white sheep in order to maximize the expected final number of black sheep. Let  $E(n)$  be the expected final number of black sheep if Peredur uses an optimal strategy.

You are given that  $E(5) = 6.871346$  rounded to 6 places behind the decimal point.

Find  $E(10\,000)$  and give your answer rounded to 6 places behind the decimal point.

### Problem 340: Crazy Function

For fixed integers  $a$ ,  $b$ ,  $c$ , define the *crazy function*  $F(n)$  as follows:

$F(n) = n - c$  for all  $n > b$

$F(n) = F(a + F(a + F(a + F(a + n))))$  for all  $n \leq b$ .

Also, define  $S(a, b, c) = \sum_{n=0}^b F(n)$ .

For example, if  $a = 50$ ,  $b = 2000$  and  $c = 40$ , then  $F(0) = 3240$  and  $F(2000) = 2040$ .

Also,  $S(50, 2000, 40) = 5204240$ .

Find the last 9 digits of  $S(21^7, 7^{21}, 12^7)$ .

### Problem 341: Golomb's self-describing sequence

The **Golomb's self-describing sequence**  $\{G(n)\}$  is the only nondecreasing sequence of natural numbers such that  $n$  appears exactly  $G(n)$  times in the sequence. The values of  $G(n)$  for the first few  $n$  are

$n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$G(n)$	1	2	2	3	3	4	4	4	5	5	5	6	6	6	6	...

You are given that  $G(10^3) = 86$ ,  $G(10^6) = 6137$ .

You are also given that  $\Sigma G(n^3) = 153506976$  for  $1 \leq n < 10^3$ .

Find  $\Sigma G(n^3)$  for  $1 \leq n < 10^6$ .

### Problem 342: The totient of a square is a cube

Consider the number 50.

$50^2 = 2500 = 2^2 \times 5^4$ , so  $\phi(2500) = 2 \times 4 \times 5^3 = 8 \times 5^3 = 2^3 \times 5^3$ .<sup>1</sup>

So 2500 is a square and  $\phi(2500)$  is a cube.

Find the sum of all numbers  $n$ ,  $1 < n < 10^{10}$  such that  $\phi(n^2)$  is a cube.

<sup>1</sup>  $\phi$  denotes **Euler's totient function**.

### Problem 343: Fractional Sequences

For any positive integer  $k$ , a finite sequence  $a_i$  of fractions  $x_i/y_i$  is defined by:

$a_1 = 1/k$  and

$a_i = (x_{i-1}+1)/(y_{i-1}-1)$  reduced to lowest terms for  $i > 1$ .

When  $a_i$  reaches some integer  $n$ , the sequence stops. (That is, when  $y=1$ .)

Define  $f(k) = n$ .

For example, for  $k = 20$ :

$1/20 \rightarrow 2/19 \rightarrow 3/18 = 1/6 \rightarrow 2/5 \rightarrow 3/4 \rightarrow 4/3 \rightarrow 5/2 \rightarrow 6/1 = 6$

So  $f(20) = 6$ .

Also  $f(1) = 1$ ,  $f(2) = 2$ ,  $f(3) = 1$  and  $\Sigma f(k^3) = 118937$  for  $1 \leq k \leq 100$ .

Find  $\Sigma f(k^3)$  for  $1 \leq k \leq 2 \times 10^6$ .

### Problem 344: Silver dollar game

One variant of N.G. de Bruijn's **silver dollar** game can be described as follows:

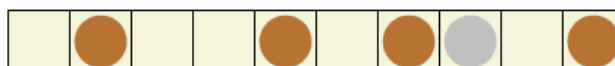
On a strip of squares a number of coins are placed, at most one coin per square. Only one coin, called the **silver dollar**, has any value. Two players take turns making moves. At each turn a player must make either a *regular* or a *special* move.

A *regular* move consists of selecting one coin and moving it one or more squares to the left. The coin cannot move out of the strip or jump on or over another coin.

Alternatively, the player can choose to make the *special* move of pocketing the leftmost coin rather than making a regular move. If no regular moves are possible, the player is forced to pocket the leftmost coin.

The winner is the player who pockets the silver dollar.

Start



A *winning configuration* is an arrangement of coins on the strip where the first player can force a win no matter what the second player does.

Let  $W(n, C)$  be the number of winning configurations for a strip of  $n$  squares,  $C$  worthless coins and one silver dollar.

You are given that  $W(10, 2) = 324$  and  $W(100, 10) = 1514704946113500$ .

Find  $W(1\,000\,000, 100)$  modulo the semiprime  $1000\,036\,000\,099 (= 1\,000\,003 \cdot 1\,000\,033)$ .

### Problem 345: Matrix Sum

We define the Matrix Sum of a matrix as the maximum sum of matrix elements with each element being the only one in his row and column. For example, the Matrix Sum of the matrix below equals 3315 ( $= 863 + 383 + 343 + 959 + 767$ ):

7	53	183	439	863
497	383	563	79	973
287	63	343	169	583
627	343	773	959	943
767	473	103	699	303

Find the Matrix Sum of:

7	53	183	439	863	497	383	563	79	973	287	63	343	169	583
627	343	773	959	943	767	473	103	699	303	957	703	583	639	913
447	283	463	29	23	487	463	993	119	883	327	493	423	159	743
217	623	3	399	853	407	103	983	89	463	290	516	212	462	350
960	376	682	962	300	780	486	502	912	800	250	346	172	812	350
870	456	192	162	593	473	915	45	989	873	823	965	425	329	803
973	965	905	919	133	673	665	235	509	613	673	815	165	992	326
322	148	972	962	286	255	941	541	265	323	925	281	601	95	973
445	721	11	525	473	65	511	164	138	672	18	428	154	448	848
414	456	310	312	798	104	566	520	302	248	694	976	430	392	198
184	829	373	181	631	101	969	613	840	740	778	458	284	760	390
821	461	843	513	17	901	711	993	293	157	274	94	192	156	574
34	124	4	878	450	476	712	914	838	669	875	299	823	329	699
815	559	813	459	522	788	168	586	966	232	308	833	251	631	107
813	883	451	509	615	77	281	613	459	205	380	274	302	35	805

### Problem 346: Strong Repunits

The number 7 is special, because 7 is 111 written in base 2, and 11 written in base 6 (i.e.  $7_{10} = 11_6 = 111_2$ ). In other words, 7 is a repunit in at least two bases  $b > 1$ .

We shall call a positive integer with this property a strong repunit. It can be verified that there are 8 strong repunits below 50:  $\{1, 7, 13, 15, 21, 31, 40, 43\}$ .

Furthermore, the sum of all strong repunits below 1000 equals 15864.

Find the sum of all strong repunits below  $10^{12}$ .

### Problem 347: Largest integer divisible by two primes

The largest integer  $\leq 100$  that is only divisible by both the primes 2 and 3 is 96, as  $96=32 \cdot 3=2^5 \cdot 3$ . For two *distinct* primes  $p$  and  $q$  let  $M(p,q,N)$  be the largest positive integer  $\leq N$  only divisible by both  $p$  and  $q$  and  $M(p,q,N)=0$  if such a positive integer does not exist.

E.g.  $M(2,3,100)=96$ .

$M(3,5,100)=75$  and not 90 because 90 is divisible by 2, 3 and 5.

Also  $M(2,73,100)=0$  because there does not exist a positive integer  $\leq 100$  that is divisible by both 2 and 73.

Let  $S(N)$  be the sum of all distinct  $M(p,q,N)$ .  $S(100)=2262$ .

Find  $S(10\,000\,000)$ .

### Problem 348: Sum of a square and a cube

Many numbers can be expressed as the sum of a square and a cube. Some of them in more than one way.

Consider the palindromic numbers that can be expressed as the sum of a square and a cube, both greater than 1, **irexactly** 4 different ways.

For example, 5229225 is a palindromic number and it can be expressed in exactly 4 different ways:

$$2285^2 + 20^3$$

$$2223^2 + 66^3$$

$$1810^2 + 125^3$$

$$1197^2 + 156^3$$

Find the sum of the five smallest such palindromic numbers.

### Problem 349: Langton's ant

An ant moves on a regular grid of squares that are coloured either black or white.

The ant is always oriented in one of the cardinal directions (left, right, up or down) and moves from square to adjacent square according to the following rules:

- if it is on a black square, it flips the color of the square to white, rotates 90 degrees counterclockwise and moves forward one square.
- if it is on a white square, it flips the color of the square to black, rotates 90 degrees clockwise and moves forward one square.

Starting with a grid that is entirely white, how many squares are black after  $10^{18}$  moves of the ant?

### Problem 350: Constraining the least greatest and the greatest least

A *list of size  $n$*  is a sequence of  $n$  natural numbers.

Examples are (2,4,6), (2,6,4), (10,6,15,6), and (11).

The **greatest common divisor**, or gcd, of a list is the largest natural number that divides all entries of the list.

Examples:  $\text{gcd}(2,6,4) = 2$ ,  $\text{gcd}(10,6,15,6) = 1$  and  $\text{gcd}(11) = 11$ .

The **least common multiple**, or lcm, of a list is the smallest natural number divisible by each entry of the list.

Examples:  $\text{lcm}(2,6,4) = 12$ ,  $\text{lcm}(10,6,15,6) = 30$  and  $\text{lcm}(11) = 11$ .

Let  $f(G, L, N)$  be the number of lists of size  $N$  with  $\text{gcd} \geq G$  and  $\text{lcm} \leq L$ . For example:

$$f(10, 100, 1) = 91.$$

$$f(10, 100, 2) = 327.$$

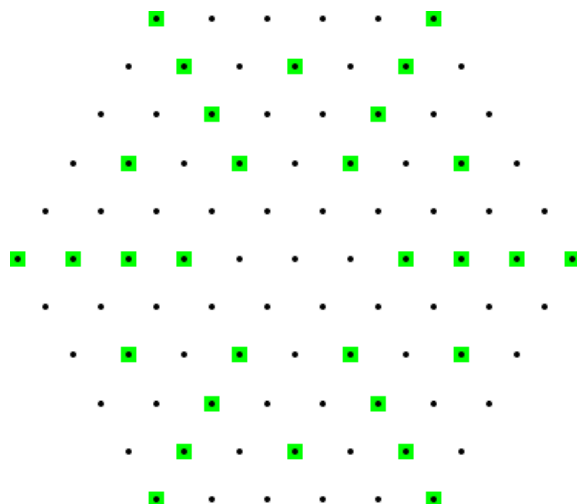
$$f(10, 100, 3) = 1135.$$

$$f(10, 100, 1000) \bmod 101^4 = 3286053.$$

Find  $f(10^6, 10^{12}, 10^{18}) \bmod 101^4$ .

### Problem 351: Hexagonal orchards

A *hexagonal orchard* of order  $n$  is a triangular lattice made up of points within a regular hexagon with side  $n$ . The following is an example of a hexagonal orchard of order 5:



Highlighted in green are the points which are hidden from the center by a point closer to it. It can be seen that for a hexagonal orchard of order 5, 30 points are hidden from the center.

Let  $H(n)$  be the number of points hidden from the center in a hexagonal orchard of order  $n$ .

$H(5) = 30$ .  $H(10) = 138$ .  $H(1\,000) = 1177848$ .

Find  $H(100\,000\,000)$ .

### Problem 352: Blood tests

Each one of the 25 sheep in a flock must be tested for a rare virus, known to affect 2% of the sheep population. An accurate and extremely sensitive PCR test exists for blood samples, producing a clear positive / negative result, but it is very time-consuming and expensive.

Because of the high cost, the vet-in-charge suggests that instead of performing 25 separate tests, the following procedure can be used instead:

The sheep are split into 5 groups of 5 sheep in each group. For each group, the 5 samples are mixed together and a single test is performed. Then,

- If the result is negative, all the sheep in that group are deemed to be virus-free.
- If the result is positive, 5 additional tests will be performed (a separate test for each animal) to determine the affected individual(s).

Since the probability of infection for any specific animal is only 0.02, the first test (on the pooled samples) for each group will be:

- Negative (and no more tests needed) with probability  $0.98^5 = 0.9039207968$ .
- Positive (5 additional tests needed) with probability  $1 - 0.9039207968 = 0.0960792032$ .

Thus, the expected number of tests for each group is  $1 + 0.0960792032 \times 5 = 1.480396016$ .

Consequently, all 5 groups can be screened using an average of only  $1.480396016 \times 5 = \mathbf{7.40198008}$  tests, which represents a huge saving of more than 70% !

Although the scheme we have just described seems to be very efficient, it can still be improved considerably (always assuming that the test is sufficiently sensitive and that there are no adverse effects caused by mixing different samples). E.g.:

- We may start by running a test on a mixture of all the 25 samples. It can be verified that in about 60.35% of the cases this test will be negative, thus no more tests will be needed. Further testing will only be required for the remaining 39.65% of the cases.
- If we know that at least one animal in a group of 5 is infected and the first 4 individual tests come out negative, there is no need to run a test on the fifth animal (we know that it must be infected).
- We can try a different number of groups / different number of animals in each group, adjusting those numbers at each level so that the total expected number of tests will be minimised.

To simplify the very wide range of possibilities, there is one restriction we place when devising the most cost-efficient testing scheme: whenever we start with a mixed sample, all the sheep contributing to that sample must be fully screened (i.e. a verdict of infected / virus-free must be reached for all of them) before we start examining any other animals.

For the current example, it turns out that the most cost-efficient testing scheme (we'll call it the *optimal strategy*) requires an average of just **4.155452** tests!

Using the optimal strategy, let  $T(S,p)$  represent the average number of tests needed to screen a flock of  $S$  sheep for a virus having probability  $p$  to be present in any individual.

Thus, rounded to six decimal places,  $T(25, 0.02) = 4.155452$  and  $T(25, 0.10) = 12.702124$ .

Find  $\Sigma T(10000, p)$  for  $p=0.01, 0.02, 0.03, \dots 0.50$ .

Give your answer rounded to six decimal places.



### Problem 353: Risky moon

A moon could be described by the sphere  $C(r)$  with centre  $(0,0,0)$  and radius  $r$ .

There are stations on the moon at the points on the surface of  $C(r)$  with integer coordinates. The station at  $(0,0,r)$  is called North Pole station, the station at  $(0,0,-r)$  is called South Pole station.

All stations are connected with each other via the shortest road on the great arc through the stations. A journey between two stations is risky. If  $d$  is the length of the road between two stations,  $d/(\pi r)^2$  is a measure for the risk of the journey (let us call it the risk of the road). If the journey includes more than two stations, the risk of the journey is the sum of risks of the used roads.

A direct journey from the North Pole station to the South Pole station has the length  $\pi r$  and risk 1. The journey from the North Pole station to the South Pole station via  $(0,r,0)$  has the same length, but a smaller risk:  $(\frac{1}{2}\pi r/(\pi r))^2 + (\frac{1}{2}\pi r/(\pi r))^2 = 0.5$ .

The minimal risk of a journey from the North Pole station to the South Pole station on  $C(r)$  is  $M(r)$ .

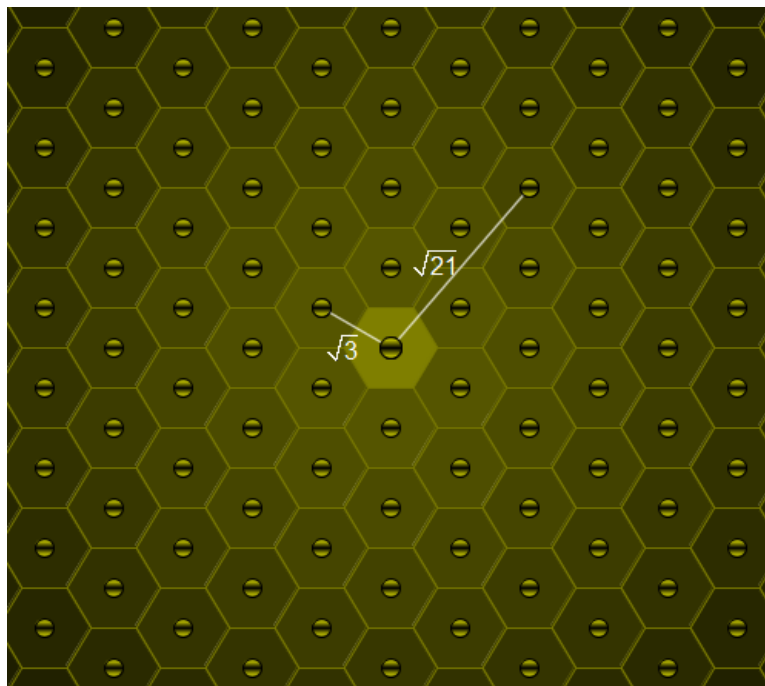
You are given that  $M(7) = 0.1784943998$  rounded to 10 digits behind the decimal point.

Find  $\sum M(2^n - 1)$  for  $1 \leq n \leq 15$ .

Give your answer rounded to 10 digits behind the decimal point in the form a.bcd efghijk.

### Problem 354: Distances in a bee's honeycomb

Consider a honey bee's honeycomb where each cell is a perfect regular hexagon with side length 1.



One particular cell is occupied by the queen bee.

For a positive real number  $L$ , let  $B(L)$  count the cells with distance  $L$  from the queen bee cell (all distances are measured from centre to centre); you may assume that the honeycomb is large enough to accommodate for any distance we wish to consider.

For example,  $B(\sqrt{3}) = 6$ ,  $B(\sqrt{21}) = 12$  and  $B(111\ 111\ 111) = 54$ .

Find the number of  $L \leq 5 \cdot 10^{11}$  such that  $B(L) = 450$ .

### Problem 355: Maximal coprime subset

Define  $\text{Co}(n)$  to be the maximal possible sum of a set of mutually co-prime elements from  $\{1, 2, \dots, n\}$ .  
For example  $\text{Co}(10)$  is 30 and hits that maximum on the subset  $\{1, 5, 7, 8, 9\}$ .

You are given that  $\text{Co}(30) = 193$  and  $\text{Co}(100) = 1356$ .

Find  $\text{Co}(200000)$ .

### Problem 356: Largest roots of cubic polynomials

Let  $a_n$  be the largest real root of a polynomial  $g(x) = x^3 - 2^n \cdot x^2 + n$ .

For example,  $a_2 = 3.86619826\dots$

Find the last eight digits of  $\sum_{i=1}^{30} \lfloor a_i^{987654321} \rfloor$ .

Note:  represents the floor function.

### Problem 357: Prime generating integers

Consider the divisors of 30: 1, 2, 3, 5, 6, 10, 15, 30.

It can be seen that for every divisor  $d$  of 30,  $d + 30/d$  is prime.

Find the sum of all positive integers  $n$  not exceeding 100 000 000  
such that for every divisor  $d$  of  $n$ ,  $d + n/d$  is prime.

### Problem 358: Cyclic numbers

A **cyclic number** with  $n$  digits has a very interesting property:

When it is multiplied by 1, 2, 3, 4, ...,  $n$ , all the products have exactly the same digits, in the same order, but rotated in a circular fashion!

The smallest cyclic number is the 6-digit number 142857 :

$$142857 \times 1 = 142857$$

$$142857 \times 2 = 285714$$

$$142857 \times 3 = 428571$$

$$142857 \times 4 = 571428$$

$$142857 \times 5 = 714285$$

$$142857 \times 6 = 857142$$

The next cyclic number is 0588235294117647 with 16 digits :

$$0588235294117647 \times 1 = 0588235294117647$$

$$0588235294117647 \times 2 = 1176470588235294$$

$$0588235294117647 \times 3 = 1764705882352941$$

...

$$0588235294117647 \times 16 = 9411764705882352$$

Note that for cyclic numbers, leading zeros are important.

There is only one cyclic number for which, the eleven leftmost digits are 00000000137 and the five rightmost digits are 56789 (i.e., it has the form 00000000137...56789 with an unknown number of digits in the middle). Find the sum of all its digits.

### Problem 359: Hilbert's New Hotel

An infinite number of people (numbered 1, 2, 3, etc.) are lined up to get a room at Hilbert's newest infinite hotel. The hotel contains an infinite number of floors (numbered 1, 2, 3, etc.), and each floor contains an infinite number of rooms (numbered 1, 2, 3, etc.).

Initially the hotel is empty. Hilbert declares a rule on how the  $n^{\text{th}}$  person is assigned a room: person  $n$  gets the first vacant room in the lowest numbered floor satisfying either of the following:

- the floor is empty
- the floor is not empty, and if the latest person taking a room in that floor is person  $m$ , then  $m + n$  is a perfect square

Person 1 gets room 1 in floor 1 since floor 1 is empty.

Person 2 does not get room 2 in floor 1 since  $1 + 2 = 3$  is not a perfect square.

Person 2 instead gets room 1 in floor 2 since floor 2 is empty.

Person 3 gets room 2 in floor 1 since  $1 + 3 = 4$  is a perfect square.

Eventually, every person in the line gets a room in the hotel.

Define  $P(f, r)$  to be 1 if person  $n$  occupies room  $r$  in floor  $f$ , and 0 if no person occupies the room. Here are a few examples:

$$P(1, 1) = 1$$

$$P(1, 2) = 3$$

$$P(2, 1) = 2$$

$$P(10, 20) = 440$$

$$P(25, 75) = 4863$$

$$P(99, 100) = 19454$$

Find the sum of all  $P(f, r)$  for all positive  $f$  and  $r$  such that  $f \times r = 71328803586048$  and give the last 8 digits as your answer.

### Problem 360: Scary Sphere

Given two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  in three dimensional space, the **Manhattan distance** between those points is defined as

$$|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|.$$

Let  $C(r)$  be a sphere with radius  $r$  and center in the origin  $O(0,0,0)$ .

Let  $I(r)$  be the set of all points with integer coordinates on the surface of  $C(r)$ .

Let  $S(r)$  be the sum of the Manhattan distances of all elements of  $I(r)$  to the origin  $O$ .

E.g.  $S(45) = 34518$ .

Find  $S(10^{10})$ .

### Problem 361: Subsequence of Thue-Morse sequence

The **Thue-Morse sequence**  $\{T_n\}$  is a binary sequence satisfying:

- $T_0 = 0$
- $T_{2n} = T_n$
- $T_{2n+1} = 1 - T_n$

The first several terms of  $\{T_n\}$  are given as follows:

01101001**1001**01101001011001101001....

We define  $\{A_n\}$  as the sorted sequence of integers such that the binary expression of each element appears as a subsequence in  $\{T_n\}$ .

For example, the decimal number 18 is expressed as 10010 in binary. 10010 appears in  $\{T_n\}$  ( $T_8$  to  $T_{12}$ ), so 18 is an element of  $\{A_n\}$ .

The decimal number 14 is expressed as 1110 in binary. 1110 never appears in  $\{T_n\}$ , so 14 is not an element of  $\{A_n\}$ .

The first several terms of  $A_n$  are given as follows:

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$A_n$	0	1	2	3	4	5	6	9	10	11	12	13	18	...

We can also verify that  $A_{100} = 3251$  and  $A_{1000} = 80852364498$ .

Find the last 9 digits of .

### Problem 362: Squarefree factors

Consider the number 54.

54 can be factored in 7 distinct ways into one or more factors larger than 1:

54,  $2 \times 27$ ,  $3 \times 18$ ,  $6 \times 9$ ,  $3 \times 3 \times 6$ ,  $2 \times 3 \times 9$  and  $2 \times 3 \times 3 \times 3$ .

If we require that the factors are all squarefree only two ways remain:  $3 \times 3 \times 6$  and  $2 \times 3 \times 3 \times 3$ .

Let's call  $Fsf(n)$  the number of ways  $n$  can be factored into one or more squarefree factors larger than 1, so  $Fsf(54)=2$ .

Let  $S(n)$  be  $\sum Fsf(k)$  for  $k=2$  to  $n$ .

$S(100)=193$ .

Find  $S(10\,000\,000\,000)$ .

### Problem 363: Bézier Curves

A cubic Bézier curve is defined by four points:  $P_0$ ,  $P_1$ ,  $P_2$  and  $P_3$ .

The curve is constructed as follows:

On the segments  $P_0P_1$ ,  $P_1P_2$  and  $P_2P_3$  the points  $Q_0$ ,  $Q_1$  and  $Q_2$  are drawn such that

$P_0Q_0 / P_0P_1 = P_1Q_1 / P_1P_2 = P_2Q_2 / P_2P_3 = t$  ( $t$  in  $[0,1]$ ).

On the segments  $Q_0Q_1$  and  $Q_1Q_2$  the points  $R_0$  and  $R_1$  are drawn such that

$Q_0R_0 / Q_0Q_1 = Q_1R_1 / Q_1Q_2 = t$  for the same value of  $t$ .

On the segment  $R_0R_1$  the point  $B$  is drawn such that  $R_0B / R_0R_1 = t$  for the same value of  $t$ .

The Bézier curve defined by the points  $P_0$ ,  $P_1$ ,  $P_2$ ,  $P_3$  is the locus of  $B$  as  $Q_0$  takes all possible positions on the segment  $P_0P_1$ .

(Please note that for all points the value of  $t$  is the same.)

At this [\(external\) web address](#) you will find an applet that allows you to drag the points  $P_0$ ,  $P_1$ ,  $P_2$  and  $P_3$  to see what the Bézier curve (green curve) defined by those points looks like. You can also drag the point  $Q_0$  along the segment  $P_0P_1$ .

From the construction it is clear that the Bézier curve will be tangent to the segments  $P_0P_1$  in  $P_0$  and  $P_2P_3$  in  $P_3$ .

A cubic Bézier curve with  $P_0=(1,0)$ ,  $P_1=(1,V)$ ,  $P_2=(V,1)$  and  $P_3=(0,1)$  is used to approximate a quarter circle.

The value  $V > 0$  is chosen such that the area enclosed by the lines  $OP_0$ ,  $OP_3$  and the curve is equal to  $\pi/4$  (the area of the quarter circle).

By how many percent does the length of the curve differ from the length of the quarter circle?

That is, if  $L$  is the length of the curve, calculate  $100 \times \frac{L - \pi/2}{\pi/2}$

Give your answer rounded to 10 digits behind the decimal point.

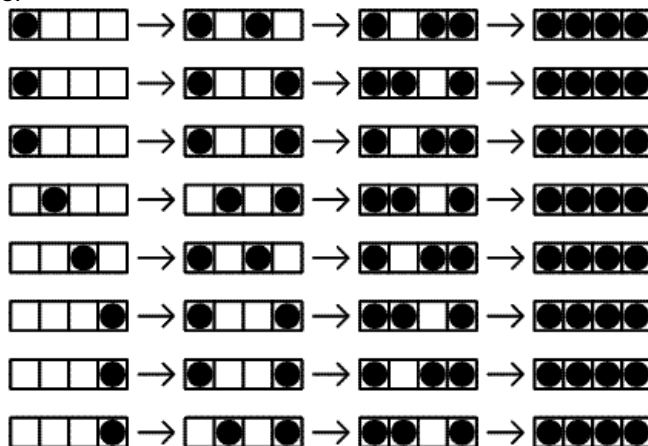
### Problem 364: Comfortable distance

There are  $N$  seats in a row.  $N$  people come after each other to fill the seats according to the following rules:

1. If there is any seat whose adjacent seat(s) are not occupied take such a seat.
2. If there is no such seat and there is any seat for which only one adjacent seat is occupied take such a seat.
3. Otherwise take one of the remaining available seats.

Let  $T(N)$  be the number of possibilities that  $N$  seats are occupied by  $N$  people with the given rules.

The following figure shows  $T(4)=8$ .



We can verify that  $T(10) = 61632$  and  $T(1\,000) \bmod 100\,000\,007 = 47255094$ .

Find  $T(1\,000\,000) \bmod 100\,000\,007$ .

### Problem 365: A huge binomial coefficient

The binomial coefficient  $C(10^{18}, 10^9)$  is a number with more than 9 billion ( $9 \times 10^9$ ) digits.

Let  $M(n,k,m)$  denote the binomial coefficient  $C(n,k)$  modulo  $m$ .

Calculate  $\sum M(10^{18}, 10^9, p \cdot q \cdot r)$  for  $1000 < p < q < r < 5000$  and  $p, q, r$  prime.

### Problem 366: Stone Game III

Two players, Anton and Bernhard, are playing the following game.  
There is one pile of  $n$  stones.  
The first player may remove any positive number of stones, but not the whole pile.  
Thereafter, each player may remove at most twice the number of stones his opponent took on the previous move.  
The player who removes the last stone wins.

E.g.  $n=5$

If the first player takes anything more than one stone the next player will be able to take all remaining stones.

If the first player takes one stone, leaving four, his opponent will take also one stone, leaving three stones.

The first player cannot take all three because he may take at most  $2 \times 1 = 2$  stones. So let's say he takes also one stone, leaving 2. The second player can take the two remaining stones and wins.

So 5 is a losing position for the first player.

For some winning positions there is more than one possible move for the first player.

E.g. when  $n=17$  the first player can remove one or four stones.

Let  $M(n)$  be the maximum number of stones the first player can take from a winning position *at his first turn* and  $M(n)=0$  for any other position.

$\sum M(n)$  for  $n \leq 100$  is 728.

Find  $\sum M(n)$  for  $n \leq 10^{18}$ . Give your answer modulo  $10^8$ .

### Problem 367: Bozo sort

**Bozo sort**, not to be confused with the slightly less efficient **bogo sort**, consists out of checking if the input sequence is sorted and if not swapping randomly two elements. This is repeated until eventually the sequence is sorted.

If we consider all permutations of the first 4 natural numbers as input the expectation value of the number of swaps, averaged over all  $4!$  input sequences is 24.75.

The already sorted sequence takes 0 steps.

In this problem we consider the following variant on bozo sort.

If the sequence is not in order we pick three elements at random and shuffle these three elements randomly.

All  $3!=6$  permutations of those three elements are equally likely.

The already sorted sequence will take 0 steps.

If we consider all permutations of the first 4 natural numbers as input the expectation value of the number of shuffles, averaged over all  $4!$  input sequences is 27.5.

Consider as input sequences the permutations of the first 11 natural numbers.

Averaged over all  $11!$  input sequences, what is the expected number of shuffles this sorting algorithm will perform?

Give your answer rounded to the nearest integer.

### Problem 368: A Kempner-like series

The **harmonic series**  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  is well known to be divergent.

If we however omit from this series every term where the denominator has a 9 in it, the series remarkably enough converges to approximately 22.9206766193.

This modified harmonic series is called the **Kempner series**.

Let us now consider another modified harmonic series by omitting from the harmonic series every term where the denominator has 3 or more equal consecutive digits. One can verify that out of the first 1200 terms of the harmonic series, only 20 terms will be omitted.

These 20 omitted terms are:

$\frac{1}{111}, \frac{1}{222}, \frac{1}{333}, \frac{1}{444}, \frac{1}{555}, \frac{1}{666}, \frac{1}{777}, \frac{1}{888},$   
 $\frac{1}{999}, \frac{1}{1000}, \frac{1}{1110}, \frac{1}{1111}, \frac{1}{1112}, \frac{1}{1113}, \frac{1}{1114}, \frac{1}{1115},$   
 $\frac{1}{1116}, \frac{1}{1117}, \frac{1}{1118}, \frac{1}{1119}$

This series converges as well.

Find the value the series converges to.

Give your answer rounded to 10 digits behind the decimal point.

### Problem 369: Badugi

In a standard 52 card deck of playing cards, a set of 4 cards is a **Badugi** if it contains 4 cards with no pairs and no two cards of the same suit.

Let  $f(n)$  be the number of ways to choose  $n$  cards with a 4 card subset that is a Badugi. For example, there are 2598960 ways to choose five cards from a standard 52 card deck, of which 514800 contain a 4 card subset that is a Badugi, so  $f(5) = 514800$ .

Find  $\sum f(n)$  for  $4 \leq n \leq 13$ .

### Problem 370: Geometric triangles

Let us define a *geometric triangle* as an integer sided triangle with sides  $a \leq b \leq c$  so that its sides form a **geometric progression**, i.e.  $b^2 = a \cdot c$ .

An example of such a geometric triangle is the triangle with sides  $a = 144$ ,  $b = 156$  and  $c = 169$ .

There are 861805 geometric triangles with perimeter  $\leq 10^6$ .

How many geometric triangles exist with perimeter  $\leq 2.5 \cdot 10^{13}$ ?

### Problem 371: Licence plates

Oregon licence plates consist of three letters followed by a three digit number (each digit can be from  $[0..9]$ ).

While driving to work Seth plays the following game:

Whenever the numbers of two licence plates seen on his trip add to 1000 that's a win.

E.g. MIC-012 and HAN-988 is a win and RYU-500 and SET-500 too. (as long as he sees them in the same trip).

Find the expected number of plates he needs to see for a win.

Give your answer rounded to 8 decimal places behind the decimal point.

**Note:** We assume that each licence plate seen is equally likely to have any three digit number on it.

### Problem 372: Pencils of rays

Let  $R(M, N)$  be the number of lattice points  $(x, y)$  which satisfy  $0 \leq x \leq M$ ,  $0 \leq y \leq N$  and  $\lfloor \frac{y^2}{x^2} \rfloor$  is odd.

We can verify that  $R(0, 100) = 3019$  and  $R(100, 10000) = 29750422$ .

Find  $R(2 \cdot 10^6, 10^9)$ .

Note:  $\lfloor x \rfloor$  represents the floor function.

### Problem 373: Circumscribed Circles

Every triangle has a circumscribed circle that goes through the three vertices. Consider all integer sided triangles for which the radius of the circumscribed circle is integral as well.

Let  $S(n)$  be the sum of the radii of the circumscribed circles of all such triangles for which the radius does not exceed  $n$ .

$S(100) = 4950$  and  $S(1200) = 1653605$ .

Find  $S(10^7)$ .

### Problem 374: Maximum Integer Partition Product

An integer partition of a number  $n$  is a way of writing  $n$  as a sum of positive integers.

Partitions that differ only in the order of their summands are considered the same. A partition of  $n$  into **distinct parts** is a partition of  $n$  in which every part occurs at most once.

The partitions of 5 into distinct parts are:  
5, 4+1 and 3+2.

Let  $f(n)$  be the maximum product of the parts of any such partition of  $n$  into distinct parts and let  $m(n)$  be the number of elements of any such partition of  $n$  with that product.

So  $f(5)=6$  and  $m(5)=2$ .

For  $n=10$  the partition with the largest product is  $10=2+3+5$ , which gives  $f(10)=30$  and  $m(10)=3$ .  
And their product,  $f(10) \cdot m(10) = 30 \cdot 3 = 90$

It can be verified that  
 $\sum f(n) \cdot m(n)$  for  $1 \leq n \leq 100 = 1683550844462$ .

Find  $\sum f(n) \cdot m(n)$  for  $1 \leq n \leq 10^{14}$ .  
Give your answer modulo 982451653, the 50 millionth prime.

### Problem 375: Minimum of subsequences

Let  $S_n$  be an integer sequence produced with the following pseudo-random number generator:

$$S_0 = 290797$$
$$S_{n+1} = S_n^2 \bmod 50515093$$

Let  $A(i, j)$  be the minimum of the numbers  $S_i, S_{i+1}, \dots, S_j$  for  $i \leq j$ .

Let  $M(N) = \sum A(i, j)$  for  $1 \leq i \leq j \leq N$ .

We can verify that  $M(10) = 432256955$  and  $M(10\,000) = 3264567774119$ .

Find  $M(2\,000\,000\,000)$ .



### Problem 376: Nontransitive sets of dice

Consider the following set of dice with nonstandard pips:

Die A: 1 4 4 4 4 4

Die B: 2 2 2 5 5 5

Die C: 3 3 3 3 3 6

A game is played by two players picking a die in turn and rolling it. The player who rolls the highest value wins.

If the first player picks die A and the second player picks die B we get

$$P(\text{second player wins}) = \frac{7}{12} > \frac{1}{2}$$

If the first player picks die B and the second player picks die C we get

$$P(\text{second player wins}) = \frac{7}{12} > \frac{1}{2}$$

If the first player picks die C and the second player picks die A we get

$$P(\text{second player wins}) = \frac{25}{36} > \frac{1}{2}$$

So whatever die the first player picks, the second player can pick another die and have a larger than 50% chance of winning. A set of dice having this property is called a **nontransitive set of dice**.

We wish to investigate how many sets of nontransitive dice exist. We will assume the following conditions:

- There are three six-sided dice with each side having between 1 and  $N$  pips, inclusive.
- Dice with the same set of pips are equal, regardless of which side on the die the pips are located.
- The same pip value may appear on multiple dice; if both players roll the same value neither player wins.
- The sets of dice  $\{A, B, C\}$ ,  $\{B, C, A\}$  and  $\{C, A, B\}$  are the same set.

For  $N = 7$  we find there are 9780 such sets.

How many are there for  $N = 30$  ?

### Problem 377: Sum of digits, experience 13

There are 16 positive integers that do not have a zero in their digits and that have a digital sum equal to 5, namely: 5, 14, 23, 32, 41, 113, 122, 131, 212, 221, 311, 1112, 1121, 1211, 2111 and 11111. Their sum is 17891.

Let  $f(n)$  be the sum of all positive integers that do not have a zero in their digits and have a digital sum equal to  $n$ .

Find  $\sum_{i=1}^{17} f(13^i)$ .

Give the last 9 digits as your answer.

### Problem 378: Triangle Triples

Let  $T(n)$  be the  $n^{\text{th}}$  triangle number, so  $T(n) = \frac{n(n+1)}{2}$ .

Let  $dT(n)$  be the number of divisors of  $T(n)$ .

E.g.:  $T(7) = 28$  and  $dT(7) = 6$ .

Let  $\text{Tr}(n)$  be the number of triples  $(i, j, k)$  such that  $1 \leq i < j < k \leq n$  and  $dT(i) > dT(j) > dT(k)$ .

$\text{Tr}(20) = 14$ ,  $\text{Tr}(100) = 5772$  and  $\text{Tr}(1000) = 11174776$ .

Find  $\text{Tr}(60\,000\,000)$ .

Give the last 18 digits of your answer.

### Problem 379: Least common multiple count

Let  $f(n)$  be the number of couples  $(x,y)$  with  $x$  and  $y$  positive integers,  $x \leq y$  and the least common multiple of  $x$  and  $y$  equal to  $n$ .

Let  $g$  be the **summatory function** of  $f$ , i.e.:  $g(n) = \sum f(i)$  for  $1 \leq i \leq n$ .

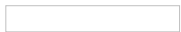
You are given that  $g(10^6) = 37429395$ .

Find  $g(10^{12})$ .

### Problem 380: Amazing Mazes!

An  $m \times n$  maze is an  $m \times n$  rectangular grid with walls placed between grid cells such that there is exactly one path from the top-left square to any other square.

The following are examples of a  $9 \times 12$  maze and a  $15 \times 20$  maze:



Let  $C(m,n)$  be the number of distinct  $m \times n$  mazes. Mazes which can be formed by rotation and reflection from another maze are considered distinct.

It can be verified that  $C(1,1) = 1$ ,  $C(2,2) = 4$ ,  $C(3,4) = 2415$ , and  $C(9,12) = 2.5720e46$  (in scientific notation rounded to 5 significant digits).

Find  $C(100,500)$  and write your answer in scientific notation rounded to 5 significant digits.

When giving your answer, use a lowercase e to separate mantissa and exponent. E.g. if the answer is 1234567891011 then the answer format would be 1.2346e12.

### Problem 381: (prime-k) factorial

For a prime  $p$  let  $S(p) = (\sum (p-k)!) \bmod(p)$  for  $1 \leq k \leq 5$ .

For example, if  $p=7$ ,

$(7-1)! + (7-2)! + (7-3)! + (7-4)! + (7-5)! = 6! + 5! + 4! + 3! + 2! = 720+120+24+6+2 = 872$ .

As  $872 \bmod(7) = 4$ ,  $S(7) = 4$ .

It can be verified that  $\sum S(p) = 480$  for  $5 \leq p < 100$ .

Find  $\sum S(p)$  for  $5 \leq p < 10^8$ .

### Problem 382: Generating polygons

A **polygon** is a flat shape consisting of straight line segments that are joined to form a closed chain or circuit. A polygon consists of at least three sides and does not self-intersect.

A set  $S$  of positive numbers is said to *generate a polygon*  $P$  if:

- no two sides of  $P$  are the same length,
- the length of every side of  $P$  is in  $S$ , and
- $S$  contains no other value.

For example:

The set  $\{3, 4, 5\}$  generates a polygon with sides 3, 4, and 5 (a triangle).

The set  $\{6, 9, 11, 24\}$  generates a polygon with sides 6, 9, 11, and 24 (a quadrilateral).

The sets  $\{1, 2, 3\}$  and  $\{2, 3, 4, 9\}$  do not generate any polygon at all.

Consider the sequence  $s$ , defined as follows:

- $s_1 = 1, s_2 = 2, s_3 = 3$
- $s_n = s_{n-1} + s_{n-3}$  for  $n > 3$ .

Let  $U_n$  be the set  $\{s_1, s_2, \dots, s_n\}$ . For example,  $U_{10} = \{1, 2, 3, 4, 6, 9, 13, 19, 28, 41\}$ .

Let  $f(n)$  be the number of subsets of  $U_n$  which generate at least one polygon.

For example,  $f(5) = 7$ ,  $f(10) = 501$  and  $f(25) = 18635853$ .

Find the last 9 digits of  $f(10^{18})$ .

### Problem 383: Divisibility comparison between factorials

Let  $f_5(n)$  be the largest integer  $X$  for which  $5^X$  divides  $n$ .

For example,  $f_5(625000) = 7$ .

Let  $T_5(n)$  be the number of integers  $i$  which satisfy  $f_5((2 \cdot i - 1)!) < 2 \cdot f_5(i!)$  and  $1 \leq i \leq n$ .

It can be verified that  $T_5(10^3) = 68$  and  $T_5(10^9) = 2408210$ .

Find  $T_5(10^{18})$ .

### Problem 384: Rudin-Shapiro sequence

Define the sequence  $a(n)$  as the number of adjacent pairs of ones in the binary expansion of  $n$  (possibly overlapping).

E.g.:  $a(5) = a(101_2) = 0$ ,  $a(6) = a(110_2) = 1$ ,  $a(7) = a(111_2) = 2$

Define the sequence  $b(n) = (-1)^{a(n)}$ .

This sequence is called the **Rudin-Shapiro** sequence.

Also consider the summatory sequence of  $b(n)$ : .

The first couple of values of these sequences are:

$n$	0	1	2	3	4	5	6	7
$a(n)$	0	0	0	1	0	0	1	2
$b(n)$	1	1	1	-1	1	1	-1	1
$s(n)$	1	2	3	2	3	4	3	4

The sequence  $s(n)$  has the remarkable property that all elements are positive and every positive integer  $k$  occurs exactly  $k$  times.

Define  $g(t, c)$ , with  $1 \leq c \leq t$ , as the index in  $s(n)$  for which  $t$  occurs for the  $c$ 'th time in  $s(n)$ .

E.g.:  $g(3, 3) = 6$ ,  $g(4, 2) = 7$  and  $g(54321, 12345) = 1220847710$ .

Let  $F(n)$  be the fibonacci sequence defined by:

$F(0) = F(1) = 1$  and

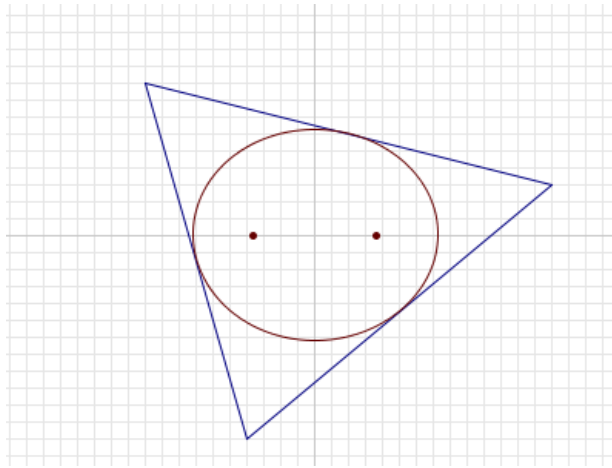
$F(n) = F(n-1) + F(n-2)$  for  $n > 1$ .

Define  $GF(t) = g(F(t), F(t-1))$ .

Find  $\sum GF(t)$  for  $2 \leq t \leq 45$ .

### Problem 385: Ellipses inside triangles

For any triangle  $T$  in the plane, it can be shown that there is a unique ellipse with largest area that is completely inside  $T$ .



For a given  $n$ , consider triangles  $T$  such that:

- the vertices of  $T$  have integer coordinates with absolute value  $\leq n$ , and
- the **foci**<sup>1</sup> of the largest-area ellipse inside  $T$  are  $(\sqrt{13}, 0)$  and  $(-\sqrt{13}, 0)$ .

Let  $A(n)$  be the sum of the areas of all such triangles.

For example, if  $n = 8$ , there are two such triangles. Their vertices are  $(-4, -3), (-4, 3), (8, 0)$  and  $(4, 3), (4, -3), (-8, 0)$ , and the area of each triangle is 36. Thus  $A(8) = 36 + 36 = 72$ .

It can be verified that  $A(10) = 252$ ,  $A(100) = 34632$  and  $A(1000) = 3529008$ .

Find  $A(1\,000\,000\,000)$ .

<sup>1</sup>The **foci** (plural of **focus**) of an ellipse are two points  $A$  and  $B$  such that for every point  $P$  on the boundary of the ellipse  $AP + PB$  is constant.

### Problem 386: Maximum length of an antichain

Let  $n$  be an integer and  $S(n)$  be the set of factors of  $n$ .

A subset  $A$  of  $S(n)$  is called an **antichain** of  $S(n)$  if  $A$  contains only one element or if none of the elements of  $A$  divides any of the other elements of  $A$ .

For example:  $S(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$\{2, 5, 6\}$  is not an antichain of  $S(30)$ .

$\{2, 3, 5\}$  is an antichain of  $S(30)$ .

Let  $N(n)$  be the maximum length of an antichain of  $S(n)$ .

Find  $\sum N(n)$  for  $1 \leq n \leq 10^8$

### Problem 387: Harshad Numbers

A **Harshad or Niven number** is a number that is divisible by the sum of its digits.

201 is a Harshad number because it is divisible by 3 (the sum of its digits.)

When we truncate the last digit from 201, we get 20, which is a Harshad number.

When we truncate the last digit from 20, we get 2, which is also a Harshad number.

Let's call a Harshad number that, while recursively truncating the last digit, always results in a Harshad number *right truncatable Harshad number*.

Also:

$201/3=67$  which is prime.

Let's call a Harshad number that, when divided by the sum of its digits, results in a prime *strong Harshad number*.

Now take the number 2011 which is prime.

When we truncate the last digit from it we get 201, a strong Harshad number that is also right truncatable.

Let's call such primes *strong, right truncatable Harshad primes*.

You are given that the sum of the strong, right truncatable Harshad primes less than 10000 is 90619.

Find the sum of the strong, right truncatable Harshad primes less than  $10^4$ .

### Problem 388: Distinct Lines

Consider all lattice points  $(a,b,c)$  with  $0 \leq a,b,c \leq N$ .

From the origin  $O(0,0,0)$  all lines are drawn to the other lattice points.

Let  $D(N)$  be the number of *distinct* such lines.

You are given that  $D(1\,000\,000) = 831909254469114121$ .

Find  $D(10^{10})$ . Give as your answer the first nine digits followed by the last nine digits.

### Problem 389: Platonic Dice

An unbiased single 4-sided die is thrown and its value,  $T$ , is noted.

$T$  unbiased 6-sided dice are thrown and their scores are added together. The sum,  $C$ , is noted.

$C$  unbiased 8-sided dice are thrown and their scores are added together. The sum,  $O$ , is noted.

$O$  unbiased 12-sided dice are thrown and their scores are added together. The sum,  $D$ , is noted.

$D$  unbiased 20-sided dice are thrown and their scores are added together. The sum,  $I$ , is noted.

Find the variance of  $I$ , and give your answer rounded to 4 decimal places.

### Problem 390: Triangles with non rational sides and integral area

Consider the triangle with sides  $\sqrt{5}$ ,  $\sqrt{65}$  and  $\sqrt{68}$ . It can be shown that this triangle has area 9.

$S(n)$  is the sum of the areas of all triangles with sides  $\sqrt{(1+b^2)}$ ,  $\sqrt{(1+c^2)}$  and  $\sqrt{(b^2+c^2)}$  (for positive integers  $b$  and  $c$ ) that have an integral area not exceeding  $n$ .

The example triangle has  $b=2$  and  $c=8$ .

$S(10^6)=18018206$ .

Find  $S(10^{10})$ .

### Problem 391: Hopping Game

Let  $S_k$  be the number of 1's when writing the numbers from 0 to  $k$  in binary.

For example, writing 0 to 5 in binary, we have 0, 1, 10, 11, 100, 101. There are seven 1's, so  $S_5 = 7$ .

The sequence  $S = \{S_k : k \geq 0\}$  starts  $\{0, 1, 2, 4, 5, 7, 9, 12, \dots\}$ .

A game is played by two players. Before the game starts, a number  $n$  is chosen. A counter  $C$  starts at 0. At each turn, the player chooses a number from 1 to  $n$  (inclusive) and increases  $C$  by that number. The resulting value of  $C$  must be a member of  $S$ . If there are no more valid moves, the player loses.

For example:

Let  $n = 5$ .  $C$  starts at 0.

Player 1 chooses 4, so  $C$  becomes  $0 + 4 = 4$ .

Player 2 chooses 5, so  $C$  becomes  $4 + 5 = 9$ .

Player 1 chooses 3, so  $C$  becomes  $9 + 3 = 12$ .

etc.

Note that  $C$  must always belong to  $S$ , and each player can increase  $C$  by at most  $n$ .

Let  $M(n)$  be the highest number the first player can choose at her first turn to force a win, and  $M(n) = 0$  if there is no such move. For example,  $M(2) = 2$ ,  $M(7) = 1$  and  $M(20) = 4$ .

Given  $\sum (M(n))^3 = 8150$  for  $1 \leq n \leq 20$ .

Find  $\sum (M(n))^3$  for  $1 \leq n \leq 1000$ .

### Problem 392: Enmeshed unit circle

A rectilinear grid is an orthogonal grid where the spacing between the gridlines does not have to be equidistant. An example of such grid is logarithmic graph paper.

Consider rectilinear grids in the Cartesian coordinate system with the following properties:

- The gridlines are parallel to the axes of the Cartesian coordinate system.
- There are  $N+2$  vertical and  $N+2$  horizontal gridlines. Hence there are  $(N+1) \times (N+1)$  rectangular cells.
- The equations of the two outer vertical gridlines are  $x = -1$  and  $x = 1$ .
- The equations of the two outer horizontal gridlines are  $y = -1$  and  $y = 1$ .
- The grid cells are colored red if they overlap with the unit circle, black otherwise.

For this problem we would like you to find the positions of the remaining  $N$  inner horizontal and  $N$  inner vertical gridlines so that the area occupied by the red cells is minimized.

E.g. here is a picture of the solution for  $N = 10$ :



The area occupied by the red cells for  $N = 10$  rounded to 10 digits behind the decimal point is 3.3469640797.

Find the positions for  $N = 400$ .

Give as your answer the area occupied by the red cells rounded to 10 digits behind the decimal point.

### Problem 393: Migrating ants

An  $n \times n$  grid of squares contains  $n^2$  ants, one ant per square.

All ants decide to move simultaneously to an adjacent square (usually 4 possibilities, except for ants on the edge of the grid or at the corners).

We define  $f(n)$  to be the number of ways this can happen without any ants ending on the same square and without any two ants crossing the same edge between two squares.

You are given that  $f(4) = 88$ .

Find  $f(10)$ .

### Problem 394: Eating pie

Jeff eats a pie in an unusual way.

The pie is circular. He starts with slicing an initial cut in the pie along a radius.

While there is at least a given fraction  $F$  of pie left, he performs the following procedure:

- He makes two slices from the pie centre to any point of what is remaining of the pie border, any point on the remaining pie border equally likely. This will divide the remaining pie into three pieces.
- Going counterclockwise from the initial cut, he takes the first two pie pieces and eats them.

When less than a fraction  $F$  of pie remains, he does not repeat this procedure. Instead, he eats all of the remaining pie.

Start



For  $X \geq 1$ , let  $E(X)$  be the expected number of times Jeff repeats the procedure above with  $F = 1/X$ .

It can be verified that  $E(1) = 1$ ,  $E(2) \approx 1.2676536759$ , and  $E(7.5) \approx 2.1215732071$ .

Find  $E(40)$  rounded to 10 decimal places behind the decimal point.

### Problem 395: Pythagorean tree

The **Pythagorean tree** is a fractal generated by the following procedure:

Start with a unit square. Then, calling one of the sides its base (in the animation, the bottom side is the base):

1. Attach a right triangle to the side opposite the base, with the hypotenuse coinciding with that side and with the sides in a 3-4-5 ratio. Note that the smaller side of the triangle must be on the 'right' side with respect to the base (see animation).
2. Attach a square to each leg of the right triangle, with one of its sides coinciding with that leg.
3. Repeat this procedure for both squares, considering as their bases the sides touching the triangle.

The resulting figure, after an infinite number of iterations, is the Pythagorean tree.

$t = 0$



It can be shown that there exists at least one rectangle, whose sides are parallel to the largest square of the Pythagorean tree, which encloses the Pythagorean tree completely.

Find the smallest area possible for such a bounding rectangle, and give your answer rounded to 10 decimal places.

### Problem 396: Weak Goodstein sequence

For any positive integer  $n$ , the  **$n$ th weak Goodstein sequence**  $\{g_1, g_2, g_3, \dots\}$  is defined as:

- $g_1 = n$
- for  $k > 1$ ,  $g_k$  is obtained by writing  $g_{k-1}$  in base  $k$ , interpreting it as a base  $k + 1$  number, and subtracting 1.

The sequence terminates when  $g_k$  becomes 0.

For example, the 6th weak Goodstein sequence is  $\{6, 11, 17, 25, \dots\}$ :

- $g_1 = 6$ .
- $g_2 = 11$  since  $6 = 11_2$ ,  $11_3 = 12$ , and  $12 - 1 = 11$ .
- $g_3 = 17$  since  $11 = 10_2$ ,  $10_4 = 18$ , and  $18 - 1 = 17$ .
- $g_4 = 25$  since  $17 = 10_4$ ,  $10_5 = 26$ , and  $26 - 1 = 25$ .

and so on.

It can be shown that every weak Goodstein sequence terminates.

Let  $G(n)$  be the number of nonzero elements in the  $n$ th weak Goodstein sequence.

It can be verified that  $G(2) = 3$ ,  $G(4) = 21$  and  $G(6) = 381$ .

It can also be verified that  $\Sigma G(n) = 2517$  for  $1 \leq n < 8$ .

Find the last 9 digits of  $\Sigma G(n)$  for  $1 \leq n < 16$ .

### Problem 397: Triangle on parabola

On the parabola  $y = x^2/k$ , three points  $A(a, a^2/k)$ ,  $B(b, b^2/k)$  and  $C(c, c^2/k)$  are chosen.

Let  $F(K, X)$  be the number of the integer quadruplets  $(k, a, b, c)$  such that at least one angle of the triangle ABC is 45-degree, with  $1 \leq k \leq K$  and  $-X \leq a < b < c \leq X$ .

For example,  $F(1, 10) = 41$  and  $F(10, 100) = 12492$ .

Find  $F(10^6, 10^9)$ .

### Problem 398: Cutting rope

Inside a rope of length  $n$ ,  $n-1$  points are placed with distance 1 from each other and from the endpoints. Among these points, we choose  $m-1$  points at random and cut the rope at these points to create  $m$  segments.

Let  $E(n, m)$  be the expected length of the second-shortest segment. For example,  $E(3, 2) = 2$  and  $E(8, 3) = 16/7$ . Note that if multiple segments have the same shortest length the length of the second-shortest segment is defined as the same as the shortest length.

Find  $E(10^7, 100)$ . Give your answer rounded to 5 decimal places behind the decimal point.



### Problem 399: Squarefree Fibonacci Numbers

The first 15 fibonacci numbers are:

1,1,2,3,5,8,13,21,34,55,89,144,233,377,610.

It can be seen that 8 and 144 are not squarefree: 8 is divisible by 4 and 144 is divisible by 4 and by 9.

So the first 13 squarefree fibonacci numbers are:

1,1,2,3,5,13,21,34,55,89,233,377 and 610.

The 200th squarefree fibonacci number is: 971183874599339129547649988289594072811608739584170445.

The last sixteen digits of this number are: 1608739584170445 and in scientific notation this number can be written as  $9.7e53$ .

Find the 100 000 000th squarefree fibonacci number.

Give as your answer its last sixteen digits followed by a comma followed by the number in scientific notation (rounded to one digit after the decimal point).

For the 200th squarefree number the answer would have been: 1608739584170445,9.7e53

Note:

For this problem, assume that for every prime  $p$ , the first fibonacci number divisible by  $p$  is not divisible by  $p^2$  (this is part of **Wall's conjecture**). This has been verified for primes  $\leq 3 \cdot 10^{15}$ , but has not been proven in general.

If it happens that the conjecture is false, then the accepted answer to this problem isn't guaranteed to be the 100 000 000th squarefree fibonacci number, rather it represents only a lower bound for that number.

### Problem 400: Fibonacci tree game

A **Fibonacci tree** is a binary tree recursively defined as:

- $T(0)$  is the empty tree.
- $T(1)$  is the binary tree with only one node.
- $T(k)$  consists of a root node that has  $T(k-1)$  and  $T(k-2)$  as children.

On such a tree two players play a take-away game. On each turn a player selects a node and removes that node along with the subtree rooted at that node.

The player who is forced to take the root node of the entire tree loses.

Here are the winning moves of the first player on the first turn for  $T(k)$  from  $k=1$  to  $k=6$ .

Let  $f(k)$  be the number of winning moves of the first player (i.e. the moves for which the second player has no winning strategy) on the first turn of the game when this game is played on  $T(k)$ .

For example,  $f(5) = 1$  and  $f(10) = 17$ .

Find  $f(10000)$ . Give the last 18 digits of your answer.

### Problem 401: Sum of squares of divisors

The divisors of 6 are 1,2,3 and 6.

The sum of the squares of these numbers is  $1+4+9+36=50$ .

Let  $\text{sigma2}(n)$  represent the sum of the squares of the divisors of  $n$ . Thus  $\text{sigma2}(6)=50$ .

Let  $\text{SIGMA2}$  represent the summatory function of  $\text{sigma2}$ , that is  $\text{SIGMA2}(n)=\sum \text{sigma2}(i)$  for  $i=1$  to  $n$ .

The first 6 values of  $\text{SIGMA2}$  are: 1,6,16,37,63 and 113.

Find  $\text{SIGMA2}(10^{15})$  modulo  $10^9$ .

### Problem 402: Integer-valued polynomials

It can be shown that the polynomial  $n^4 + 4n^3 + 2n^2 + 5n$  is a multiple of 6 for every integer  $n$ . It can also be shown that 6 is the largest integer satisfying this property.

Define  $M(a, b, c)$  as the maximum  $m$  such that  $n^4 + an^3 + bn^2 + cn$  is a multiple of  $m$  for all integers  $n$ . For example,  $M(4, 2, 5) = 6$ .

Also, define  $S(N)$  as the sum of  $M(a, b, c)$  for all  $0 < a, b, c \leq N$ .

We can verify that  $S(10) = 1972$  and  $S(10000) = 2024258331114$ .

Let  $F_k$  be the Fibonacci sequence:

$F_0 = 0, F_1 = 1$  and

$F_k = F_{k-1} + F_{k-2}$  for  $k \geq 2$ .

Find the last 9 digits of  $\sum S(F_k)$  for  $2 \leq k \leq 1234567890123$ .

### Problem 403: Lattice points enclosed by parabola and line

For integers  $a$  and  $b$ , we define  $D(a, b)$  as the domain enclosed by the parabola  $y = x^2$  and the line  $y = a \cdot x + b$ :

$D(a, b) = \{ (x, y) \mid x^2 \leq y \leq a \cdot x + b \}$ .

$L(a, b)$  is defined as the number of lattice points contained in  $D(a, b)$ .

For example,  $L(1, 2) = 8$  and  $L(2, -1) = 1$ .

We also define  $S(N)$  as the sum of  $L(a, b)$  for all the pairs  $(a, b)$  such that the area of  $D(a, b)$  is a rational number and  $|a|, |b| \leq N$ .

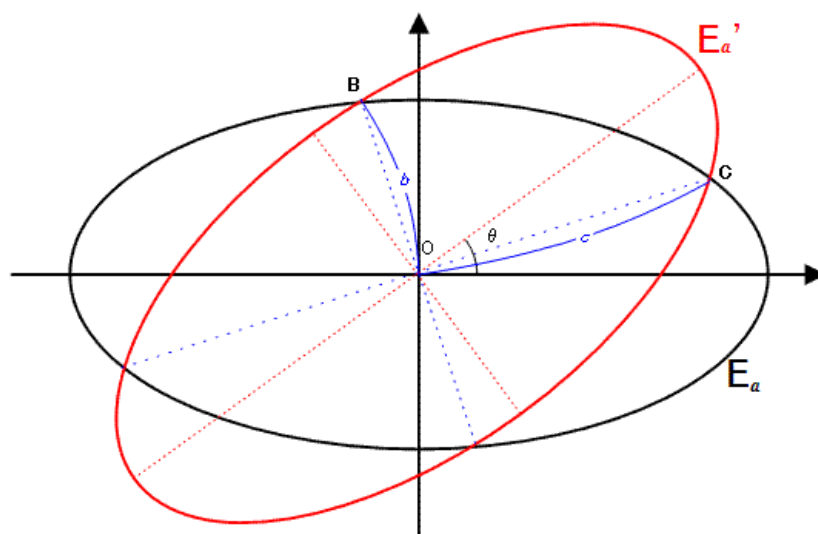
We can verify that  $S(5) = 344$  and  $S(100) = 26709528$ .

Find  $S(10^{12})$ . Give your answer mod  $10^8$ .

#### Problem 404: Crisscross Ellipses

$E_a$  is an ellipse with an equation of the form  $x^2 + 4y^2 = 4a^2$ .

$E_a'$  is the rotated image of  $E_a$  by  $\theta$  degrees counterclockwise around the origin  $O(0, 0)$  for  $0^\circ < \theta < 90^\circ$ .



$b$  is the distance to the origin of the two intersection points closest to the origin and  $c$  is the distance of the two other intersection points.

We call an ordered triplet  $(a, b, c)$  a *canonical ellipsoidal triplet* if  $a, b$  and  $c$  are positive integers.

For example,  $(209, 247, 286)$  is a canonical ellipsoidal triplet.

Let  $C(N)$  be the number of distinct canonical ellipsoidal triplets  $(a, b, c)$  for  $a \leq N$ .

It can be verified that  $C(10^3) = 7$ ,  $C(10^4) = 106$  and  $C(10^6) = 11845$ .

Find  $C(10^{17})$ .

#### Problem 405: A rectangular tiling

We wish to tile a rectangle whose length is twice its width.

Let  $T(0)$  be the tiling consisting of a single rectangle.

For  $n > 0$ , let  $T(n)$  be obtained from  $T(n-1)$  by replacing all tiles in the following manner:



The following animation demonstrates the tilings  $T(n)$  for  $n$  from 0 to 5:



Let  $f(n)$  be the number of points where four tiles meet in  $T(n)$ .

For example,  $f(1) = 0$ ,  $f(4) = 82$  and  $f(10^9) \bmod 17^7 = 126897180$ .

Find  $f(10^k)$  for  $k = 10^{18}$ , give your answer modulo  $17^7$ .

### Problem 406: Guessing Game

We are trying to find a hidden number selected from the set of integers  $\{1, 2, \dots, n\}$  by asking questions. Each number (question) we ask, we get one of three possible answers:

- "Your guess is lower than the hidden number" (and you incur a cost of  $a$ ), or
- "Your guess is higher than the hidden number" (and you incur a cost of  $b$ ), or
- "Yes, that's it!" (and the game ends).

Given the value of  $n$ ,  $a$ , and  $b$ , an *optimal strategy* minimizes the total cost for the worst possible case.

For example, if  $n = 5$ ,  $a = 2$ , and  $b = 3$ , then we may begin by asking **2** as our first question.

If we are told that 2 is higher than the hidden number (for a cost of  $b=3$ ), then we are sure that **1** is the hidden number (for a total cost of **3**).

If we are told that 2 is lower than the hidden number (for a cost of  $a=2$ ), then our next question will be **4**.

If we are told that 4 is higher than the hidden number (for a cost of  $b=3$ ), then we are sure that **3** is the hidden number (for a total cost of  $2+3=5$ ).

If we are told that 4 is lower than the hidden number (for a cost of  $a=2$ ), then we are sure that **5** is the hidden number (for a total cost of  $2+2=4$ ).

Thus, the worst-case cost achieved by this strategy is **5**. It can also be shown that this is the lowest worst-case cost that can be achieved. So, in fact, we have just described an optimal strategy for the given values of  $n$ ,  $a$ , and  $b$ .

Let  $C(n, a, b)$  be the worst-case cost achieved by an optimal strategy for the given values of  $n$ ,  $a$ , and  $b$ .

Here are a few examples:

$$C(5, 2, 3) = 5$$

$$C(500, \sqrt{2}, \sqrt{3}) = 13.22073197\dots$$

$$C(20000, 5, 7) = 82$$

$$C(2000000, \sqrt{5}, \sqrt{7}) = 49.63755955\dots$$

Let  $F_k$  be the Fibonacci numbers:  $F_k = F_{k-1} + F_{k-2}$  with base cases  $F_1 = F_2 = 1$ .

Find  $\sum_{1 \leq k \leq 30} C(10^{12}, \sqrt{k}, \sqrt{F_k})$ , and give your answer rounded to 8 decimal places behind the decimal point.

### Problem 407: Idempotents

If we calculate  $a^2 \bmod 6$  for  $0 \leq a \leq 5$  we get: 0,1,4,3,4,1.

The largest value of  $a$  such that  $a^2 \equiv a \bmod 6$  is 4.

Let's call  $M(n)$  the largest value of  $a < n$  such that  $a^2 \equiv a \pmod{n}$ .

So  $M(6) = 4$ .

Find  $\sum M(n)$  for  $1 \leq n \leq 10^7$ .

### Problem 408: Admissible paths through a grid

Let's call a lattice point  $(x, y)$  *inadmissible* if  $x$ ,  $y$  and  $x + y$  are all positive perfect squares.

For example,  $(9, 16)$  is inadmissible, while  $(0, 4)$ ,  $(3, 1)$  and  $(9, 4)$  are not.

Consider a path from point  $(x_1, y_1)$  to point  $(x_2, y_2)$  using only unit steps north or east.

Let's call such a path *admissible* if none of its intermediate points are inadmissible.

Let  $P(n)$  be the number of admissible paths from  $(0, 0)$  to  $(n, n)$ .

It can be verified that  $P(5) = 252$ ,  $P(16) = 596994440$  and  $P(1000) \bmod 1\,000\,000\,007 = 341920854$ .

Find  $P(10\,000\,000) \bmod 1\,000\,000\,007$ .

### Problem 409: Nim Extreme

Let  $n$  be a positive integer. Consider **nim** positions where:

- There are  $n$  non-empty piles.
- Each pile has size less than  $2^n$ .
- No two piles have the same size.

Let  $W(n)$  be the number of winning nim positions satisfying the above conditions (a position is winning if the first player has a winning strategy). For example,  $W(1) = 1$ ,  $W(2) = 6$ ,  $W(3) = 168$ ,  $W(5) = 19764360$  and  $W(100) \bmod 1\,000\,000\,007 = 384777056$ .

Find  $W(10\,000\,000) \bmod 1\,000\,000\,007$ .

### Problem 410: Circle and tangent line

Let  $C$  be the circle with radius  $r$ ,  $x^2 + y^2 = r^2$ . We choose two points  $P(a, b)$  and  $Q(-a, c)$  so that the line passing through  $P$  and  $Q$  is tangent to  $C$ .

For example, the quadruplet  $(r, a, b, c) = (2, 6, 2, -7)$  satisfies this property.

Let  $F(R, X)$  be the number of the integer quadruplets  $(r, a, b, c)$  with this property, and with  $0 < r \leq R$  and  $0 < a \leq X$ .

We can verify that  $F(1, 5) = 10$ ,  $F(2, 10) = 52$  and  $F(10, 100) = 3384$ .

Find  $F(10^8, 10^9) + F(10^9, 10^8)$ .

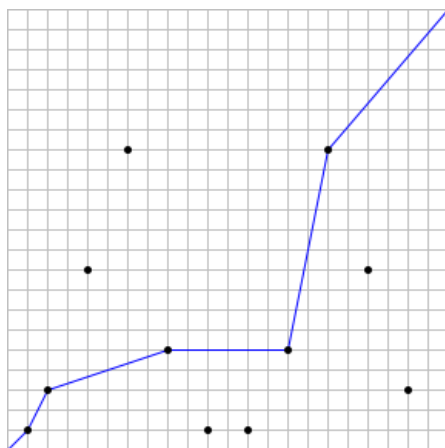
### Problem 411: Uphill paths

Let  $n$  be a positive integer. Suppose there are stations at the coordinates  $(x, y) = (2^i \bmod n, 3^i \bmod n)$  for  $0 \leq i \leq 2n$ . We will consider stations with the same coordinates as the same station.

We wish to form a path from  $(0, 0)$  to  $(n, n)$  such that the  $x$  and  $y$  coordinates never decrease.

Let  $S(n)$  be the maximum number of stations such a path can pass through.

For example, if  $n = 22$ , there are 11 distinct stations, and a valid path can pass through at most 5 stations. Therefore,  $S(22) = 5$ . The case is illustrated below, with an example of an optimal path:



It can also be verified that  $S(123) = 14$  and  $S(10000) = 48$ .

Find  $\sum S(k^5)$  for  $1 \leq k \leq 30$ .

### Problem 412: Gnomon numbering

For integers  $m, n$  ( $0 \leq n < m$ ), let  $L(m, n)$  be an  $m \times m$  grid with the top-right  $n \times n$  grid removed.

For example,  $L(5, 3)$  looks like this:


We want to number each cell of  $L(m, n)$  with consecutive integers  $1, 2, 3, \dots$  such that the number in every cell is smaller than the number below it and to the left of it.

For example, here are two valid numberings of  $L(5, 3)$ :

6	4			
7	5			
14	9			
15	11	8	3	1
16	13	12	10	2

8	1			
11	2			
13	7			
14	12	9	5	3
16	15	10	6	4

Let  $LC(m, n)$  be the number of valid numberings of  $L(m, n)$ .

It can be verified that  $LC(3, 0) = 42$ ,  $LC(5, 3) = 250250$ ,  $LC(6, 3) = 406029023400$  and  $LC(10, 5) \bmod 76543217 = 61251715$ .

Find  $LC(10000, 5000) \bmod 76543217$ .

### Problem 413: One-child Numbers

We say that a  $d$ -digit positive number (no leading zeros) is a one-child number if exactly one of its sub-strings is divisible by  $d$ .

For example, 5671 is a 4-digit one-child number. Among all its sub-strings 5, 6, 7, 1, 56, 67, 71, 567, 671 and 5671, only 56 is divisible by 4.

Similarly, 104 is a 3-digit one-child number because only 0 is divisible by 3.

1132451 is a 7-digit one-child number because only 245 is divisible by 7.

Let  $F(N)$  be the number of the one-child numbers less than  $N$ .

We can verify that  $F(10) = 9$ ,  $F(10^3) = 389$  and  $F(10^7) = 277674$ .

Find  $F(10^{19})$ .

#### Problem 414: Kaprekar constant

6174 is a remarkable number; if we sort its digits in increasing order and subtract that number from the number you get when you sort the digits in decreasing order, we get  $7641-1467=6174$ .

Even more remarkable is that if we start from any 4 digit number and repeat this process of sorting and subtracting, we'll eventually end up with 6174 or immediately with 0 if all digits are equal.

This also works with numbers that have less than 4 digits if we pad the number with leading zeroes until we have 4 digits.

E.g. let's start with the number 0837:

$8730-0378=8352$

$8532-2358=6174$

6174 is called the **Kaprekar constant**. The process of sorting and subtracting and repeating this until either 0 or the Kaprekar constant is reached is called the **Kaprekar routine**.

We can consider the Kaprekar routine for other bases and number of digits.

Unfortunately, it is not guaranteed a Kaprekar constant exists in all cases; either the routine can end up in a cycle for some input numbers or the constant the routine arrives at can be different for different input numbers.

However, it can be shown that for 5 digits and a base  $b = 6t+3 \neq 9$ , a Kaprekar constant exists.

E.g. base 15:  $(10,4,14,9,5)_{15}$

base 21:  $(14,6,20,13,7)_{21}$

Define  $C_b$  to be the Kaprekar constant in base  $b$  for 5 digits. Define the function  $Sb(i)$  to be

- 0 if  $i = C_b$  or if  $i$  written in base  $b$  consists of 5 identical digits
- the number of iterations it takes the Kaprekar routine in base  $b$  to arrive at  $C_b$ , otherwise

Note that we can define  $Sb(i)$  for all integers  $i < b^5$ . If  $i$  written in base  $b$  takes less than 5 digits, the number is padded with leading zero digits until we have 5 digits before applying the Kaprekar routine.

Define  $S(b)$  as the sum of  $Sb(i)$  for  $0 < i < b^5$ .

E.g.  $S(15) = 5274369$

$S(111) = 400668930299$

Find the sum of  $S(6k+3)$  for  $2 \leq k \leq 300$ .

Give the last 18 digits as your answer.

#### Problem 415: Titanic sets

A set of lattice points  $S$  is called a *titanic set* if there exists a line passing through exactly two points in  $S$ .

An example of a titanic set is  $S = \{(0, 0), (0, 1), (0, 2), (1, 1), (2, 0), (1, 0)\}$ , where the line passing through  $(0, 1)$  and  $(2, 0)$  does not pass through any other point in  $S$ .

On the other hand, the set  $\{(0, 0), (1, 1), (2, 2), (4, 4)\}$  is not a titanic set since the line passing through any two points in the set also passes through the other two.

For any positive integer  $N$ , let  $T(N)$  be the number of titanic sets  $S$  whose every point  $(x, y)$  satisfies  $0 \leq x, y \leq N$ . It can be verified that  $T(1) = 11$ ,  $T(2) = 494$ ,  $T(4) = 33554178$ ,  $T(111) \bmod 10^8 = 13500401$  and  $T(10^5) \bmod 10^8 = 63259062$ .

Find  $T(10^{11}) \bmod 10^8$ .

#### Problem 416: A frog's trip

A row of  $n$  squares contains a frog in the leftmost square. By successive jumps the frog goes to the rightmost square and then back to the leftmost square. On the outward trip he jumps one, two or three squares to the right, and on the homeward trip he jumps to the left in a similar manner. He cannot jump outside the squares. He repeats the round-trip travel  $m$  times.

Let  $F(m, n)$  be the number of the ways the frog can travel so that at most one square remains unvisited.

For example,  $F(1, 3) = 4$ ,  $F(1, 4) = 15$ ,  $F(1, 5) = 46$ ,  $F(2, 3) = 16$  and  $F(2, 100) \bmod 10^9 = 429619151$ .

Find the last 9 digits of  $F(10, 10^{12})$ .

### Problem 417: Reciprocal cycles II

A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given:

$$\frac{1}{2} = 0.5$$

$$\frac{1}{3} = 0.(3)$$

$$\frac{1}{4} = 0.25$$

$$\frac{1}{5} = 0.2$$

$$\frac{1}{6} = 0.1(6)$$

$$\frac{1}{7} = 0.(142857)$$

$$\frac{1}{8} = 0.125$$

$$\frac{1}{9} = 0.(1)$$

$$\frac{1}{10} = 0.1$$

Where 0.1(6) means 0.16666..., and has a 1-digit recurring cycle. It can be seen that  $\frac{1}{7}$  has a 6-digit recurring cycle.

Unit fractions whose denominator has no other prime factors than 2 and/or 5 are not considered to have a recurring cycle. We define the length of the recurring cycle of those unit fractions as 0.

Let  $L(n)$  denote the length of the recurring cycle of  $1/n$ . You are given that  $\sum L(n)$  for  $3 \leq n \leq 1\,000\,000$  equals 55535191115.

Find  $\sum L(n)$  for  $3 \leq n \leq 100\,000\,000$

### Problem 418: Factorisation triples

Let  $n$  be a positive integer. An integer triple  $(a, b, c)$  is called a *factorisation triple* of  $n$  if:

- $1 \leq a \leq b \leq c$
- $a \cdot b \cdot c = n$ .

Define  $f(n)$  to be  $a + b + c$  for the factorisation triple  $(a, b, c)$  of  $n$  which minimises  $c / a$ . One can show that this triple is unique.

For example,  $f(165) = 19$ ,  $f(100100) = 142$  and  $f(20!) = 4034872$ .

Find  $f(43!)$ .

### Problem 419: Look and say sequence

The **look and say** sequence goes 1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, ...

The sequence starts with 1 and all other members are obtained by describing the previous member in terms of consecutive digits.

It helps to do this out loud:

1 is 'one one'  $\rightarrow$  11

11 is 'two ones'  $\rightarrow$  21

21 is 'one two and one one'  $\rightarrow$  1211

1211 is 'one one, one two and two ones'  $\rightarrow$  111221

111221 is 'three ones, two twos and one one'  $\rightarrow$  312211

...

Define  $A(n)$ ,  $B(n)$  and  $C(n)$  as the number of ones, twos and threes in the  $n$ 'th element of the sequence respectively.

One can verify that  $A(40) = 31254$ ,  $B(40) = 20259$  and  $C(40) = 11625$ .

Find  $A(n)$ ,  $B(n)$  and  $C(n)$  for  $n = 10^{12}$ .

Give your answer modulo  $2^{30}$  and separate your values for  $A$ ,  $B$  and  $C$  by a comma.

E.g. for  $n = 40$  the answer would be 31254,20259,11625



**Problem 420: 2x2 positive integer matrix**

A *positive integer matrix* is a matrix whose elements are all positive integers.

Some positive integer matrices can be expressed as a square of a positive integer matrix in two different ways. Here is an example:

$$\begin{pmatrix} 40 & 12 \\ 48 & 40 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 12 & 2 \end{pmatrix}^2 = \begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix}^2$$

We define  $F(N)$  as the number of the 2x2 positive integer matrices which have trace less than  $N$  and which can be expressed as a square of a positive integer matrix in two different ways.

We can verify that  $F(50) = 7$  and  $F(1000) = 1019$ .

Find  $F(10^7)$ .

**Problem 421: Prime factors of  $n^{15}+1$** 

Numbers of the form  $n^{15}+1$  are composite for every integer  $n > 1$ .

For positive integers  $n$  and  $m$  let  $S(n,m)$  be defined as the sum of the *distinct* prime factors of  $n^{15}+1$  not exceeding  $m$ .

E.g.  $2^{15}+1 = 3 \times 3 \times 11 \times 331$ .

So  $S(2,10) = 3$  and  $S(2,1000) = 3+11+331 = 345$ .

Also  $10^{15}+1 = 7 \times 11 \times 13 \times 211 \times 241 \times 2161 \times 9091$ .

So  $S(10,100) = 31$  and  $S(10,1000) = 483$ .

Find  $\sum S(n,10^8)$  for  $1 \leq n \leq 10^{11}$ .

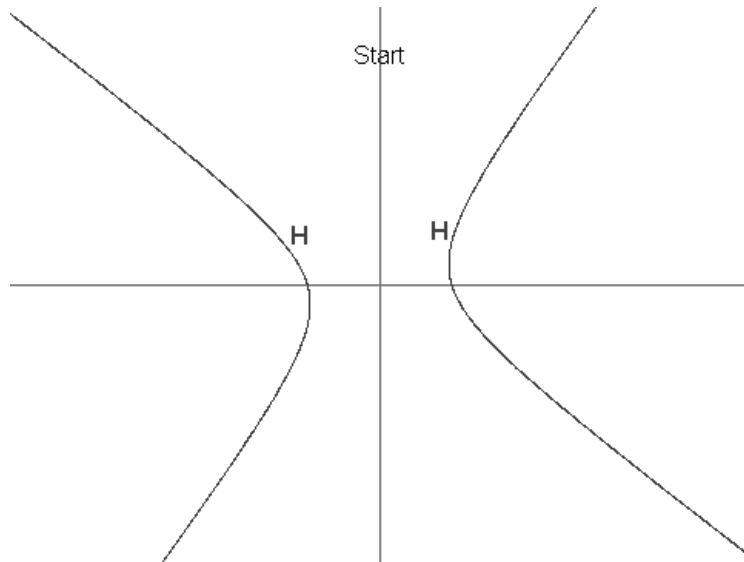
### Problem 422: Sequence of points on a hyperbola

Let  $H$  be the hyperbola defined by the equation  $12x^2 + 7xy - 12y^2 = 625$ .

Next, define  $X$  as the point  $(7, 1)$ . It can be seen that  $X$  is in  $H$ .

Now we define a sequence of points in  $H$ ,  $\{P_i : i \geq 1\}$ , as:

- $P_1 = (13, 61/4)$ .
- $P_2 = (-43/6, -4)$ .
- For  $i > 2$ ,  $P_i$  is the unique point in  $H$  that is different from  $P_{i-1}$  and such that line  $P_i P_{i-1}$  is parallel to line  $P_{i-2} X$ . It can be shown that  $P_i$  is well-defined, and that its coordinates are always rational.



You are given that  $P_3 = (-19/2, -229/24)$ ,  $P_4 = (1267/144, -37/12)$  and  $P_7 = (17194218091/143327232, 274748766781/1719926784)$ .

Find  $P_n$  for  $n = 11^{14}$  in the following format:

If  $P_n = (a/b, c/d)$  where the fractions are in lowest terms and the denominators are positive, then the answer is  $a + b + c + d \pmod{1\,000\,000\,007}$ .

For  $n = 7$ , the answer would have been: 806236837.

### Problem 423: Consecutive die throws

Let  $n$  be a positive integer.

A 6-sided die is thrown  $n$  times. Let  $C$  be the number of pairs of consecutive throws that give the same value.

For example, if  $n = 7$  and the values of the die throws are  $(1, 1, 5, 6, 6, 6, 3)$ , then the following pairs of consecutive throws give the same value:

(1, 1), 5, 6, 6, 6, 3)

(1, 1, 5, 6), 6, 3)

(1, 1, 5, 6, 6, 6), 3)

Therefore,  $C = 3$  for  $(1, 1, 5, 6, 6, 6, 3)$ .

Define  $C(n)$  as the number of outcomes of throwing a 6-sided die  $n$  times such that  $C$  does not exceed  $\pi(n)$ .<sup>1</sup>

For example,  $C(3) = 216$ ,  $C(4) = 1290$ ,  $C(11) = 361912500$  and  $C(24) = 4727547363281250000$ .

Define  $S(L)$  as  $\sum C(n)$  for  $1 \leq n \leq L$ .

For example,  $S(50) \pmod{1\,000\,000\,007} = 832833871$ .

Find  $S(50\,000\,000) \pmod{1\,000\,000\,007}$ .

<sup>1</sup>  $\pi$  denotes the **prime-counting function**, i.e.  $\pi(n)$  is the number of primes  $\leq n$ .

## Problem 424: Kakuro

			CC	I					
		H	B		CA	JE			
		FE							
		D							
	A		I	JC					
	JC	H		B					
			JE						

		22	7						
	5	4	1	28	10				
	30	9	6	8	7				
	6								
8	1	7	12	9	3				
		4							
12	5	2	1	4					
			10	3	7				

A	B	C	D	E	F	G	H	I	J
8	4	2	6	0	3	9	5	7	1

The above is an example of a cryptic kakuro (also known as cross sums, or even sums cross) puzzle, with its final solution on the right. (The common rules of kakuro puzzles can be found easily on numerous internet sites. Other related information can also be currently found at [krazydad.com](http://krazydad.com) whose author has provided the puzzle data for this challenge.)

The downloadable text file ([kakuro200.txt](#)) contains the description of 200 such puzzles, a mix of 5x5 and 6x6 types. The first puzzle in the file is the above example which is coded as follows:

6,X,X,(vCC),(vI),X,X,X,(hH),B,O,(vCA),(vJE),X,(hFE,vD),O,O,O,O,(hA),O,I,(hJC,vB),O,O,(hJC),H,O,O,O,X,X,X,(hJE),O,O,X

The first character is a numerical digit indicating the size of the information grid. It would be either a 6 (for a 5x5 kakuro puzzle) or a 7 (for a 6x6 puzzle) followed by a comma (.). The extra top line and left column are needed to insert information.

The content of each cell is then described and followed by a comma, going left to right and starting with the top line.

X = Gray cell, not required to be filled by a digit.

O (upper case letter) = White empty cell to be filled by a digit.

A = Or any one of the upper case letters from A to J to be replaced by its equivalent digit in the solved puzzle.

( ) = Location of the encrypted sums. Horizontal sums are preceded by a lower case "h" and vertical sums are preceded by a lower case "v". Those are followed by one or two upper case letters depending if the sum is a single digit or double digit one. For double digit sums, the first letter would be for the "tens" and the second one for the "units". When the cell must contain information for both a horizontal and a vertical sum, the first one is always for the horizontal sum and the two are separated by a comma within the same set of brackets, ex.: (hFE,vD). Each set of brackets is also immediately followed by a comma.

The description of the last cell is followed by a Carriage Return/Line Feed (CRLF) instead of a comma.

The required answer to each puzzle is based on the value of each letter necessary to arrive at the solution and according to the alphabetical order. As indicated under the example puzzle, its answer would be 8426039571. At least 9 out of the 10 encrypting letters are always part of the problem description. When only 9 are given, the missing one must be assigned the remaining digit.

You are given that the sum of the answers for the first 10 puzzles in the file is 64414157580.

Find the sum of the answers for the 200 puzzles.

## Problem 425: Prime connection

Two positive numbers A and B are said to be *connected* (denoted by "A ↔ B") if one of these conditions holds:

- (1) A and B have the same length and differ in exactly one digit; for example, 123 ↔ 173.
- (2) Adding one digit to the left of A (or B) makes B (or A); for example, 23 ↔ 223 and 123 ↔ 23.

We call a prime P a 2's *relative* if there exists a chain of connected primes between 2 and P and no prime in the chain exceeds P.

For example, 127 is a 2's relative. One of the possible chains is shown below:

2 ↔ 3 ↔ 13 ↔ 113 ↔ 103 ↔ 107 ↔ 127

However, 11 and 103 are not 2's relatives.

Let F(N) be the sum of the primes ≤ N which are not 2's relatives.

We can verify that F(10<sup>3</sup>) = 431 and F(10<sup>4</sup>) = 78728.

Find F(10<sup>7</sup>).

### Problem 426: Box-ball system

Consider an infinite row of boxes. Some of the boxes contain a ball. For example, an initial configuration of 2 consecutive occupied boxes followed by 2 empty boxes, 2 occupied boxes, 1 empty box, and 2 occupied boxes can be denoted by the sequence (2, 2, 2, 1, 2), in which the number of consecutive occupied and empty boxes appear alternately.

A turn consists of moving each ball exactly once according to the following rule: Transfer the leftmost ball which has not been moved to the nearest empty box to its right.

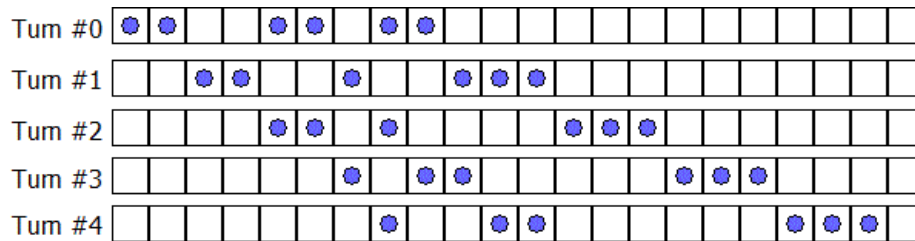
After one turn the sequence (2, 2, 2, 1, 2) becomes (2, 2, 1, 2, 3) as can be seen below; note that we begin the new sequence starting at the first occupied box.

Turn begin



A system like this is called a **Box-Ball System** or **BBS** for short.

It can be shown that after a sufficient number of turns, the system evolves to a state where the consecutive numbers of occupied boxes is invariant. In the example below, the consecutive numbers of **occupied boxes** evolves to [1, 2, 3]; we shall call this the final state.



We define the sequence  $\{t_i\}$ :

- $S_0 = 290797$
- $S_{k+1} = S_k^2 \bmod 50515093$
- $t_k = (S_k \bmod 64) + 1$

Starting from the initial configuration  $(t_0, t_1, \dots, t_{10})$ , the final state becomes [1, 3, 10, 24, 51, 75].

Starting from the initial configuration  $(t_0, t_1, \dots, t_{10\,000\,000})$ , find the final state.

Give as your answer the sum of the squares of the elements of the final state. For example, if the final state is [1, 2, 3] then  $14 (= 1^2 + 2^2 + 3^2)$  is your answer.

### Problem 427: n-sequences

A sequence of integers  $S = \{s_i\}$  is called an *n-sequence* if it has  $n$  elements and each element  $s_i$  satisfies  $1 \leq s_i \leq n$ . Thus there are  $n^n$  distinct  $n$ -sequences in total. For example, the sequence  $S = \{1, 5, 5, 10, 7, 7, 7, 2, 3, 7\}$  is a 10-sequence.

For any sequence  $S$ , let  $L(S)$  be the length of the longest contiguous subsequence of  $S$  with the same value. For example, for the given sequence  $S$  above,  $L(S) = 3$ , because of the three consecutive 7's.

Let  $f(n) = \sum L(S)$  for all  $n$ -sequences  $S$ .

For example,  $f(3) = 45$ ,  $f(7) = 1403689$  and  $f(11) = 481496895121$ .

Find  $f(7\,500\,000) \bmod 1\,000\,000\,009$ .

### Problem 428: Necklace of circles

Let  $a, b$  and  $c$  be positive numbers.

Let  $W, X, Y, Z$  be four collinear points where  $|WX| = a$ ,  $|XY| = b$ ,  $|YZ| = c$  and  $|WZ| = a + b + c$ .

Let  $C_{\text{in}}$  be the circle having the diameter  $XY$ .

Let  $C_{\text{out}}$  be the circle having the diameter  $WZ$ .

The triplet  $(a, b, c)$  is called a *necklace triplet* if you can place  $k \geq 3$  distinct circles  $C_1, C_2, \dots, C_k$  such that:

- $C_j$  has no common interior points with any  $C_j$  for  $1 \leq i, j \leq k$  and  $i \neq j$ ,
- $C_j$  is tangent to both  $C_{\text{in}}$  and  $C_{\text{out}}$  for  $1 \leq i \leq k$ ,
- $C_j$  is tangent to  $C_{j+1}$  for  $1 \leq i < k$ , and
- $C_k$  is tangent to  $C_1$ .

For example,  $(5, 5, 5)$  and  $(4, 3, 21)$  are necklace triplets, while it can be shown that  $(2, 2, 5)$  is not.

Let  $T(n)$  be the number of necklace triplets  $(a, b, c)$  such that  $a, b$  and  $c$  are positive integers, and  $b \leq n$ . For example,  $T(1) = 9$ ,  $T(20) = 732$  and  $T(3000) = 438106$ .

Find  $T(1\,000\,000\,000)$ .

### Problem 429: Sum of squares of unitary divisors

A unitary divisor  $d$  of a number  $n$  is a divisor of  $n$  that has the property  $\gcd(d, n/d) = 1$ .

The unitary divisors of  $4! = 24$  are 1, 3, 8 and 24.

The sum of their squares is  $1^2 + 3^2 + 8^2 + 24^2 = 650$ .

Let  $S(n)$  represent the sum of the squares of the unitary divisors of  $n$ . Thus  $S(4!) = 650$ .

Find  $S(100\,000\,000!)$  modulo  $1\,000\,000\,009$ .

### Problem 430: Range flips

$N$  disks are placed in a row, indexed 1 to  $N$  from left to right.

Each disk has a black side and white side. Initially all disks show their white side.

At each turn, two, not necessarily distinct, integers  $A$  and  $B$  between 1 and  $N$  (inclusive) are chosen uniformly at random.

All disks with an index from  $A$  to  $B$  (inclusive) are flipped.

The following example shows the case  $N = 8$ . At the first turn  $A = 5$  and  $B = 2$ , and at the second turn  $A = 4$  and  $B = 6$ .

Let  $E(N, M)$  be the expected number of disks that show their white side after  $M$  turns.

We can verify that  $E(3, 1) = 10/9$ ,  $E(3, 2) = 5/3$ ,  $E(10, 4) \approx 5.157$  and  $E(100, 10) \approx 51.893$ .

Find  $E(10^{10}, 4000)$ .

Give your answer rounded to 2 decimal places behind the decimal point.

### Problem 431: Square Space Silo

Fred the farmer arranges to have a new storage silo installed on his farm and having an obsession for all things square he is absolutely devastated when he discovers that it is circular. Quentin, the representative from the company that installed the silo, explains that they only manufacture cylindrical silos, but he points out that it is resting on a square base. Fred is not amused and insists that it is removed from his property.

Quick thinking Quentin explains that when granular materials are delivered from above a conical slope is formed and the natural angle made with the horizontal is called the angle of repose. For example if the angle of repose,  $\alpha = 30$  degrees, and grain is delivered at the centre of the silo then a perfect cone will form towards the top of the cylinder. In the case of this silo, which has a diameter of 6m, the amount of space wasted would be approximately  $32.648388556 \text{ m}^3$ . However, if grain is delivered at a point on the top which has a horizontal distance of  $x$  metres from the centre then a cone with a strangely curved and sloping base is formed. He shows Fred a picture.



We shall let the amount of space wasted in cubic metres be given by  $V(x)$ . If  $x = 1.114785284$ , which happens to have three squared decimal places, then the amount of space wasted,  $V(1.114785284) \approx 36$ . Given the range of possible solutions to this problem there is exactly one other option:  $V(2.511167869) \approx 49$ . It would be like knowing that the square is king of the silo, sitting in splendid glory on top of your grain.

Fred's eyes light up with delight at this elegant resolution, but on closer inspection of Quentin's drawings and calculations his happiness turns to despondency once more. Fred points out to Quentin that it's the radius of the silo that is 6 metres, not the diameter, and the angle of repose for his grain is 40 degrees. However, if Quentin can find a set of solutions for this particular silo then he will be more than happy to keep it.

If Quick thinking Quentin is to satisfy frustratingly fussy Fred the farmer's appetite for all things square then determine the values of  $x$  for all possible square space wastage options and calculate  $\sum x$  correct to 9 decimal places.

### Problem 432: Totient sum

Let  $S(n, m) = \sum \varphi(n \times i)$  for  $1 \leq i \leq m$ . ( $\varphi$  is Euler's totient function)

You are given that  $S(510510, 10^6) = 45480596821125120$ .

Find  $S(510510, 10^{11})$ .

Give the last 9 digits of your answer.

### Problem 433: Steps in Euclid's algorithm

Let  $E(x_0, y_0)$  be the number of steps it takes to determine the greatest common divisor of  $x_0$  and  $y_0$  with **Euclid's algorithm**. More formally:

$$x_1 = y_0, y_1 = x_0 \bmod y_0$$

$$x_n = y_{n-1}, y_n = x_{n-1} \bmod y_{n-1}$$

$E(x_0, y_0)$  is the smallest  $n$  such that  $y_n = 0$ .

We have  $E(1, 1) = 1$ ,  $E(10, 6) = 3$  and  $E(6, 10) = 4$ .

Define  $S(N)$  as the sum of  $E(x, y)$  for  $1 \leq x, y \leq N$ .

We have  $S(1) = 1$ ,  $S(10) = 221$  and  $S(100) = 39826$ .

Find  $S(5 \cdot 10^6)$ .

### Problem 434: Rigid graphs

Recall that a graph is a collection of vertices and edges connecting the vertices, and that two vertices connected by an edge are called adjacent.

Graphs can be embedded in Euclidean space by associating each vertex with a point in the Euclidean space.

A **flexible** graph is an embedding of a graph where it is possible to move one or more vertices continuously so that the distance between at least two nonadjacent vertices is altered while the distances between each pair of adjacent vertices is kept constant.

A **rigid** graph is an embedding of a graph which is not flexible.

Informally, a graph is rigid if by replacing the vertices with fully rotating hinges and the edges with rods that are unbending and inelastic, no parts of the graph can be moved independently from the rest of the graph.

The **grid graphs** embedded in the Euclidean plane are not rigid, as the following animation demonstrates:



However, one can make them rigid by adding diagonal edges to the cells. For example, for the  $2 \times 3$  grid graph, there are 19 ways to make the graph rigid:



Note that for the purposes of this problem, we do not consider changing the orientation of a diagonal edge or adding both diagonal edges to a cell as a different way of making a grid graph rigid.

Let  $R(m, n)$  be the number of ways to make them  $m \times n$  grid graph rigid.

E.g.  $R(2, 3) = 19$  and  $R(5, 5) = 23679901$

Define  $S(N)$  as  $\sum R(i, j)$  for  $1 \leq i, j \leq N$ .

E.g.  $S(5) = 25021721$ .

Find  $S(100)$ , give your answer modulo 1000000033

### Problem 435: Polynomials of Fibonacci numbers

The **Fibonacci numbers**  $\{f_n, n \geq 0\}$  are defined recursively as  $f_n = f_{n-1} + f_{n-2}$  with base cases  $f_0 = 0$  and  $f_1 = 1$ .

Define the polynomials  $\{F_n, n \geq 0\}$  as  $F_n(x) = \sum f_i x^i$  for  $0 \leq i \leq n$ .

For example,  $F_7(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7$ , and  $F_7(11) = 268357683$ .

Let  $n = 10^{15}$ . Find the sum  $[\sum_{0 \leq x \leq 100} F_n(x)] \bmod 1307674368000 (= 15!)$ .

### Problem 436: Unfair wager

Julie proposes the following wager to her sister Louise.

She suggests they play a game of chance to determine who will wash the dishes.

For this game, they shall use a generator of independent random numbers uniformly distributed between 0 and 1.

The game starts with  $S = 0$ .

The first player, Louise, adds to  $S$  different random numbers from the generator until  $S > 1$  and records her last random number ' $x$ '.

The second player, Julie, continues adding to  $S$  different random numbers from the generator until  $S > 2$  and records her last random number ' $y$ '.

The player with the highest number wins and the loser washes the dishes, i.e. if  $y > x$  the second player wins.

For example, if the first player draws 0.62 and 0.44, the first player turn ends since  $0.62 + 0.44 > 1$  and  $x = 0.44$ .

If the second player draws 0.1, 0.27 and 0.91, the second player turn ends since  $0.62 + 0.44 + 0.1 + 0.27 + 0.91 > 2$  and  $y = 0.91$ .

Since  $y > x$ , the second player wins.

Louise thinks about it for a second, and objects: "That's not fair".

What is the probability that the second player wins?

Give your answer rounded to 10 places behind the decimal point in the form 0.abcdefghij

### Problem 437: Fibonacci primitive roots

When we calculate  $8^n$  modulo 11 for  $n=0$  to 9 we get: 1, 8, 9, 6, 4, 10, 3, 2, 5, 7.

As we see all possible values from 1 to 10 occur. So 8 is a **primitive root** of 11.

But there is more:

If we take a closer look we see:

$$1+8=9$$

$$8+9=17 \equiv 6 \pmod{11}$$

$$9+6=15 \equiv 4 \pmod{11}$$

$$6+4=10$$

$$4+10=14 \equiv 3 \pmod{11}$$

$$10+3=13 \equiv 2 \pmod{11}$$

$$3+2=5$$

$$2+5=7$$

$$5+7=12 \equiv 1 \pmod{11}.$$

So the powers of 8 mod 11 are cyclic with period 10, and  $8^n + 8^{n+1} \equiv 8^{n+2} \pmod{11}$ .

8 is called a **Fibonacci primitive root** of 11.

Not every prime has a Fibonacci primitive root.

There are 323 primes less than 10000 with one or more Fibonacci primitive roots and the sum of these primes is 1480491.

Find the sum of the primes less than 100,000 with at least one Fibonacci primitive root.

### Problem 438: Integer part of polynomial equation's solutions

For an  $n$ -tuple of integers  $t = (a_1, \dots, a_n)$ , let  $(x_1, \dots, x_n)$  be the solutions of the polynomial equation  $x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$ .

Consider the following two conditions:

- $x_1, \dots, x_n$  are all real.
- If  $x_1, \dots, x_n$  are sorted,  $\lfloor x_i \rfloor = i$  for  $1 \leq i \leq n$ . ( $\lfloor \cdot \rfloor$ : floor function.)

In the case of  $n = 4$ , there are 12  $n$ -tuples of integers which satisfy both conditions.

We define  $S(t)$  as the sum of the absolute values of the integers in  $t$ .

For  $n = 4$  we can verify that  $\sum S(t) = 2087$  for all  $n$ -tuples  $t$  which satisfy both conditions.

Find  $\sum S(t)$  for  $n = 7$ .

### Problem 439: Sum of sum of divisors

Let  $d(k)$  be the sum of all divisors of  $k$ .

We define the function  $S(N) = \sum_{i=1}^N \sum_{j=1}^{Nd(i)} (i \cdot j)$ .

For example,  $S(3) = d(1) + d(2) + d(3) + d(2) + d(4) + d(6) + d(3) + d(6) + d(9) = 59$ .

You are given that  $S(10^3) = 563576517282$  and  $S(10^5) \bmod 10^9 = 215766508$ .

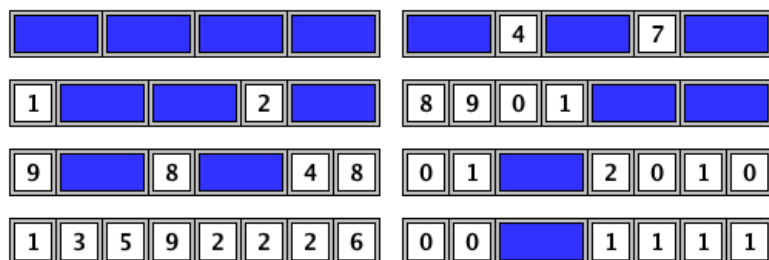
Find  $S(10^{11}) \bmod 10^9$ .



### Problem 440: GCD and Tiling

We want to tile a board of length  $n$  and height 1 completely, with either  $1 \times 2$  blocks or  $1 \times 1$  blocks with a single decimal digit on top:

For example, here are some of the ways to tile a board of length  $n = 8$ :



Let  $T(n)$  be the number of ways to tile a board of length  $n$  as described above.

For example,  $T(1) = 10$  and  $T(2) = 101$ .

Let  $S(L)$  be the triple sum  $\sum_{a,b,c} \gcd(T(c^a), T(c^b))$  for  $1 \leq a, b, c \leq L$ .

For example:

$S(2) = 10444$

$S(3) = 1292115238446807016106539989$

$S(4) \bmod 987\,898\,789 = 670616280$ .

Find  $S(2000) \bmod 987\,898\,789$ .

### Problem 441: The inverse summation of coprime couples

For an integer  $M$ , we define  $R(M)$  as the sum of  $1/(p \cdot q)$  for all the integer pairs  $p$  and  $q$  which satisfy all of these conditions:

- $1 \leq p < q \leq M$
- $p + q \geq M$
- $p$  and  $q$  are coprime.

We also define  $S(N)$  as the sum of  $R(i)$  for  $2 \leq i \leq N$ .

We can verify that  $S(2) = R(2) = 1/2$ ,  $S(10) \approx 6.9147$  and  $S(100) \approx 58.2962$ .

Find  $S(10^7)$ . Give your answer rounded to four decimal places.

### Problem 442: Eleven-free integers

An integer is called *eleven-free* if its decimal expansion does not contain any substring representing a power of 11 except 1.

For example, 2404 and 13431 are eleven-free, while 911 and 4121331 are not.

Let  $E(n)$  be the  $n$ th positive eleven-free integer. For example,  $E(3) = 3$ ,  $E(200) = 213$  and  $E(500\,000) = 531563$ .

Find  $E(10^{18})$ .

#### Problem 443: GCD sequence

Let  $g(n)$  be a sequence defined as follows:

$$g(4) = 13,$$

$$g(n) = g(n-1) + \gcd(n, g(n-1)) \text{ for } n > 4.$$

The first few values are:

$n$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
$g(n)$	13	14	16	17	18	27	28	29	30	31	32	33	34	51	54	55	60	...

You are given that  $g(1\,000) = 2524$  and  $g(1\,000\,000) = 2624152$ .

Find  $g(10^{15})$ .

#### Problem 444: The Roundtable Lottery

A group of  $p$  people decide to sit down at a round table and play a lottery-ticket trading game. Each person starts off with a randomly-assigned, unscratched lottery ticket. Each ticket, when scratched, reveals a whole-pound prize ranging anywhere from £1 to £ $p$ , with no two tickets alike. The goal of the game is for each person to maximize his ticket winnings upon leaving the game.

An arbitrary person is chosen to be the first player. Going around the table, each player has only one of two options:

1. The player can scratch his ticket and reveal its worth to everyone at the table.
2. The player can trade his unscratched ticket for a previous player's scratched ticket, and then leave the game with that ticket. The previous player then scratches his newly-acquired ticket and reveals its worth to everyone at the table.

The game ends once all tickets have been scratched. All players still remaining at the table must leave with their currently-held tickets.

Assume that each player uses the optimal strategy for maximizing the expected value of his ticket winnings.

Let  $E(p)$  represent the expected number of players left at the table when the game ends in a game consisting of  $p$  players (e.g.  $E(111) = 5.2912$  when rounded to 5 significant digits).

Let  $S_1(N) = \text{[input box]} E(p)$

Let  $S_k(N) = \text{[input box]} S_{k-1}(p)$  for  $k > 1$

Find  $S_{20}(10^{14})$  and write the answer in scientific notation rounded to 10 significant digits. Use a lowercase e to separate mantissa and exponent (e.g.  $S_3(100) = 5.983679014e5$ ).

#### Problem 445: Retractions A

For every integer  $n > 1$ , the family of functions  $f_{n,a,b}$  is defined by  $f_{n,a,b}(x) \equiv ax + b \pmod n$  for  $a, b, x$  integer and  $0 < a < n$ ,  $0 \leq b < n$ ,  $0 \leq x < n$ .

We will call  $f_{n,a,b}$  a *retraction* if  $f_{n,a,b}(f_{n,a,b}(x)) \equiv f_{n,a,b}(x) \pmod n$  for every  $0 \leq x < n$ .

Let  $R(n)$  be the number of retractions for  $n$ .

You are given that

$$\sum R(c) \text{ for } c = C(100\,000, k), \text{ and } 1 \leq k \leq 99\,999 \equiv 628701600 \pmod{1\,000\,000\,007}.$$

( $C(n, k)$  is the binomial coefficient).

Find  $\sum R(c)$  for  $c = C(10\,000\,000, k)$ , and  $1 \leq k \leq 9\,999\,999$ .

Give your answer modulo  $1\,000\,000\,007$ .

#### Problem 446: Retractions B

For every integer  $n > 1$ , the family of functions  $f_{n,a,b}$  is defined by  $f_{n,a,b}(X) \equiv aX + b \pmod n$  for  $a, b, X$  integer and  $0 < a < n$ ,  $0 \leq b < n$ ,  $0 \leq X < n$ .

We will call  $f_{n,a,b}$  a *retraction* if  $f_{n,a,b}(f_{n,a,b}(X)) \equiv f_{n,a,b}(X) \pmod n$  for every  $0 \leq X < n$ .

Let  $R(n)$  be the number of retractions for  $n$ .

$$F(N) = \sum R(n^4 + 4) \text{ for } 1 \leq n \leq N.$$
$$F(1024) = 77532377300600.$$

Find  $F(10^7) \pmod{1\,000\,000\,007}$

#### Problem 447: Retractions C

For every integer  $n > 1$ , the family of functions  $f_{n,a,b}$  is defined by  $f_{n,a,b}(X) \equiv aX + b \pmod n$  for  $a, b, X$  integer and  $0 < a < n$ ,  $0 \leq b < n$ ,  $0 \leq X < n$ .

We will call  $f_{n,a,b}$  a *retraction* if  $f_{n,a,b}(f_{n,a,b}(X)) \equiv f_{n,a,b}(X) \pmod n$  for every  $0 \leq X < n$ .

Let  $R(n)$  be the number of retractions for  $n$ .

$$F(N) = \sum R(n) \text{ for } 2 \leq n \leq N.$$
$$F(10^7) \equiv 638042271 \pmod{1\,000\,000\,007}.$$

Find  $F(10^{14}) \pmod{1\,000\,000\,007}$ .

#### Problem 448: Average least common multiple

The function **lcm**(a,b) denotes the least common multiple of a and b.

Let  $A(n)$  be the average of the values of  $\text{lcm}(n,i)$  for  $1 \leq i \leq n$ .

E.g:  $A(2) = (2+2)/2 = 2$  and  $A(10) = (10+10+30+20+10+30+70+40+90+10)/10 = 32$ .

$$\text{Let } S(n) = \sum A(k) \text{ for } 1 \leq k \leq n.$$
$$S(100) = 122726.$$

Find  $S(99999999019) \pmod{999999017}$ .

#### Problem 449: Chocolate covered candy

Phil the confectioner is making a new batch of chocolate covered candy. Each candy centre is shaped like an ellipsoid of revolution defined by the equation:  $b^2x^2 + b^2y^2 + a^2z^2 = a^2b^2$ .

Phil wants to know how much chocolate is needed to cover one candy centre with a uniform coat of chocolate one millimeter thick.

If  $a=1$  mm and  $b=1$  mm, the amount of chocolate required is  $\frac{28}{3} \pi \text{ mm}^3$

If  $a=2$  mm and  $b=1$  mm, the amount of chocolate required is approximately  $60.35475635 \text{ mm}^3$ .

Find the amount of chocolate in  $\text{mm}^3$  required if  $a=3$  mm and  $b=1$  mm. Give your answer as the number rounded to 8 decimal places behind the decimal point.

### Problem 450: Hypocycloid and Lattice points

A hypocycloid is the curve drawn by a point on a small circle rolling inside a larger circle. The parametric equations of a hypocycloid centered at the origin, and starting at the right most point is given by:

$$\begin{aligned}x(t) &= (R - r) \cos(t) + r \cos\left(\frac{R - r}{r} t\right) \\y(t) &= (R - r) \sin(t) - r \sin\left(\frac{R - r}{r} t\right)\end{aligned}$$

Where  $R$  is the radius of the large circle and  $r$  the radius of the small circle.

Let  $C(R, r)$  be the set of distinct points with integer coordinates on the hypocycloid with radius  $R$  and  $r$  and for which there is a corresponding value of  $t$  such that  $\sin(t)$  and  $\cos(t)$  are rational numbers.

Let  $S(R, r) = \sum_{(x,y) \in C(R, r)} |x| + |y|$  be the sum of the absolute values of the  $x$  and  $y$  coordinates of the points in  $C(R, r)$ .

Let  $T(N) = \sum_{R=3}^N \sum_{r=1}^{\lfloor \frac{R-1}{2} \rfloor} S(R, r)$  be the sum of  $S(R, r)$  for  $R$  and  $r$  positive integers,  $R \leq N$  and  $2r < R$ .

You are given:

$$C(3, 1) = \{(3, 0), (-1, 2), (-1, 0), (-1, -2)\}$$

$$C(2500, 1000) = \{(2500, 0), (772, 2376), (772, -2376), (516, 1792), (516, -1792), (500, 0), (68, 504), (68, -504), (-1356, 1088), (-1356, -1088), (-1500, 1000), (-1500, -1000)\}$$

*Note:  $(-625, 0)$  is not an element of  $C(2500, 1000)$  because  $\sin(t)$  is not a rational number for the corresponding values of  $t$ .*

$$S(3, 1) = (|3| + |0|) + (|-1| + |2|) + (|-1| + |0|) + (|-1| + |-2|) = 10$$

$$T(3) = 10; T(10) = 524; T(100) = 580442; T(10^3) = 583108600.$$

Find  $T(10^6)$ .

### Problem 451: Modular inverses

Consider the number 15.

There are eight positive numbers less than 15 which are coprime to 15: 1, 2, 4, 7, 8, 11, 13, 14.

The modular inverses of these numbers modulo 15 are: 1, 8, 4, 13, 2, 11, 7, 14 because

$$1 \cdot 1 \bmod 15 = 1$$

$$2 \cdot 8 \bmod 15 = 1$$

$$4 \cdot 4 \bmod 15 = 1$$

$$7 \cdot 13 \bmod 15 = 1$$

$$11 \cdot 11 \bmod 15 = 1$$

$$14 \cdot 14 \bmod 15 = 1$$

Let  $I(n)$  be the largest positive number  $m$  smaller than  $n-1$  such that the modular inverse of  $m$  modulo  $n$  equals  $m$  itself. So  $I(15)=11$ .

Also  $I(100)=51$  and  $I(7)=1$ .

Find  $\sum I(n)$  for  $3 \leq n \leq 2 \cdot 10^7$

### Problem 452: Long Products

Define  $F(m, n)$  as the number of  $n$ -tuples of positive integers for which the product of the elements doesn't exceed  $m$ .

$$F(10, 10) = 571.$$

$$F(10^6, 10^6) \bmod 1\,234\,567\,891 = 252903833.$$

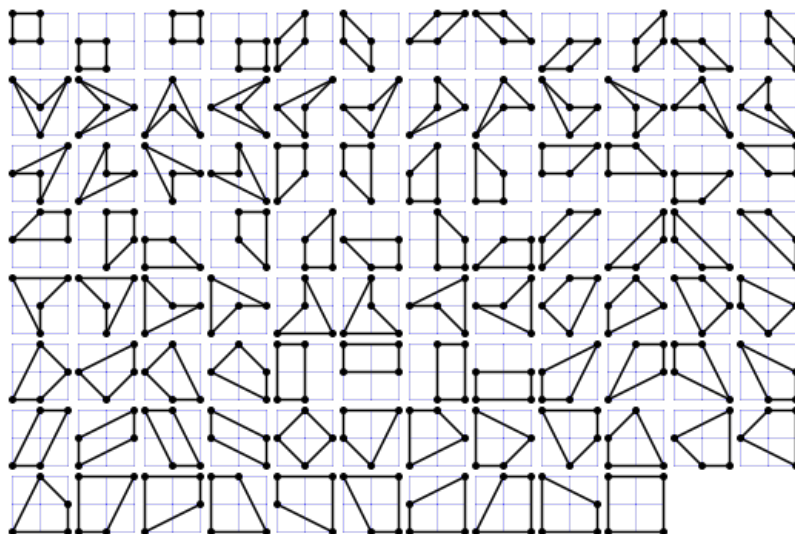
Find  $F(10^9, 10^9) \bmod 1\,234\,567\,891$ .

### Problem 453: Lattice Quadrilaterals

A **simple quadrilateral** is a polygon that has four distinct vertices, has no straight angles and does not self-intersect.

Let  $Q(m, n)$  be the number of simple quadrilaterals whose vertices are lattice points with coordinates  $(x, y)$  satisfying  $0 \leq x \leq m$  and  $0 \leq y \leq n$ .

For example,  $Q(2, 2) = 94$  as can be seen below:



It can also be verified that  $Q(3, 7) = 39590$ ,  $Q(12, 3) = 309000$  and  $Q(123, 45) = 70542215894646$ .

Find  $Q(12345, 6789) \bmod 135707531$ .

### Problem 454: Diophantine reciprocals III

In the following equation  $x, y$ , and  $n$  are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

For a limit  $L$  we define  $F(L)$  as the number of solutions which satisfy  $x < y \leq L$ .

We can verify that  $F(15) = 4$  and  $F(1000) = 1069$ .

Find  $F(10^{12})$ .

### Problem 455: Powers With Trailing Digits

Let  $f(n)$  be the largest positive integer  $x$  less than  $10^9$  such that the last 9 digits of  $n^x$  form the number  $x$  (including leading zeros), or zero if no such integer exists.

For example:

- $f(4) = 411728896$  ( $4^{411728896} = \dots 490411728896$ )
- $f(10) = 0$
- $f(157) = 743757$  ( $157^{743757} = \dots 567000743757$ )
- $\sum f(n), 2 \leq n \leq 10^3 = 442530011399$

Find  $\sum f(n), 2 \leq n \leq 10^6$ .

**Problem 456: Triangles containing the origin II**

Define:

$$X_n = (1248^n \bmod 32323) - 16161$$

$$Y_n = (8421^n \bmod 30103) - 15051$$

$$P_n = \{(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)\}$$

For example,  $P_8 = \{(-14913, -6630), (-10161, 5625), (5226, 11896), (8340, -10778), (15852, -5203), (-15165, 11295), (-1427, -14495), (12407, 1060)\}$ .

Let  $C(n)$  be the number of triangles whose vertices are in  $P_n$  which contain the origin in the interior.

Examples:

$$C(8) = 20$$

$$C(600) = 8950634$$

$$C(40\,000) = 2666610948988$$

Find  $C(2\,000\,000)$ .

**Problem 457: A polynomial modulo the square of a prime**

Let  $f(n) = n^2 - 3n - 1$ .

Let  $p$  be a prime.

Let  $R(p)$  be the smallest positive integer  $n$  such that  $f(n) \bmod p^2 = 0$  if such an integer  $n$  exists, otherwise  $R(p) = 0$ .

Let  $SR(L)$  be  $\sum R(p)$  for all primes not exceeding  $L$ .

Find  $SR(10^7)$ .

**Problem 458: Permutations of Project**

Consider the alphabet  $A$  made out of the letters of the word "project":  $A = \{c, e, j, o, p, r, t\}$ .

Let  $T(n)$  be the number of strings of length  $n$  consisting of letters from  $A$  that do not have a substring that is one of the 5040 permutations of "project".

$$T(7) = 7^7 - 7! = 818503.$$

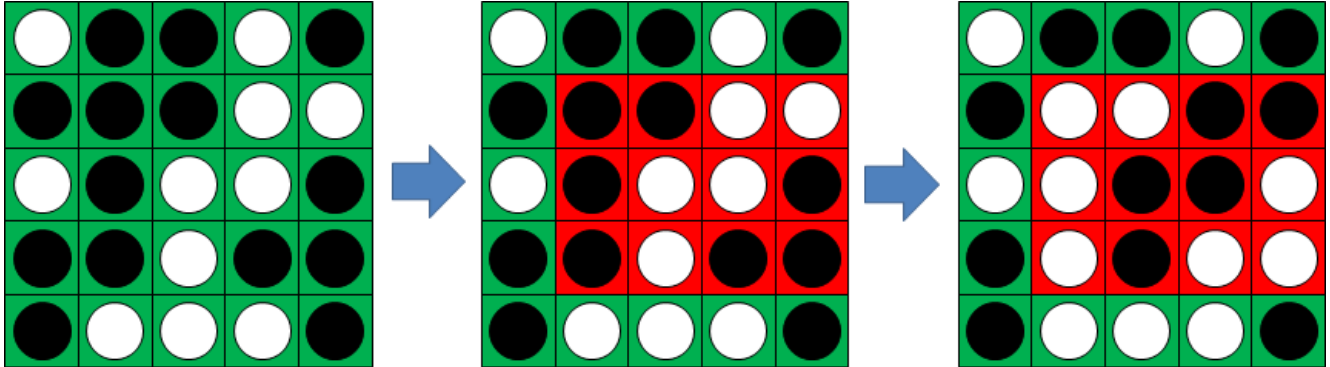
Find  $T(10^{12})$ . Give the last 9 digits of your answer.

### Problem 459: Flipping game

The flipping game is a two player game played on a  $N$  by  $N$  square board. Each square contains a disk with one side white and one side black. The game starts with all disks showing their white side.

A turn consists of flipping all disks in a rectangle with the following properties:

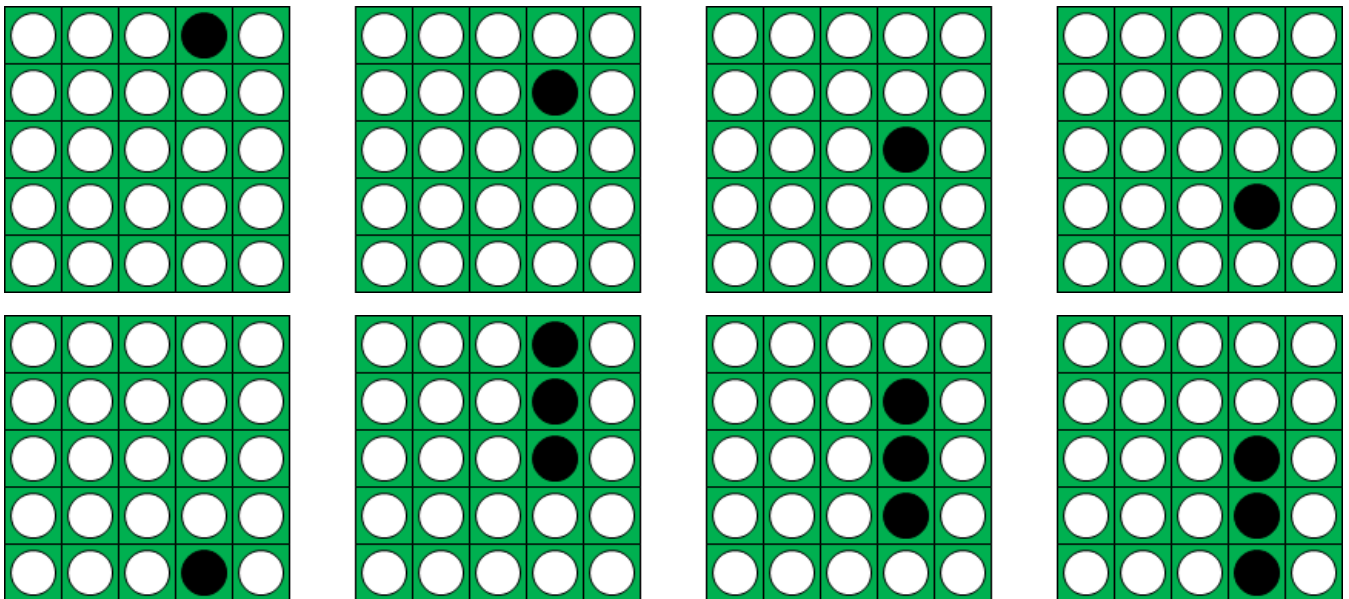
- the upper right corner of the rectangle contains a white disk
- the rectangle width is a perfect square (1, 4, 9, 16, ...)
- the rectangle height is a triangular number (1, 3, 6, 10, ...)



Players alternate turns. A player wins by turning the grid all black.

Let  $W(N)$  be the number of winning moves for the first player on a  $N$  by  $N$  board with all disks white, assuming perfect play.  $W(1) = 1$ ,  $W(2) = 0$ ,  $W(5) = 8$  and  $W(10^2) = 31395$ .

For  $N=5$ , the first player's eight winning first moves are:



Find  $W(10^6)$ .

### Problem 460: An ant on the move

On the Euclidean plane, an ant travels from point  $A(0, 1)$  to point  $B(d, 1)$  for an integer  $d$ .

In each step, the ant at point  $(X_0, Y_0)$  chooses one of the lattice points  $(X_1, Y_1)$  which satisfy  $X_1 \geq 0$  and  $Y_1 \geq 1$  and goes straight to  $(X_1, Y_1)$  at a constant velocity  $V$ . The value of  $V$  depends on  $Y_0$  and  $Y_1$  as follows:

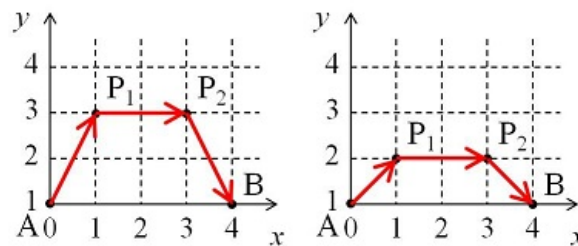
- If  $Y_0 = Y_1$ , the value of  $V$  equals  $Y_0$ .
- If  $Y_0 \neq Y_1$ , the value of  $V$  equals  $(Y_1 - Y_0) / (\ln(Y_1) - \ln(Y_0))$ .

The left image is one of the possible paths for  $d = 4$ . First the ant goes from  $A(0, 1)$  to  $P_1(1, 3)$  at velocity  $(3 - 1) / (\ln(3) - \ln(1)) \approx 1.8205$ . Then the required time is  $\sqrt{5} / 1.8205 \approx 1.2283$ .

From  $P_1(1, 3)$  to  $P_2(3, 3)$  the ant travels at velocity 3 so the required time is  $2 / 3 \approx 0.6667$ . From  $P_2(3, 3)$  to  $B(4, 1)$  the ant travels at velocity  $(1 - 3) / (\ln(1) - \ln(3)) \approx 1.8205$  so the required time is  $\sqrt{5} / 1.8205 \approx 1.2283$ .

Thus the total required time is  $1.2283 + 0.6667 + 1.2283 = 3.1233$ .

The right image is another path. The total required time is calculated as  $0.98026 + 1 + 0.98026 = 2.96052$ . It can be shown that this is the quickest path for  $d = 4$ .



Let  $F(d)$  be the total required time if the ant chooses the quickest path. For example,  $F(4) \approx 2.960516287$ .

We can verify that  $F(10) \approx 4.668187834$  and  $F(100) \approx 9.217221972$ .

Find  $F(10000)$ . Give your answer rounded to nine decimal places.

### Problem 461: Almost Pi

Let  $f_n(k) = e^{k/n} - 1$ , for all non-negative integers  $k$ .

Remarkably,  $f_{200}(6) + f_{200}(75) + f_{200}(89) + f_{200}(226) = \underline{3.141592644529} \dots \approx \pi$ .

In fact, it is the best approximation of  $\pi$  of the form  $f_n(a) + f_n(b) + f_n(c) + f_n(d)$  for  $n = 200$ .

Let  $g(n) = a^2 + b^2 + c^2 + d^2$  for  $a, b, c, d$  that minimize the error:  $|f_n(a) + f_n(b) + f_n(c) + f_n(d) - \pi|$  (where  $|X|$  denotes the absolute value of  $X$ ).

You are given  $g(200) = 6^2 + 75^2 + 89^2 + 226^2 = 64658$ .

Find  $g(10000)$ .



#### Problem 462: Permutation of 3-smooth numbers

A **3-smooth number** is an integer which has no prime factor larger than 3. For an integer  $N$ , we define  $S(N)$  as the set of 3-smooth numbers less than or equal to  $N$ . For example,  $S(20) = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18\}$ .

We define  $F(N)$  as the number of permutations of  $S(N)$  in which each element comes after all of its proper divisors.

This is one of the possible permutations for  $N = 20$ .

- 1, 2, 4, 3, 9, 8, 16, 6, 18, 12.

This is not a valid permutation because 12 comes before its divisor 6.

- 1, 2, 4, 3, 9, 8, **12**, 16, **6**, 18.

We can verify that  $F(6) = 5$ ,  $F(8) = 9$ ,  $F(20) = 450$  and  $F(1000) \approx 8.8521816557e21$ .

Find  $F(10^{18})$ . Give as your answer its scientific notation rounded to ten digits after the decimal point.

When giving your answer, use a lowercase e to separate mantissa and exponent. E.g. if the answer is 112,233,445,566,778,899 then the answer format would be 1.1223344557e17.

#### Problem 463: A weird recurrence relation

The function  $f$  is defined for all positive integers as follows:

- $f(1)=1$
- $f(3)=3$
- $f(2n)=f(n)$
- $f(4n+1)=2f(2n+1) - f(n)$
- $f(4n+3)=3f(2n+1) - 2f(n)$

The function  $S(n)$  is defined as  $\sum_{i=1}^n f(i)$ .

$S(8)=22$  and  $S(100)=3604$ .

Find  $S(3^{37})$ . Give the last 9 digits of your answer.

#### Problem 464: Möbius function and intervals

The **Möbius function**, denoted  $\mu(n)$ , is defined as:

- $\mu(n) = (-1)^{\omega(n)}$  if  $n$  is squarefree (where  $\omega(n)$  is the number of distinct prime factors of  $n$ )
- $\mu(n) = 0$  if  $n$  is not squarefree.

Let  $P(a,b)$  be the number of integers  $n$  in the interval  $[a,b]$  such that  $\mu(n) = 1$ .

Let  $N(a,b)$  be the number of integers  $n$  in the interval  $[a,b]$  such that  $\mu(n) = -1$ .

For example,  $P(2,10) = 2$  and  $N(2,10) = 4$ .

Let  $C(n)$  be the number of integer pairs  $(a,b)$  such that:

- $1 \leq a \leq b \leq n$ ,
- $99 \cdot N(a,b) \leq 100 \cdot P(a,b)$ , and
- $99 \cdot P(a,b) \leq 100 \cdot N(a,b)$ .

For example,  $C(10) = 13$ ,  $C(500) = 16676$  and  $C(10\,000) = 20155319$ .

Find  $C(20\,000\,000)$ .

### Problem 465: Polar polygons

The *kernel* of a polygon is defined by the set of points from which the entire polygon's boundary is visible. We define a *polar polygon* as a polygon for which the origin is strictly contained inside its kernel.

For this problem, a polygon can have collinear consecutive vertices. However, a polygon still cannot have self-intersection and cannot have zero area.

For example, only the first of the following is a polar polygon (the kernels of the second, third, and fourth do not strictly contain the origin, and the fifth does not have a kernel at all):



Notice that the first polygon has three consecutive collinear vertices.

Let  $P(n)$  be the number of polar polygons such that the vertices  $(x, y)$  have integer coordinates whose absolute values are not greater than  $n$ .

Note that polygons should be counted as different if they have different set of edges, even if they enclose the same area. For example, the polygon with vertices  $[(0,0),(0,3),(1,1),(3,0)]$  is distinct from the polygon with vertices  $[(0,0),(0,3),(1,1),(3,0),(1,0)]$ .

For example,  $P(1) = 131$ ,  $P(2) = 1648531$ ,  $P(3) = 1099461296175$  and  $P(343) \bmod 1\,000\,000\,007 = 937293740$ .

Find  $P(7^{13}) \bmod 1\,000\,000\,007$ .

### Problem 466: Distinct terms in a multiplication table

Let  $P(m,n)$  be the number of *distinct* terms in an  $m \times n$  multiplication table.

For example, a  $3 \times 4$  multiplication table looks like this:

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12

There are 8 distinct terms  $\{1,2,3,4,6,8,9,12\}$ , therefore  $P(3,4) = 8$ .

You are given that:

$P(64,64) = 1263$ ,

$P(12,345) = 1998$ , and

$P(32,10^{15}) = 13826382602124302$ .

Find  $P(64,10^{16})$ .

### Problem 467: Superinteger

An integer  $S$  is called a *superinteger* of another integer  $n$  if the digits of  $n$  form a subsequence of the digits of  $S$ . For example, 2718281828 is a superinteger of 18828, while 314159 is not a superinteger of 151.

Let  $p(n)$  be the  $n$ th prime number, and let  $c(n)$  be the  $n$ th composite number. For example,  $p(1) = 2$ ,  $p(10) = 29$ ,  $c(1) = 4$  and  $c(10) = 18$ .

$\{p(i) : i \geq 1\} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots\}$

$\{c(i) : i \geq 1\} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \dots\}$

Let  $P^D$  the sequence of the **digital roots** of  $\{p(i)\}$  ( $C^D$  is defined similarly for  $\{c(i)\}$ ):

$P^D = \{2, 3, 5, 7, 2, 4, 8, 1, 5, 2, \dots\}$

$C^D = \{4, 6, 8, 9, 1, 3, 5, 6, 7, 9, \dots\}$

Let  $P_n$  be the integer formed by concatenating the first  $n$  elements of  $P^D$  ( $C_n$  is defined similarly for  $C^D$ ).

$P_{10} = 2357248152$

$C_{10} = 4689135679$

Let  $f(n)$  be the smallest positive integer that is a common superinteger of  $P_n$  and  $C_n$ .

For example,  $f(10) = 2357246891352679$ , and  $f(100) \bmod 1\,000\,000\,007 = 771661825$ .

Find  $f(10\,000) \bmod 1\,000\,000\,007$ .

### Problem 468: Smooth divisors of binomial coefficients

An integer is called  **$B$ -smooth** if none of its prime factors is greater than  $B$ .

Let  $S_B(n)$  be the largest  $B$ -smooth divisor of  $n$ .

Examples:

$S_1(10) = 1$

$S_4(2100) = 12$

$S_{17}(2496144) = 5712$

Define  $F(n) = \sum_{1 \leq B \leq n} \sum_{0 \leq r \leq n} S_B(C(n, r))$ . Here,  $C(n, r)$  denotes the binomial coefficient.

Examples:

$F(11) = 3132$

$F(1\,111) \bmod 1\,000\,000\,993 = 706036312$

$F(111\,111) \bmod 1\,000\,000\,993 = 22156169$

Find  $F(11\,111\,111) \bmod 1\,000\,000\,993$ .

### Problem 469: Empty chairs

In a room  $N$  chairs are placed around a round table.

Knights enter the room one by one and choose at random an available empty chair.

To have enough elbow room the knights always leave at least one empty chair between each other.

When there aren't any suitable chairs left, the fraction  $C$  of empty chairs is determined.

We also define  $E(N)$  as the expected value of  $C$ .

We can verify that  $E(4) = 1/2$  and  $E(6) = 5/9$ .

Find  $E(10^{18})$ . Give your answer rounded to fourteen decimal places in the form 0.abcdefghijklmn.

### Problem 470: Super Ramvok

Consider a single game of Ramvok:

Let  $t$  represent the maximum number of turns the game lasts. If  $t = 0$ , then the game ends immediately. Otherwise, on each turn  $i$ , the player rolls a die. After rolling, if  $i < t$  the player can either stop the game and receive a prize equal to the value of the current roll, or discard the roll and try again next turn. If  $i = t$ , then the roll cannot be discarded and the prize must be accepted. Before the game begins,  $t$  is chosen by the player, who must then pay an up-front cost  $Ct$  for some constant  $C$ . For  $C = 0$ ,  $t$  can be chosen to be infinite (with an up-front cost of 0). Let  $R(d, C)$  be the expected profit (i.e. net gain) that the player receives from a single game of optimally-played Ramvok, given a fair  $d$ -sided die and cost constant  $C$ . For example,  $R(4, 0.2) = 2.65$ . Assume that the player has sufficient funds for paying any/all up-front costs.

Now consider a game of Super Ramvok:

In Super Ramvok, the game of Ramvok is played repeatedly, but with a slight modification. After each game, the die is altered. The alteration process is as follows: The die is rolled once, and if the resulting face has its pips visible, then that face is altered to be blank instead. If the face is already blank, then it is changed back to its original value. After the alteration is made, another game of Ramvok can begin (and during such a game, at each turn, the die is rolled until a face with a value on it appears). The player knows which faces are blank and which are not at all times. The game of Super Ramvok ends once all faces of the die are blank.

Let  $S(d, C)$  be the expected profit that the player receives from an optimally-played game of Super Ramvok, given a fair  $d$ -sided die to start (with all sides visible), and cost constant  $C$ . For example,  $S(6, 1) = 208.3$ .

Let  $F(n) = \sum_{4 \leq d \leq n} \sum_{0 \leq C \leq n} S(d, C)$ .

Calculate  $F(20)$ , rounded to the nearest integer.

### Problem 471: Triangle inscribed in ellipse

The triangle  $\triangle ABC$  is inscribed in an ellipse with equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,  $0 < 2b < a$ ,  $a$  and  $b$  integers.

Let  $r(a, b)$  be the radius of the incircle of  $\triangle ABC$  when the incircle has center  $(\sqrt{3}, 0)$  and A has coordinates  $(\frac{a}{2}, \frac{\sqrt{3}}{2}b)$ .

For example,  $r(3, 1) = \frac{1}{2}$ ,  $r(6, 2) = 1$ ,  $r(12, 3) = 2$ .

Let  $G(n) = \sum_{a=3}^n \sum_{b=1}^{\lfloor \frac{a-1}{2} \rfloor} r(a, b)$

You are given  $G(10) = 20.59722222$ ,  $G(100) = 19223.60980$  (rounded to 10 significant digits).

Find  $G(10^{11})$ .

Give your answer in scientific notation rounded to 10 significant digits. Use a lowercase e to separate mantissa and exponent.

For  $G(10)$  the answer would have been  $2.059722222e1$ .

### Problem 472: Comfortable Distance II

There are  $N$  seats in a row.  $N$  people come one after another to fill the seats according to the following rules:

1. No person sits beside another.
2. The first person chooses any seat.
3. Each subsequent person chooses the seat furthest from anyone else already seated, as long as it does not violate rule 1. If there is more than one choice satisfying this condition, then the person chooses the leftmost choice.

Note that due to rule 1, some seats will surely be left unoccupied, and the maximum number of people that can be seated is less than  $N$  (for  $N > 1$ ).

Here are the possible seating arrangements for  $N = 15$ :



We see that if the first person chooses correctly, the 15 seats can seat up to 7 people.

We can also see that the first person has 9 choices to maximize the number of people that may be seated.

Let  $f(N)$  be the number of choices the first person has to maximize the number of occupants for  $N$  seats in a row. Thus,  $f(1) = 1$ ,  $f(15) = 9$ ,  $f(20) = 6$ , and  $f(500) = 16$ .

Also,  $\sum f(N) = 83$  for  $1 \leq N \leq 20$  and  $\sum f(N) = 13343$  for  $1 \leq N \leq 500$ .

Find  $\sum f(N)$  for  $1 \leq N \leq 10^{12}$ . Give the last 8 digits of your answer.

### Problem 473: Phigital number base

Let  $\varphi$  be the golden ratio:  $\varphi = \frac{1+\sqrt{5}}{2}$ .

Remarkably it is possible to write every positive integer as a sum of powers of  $\varphi$  even if we require that every power of  $\varphi$  is used at most once in this sum.

Even then this representation is not unique.

We can make it unique by requiring that no powers with consecutive exponents are used and that the representation is finite.

E.g:  $2 = \varphi + \varphi^{-2}$  and  $3 = \varphi^2 + \varphi^{-2}$

To represent this sum of powers of  $\varphi$  we use a string of 0's and 1's with a point to indicate where the negative exponents start.

We call this the representation in the **phigital numberbase**.

So  $1 = 1_{\varphi}$ ,  $2 = 10.01_{\varphi}$ ,  $3 = 100.01_{\varphi}$  and  $14 = 100100.001001_{\varphi}$ .

The strings representing 1, 2 and 14 in the phigital number base are palindromic, while the string representing 3 is not. (the phigital point is not the middle character).

The sum of the positive integers not exceeding 1000 whose phigital representation is palindromic is 4345.

Find the sum of the positive integers not exceeding  $10^{10}$  whose phigital representation is palindromic.

### Problem 474: Last digits of divisors

For a positive integer  $n$  and digits  $d$ , we define  $F(n, d)$  as the number of the divisors of  $n$  whose last digits equal  $d$ .

For example,  $F(84, 4) = 3$ . Among the divisors of 84 (1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84), three of them (4, 14, 84) have the last digit 4.

We can also verify that  $F(12!, 12) = 11$  and  $F(50!, 123) = 17888$ .

Find  $F(10^6!, 65432)$  modulo  $(10^{16} + 61)$ .

#### Problem 475: Music festival

$12n$  musicians participate at a music festival. On the first day, they form  $3n$  quartets and practice all day.

It is a disaster. At the end of the day, all musicians decide they will never again agree to play with any member of their quartet.

On the second day, they form  $4n$  trios, each musician avoiding his previous quartet partners.

Let  $f(12n)$  be the number of ways to organize the trios amongst the  $12n$  musicians.

You are given  $f(12) = 576$  and  $f(24) \bmod 1\,000\,000\,007 = 509089824$ .

Find  $f(600) \bmod 1\,000\,000\,007$ .

#### Problem 476: Circle Packing II

Let  $R(a, b, c)$  be the maximum area covered by three non-overlapping circles inside a triangle with edge lengths  $a, b$  and  $c$ .

Let  $S(n)$  be the average value of  $R(a, b, c)$  over all integer triplets  $(a, b, c)$  such that  $1 \leq a \leq b \leq c < a + b \leq n$

You are given  $S(2) = R(1, 1, 1) \approx 0.31998$ ,  $S(5) \approx 1.25899$ .

Find  $S(1803)$  rounded to 5 decimal places behind the decimal point.

#### Problem 477: Number Sequence Game

The number sequence game starts with a sequence  $S$  of  $N$  numbers written on a line.

Two players alternate turns. At his turn, a player must select and remove either the first or the last number remaining in the sequence.

The player score is the sum of all the numbers he has taken. Each player attempts to maximize his own sum.

If  $N = 4$  and  $S = \{1, 2, 10, 3\}$ , then each player maximizes his score as follows:

- Player 1: removes the first number (1)
- Player 2: removes the last number from the remaining sequence (3)
- Player 1: removes the last number from the remaining sequence (10)
- Player 2: removes the remaining number (2)

Player 1 score is  $1 + 10 = 11$ .

Let  $F(N)$  be the score of player 1 if both players follow the optimal strategy for the sequence  $S = \{s_1, s_2, \dots, s_N\}$  defined as:

- $s_1 = 0$
- $s_{j+1} = (s_j^2 + 45) \bmod 1\,000\,000\,007$

The sequence begins with  $S = \{0, 45, 2070, 4284945, 753524550, 478107844, 894218625, \dots\}$ .

You are given  $F(2) = 45$ ,  $F(4) = 4284990$ ,  $F(100) = 26365463243$ ,  $F(10^4) = 2495838522951$ .

Find  $F(10^8)$ .

### Problem 478: Mixtures

Let us consider **mixtures** of three substances: **A**, **B** and **C**. A mixture can be described by a ratio of the amounts of **A**, **B**, and **C** in it, i.e.,  $(a : b : c)$ . For example, a mixture described by the ratio  $(2 : 3 : 5)$  contains 20% **A**, 30% **B** and 50% **C**.

For the purposes of this problem, we cannot separate the individual components from a mixture. However, we can combine different amounts of different mixtures to form mixtures with new ratios.

For example, say we have three mixtures with ratios  $(3 : 0 : 2)$ ,  $(3 : 6 : 11)$  and  $(3 : 3 : 4)$ . By mixing 10 units of the first, 20 units of the second and 30 units of the third, we get a new mixture with ratio  $(6 : 5 : 9)$ , since:

$$(10 \cdot \frac{3}{5} + 20 \cdot \frac{3}{20} + 30 \cdot \frac{3}{10} : 10 \cdot \frac{0}{5} + 20 \cdot \frac{6}{20} + 30 \cdot \frac{3}{10} : 10 \cdot \frac{2}{5} + 20 \cdot \frac{11}{20} + 30 \cdot \frac{4}{10}) = (18 : 15 : 27) = (6 : 5 : 9)$$

However, with the same three mixtures, it is impossible to form the ratio  $(3 : 2 : 1)$ , since the amount of **B** is always less than the amount of **C**.

Let  $n$  be a positive integer. Suppose that for every triple of integers  $a, b, c$  with  $0 \leq a, b, c \leq n$  and  $\gcd(a, b, c) = 1$ , we have a mixture with ratio  $(a : b : c)$ . Let  $M(n)$  be the set of all such mixtures.

For example,  $M(2)$  contains the 19 mixtures with the following ratios:

$\{(0 : 0 : 1), (0 : 1 : 0), (0 : 1 : 1), (0 : 1 : 2), (0 : 2 : 1),$   
 $(1 : 0 : 0), (1 : 0 : 1), (1 : 0 : 2), (1 : 1 : 0), (1 : 1 : 1),$   
 $(1 : 1 : 2), (1 : 2 : 0), (1 : 2 : 1), (1 : 2 : 2), (2 : 0 : 1),$   
 $(2 : 1 : 0), (2 : 1 : 1), (2 : 1 : 2), (2 : 2 : 1)\}.$

Let  $E(n)$  be the number of subsets of  $M(n)$  which can produce the mixture with ratio  $(1 : 1 : 1)$ , i.e., the mixture with equal parts **A**, **B** and **C**.

We can verify that  $E(1) = 103$ ,  $E(2) = 520447$ ,  $E(10) \bmod 11^8 = 82608406$  and  $E(500) \bmod 11^8 = 13801403$ .

Find  $E(10\,000\,000) \bmod 11^8$ .

### Problem 479: Roots on the Rise

Let  $a_k, b_k$ , and  $c_k$  represent the three solutions (real or complex numbers) to the expression  $1x = (k/x)^2(k+x^2) - kx$ .

For instance, for  $k = 5$ , we see that  $\{a_5, b_5, c_5\}$  is approximately  $\{5.727244, -0.363622+2.057397i, -0.363622-2.057397i\}$ .

Let  $S(n) = \sum (a_k + b_k)^p (b_k + c_k)^p (c_k + a_k)^p$  for all integers  $p, k$  such that  $1 \leq p, k \leq n$ .

Interestingly,  $S(n)$  is always an integer. For example,  $S(4) = 51160$ .

Find  $S(10^6)$  modulo  $1\,000\,000\,007$ .

## Problem 480: The Last Question

Consider all the words which can be formed by selecting letters, in any order, from the phrase:

**thereisasyetinsufficientdataforameaningfulanswer**

Suppose those with 15 letters or less are listed in **alphabetical order** and numbered sequentially starting at 1. The list would include:

```
1 : a
2 : aa
3 : aaa
4 : aaaa
5 : aaaaa
6 : aaaaaa
7 : aaaaaac
8 : aaaaaacd
9 : aaaaaacde
10 : aaaaaacdee
11 : aaaaaacdeee
12 : aaaaaacdeeee
13 : aaaaaacdeeeee
14 : aaaaaacdeeeeee
15 : aaaaaacdeeeeeef
16 : aaaaaacdeeeeeeg
17 : aaaaaacdeeeeeeh
...
28 : aaaaaacdeeeeeey
29 : aaaaaacdeeeeeef
30 : aaaaaacdeeeeeefe
...
115246685191495242: euleoywuttttsss
115246685191495243: euler
115246685191495244: eulera
...
525069350231428029: ywuuttttssssrrr
```

Define  $P(w)$  as the position of the word  $w$ .

Define  $W(p)$  as the word in position  $p$ .

We can see that  $P(w)$  and  $W(p)$  are inverses:  $P(W(p)) = p$  and  $W(P(w)) = w$ .

Examples:

```
W(10) = aaaaaacdee
P(aaaaaacdee) = 10
W(115246685191495243) = euler
P(euler) = 115246685191495243
```

Find  $W(P(\text{legionary}) + P(\text{calorimeters}) - P(\text{annihilate}) + P(\text{orchestrated}) - P(\text{fluttering}))$ .  
Give your answer using lowercase characters (no punctuation or space).



### Problem 481: Chef Showdown

A group of chefs (numbered #1, #2, etc) participate in a turn-based strategic cooking competition. On each chef's turn, he/she cooks up a dish to the best of his/her ability and gives it to a separate panel of judges for taste-testing. Let  $S(k)$  represent chef # $k$ 's skill level (which is publicly known). More specifically,  $S(k)$  is the probability that chef # $k$ 's dish will be assessed favorably by the judges (on any/all turns). If the dish receives a favorable rating, then the chef must choose one other chef to be eliminated from the competition. The last chef remaining in the competition is the winner.

The game always begins with chef #1, with the turn order iterating sequentially over the rest of the chefs still in play. Then the cycle repeats from the lowest-numbered chef. All chefs aim to optimize their chances of winning within the rules as stated, assuming that the other chefs behave in the same manner. In the event that a chef has more than one equally-optimal elimination choice, assume that the chosen chef is always the one with the next-closest turn.

Define  $W_n(k)$  as the probability that chef # $k$  wins in a competition with  $n$  chefs. If we have  $S(1) = 0.25$ ,  $S(2) = 0.5$ , and  $S(3) = 1$ , then  $W_3(1) = 0.29375$ .

Going forward, we assign  $S(k) = F_k/F_{n+1}$  over all  $1 \leq k \leq n$ , where  $F_k$  is a Fibonacci number:  $F_k = F_{k-1} + F_{k-2}$  with base cases  $F_1 = F_2 = 1$ . Then, for example, when considering a competition with  $n = 7$  chefs, we have  $W_7(1) = 0.08965042$ ,  $W_7(2) = 0.20775702$ ,  $W_7(3) = 0.15291406$ ,  $W_7(4) = 0.14554098$ ,  $W_7(5) = 0.15905291$ ,  $W_7(6) = 0.10261412$ , and  $W_7(7) = 0.14247050$ , rounded to 8 decimal places each.

Let  $E(n)$  represent the expected number of dishes cooked in a competition with  $n$  chefs. For instance,  $E(7) = 42.28176050$ .

Find  $E(14)$  rounded to 8 decimal places.

### Problem 482: The incenter of a triangle

ABC is an integer sided triangle with incenter I and perimeter p. The segments IA, IB and IC have integral length as well.

Let  $L = p + |IA| + |IB| + |IC|$ .

Let  $S(P) = \sum L$  for all such triangles where  $p \leq P$ . For example,  $S(10^3) = 3619$ .

Find  $S(10^7)$ .

### Problem 483: Repeated permutation

We define a *permutation* as an operation that rearranges the order of the elements  $\{1, 2, 3, \dots, n\}$ . There are  $n!$  such permutations, one of which leaves the elements in their initial order. For  $n = 3$  we have  $3! = 6$  permutations:

- $P_1$  = keep the initial order
- $P_2$  = exchange the 1<sup>st</sup> and 2<sup>nd</sup> elements
- $P_3$  = exchange the 1<sup>st</sup> and 3<sup>rd</sup> elements
- $P_4$  = exchange the 2<sup>nd</sup> and 3<sup>rd</sup> elements
- $P_5$  = rotate the elements to the right
- $P_6$  = rotate the elements to the left

If we select one of these permutations, and we re-apply the same permutation repeatedly, we eventually restore the initial order.

For a permutation  $P_i$ , let  $f(P_i)$  be the number of steps required to restore the initial order by applying the permutation  $P$  repeatedly.

For  $n = 3$ , we obtain:

- $f(P_1) = 1 : (1,2,3) \rightarrow (1,2,3)$
- $f(P_2) = 2 : (1,2,3) \rightarrow (2,1,3) \rightarrow (1,2,3)$
- $f(P_3) = 2 : (1,2,3) \rightarrow (3,2,1) \rightarrow (1,2,3)$
- $f(P_4) = 2 : (1,2,3) \rightarrow (1,3,2) \rightarrow (1,2,3)$
- $f(P_5) = 3 : (1,2,3) \rightarrow (3,1,2) \rightarrow (2,3,1) \rightarrow (1,2,3)$
- $f(P_6) = 3 : (1,2,3) \rightarrow (2,3,1) \rightarrow (3,1,2) \rightarrow (1,2,3)$

Let  $g(n)$  be the average value of  $f^2(P_i)$  over all permutations  $P_i$  of length  $n$ .

$$g(3) = (1^2 + 2^2 + 2^2 + 2^2 + 3^2 + 3^2)/3! = 31/6 \approx 5.166666667e0$$

$$g(5) = 2081/120 \approx 1.734166667e1$$

$$g(20) = 12422728886023769167301/2432902008176640000 \approx 5.106136147e3$$

Find  $g(350)$  and write the answer in scientific notation rounded to 10 significant digits, using a lowercase e to separate mantissa and exponent, as in the examples above.

### Problem 484: Arithmetic Derivative

The **arithmetic derivative** is defined by

- $p' = 1$  for any prime  $p$
- $(ab)' = a'b + ab'$  for all integers  $a, b$  (Leibniz rule)

For example,  $20' = 24$

Find  $\sum \text{gcd}(k, k')$  for  $1 < k \leq 5 \cdot 10^{15}$

Note:  $\text{gcd}(x, y)$  denotes the greatest common divisor of  $x$  and  $y$ .

### Problem 485: Maximum number of divisors

Let  $d(n)$  be the number of divisors of  $n$ .

Let  $M(n, k)$  be the maximum value of  $d(j)$  for  $n \leq j \leq n+k-1$ .

Let  $S(u, k)$  be the sum of  $M(n, k)$  for  $1 \leq n \leq u-k+1$ .

You are given that  $S(1000, 10) = 17176$ .

Find  $S(100\,000\,000, 100\,000)$ .

#### Problem 486: Palindrome-containing strings

Let  $F_5(n)$  be the number of strings  $S$  such that:

- $S$  consists only of '0's and '1's,
- $S$  has length at most  $n$ , and
- $S$  contains a palindromic substring of length at least 5.

For example,  $F_5(4) = 0$ ,  $F_5(5) = 8$ ,  $F_5(6) = 42$  and  $F_5(11) = 3844$ .

Let  $D(L)$  be the number of integers  $n$  such that  $5 \leq n \leq L$  and  $F_5(n)$  is divisible by 87654321.

For example,  $D(10^7) = 0$  and  $D(5 \cdot 10^9) = 51$ .

Find  $D(10^{18})$ .

#### Problem 487: Sums of power sums

Let  $f_k(n)$  be the sum of the  $k^{\text{th}}$  powers of the first  $n$  positive integers.

For example,  $f_2(10) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 = 385$ .

Let  $S_k(n)$  be the sum of  $f_k(i)$  for  $1 \leq i \leq n$ . For example,  $S_4(100) = 35375333830$ .

What is  $\sum (S_{10000}(10^{12}) \bmod p)$  over all primes  $p$  between  $2 \cdot 10^9$  and  $2 \cdot 10^9 + 2000$ ?

#### Problem 488: Unbalanced Nim

Alice and Bob have enjoyed playing **Nim** every day. However, they finally got bored of playing ordinary three-heap Nim. So, they added an extra rule:

- Must not make two heaps of the same size.

The triple  $(a, b, c)$  indicates the size of three heaps.

Under this extra rule,  $(2, 4, 5)$  is one of the losing positions for the next player.

To illustrate:

- Alice moves to  $(2, 4, 3)$
- Bob moves to  $(0, 4, 3)$
- Alice moves to  $(0, 2, 3)$
- Bob moves to  $(0, 2, 1)$

Unlike ordinary three-heap Nim,  $(0, 1, 2)$  and its permutations are the end states of this game.

For an integer  $N$ , we define  $F(N)$  as the sum of  $a+b+c$  for all the losing positions for the next player, with  $0 < a < b < c < N$ .

For example,  $F(8) = 42$ , because there are 4 losing positions for the next player,  $(1, 3, 5)$ ,  $(1, 4, 6)$ ,  $(2, 3, 6)$  and  $(2, 4, 5)$ .

We can also verify that  $F(128) = 496062$ .

Find the last 9 digits of  $F(10^{18})$ .

#### Problem 489: Common factors between two sequences

Let  $G(a, b)$  be the smallest non-negative integer  $n$  for which  $\gcd(n^3 + b, (n + a)^3 + b)$  is maximized.

For example,  $G(1, 1) = 5$  because  $\gcd(n^3 + 1, (n + 1)^3 + 1)$  reaches its maximum value of 7 for  $n = 5$ , and is smaller for  $0 \leq n < 5$ .

Let  $H(m, n) = \sum G(a, b)$  for  $1 \leq a \leq m$ ,  $1 \leq b \leq n$ .

You are given  $H(5, 5) = 128878$  and  $H(10, 10) = 32936544$ .

Find  $H(18, 1900)$ .

### Problem 490: Jumping frog

There are  $n$  stones in a pond, numbered 1 to  $n$ . Consecutive stones are spaced one unit apart.

A frog sits on stone 1. He wishes to visit each stone exactly once, stopping on stone  $n$ . However, he can only jump from one stone to another if they are at most 3 units apart. In other words, from stone  $i$ , he can reach a stone  $j$  if  $1 \leq j \leq n$  and  $j$  is in the set  $\{i-3, i-2, i-1, i+1, i+2, i+3\}$ .

Let  $f(n)$  be the number of ways he can do this. For example,  $f(6) = 14$ , as shown below:

1  $\rightarrow$  2  $\rightarrow$  3  $\rightarrow$  4  $\rightarrow$  5  $\rightarrow$  6  
1  $\rightarrow$  2  $\rightarrow$  3  $\rightarrow$  5  $\rightarrow$  4  $\rightarrow$  6  
1  $\rightarrow$  2  $\rightarrow$  4  $\rightarrow$  3  $\rightarrow$  5  $\rightarrow$  6  
1  $\rightarrow$  2  $\rightarrow$  4  $\rightarrow$  5  $\rightarrow$  3  $\rightarrow$  6  
1  $\rightarrow$  2  $\rightarrow$  5  $\rightarrow$  3  $\rightarrow$  4  $\rightarrow$  6  
1  $\rightarrow$  2  $\rightarrow$  5  $\rightarrow$  4  $\rightarrow$  3  $\rightarrow$  6  
1  $\rightarrow$  3  $\rightarrow$  2  $\rightarrow$  4  $\rightarrow$  5  $\rightarrow$  6  
1  $\rightarrow$  3  $\rightarrow$  2  $\rightarrow$  5  $\rightarrow$  4  $\rightarrow$  6  
1  $\rightarrow$  3  $\rightarrow$  4  $\rightarrow$  2  $\rightarrow$  5  $\rightarrow$  6  
1  $\rightarrow$  3  $\rightarrow$  5  $\rightarrow$  2  $\rightarrow$  4  $\rightarrow$  6  
1  $\rightarrow$  4  $\rightarrow$  2  $\rightarrow$  3  $\rightarrow$  5  $\rightarrow$  6  
1  $\rightarrow$  4  $\rightarrow$  2  $\rightarrow$  5  $\rightarrow$  3  $\rightarrow$  6  
1  $\rightarrow$  4  $\rightarrow$  3  $\rightarrow$  2  $\rightarrow$  5  $\rightarrow$  6  
1  $\rightarrow$  4  $\rightarrow$  5  $\rightarrow$  2  $\rightarrow$  3  $\rightarrow$  6

Other examples are  $f(10) = 254$  and  $f(40) = 1439682432976$ .

Let  $S(L) = \sum f(n)^3$  for  $1 \leq n \leq L$ .

Examples:

$$S(10) = 18230635$$

$$S(20) = 104207881192114219$$

$$S(1\,000) \bmod 10^9 = 225031475$$

$$S(1\,000\,000) \bmod 10^9 = 363486179$$

Find  $S(10^{14}) \bmod 10^9$ .

### Problem 491: Double pandigital number divisible by 11

We call a positive integer *double pandigital* if it uses all the digits 0 to 9 exactly twice (with no leading zero). For example, 40561817703823564929 is one such number.

How many double pandigital numbers are divisible by 11?

### Problem 492: Exploding sequence

Define the sequence  $a_1, a_2, a_3, \dots$  as:

- $a_1 = 1$
- $a_{n+1} = 6a_n^2 + 10a_n + 3$  for  $n \geq 1$ .

Examples:

$$a_3 = 2359$$

$$a_6 = 269221280981320216750489044576319$$

$$a_6 \bmod 1\,000\,000\,007 = 203064689$$

$$a_{100} \bmod 1\,000\,000\,007 = 456482974$$

Define  $B(X, Y, n)$  as  $\sum (a_n \bmod p)$  for every prime  $p$  such that  $X \leq p \leq X+Y$ .

Examples:

$$B(10^9, 10^3, 10^3) = 23674718882$$

$$B(10^9, 10^3, 10^{15}) = 20731563854$$

Find  $B(10^9, 10^7, 10^{15})$ .

#### Problem 493: Under The Rainbow

70 colored balls are placed in an urn, 10 for each of the seven rainbow colors.

What is the expected number of distinct colors in 20 randomly picked balls?

Give your answer with nine digits after the decimal point (a.bcd efghij).

#### Problem 494: Collatz prefix families

The Collatz sequence is defined as:  $a_{i+1} = \begin{cases} \frac{a_i}{2} & \text{if } a_i \text{ is even} \\ 3a_i + 1 & \text{if } a_i \text{ is odd} \end{cases}$ .

The Collatz conjecture states that starting from any positive integer, the sequence eventually reaches the cycle 1,4,2,1....

We shall define the sequence prefix  $p(n)$  for the Collatz sequence starting with  $a_1 = n$  as the sub-sequence of all numbers not a power of 2 ( $2^0=1$  is considered a power of 2 for this problem). For example:

$p(13) = \{13, 40, 20, 10, 5\}$

$p(8) = \{\}$

Any number invalidating the conjecture would have an infinite length sequence prefix.

Let  $S_m$  be the set of all sequence prefixes of length  $m$ . Two sequences  $\{a_1, a_2, \dots, a_m\}$  and  $\{b_1, b_2, \dots, b_m\}$  in  $S_m$  are said to belong to the same prefix family if  $a_i < a_j$  if and only if  $b_i < b_j$  for all  $1 \leq i, j \leq m$ .

For example, in  $S_4$ ,  $\{6, 3, 10, 5\}$  is in the same family as  $\{454, 227, 682, 341\}$ , but not  $\{113, 340, 170, 85\}$ .

Let  $f(m)$  be the number of distinct prefix families in  $S_m$ .

You are given  $f(5) = 5$ ,  $f(10) = 55$ ,  $f(20) = 6771$ .

Find  $f(90)$ .

#### Problem 495: Writing $n$ as the product of $k$ distinct positive integers

Let  $W(n, k)$  be the number of ways in which  $n$  can be written as the product of  $k$  distinct positive integers.

For example,  $W(144, 4) = 7$ . There are 7 ways in which 144 can be written as a product of 4 distinct positive integers:

- $144 = 1 \times 2 \times 4 \times 18$
- $144 = 1 \times 2 \times 8 \times 9$
- $144 = 1 \times 2 \times 3 \times 24$
- $144 = 1 \times 2 \times 6 \times 12$
- $144 = 1 \times 3 \times 4 \times 12$
- $144 = 1 \times 3 \times 6 \times 8$
- $144 = 2 \times 3 \times 4 \times 6$

Note that permutations of the integers themselves are not considered distinct.

Furthermore,  $W(100!, 10)$  modulo 1 000 000 007 = 287549200.

Find  $W(10000!, 30)$  modulo 1 000 000 007.

#### Problem 496: Incenter and circumcenter of triangle

Given an integer sided triangle ABC:

Let I be the incenter of ABC.

Let D be the intersection between the line AI and the circumcircle of ABC ( $A \neq D$ ).

We define  $F(L)$  as the sum of BC for the triangles ABC that satisfy  $AC = DI$  and  $BC \perp AI$ .

For example,  $F(15) = 45$  because the triangles ABC with  $(BC, AC, AB) = (6, 4, 5)$ ,  $(12, 8, 10)$ ,  $(12, 9, 7)$ ,  $(15, 9, 16)$  satisfy the conditions.

Find  $F(10^9)$ .

### Problem 497: Drunken Tower of Hanoi

Bob is very familiar with the famous mathematical puzzle/game, "Tower of Hanoi," which consists of three upright rods and disks of different sizes that can slide onto any of the rods. The game begins with a stack of  $n$  disks placed on the leftmost rod in descending order by size. The objective of the game is to move all of the disks from the leftmost rod to the rightmost rod, given the following restrictions:

1. Only one disk can be moved at a time.
2. A valid move consists of taking the top disk from one stack and placing it onto another stack (or an empty rod).
3. No disk can be placed on top of a smaller disk.

Moving on to a variant of this game, consider a long room  $k$  units (square tiles) wide, labeled from 1 to  $k$  in ascending order. Three rods are placed at squares  $a$ ,  $b$ , and  $c$ , and a stack of  $n$  disks is placed on the rod at square  $a$ .

Bob begins the game standing at square  $b$ . His objective is to play the Tower of Hanoi game by moving all of the disks to the rod at square  $c$ . However, Bob can only pick up or set down a disk if he is on the same square as the rod / stack in question.

Unfortunately, Bob is also drunk. On a given move, Bob will either stumble one square to the left or one square to the right with equal probability, unless Bob is at either end of the room, in which case he can only move in one direction. Despite Bob's inebriated state, he is still capable of following the rules of the game itself, as well as choosing when to pick up or put down a disk.

The following animation depicts a side-view of a sample game for  $n = 3$ ,  $k = 7$ ,  $a = 2$ ,  $b = 4$ , and  $c = 6$ :



Let  $E(n, k, a, b, c)$  be the expected number of squares that Bob travels during a single optimally-played game. A game is played optimally if the number of disk-pickups is minimized.

Interestingly enough, the result is always an integer. For example,  $E(2, 5, 1, 3, 5) = 60$  and  $E(3, 20, 4, 9, 17) = 2358$ .

Find the last nine digits of  $\sum_{1 \leq n \leq 10000} E(n, 10^n, 3^n, 6^n, 9^n)$ .

### Problem 498: Remainder of polynomial division

For positive integers  $n$  and  $m$ , we define two polynomials  $F_n(x) = x^n$  and  $G_m(x) = (x-1)^m$ .

We also define a polynomial  $R_{n,m}(x)$  as the remainder of the division of  $F_n(x)$  by  $G_m(x)$ .

For example,  $R_{6,3}(x) = 15x^2 - 24x + 10$ .

Let  $C(n, m, d)$  be the absolute value of the coefficient of the  $d$ -th degree term of  $R_{n,m}(x)$ .

We can verify that  $C(6, 3, 1) = 24$  and  $C(100, 10, 4) = 227197811615775$ .

Find  $C(10^{13}, 10^{12}, 10^4) \bmod 999999937$ .

### Problem 499: St. Petersburg Lottery

A gambler decides to participate in a special lottery. In this lottery the gambler plays a series of one or more games.

Each game costs  $m$  pounds to play and starts with an initial pot of 1 pound. The gambler flips an unbiased coin. Every time a head appears, the pot is doubled and the gambler continues. When a tail appears, the game ends and the gambler collects the current value of the pot. The gambler is certain to win at least 1 pound, the starting value of the pot, at the cost of  $m$  pounds, the initial fee.

The gambler cannot continue to play if his fortune falls below  $m$  pounds. Let  $p_m(S)$  denote the probability that the gambler will never run out of money in this lottery given his initial fortune  $S$  and the cost per game  $m$ .

For example  $p_2(2) \approx 0.2522$ ,  $p_2(5) \approx 0.6873$  and  $p_6(10\,000) \approx 0.9952$  (note:  $p_m(S) = 0$  for  $S < m$ ).

Find  $p_{15}(10^9)$  and give your answer rounded to 7 decimal places behind the decimal point in the form 0.abcdefg.

### Problem 500: Problem 500!!!

The number of divisors of 120 is 16.  
In fact 120 is the smallest number having 16 divisors.

Find the smallest number with  $2^{500500}$  divisors.  
Give your answer modulo 500500507.

### Problem 501: Eight Divisors

The eight divisors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24. The ten numbers not exceeding 100 having exactly eight divisors are 24, 30, 40, 42, 54, 56, 66, 70, 78 and 88. Let  $f(n)$  be the count of numbers not exceeding  $n$  with exactly eight divisors. You are given  $f(100) = 10$ ,  $f(1000) = 180$  and  $f(10^6) = 224427$ . Find  $f(10^{12})$ .

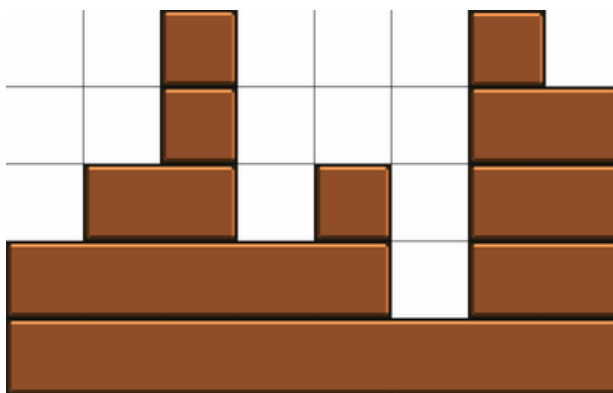
### Problem 502: Counting Castles

We define a *block* to be a rectangle with a height of 1 and an integer-valued length. Let  $acastle$  be a configuration of stacked blocks.

Given a game grid that is  $W$  units wide and  $h$  units tall, a castle is generated according to the following rules:

1. Blocks can be placed on top of other blocks as long as nothing sticks out past the edges or hangs out over open space.
2. All blocks are aligned/snapped to the grid.
3. Any two neighboring blocks on the same row have at least one unit of space between them.
4. The bottom row is occupied by a block of length  $W$ .
5. The maximum achieved height of the entire castle is exactly  $h$ .
6. The castle is made from an even number of blocks.

The following is a sample castle for  $W=8$  and  $h=5$ :



Let  $F(W, h)$  represent the number of valid castles, given grid parameters  $W$  and  $h$ .

For example,  $F(4, 2) = 10$ ,  $F(13, 10) = 3729050610636$ ,  $F(10, 13) = 37959702514$ , and  $F(100, 100) \bmod 1\,000\,000\,007 = 841913936$ .

Find  $(F(10^{12}, 100) + F(10000, 10000) + F(100, 10^{12})) \bmod 1\,000\,000\,007$ .

### Problem 503: Compromise or persist

Alice is playing a game with  $n$  cards numbered 1 to  $n$ .

A game consists of iterations of the following steps.

- (1) Alice picks one of the cards at random.
- (2) Alice cannot see the number on it. Instead, Bob, one of her friends, sees the number and tells Alice how many previously-seen numbers are bigger than the number which he is seeing.
- (3) Alice can end or continue the game. If she decides to end, the number becomes her score. If she decides to continue, the card is removed from the game and she returns to (1). If there is no card left, she is forced to end the game.

Let  $F(n)$  be the Alice's expected score if she takes the optimized strategy to minimize her score.

For example,  $F(3) = 5/3$ . At the first iteration, she should continue the game. At the second iteration, she should end the game if Bob says that one previously-seen number is bigger than the number which he is seeing, otherwise she should continue the game.

We can also verify that  $F(4) = 15/8$  and  $F(10) \approx 2.5579365079$ .

Find  $F(10^6)$ . Give your answer rounded to 10 decimal places behind the decimal point.

### Problem 504: Square on the Inside

Let  $ABCD$  be a quadrilateral whose vertices are lattice points lying on the coordinate axes as follows:

$A(a, 0)$ ,  $B(0, b)$ ,  $C(-c, 0)$ ,  $D(0, -d)$ , where  $1 \leq a, b, c, d \leq m$  and  $a, b, c, d, m$  are integers.

It can be shown that for  $m = 4$  there are exactly 256 valid ways to construct  $ABCD$ . Of these 256 quadrilaterals, 42 of them strictly contain a square number of lattice points.

How many quadrilaterals  $ABCD$  strictly contain a square number of lattice points for  $m = 100$ ?

### Problem 505: Bidirectional Recurrence

Let:

$$\begin{array}{l} x(0)=0 \quad x(1)=1 \quad x(2k)=(3x(k)+2x(\lfloor \frac{k}{2} \rfloor)) \pmod{2^{60}} \text{ for } k \geq 1 \\ \text{where } \lfloor \cdot \rfloor \text{ is the floor function} \\ x(2k+1)=(2x(k)+3x(\lfloor \frac{k}{2} \rfloor)) \pmod{2^{60}} \text{ for } k \geq 1 \\ y_n(k)=\left\lfloor \begin{array}{l} x(k) \text{ if } k \geq n \\ 2^{60} - 1 - \max(y_n(2k), y_n(2k+1)) \text{ if } k < n \end{array} \right\rfloor \end{array}$$

You are given:

$$\begin{array}{l} x(2)=3 \quad x(3)=2 \quad x(4)=11 \quad y_4(4)=11 \quad y_4(3)=2^{60}-9 \quad y_4(2)=2^{60}-12 \quad y_4(1)=A(4)=8 \\ A(10)=2^{60}-34 \quad A(10^3)=101881 \end{array}$$

Find  $A(10^{12})$ .

### Problem 506: Clock sequence

Consider the infinite repeating sequence of digits:  
1234321234321234321...

Amazingly, you can break this sequence of digits into a sequence of integers such that the sum of the digits in the  $n$ 'th value is  $n$ .

The sequence goes as follows:

1, 2, 3, 4, 32, 123, 43, 2123, 432, 1234, 32123, ...

Let  $v_n$  be the  $n$ 'th value in this sequence. For example,  $v_2 = 2$ ,  $v_5 = 32$  and  $v_{11} = 32123$ .

Let  $S(n)$  be  $v_1 + v_2 + \dots + v_n$ . For example,  $S(11) = 36120$ , and  $S(1000) \bmod 123454321 = 18232686$ .

Find  $S(10^{14}) \bmod 123454321$ .



### Problem 507: Shortest Lattice Vector

Let  $t_n$  be the **tribonacci numbers** defined as:

$$t_0 = t_1 = 0;$$

$$t_2 = 1;$$

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} \text{ for } n \geq 3$$

$$\text{and let } r_n = t_n \bmod 10^7.$$

For each pair of Vectors  $V_n = (v_1, v_2, v_3)$  and  $W_n = (w_1, w_2, w_3)$  with  $v_1 = r_{12n-11} - r_{12n-10}$ ,  $v_2 = r_{12n-9} + r_{12n-8}$ ,  $v_3 = r_{12n-7} \cdot r_{12n-6}$  and

$$w_1 = r_{12n-5} - r_{12n-4}, w_2 = r_{12n-3} + r_{12n-2}, w_3 = r_{12n-1} \cdot r_{12n}$$

we define  $S(n)$  as the minimal value of the manhattan length of the vector  $D = k \cdot V_n + l \cdot W_n$  measured as  $|k \cdot v_1 + l \cdot w_1| + |k \cdot v_2 + l \cdot w_2| + |k \cdot v_3 + l \cdot w_3|$  for any integers  $k$  and  $l$  with  $(k, l) \neq (0, 0)$ .

The first vector pair is  $(-1, 3, 28), (-11, 125, 40826)$ .

You are given that  $S(1) = 32$  and  $\sum_{n=1}^{10} S(n) = 130762273722$ .

Find  $\sum_{n=1}^{20000000} S(n)$ .

### Problem 508: Integers in base i-1

Consider the Gaussian integer  $i-1$ . A **base i-1 representation** of a Gaussian integer  $a+bi$  is a finite sequence of digits  $d_{n-1}d_{n-2} \dots d_1d_0$  such that:

- $a+bi = d_{n-1}(i-1)^{n-1} + d_{n-2}(i-1)^{n-2} + \dots + d_1(i-1) + d_0$
- Each  $d_k$  is in  $\{0, 1\}$
- There are no leading zeroes, i.e.  $d_{n-1} \neq 0$ , unless  $a+bi$  is itself 0

Here are base i-1 representations of a few Gaussian integers:

$$11+24i \rightarrow 111010110001101$$

$$24-11i \rightarrow 110010110011$$

$$8+0i \rightarrow 111000000$$

$$-5+0i \rightarrow 11001101$$

$$0+0i \rightarrow 0$$

Remarkably, every Gaussian integer has a unique base i-1 representation!

Define  $f(a+bi)$  as the number of 1s in the unique base i-1 representation of  $a+bi$ . For example,  $f(11+24i) = 9$  and  $f(24-11i) = 7$ .

Define  $B(L)$  as the sum of  $f(a+bi)$  for all integers  $a, b$  such that  $|a| \leq L$  and  $|b| \leq L$ . For example,  $B(500) = 10795060$ .

Find  $B(10^{15}) \bmod 1\,000\,000\,007$ .

### Problem 509: Divisor Nim

Anton and Bertrand love to play three pile Nim.

However, after a lot of games of Nim they got bored and changed the rules somewhat.

They may only take a number of stones from a pile that is a proper divisor of the number of stones present in the pile.

E.g. if a pile at a certain moment contains 24 stones they may take only 1, 2, 3, 4, 6, 8 or 12 stones from that pile.

So if a pile contains one stone they can't take the last stone from it as 1 isn't a proper divisor of 1.

The first player that can't make a valid move loses the game.

Of course both Anton and Bertrand play optimally.

The triple  $(a, b, c)$  indicates the number of stones in the three piles.

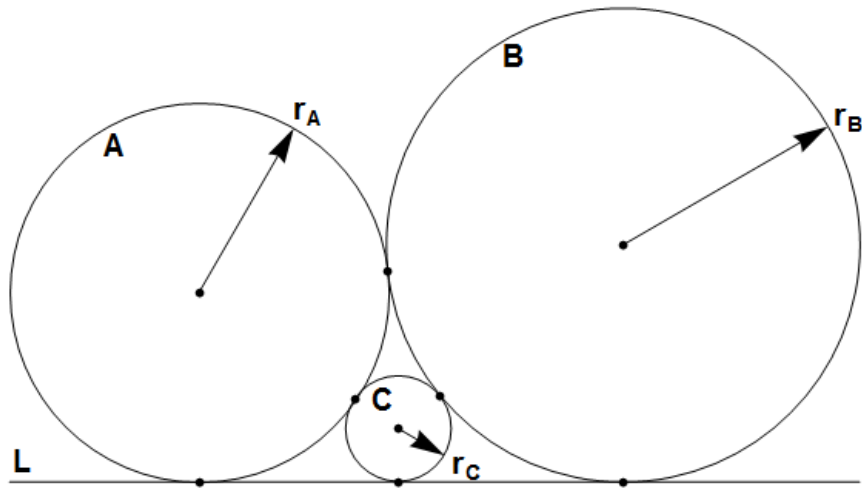
Let  $S(n)$  be the number of winning positions for the next player for  $1 \leq a, b, c \leq n$ .

$$S(10) = 692 \text{ and } S(100) = 735494.$$

Find  $S(123456787654321)$  modulo 1234567890.

### Problem 510: Tangent Circles

Circles A and B are tangent to each other and to line L at three distinct points. Circle C is inside the space between A, B and L, and tangent to all three. Let  $r_A$ ,  $r_B$  and  $r_C$  be the radii of A, B and C respectively.



Let  $S(n) = \sum r_A + r_B + r_C$ , for  $0 < r_A \leq r_B \leq n$  where  $r_A$ ,  $r_B$  and  $r_C$  are integers. The only solution for  $0 < r_A \leq r_B \leq 5$  is  $r_A = 4$ ,  $r_B = 4$  and  $r_C = 1$ , so  $S(5) = 4 + 4 + 1 = 9$ . You are also given  $S(100) = 3072$ .

Find  $S(10^9)$ .

### Problem 511: Sequences with nice divisibility properties

Let  $Seq(n, k)$  be the number of positive-integer sequences  $\{a_i\}_{1 \leq i \leq n}$  of length  $n$  such that:

- $n$  is divisible by  $a_i$  for  $1 \leq i \leq n$ , and
- $n + a_1 + a_2 + \dots + a_n$  is divisible by  $k$ .

Examples:

$Seq(3, 4) = 4$ , and the 4 sequences are:

$\{1, 1, 3\}$   
 $\{1, 3, 1\}$   
 $\{3, 1, 1\}$   
 $\{3, 3, 3\}$

$Seq(4, 11) = 8$ , and the 8 sequences are:

$\{1, 1, 1, 4\}$   
 $\{1, 1, 4, 1\}$   
 $\{1, 4, 1, 1\}$   
 $\{4, 1, 1, 1\}$   
 $\{2, 2, 2, 1\}$   
 $\{2, 2, 1, 2\}$   
 $\{2, 1, 2, 2\}$   
 $\{1, 2, 2, 2\}$

The last nine digits of  $Seq(1111, 24)$  are 840643584.

Find the last nine digits of  $Seq(1234567898765, 4321)$ .

### Problem 512: Sums of totients of powers

Let  $\varphi(n)$  be Euler's totient function.

Let  $f(n) = \left( \sum_{i=1}^n \varphi(n^i) \right) \bmod (n+1)$ .

Let  $g(n) = \sum_{i=1}^n f(i)$ .

$g(100) = 2007$ .

Find  $g(5 \times 10^8)$ .

### Problem 513: Integral median

ABC is an integral sided triangle with sides  $a \leq b \leq c$ .

$m_c$  is the median connecting C and the midpoint of AB.

$F(n)$  is the number of such triangles with  $c \leq n$  for which  $m_c$  has integral length as well.

$F(10) = 3$  and  $F(50) = 165$ .

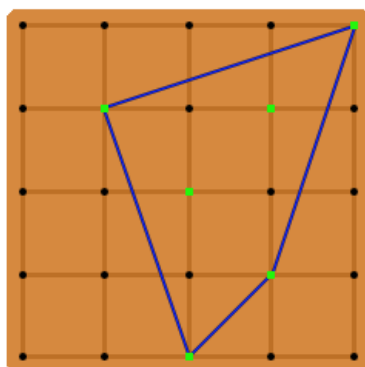
Find  $F(100000)$ .

### Problem 514: Geoboard Shapes

A **geoboard** (of order  $N$ ) is a square board with equally-spaced pins protruding from the surface, representing an integer point lattice for coordinates  $0 \leq x, y \leq N$ .

John begins with a pinless geoboard. Each position on the board is a hole that can be filled with a pin. John decides to generate a random integer between 1 and  $N+1$  (inclusive) for each hole in the geoboard. If the random integer is equal to 1 for a given hole, then a pin is placed in that hole.

After John is finished generating numbers for all  $(N+1)^2$  holes and placing any/all corresponding pins, he wraps a tight rubberband around the entire group of pins protruding from the board. Let  $S$  represent the shape that is formed.  $S$  can also be defined as the smallest convex shape that contains all the pins.



The above image depicts a sample layout for  $N = 4$ . The green markers indicate positions where pins have been placed, and the blue lines collectively represent the rubberband. For this particular arrangement,  $S$  has an area of 6. If there are fewer than three pins on the board (or if all pins are collinear),  $S$  can be assumed to have zero area.

Let  $E(N)$  be the expected area of  $S$  given a geoboard of order  $N$ . For example,  $E(1) = 0.18750$ ,  $E(2) = 0.94335$ , and  $E(10) = 55.03013$  when rounded to five decimal places each.

Calculate  $E(100)$  rounded to five decimal places.

### Problem 515: Dissonant Numbers

Let  $d(p,n,0)$  be the multiplicative inverse of  $n$  modulo prime  $p$ , defined as  $n \times d(p,n,0) = 1 \pmod p$ .

Let  $d(p,n,k) = \sum_{i=1}^n d(p,i,k-1)$  for  $k \geq 1$ .

Let  $D(a,b,k) = \sum (d(p,p-1,k) \pmod p)$  for all primes  $a \leq p < a+b$ .

You are given:

- $D(101,1,10) = 45$
- $D(10^3,10^2,10^2) = 8334$
- $D(10^6,10^3,10^3) = 38162302$

Find  $D(10^9,10^5,10^5)$ .

### Problem 516: 5-smooth totients

5-smooth numbers are numbers whose largest prime factor doesn't exceed 5.

5-smooth numbers are also called Hamming numbers.

Let  $S(L)$  be the sum of the numbers  $n$  not exceeding  $L$  such that Euler's totient function  $\phi(n)$  is a Hamming number.  
 $S(100)=3728$ .

Find  $S(10^{12})$ . Give your answer modulo  $2^{32}$ .

### Problem 517: A real recursion

For every real number  $a \geq 1$  is given the sequence  $g_a$  by:

$g_a(x)=1$  for  $x \leq a$

$g_a(x)=g_a(x-1)+g_a(x-a)$  for  $x \geq a$

$G(n)=g_{\sqrt{n}}(n)$

$G(90)=7564511$ .

Find  $\sum G(p)$  for  $p$  prime and  $10000000 \leq p \leq 10010000$

Give your answer modulo 1000000007.

### Problem 518: Prime triples and geometric sequences

Let  $S(n) = \sum a+b+c$  over all triples  $(a,b,c)$  such that:

- $a, b$ , and  $c$  are prime numbers.
- $a < b < c < n$ .
- $a+1, b+1$ , and  $c+1$  form a **geometric sequence**.

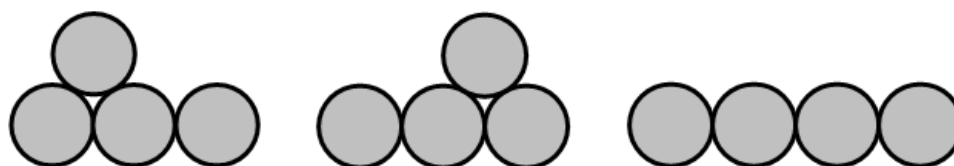
For example,  $S(100) = 1035$  with the following triples:

(2, 5, 11), (2, 11, 47), (5, 11, 23), (5, 17, 53), (7, 11, 17), (7, 23, 71), (11, 23, 47), (17, 23, 31), (17, 41, 97), (31, 47, 71), (71, 83, 97)

Find  $S(10^8)$ .

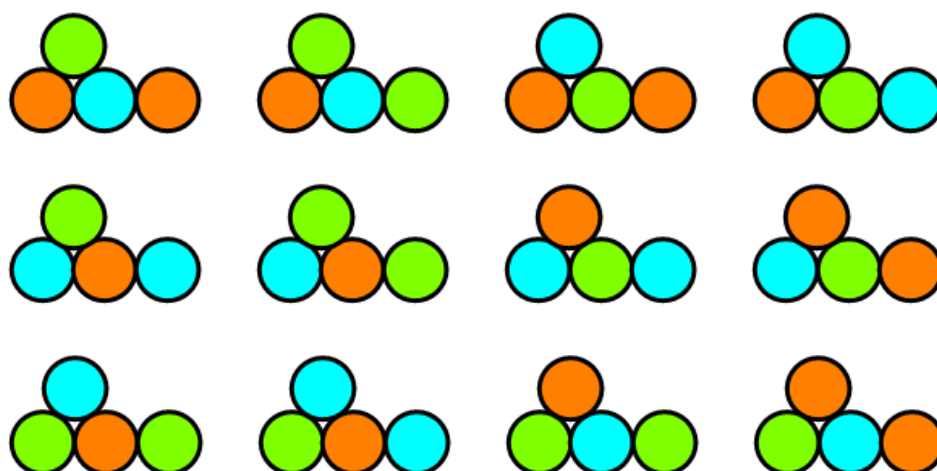
### Problem 519: Tricolored Coin Fountains

An arrangement of coins in one or more rows with the bottom row being a block without gaps and every coin in a higher row touching exactly two coins in the row below is called a **fountain** of coins. Let  $f(n)$  be the number of possible fountains with  $n$  coins. For 4 coins there are three possible arrangements:



Therefore  $f(4) = 3$  while  $f(10) = 78$ .

Let  $T(n)$  be the number of all possible colorings with three colors for all  $f(n)$  different fountains with  $n$  coins, given the condition that no two touching coins have the same color. Below you see the possible colorings for one of the three valid fountains for 4 coins:



You are given that  $T(4) = 48$  and  $T(10) = 17760$ .

Find the last 9 digits of  $T(20000)$ .

### Problem 520: Simbers

We define a *simber* to be a positive integer in which any odd digit, if present, occurs an odd number of times, and any even digit, if present, occurs an even number of times.

For example, 141221242 is a 9-digit simber because it has three 1's, four 2's and two 4's.

Let  $Q(n)$  be the count of all simbers with at most  $n$  digits.

You are given  $Q(7) = 287975$  and  $Q(100) \bmod 1\,000\,000\,123 = 123864868$ .

Find  $(\sum_{1 \leq u \leq 39} Q(2^u)) \bmod 1\,000\,000\,123$ .

### Problem 521: Smallest prime factor

Let  $\text{smpf}(n)$  be the smallest prime factor of  $n$ .

$\text{smpf}(91)=7$  because  $91=7 \times 13$  and  $\text{smpf}(45)=3$  because  $45=3 \times 3 \times 5$ .

Let  $S(n)$  be the sum of  $\text{smpf}(i)$  for  $2 \leq i \leq n$ .

E.g.  $S(100)=1257$ .

Find  $S(10^{12}) \bmod 10^9$ .

### Problem 522: Hilbert's Blackout

Despite the popularity of Hilbert's infinite hotel, Hilbert decided to try managing extremely large finite hotels, instead.

To cut costs, Hilbert wished to power the new hotel with his own special generator. Each floor would send power to the floor above it, with the top floor sending power back down to the bottom floor. That way, Hilbert could have the generator placed on any given floor (as he likes having the option) and have electricity flow freely throughout the entire hotel.

Unfortunately, the contractors misinterpreted the schematics when they built the hotel. They informed Hilbert that each floor sends power to another floor at random, instead. This may compromise Hilbert's freedom to have the generator placed anywhere, since blackouts could occur on certain floors.

For example, consider a sample flow diagram for a three-story hotel:



If the generator were placed on the first floor, then every floor would receive power. But if it were placed on the second or third floors instead, then there would be a blackout on the first floor. Note that while a given floor can *receive* power from many other floors at once, it can only *send* power to one other floor.

To resolve the blackout concerns, Hilbert decided to have a minimal number of floors rewired. To rewire a floor is to change the floor it sends power to. In the sample diagram above, all possible blackouts can be avoided by rewiring the second floor to send power to the first floor instead of the third floor.

Let  $F(n)$  be the sum of the minimum number of floor rewirings needed over all possible power-flow arrangements in a hotel of  $n$  floors. For example,  $F(3) = 6$ ,  $F(8) = 16276736$ , and  $F(100) \bmod 135707531 = 84326147$ .

Find  $F(12344321) \bmod 135707531$ .

### Problem 523: First Sort I

Consider the following algorithm for sorting a list:

1. Starting from the beginning of the list, check each pair of adjacent elements in turn.
2. If the elements are out of order:
  - a. Move the smallest element of the pair at the beginning of the list.
  - b. Restart the process from step 1.
3. If all pairs are in order, stop.

For example, the list  $\{4\ 1\ 3\ 2\}$  is sorted as follows:

4 1 3 2 (4 and 1 are out of order so move 1 to the front of the list)  
1 4 3 2 (4 and 3 are out of order so move 3 to the front of the list)  
3 1 4 2 (3 and 1 are out of order so move 1 to the front of the list)  
1 3 4 2 (4 and 2 are out of order so move 2 to the front of the list)  
2 1 3 4 (2 and 1 are out of order so move 1 to the front of the list)  
1 2 3 4 (The list is now sorted)

Let  $F(L)$  be the number of times step 2a is executed to sort list  $L$ . For example,  $F(\{4\ 1\ 3\ 2\}) = 5$ .

Let  $E(n)$  be the **expected value** of  $F(P)$  over all permutations  $P$  of the integers  $\{1, 2, \dots, n\}$ .

You are given  $E(4) = 3.25$  and  $E(10) = 115.725$ .

Find  $E(30)$ . Give your answer rounded to two digits after the decimal point.

### Problem 524: First Sort II

Consider the following algorithm for sorting a list:

1. Starting from the beginning of the list, check each pair of adjacent elements in turn.
2. If the elements are out of order:
  - a. Move the smallest element of the pair at the beginning of the list.
  - b. Restart the process from step 1.
3. If all pairs are in order, stop.

For example, the list  $\{4\ 1\ 3\ 2\}$  is sorted as follows:

4 1 3 2 (4 and 1 are out of order so move 1 to the front of the list)  
1 4 3 2 (4 and 3 are out of order so move 3 to the front of the list)  
3 1 4 2 (3 and 1 are out of order so move 1 to the front of the list)  
1 3 4 2 (4 and 2 are out of order so move 2 to the front of the list)

2 1 3 4 (2 and 1 are out of order so move 1 to the front of the list)  
 1 2 3 4 (The list is now sorted)

Let  $F(L)$  be the number of times step 2a is executed to sort list  $L$ . For example,  $F(\{4\ 1\ 3\ 2\}) = 5$ .

We can list all permutations  $P$  of the integers  $\{1, 2, \dots, n\}$  in **lexicographical order**, and assign to each permutation an index  $I_n(P)$  from 1 to  $n!$  corresponding to its position in the list.

Let  $Q(n, k) = \min(I_n(P))$  for  $F(P) = k$ , the index of the first permutation requiring exactly  $k$  steps to sort with First Sort. If there is no permutation for which  $F(P) = k$ , then  $Q(n, k)$  is undefined.

For  $n = 4$  we have:

$P$	$I_4(P)$	$F(P)$	
{1, 2, 3, 4}	1	0	$Q(4, 0) = 1$
{1, 2, 4, 3}	2	4	$Q(4, 4) = 2$
{1, 3, 2, 4}	3	2	$Q(4, 2) = 3$
{1, 3, 4, 2}	4	2	
{1, 4, 2, 3}	5	6	$Q(4, 6) = 5$
{1, 4, 3, 2}	6	4	
{2, 1, 3, 4}	7	1	$Q(4, 1) = 7$
{2, 1, 4, 3}	8	5	$Q(4, 5) = 8$
{2, 3, 1, 4}	9	1	
{2, 3, 4, 1}	10	1	
{2, 4, 1, 3}	11	5	
{2, 4, 3, 1}	12	3	$Q(4, 3) = 12$
{3, 1, 2, 4}	13	3	
{3, 1, 4, 2}	14	3	
{3, 2, 1, 4}	15	2	
{3, 2, 4, 1}	16	2	
{3, 4, 1, 2}	17	3	
{3, 4, 2, 1}	18	2	
{4, 1, 2, 3}	19	7	$Q(4, 7) = 19$
{4, 1, 3, 2}	20	5	
{4, 2, 1, 3}	21	6	
{4, 2, 3, 1}	22	4	
{4, 3, 1, 2}	23	4	
{4, 3, 2, 1}	24	3	

Let  $R(k) = \min(Q(n, k))$  over all  $n$  for which  $Q(n, k)$  is defined.

Find  $R(12^{12})$ .

### Problem 525: Rolling Ellipse

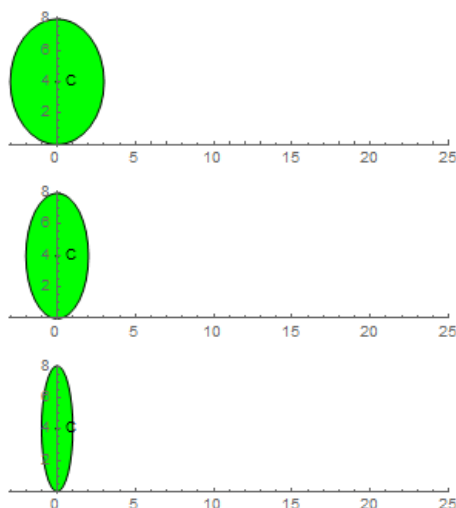
An ellipse  $E(a, b)$  is given at its initial position by equation:

$$\frac{x^2}{a^2} + \frac{(y - b)^2}{b^2} = 1$$

The ellipse rolls without slipping along the  $x$  axis for one complete turn. Interestingly, the length of the curve generated by a focus is independent from the size of the minor axis:

$$F(a, b) = 2\pi \max(a, b)$$

This is not true for the curve generated by the ellipse center. Let  $C(a, b)$  be the length of the curve generated by the center of the ellipse as it rolls without slipping for one turn.



You are given  $C(2, 4) \sim 21.38816906$ .

Find  $C(1, 4) + C(3, 4)$ . Give your answer rounded to 8 digits behind the decimal point in the form  $ab.cdefghij$ .

### Problem 526: Largest prime factors of consecutive numbers

Let  $f(n)$  be the largest prime factor of  $n$ .

Let  $g(n) = f(n) + f(n+1) + f(n+2) + f(n+3) + f(n+4) + f(n+5) + f(n+6) + f(n+7) + f(n+8)$ , the sum of the largest prime factor of each of nine consecutive numbers starting with  $n$ .

Let  $h(n)$  be the maximum value of  $g(k)$  for  $2 \leq k \leq n$ .

You are given:

- $f(100) = 5$
- $f(101) = 101$
- $g(100) = 409$
- $h(100) = 417$
- $h(10^9) = 4896292593$

Find  $h(10^{16})$ .



### Problem 527: Randomized Binary Search

A secret integer  $t$  is selected at random within the range  $1 \leq t \leq n$ .

The goal is to guess the value of  $t$  by making repeated guesses, via integer  $g$ . After a guess is made, there are three possible outcomes, in which it will be revealed that either  $g < t$ ,  $g = t$ , or  $g > t$ . Then the process can repeat as necessary.

Normally, the number of guesses required on average can be minimized with a binary search: Given a lower bound  $L$  and upper bound  $H$  (initialized to  $L = 1$  and  $H = n$ ), let  $g = \lfloor (L+H)/2 \rfloor$  where  $\lfloor \cdot \rfloor$  is the integer floor function. If  $g = t$ , the process ends. Otherwise, if  $g < t$ , set  $L = g+1$ , but if  $g > t$  instead, set  $H = g-1$ . After setting the new bounds, the search process repeats, and ultimately ends once  $t$  is found. Even if  $t$  can be deduced without searching, assume that a search will be required anyway to confirm the value.

Your friend Bob believes that the standard binary search is not that much better than his randomized variant: Instead of setting  $g = \lfloor (L+H)/2 \rfloor$ , simply let  $g$  be a random integer between  $L$  and  $H$ , inclusive. The rest of the algorithm is the same as the standard binary search. This new search routine will be referred to as a *random binary search*.

Given that  $1 \leq t \leq n$  for random  $t$ , let  $B(n)$  be the expected number of guesses needed to find  $t$  using the standard binary search, and let  $R(n)$  be the expected number of guesses needed to find  $t$  using the random binary search. For example,  $B(6) = 2.33333333$  and  $R(6) = 2.71666667$  when rounded to 8 decimal places.

Find  $R(10^{10}) - B(10^{10})$  rounded to 8 decimal places.

### Problem 528: Constrained Sums

Let  $S(n, k, b)$  represent the number of valid solutions to  $x_1 + x_2 + \dots + x_k \leq n$ , where  $0 \leq x_m \leq b^m$  for all  $1 \leq m \leq k$ .

For example,  $S(14, 3, 2) = 135$ ,  $S(200, 5, 3) = 12949440$ , and  $S(1000, 10, 5) \bmod 1\,000\,000\,007 = 624839075$ .

Find  $(\sum_{10 \leq k \leq 15} S(10^k, k, k)) \bmod 1\,000\,000\,007$ .

### Problem 529: 10-substrings

A *10-substring* of a number is a substring of its digits that sum to 10. For example, the 10-substrings of the number 3523014 are:

3523014  
3523014  
3523014  
3523014  
3523014  
3523014  
3523014

A number is called *10-substring-friendly* if every one of its digits belongs to a 10-substring. For example, 3523014 is 10-substring-friendly, but 28546 is not.

Let  $T(n)$  be the number of 10-substring-friendly numbers from 1 to  $10^n$  (inclusive).

For example  $T(2) = 9$  and  $T(5) = 3492$ .

Find  $T(10^{18}) \bmod 1\,000\,000\,007$ .

### Problem 530: GCD of Divisors

Every divisor  $d$  of a number  $n$  has a **complementary divisor**  $n/d$ .

Let  $f(n)$  be the sum of the **greatest common divisor** of  $d$  and  $n/d$  over all positive divisors  $d$  of  $n$ , that is  $f(n) = \sum_{d|n} \gcd(d, n/d)$ .

Let  $F$  be the summatory function of  $f$ , that is  $F(k) = \sum_{n=1}^k f(n)$ .

You are given that  $F(10) = 32$  and  $F(1000) = 12776$ .

Find  $F(10^{15})$ .

### Problem 531: Chinese leftovers

Let  $g(a, n, b, m)$  be the smallest non-negative solution  $x$  to the system:

$$x = a \bmod n$$

$$x = b \bmod m$$

if such a solution exists, otherwise 0.

E.g.  $g(2, 4, 4, 6) = 10$ , but  $g(3, 4, 4, 6) = 0$ .

Let  $\varphi(n)$  be Euler's totient function.

Let  $f(n, m) = g(\varphi(n), n, \varphi(m), m)$

Find  $\sum f(n, m)$  for  $1000000 \leq n < m < 1005000$

### Problem 532: Nanobots on Geodesics

Bob is a manufacturer of nanobots and wants to impress his customers by giving them a ball colored by his new nanobots as a present.

His nanobots can be programmed to select and locate exactly one other bot precisely and, after activation, move towards this bot along the shortest possible path and draw a colored line onto the surface while moving. Placed on a plane, the bots will start to move towards their selected bots in a straight line. In contrast, being placed on a ball, they will start to move along a geodesic as the shortest possible path. However, in both cases, whenever their target moves they will adjust their direction instantaneously to the new shortest possible path. All bots will move at the same speed after their simultaneous activation until each bot reaches its goal.

Now Bob places  $n$  bots on the ball (with radius 1) equidistantly on a small circle with radius 0.999 and programs each of them to move toward the next nanobot sitting counterclockwise on that small circle. After activation, the bots move in a sort of spiral until they finally meet at one point on the ball.

Using three bots, Bob finds that every bot will draw a line of length 2.84, resulting in a total length of 8.52 for all three bots, each time rounded to two decimal places. The colored ball looks like this:



In order to show off a little with his presents, Bob decides to use just enough bots to make sure that the line each bot draws is longer than 1000. What is the total length of all lines drawn with this number of bots, rounded to two decimal places?

### Problem 533: Minimum values of the Carmichael function

The **Carmichael function**  $\lambda(n)$  is defined as the smallest positive integer  $m$  such that  $a^m = 1$  modulo  $n$  for all integers  $a$  coprime with  $n$ .

For example  $\lambda(8) = 2$  and  $\lambda(240) = 4$ .

Define  $L(n)$  as the smallest positive integer  $m$  such that  $\lambda(k) \geq n$  for all  $k \geq m$ .

For example,  $L(6) = 241$  and  $L(100) = 20\,174\,525\,281$ .

Find  $L(20\,000\,000)$ . Give the last 9 digits of your answer.

### Problem 534: Weak Queens

The classical **eight queens puzzle** is the well known problem of placing eight chess queens on a  $8 \times 8$  chessboard so that no two queens threaten each other. Allowing configurations to reappear in rotated or mirrored form, a total of 92 distinct configurations can be found for eight queens. The general case asks for the number of distinct ways of placing  $n$  queens on a  $n \times n$  board, e.g. you can find 2 distinct configurations for  $n=4$ .

Lets define a *weak queen* on a  $n \times n$  board to be a piece which can move any number of squares if moved horizontally, but a maximum of  $n-1-W$  squares if moved vertically or diagonally,  $0 \leq w < n$  being the "weakness factor". For example, a weak queen on a  $n \times n$  board with a weakness factor of  $W=1$  located in the bottom row will not be able to threaten any square in the top row as the weak queen would need to move  $n-1$  squares vertically or diagonally to get there, but may move  $n-2$  squares in these directions. In contrast, the weak queen is not handicapped horizontally, thus threatening every square in its own row, independently from its current position in that row.

Let  $Q(n, W)$  be the number of ways  $n$  weak queens with weakness factor  $W$  can be placed on a  $n \times n$  board so that no two queens threaten each other. It can be shown, for example, that  $Q(4, 0)=2$ ,  $Q(4, 2)=16$  and  $Q(4, 3)=256$ .

Let  $S(n) = \sum_{w=0}^{n-1} Q(n, w)$ .

You are given that  $S(4)=276$  and  $S(5)=3347$ .

Find  $S(14)$ .

### Problem 535: Fractal Sequence

Consider the infinite integer sequence  $S$  starting with:

$S = 1, 1, 2, 1, 3, 2, 4, 1, 5, 3, 6, 2, 7, 8, 4, 9, 1, 10, 11, 5, \dots$

Circle the first occurrence of each integer.

$S = \textcircled{1}, 1, \textcircled{2}, 1, \textcircled{3}, 2, \textcircled{4}, 1, \textcircled{5}, 3, \textcircled{6}, 2, \textcircled{7}, \textcircled{8}, 4, \textcircled{9}, 1, \textcircled{10}, \square, 5, \dots$

The sequence is characterized by the following properties:

- The circled numbers are consecutive integers starting with 1.
- Immediately preceding each non-circled numbers  $a_i$ , there are exactly  $\lfloor \sqrt{a_i} \rfloor$  adjacent circled numbers, where  $\lfloor \cdot \rfloor$  is the floor function.
- If we remove all circled numbers, the remaining numbers form a sequence identical to  $S$ , so  $S$  is a **fractal sequence**.

Let  $T(n)$  be the sum of the first  $n$  elements of the sequence.

You are given  $T(1) = 1$ ,  $T(20) = 86$ ,  $T(10^3) = 364089$  and  $T(10^9) = 498676527978348241$ .

Find  $T(10^{18})$ . Give the last 9 digits of your answer.

### Problem 536: Modulo power identity

Let  $S(n)$  be the sum of all positive integers  $m$  not exceeding  $n$  having the following property:

$a^{m+4} \equiv a \pmod{m}$  for all integers  $a$ .

The values of  $m \leq 100$  that satisfy this property are 1, 2, 3, 5 and 21, thus  $S(100) = 1+2+3+5+21 = 32$ .

You are given  $S(10^6) = 22868117$ .

Find  $S(10^{12})$ .

### Problem 537: Counting tuples

Let  $\pi(x)$  be the prime counting function, i.e. the number of prime numbers less than or equal to  $x$ .  
For example,  $\pi(1)=0$ ,  $\pi(2)=1$ ,  $\pi(100)=25$ .

Let  $T(n,k)$  be the number of  $k$ -tuples  $(x_1, \dots, x_k)$  which satisfy:

1. every  $x_i$  is a positive integer;
2.  $\sum_{i=1}^k \pi(x_i) = n$

For example  $T(3,3)=19$ .

The 19 tuples are (1,1,5), (1,5,1), (5,1,1), (1,1,6), (1,6,1), (6,1,1), (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (1,2,4), (1,4,2), (2,1,4), (2,4,1), (4,1,2), (4,2,1), (2,2,2).

You are given  $T(10,10) = 869\,985$  and  $T(10^3, 10^3) \equiv 578\,270\,566 \pmod{1\,004\,535\,809}$ .

Find  $T(20\,000, 20\,000) \pmod{1\,004\,535\,809}$ .

### Problem 538: Maximum quadrilaterals

Consider a positive integer sequence  $S = (s_1, s_2, \dots, s_n)$ .

Let  $f(S)$  be the perimeter of the maximum-area quadrilateral whose side lengths are 4 elements  $s_i, s_j, s_k, s_l$  of  $S$  (all  $i, j, k, l$  distinct). If there are many quadrilaterals with the same maximum area, then choose the one with the largest perimeter.

For example, if  $S = (8, 9, 14, 9, 27)$ , then we can take the elements (9, 14, 9, 27) and form an isosceles trapezium with parallel side lengths 14 and 27 and both leg lengths 9. The area of this quadrilateral is 127.611470879... It can be shown that this is the largest area for any quadrilateral that can be formed using side lengths from  $S$ . Therefore,  
 $f(S) = 9 + 14 + 9 + 27 = 59$ .

Let  $u_n = 2^{B(3n)} + 3^{B(2n)} + B(n+1)$ , where  $B(k)$  is the number of 1 bits of  $k$  in base 2.

For example,  $B(6) = 2$ ,  $B(10) = 2$  and  $B(15) = 4$ , and  $u_5 = 2^4 + 3^2 + 2 = 27$ .

Also, let  $U_n$  be the sequence  $(u_1, u_2, \dots, u_n)$ .

For example,  $U_{10} = (8, 9, 14, 9, 27, 16, 36, 9, 27, 28)$ .

It can be shown that  $f(U_5) = 59$ ,  $f(U_{10}) = 118$ ,  $f(U_{150}) = 3223$ .

It can also be shown that  $\sum f(U_n) = 234761$  for  $4 \leq n \leq 150$ .

Find  $\sum f(U_n)$  for  $4 \leq n \leq 3\,000\,000$ .

### Problem 539: Odd elimination

Start from an ordered list of all integers from 1 to  $n$ . Going from left to right, remove the first number and every other number afterward until the end of the list. Repeat the procedure from right to left, removing the right most number and every other number from the numbers left. Continue removing every other numbers, alternating left to right and right to left, until a single number remains.

Starting with  $n = 9$ , we have:

1 2 3 4 5 6 7 8 9  
2 4 6 8  
2 6  
6

Let  $P(n)$  be the last number left starting with a list of length  $n$ .

Let  $S(n) = \sum_{k=1}^n P(k)$ .

You are given  $P(1)=1$ ,  $P(9) = 6$ ,  $P(1000)=510$ ,  $S(1000)=268271$ .

Find  $S(10^{18}) \pmod{987654321}$ .

#### Problem 540: Counting primitive Pythagorean triples

A **Pythagorean triple** consists of three positive integers  $a$ ,  $b$  and  $c$  satisfying  $a^2 + b^2 = c^2$ .

The triple is called primitive if  $a$ ,  $b$  and  $c$  are relatively prime.

Let  $P(n)$  be the number of **primitive Pythagorean triples** with  $a < b < c \leq n$ .

For example  $P(20) = 3$ , since there are three triples:  $(3, 4, 5)$ ,  $(5, 12, 13)$  and  $(8, 15, 17)$ .

You are given that  $P(10^6) = 159139$ .

Find  $P(3141592653589793)$ .

#### Problem 541: Divisibility of Harmonic Number Denominators

The  $n^{\text{th}}$  **harmonic number**  $H_n$  is defined as the sum of the multiplicative inverses of the first  $n$  positive integers, and can be written as a **reduced fraction**  $a_n/b_n$ .

$$H_n = \sum_{k=1}^n \frac{1}{k} = \frac{a_n}{b_n} \text{ with } \gcd(a_n, b_n) = 1.$$

Let  $M(p)$  be the largest value of  $n$  such that  $b_n$  is not divisible by  $p$ .

For example,  $M(3) = 68$  because  $H_{68} = \frac{a_{68}}{b_{68}} = \frac{14094018321907827923954201611}{2933773379069966367528193600}$ ,  $b_{68} = 2933773379069966367528193600$  is not divisible by 3, but all larger harmonic numbers have denominators divisible by 3.

You are given  $M(7) = 719102$ .

Find  $M(137)$ .

#### Problem 542: Geometric Progression with Maximum Sum

Let  $S(k)$  be the sum of three or more distinct positive integers having the following properties:

- No value exceeds  $k$ .
- The values form a **geometric progression**.
- The sum is maximal.

$$S(4) = 4 + 2 + 1 = 7$$

$$S(10) = 9 + 6 + 4 = 19$$

$$S(12) = 12 + 6 + 3 = 21$$

$$S(1000) = 1000 + 900 + 810 + 729 = 3439$$

$$\text{Let } T(n) = \sum_{k=4}^n (-1)^k S(k).$$

$$T(1000) = 2268$$

Find  $T(10^{17})$ .

#### Problem 543: Prime-Sum Numbers

Define function  $P(n, k) = 1$  if  $n$  can be written as the sum of  $k$  prime numbers (with repetitions allowed), and  $P(n, k) = 0$  otherwise.

For example,  $P(10, 2) = 1$  because 10 can be written as either  $3 + 7$  or  $5 + 5$ , but  $P(11, 2) = 0$  because no two primes can sum to 11.

Let  $S(n)$  be the sum of all  $P(i, k)$  over  $1 \leq i, k \leq n$ .

For example,  $S(10) = 20$ ,  $S(100) = 2402$ , and  $S(1000) = 248838$ .

Let  $F(k)$  be the  $k$ th Fibonacci number (with  $F(0) = 0$  and  $F(1) = 1$ ).

Find the sum of all  $S(F(k))$  over  $3 \leq k \leq 44$ .

#### Problem 544: Chromatic Conundrum

Let  $F(r, c, n)$  be the number of ways to color a rectangular grid with  $r$  rows and  $c$  columns using at most  $n$  colors such that no two adjacent cells share the same color. Cells that are diagonal to each other are not considered adjacent.

For example,  $F(2, 2, 3) = 18$ ,  $F(2, 2, 20) = 130340$ , and  $F(3, 4, 6) = 102923670$ .

Let  $S(r, c, n) = \sum_{k=1}^n F(r, c, k)$ .

For example,  $S(4, 4, 15) \bmod 10^9 + 7 = 325951319$ .

Find  $S(9, 10, 1112131415) \bmod 10^9 + 7$ .

#### Problem 545: Faulhaber's Formulas

The sum of the  $k^{\text{th}}$  powers of the first  $n$  positive integers can be expressed as a polynomial of degree  $k+1$  with rational coefficients, the **Faulhaber's Formulas**:

$1^k + 2^k + \dots + n^k = \sum_{i=1}^n i^k = \sum_{i=1}^{k+1} a_i n^i = a_1 n + a_2 n^2 + \dots + a_k n^k + a_{k+1} n^{k+1}$  where  $a_i$ 's are rational coefficients that can be written as reduced fractions  $p_i/q_i$  (if  $a_i = 0$ , we shall consider  $q_i = 1$ ).

For example,  $1^4 + 2^4 + \dots + n^4 = \frac{1}{30} n + \frac{1}{3} n^3 + \frac{1}{2} n^4 + \frac{1}{5} n^5$ .

Define  $D(k)$  as the value of  $q_1$  for the sum of  $k^{\text{th}}$  powers (i.e. the denominator of the reduced fraction  $a_1$ ).

Define  $F(m)$  as the  $m^{\text{th}}$  value of  $k \geq 1$  for which  $D(k) = 20010$ .

You are given  $D(4) = 30$  (since  $a_1 = -1/30$ ),  $D(308) = 20010$ ,  $F(1) = 308$ ,  $F(10) = 96404$ .

Find  $F(10^5)$ .

#### Problem 546: The Floor's Revenge

Define  $f_k(n) = \sum_{i=0}^n f_k(\lfloor \frac{i}{k} \rfloor)$  where  $f_k(0) = 1$  and  $\lfloor x \rfloor$  denotes the floor function.

For example,  $f_5(10) = 18$ ,  $f_7(100) = 1003$ , and  $f_2(10^3) = 264830889564$ .

Find  $(\sum_{k=2}^{10} f_k(10^{14})) \bmod (10^9 + 7)$ .

#### Problem 547: Distance of random points within hollow square laminae

Assuming that two points are chosen randomly (with **uniform distribution**) within a rectangle, it is possible to determine the **expected value** of the distance between these two points.

For example, the expected distance between two random points in a unit square is about 0.521405, while the expected distance between two random points in a rectangle with side lengths 2 and 3 is about 1.317067.

Now we define a *hollow square lamina* of size  $n$  to be an integer sized square with side length  $n \geq 3$  consisting of  $n^2$  unit squares from which a rectangle consisting of  $X \times Y$  unit squares ( $1 \leq X, Y \leq n - 2$ ) within the original square has been removed.

For  $n = 3$  there exists only one hollow square lamina:



For  $n = 4$  you can find 9 distinct hollow square laminae, allowing shapes to reappear in rotated or mirrored form:



Let  $S(n)$  be the sum of the expected distance between two points chosen randomly within each of the possible hollow square laminae of size  $n$ . The two points have to lie within the area left after removing the inner rectangle, i.e. the gray-colored areas in the illustrations above.

For example,  $S(3) = 1.6514$  and  $S(4) = 19.6564$ , rounded to four digits after the decimal point.

Find  $S(40)$  rounded to four digits after the decimal point.

#### Problem 548: Gozinta Chains

A **gozinta chain** for  $n$  is a sequence  $\{1, a, b, \dots, n\}$  where each element properly divides the next.

There are eight gozinta chains for 12:

$\{1, 12\}$ ,  $\{1, 2, 12\}$ ,  $\{1, 2, 4, 12\}$ ,  $\{1, 2, 6, 12\}$ ,  $\{1, 3, 12\}$ ,  $\{1, 3, 6, 12\}$ ,  $\{1, 4, 12\}$  and  $\{1, 6, 12\}$ .

Let  $g(n)$  be the number of gozinta chains for  $n$ , so  $g(12)=8$ .

$g(48)=48$  and  $g(120)=132$ .

Find the sum of the numbers  $n$  not exceeding  $10^{16}$  for which  $g(n)=n$ .

#### Problem 549: Divisibility of factorials

The smallest number  $m$  such that 10 divides  $m!$  is  $m=5$ .

The smallest number  $m$  such that 25 divides  $m!$  is  $m=10$ .

Let  $s(n)$  be the smallest number  $m$  such that  $n$  divides  $m!$ .

So  $s(10)=5$  and  $s(25)=10$ .

Let  $S(n)$  be  $\sum s(i)$  for  $2 \leq i \leq n$ .

$S(100)=2012$ .

Find  $S(10^8)$ .

#### Problem 550: Divisor game

Two players are playing a game. There are  $k$  piles of stones. When it is his turn a player has to choose a pile and replace it by two piles of stones under the following two conditions:

- Both new piles must have a number of stones more than one and less than the number of stones of the original pile.
- The number of stones of each of the new piles must be a divisor of the number of stones of the original pile.

The first player unable to make a valid move loses.

Let  $f(n, k)$  be the number of winning positions for the first player, assuming perfect play, when the game is played with  $k$  piles each having between 2 and  $n$  stones (inclusively).

$f(10, 5)=40085$ .

Find  $f(10^7, 10^{12})$ .

Give your answer modulo 987654321.

#### Problem 551: Sum of digits sequence

Let  $a_0, a_1, a_2, \dots$  be an integer sequence defined by:

- $a_0 = 1$ ;
- for  $n \geq 1$ ,  $a_n$  is the sum of the digits of all preceding terms.

The sequence starts with 1, 1, 2, 4, 8, 16, 23, 28, 38, 49, ...

You are given  $a_{10^6} = 31054319$ .

Find  $a_{10^{15}}$ .

### Problem 552: Chinese leftovers II

Let  $A_n$  be the smallest positive integer satisfying  $A_n \bmod p_i = i$  for all  $1 \leq i \leq n$ , where  $p_i$  is the  $i$ -th prime.  
For example  $A_2 = 5$ , since this is the smallest positive solution of the system of equations

$$A_2 \bmod 2 = 1$$

$$A_2 \bmod 3 = 2$$

The system of equations for  $A_3$  adds another constraint. That is,  $A_3$  is the smallest positive solution of

$$A_3 \bmod 2 = 1$$

$$A_3 \bmod 3 = 2$$

$$A_3 \bmod 5 = 3$$

and hence  $A_3 = 23$ . Similarly, one gets  $A_4 = 53$  and  $A_5 = 1523$ .

Let  $S(n)$  be the sum of all primes up to  $n$  that divide at least one element in the sequence  $A$ .

For example,  $S(50) = 69 = 5 + 23 + 41$ , since 5 divides  $A_2$ , 23 divides  $A_3$  and 41 divides  $A_{10} = 5765999453$ . No other prime number up to 50 divides an element in  $A$ .

Find  $S(300000)$ .

### Problem 553: Power sets of power sets

Let  $P(n)$  be the set of the first  $n$  positive integers  $\{1, 2, \dots, n\}$ .

Let  $Q(n)$  be the set of all the non-empty subsets of  $P(n)$ .

Let  $R(n)$  be the set of all the non-empty subsets of  $Q(n)$ .

An element  $X \in R(n)$  is a non-empty subset of  $Q(n)$ , so it is itself a set.

From  $X$  we can construct a graph as follows:

- Each element  $Y \in X$  corresponds to a vertex and labeled with  $Y$ ;
- Two vertices  $Y_1$  and  $Y_2$  are connected if  $Y_1 \cap Y_2 \neq \emptyset$ .

For example,  $X = \{\{1\}, \{1,2,3\}, \{3\}, \{5,6\}, \{6,7\}\}$  results in the following graph:



This graph has two **connected components**.

Let  $C(n,k)$  be the number of elements of  $R(n)$  that have exactly  $k$  connected components in their graph.

You are given  $C(2,1) = 6$ ,  $C(3,1) = 111$ ,  $C(4,2) = 486$ ,  $C(100,10) \bmod 1\,000\,000\,007 = 728209718$ .

Find  $C(10^4, 10) \bmod 1\,000\,000\,007$ .

### Problem 554: Centaurs on a chess board

On a chess board, a centaur moves like a king or a knight. The diagram below shows the valid moves of a centaur (represented by an inverted king) on an 8x8 board.



It can be shown that at most  $n^2$  non-attacking centaurs can be placed on a board of size  $2n \times 2n$ .

Let  $C(n)$  be the number of ways to place  $n^2$  centaurs on a  $2n \times 2n$  board so that no centaur attacks another directly.

For example  $C(1) = 4$ ,  $C(2) = 25$ ,  $C(10) = 1477721$ .

Let  $F_i$  be the  $i^{\text{th}}$  Fibonacci number defined as  $F_1 = F_2 = 1$  and  $F_i = F_{i-1} + F_{i-2}$  for  $i > 2$ .

Find  $\left( \sum_{i=2}^{90} C(F_i) \right) \bmod (10^8+7)$



### Problem 555: McCarthy 91 function

The McCarthy 91 function is defined as follows:  $M_{91}(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ M_{91}(M_{91}(n+11)) & \text{if } 0 \leq n \leq 100 \end{cases}$

We can generalize this definition by abstracting away the constants into new variables:  $M_{m,k,s}(n) = \begin{cases} n - s & \text{if } n > m \\ M_{m,k,s}(M_{m,k,s}(n+k)) & \text{if } 0 \leq n \leq m \end{cases}$

This way, we have  $M_{91} = M_{100,11,10}$ .

Let  $F_{m,k,s}$  be the set of fixed points of  $M_{m,k,s}$ . That is,  $F_{m,k,s} = \{n \in \mathbb{N} \mid M_{m,k,s}(n) = n\}$

For example, the only fixed point of  $M_{91}$  is  $n = 91$ . In other words,  $F_{100,11,10} = \{91\}$ .

Now, define  $SF(m,k,s)$  as the sum of the elements in  $F_{m,k,s}$  and let  $S(p,m) = \sum_{1 \leq s < k \leq p} \{SF(m,k,s)\}$ .

For example,  $S(10, 10) = 225$  and  $S(1000, 1000) = 208724467$ .

Find  $S(10^6, 10^6)$ .

### Problem 556: Squarefree Gaussian Integers

A **Gaussian integer** is a number  $Z = a + bi$  where  $a, b$  are integers and  $i^2 = -1$ .

Gaussian integers are a subset of the complex numbers, and the integers are the subset of Gaussian integers for which  $b = 0$ .

A Gaussian integer **unit** is one for which  $a^2 + b^2 = 1$ , i.e. one of  $1, i, -1, -i$ .

Let's define a *proper* Gaussian integer as one for which  $a > 0$  and  $b \geq 0$ .

A Gaussian integer  $z_1 = a_1 + b_1 i$  is said to be divisible by  $z_2 = a_2 + b_2 i$  if  $z_3 = a_3 + b_3 i = z_1/z_2$  is a Gaussian integer.

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)} = \frac{a_1 a_2 + b_1 b_2 + i(a_1 b_2 - a_2 b_1)}{a_2^2 + b_2^2} = \frac{a_3 + b_3 i}{1}$$

So,  $z_1$  is divisible by  $z_2$  if  $\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}$  and  $\frac{a_1 b_2 - a_2 b_1}{a_2^2 + b_2^2}$  are integers.

For example, 2 is divisible by  $1 + i$  because  $2/(1 + i) = 1 - i$  is a Gaussian integer.

A **Gaussian prime** is a Gaussian integer that is divisible only by a unit, itself or itself times a unit.

For example,  $1 + 2i$  is a Gaussian prime, because it is only divisible by  $1, i, -1, -i, 1 + 2i, i(1 + 2i) = i - 2, -(1 + 2i) = -1 - 2i$  and  $-i(1 + 2i) = 2 - i$ .

2 is not a Gaussian prime as it is divisible by  $1 + i$ .

A Gaussian integer can be uniquely factored as the product of a unit and proper Gaussian primes.

For example  $2 = -i(1 + i)^2$  and  $1 + 3i = (1 + i)(2 + i)$ .

A Gaussian integer is said to be squarefree if its prime factorization does not contain repeated proper Gaussian primes.

So 2 is not squarefree over the Gaussian integers, but  $1 + 3i$  is.

Units and Gaussian primes are squarefree by definition.

Let  $f(n)$  be the count of proper squarefree Gaussian integers with  $a^2 + b^2 \leq n$ .

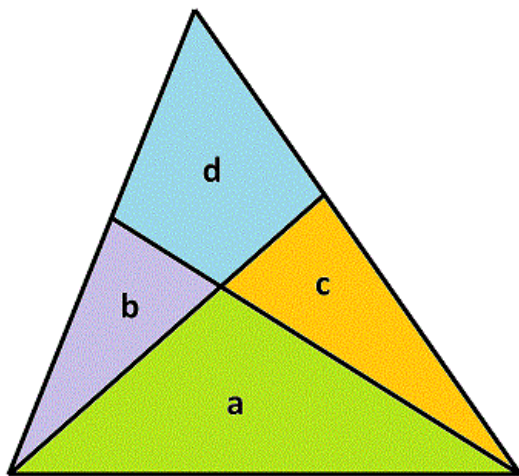
For example  $f(10) = 7$  because  $1, 1 + i, 1 + 2i, 1 + 3i = (1 + i)(2 + i), 2 + i, 3$  and  $3 + i = -i(1 + i)(1 + 2i)$  are squarefree, while  $2 = -i(1 + i)^2$  and  $2 + 2i = -i(1 + i)^3$  are not.

You are given  $f(10^2) = 54$ ,  $f(10^4) = 5218$  and  $f(10^8) = 52126906$ .

Find  $f(10^{14})$ .

### Problem 557: Cutting triangles

A triangle is cut into four pieces by two straight lines, each starting at one vertex and ending on the opposite edge. This results in forming three smaller triangular pieces, and one quadrilateral. If the original triangle has an integral area, it is often possible to choose cuts such that all of the four pieces also have integral area. For example, the diagram below shows a triangle of area 55 that has been cut in this way.



Representing the areas as  $a$ ,  $b$ ,  $c$  and  $d$ , in the example above, the individual areas are  $a = 22$ ,  $b = 8$ ,  $c = 11$  and  $d = 14$ . It is also possible to cut a triangle of area 55 such that  $a = 20$ ,  $b = 2$ ,  $c = 24$ ,  $d = 9$ .

Define a triangle cutting quadruple  $(a, b, c, d)$  as a valid integral division of a triangle, where  $a$  is the area of the triangle between the two cut vertices,  $d$  is the area of the quadrilateral and  $b$  and  $c$  are the areas of the two other triangles, with the restriction that  $b \leq c$ . The two solutions described above are  $(22, 8, 11, 14)$  and  $(20, 2, 24, 9)$ . These are the only two possible quadruples that have a total area of 55.

Define  $S(n)$  as the sum of the area of the uncut triangles represented by all valid quadruples with  $a+b+c+d \leq n$ . For example,  $S(20) = 259$ .

Find  $S(10000)$ .

### Problem 558: Irrational base

Let  $r$  be the real root of the equation  $x^3 = x^2 + 1$ .

Every positive integer can be written as the sum of distinct increasing powers of  $r$ .

If we require the number of terms to be finite and the difference between any two exponents to be three or more, then the representation is unique.

For example,  $3 = r^{-10} + r^{-5} + r^{-1} + r^2$  and  $10 = r^{-10} + r^{-7} + r^6$ .

Interestingly, the relation holds for the complex roots of the equation.

Let  $w(n)$  be the number of terms in this unique representation of  $n$ . Thus  $w(3) = 4$  and  $w(10) = 3$ .

More formally, for all positive integers  $n$ , we have:

$$n = \sum_{k=-\infty}^{\infty} b_k r^k$$

under the conditions that:

$b_k$  is 0 or 1 for all  $k$ ;

$b_k + b_{k+1} + b_{k+2} \leq 1$  for all  $k$ ;

$w(n) = \sum_{k=-\infty}^{\infty} b_k$  is finite.

Let  $S(m) = \sum_{j=1}^m w(j^2)$ .

You are given  $S(10) = 61$  and  $S(1000) = 19403$ .

Find  $S(5\,000\,000)$ .

### Problem 559: Permuted Matrices

An **ascent** of a column  $j$  in a matrix occurs if the value of column  $j$  is smaller than the value of column  $j+1$  in all rows.

Let  $P(k, r, n)$  be the number of  $r \times n$  matrices with the following properties:

- The rows are permutations of  $\{1, 2, 3, \dots, n\}$ .
- Numbering the first column as 1, a column ascent occurs at column  $j < n$  if and only if  $j$  is not a multiple of  $k$ .

For example,  $P(1, 2, 3) = 19$ ,  $P(2, 4, 6) = 65508751$  and  $P(7, 5, 30) \bmod 1000000123 = 161858102$ .

Let  $Q(n) = \sum_{k=1}^n P(k, n, n)$ .

For example,  $Q(5) = 21879393751$  and  $Q(50) \bmod 1000000123 = 819573537$ .

Find  $Q(50000) \bmod 1000000123$ .

### Problem 560: Coprime Nim

Coprime Nim is just like ordinary normal play Nim, but the players may only remove a number of stones from a pile that is **coprime** with the current size of the pile. Two players remove stones in turn. The player who removes the last stone wins.

Let  $L(n, k)$  be the number of **losing** starting positions for the first player, assuming perfect play, when the game is played with  $k$  piles, each having between 1 and  $n - 1$  stones inclusively.

For example,  $L(5, 2) = 6$  since the losing initial positions are (1, 1), (2, 2), (2, 4), (3, 3), (4, 2) and (4, 4).

You are also given  $L(10, 5) = 9964$ ,  $L(10, 10) = 472400303$ ,  $L(10^3, 10^3) \bmod 1\,000\,000\,007 = 954021836$ .

Find  $L(10^7, 10^7) \bmod 1\,000\,000\,007$ .

### Problem 561: Divisor Pairs

Let  $S(n)$  be the number of pairs  $(a, b)$  of distinct divisors of  $n$  such that  $a$  divides  $b$ .

For  $n=6$  we get the following pairs: (1,2), (1,3), (1,6), (2,6) and (3,6). So  $S(6)=5$ .

Let  $p_m\#$  be the product of the first  $m$  prime numbers, so  $p_2\# = 2*3 = 6$ .

Let  $E(m, n)$  be the highest integer  $k$  such that  $2^k$  divides  $S((p_m\#)^n)$ .

$E(2,1) = 0$  since  $2^0$  is the highest power of 2 that divides  $S(6)=5$ .

Let  $Q(n) = \sum_{i=1}^n E(904961, i)$

$Q(8)=2714886$ .

Evaluate  $Q(10^{12})$ .

### Problem 562: Maximal perimeter

Construct triangle ABC such that:

- Vertices A, B and C are lattice points inside or on the circle of radius  $r$  centered at the origin;
- the triangle contains no other lattice point inside or on its edges;
- the perimeter is maximum.

Let  $R$  be the circumradius of triangle ABC and  $T(r) = R/r$ .

For  $r = 5$ , one possible triangle has vertices  $(-4, -3)$ ,  $(4, 2)$  and  $(1, 0)$  with perimeter  $\sqrt{13} + \sqrt{34} + \sqrt{89}$  and

circumradius  $R = \sqrt{\frac{19669}{2}}$ , so  $T(5) = \sqrt{\frac{19669}{50}}$ .

You are given  $T(10) \sim 97.26729$  and  $T(100) \sim 9157.64707$ .

Find  $T(10^7)$ . Give your answer rounded to the nearest integer.

### Problem 563: Robot Welders

A company specialises in producing large rectangular metal sheets, starting from unit square metal plates. The welding is performed by a range of robots of increasing size. Unfortunately, the programming options of these robots are rather limited. Each one can only process up to 25 identical rectangles of metal, which they can weld along either edge to produce a larger rectangle. The only programmable variables are the number of rectangles to be processed (up to and including 25), and whether to weld the long or short edge.

For example, the first robot could be programmed to weld together 11 raw unit square plates to make a  $11 \times 1$  strip. The next could take 10 of these  $11 \times 1$  strips, and weld them either to make a longer  $110 \times 1$  strip, or a  $11 \times 10$  rectangle. Many, but not all, possible dimensions of metal sheets can be constructed in this way.

One regular customer has a particularly unusual order. He always demands that the finished product should have an exact area, and that the long side must not be more than 10% larger than the short side. If these requirements can be met in more than one way, in terms of the exact dimensions of the two sides, then he demands that all variants are produced. For example, if he were to ask for metal sheet of area 889200, then there are three final dimensions that can be produced:  $900 \times 988$ ,  $912 \times 975$  and  $936 \times 950$ . The target area of 889200 is the smallest area which can be manufactured in three different variants, within the limitations of the robot welders.

Let  $M(n)$  be the minimal area that can be manufactured inexactly  $n$  variants with the longer edge not greater than 10% bigger than the shorter edge. Hence  $M(3) = 889200$ .

Find  $\sum_{n=2}^{100} M(n)$ .

### Problem 564: Maximal polygons

A line segment of length  $2n-3$  is randomly split into  $n$  segments of integer length ( $n \geq 3$ ). In the sequence given by this split, the segments are then used as consecutive sides of a convex  $n$ -polygon, formed in such a way that its area is maximal. All of the  $\binom{2n-4}{n-1}$  possibilities for splitting up the initial line segment occur with the same probability.

Let  $E(n)$  be the expected value of the area that is obtained by this procedure.

For example, for  $n=3$  the only possible split of the line segment of length 3 results in three line segments with length 1, that form an equilateral triangle with an area of  $\frac{1}{4}\sqrt{3}$ . Therefore  $E(3)=0.433013$ , rounded to 6 decimal places.

For  $n=4$  you can find 4 different possible splits, each of which is composed of three line segments with length 1 and one line segment with length 2. All of these splits lead to the same maximal quadrilateral with an area of  $\frac{3}{4}\sqrt{3}$ , thus  $E(4)=1.299038$ , rounded to 6 decimal places.

Let  $S(k) = \sum_{n=3}^k E(n)$ .

For example,  $S(3)=0.433013$ ,  $S(4)=1.732051$ ,  $S(5)=4.604767$  and  $S(10)=66.955511$ , rounded to 6 decimal places each.

Find  $S(50)$ , rounded to 6 decimal places.

### Problem 565: Divisibility of sum of divisors

Let  $\sigma(n)$  be the sum of the divisors of  $n$ .

E.g. the divisors of 4 are 1, 2 and 4, so  $\sigma(4)=7$ .

The numbers  $n$  not exceeding 20 such that 7 divides  $\sigma(n)$  are: 4, 12, 13 and 20, the sum of these numbers being 49.

Let  $S(n, d)$  be the sum of the numbers  $i$  not exceeding  $n$  such that  $d$  divides  $\sigma(i)$ .

So  $S(20, 7)=49$ .

You are given:  $S(10^6, 2017)=150850429$  and  $S(10^9, 2017)=249652238344557$ .

Find  $S(10^{11}, 2017)$

### Problem 566: Cake Icing Puzzle

Adam plays the following game with his birthday cake.

He cuts a piece forming a circular sector of 60 degrees and flips the piece upside down, with the icing on the bottom. He then rotates the cake by 60 degrees counterclockwise, cuts an adjacent 60 degree piece and flips it upside down. He keeps repeating this, until after a total of twelve steps, all the icing is back on top.

Amazingly, this works for any piece size, even if the cutting angle is an irrational number: all the icing will be back on top after a finite number of steps.

Now, Adam tries something different: he alternates cutting pieces of size  $x = \frac{360}{9}$  degrees,  $y = \frac{360}{10}$  degrees and  $z = \frac{360}{\sqrt{11}}$  degrees. The first piece he cuts has size  $x$  and he flips it. The second has size  $y$  and he flips it. The third has size  $z$  and he flips it. He repeats this with pieces of size  $x$ ,  $y$  and  $z$  in that order until all the icing is back on top, and discovers he needs 60 flips altogether.



Let  $F(a, b, c)$  be the minimum number of piece flips needed to get all the icing back on top for pieces of size  $x = \frac{360}{a}$  degrees,  $y = \frac{360}{b}$  degrees and  $z = \frac{360}{\sqrt{c}}$  degrees.

Let  $G(n) = \sum_{9 \leq a < b < c \leq n} F(a, b, c)$  for integers  $a$ ,  $b$  and  $c$ .

You are given that  $F(9, 10, 11) = 60$ ,  $F(10, 14, 16) = 506$ ,  $F(15, 16, 17) = 785232$ .

You are also given  $G(11) = 60$ ,  $G(14) = 58020$  and  $G(17) = 1269260$ .

Find  $G(53)$ .

### Problem 567: Reciprocal games I

Tom has built a random generator that is connected to a row of  $n$  light bulbs. Whenever the random generator is activated each of the  $n$  lights is turned on with the probability of  $\frac{1}{2}$ , independently of its former state or the state of the other light bulbs.

While discussing with his friend Jerry how to use his generator, they invent two different games, they call them *reciprocal games*:

Both games consist of  $n$  turns. Each turn is started by choosing a number  $k$  randomly between (and including) 1 and  $n$ , with equal probability of  $\frac{1}{n}$  for each number, while the possible win for that turn is the reciprocal of  $k$ , that is  $\frac{1}{k}$ .

In game A, Tom activates his random generator once in each turn. If the number of lights turned on is the same as the previously chosen number  $k$ , Jerry wins and gets  $\frac{1}{k}$ , otherwise he will receive nothing for that turn. Jerry's expected win after playing the total game A consisting of  $n$  turns is called  $J_A(n)$ . For example  $J_A(6) = 0.39505208$ , rounded to 8 decimal places.

For each turn in game B, after  $k$  has been randomly selected, Tom keeps reactivating his random generator until exactly  $k$  lights are turned on. After that Jerry takes over and reactivates the random generator until he, too, has generated a pattern with exactly  $k$  lights turned on. If this pattern is identical to Tom's last pattern, Jerry wins and gets  $\frac{1}{k}$ , otherwise he will receive nothing. Jerry's expected win after the total game B consisting of  $n$  turns is called  $J_B(n)$ . For example  $J_B(6) = 0.43333333$ , rounded to 8 decimal places.

Let 
$$S(m) = \sum_{n=1}^m (J_A(n) + J_B(n)).$$
 For example  $S(6) = 7.58932292$ , rounded to 8 decimal places.

Find  $S(123456789)$ , rounded to 8 decimal places.

### Problem 568: Reciprocal games II

Tom has built a random generator that is connected to a row of  $n$  light bulbs. Whenever the random generator is activated each of the  $n$  lights is turned on with the probability of  $\frac{1}{2}$ , independently of its former state or the state of the other light bulbs.

While discussing with his friend Jerry how to use his generator, they invent two different games, they call them *reciprocal games*:

Both games consist of  $n$  turns. Each turn is started by choosing a number  $k$  randomly between (and including) 1 and  $n$ , with equal probability of  $\frac{1}{n}$  for each number, while the possible win for that turn is the reciprocal of  $k$ , that is  $\frac{1}{k}$ .

In game A, Tom activates his random generator once in each turn. If the number of lights turned on is the same as the previously chosen number  $k$ , Jerry wins and gets  $\frac{1}{k}$ , otherwise he will receive nothing for that turn. Jerry's expected win after playing the total game A consisting of  $n$  turns is called  $J_A(n)$ . For example  $J_A(6)=0.39505208$ , rounded to 8 decimal places.

For each turn in game B, after  $k$  has been randomly selected, Tom keeps reactivating his random generator until exactly  $k$  lights are turned on. After that Jerry takes over and reactivates the random generator until he, too, has generated a pattern with exactly  $k$  lights turned on. If this pattern is identical to Tom's last pattern, Jerry wins and gets  $\frac{1}{k}$ , otherwise he will receive nothing. Jerry's expected win after the total game B consisting of  $n$  turns is called  $J_B(n)$ . For example  $J_B(6)=0.43333333$ , rounded to 8 decimal places.

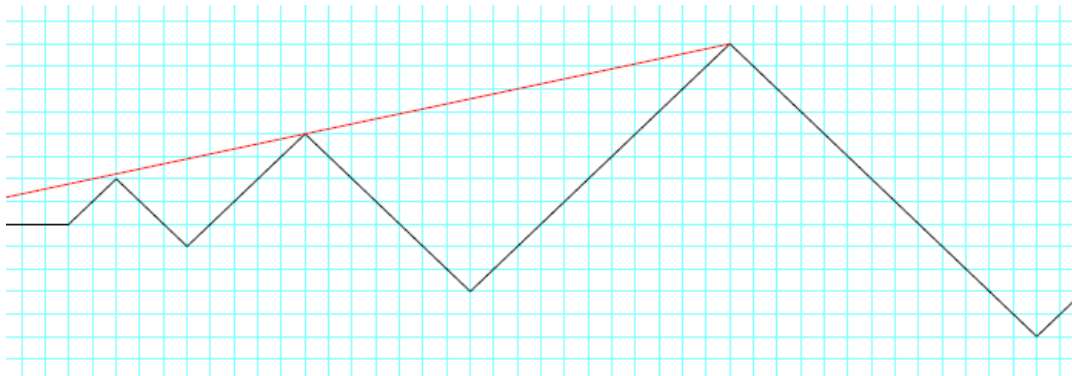
Let  $D(n)=J_B(n)-J_A(n)$ . For example,  $D(6) = 0.03828125$ .

Find the 7 most significant digits of  $D(123456789)$  after removing all leading zeros.

(If, for example, we had asked for the 7 most significant digits of  $D(6)$ , the answer would have been 3828125.)

### Problem 569: Prime Mountain Range

A *mountain range* consists of a line of mountains with slopes of exactly  $45^\circ$ , and heights governed by the prime numbers  $p_n$ . The up-slope of the  $k^{\text{th}}$  mountain is of height  $p_{2k-1}$ , and the downslope is  $p_{2k}$ . The first few foot-hills of this range are illustrated below.



Tenzing sets out to climb each one in turn, starting from the lowest. At the top of each peak, he looks back and counts how many of the previous peaks he can see. In the example above, the eye-line from the third mountain is drawn in red, showing that he can only see the peak of the second mountain from this viewpoint. Similarly, from the 9<sup>th</sup> mountain, he can see three peaks, those of the 5<sup>th</sup>, 7<sup>th</sup> and 8<sup>th</sup> mountain.

Let  $P(k)$  be the number of peaks that are visible looking back from the  $k^{\text{th}}$  mountain. Hence  $P(3)=1$  and  $P(9)=3$ .

Also  $\displaystyle \sum_{k=1}^{100} P(k) = 227$ .

Find  $\displaystyle \sum_{k=1}^{2500000} P(k)$ .

### Problem 570: Snowflakes

A snowflake of order  $n$  is formed by overlaying an equilateral triangle (rotated by 180 degrees) onto each equilateral triangle of the same size in a snowflake of order  $n-1$ . A snowflake of order 1 is a single equilateral triangle.



Some areas of the snowflake are overlaid repeatedly. In the above picture, blue represents the areas that are one layer thick, red two layers thick, yellow three layers thick, and so on.

For an order  $n$  snowflake, let  $A(n)$  be the number of triangles that are one layer thick, and let  $B(n)$  be the number of triangles that are three layers thick. Define  $G(n) = \gcd(A(n), B(n))$ .

E.g.  $A(3) = 30$ ,  $B(3) = 6$ ,  $G(3) = 6$

$A(11) = 3027630$ ,  $B(11) = 19862070$ ,  $G(11) = 30$

Further,  $G(500) = 186$  and  $\sum_{n=3}^{500} G(n) = 5124$

Find  $\sum_{n=3}^{10^7} G(n)$ .

### Problem 571: Super Pandigital Numbers

A positive number is **pandigital** in base  $b$  if it contains all digits from 0 to  $b-1$  at least once when written in base  $b$ .

A *n-super-pandigital* number is a number that is simultaneously pandigital in all bases from 2 to  $n$  inclusively.

For example  $978 = 1111010010_2 = 1100020_3 = 33102_4 = 12403_5$  is the smallest 5-super-pandigital number.

Similarly, 1093265784 is the smallest 10-super-pandigital number.

The sum of the 10 smallest 10-super-pandigital numbers is 20319792309.

What is the sum of the 10 smallest 12-super-pandigital numbers?

### Problem 572: Idempotent matrices

A matrix  $M$  is called idempotent if  $M^2 = M$ .

Let  $M$  be a three by three matrix :  $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ .

Let  $C(n)$  be the number of idempotent three by three matrices  $M$  with integer elements such that

$-n \leq a, b, c, d, e, f, g, h, i \leq n$ .

$C(1) = 164$  and  $C(2) = 848$ .

Find  $C(200)$ .

### Problem 573: Unfair race

$n$  runners in very different training states want to compete in a race. Each one of them is given a different starting number  $k$  ( $1 \leq k \leq n$ ) according to his (constant) individual racing speed being  $v_k = \frac{k}{n}$ .

In order to give the slower runners a chance to win the race,  $n$  different starting positions are chosen randomly (with uniform distribution) and independently from each other within the racing track of length 1. After this, the starting position nearest to the goal is assigned to runner 1, the next nearest starting position to runner 2 and so on, until finally the starting position furthest away from the goal is assigned to runner  $n$ . The winner of the race is the runner who reaches the goal first.

Interestingly, the expected running time for the winner is  $\frac{1}{2}$ , independently of the number of runners. Moreover, while it can be shown that all runners will have the same expected running time of  $\frac{n}{n+1}$ , the race is still unfair, since the winning chances may differ significantly for different starting numbers:

Let  $P_{n,k}$  be the probability for runner  $k$  to win a race with  $n$  runners and  $E_n = \sum_{k=1}^n k P_{n,k}$  be the expected starting number of the winner in that race. It can be shown that, for example,  $P_{3,1} = \frac{4}{9}$ ,  $P_{3,2} = \frac{2}{9}$ ,  $P_{3,3} = \frac{1}{9}$  and  $E_3 = \frac{17}{9}$  for a race with 3 runners.

You are given that  $E_4 = 2.21875$ ,  $E_5 = 2.5104$  and  $E_{10} = 3.66021568$ .

Find  $E_{1000000}$  rounded to 4 digits after the decimal point.

### Problem 574: Verifying Primes

Let  $q$  be a prime and  $A \geq B > 0$  be two integers with the following properties:

- $A$  and  $B$  have no prime factor in common, that is  $\text{gcd}(A, B) = 1$ .
- The product  $AB$  is divisible by every prime less than  $q$ .

It can be shown that, given these conditions, any sum  $A+B < q^2$  and any difference  $1 < A-B < q^2$  has to be a prime number. Thus you can verify that a number  $p$  is prime by showing that either  $p = A+B < q^2$  or  $p = A-B < q^2$  for some  $A, B, q$  fulfilling the conditions listed above.

Let  $V(p)$  be the smallest possible value of  $A$  in any sum  $p = A+B$  and any difference  $p = A-B$ , that verifies  $p$  being prime.

Examples:

$V(2) = 1$ , since  $2 = 1 + 1 < 2^2$ .

$V(37) = 22$ , since  $37 = 22 + 15 = 2 \cdot 11 + 3 \cdot 5 < 7^2$  is the associated sum with the smallest possible  $A$ .

$V(151) = 165$  since  $151 = 165 - 14 = 3 \cdot 5 \cdot 11 - 2 \cdot 7 < 13^2$  is the associated difference with the smallest possible  $A$ .

Let  $S(n)$  be the sum of  $V(p)$  for all primes  $p < n$ . For example,  $S(10) = 10$  and  $S(200) = 7177$ .

Find  $S(3800)$ .



Problem 575: Wandering Robots

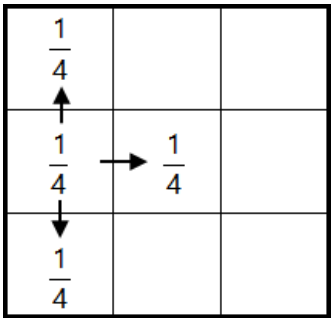
It was quite an ordinary day when a mysterious alien vessel appeared as if from nowhere. After waiting several hours and receiving no response it is decided to send a team to investigate, of which you are included. Upon entering the vessel you are met by a friendly holographic figure, Katharina, who explains the purpose of the vessel, Eulertopia.

She claims that Eulertopia is almost older than time itself. Its mission was to take advantage of a combination of incredible computational power and vast periods of time to discover the answer to life, the universe, and everything. Hence the resident cleaning robot, Leonhard, along with his housekeeping responsibilities, was built with a powerful computational matrix to ponder the meaning of life as he wanders through a massive 1000 by 1000 square grid of rooms. She goes on to explain that the rooms are numbered sequentially from left to right, row by row. So, for example, if Leonhard was wandering around a 5 by 5 grid then the rooms would be numbered in the following way.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Many millenia ago Leonhard reported to Katharina to have found the answer and he is willing to share it with any life form who proves to be worthy of such knowledge.

Katharina further explains that the designers of Leonhard were given instructions to program him with equal probability of remaining in the same room or travelling to an adjacent room. However, it was not clear to them if this meant (i) an equal probability being split equally between remaining in the room and the number of available routes, or, (ii) an equal probability (50%) of remaining in the same room and then the other 50% was to be split equally between the number of available routes.



(i) Probability of remaining related to number of exits

(ii) Fixed 50% probability of remaining

The records indicate that they decided to flip a coin. Heads would mean that the probability of remaining was dynamically related to the number of exits whereas tails would mean that they program Leonhard with a fixed 50% probability of remaining in a particular room. Unfortunately there is no record of the outcome of the coin, so without further information we would need to assume that there is equal probability of either of the choices being implemented.

Katharina suggests it should not be too challenging to determine that the probability of finding him in a square numbered room in a 5 by 5 grid after unfathomable periods of time would be approximately 0.177976190476 [12 d.p.].

In order to prove yourself worthy of visiting the great oracle you must calculate the probability of finding him in a square numbered room in the 1000 by 1000 lair in which he has been wandering.  
(Give your answer rounded to 12 decimal places)

### Problem 576: Irrational jumps

A bouncing point moves counterclockwise along a circle with circumference 1 with jumps of constant length  $l < 1$ , until it hits a gap of length  $g < 1$ , that is placed in a distance  $d$  counterclockwise from the starting point. The gap does not include the starting point, that is  $g + d < 1$ .

Let  $S(l, g, d)$  be the sum of the length of all jumps, until the point falls into the gap. It can be shown that  $S(l, g, d)$  is finite for any irrational jump size  $l$ , regardless of the values of  $g$  and  $d$ .

Examples:

$S(\sqrt{\frac{1}{2}}, 0.06, 0.7) = 0.7071 \dots$ ,  $S(\sqrt{\frac{1}{2}}, 0.06, 0.3543) = 1.4142 \dots$  and  $S(\sqrt{\frac{1}{2}}, 0.06, 0.2427) = 16.2634 \dots$ .

Let  $M(n, g)$  be the maximum of  $\sum S(\sqrt{\frac{1}{p}}, g, d)$  for all primes  $p \leq n$  and any valid value of  $d$ .

Examples:

$M(3, 0.06) = 29.5425 \dots$ , since  $S(\sqrt{\frac{1}{2}}, 0.06, 0.2427) + S(\sqrt{\frac{1}{3}}, 0.06, 0.2427) = 29.5425 \dots$  is the maximal reachable sum for  $g = 0.06$ .

$M(10, 0.01) = 266.9010 \dots$

Find  $M(100, 0.00002)$ , rounded to 4 decimal places.

### Problem 577: Counting hexagons

An equilateral triangle with integer side length  $n \geq 3$  is divided into  $n^2$  equilateral triangles with side length 1 as shown in the diagram below.

The vertices of these triangles constitute a triangular lattice with  $\frac{(n+1)(n+2)}{2}$  lattice points.

Let  $H(n)$  be the number of all regular hexagons that can be found by connecting 6 of these points.



For example,  $H(3) = 1$ ,  $H(6) = 12$  and  $H(20) = 966$ .

Find  $\sum_{n=3}^{12345} H(n)$ .

### Problem 578: Integers with decreasing prime powers

Any positive integer can be written as a product of prime powers:  $p_1^{a_1} \times p_2^{a_2} \times \dots \times p_k^{a_k}$ , where  $p_i$  are distinct prime integers,  $a_i > 0$  and  $p_i < p_j$  if  $i < j$ .

A *decreasing prime power* positive integer is one for which  $a_i \geq a_j$  if  $i < j$ .

For example, 1, 2,  $15 = 3 \times 5$ ,  $360 = 2^3 \times 3^2 \times 5$  and  $1000 = 2^3 \times 5^3$  are decreasing prime power integers.

Let  $C(n)$  be the count of decreasing prime power positive integers not exceeding  $n$ .

$C(100) = 94$  since all positive integers not exceeding 100 have decreasing prime powers except 18, 50, 54, 75, 90 and 98.

You are given  $C(10^6) = 922052$ .

Find  $C(10^{13})$ .

### Problem 579: Lattice points in lattice cubes

A **lattice cube** is a cube in which all vertices have integer coordinates. Let  $C(n)$  be the number of different lattice cubes in which the coordinates of all vertices range between (and including) 0 and  $n$ . Two cubes are hereby considered different if any of their vertices have different coordinates.

For example,  $C(1)=1$ ,  $C(2)=9$ ,  $C(4)=100$ ,  $C(5)=229$ ,  $C(10)=4469$  and  $C(50)=8154671$ .

Different cubes may contain different numbers of lattice points.

For example, the cube with the vertices

$(0, 0, 0)$ ,  $(3, 0, 0)$ ,  $(0, 3, 0)$ ,  $(0, 0, 3)$ ,  $(0, 3, 3)$ ,  $(3, 0, 3)$ ,  $(3, 3, 0)$ ,  $(3, 3, 3)$  contains 64 lattice points (56 lattice points on the surface including the 8 vertices and 8 points within the cube).

In contrast, the cube with the vertices

$(0, 2, 2)$ ,  $(1, 4, 4)$ ,  $(2, 0, 3)$ ,  $(2, 3, 0)$ ,  $(3, 2, 5)$ ,  $(3, 5, 2)$ ,  $(4, 1, 1)$ ,  $(5, 3, 3)$  contains only 40 lattice points (20 points on the surface and 20 points within the cube), although both cubes have the same side length 3.

Let  $S(n)$  be the sum of the lattice points contained in the different lattice cubes in which the coordinates of all vertices range between (and including) 0 and  $n$ .

For example,  $S(1)=8$ ,  $S(2)=91$ ,  $S(4)=1878$ ,  $S(5)=5832$ ,  $S(10)=387003$  and  $S(50)=29948928129$ .

Find  $S(5000) \bmod 10^9$ .

### Problem 580: Squarefree Hilbert numbers

A **Hilbert number** is any positive integer of the form  $4k+1$  for integer  $k \geq 0$ . We shall define a *squarefree Hilbert number* as a Hilbert number which is not divisible by the square of any Hilbert number other than one. For example, 117 is a squarefree Hilbert number, equaling  $9 \times 13$ . However 6237 is a Hilbert number that is not squarefree in this sense, as it is divisible by  $9^2$ . The number 3969 is also not squarefree, as it is divisible by both  $9^2$  and  $21^2$ .

There are 2327192 squarefree Hilbert numbers below  $10^7$ .

How many squarefree Hilbert numbers are there below  $10^{16}$ ?

### Problem 581: 47-smooth triangular numbers

A number is  $p$ -smooth if it has no prime factors larger than  $p$ .

Let  $T$  be the sequence of triangular numbers, ie  $T(n)=n(n+1)/2$ .

Find the sum of all indices  $n$  such that  $T(n)$  is 47-smooth.

### Problem 582: Nearly isosceles 120 degree triangles

Let  $a$ ,  $b$  and  $c$  be the sides of an integer sided triangle with one angle of 120 degrees,  $a \leq b \leq c$  and  $b-a \leq 100$ .

Let  $T(n)$  be the number of such triangles with  $c \leq n$ .

$T(1000)=235$  and  $T(10^8)=1245$ .

Find  $T(10^{100})$ .

### Problem 583: Heron Envelopes

A standard envelope shape is a convex figure consisting of an isosceles triangle (the flap) placed on top of a rectangle. An example of an envelope with integral sides is shown below. Note that to form a sensible envelope, the perpendicular height of the flap (BCD) must be smaller than the height of the rectangle (ABDE).



In the envelope illustrated, not only are all the sides integral, but also all the diagonals (AC, AD, BD, BE and CE) are integral too. Let us call an envelope with these properties a *Heron envelope*.

Let  $S(p)$  be the sum of the perimeters of all the Heron envelopes with a perimeter less than or equal to  $p$ .

You are given that  $S(10^4) = 884680$ . Find  $S(10^7)$ .

### Problem 584: Birthday Problem Revisited

A long long time ago in a galaxy far far away, the Wimwians, inhabitants of planet WimWi, discovered an unmanned drone that had landed on their planet. On examining the drone, they uncovered a device that sought the answer for the so called "**Birthday Problem**". The description of the problem was as follows:

*If people on your planet were to enter a very large room one by one, what will be the expected number of people in the room when you first find **3 people with Birthdays within 1 day from each other***

The description further instructed them to enter the answer into the device and send the drone into space again. Startled by this turn of events, the Wimwians consulted their best mathematicians. Each year on Wimwi has 10 days and the mathematicians assumed equally likely birthdays and ignored leap years (leap years in Wimwi have 11 days), and found 5.78688636 to be the required answer. As such, the Wimwians entered this answer and sent the drone back into space.

After traveling light years away, the drone then landed on planet Joka. The same events ensued except this time, the numbers in the device had changed due to some unknown technical issues. The description read:

*If people on your planet were to enter a very large room one by one, what will be the expected number of people in the room when you first find **3 people with Birthdays within 7 days from each other***

With a 100-day year on the planet, the Jokars (inhabitants of Joka) found the answer to be 8.48967364 (rounded to 8 decimal places because the device allowed only 8 places after the decimal point) assuming equally likely birthdays. They too entered the answer into the device and launched the drone into space again.

This time the drone landed on planet Earth. As before the numbers in the problem description had changed. It read:

*If people on your planet were to enter a very large room one by one, what will be the expected number of people in the room when you first find **4 people with Birthdays within 7 days from each other***

What would be the answer (rounded to eight places after the decimal point) the people of Earth have to enter into the device for a year with 365 days? Ignore leap years. Also assume that all birthdays are equally likely and independent of each other.

### Problem 585: Nested square roots

Consider the term  $\sqrt{x + \sqrt{y + \sqrt{z}}}$  that is representing a **nested square root**.  $x$ ,  $y$  and  $z$  are positive integers and  $y$  and  $z$  are not allowed to be perfect squares, so the number below the outer square root is irrational. Still it can be shown that for some combinations of  $x$ ,  $y$  and  $z$  the given term can be simplified into a sum and/or difference of simple square roots of integers, actually **denesting** the square roots in the initial expression.

Here are some examples of this denesting:

$$\sqrt{3 + \sqrt{2} + \sqrt{2}} = \sqrt{2} + \sqrt{1} = \sqrt{2} + 1$$

$$\sqrt{8 + \sqrt{15} + \sqrt{15}} = \sqrt{5} + \sqrt{3}$$

$$\sqrt{20 + \sqrt{96} + \sqrt{12}} = \sqrt{9} + \sqrt{6} + \sqrt{3} - \sqrt{2} = 3 + \sqrt{6} + \sqrt{3} - \sqrt{2}$$

$$\sqrt{28 + \sqrt{160} + \sqrt{108}} = \sqrt{15} + \sqrt{6} + \sqrt{5} - \sqrt{2}$$

As you can see the integers used in the denested expression may also be perfect squares resulting in further simplification.

Let  $F(n)$  be the number of different terms  $\sqrt{x + \sqrt{y + \sqrt{z}}}$ , that can be denested into the sum and/or difference of a finite number of square roots, given the additional condition that  $0 < x \leq n$ . That is,

$$\sqrt{x + \sqrt{y + \sqrt{z}}} = \sum_{i=1}^k s_i \sqrt{a_i}$$

with  $k, x, y, z$  and all  $a_i$  being positive integers, all  $s_i = \pm 1$  and  $x \leq n$ .

Furthermore  $y$  and  $z$  are not allowed to be perfect squares.

Nested roots with the same value are not considered different, for example  $\sqrt{7 + \sqrt{3} + \sqrt{27}}$ ,  $\sqrt{7 + \sqrt{12} + \sqrt{12}}$  and  $\sqrt{7 + \sqrt{27} + \sqrt{3}}$ , that can all three be denested into  $2 + \sqrt{3}$ , would only be counted once.

You are given that  $F(10)=17$ ,  $F(15)=46$ ,  $F(20)=86$ ,  $F(30)=213$  and  $F(100)=2918$  and  $F(5000)=11134074$ . Find  $F(5000000)$ .

### Problem 586: Binary Quadratic Form

The number 209 can be expressed as  $a^2 + 3ab + b^2$  in two distinct ways:

$$\begin{aligned} 209 &= 8^2 + 3 \cdot 8 \cdot 5 + 5^2 \\ 209 &= 13^2 + 3 \cdot 13 \cdot 1 + 1^2 \end{aligned}$$

Let  $f(n, r)$  be the number of integers  $k$  not exceeding  $n$  that can be expressed as  $k = a^2 + 3ab + b^2$ , with  $a, b > 0$  integers, in exactly  $r$  different ways.

You are given that  $f(10^5, 4) = 237$  and  $f(10^8, 6) = 59517$ .

Find  $f(10^{15}, 40)$ .

### Problem 587: Concave triangle

A square is drawn around a circle as shown in the diagram below on the left.

We shall call the blue shaded region the L-section.

A line is drawn from the bottom left of the square to the top right as shown in the diagram on the right.

We shall call the orange shaded region a concave triangle.



It should be clear that the concave triangle occupies exactly half of the L-section.

Two circles are placed next to each other horizontally, a rectangle is drawn around both circles, and a line is drawn from the bottom left to the top right as shown in the diagram below.



This time the concave triangle occupies approximately 36.46% of the L-section.

If  $n$  circles are placed next to each other horizontally, a rectangle is drawn around the  $n$  circles, and a line is drawn from the bottom left to the top right, then it can be shown that the least value of  $n$  for which the concave triangle occupies less than 10% of the L-section is  $n = 15$ .

What is the least value of  $n$  for which the concave triangle occupies less than 0.1% of the L-section?

### Problem 588: Quintinomial coefficients

The coefficients in the expansion of  $(x+1)^k$  are called **binomial coefficients**.

Analogously the coefficients in the expansion of  $(x^4+x^3+x^2+x+1)^k$  are called **quintinomial coefficients**. (quintus= Latin for fifth).

Consider the expansion of  $(x^4+x^3+x^2+x+1)^3$ :

$$x^{12} + 3x^{11} + 6x^{10} + 10x^9 + 15x^8 + 18x^7 + 19x^6 + 18x^5 + 15x^4 + 10x^3 + 6x^2 + 3x + 1$$

As we can see 7 out of the 13 quintinomial coefficients for  $k=3$  are odd.

Let  $Q(k)$  be the number of odd coefficients in the expansion of  $(x^4+x^3+x^2+x+1)^k$ .

So  $Q(3)=7$ .

You are given  $Q(10)=17$  and  $Q(100)=35$ .

Find  $\sum_{k=1}^{18} Q(10^k)$ .

### Problem 589: Poohsticks Marathon

Christopher Robin and Pooh Bear love the game of Poohsticks so much that they invented a new version which allows them to play for longer before one of them wins and they have to go home for tea. The game starts as normal with both dropping a stick simultaneously on the upstream side of a bridge. But rather than the game ending when one of the sticks emerges on the downstream side, instead they fish their sticks out of the water, and drop them back in again on the upstream side. The game only ends when one of the sticks emerges from under the bridge ahead of the other one having also 'lapped' the other stick - that is, having made one additional journey under the bridge compared to the other stick.

On a particular day when playing this game, the time taken for a stick to travel under the bridge varies between a minimum of 30 seconds, and a maximum of 60 seconds. The time taken to fish a stick out of the water and drop it back in again on the other side is 5 seconds. The current under the bridge has the unusual property that the sticks' journey time is always an integral number of seconds, and it is equally likely to emerge at any of the possible times between 30 and 60 seconds (inclusive). It turns out that under these circumstances, the expected time for playing a single game is 1036.15 seconds (rounded to 2 decimal places). This time is measured from the point of dropping the sticks for the first time, to the point where the winning stick emerges from under the bridge having lapped the other.

The stream flows at different rates each day, but maintains the property that the journey time in seconds is equally distributed amongst the integers from a minimum,  $n$ , to a maximum,  $m$ , inclusive. Let the expected time of play in seconds be  $E(m,n)$ . Hence  $E(60,30)=1036.15\dots$

Let  $S(k)=\sum_{m=2}^k\sum_{n=1}^{m-1}E(m,n)$ .

For example  $S(5)=7722.82$  rounded to 2 decimal places.

Find  $S(100)$  and give your answer rounded to 2 decimal places.

### Problem 590: Sets with a given Least Common Multiple

Let  $H(n)$  denote the number of sets of positive integers such that the least common multiple of the integers in the set equals  $n$ .

E.g.:

The integers in the following ten sets all have a least common multiple of 6:

$\{2,3\}$ ,  $\{1,2,3\}$ ,  $\{6\}$ ,  $\{1,6\}$ ,  $\{2,6\}$ ,  $\{1,2,6\}$ ,  $\{3,6\}$ ,  $\{1,3,6\}$ ,  $\{2,3,6\}$  and  $\{1,2,3,6\}$ .

Thus  $H(6)=10$ .

Let  $L(n)$  denote the least common multiple of the numbers 1 through  $n$ .

E.g.  $L(6)$  is the least common multiple of the numbers 1,2,3,4,5,6 and  $L(6)$  equals 60.

Let  $HL(n)$  denote  $H(L(n))$ .

You are given  $HL(4)=H(12)=44$ .

Find  $HL(50000)$ . Give your answer modulo  $10^9$ .

### Problem 591: Best Approximations by Quadratic Integers

Given a non-square integer  $d$ , any real  $x$  can be approximated arbitrarily close by **quadratic integers**  $a+b\sqrt{d}$ , where  $a,b$  are integers. For example, the following inequalities approximate  $\pi$  with precision  $10^{-13}$ :

$4375636191520\sqrt{2}-6188084046055 < \pi < 721133315582\sqrt{2}-1019836515172$

We call  $BQA_d(x,n)$  the quadratic integer closest to  $x$  with the absolute values of  $a,b$  not exceeding  $n$ .

We also define the integral part of a quadratic integer as  $I_d(a+b\sqrt{d}) = a$ .

You are given that:

- $BQA_2(\pi,10) = 6 - 2\sqrt{2}$
- $BQA_5(\pi,100)=26\sqrt{5}-55$
- $BQA_7(\pi,10^6)=560323 - 211781\sqrt{7}$
- $I_2(BQA_2(\pi,10^{13}))=-6188084046055$

Find the sum of  $|I_d(BQA_d(\pi,10^{13}))|$  for all non-square positive integers less than 100.

### Problem 592: Factorial trailing digits 2

For any  $N$ , let  $f(N)$  be the last twelve hexadecimal digits before the trailing zeroes in  $N!$ .

For example, the hexadecimal representation of  $20!$  is  $21C3677C82B40000$ , so  $f(20)$  is the digit sequence  $21C3677C82B4$ .

Find  $f(20!)$ . Give your answer as twelve hexadecimal digits, using uppercase for the digits A to F.

### Problem 593: Fleeting Medians

We define two sequences  $S = \{S(1), S(2), \dots, S(n)\}$  and  $S_2 = \{S_2(1), S_2(2), \dots, S_2(n)\}$

$S(k) = (p_k)^k \bmod 10007$  where  $p_k$  is the  $k$ th prime number.

$S_2(k) = S(k) + S(\lfloor \frac{k}{10000} \rfloor + 1)$  where  $\lfloor \cdot \rfloor$  denotes the floor function.

Then let  $M(i, j)$  be the median of elements  $S_2(i)$  through  $S_2(j)$ , inclusive. For example,  $M(1, 10) = 2021.5$  and  $M(10^2, 10^3) = 4715.0$ .

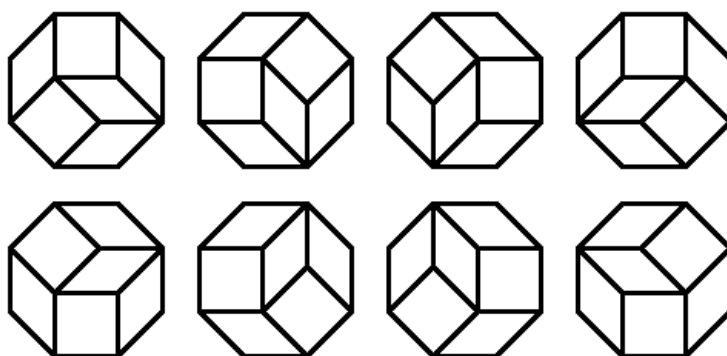
Let  $F(n, k) = \sum_{i=1}^{n-k+1} M(i, i+k-1)$  For example,  $F(100, 10) = 463628.5$  and  $F(10^5, 10^4) = 675348207.5$ .

Find  $F(10^7, 10^5)$ . If the sum is not an integer, use .5 to denote a half. Otherwise, use .0 instead.

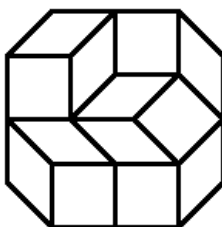
### Problem 594: Rhombus Tilings

For a polygon  $P$ , let  $t(P)$  be the number of ways in which  $P$  can be tiled using rhombi and squares with edge length 1. Distinct rotations and reflections are counted as separate tilings.

For example, if  $O$  is a regular octagon with edge length 1, then  $t(O) = 8$ . As it happens, all these 8 tilings are rotations of one another:



Let  $O_{\{a,b\}}$  be the equal-angled convex octagon whose edges alternate in length between  $a$  and  $b$ . For example, here is  $O_{\{2,1\}}$ , with one of its tilings:



You are given that  $t(O_{\{1,1\}}) = 8$ ,  $t(O_{\{2,1\}}) = 76$  and  $t(O_{\{3,2\}}) = 456572$ .

Find  $t(O_{\{4,2\}})$ .

### Problem 595: Incremental Random Sort

A deck of cards numbered from 1 to  $n$  is shuffled randomly such that each permutation is equally likely.

The cards are to be sorted into ascending order using the following technique:

1. Look at the initial sequence of cards. If it is already sorted, then there is no need for further action. Otherwise, if any subsequences of cards happen to be in the correct place relative to one another (ascending with no gaps), then those subsequences are fixed by attaching the cards together. For example, with 7 cards initially in the order 4123756, the cards labelled 1, 2 and 3 would be attached together, as would 5 and 6.
2. The cards are 'shuffled' by being thrown into the air, but note that any correctly sequenced cards remain attached, so their orders are maintained. The cards (or bundles of attached cards) are then picked up randomly. You should assume that this randomisation is unbiased, despite the fact that some cards are single, and others are grouped together.
3. Repeat steps 1 and 2 until the cards are sorted.

Let  $S(n)$  be the expected number of shuffles needed to sort the cards. Since the order is checked before the first shuffle,  $S(1) = 0$ . You are given that  $S(2) = 1$ , and  $S(5) = 4213/871$ .

Find  $S(52)$ , and give your answer rounded to 8 decimal places.

### Problem 596: Number of lattice points in a hyperball

Let  $T(r)$  be the number of integer quadruplets  $x, y, z, t$  such that  $x^2 + y^2 + z^2 + t^2 \leq r^2$ . In other words,  $T(r)$  is the number of lattice points in the four-dimensional hyperball of radius  $r$ .

You are given that  $T(2) = 89$ ,  $T(5) = 3121$ ,  $T(100) = 493490641$  and  $T(10^6) = 49348022079085897$ .

Find  $T(10^8) \bmod 1000000007$ .



## Problem 597: Torpids

The Torpids are rowing races held annually in Oxford, following some curious rules:

- A division consists of  $n$  boats (typically 13), placed in order based on past performance.
- All boats within a division start at 40 metre intervals along the river, in order with the highest-placed boat starting furthest upstream.
- The boats all start rowing simultaneously, upstream, trying to catch the boat in front while avoiding being caught by boats behind.
- Each boat continues rowing until *either* it reaches the finish line *or* it catches up with ("bumps") a boat in front.
- The finish line is a distance  $L$  metres (the course length, in reality about 1800 metres) upstream from the starting position of the lowest-placed boat. (Because of the staggered starting positions, higher-placed boats row a slightly shorter course than lower-placed boats.)
- When a "bump" occurs, the "bumping" boat takes no further part in the race. The "bumped" boat must continue, however, and may even be "bumped" again by boats that started two or more places behind it.
- After the race, boats are assigned new places within the division, based on the bumps that occurred. Specifically, for any boat  $A$  that started in a lower place than  $B$ , then  $A$  will be placed higher than  $B$  in the new order if and only if one of the following occurred:
  1.  $A$  bumped  $B$  directly
  2.  $A$  bumped another boat that went on to bump  $B$
  3.  $A$  bumped another boat, that bumped yet another boat, that bumped  $B$
  4. etc

**NOTE:** For the purposes of this problem you may disregard the boats' lengths, and assume that a bump occurs precisely when the two boats draw level. (In reality, a bump is awarded as soon as physical contact is made, which usually occurs when there is much less than a full boat length's overlap.)

Suppose that, in a particular race, each boat  $B_j$  rows at a steady speed  $v_j = -\log X_j$  metres per second, where the  $X_j$  are chosen randomly (with uniform distribution) between 0 and 1, independently from one another. These speeds are relative to the riverbank: you may disregard the flow of the river.

Let  $p(n, L)$  be the probability that the new order is an **even permutation** of the starting order, when there are  $n$  boats in the division and  $L$  is the course length.

For example, with  $n=3$  and  $L=160$ , labelling the boats as  $A, B, C$  in starting order with  $C$  highest, the different possible outcomes of the race are as follows:

Bumps occurring	New order	Permutation	Probability
none	A, B, C	even	$4/15$
B bumps C	A, C, B	odd	$8/45$
A bumps B	B, A, C	odd	$1/3$
B bumps C, then A bumps C	C, A, B	even	$4/27$
A bumps B, then B bumps C	C, B, A	odd	$2/27$

Therefore,  $p(3, 160) = 4/15 + 4/27 = 56/135$

You are also given that  $p(4, 400) = 0.5107843137$ , rounded to 10 digits after the decimal point.

Find  $p(13, 1800)$  rounded to 10 digits after the decimal point.

### Problem 598: Split Divisibilities

Consider the number 48.

There are five pairs of integers  $a$  and  $b$  ( $a \leq b$ ) such that  $a \times b = 48$ : (1,48), (2,24), (3,16), (4,12) and (6,8).

It can be seen that both 6 and 8 have 4 divisors.

So of those five pairs one consists of two integers with the same number of divisors.

In general:

Let  $C(n)$  be the number of pairs of positive integers  $a \times b = n$ , ( $a \leq b$ ) such that  $a$  and  $b$  have the same number of divisors;

so  $C(48)=1$ .

You are given  $C(10!)=3$ : (1680, 2160), (1800, 2016) and (1890,1920).

Find  $C(100!)$

### Problem 599: Distinct Colourings of a Rubik's Cube

The well-known **Rubik's Cube** puzzle has many fascinating mathematical properties. The  $2 \times 2 \times 2$  variant has 8 cubelets with a total of 24 visible faces, each with a coloured sticker. Successively turning faces will rearrange the cubelets, although not all arrangements of cubelets are reachable without dismantling the puzzle.

Suppose that we wish to apply new stickers to a  $2 \times 2 \times 2$  Rubik's cube in a non-standard colouring. Specifically, we have  $n$  different colours available (with an unlimited supply of stickers of each colour), and we place one sticker on each of the 24 faces in any arrangement that we please. We are not required to use all the colours, and if desired the same colour may appear in more than one face of a single cubelet.

We say that two such colourings  $c_1, c_2$  are *essentially distinct* if a cube coloured according to  $c_1$  cannot be made to match a cube coloured according to  $c_2$  by performing mechanically possible Rubik's Cube moves.

For example, with two colours available, there are 183 essentially distinct colourings.

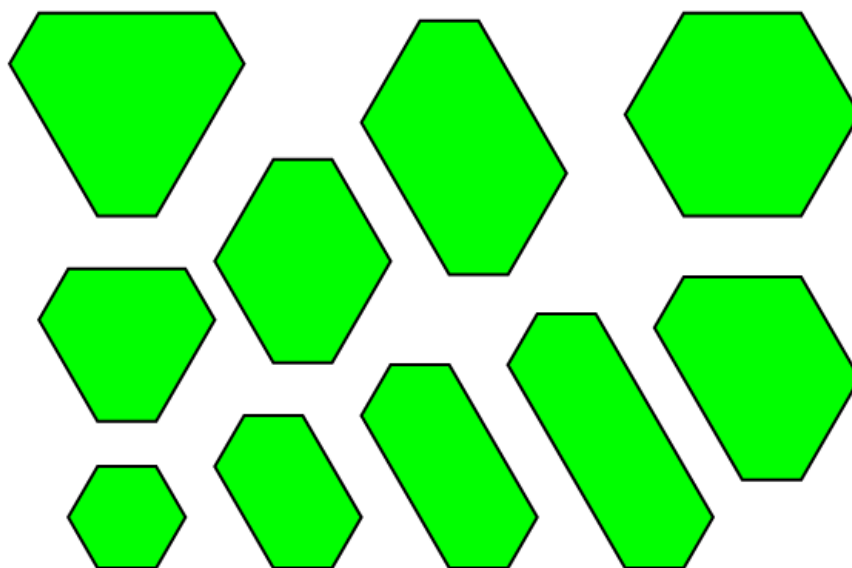
How many essentially distinct colourings are there with 10 different colours available?

### Problem 600: Integer sided equiangular hexagons

Let  $H(n)$  be the number of distinct integer sided **equiangular** convex hexagons with perimeter not exceeding  $n$ . Hexagons are distinct if and only if they are not **congruent**.

You are given  $H(6) = 1$ ,  $H(12) = 10$ ,  $H(100) = 31248$ .

Find  $H(55106)$ .



*Equiangular hexagons with perimeter not exceeding 12*

### Problem 601: Divisibility streaks

For every positive number  $n$  we define the function  $\text{streak}(n)=k$  as the smallest positive integer  $k$  such that  $n+k$  is not divisible by  $k+1$ .

E.g:

13 is divisible by 1

14 is divisible by 2

15 is divisible by 3

16 is divisible by 4

17 is NOT divisible by 5

So  $\text{streak}(13) = 4$ .

Similarly:

120 is divisible by 1

121 is NOT divisible by 2

So  $\text{streak}(120) = 1$ .

Define  $P(s, N)$  to be the number of integers  $n$ ,  $1 < n < N$ , for which  $\text{streak}(n) = s$ .

So  $P(3, 14) = 1$  and  $P(6, 10^6) = 14286$ .

Find the sum, as  $i$  ranges from 1 to 31, of  $P(i, 4^i)$ .

### Problem 602: Product of Head Counts

Alice enlists the help of some friends to generate a random number, using a single unfair coin. She and her friends sit around a table and, starting with Alice, they take it in turns to toss the coin. Everyone keeps a count of how many heads they obtain individually. The process ends as soon as Alice obtains a Head. At this point, Alice multiplies all her friends' Head counts together to obtain her random number.

As an illustration, suppose Alice is assisted by Bob, Charlie, and Dawn, who are seated round the table in that order, and that they obtain the sequence of Head/Tail outcomes **THHH—TTTT—THHT—H** beginning and ending with Alice. Then Bob and Charlie each obtain 2 heads, and Dawn obtains 1 head. Alice's random number is therefore  $2 \times 2 \times 1 = 4$ .

Define  $e(n, p)$  to be the expected value of Alice's random number, where  $n$  is the number of friends helping (excluding Alice herself), and  $p$  is the probability of the coin coming up Tails.

It turns out that, for any fixed  $n$ ,  $e(n, p)$  is always a polynomial in  $p$ . For example,  $e(3, p) = p^3 + 4p^2 + p$

Define  $c(n, k)$  to be the coefficient of  $p^k$  in the polynomial  $e(n, p)$ . So  $c(3, 1) = 1$ ,  $c(3, 2) = 4$ , and  $c(3, 3) = 1$ .

You are given that  $c(100, 40) \equiv 986699437 \pmod{10^9+7}$

Find  $c(10000000, 4000000) \pmod{10^9+7}$ .

### Problem 603: Substring sums of prime concatenations

Let  $S(n)$  be the sum of all contiguous integer-substrings that can be formed from the integer  $n$ . The substrings need not be distinct.

For example,  $S(2024) = 2 + 0 + 2 + 4 + 20 + 02 + 24 + 202 + 024 + 2024 = 2304$

Let  $P(n)$  be the integer formed by concatenating the first  $n$  primes together. For example,  $P(7) = 2357111317$ .

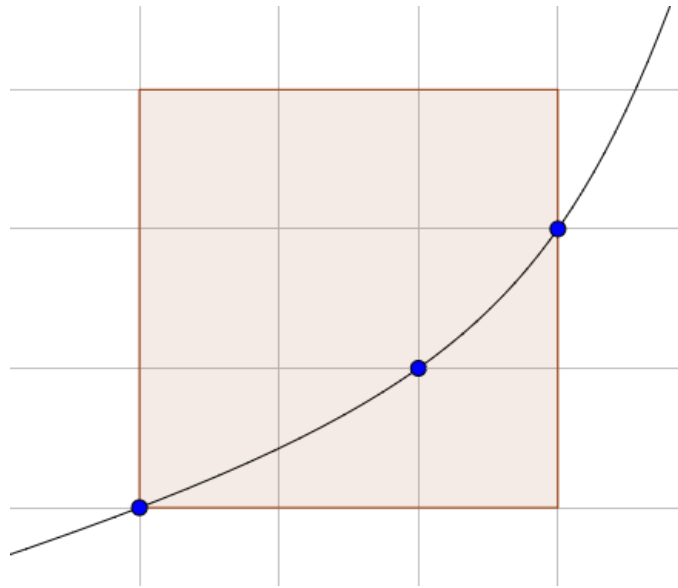
Let  $C(n, k)$  be the integer formed by concatenating  $k$  copies of  $P(n)$  together. For example,  $C(7, 3) = 235711131723571113172357111317$ .

Evaluate  $S(C(10^6, 10^{12})) \pmod{10^9 + 7}$ .

#### Problem 604: Convex path in square

Let  $F(N)$  be the maximum number of lattice points in an axis-aligned  $N \times N$  square that the graph of a single **strictly convex** increasing function can pass through.

You are given that  $F(1) = 2$ ,  $F(3) = 3$ ,  $F(9) = 6$ ,  $F(11) = 7$ ,  $F(100) = 30$  and  $F(50000) = 1898$ . Below is the graph of a function reaching the maximum 3 for  $N=3$ :



Find  $F(10^{18})$ .

#### Problem 605: Pairwise Coin-Tossing Game

Consider an  $n$ -player game played in consecutive pairs: Round 1 takes place between players 1 and 2, round 2 takes place between players 2 and 3, and so on and so forth, all the way up to round  $n$ , which takes place between players  $n$  and 1. Then round  $n+1$  takes place between players 1 and 2 as the entire cycle starts again.

In other words, during round  $r$ , player  $((r-1) \bmod n) + 1$  faces off against player  $(r \bmod n) + 1$ .

During each round, a fair coin is tossed to decide which of the two players wins that round. If any given player wins both rounds  $r$  and  $r+1$ , then that player wins the entire game.

Let  $P_n(k)$  be the probability that player  $k$  wins in an  $n$ -player game, in the form of a reduced fraction. For example,  $P_3(1) = 12/49$  and  $P_6(2) = 368/1323$ .

Let  $M_n(k)$  be the product of the reduced numerator and denominator of  $P_n(k)$ . For example,  $M_3(1) = 588$  and  $M_6(2) = 486864$ .

Find the last 8 digits of  $M_{10^8+7}(10^4+7)$ .

#### Problem 606: Gozinta Chains II

A **gozinta chain** for  $n$  is a sequence  $\{1, a, b, \dots, n\}$  where each element properly divides the next.

For example, there are eight distinct gozinta chains for 12:

$\{1, 12\}$ ,  $\{1, 2, 12\}$ ,  $\{1, 2, 4, 12\}$ ,  $\{1, 2, 6, 12\}$ ,  $\{1, 3, 12\}$ ,  $\{1, 3, 6, 12\}$ ,  $\{1, 4, 12\}$  and  $\{1, 6, 12\}$ .

Let  $S(n)$  be the sum of all numbers  $k$ , not exceeding  $n$ , which have 252 distinct gozinta chains.

You are given  $S(10^6) = 8462952$  and  $S(10^{12}) = 623291998881978$ .

Find  $S(10^{36})$ , giving the last nine digits of your answer.

### Problem 607: Marsh Crossing

Frodo and Sam need to travel 100 leagues due East from point A to point B. On normal terrain, they can cover 10 leagues per day, and so the journey would take 10 days. However, their path is crossed by a long marsh which runs exactly South-West to North-East, and walking through the marsh will slow them down. The marsh is 50 leagues wide at all points, and the mid-point of AB is located in the middle of the marsh. A map of the region is shown in the diagram below:



The marsh consists of 5 distinct regions, each 10 leagues across, as shown by the shading in the map. The strip closest to point A is relatively light marsh, and can be crossed at a speed of 9 leagues per day. However, each strip becomes progressively harder to navigate, the speeds going down to 8, 7, 6 and finally 5 leagues per day for the final region of marsh, before it ends and the terrain becomes easier again, with the speed going back to 10 leagues per day.

If Frodo and Sam were to head directly East for point B, they would travel exactly 100 leagues, and the journey would take approximately 13.4738 days. However, this time can be shortened if they deviate from the direct path.

Find the shortest possible time required to travel from point A to B, and give your answer in days, rounded to 10 decimal places.

### Problem 608: Divisor Sums

Let  $D(m,n) = \sum_{d|m} \sum_{k=1}^n \sigma_0(kd)$  where  $d$  runs through all divisors of  $m$  and  $\sigma_0(n)$  is the number of divisors of  $n$ .

You are given  $D(3!, 10^2) = 3398$  and  $D(4!, 10^6) = 268882292$ .

Find  $D(200!, 10^{12}) \pmod{10^9 + 7}$ .

### Problem 609: $\pi$ sequences

For every  $n \geq 1$  the **prime-counting** function  $\pi(n)$  is equal to the number of primes not exceeding  $n$ .  
E.g.  $\pi(6) = 3$  and  $\pi(100) = 25$ .

We say that a sequence of integers  $u = (u_0, \dots, u_m)$  is a  *$\pi$  sequence* if

- $u_n \geq 1$  for every  $n$
- $u_{n+1} = \pi(u_n)$
- $u$  has two or more elements

For  $u_0 = 10$  there are three distinct  $\pi$  sequences:  $(10, 4)$ ,  $(10, 4, 2)$  and  $(10, 4, 2, 1)$ .

Let  $c(u)$  be the number of elements of  $u$  that are not prime.

Let  $p(n, k)$  be the number of  $\pi$  sequences  $u$  for which  $u_0 \leq n$  and  $c(u) = k$ .

Let  $P(n)$  be the product of all  $p(n, k)$  that are larger than 0.

You are given:  $P(10) = 3 \times 8 \times 9 \times 3 = 648$  and  $P(100) = 31038676032$ .

Find  $P(10^8)$ . Give your answer modulo 1000000007.

### Problem 610: Roman Numerals II

A random generator produces a sequence of symbols drawn from the set  $\{V, X, L, C, D, M, \#\}$ . Each item in the sequence is determined by selecting one of these symbols at random, independently of the other items in the sequence. At each step, the seven letters are equally likely to be selected, with probability 14% each, but the  $\#$  symbol only has a 2% chance of selection.

We write down the sequence of letters from left to right as they are generated, and we stop at the first occurrence of the  $\#$  symbol (without writing it). However, we stipulate that what we have written down must always (when non-empty) be a valid Roman numeral representation in minimal form. If appending the next letter would contravene this then we simply skip it and try again with the next symbol generated.

Please take careful note of [About... Roman Numerals](#) for the definitive rules for this problem on what constitutes a "valid Roman numeral representation" and "minimal form". For example, the (only) sequence that represents 49 is XLIX. The subtractive combination IL is invalid because of rule (ii), while XXXIX is valid but not minimal. The rules do not place any restriction on the number of occurrences of M, so all integers have a valid representation. These are the same rules as were used in Problem 89, and members are invited to solve that problem first.

Find the expected value of the number represented by what we have written down when we stop. (If nothing is written down then count that as zero.) Give your answer rounded to 8 places after the decimal point.

### Problem 611: Hallway of square steps

Peter moves in a hallway with  $N+1$  doors consecutively numbered from 0 through  $N$ . All doors are initially closed. Peter starts in front of door 0, and repeatedly performs the following steps:

- First, he walks a positive square number of doors away from his position.
- Then he walks another, larger square number of doors away from his new position.
- He toggles the door he faces (opens it if closed, closes it if open).
- And finally returns to door 0.

We call an action any sequence of those steps. Peter never performs the exact same action twice, and makes sure to perform all possible actions that don't bring him past the last door.

Let  $F(N)$  be the number of doors that are open after Peter has performed all possible actions. You are given that  $F(5) = 1$ ,  $F(100) = 27$ ,  $F(1000) = 233$  and  $F(10^6) = 112168$ .

Find  $F(10^{12})$ .

### Problem 612: Friend numbers

Let's call two numbers *friend numbers* if their representation in base 10 has at least one common digit. E.g. 1123 and 3981 are friend numbers.

Let  $f(n)$  be the number of pairs  $(p, q)$  with  $1 \leq p \leq q \leq n$  such that  $p$  and  $q$  are friend numbers.  $f(100) = 1539$ .

Find  $f(10^{18}) \bmod 1000267129$ .

### Problem 613: Pythagorean Ant

Dave is doing his homework on the balcony and, preparing a presentation about Pythagorean triangles, has just cut out a triangle with side lengths 30cm, 40cm and 50cm from some cardboard, when a gust of wind blows the triangle down into the garden.

Another gust blows a small ant straight onto this triangle. The poor ant is completely disoriented and starts to crawl straight ahead in random direction in order to get back into the grass.

Assuming that all possible positions of the ant within the triangle and all possible directions of moving on are equiprobable, what is the probability that the ant leaves the triangle along its longest side? Give your answer rounded to 10 digits after the decimal point.

### Problem 614: Special partitions 2

An **integer partition** of a number  $n$  is a way of writing  $n$  as a sum of positive integers. Partitions that differ only by the order of their summands are considered the same.

We call an integer partition *special* if 1) all its summands are distinct, and 2) all its even summands are also divisible by 4. For example, the special partitions of 10 are:  $10 = 1+4+5 = 3+7 = 1+9$ . The number 10 admits many more integer partitions (a total of 42), but only those three are special.

Let be  $P(n)$  the number of special integer partitions of  $n$ . You are given that  $P(1) = 1$ ,  $P(2) = 0$ ,  $P(3) = 1$ ,  $P(6) = 1$ ,  $P(10) = 3$ ,  $P(100) = 37076$  and  $P(1000) = 3699177285485660336$ .

Find  $\sum_{i=1}^{10^7} P(i)$ . Give the result modulo  $10^9+7$ .

### Problem 615: The millionth number with at least one million prime factors

Consider the natural numbers having at least 5 prime factors, which don't have to be distinct.  
Sorting these numbers by size gives a list which starts with:

32=2·2·2·2·2  
48=2·2·2·2·3  
64=2·2·2·2·2  
72=2·2·2·3·3  
80=2·2·2·2·5  
96=2·2·2·2·2·3  
...

So, for example, the fifth number with at least 5 prime factors is 80.

Find the millionth number with at least one million prime factors.  
Give your answer modulo 123454321.

### Problem 616: Creative numbers

Alice plays the following game, she starts with a list of integers  $L$  and on each step she can either:

- remove two elements  $a$  and  $b$  from  $L$  and add  $a^b$  to  $L$
- or conversely remove an element  $c$  from  $L$  that can be written as  $a^b$ , with  $a$  and  $b$  being two integers such that  $a, b > 1$ , and add both  $a$  and  $b$  to  $L$

For example starting from the list  $L=\{8\}$ , Alice can remove 8 and add 2 and 3 resulting in  $L=\{2,3\}$  in a first step. Then she can obtain  $L=\{9\}$  in a second step.

Note that the same integer is allowed to appear multiple times in the list.

An integer  $n > 1$  is said to be *creative* if for any integer  $m > 1$  Alice can obtain a list that contains  $m$  starting from  $L=\{n\}$ .

Find the sum of all creative integers less than or equal to  $10^{12}$ .

### Problem 617: Mirror Power Sequence

For two integers  $n, e > 1$ , we define a  $(n, e)$ -MPS (*Mirror Power Sequence*) to be an infinite sequence of integers  $(a_i)_{i \geq 0}$  such that for all  $i \geq 0$ ,  $a_{i+1} = \min(a_i^e, n - a_i^e)$  and  $a_i > 1$ .

Examples of such sequences are the two  $(18, 2)$ -MPS sequences made of alternating 2 and 4.

Note that even though such a sequence is uniquely determined by  $n, e$  and  $a_0$ , for most values such a sequence does not exist. For example, no  $(n, e)$ -MPS exists for  $n < 6$ .

Define  $C(n)$  to be the number of  $(n, e)$ -MPS for some  $e$ , and 
$$D(N) = \sum_{n=2}^N C(n).$$
 You are given that  $D(10) = 2$ ,  $D(100) = 21$ ,  $D(1000) = 69$ ,  $D(10^6) = 1303$  and  $D(10^{12}) = 1014800$ .

Find  $D(10^{18})$ .

### Problem 618: Numbers with a given prime factor sum

Consider the numbers 15, 16 and 18:

$15 = 3 \times 5$  and  $3 + 5 = 8$ .

$16 = 2 \times 2 \times 2 \times 2$  and  $2 + 2 + 2 + 2 = 8$ .

$18 = 2 \times 3 \times 3$  and  $2 + 3 + 3 = 8$ .

15, 16 and 18 are the only numbers that have 8 as sum of the prime factors (counted with multiplicity).

We define  $S(k)$  to be the sum of all numbers  $n$  where the sum of the prime factors (with multiplicity) of  $n$  is  $k$ .

Hence  $S(8) = 15 + 16 + 18 = 49$ .

Other examples:  $S(1) = 0$ ,  $S(2) = 2$ ,  $S(3) = 3$ ,  $S(5) = 5 + 6 = 11$ .

The Fibonacci sequence is  $F_1 = 1$ ,  $F_2 = 1$ ,  $F_3 = 2$ ,  $F_4 = 3$ ,  $F_5 = 5$ ....

Find the last nine digits of 
$$\sum_{k=2}^{24} S(F_k).$$

### Problem 619: Square subsets

For a set of positive integers  $\{a, a+1, a+2, \dots, b\}$ , let  $C(a,b)$  be the number of non-empty subsets in which the product of all elements is a perfect square.

For example  $C(5,10)=3$ , since the products of all elements of  $\{5, 8, 10\}$ ,  $\{5, 8, 9, 10\}$  and  $\{9\}$  are perfect squares, and no other subsets of  $\{5, 6, 7, 8, 9, 10\}$  have this property.

You are given that  $C(40,55) = 15$ , and  $C(1000,1234) \bmod 1000000007 = 975523611$ .

Find  $C(1000000,1234567) \bmod 1000000007$ .

### Problem 620: Planetary Gears

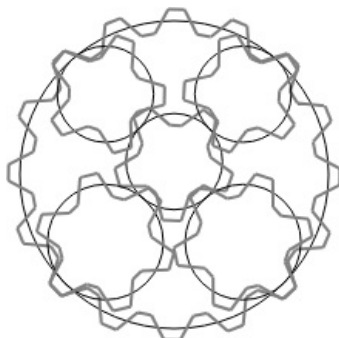
A circle  $C$  of circumference  $c$  centimetres has a smaller circle  $S$  of circumference  $s$  centimetres lying off-centre within it. Four other distinct circles, which we call "planets", with circumferences  $p, p, q, q$  centimetres respectively ( $p < q$ ), are inscribed within  $C$  but outside  $S$ , with each planet touching both  $C$  and  $S$  tangentially. The planets are permitted to overlap one another, but the boundaries of  $S$  and  $C$  must be at least 1cm apart at their closest point.

Now suppose that these circles are actually gears with perfectly meshing teeth at a pitch of 1cm.  $C$  is an internal gear with teeth on the inside. We require that  $c, s, p, q$  are all integers (as they are the numbers of teeth), and we further stipulate that any gear must have at least 5 teeth.

Note that "perfectly meshing" means that as the gears rotate, the ratio between their angular velocities remains constant, and the teeth of one gear perfectly align with the grooves of the other gear and vice versa. Only for certain gear sizes and positions will it be possible for  $S$  and  $C$  each to mesh perfectly with all the planets. Arrangements where not all gears mesh perfectly are not valid.

Define  $g(c,s,p,q)$  to be the number of such gear arrangements for given values of  $c, s, p, q$ : it turns out that this is finite as only certain discrete arrangements are possible satisfying the above conditions. For example,  $g(16,5,5,6)=9$ .

Here is one such arrangement:



Let  $G(n) = \sum_{\{s+p+q \leq n\}} g(s+p+q, s, p, q)$  where the sum only includes cases with  $p < q$ ,  $p \geq 5$ , and  $s \geq 5$ , all integers. You are given that  $G(16)=9$  and  $G(20)=205$ .

Find  $G(500)$ .

### Problem 621: Expressing an integer as the sum of triangular numbers

Gauss famously proved that every positive integer can be expressed as the sum of three **triangular numbers** (including 0 as the lowest triangular number). In fact most numbers can be expressed as a sum of three triangular numbers in several ways.

Let  $G(n)$  be the number of ways of expressing  $n$  as the sum of three triangular numbers, regarding different arrangements of the terms of the sum as distinct.

For example,  $G(9) = 7$ , as 9 can be expressed as:  $3+3+3$ ,  $0+3+6$ ,  $0+6+3$ ,  $3+0+6$ ,  $3+6+0$ ,  $6+0+3$ ,  $6+3+0$ .

You are given  $G(1000) = 78$  and  $G(10^6) = 2106$ .

Find  $G(17\,526 \times 10^9)$ .



## Problem 622: Riffle Shuffles

A riffle shuffle is executed as follows: a deck of cards is split into two equal halves, with the top half taken in the left hand and the bottom half taken in the right hand. Next, the cards are interleaved exactly, with the top card in the right half inserted just after the top card in the left half, the 2nd card in the right half just after the 2nd card in the left half, etc. (Note that this process preserves the location of the top and bottom card of the deck)

Let  $s(n)$  be the minimum number of consecutive riffle shuffles needed to restore a deck of size  $n$  to its original configuration, where  $n$  is a positive even number.

Amazingly, a standard deck of 52 cards will first return to its original configuration after only 8 perfect shuffles, so  $s(52) = 8$ . It can be verified that a deck of 86 cards will also return to its original configuration after exactly 8 shuffles, and the sum of all values of  $n$  that satisfy  $s(n) = 8$  is 412.

Find the sum of all values of  $n$  that satisfy  $s(n) = 60$ .

## Problem 623: Lambda Count

The *lambda-calculus* is a universal model of computation at the core of functional programming languages. It is based on *lambda-terms*, a minimal programming language featuring only function definitions, function calls and variables. Lambda-terms are built according to the following rules:

- Any variable  $x$  (single letter, from some infinite alphabet) is a lambda-term.
- If  $M$  and  $N$  are lambda-terms, then  $(M N)$  is a lambda-term, called the *application* of  $M$  to  $N$ .
- If  $x$  is a variable and  $M$  is a term, then  $(\lambda x. M)$  is a lambda-term, called an *abstraction*. An abstraction defines an anonymous function, taking  $x$  as parameter and sending back  $M$ .

A lambda-term  $T$  is said to be *closed* if for all variables  $x$ , all occurrences of  $x$  within  $T$  are contained within some abstraction  $(\lambda x. M)$  in  $T$ . The smallest such abstraction is said to *bind* the occurrence of the variable  $x$ . In other words, a lambda-term is closed if all its variables are bound to parameters of enclosing functions definitions. For example, the term  $(\lambda x. x)$  is closed, while the term  $(\lambda x. (x y))$  is not because  $y$  is not bound.

Also, we can rename variables as long as no binding abstraction changes. This means that  $(\lambda x. x)$  and  $(\lambda y. y)$  should be considered equivalent since we merely renamed a parameter. Two terms equivalent modulo such renaming are called *alpha-equivalent*. Note that  $(\lambda x. (\lambda y. (x y)))$  and  $(\lambda x. (\lambda x. (x x)))$  are not alpha-equivalent, since the abstraction binding the first variable was the outer one and becomes the inner one. However,  $(\lambda x. (\lambda y. (x y)))$  and  $(\lambda y. (\lambda x. (y x)))$  are alpha-equivalent.

The following table regroups the lambda-terms that can be written with at most 15 symbols, symbols being parenthesis, lambda, dot and variables.

$(\lambda x. x)$	$(\lambda x. (x x))$	$(\lambda x. (\lambda y. x))$	$(\lambda x. (\lambda y. y))$
$(\lambda x. (x (x x)))$	$(\lambda x. ((x x) x))$	$(\lambda x. (\lambda y. (x x)))$	$(\lambda x. (\lambda y. (x y)))$
$(\lambda x. (\lambda y. (y x)))$	$(\lambda x. (\lambda y. (y y)))$	$(\lambda x. (x (\lambda y. x)))$	$(\lambda x. (x (\lambda y. y)))$
$(\lambda x. ((\lambda y. x) x))$	$(\lambda x. ((\lambda y. y) x))$	$((\lambda x. x) (\lambda x. x))$	$(\lambda x. (x (x x)))$
$(\lambda x. (x ((x x) x)))$	$(\lambda x. ((x x) (x x)))$	$(\lambda x. ((x (x x) x))$	$(\lambda x. (((x x) x) x))$

Let be  $\Lambda(n)$  the number of distinct closed lambda-terms that can be written using at most  $n$  symbols, where terms that are alpha-equivalent to one another should be counted only once. You are given that  $\Lambda(6) = 1$ ,  $\Lambda(9) = 2$ ,  $\Lambda(15) = 20$  and  $\Lambda(35) = 3166438$ .

Find  $\Lambda(2000)$ . Give the answer modulo 1,000,000,007.

## Problem 624: Two heads are better than one

An unbiased coin is tossed repeatedly until two consecutive heads are obtained. Suppose these occur on the  $(M-1)$ th and  $M$ th toss.

Let  $P(n)$  be the probability that  $M$  is divisible by  $n$ . For example, the outcomes HH, HTHH, and THTTHH all count towards  $P(2)$ , but THH and HTTHH do not.

You are given that  $P(2) = \frac{3}{5}$  and  $P(3) = \frac{9}{31}$ . Indeed, it can be shown that  $P(n)$  is always a rational number.

For a prime  $p$  and a fully reduced fraction  $\frac{a}{b}$ , define  $Q(\frac{a}{b}, p)$  to be the smallest positive  $q$  for which  $a \leq b q \pmod{p}$ .

For example  $Q(P(2), 109) = Q(\frac{3}{5}, 109) = 66$  because  $5 \cdot 66 = 330 \equiv 3 \pmod{109}$  and 66 is the smallest positive such number.

Similarly  $Q(P(3), 109) = 46$ .

Find  $Q(P(10^{18}), 1,000,000,009)$ .

### Problem 625: Gcd sum

$G(N) = \sum_{j=1}^N \sum_{i=1}^j \text{gcd}(i, j)$ .

You are given:  $G(10) = 122$ .

Find  $G(10^{11})$ . Give your answer modulo 998244353

### Problem 626: Counting Binary Matrices

A binary matrix is a matrix consisting entirely of 0s and 1s. Consider the following transformations that can be performed on a binary matrix:

- Swap any two rows
- Swap any two columns
- Flip all elements in a single row (1s become 0s, 0s become 1s)
- Flip all elements in a single column

Two binary matrices  $A$  and  $B$  will be considered *equivalent* if there is a sequence of such transformations that when applied to  $A$  yields  $B$ . For example, the following two matrices are equivalent:

$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

via the sequence of two transformations "Flip all elements in column 3" followed by "Swap rows 1 and 2".

Define  $c(n)$  to be the maximum number of  $n \times n$  binary matrices that can be found such that no two are equivalent. For example,  $c(3) = 3$ . You are also given that  $c(5) = 39$  and  $c(8) = 656108$ .

Find  $c(20)$ , and give your answer modulo 1,001,001,011.

### Problem 627: Counting products

Consider the set  $S$  of all possible products of  $n$  positive integers not exceeding  $m$ , that is

$S = \{x_1 x_2 \dots x_n \mid 1 \leq x_1, x_2, \dots, x_n \leq m\}$

Let  $F(m, n)$  be the number of the distinct elements of the set  $S$ .

For example,  $F(9, 2) = 36$  and  $F(30, 2) = 308$ .

Find  $F(30, 10001) \bmod 1,000,000,007$ .

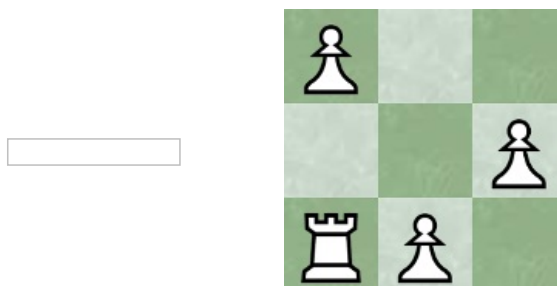
### Problem 628: Open chess positions

A position in chess is an (orientated) arrangement of chess pieces placed on a chessboard of given size. In the following, we consider all positions in which  $n$  pawns are placed on a  $n \times n$  board in such a way, that there is a single pawn in every row and every column.

We call such a position an *open* position, if a rook, starting at the (empty) lower left corner and using only moves towards the right or upwards, can reach the upper right corner without moving onto any field occupied by a pawn.

Let  $f(n)$  be the number of open positions for an  $n \times n$  chessboard.

For example,  $f(3) = 2$ , illustrated by the two open positions for a  $3 \times 3$  chessboard below.



You are also given  $f(5) = 70$ .

Find  $f(10^8) \bmod 1,008,691,207$ .

### Problem 629: Scatterstone Nim

Alice and Bob are playing a modified game of Nim called Scatterstone Nim, with Alice going first, alternating turns with Bob. The game begins with an arbitrary set of stone piles with a total number of stones equal to  $n$ .

During a player's turn, he/she must pick a pile having at least 2 stones and perform a split operation, dividing the pile into an arbitrary set of  $p$  non-empty, arbitrarily-sized piles where  $2 \leq p \leq k$  for some fixed constant  $k$ . For example, a pile of size 4 can be split into  $\{1, 3\}$  or  $\{2, 2\}$ , or  $\{1, 1, 2\}$  if  $k = 3$  and in addition  $\{1, 1, 1, 1\}$  if  $k = 4$ .

If no valid move is possible on a given turn, then the other player wins the game.

A winning position is defined as a set of stone piles where a player can ultimately ensure victory no matter what the other player does.

Let  $f(n, k)$  be the number of winning positions for Alice on her first turn, given parameters  $n$  and  $k$ . For example,  $f(5, 2) = 3$  with winning positions  $\{1, 1, 1, 2\}$ ,  $\{1, 4\}$ ,  $\{2, 3\}$ . In contrast,  $f(5, 3) = 5$  with winning positions  $\{1, 1, 1, 2\}$ ,  $\{1, 1, 3\}$ ,  $\{1, 4\}$ ,  $\{2, 3\}$ ,  $\{5\}$ .

Let  $g(n)$  be the sum of  $f(n, k)$  over all  $2 \leq k \leq n$ . For example,  $g(7) = 66$  and  $g(10) = 291$ .

Find  $g(200) \bmod (10^9 + 7)$ .

### Problem 630: Crossed lines

Given a set,  $L$ , of unique lines, let  $M(L)$  be the number of lines in the set and let  $S(L)$  be the sum over every line of the number of times that line is crossed by another line in the set. For example, two sets of three lines are shown below:



In both cases  $M(L)$  is 3 and  $S(L)$  is 6: each of the three lines is crossed by two other lines. Note that even if the lines cross at a single point, all of the separate crossings of lines are counted.

Consider points  $(T_{2k-1}, T_{2k})$ , for integer  $k \geq 1$ , generated in the following way:

$$S_0 = 290797$$

$$S_{n+1} = \{S_n^2 \bmod 50515093$$

$$T_n = (S_n \bmod 2000) - 1000$$

For example, the first three points are:  $(527, 144)$ ,  $(-488, 732)$ ,  $(-454, -947)$ . Given the first  $n$  points generated in this manner, let  $L_n$  be the set of **unique** lines that can be formed by joining each point with every other point, the lines being extended indefinitely in both directions. We can then define  $M(L_n)$  and  $S(L_n)$  as described above.

For example,  $M(L_3) = 3$  and  $S(L_3) = 6$ . Also  $M(L_{100}) = 4948$  and  $S(L_{100}) = 24477690$ .

Find  $S(L_{2500})$ .

### Problem 631: Constrained Permutations

Let  $(p_1 p_2 \dots p_k)$  denote the permutation of the set  $\{1, \dots, k\}$  that maps  $p_i$  to  $i$ . Define the length of the permutation to be  $k$ ; note that the empty permutation  $()$  has length zero.

Define an *occurrence* of a permutation  $p = (p_1 p_2 \dots p_k)$  in a permutation  $P = (P_1 P_2 \dots P_n)$  to be a sequence  $1 \leq t_1 < t_2 < \dots < t_k \leq n$  such that  $p_i < p_j$  if and only if  $P_{t_i} < P_{t_j}$  for all  $i, j \in \{1, \dots, k\}$ .

For example,  $(1243)$  occurs twice in the permutation  $(314625)$ : once as the 1st, 3rd, 4th and 6th elements  $(3, 1, 4, 6)$ , and once as the 2nd, 3rd, 4th and 6th elements  $(1, 1, 4, 6)$ .

Let  $f(n, m)$  be the number of permutations  $P$  of length at most  $n$  such that there is no occurrence of the permutation  $1243$  in  $P$  and there are at most  $m$  occurrences of the permutation  $21$  in  $P$ .

For example,  $f(2, 0) = 3$ , with the permutations  $()$ ,  $(1)$ ,  $(1, 2)$  but not  $(2, 1)$ .

You are also given that  $f(4, 5) = 32$  and  $f(10, 25) = 294,400$ .

Find  $f(10^{18}, 40) \bmod 1,000,000,007$ .

Problem 632: Square prime factors

For an integer  $n$ , we define the *square prime factors* of  $n$  to be the primes whose square divides  $n$ . For example, the square prime factors of  $1500=2^2 \times 3 \times 5^3$  are 2 and 5.

Let  $C_k(N)$  be the number of integers between 1 and  $N$  inclusive with exactly  $k$  square prime factors. You are given some values of  $C_k(N)$  in the table below.

$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$N=10$	7	3	0	0	0
$N=10^2$	61	36	3	0	0
$N=10^3$	608	343	48	1	0
$N=10^4$	6083	3363	533	21	0
$N=10^5$	60794	33562	5345	297	2
$N=10^6$	607926	335438	53358	3218	60
$N=10^7$	6079291	3353956	533140	32777	834
$N=10^8$	60792694	33539196	5329747	329028	9257

Find the product of all non-zero  $C_k(10^{16})$ . Give the result reduced modulo 1,000,000,007.

Problem 633: Square prime factors II

For an integer  $n$ , we define the *square prime factors* of  $n$  to be the primes whose square divides  $n$ . For example, the square prime factors of  $1500=2^2 \times 3 \times 5^3$  are 2 and 5.

Let  $C_k(N)$  be the number of integers between 1 and  $N$  inclusive with exactly  $k$  square prime factors. It can be shown that with growing  $N$  the ratio  $\frac{C_k(N)}{N}$  gets arbitrarily close to a constant  $c_k$ , as suggested by the table below.

$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$C_k(10)$	7	3	0	0
$C_k(10^2)$	61	36	3	0
$C_k(10^3)$	608	343	48	1
$C_k(10^4)$	6083	3363	533	21
$C_k(10^5)$	60794	33562	5345	297
$C_k(10^6)$	607926	335438	53358	3218
$C_k(10^7)$	6079291	3353956	533140	32777
$C_k(10^8)$	60792694	33539196	5329747	329028
$C_k(10^9)$	607927124	335389706	53294365	3291791
$c_k$	$\frac{6}{\pi^2}$	$3.3539$	$3.2921$	$9.7046$

Find  $c_7$ . Give the result in scientific notation rounded to 5 significant digits, using  $e$  to separate mantissa and exponent. E.g. if the answer were 0.000123456789, then the answer format would be 1.23456e-4.

Problem 634: Numbers of the form

Define  $F(n)$  to be the number of integers  $x \leq n$  that can be written in the form  $x=a^2b^3$ , where  $a$  and  $b$  are integers not necessarily different and both greater than 1.

For example,  $32=2^2 \times 2^3$  and  $72=3^2 \times 2^3$  are the only two integers less than 100 that can be written in this form. Hence,  $F(100)=2$ .

Further you are given  $F(10^4)=130$  and  $F(10^6)=2014$ .

Find  $F(10^{18})$ .