QUANTITATIVE FINANCE

SUMMARY REPORT

Paper : More stochastic expansions for the pricing of vanilla options with cash dividends

(https://arxiv.org/pdf/2106.12051v1.pdf)

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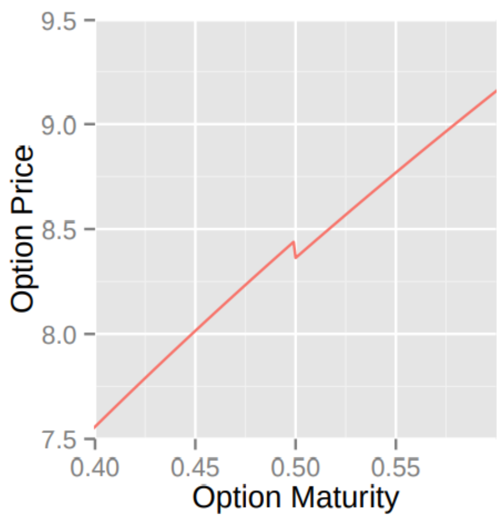
Summary

Introduction

Dividends are better modeled as fixed cash amounts, especially for short maturities. For very short maturities (a quarter), the dividend amount is known exactly. For longer maturities, proportional dividends are more appropriate. From the arbitrage-free hypothesis and ignoring transaction costs, a discrete dividend implies that the stock jumps from the dividend amount at the dividend ex-date. To stay in Black-Scholes-Merton framework, we compute the equivalent dividend yield but this causes the model to be inconsistent around a dividend date. To obey the continuity relation, the volatility must jump on the dividend date.



Another issue is an exploding forward dividend yield which will impact numerical accuracy when solving the Black-Scholes PDE. A natural model is to represent the equity as a piecewise lognormal process, which jumps from the dividend amount at each exercise date. However no exact analytical formula exists for this problem. Hence we take interest in the stochastic Taylor expansion technique around a shifted lognormal model by Etore and Gobet (2012) leading to first-, second- and third-order formulas. The paper applies their Taylor expansion technique, but on a different proxy, aiming for more accurate approximations of European options prices.

Notations

The asset price with proportional dividends $y_i \in [0, 1)$ and cash dividends $\delta_i \geq 0$ is represented by the process $S^{(y,\delta)}$. The piecewise lognormal dynamics of the stock price under the risk-neutral measure Q between two dividend dates is:

$$dS_t^{(y,\delta)} = \sigma_t S_t^{(y,\delta)} dW_t + (r_t - q_t) S_t^{(y,\delta)} dt$$

where r_t is the risk free deterministic interest rate and q_t a deterministic repo spread. And it jumps at the dividend dates t_i:

$$S_{t_i}^{(y,\delta)} = S_{t_i^-}^{(y,\delta)} - (\delta_i + y_i S_{t_i^-}^{(y,\delta)}) = S_{t_i^-}^{(y,\delta)}(1-y_i) - \delta_i$$

The zero discount factor B_t, the discount factor D_t and the lognormal martingale M_t are defined as:

$$B_t = e^{-\int_0^t r_s ds},\ D_t = e^{-\int_0^t (r_s - q_s)ds},\ M_t = e^{\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds}$$

For initial value S_0 we have, $S_t = S_0 \frac{M_t}{D_t}$. We set $\pi_{i,n} = \prod_{j=i+1}^{n} (1 - y_j)$ and will make use of the simplified notation $\widehat{\delta}_i = \delta_i \pi_{i,n} \frac{D_{t_i}}{D_T}$, where T is option maturity.

Lemma 2.1 :

$$S_t^{(y,\delta)} = \pi_{0,n} S_t - \sum_{i=1}^{n} \delta_i \pi_{i,n} \frac{S_t}{S_{t_i}}.$$

Expansion around the Forward

Etore and Gobet (2012) considers an expansion around a shifted lognormal spot model where the asset price follows:

$$\bar{S}_t^{(y,\delta)} = S_t - \mathbb{E}\left[\sum_{i=1}^{n} \delta_i \pi_{i,n} \frac{S_t}{S_{t_i}}\right] = \pi_{0,n} S_t - \sum_{i=1}^{n} \hat{\delta}_i$$

The paper instead does an expansion on the forward model:

$$F_t = \left(\pi_{0,n} S_0 - \sum_{i=1}^{n} \delta_i \pi_{i,n} D_{t_i}\right) \frac{M_t}{D_t}$$

This leads to the following stochastic Taylor expansions:

First-order expansion : For a smooth function h satisfying H_2, we have

$$\begin{aligned} \mathbb{E}\left[B_T h(S_T^{(y,\delta)} - K)\right] =& \mathbb{E}\left[B_T h(F_T - K)\right] \\ &+ \sum_{i=1}^{n} \hat{\delta}_i \left(\partial_K \mathbb{E}\left[B_T h(F_T e^{\int_{t_i}^T \sigma_s^2 ds} - K)\right]\right. \\ &\left. - \partial_K \mathbb{E}\left[B_T h(F_T e^{\int_0^T \sigma_s^2 ds} - K)\right]\right) \\ &+ \mathrm{Error}_2(h) \end{aligned}$$

$$\textit{where}\ |\,\mathrm{Error}_2(h)| \le c(1 + S_0^p) \sup_i \left(\delta_i \sigma \sqrt{t_i}\right)^2.$$

Second-order expansion : For a smooth function h satisfying H_3, we have

$$
\begin{aligned}
&\mathbb{E}\left[B_T h(S_T^{(y,\delta)} - K)\right] \\
=&\mathbb{E}\left[B_T h(F_T - K)\right] \\
&+ \sum_{i=1}^{n} \hat{\delta}_i \left(\partial_K \mathbb{E}\left[B_T h(F_T e^{\int_{t_i}^{T} \sigma_s^2 ds} - K)\right] - \partial_K \mathbb{E}\left[B_T h(F_T e^{\int_0^T \sigma_s^2 ds} - K)\right] \right) \\
&+ \frac{1}{2} \sum_{1 \leq i,j \leq n} \hat{\delta}_i \hat{\delta}_j \partial_K^2 \mathbb{E}\left[B_T h(F_T e^{\int_{t_i}^{T} \sigma_s^2 ds + \int_{t_j}^{T} \sigma_s^2 ds} - K)\right] e^{\int_{\max(t_i,t_j)}^{T} \sigma_s^2 ds} \\
&- \left(\sum_{j=1}^{n} \hat{\delta}_j \right) \sum_{i=1}^{n} \hat{\delta}_i \partial_K^2 \mathbb{E}\left[B_T h(F_T e^{\int_0^T \sigma_s^2 ds + \int_{t_i}^{T} \sigma_s^2 ds} - K)\right] e^{\int_{t_i}^{T} \sigma_s^2 ds} \\
&+ \frac{1}{2} \left(\sum_{j=1}^{n} \hat{\delta}_j \right)^2 \partial_K^2 \mathbb{E}\left[B_T h(F_T e^{2\int_0^T \sigma_s^2 ds} - K)\right] e^{\int_0^T \sigma_s^2 ds} \\
&+ \mathrm{Error}_3(h)
\end{aligned}
$$

$$
where\ |\mathrm{Error}_3(h)| \leq c(1 + S_0^p) \sup_i \left(\delta_i \sigma \sqrt{t_i}\right)^3.
$$

Applying to $h(x) = |x|^+$ would result in an approximation for the call option prices of second-order. Even though this choice of h does not obey H_3, Etore and Gobet (2012) have proven that the results stay valid. Under the forward model, the option price of strike k, forward $f = E(F_T)$ and maturity T is obtained by the Black formula (Black, 1976),

$$
V_B(f,k) = \eta B_T \left[x \Phi(\eta d_1(f,k)) - k \Phi(\eta d_2(f,k))\right]
$$

Expansion around the Lehman model

Here an adjustment of the spot and the strike is made in the Black Scholes formula. The spot is adjusted by the present value of near-term dividends, strike is adjusted by far-away dividends. Let the near and far parts be,

$$
X_T^n = \sum_i \frac{T - t_i}{T} \delta_i \pi_{i,n} \frac{D_{t_i}}{D_T}
$$

$$
X_T^f = \sum_i \frac{t_i}{T} \delta_i \pi_{i,n} \frac{D_{t_i}}{D_T} .
$$

In the Lehman model, the asset follows,

$$
\bar{S}_t = \left(\pi_{0,n} \frac{S_0}{D_t} - X_t^n \right) M_t - X_t^f
$$

The vanilla option price can be obtained through the Black formula $V_B(F, K)$ for a forward F, strike K, expiry T. We also define,

$$\bar{F}_t = \left(\pi_{0,n}\frac{S_0}{D_t} - X_t^n\right) M_t \ ,$$
$$\bar{f} = \frac{S_0}{D_T}\pi_{0,n} - X_T^n \ ,$$
$$\bar{k} = K + X_T^f \ .$$

This leads to the following stochastic Taylor expansions:

First-order expansion : For a smooth function h satisfying H_2, we have

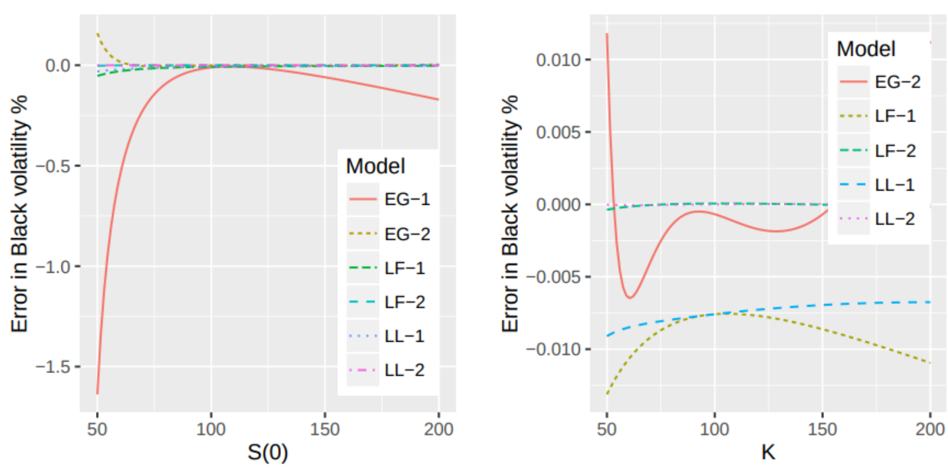
$$\begin{aligned}
&\mathbb{E}\left[B_T h(S_T^{(y,\delta)} - K)\right] \\
=&\mathbb{E}\left[B_T h\left(\bar{F}_T - \bar{k}\right)\right] \\
&+ \sum_{i=1}^{n} \frac{T-t_i}{T}\hat{\delta}_i \left(\partial_K \mathbb{E}\left[B_T h\left(\bar{F}_T e^{\int_{t_i}^T \sigma_s^2 ds} - \bar{k}\right)\right] - \partial_K \mathbb{E}\left[B_T h\left(\bar{F}_T e^{\int_0^T \sigma_s^2 ds} - \bar{k}\right)\right]\right) \\
&+ \sum_{i=1}^{n} \frac{t_i}{T}\hat{\delta}_i \left(\partial_K \mathbb{E}\left[B_T h\left(\bar{F}_T e^{\int_{t_i}^T \sigma_s^2 ds} - \bar{k}\right)\right] - \partial_K \mathbb{E}\left[B_T h\left(\bar{F}_T - \bar{k}\right)\right]\right) + \mathrm{Error}_2(h)
\end{aligned}$$

$$where \ |\mathrm{Error}_2(h)| \leq c(1 + S_0^p)\sup_i \left(\left(\delta_i \sigma \tfrac{T-t_i}{T}\sqrt{t_i}\right)^2 + \left(\delta_i \sigma \tfrac{t_i}{T}\sqrt{T-t_i}\right)^2\right).$$

Similarly, the paper details second and third order expansions. The expansion around the Lehman model should provide a more stable approximation over the full range of dividend dates.

Numerical Experiments

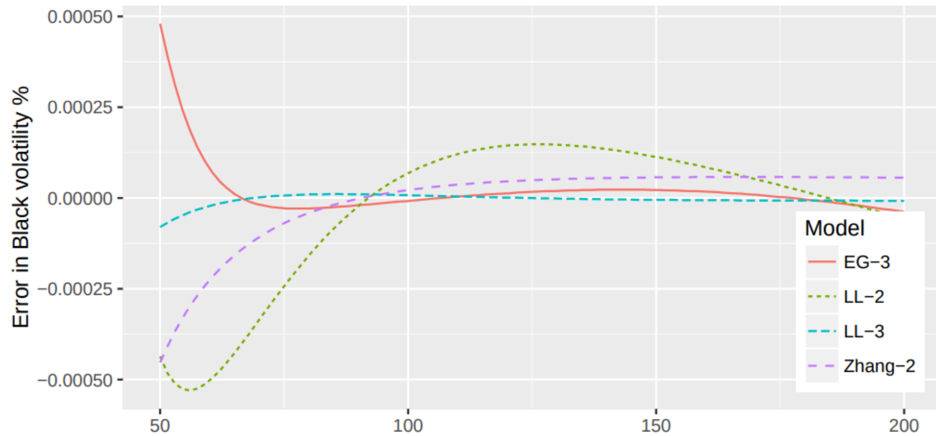
Considering a Single Dividend case, for vanilla call options of maturity T = 1 on an asset of spot S(0) = 100 and look at the error in implied Black volatility for a range of strikes, we get



(a) $t_1 = 0.1$

(b) $t_1 = 0.1$ removing EG-1

The expansions on the forward LF-1 and LF-2 are more accurate when the dividend ex-date is close to the valuation date. For low strikes, LF-1 becomes more accurate than the second-order expansion on the displaced strike EG-2.
In the Many Dividends case,



The second-order expansion LL-2 is good and improves greatly over the second order expansion on the displaced strike. The third-order expansion on the displaced strike EG-3 does not really improve over LL-2 or Zhang-2 expansions on this example. But the third-order expansion on the Lehman model LL-3 is significantly more accurate.

Conclusion

Among the first-order expansions, the Lehman proxy is as robust as Zhang first-order expansion, but purely analytical.
Among the second-order methods, the Zhang method is found to be the most accurate. LL-2 is as robust as Zhang method.
Among third-order expansion using the Lehman model as proxy is the most accurate in practical use cases. Its accuracy is not as good for very long term options where dividends would be modeled as cash. Furthermore, it will be significantly slower for a very large number of dividends. In reality however, the practice is to model long term dividends as proportional and those drawbacks may not be relevant.