# Factorizating the Brauer monoid in polynomial time

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#### Abstract

Finding a minimal factorization for a generic semigroup can be done by using the Froidure-Pin Algorithm, which is not feasible for semigroups of large sizes. On the other hand, if we restrict our attention to just a particular semigroup, we could leverage its structure to obtain a much faster algorithm. In particular,  $\mathcal{O}(N^2)$  algorithms are known for factorizing the Symmetric group  $S_N$  and the Temperley-Lieb monoid  $\mathcal{TL}_N$ , but none for their superset the Brauer monoid  $\mathcal{B}_N$ . In this paper we hence propose a  $\mathcal{O}(N^4)$  factorization algorithm for  $\mathcal{B}_N$ . At each iteration, the algorithm rewrites the input  $X \in \mathcal{B}_N$  as  $X = X' \circ p_i$  such that  $\ell(X') = \ell(X) - 1$ , where  $p_i$  is a factor for X and  $\ell$  is a length function that returns the minimal number of factors needed to generate X.

### 1 Introduction

The Brauer monoid  $\mathcal{B}_N$  is a diagram algebra introduced by Brauer in 1937 [2], and later utilized by Kauffman and Magarshak in 1995 to define a mapping between RNA secondary structures and elements of  $\mathcal{B}_N$  (which they call "tangles") [10]. The correspondence between properties of RNA secondary structures, and properties of the factorizations of the tangles they get mapped to, can then be studied [13]. But for this, factorization of Brauer tangles is necessary, and this is the major topic of the present paper.

It can be easily proven that if there exists a length function for  $\mathcal{B}_N$  (i.e. a function that returns the minimal amount of factors the input tangle is generated by) computable in polynomial time, then there exists a polynomial time factorization algorithm (see Corollary 4). In this paper we will find two length functions computable in quadratic time, one of which is computable in linear time if we assume some precomputation steps are already done. The key insight is that, since the length function for tangles in the symmetric group  $S_N$  (which is a submonoid of  $\mathcal{B}_N$ ) just involves counting the number of crossings they have, it would be interesting to map every tangle in  $\mathcal{B}_N$  with length l to a tangle in  $S_N$  with exactly l crossings. In this way, if this mapping could be done in polynomial time, then we would have found our length function.

The paper is divided in the following way. In Section 2 we begin by recalling some related work. Section 3 is dedicated to giving some preliminaries on the Brauer monoid, setting up the problem,

illustrating a naive  $\mathcal{O}(N^5)$  algorithm and we outline two assumptions we are basing some proofs on. In Section 4 we go into more detail on the idea of defining a mapping  $\tau: \mathcal{B}_N \to S_N$ , and some of its properties. Section 5 will focus on just proving one theorem: i.e. that some edges in a tangle "cannot come back" if they satisfy a particular property. This will be the base for Section 6, in which we finally outline a quadratic time algorithm for  $\tau$ . In Section 7 we will propose the two length functions that will then be used in Section 8 to illustrate the final factorization algorithm with  $\mathcal{O}(N^4)$  time complexity.

### 2 Related works

In a previous work [13] we proposed a factorization algorithm that uses a set of polynomial-time heuristics for finding a possibly non-minimal factorization, we then refine it by applying the axioms for the Brauer monoid as a Term Rewriting System (TRS). This will eventually ensure minimality, but since the TRS is not confluent the overall time complexity is difficult to calculate and likely to be huge. This is because for the symmetric group, it is known [17] that the maximal number of minimal factorizations is

$$\frac{\binom{N}{2}!}{1^{N-1}3^{N-2}\cdots(2N-3)^1}$$

and the TRS will have to check all of them before deciding that there are no more reductions possible.

The Froidure-Pin Algorithm can find the minimal factorization for any element in a finite semigroup S by performing |S| + |A| - |G| - 1 operations, where A is the set of axioms of S, and G the set of generators [9]. For the Brauer monoid the resulting time complexity lower-bound is therefore  $\Omega(|\mathcal{B}_N|) = \Omega((2N-1)!!)^1$ .

Algorithm 13 of "Computing with semigroups" [6] is capable of finding a factorization, but it is not guaranteed to be minimal. It computes two sets  $\mathfrak{R}$  (the  $\mathcal{R}$ -classes of  $\mathcal{B}_N$ ) and  $(\mathcal{B}_N)\lambda$ . The size of  $\mathfrak{R}$  for  $\mathcal{B}_N$  can be calculated by the following recurrence relation:

$$a(0) = 1$$
  
 $a(1) = 1$   
 $a(N) = a(N-1) + (N-1)a(N-2)$ 

which is the number of ways to partition a set of size N into subsets of size one or two [5]. This recurrent relation is clearly bounded below by N!.

The authors also say that for regular semigroups (as in the case for  $\mathcal{B}_N$ ),  $|\mathfrak{R}| = |(\mathcal{B}_N)\lambda|$ , and that the calculation for  $\mathfrak{R}$  is redundant. This does not reduce the time complexity because  $(\mathcal{B}_N)\lambda$  still needs to be computed.

Lastly, two submonoids of  $\mathcal{B}_N$  can be factorized in quadratic time. The symmetric group can be factorized by using the BUBBLESORT algorithm, and the Temperley-Lieb monoid can be factorized by using the algorithm proposed by Ernst et al. [8]. We will discuss these two algorithms in Appendix A.

 $<sup>^{1}(2</sup>N-1)!!$  is the "odd double factorial", defined as  $(2N-1)!!=1\cdot 3\cdots (2N-3)\cdot (2N-1)$ .

### 3 Preliminaries

Given  $N \geq 0$ , arrange 2N nodes in two rows of N nodes each. Nodes in the upper row are labelled with  $[N] = \{1, 2, \dots, N\}$  while nodes in the bottom row are labelled with  $[N'] = \{1', 2', \dots, N'\}$ . A tangle is a set of N edges connecting any two distinct nodes in  $[N] \cup [N']$  such that each node is in exactly one edge. We will represent edges as e = (x, y) in a canonical form in which x < y if both x and y are nodes in the same row, while in the case that e connects nodes from in different rows,  $x \in [N]$  and  $y \in [N']$ .

Given two tangles X and Y we define their composition  $X \circ Y$  by stacking X on top of Y (matching the bottom row of X with the top row of Y) and then tracing the path of each edge (we will ignore internal loops in our setup). The set of all tangles on 2N nodes under composition is called the Brauer monoid  $\mathcal{B}_N$  [2] and the identity tangle is  $I_N = (1, 1')(2, 2') \cdots (i, i') \cdots (N, N')$  (see Figure 1).

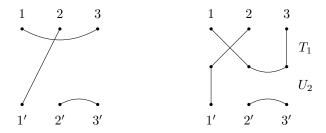


Figure 1: On the left a tangle X = (1,3)(2,1')(2',3') in  $\mathcal{B}_3$ , on the right its unique minimal factorization  $T_1 \circ U_2$ .

For our purposes, it will be useful to classify edges by where they are connected. Tangles have two types of edges [4]:

- hooks are edges in which both nodes are on the top or on the bottom row. The former ones are called *upper hooks* and the latter *lower hooks*;
- transversals are edges in which one node is on the top row while the other one is on the bottom. We further classify transversal edges as:
  - positive transversal are in the form x > y'
  - zero transversal are in the form x = y'
  - negative transversal are in the form x < y'

where x is the upper node and y' is the lower node.

Given a tangle X, we say that two edges cross if they intersect each other in the diagrammatic representation of X (assuming the edges are drawn in a way that minimizes the number of crossings). The size of an edge e = (x, y) is defined as |e| = |x - y| (x and y are arbitrary nodes).

Given a tangle X with an upper hook h = (i, i+1) of size one and another distinct edge e = (x, y), we say that we merge h with e by removing them from X and connecting their respective nodes such that the newly added edges  $e_1$  and  $e_2$  do not cross (see Figure 2). In particular:

• if e is a upper hook such that x < i and i + 1 < y, then  $e_1 = (x, i)$  and  $e_2 = (i + 1, y)$ ;

- if e is a lower hook such that  $x \le i$  and  $i + 1 \le y$ , then  $e_1 = (i, x)$  and  $e_2 = (i + 1, y)$ ;
- if e is a negative transversal such that x < i and  $i+1 \le y$ , then  $e_1 = (x,i)$  and  $e_2 = (i+1,y)$ ;
- if e is a positive transversal such that x > i + 1 and  $i \ge y$ , then  $e_1 = (i, y)$  and  $e_2 = (i + 1, x)$ ;
- for all other combinations, the merging of h and e is undefined, since it will not be useful for our use case (at the end of Section 5 it will become clear why).

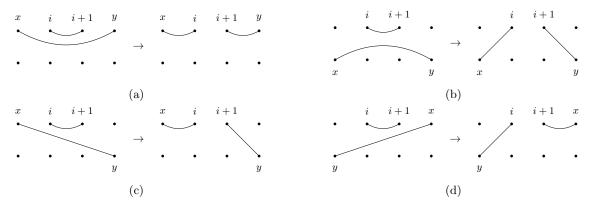


Figure 2: The four cases for which we define the merge operation. In all cases, e = (x, y) and h = (i, i + 1). (a) e is an upper hook. (b) e is a lower hook. (c) e is a negative transversal. (d) e is a positive transversal.

The Brauer monoid can also be defined as the set of tangles generated by the composition of prime tangles, which we call T-primes and U-primes, defined as:

- $T_i = (1, 1')(2, 2') \cdots (i, i' + 1)(i + 1, i') \cdots (N, N');$
- $U_i = (1, 1')(2, 2') \cdots (i, i+1)(i', i'+1) \cdots (N, N').$

where i is called the index of that prime tangle. These primes also satisfy the axioms in Table 1. We classify these axioms into three types:

- Delete rules: because they decrease the number of primes to be composed;
- Braid rules: because they resemble the braid relation in the symmetric group;
- Swap rules: because they allow to commute primes without changing the resulting tangle.

We call a word  $F \in \{T_i, U_i\}^*$  a factorization (the empty word coincides with the identity tangle  $I_N$ ). If this factorization is reduced, i.e. there is no other equivalent word of shorter length, then it is called *minimal*.

Define  $\otimes : \mathcal{B}_N \times \mathcal{B}_M \to \mathcal{B}_{N+M}$  to be the tensor product such that  $Z = X \otimes Y$  is the tangle in which Y is placed on the right of X. We call X and Y the components of Z [16]. If we assume  $F_X$  and  $F_Y$  to be minimal factorizations for X and Y, then we have that  $F_Z = F_X \circ F_Y$  is a minimal factorization for Z, therefore we can factorize each component separately and, for the rest of the paper, we will assume that every tangle has only one component.

```
Delete
                    T_i \circ T_i
1.
                    U_i \circ U_i
2.
                    T_i \circ U_i
3.
                   4.
5.
6.
7.
8.
9.
10.
Braid
                    \begin{array}{lcl} T_i \circ T_j \circ T_i & = & T_j \circ T_i \circ T_j \iff |i-j| = 1 \\ T_i \circ U_j \circ T_i & = & T_j \circ U_i \circ T_j \iff |i-j| = 1 \end{array}
11.
12.
Swap
                    \begin{array}{lll} T_i \circ T_j & = & T_j \circ T_i \iff |i-j| > 1 \\ T_i \circ U_j & = & U_j \circ T_i \iff |i-j| > 1 \\ U_i \circ T_j & = & T_j \circ U_i \iff |i-j| > 1 \\ U_i \circ U_j & = & U_j \circ U_i \iff |i-j| > 1 \end{array}
13.
14.
15.
16.
```

Table 1: Axioms for the Brauer monoid. Axioms from 1 to 10 are called *delete rules*, 11 and 12 are braid rules and 13-16 are swap rules.

A length function  $\ell(X): \mathcal{B}_N \to \mathbb{N}$  is a function that returns the minimal number of prime factors required to compose the tangle X, in other words, it returns the length of a minimal factorization for X (define  $\ell(I_N) = 0$ ). If  $\ell(X)$  can be computed in polynomial time, say t(N), then Algorithm 1 returns a minimal factorization in  $\mathcal{O}(N^3t(N))$  (see Corollary 4 for the proof).

Lastly, we list two assumptions we believe are true but were not able to prove. They will be useful for some proofs in the following Sections.

**Assumption 1.** Every factorization in the Brauer monoid can be reduced to a minimal one by a sequence of "delete", "braid" and "swap" rules. In other words, we do not need to increase the factorization length in order to find a shorter one.

**Assumption 2.** If a tangle X has k crossings, then there exists a minimal factorization F with exactly k T-primes and no other factorization with fewer T-primes exists.

We empirically tested these assumptions, for the methodology we refer to Appendix B.

# 4 Mapping $\mathcal{B}_N$ to $S_N$

Tangles in the symmetric groups  $S_N$  are generated only by T-primes. This implies that, given a tangle X in  $S_N$ , calculating  $\ell(X)$  will amount to just counting its number of crossings, which can be done in  $\mathcal{O}(N^2)$  time<sup>2</sup>. With this in mind, it would be useful to have a function  $\tau: \mathcal{B}_N \to S_N$ 

<sup>&</sup>lt;sup>2</sup>There is a faster approach that brings down the time complexity to  $\mathcal{O}(N \log N)$  [11], but as we will see in Section 6, for our purposes it will be much more useful to have the actual factorization for X, which cannot be done faster than  $\mathcal{O}(N^2)$ .

**Algorithm 1** Finds the minimal factorization for any tangle in  $\mathcal{B}_N$ .

```
Require: X \in \mathcal{B}_N
   function Factorize \mathcal{B}_N(X)
       l \leftarrow \ell(X)
       F \leftarrow \text{empty list}
       while l \neq 0 do
            if (i, i+1) \in X then
                 h \leftarrow (i, i+1)
                 for e \in X, e \neq h do
                      X' \leftarrow \text{merge } h \text{ with } e \text{ in } X \text{ (if defined)}
                      if \ell(X') = l - 1 then
                          X \leftarrow X'
                          Append U_i to F
                          break
            else
                 for i \in [1 ... N - 1] do
                      X' \leftarrow T_i \circ X
                      if \ell(X') = l - 1 then
                          X \leftarrow X'
                          Append T_i to F
                          break
            l \leftarrow l - 1
       return F
```

that maps tangles  $X \in \mathcal{B}_N$  to tangles in  $S_N$  such that  $\ell(X) = \ell(\tau(X))$ . In this way, we could define  $\ell(X)$  by just computing  $\tau(X)$  and counting its number of crossings. Therefore if  $\tau$  can be computed in polynomial time, then  $\ell$  can be computed in polynomial time too. We now proceed to define  $\tau$ .

**Definition 1.** Given a factorization F for a tangle  $X \in \mathcal{B}_N$ , we define  $\tau$  as the function that maps each prime factor  $p_i$  of F to  $T_i$ . In other words,  $T_i \mapsto T_i$  and  $U_i \mapsto T_i$  for each T and U prime in F.

**Theorem 1.** If F is a minimal factorization, then  $\tau(F)$  is minimal too.

*Proof.* We prove this by contradiction (see Figure 3).

Let F be a minimal factorization and assume  $\tau(F)$  is not minimal. Therefore it can be rewritten to contain the subword  $T_i \circ T_i$  by using Axioms 11 and 13 ([1], Theorem 3.3.1).

Let  $s = r_1 r_2 \dots r_{k-1}$  be a sequence of rewritings for  $\tau(F)$  using Axioms 11 or 13 and  $r_k$  be a rewriting step using Axiom 1. For each  $r_i \in s$ , there exists at least one rewriting  $\bar{r}_i$  in the preimage  $\tau^{-1}(r_i)$ , meaning that  $r_i$  applied to  $\tau(F)$  maps to a rewriting step  $\bar{r}_i \in \tau^{-1}(r_i)$ , applied to F, that uses Axioms 11 to 16.

This implies that there exists a sequence of rewritings  $\bar{s} = \bar{r}_1 \bar{r}_2 \dots \bar{r}_{k-1}$  for F such that, after  $\bar{r}_{k-1}$ , F will contain one of the following subwords:  $T_i \circ T_i$ ,  $U_i \circ U_i$ ,  $T_i \circ U_i$  or  $U_i \circ T_i$ . Therefore,  $\bar{r}_k$  will use one of the Axioms from 13 to 16 to reduce F to a shorter factorization.

This implies that F was not minimal, which is a contradiction.

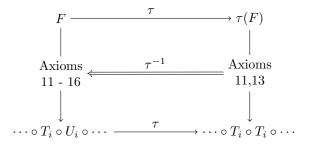


Figure 3: Diagram for the proof of Theorem 1. Assume F is minimal and  $\tau(F)$  is not. Axioms 11 and 13 applied to  $\tau(F)$  have preimage to axioms 11-16 applied to F. In the end,  $\tau(F)$  will contain a  $T_i \circ T_i$  because we assumed it was not minimal, which implies F was not minimal too and thus we have a contradiction.

Corollary 1. Let F be an arbitrary factorization. If  $\tau(F)$  is not minimal, then F is not minimal too.

*Proof.* This is the contrapositive of Theorem 1.

Corollary 2. Assuming F to be minimal, then  $|F| = |\tau(F)|$ .

Corollary 2 allows us to determine the maximal length a minimal factorization in  $\mathcal{B}_N$  can have.

Corollary 3. The longest minimal factorization in  $\mathcal{B}_N$  has length  $\frac{N(N-1)}{2}$ .

*Proof.* By Theorem 1 we know that every factorization F has another factorization  $\tau(F)$  with the same length composed only by T-primes. Therefore we only need to check the longest minimal factorization in  $S_N$ , which is well-known to be  $\frac{N(N-1)}{2}$ .

Corollary 4. Algorithm 1 has time complexity  $\mathcal{O}(N^3t(N))$ , where t(N) is the time complexity for the length function  $\ell$ .

*Proof.* In the case in which there exists edge h = (i, i + 1) in X, Algorithm 1 iterates through all edges, merges them with h and then calculates  $\ell$  for each of the resulting tangle. The overall complexity for this case is therefore  $\mathcal{O}(Nt(N))$ . By Corollary 3 we know that the longest factorization possible is quadratic and Algorithm 1 removes each of them one at a time. Therefore the time complexity for Algorithm 1 is  $\mathcal{O}(N^3t(N))$ .

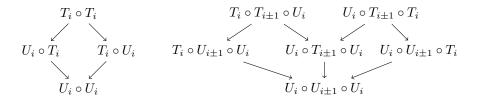
The function  $\tau$  we defined can be generalized to operate on an arbitrary subword of a factorization F.

**Definition 2.** Let F be a factorization for a tangle  $X \in \mathcal{B}_N$ , we define  $\tau^*$  be the function that applies  $\tau$  to only a subword of F.

**Theorem 2.** If F is a minimal factorization, then  $\tau^*(F)$  is minimal too.

*Proof.* The proof is similar to the one for Theorem 1 but it requires Assumption 1 because  $\tau^*(F)$  can be any factorization in  $\mathcal{B}_N$ , and every braid/swap r applied to  $\tau^*(F)$  has to correspond to a

braid/swap in F that is in the preimage of  $\tau^*(r)$ . There are also more ways in which  $\tau^*(F)$  can be not minimal. See the following Hasse diagram:



The arrows represent the preimage of  $\tau^*$ , each element has also a self-loop that was not drawn. For the proof of Theorem 1 only the left Hasse Diagram was applicable, but now also the right one has to be taken into consideration. Since we assumed that  $\tau^*(F)$  was not minimal, then by Assumption 1 it can be rewritten by a sequence of swaps and braids to contain any element in the above Hasse Diagram, but for all such elements there exists a preimage that must be in F and, since all preimages are not minimal, it implies that F was not minimal in the first place, hence we have a contradiction and  $\tau^*(F)$  must have been minimal too.

Corollary 5. Let F be an arbitrary factorization. If  $\tau^*(F)$  is not minimal, then F is not minimal too.

*Proof.* This is the contrapositive of Theorem 2.

Corollary 5 will be very helpful in Section 5, where we will use it to prove that some undesirable factorizations are always not minimal.

We can now use the above theorems to find a (not very useful) length function for  $\mathcal{B}_N$ . Given a tangle  $X \in \mathcal{B}_N$ , assume to have a minimal factorization F for X. Now compute  $\tau(F)$ , which corresponds to another tangle  $\tau(X) \in S_N$ . By Corollary 2 we know that the number of crossings of  $\tau(X) = |F|$  so we just count them and we obtain the number of factors for X. We can represent this mapping as the diagram in Figure 4.

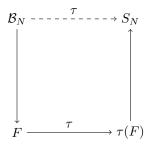


Figure 4: From a tangle in  $\mathcal{B}_N$  we obtain a minimal factorization F, we then compute  $\tau(F)$  which corresponds to a tangle in  $S_N$ .

Admittedly, this is not a very useful mapping because we are assuming to have F in the first place (which is what we are ultimately looking for). What we would like to find now is a way to extend  $\tau$  to arbitrary tangles, not just factorizations (the dashed arrow in Figure 4). In this way,

if we find a polynomial time algorithm for finding a tangle  $\tau(X) \in S_N$  such that its number of crossings is equal to  $\ell(X)$ , we would then have a polynomial time length function. We will present such an algorithm in Section 6.

## 5 Passing through and coming back

This Section is entirely dedicated to proving that if the number of T-primes an edge e "passes through" is greater or equal to |e|, then it does not pass through any U-prime. This is a key property we will leverage in Section 6, where at the end of it, we will have all the necessary components for computing  $\tau(X)$  in polynomial time. In this Section we will also argue why the merge operation presented in Section 3 is undefined for some edges.

**Definition 3** (Passing through). Given a factorization F for a tangle X, a prime tangle  $p_i \in F$  and an edge  $e \in X$ , we say that e "passes through"  $p_i$  if e is connected to the node i or i' of  $p_i$ . We indicate with #T(e) and #U(e) the number of T-primes and U-primes respectively e passes through in F.

We will now define what "coming back" means. The goal for these definitions is to prove that if #U(e) > 0 then e cannot come back. This will imply that  $\#U(e) > 0 \implies \#T(e) < |e|$  and therefore, by contraposition,  $\#T(e) \ge |e| \implies \#U(e) = 0$ , which is what we want to prove.

**Definition 4** (Coming back at p). Given a factorization F for a tangle X, a prime tangle  $p \in F$  and an edge  $e \in X$ , we say that e "comes back" at p if |e| decreases when passing through p.

**Definition 5** (Coming back in F). Given a factorization F for a tangle X and an edge  $e \in X$ , we say that e "comes back" in F if there exists a prime p at which e comes back.

See Figure 5 for an example.

**Theorem 3.** Let  $X \in \mathcal{B}_N$ ,  $e \in X$  and F any minimal factorization for X. Then e comes back in F if and only if #T(e) + #U(e) > |e|.

*Proof.* Forward direction.

Assume e comes back in F, then there exists a prime  $p_i$  at which e comes back, but if it comes back at  $p_i$  it means e passed through another prime  $p'_i$  with the same index i, and since every edge passes through at least |e| distinct primes, it must be that #T(e) + #U(e) > |e|.

The backward direction uses the same argument.

We will now use Theorem 3 to prove that a tangle in  $S_N$  has a pair of edges e and  $e_2$  as in Figure 6a if and only if e comes back. This theorem will allow us to ignore these tangles in the following theorems, because later on we will assume that a given edge does not come back.

**Theorem 4.** Given  $X \in S_N$  with a negative transversal  $e_1 = (x_1, y'_1)$ , if there exists another edge  $e_2 = (x_2, y'_2)$  such that  $x_2 < x_1$  and  $y'_2 > y'_1$  then  $e_1$  comes back in any factorization for X.

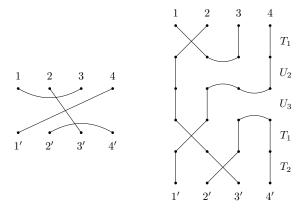


Figure 5: A tangle along with one of its minimal factorizations  $F = T_1 \circ U_2 \circ U_3 \circ T_1 \circ T_2$ . The edge (1,3) passes through  $T_1$  and  $U_2$ , edge (2,3') passes through two  $T_1$ s and one  $T_2$ , edge (4,1') passes through  $U_3, U_2$  and  $U_3$  and  $U_4$  and  $U_5$  are through  $U_5$  and  $U_7$  and  $U_8$  are the bottom-most  $U_8$  because it decreases in size, this also means that it comes back in  $U_8$ . Note also that  $U_8$  is the only edge that passes through only  $U_8$ -primes and satisfies the property  $U_8$ -primes are the property  $U_8$ -primes and satisfies the property  $U_8$ -primes are the property  $U_8$ -primes and satisfies the property  $U_8$ -primes are the primes  $U_8$ -primes are the primes  $U_8$ -primes  $U_8$ -primes

*Proof.* Let's set some variables:

$$\begin{array}{rcl} a & = & x_2 - 1 \\ b & = & x_1 - x_2 - 1 \\ c & = & N - x_1 \\ a' & = & y'_1 - 1 \\ b' & = & y'_2 - y'_1 - 1 \\ c' & = & N - y'_2 \end{array}$$

where a, b and c are the number of nodes with indices less than  $x_2$ , in between  $x_2$  and  $x_1$ , and greater than  $x_1$  respectively (same for a', b' and c'). Therefore we have that

$$a+b+c=a'+b'+c'$$

implying that

$$a' - a - b = c - b' - c'$$

which counts the least amount of edges that have to cross e from right to left. Notice now that

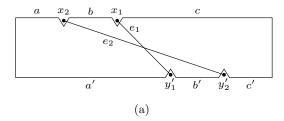
$$x_1 = b + x_2 + 1 = b + a + 2$$
  
 $y'_1 = a' + 1$ 

and therefore

$$|e_1| = y_1' - x_1 = a' + 1 - (b + a + 2) = a' - a - b - 1$$

By substitution, we can now obtain

$$a' - a - b = c - b' - c'$$
  
 $|e_1| + 1 = c - b' - c'$   
 $|e_1| < c - b' - c'$ 



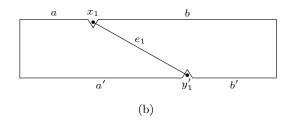


Figure 6: (a) Illustration for Theorem 4. If a tangle X in  $S_N$  contains an edge  $e_1$  and an edge  $e_2$  as shown, then  $e_1$  will come back in all minimal factorizations for X. (b) Illustration for Theorem 5. If  $e_1$  comes back in a minimal factorization, then there exists another edge  $e_2$  like in (a).

which shows that the least amount of edges that have to cross  $e_1$  is bigger than  $|e_1|$ , which implies that  $\#T(e_1) > |e_1|$  and therefore that  $e_1$  comes back in F (Theorem 3).

Note that this proof is not dependent on the factorization chosen, therefore  $e_1$  comes back in all minimal factorization for X.

**Theorem 5.** Given  $X \in S_N$  with a negative transversal  $e_1 = (x_1, y'_1)$  and minimal factorization F for X, if  $e_1$  comes back in F, then there exists another edge  $e_2 = (x_2, y'_2)$  such that  $x_2 < x_1$  and  $y'_2 > y'_1$  (Figure 6b).

*Proof.* Since  $e_1$  comes back in F, then  $\#T(e_1) > |e_1|$  (Theorem 3). Therefore  $\#T(e_1) = |e_1| + c$  where  $c \ge 1$ .

Let's set some variables:

$$\begin{array}{rcl}
a & = & x_1 - 1 \\
a' & = & y'_1 - 1 = a + |e_1|
\end{array}$$

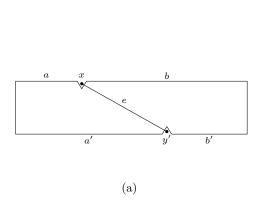
Where a is the number of nodes before  $x_1$  and a' is the number of nodes before y'. Suppose now that there is no edge  $e_2 = (x_2, y_2')$  such that  $x_2 < x_1$  and  $y_2' > y_1'$ , therefore we are assuming that all  $|e_1| + c$  edges that cross  $e_1$  connect nodes with indices greater than  $x_1$  to nodes with indices less than  $y_1'$ .

The number of nodes with indices less than  $y'_1$  that do not cross e is therefore

$$a' - (|e_1| + c) = a + |e_1| - |e_1| - c = a - c$$

Since we know that  $c \ge 1$ , we have that a - c < a, which implies that not all nodes with indices less than  $x_1$  can connect to lower nodes with indices less than  $y'_1$ . Therefore there must be another edge  $e_2 = (x_2, y'_2)$  that crosses  $e_1$  with  $x_2 < x_1$  and  $y'_2 > y'_1$ .

Corollary 6. Given  $X \in S_N$  with a negative transversal  $e = (x_1, y'_1)$ , then e comes back in every minimal factorization for X if and only if there exists another edge  $e_2 = (x_2, y'_2)$  such that  $x_2 < x_1$  and  $y'_2 > y'_1$ .



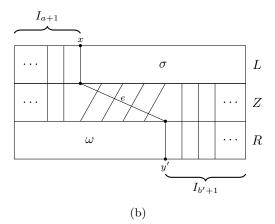


Figure 7: (a) A tangle  $X \in S_N$  with an edge e = (x, y') that does not come back. (b) The LZR decomposition of X.  $\sigma$  and  $\omega$  are tangles in  $S_b$  and  $S_{a'}$ , while  $Z = T_{a+1} \circ T_{a+2} \circ \cdots \circ T_{a+|e|}$ .

In light of Corollary 6, from now on we will just say that e does not come back in X, since it is independent of the factorization.

Assuming that we now have a tangle  $X \in S_N$  with an edge e that does not come back, we can always decompose it as  $X = L \circ Z \circ R$  (see Figure 7). This decomposition will come in handy when we will have to prove that a particular factorization is always not minimal, and it will turn out that the two tangles L and R can be safely ignored, simplifying the proof.

**Theorem 6** (LZR Decomposition). Let  $a, b \ge 0$  be natural integers. Let  $a', b' \ge 0$  be natural integers such that a' > a and a + b = a' + b'. Let  $X \in S_{a+b+1}$  be a tangle containing the edge e = (a+1, a'+1) that does not come back. Then X can be decomposed as a composition of three tangles  $L, Z, R \in S_{a+b+1}$ , where  $L = I_{a+1} \otimes \sigma$  with  $\sigma \in S_b$ ,  $Z = T_{a+1} \circ T_{a+2} \circ \cdots T_{a+|e|}$  (containing the edge e) and  $R = \omega \otimes I_{b'+1}$  with  $\omega \in S_{a'}$ . This decomposition satisfies the property  $\ell(X) = \ell(L) + \ell(Z) + \ell(R)$ . In the special case in which a' + 1 = a + b + 1 then  $L = I_N$ .

Proof. Suppose a minimal factorization F for X is given. Since we know that e does not come back, then we also know that #T(e) = |e|. This implies that F contains the sub-word  $Z = T_{a+1} \circ T_{a+2} \circ \cdots T_{a+|e|}$ . Now use the axioms to rewrite F so that  $F = L' \circ Z \circ R'$ , where L' and R' contain the rest of the factorization. Finally, use Axiom 13 to move every prime  $p_i$  in L' with i < a into R' and move every prime  $p_i$  in R' with i > a' + 1 into L'. Thus obtaining  $F = L \circ Z \circ R$ . This obviously satisfies  $\ell(X) = \ell(L) + \ell(Z) + \ell(R)$ .

In the case in which N=y'=a+b+1, we can construct R by removing edge e from X, relabelling all upper nodes d such that  $a+2 \le d \le N$  as d-1 and adding the edge (N, N'). By construction, we have then that  $X=Z\circ R$  and therefore we can set  $L=I_N$ .

Finally, we can now prove that if an edge e passes through at least one U-prime then it cannot come back (see Figure 8). We assume  $U_i$  to be the last U-prime edge e passes through before passing through prime tangle p. This is because in the case in which e passed through another U-prime, say  $U_j$ , in a different factorization, we could pick  $U_j$  as the focus of the proof. This implies that, after  $U_i$ , e passes through only T-primes, in a generic tangle A, before p. We will also assume that

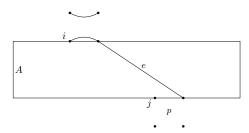


Figure 8: The general case in which an edge e "comes back". e passes through a U-prime  $U_i$ , then, after a series of T-primes contained in a tangle  $A \in \mathcal{B}_N$ , it passes through a prime tangle  $p_j$  that reduces it size. We would like to prove that the factorization  $U_i \circ F_A \circ p_j$  (where  $F_A$  is a minimal factorization for A) is always not minimal.

it is the first time that e comes back. This assumption also allows us to say that, since e cannot come back the first time, it can never come back.

Lastly, we prove only the case in which e is a negative transversal in A and  $U_i$  is on top of A. This is because we can reflect  $U_i \circ A \circ T_j$  vertically or horizontally to cover any case, and since reflecting a tangle does not change the length of its factorization, then if one of them is not minimal, then all reflections are not minimal too. This implies that an edge cannot come back if at *any time* it passes through a U-prime.

We now prove that the configuration seen in Figure 8 always results in a not minimal factorization. By using Theorem 2 and Theorem 6 we can now reduce the general case of Figure 8 to a simpler one, in which we prove the non-minimality of the factorization  $U_i \circ Z \circ Z' \circ T_j$ , where Z and Z' are the results of two LZR decompositions and thus have the form  $Z = T_{i+1} \circ T_{i+2} \circ \cdots \circ T_j$  and  $Z' = T_k \circ T_{k+1} \circ \cdots \circ T_{j-1}$  for some k < j. Follow Figure 9 for the steps of the proof.

Once we are in the case of Figure 9i we will have a factorization in this form:

$$U_i \circ \underbrace{T_{i+1} \circ T_{i+2} \circ \cdots \circ T_j}_{Z} \circ \underbrace{T_k \circ T_{k+1} \circ \cdots \circ T_{j-1}}_{Z'} \circ T_j$$

We will go through all cases for k and prove that this factorization is always not minimal:

- if k > i, then there are too many crossings (Figure 10):
  - the factorization  $Z \circ Z' \circ T_j$  has  $\ell(Z) + \ell(Z') + 1$  crossings, but edge e and edge u can be redrawn so that they do not cross, thus obtaining the same tangle in  $S_N$  but with fewer crossings. This implies that this factorization is not minimal;
- if  $k \leq i$ , then the factorization contains  $U_i \circ T_{i+1} \circ T_i$  (Figure 11):
  - every factor in Z' with index k < i 1 can be ignored because

$$U_{i} \circ \underbrace{T_{i+1} \circ T_{i+2} \circ \cdots \circ T_{j}}_{\substack{k < i-1}} \circ \underbrace{T_{k} \circ T_{k+1} \circ \cdots \circ T_{i-2}}_{\substack{k < i-1}} \circ T_{i-1} \circ T_{i} \circ T_{i+1} \circ \cdots \circ T_{j-1}}_{\substack{j \\ j \\ j}} \circ T_{j} = \underbrace{T_{k} \circ T_{k+1} \circ \cdots \circ T_{i-2}}_{\substack{k < i-1}} \circ U_{i} \circ T_{i+1} \circ T_{i+2} \circ \cdots \circ T_{j} \circ T_{i-1} \circ T_{i} \circ T_{i+1} \circ \cdots \circ T_{j-1} \circ T_{j}$$

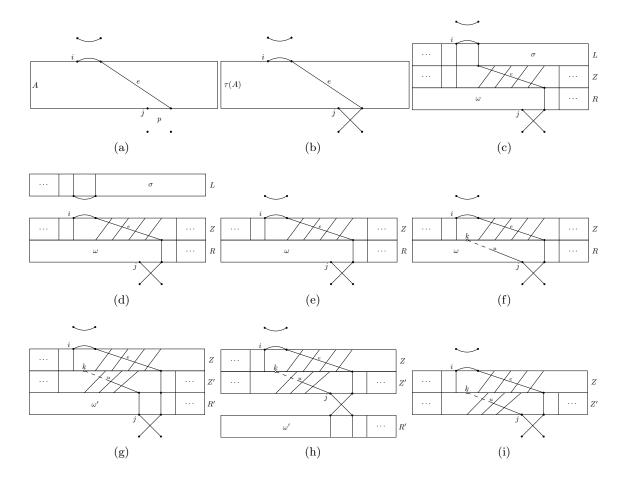


Figure 9: (a) The original tangle X, which is a composition of  $U_i$ , a tangle  $A \in \mathcal{B}_N$  and a prime tangle  $p_j$ . (b) Apply  $\tau$  on A and  $p_j$  (Theorem 2). Remember that e passes through only T-primes, therefore it is still present in  $\tau(A)$ . (c) Perform a LZR decomposition on  $\tau(A)$  in which a=i, b=N-i-1, a'=j and b'=N-j-1. (d) By Theorem 6,  $L=I_{i+1}\otimes \sigma$  with  $\sigma\in S_b$ . This implies that we can move  $U_i$  to be under L. (e) Since L no longer affects edge e, we can delete it. (f) Since  $R=\omega\otimes I_{N-j}$  with  $\omega\in S_j$ , then there exists an edge u=(k,j') for some k< j. The white circle and the dashed line indicate that we do not know where k lies on the top row. (g) Perform a LZR decomposition on R, but since we have that u is connected to j, we then have that L' is the identity tangle and can be ignored (remember the special case for Theorem 6). We thus now have  $R=Z'\circ R'$ . (h) As before, we can move  $T_j$  above R'. (i) We can now delete R', and this is the simplified case we will prove will always imply a not minimal factorization.

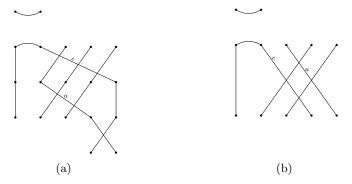


Figure 10: Example for the case in which k > i. (a) The factorization  $U_i \circ Z \circ Z' \circ T_j$  after LZR decomposition. There are six crossings. (b) The tangle obtained by composing Z, Z' and  $T_j$ . Edges e and u were drawn again so that they do not cross. Note how the resulting tangle has only four crossings, meaning that the previous one was not minimal.

therefore reducing the factorization to

$$U_i \circ \underbrace{T_{i+1} \circ T_{i+2} \circ \cdots \circ T_j}_{Z} \circ \underbrace{T_{i-1} \circ T_i \circ T_{i+1} \circ \cdots \circ T_{j-1}}_{Z'} \circ T_j$$

By construction, we have that  $Z \circ Z' \circ T_j$  will contain an edge e = (i - 1, j' + 1), which falls into the special case of the LZR decomposition. This implies that we can rewrite the above factorization as

$$U_i \circ Z'' \circ R$$

where  $Z'' = T_{i-1} \circ T_i \circ \cdots \circ T_j$  and  $R = \omega \otimes I_{N-j}$ , where  $\omega \in S_j$ . This is clearly not minimal because

$$\begin{array}{lll} U_i \circ Z'' & = \\ U_i \circ T_{i-1} \circ T_i \circ \cdots & = \\ U_i \circ U_{i-1} \circ \cdots & \end{array}$$

in both cases, we have shown that the factorization was not minimal.

There are two more special cases to address. The first one is when edge e is a zero transversal (Figure 12). In this case, the procedure is the same as in Figure 8 but we perform the LZR decomposition only once.

The last special case is when not only e is a zero transversal, but A has another zero transversal at (N-1, N'-1) (Figure 13). In this case, no simplification step is required because the factorization is trivially not minimal for any prime tangle p.

This whole argument proves that  $\#U(e) > 0 \implies \#T(e) < |e|$ . This is because if it was the case that if #U(e) > 0 but  $\#T(e) \ge |e|$ , then it would imply that there are more T-primes than |e| and therefore, at some point, e must come back, which we just proved is not possible given that #U(e) > 0. This actually proves that  $\#T(e) \ge |e| \implies \#U(e) = 0$  by contraposition.

This also explains why the merge operation is undefined for some edges. If h = (i, i + 1) is an upper hook of size one and e is another distinct edge, then e can pass through  $U_i$  only if it satisfies one of the conditions presented in Section 3. If it doesn't, then it must come back to pass through  $U_i$ , but this would imply that that particular factorization is not minimal, and therefore can be ignored.

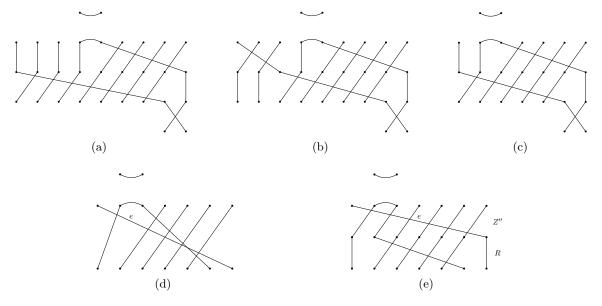


Figure 11: Example for the case in which  $k \leq i$ . (a) The factorization  $U_i \circ Z \circ Z' \circ T_j$  after LZR decomposition. (b) All T-primes with an index less than i-1 are removed from their tangle. (c) The T-primes are removed. (d) Z and Z' are composed together. By construction, there is an edge e = (i-1, j'+1). (e) Perform a LZR decomposition (special case). By the structure of Z'', the resulting factorization is not minimal because it contains  $U_i \circ T_{i-1} \circ T_i$ .

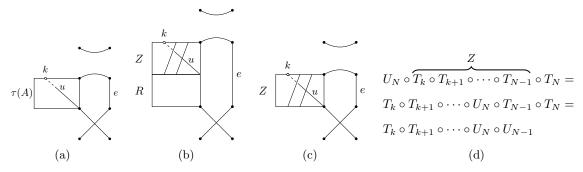


Figure 12: The special case in which e is a zero transversal. (a) Given a tangle  $A \in \mathcal{B}_N$  and a prime tangle  $p_N$ , apply  $\tau$  to A and p. In this case, the index for the U-prime is N. (b) LZR decomposition of  $\tau(A)$ . (c) Move R below  $T_N$  and remove it. (d) Proof that the factorization is not minimal.

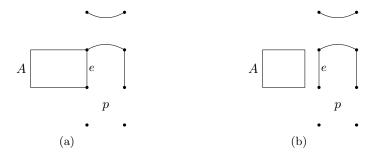


Figure 13: The factorization is not minimal because  $U_i \circ p$  is not minimal for any p.

## 6 Node polarity

In this Section we will prove that  $\tau(X)$  is unique and computable in polynomial time. To do this, we will introduce the concept of "node polarity", a property preserved by  $\tau$ .

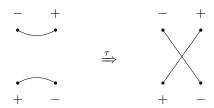
Given a node x, we define the *polarity* of x as follows:

- if x is connected to a transversal edge e, then x is positive (+) if e is positive transversal, negative (-) if it is negative transversal and zero (0) if it is a zero transversal
- if x is connected to an upper hook h, then x is negative (-) if it is the left node of h and positive (+) if it is its right node. The polarity is reversed for lower hooks.

**Theorem 7.**  $\tau$  preserves node polarity.

*Proof.* For all edges e such that  $\#T(e) \ge |e|$  this is trivially true because they are present in both in X and  $\tau(X)$ , this includes all edges with node polarity 0. Therefore we need to prove that  $\tau$  preserves node polarity for all nodes that are connected to edges e such that #U(e) > 0.

As we can see in the following diagram, after  $\tau$  every node of every factor in the factorization will be connected to another one with the same polarity.



Since we are assuming that #U(e) > 0, then it implies that e does not come back, and therefore node polarity is preserved.

Polarity preservation is the first property that  $\tau(X)$  must satisfy, but it is not enough because there are many tangles in  $S_N$  with the same polarity as X. Lemma 1 and Theorem 8 will state that edges with the same polarity in  $\tau(X)$  do not cross, which will imply the uniqueness of  $\tau(X)$ .

**Lemma 1.** Let  $\sigma$  be a permutation containing an inversion  $(\sigma(i), \sigma(j))$ , with i < j. After swapping i with j, the new permutation will have fewer inversions than  $\sigma$ .

*Proof.* For readability's sake, we will define  $y = \sigma(i)$  and  $x = \sigma(j)$ . We also define the notation  $\sigma_I$  for the set of elements in  $\sigma$  having indices in the set I.

Let L, C and R be three disjoint sets of indices satisfying i < L, C, R < j. Let's also assume that

$$\sigma_L > y > \sigma_C > x > \sigma_R$$

meaning that the elements between y and x can take any value. We do not need to check the elements outside the rage [i, j] because their number of inversions will stay fixed.

We can now observe what happens after swapping y with x. Moving y will add  $|\sigma_L|$  inversions because y is smaller than every element in  $\sigma_L$ , while it will remove  $|\sigma_C| + |\sigma_R|$  inversions because y is bigger than the elements contained in  $\sigma_C$  and  $\sigma_R$ .

On the other hand, moving x will remove  $|\sigma_L| + |\sigma_C|$  inversions because they contain bigger elements, and it will add  $|\sigma_R|$  because it contains smaller elements.

Finally, swapping y and x will remove one inversion because we assumed y > x.

By summing everything together we obtain that the new permutation will have a different amount of inversions, i.e. :

$$|\sigma_L| - |\sigma_C| - |\sigma_R| - |\sigma_L| - |\sigma_C| + |\sigma_R| - 1 = -2|\sigma_C| - 1$$

We can see that the difference in the number of inversions is always negative, and therefore the new permutation will have fewer inversions.  $\Box$ 

**Theorem 8.** Let X be a tangle with minimal factorization F. If the nodes of two edges  $e_1, e_2 \in \tau(X)$  have the same polarity and  $e_1$  and  $e_2$  are not edges of X too, then they do not cross.

*Proof.* We know that  $\tau$  preserves node polarity and also that since  $|\tau(F)|$  is minimal, then it will have the minimal amount of T-primes and therefore the minimal amount of crossings in  $\tau(X)$ . By Lemma 1 we know that two edges that do not cross add fewer T-primes compared to edges that do cross. Therefore  $e_1$  and  $e_2$  do not cross in  $\tau(X)$ .

We can now finally prove that there exists only one  $\tau(X)$ .

**Theorem 9.** Given a tangle X, there exists only one  $\tau(X)$  that preserves node polarity and minimizes the number of crossings.

*Proof.* For the sake of brevity, we will assume all mentioned nodes are not connected to edges in both X and  $\tau(X)$ . We will also just focus on nodes with negative polarity, because for positive polarity the argument is basically the same.

Suppose two upper nodes  $x_1 < x_2$  have negative polarity. Assume also that they are the last two upper nodes with negative polarity. Assume now the same for two lower nodes  $y'_1 < y'_2$ .

If we connect  $x_1$  with  $y_2'$  then it must be the case that  $x_2$  has to be connected to  $y_1'$ , thus introducing a crossing, which is forbidden by Theorem 8. Therefore we have that  $x_2$  must be connected to  $y_2'$  in  $\tau(X)$ . We are now in a situation in which  $x_1$  and  $y_1'$  are the last nodes that are not connected, and by the previous argument, they must form an edge in  $\tau(X)$ .

We can repeat this argument until there are no more edges to connect. Since at each step there was only one possible connection to make, it implies that  $\tau(X)$  is unique.

Corollary 7. For all minimal factorizations F and F' for X, the tangles corresponding to  $\tau(F)$  and  $\tau(F')$  are equal.

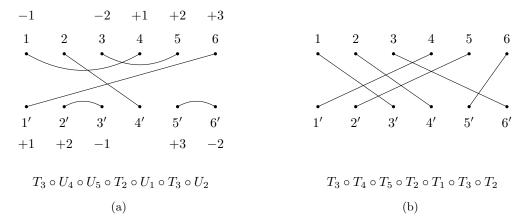


Figure 14: (a) A tangle X for which we have computed the node polarities along with one of its minimal factorizations. Note that the edge (2,4') is not considered because  $\#T(2,4') \ge |(2,4')|$  and therefore passes through only T-primes and it is not affected by  $\tau$ , therefore it is present in both X and  $\tau(X)$ . (b) The tangle corresponding to  $\tau(X)$  and one of it minimal factorizations. To compute it, add the edge (2,4') and connect from top to bottom the nodes that in X have the same node polarity label. Note how the indices for both factorizations coincide.

The proof for Theorem 9 gives also an idea of how  $\tau$  could be computed. We first start with calculating the node polarity for each node in X from left to right, but every time we find a node with a certain polarity, say "+", we will label that node with +j, where j is a counter that keeps track of how many nodes with polarity "+" we have encountered so far. We can then compute  $\tau(X)$  by adding every edge e in X such that  $\#T(e) \geq |e|$ , and then connecting nodes from top to bottom in  $\tau(X)$  if and only if they have the same node polarity label in X (Algorithm 2). See Figure 14 for an example. Algorithm 2 has therefore quadratic time complexity because we have to calculate #T(e) for each edge.

Now that we know that  $\tau(X)$  is unique, it is not difficult to see that not only we could count the number of crossings of  $\tau(X)$  to find the number of prime tangles for X, but we can also factorize  $\tau(X)$  to find the indices of those primes.

**Theorem 10.** Given a tangle X with minimal factorization  $F_X$ , then there exists a minimal factorization  $F_Y$  for  $Y = \tau(X)$  such that  $F_Y$  has T-primes with the same indices in the same order as in  $F_X$ .

*Proof.* Since  $\tau$  does not change the indices, then this is a direct implication of the fact that  $\tau(X)$  is the tangle we obtain by composing the factors in  $\tau(F_X)$ . This factorization can be found by using Algorithm 4.

# 7 Length function

We are now ready to define two length functions for the Brauer monoid. We will use different subscripts to differentiate between them. The first one,  $\ell_P$ , is trivially  $\ell_{\tau}(X) = \ell(\tau(X))$ , where

#### **Algorithm 2** Compute $\tau(X)$

```
Require: X \in \mathcal{B}_N function \tau(X)

Let \rho(i) \in \{+, -\} be the polarity for node i in X
S \leftarrow all nodes from 1 to N connected to edge e s.t. \#T(e) < |e|
S' \leftarrow all nodes from 1' to N' connected to edge e s.t. \#T(e) < |e|
For i \in S, label with "\rho(i)j" the jth node with polarity \rho(i)
For i' \in S', label with "\rho(i')j" the jth node with polarity \rho(i')
Z \leftarrow empty tangle in \mathcal{B}_N
Add to Z all edges in X s.t. \#T(e) \geq |e|
for nodes x, y' s.t. x and y' have the same node polarity label in X do
Add to Z the edge (x, y')
```

 $\ell(\tau(X))$  is just the number of crossings for  $\tau(X)$ . This function has quadratic complexity because it has to calculate the node polarity first. But even if node polarity could be computed faster, Theorem 10 tells us that we can obtain the indices for a factorization of X by factorizing  $\tau(X)$  (which is very useful information, as we will see in Section 8), and factorizing a tangle in  $S_N$  cannot be done faster than  $\mathcal{O}(N^2)$  (see Appendix A).

The second length function is a direct corollary of the fact that  $\#T(e) \ge |e| \implies \#U(e) = 0$  that we proved in Section 5, because it implies that  $\#U(e) > 0 \implies \#T(e) < |e|$  and therefore  $\#U(e) > 0 \implies \#T(e) + \#U(e) = |e|$ . Let's define the function #P(e) to be the number of prime tangles the edge e passes through in a factorization F.

Since if  $\#T(e) \ge |e|$  then e passes through only T-primes, and otherwise we have that #T(e) + #U(e) = |e|, we can define #P(e) to be:

$$\#P(e) = \left\{ \begin{array}{ccc} \#T(e) & : \ \#T(e) \geq |e| \\ |e| & : \ otherwise \end{array} \right. = \max(\#T(e), |e|)$$

Remember that #T(e) is just the number of crossings edge e has (Assumption 2). Since now both #T(e) and |e| are values that are independent of the factorization considered, so is #P(e). This allows us to use #P(e) to find a length function for the Brauer monoid by just summing crossings and edge sizes. It is defined as follows:

$$\ell_P(X) = \frac{1}{2} \sum_{e \in X} \#P(e)$$

Since #P(e) counts the number of primes e passes through, we have that the sum will count every prime twice. We then have to divide by two to obtain the minimal number of primes for the tangle X.

This function still has a quadratic time complexity, but if we assume we already have calculated #T(e) for all  $e \in X$ , then it can be computed in linear time (see Section 8). We will use this trick in the following Section to bring down the time complexity from  $\mathcal{O}(N^5)$  to  $\mathcal{O}(N^4)$ .

## 8 Factorization algorithm

As we have seen in the previous Section, every length function has to be computed in quadratic time, therefore by Corollary 4 we have that Algorithm 1 runs in  $\mathcal{O}(N^5)$ . It turns out however that we can do better. Instead of calculating #T(e) every time we have to compute  $\ell_P$ , we can store the number of crossings for each edge at the beginning, and then we can just update them every time we merge a lower hook or compose the tangle with a T-prime. In this way, updating will have linear time complexity and therefore  $\ell_P$  will be linear, which brings the overall time complexity to  $\mathcal{O}(N^4)$ .

We update #T(e) in different ways depending on if we composed with a T-prime or merged two edges. In the first case, composing with  $T_i$  will modify just two edges, i.e. the ones connected to i and i+1. In this case, we just decrease #T by one for each of them.

In the second case, let's say we merged the lower hook h of size one with the edge e. Since we are going to remove them from X, we need to decrease #T for all edges that cross with e (no edge will cross with e). Then, after merging e0 with e1, two new edges will be in e1, let's call them e1 and e2. At this point we iterate again through all edges of e3, and if another edge e4 crosses with e5, then we increase e7, we then do the same for e2.

In both cases updating #T takes at most linear time, therefore  $\ell_P$  is computable in linear time and we have a  $\mathcal{O}(N^4)$  factorization algorithm, we just have to compute #T once at the beginning (Algorithm 3). See Figure 15 for an example. A Python implementation can be found at https://github.com/DanieleMarchei/BrauerMonoidFactorization.

Algorithm 3 could be altered to output a factorization that minimizes the number of T-primes without affecting the time complexity by just keeping track of the tangle with the least amount of crossings when merging e and h. However, the nested for-loops that search for the edge to merge h with are still the bottleneck of the algorithm. One way to bring down the complexity to  $\mathcal{O}(N^3)$  could be finding a constant time decision algorithm that determines if a particular edge passes through  $U_i$ . In this way, finding the edges that can be merged would take linear time and hence reach  $\mathcal{O}(N^3)$ . We were not able to find such an algorithm, so we leave it as a future research direction.

### 9 Discussion

The Brauer monoid can be factorized in polynomial time, specifically, with a time complexity of  $\mathcal{O}(N^4)$ . To our knowledge, this is the first polynomial time algorithm proposed to solve this problem. We are not sure if it has an optimal running time, since there might be some room for improvements we leave as further research directions. In parallel, we also found two length functions that can be computed in quadratic time, one of which can be computed in linear time assuming the crossing numbers are known.

Some proofs for this paper rely on two assumptions we were not able to prove nor find in the literature, but their validity has been empirically checked (see Appendix B). We leave their proof as another future research problem.

The factorization problem could also lead to some interesting combinatorial problems. For example, what is the maximum amount of edges we can merge for a tangle X with a hook h of size one, such that  $\ell(X) = \ell(X') + 1$ , where X' is the tangle obtained after the merge? This is important to ask because, as we already discussed, finding the right edge to be merged is the bottleneck of Algorithm 3. By means of enumeration, we obtained Table 2, and it seems the case that the above

# Algorithm 3 $\mathcal{O}(N^4)$ factorization

```
Require: X \in \mathcal{B}_N
  function Factorize \mathcal{B}_N(X)
      Calculate \#T(e) for all e \in X
      I \leftarrow \text{factorization indices of } \tau(X)
                                                                                   \triangleright Algorithm 2 and Algorithm 4
      F \leftarrow \text{empty list}
      for i \in I do
          h \leftarrow (i, i+1)
          if h \in X then
               l \leftarrow \ell_P(X)
               for e \in X : e \neq h, \#T(e) < |e| do
                   X' \leftarrow X
                   for d \in X' : d \neq h, e do
                       if d crosses e then
                            \#T(d) \leftarrow \#T(d) - 1
                   Merge e and h in X', creating edges e_1 and e_2
                   for d \in X' : d \neq e_1 do
                       if d crosses e_1 then
                            \#T(e_1) \leftarrow \#T(e_1) + 1
                            \#T(d) \leftarrow \#T(d) + 1
                   for d \in X' : d \neq e_2 do
                       if d crosses e_2 then
                            \#T(e_2) \leftarrow \#T(e_2) + 1
                            \#T(d) \leftarrow \#T(d) + 1
                   if \ell_P(X') = l - 1 then
                        Append U_i to F
                        X \leftarrow X'
                        break
           else
               X \leftarrow T_i \circ X
               Append T_i to F
               Decrease the number of crossings for the two edges connected at i and i+1
      return F
```

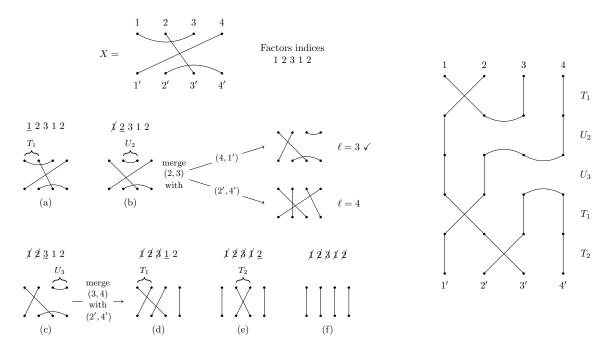


Figure 15: High level illustration for how Algorithm 3 works. On the left, we have a tangle X from which we have extracted its factorization indices. (a) The first factor index is 1, therefore we look at upper nodes 1 and 2 of our tangle and see that they are not connected. We record  $T_1$  and compute  $T_1 \circ X$  for the second step. (b) The second index is 2 and the upper nodes 2 and 3 are connected, therefore we record  $U_2$ . For the next step, we have to decide which edge we have to merge (2,3) with. We have two options: (4,1') and (2',4') (edge (1,3') satisfies  $\#T(e) \ge |e|$  so it does not pass through a U-prime). If we merge it with (4,1') we obtain a tangle with only three factors, while the other will have four, therefore we select the first one for the next step. (c) We are in the same situation of step (b) but now we can only merge (3,4) with (2',4') and we record  $U_3$ . (d) Same situation of step (a), we record  $U_3$ . (e) The factor index is 2 and the upper nodes 2 and 3 are not connected, therefore we record  $U_3$ . (f) We have reached the identity tangle so we stop the algorithm and output the factorization  $U_3 \circ U_3 \circ U_3$ 

questions is answered by  $\lfloor \frac{N}{2} \rfloor$ , while the number of tangles that have that number of merges is much more difficult to count. For example, the even entries match with the A132911<sup>3</sup> sequence of the OEIS [15], here we call it C(k):

$$C(k) = (k+1)\frac{(2k)!}{2^k}$$

where k starts from zero. To obtain an exact match with the even entries of Table 2, we will call them B(2k), we have to modify it as follows:

$$B(2k) = C(k-1) = k! |\mathcal{B}_{k-1}|$$

where k starts from one.

N	max amount of merges	n. tangles $B(N)$
2	1	1
3	1	6
4	2	2
5	2	46
6	3	18
7	3	900
8	4	360
9	4	31320
10	5	12600

Table 2: The maximum amount of possible merges in  $\mathcal{B}_N$  and the number of tangles X with a hook h of size one that can be merged with other edges such that  $\ell(X) = \ell(X') + 1$ , where X' is the tangle obtained after the merge.

Another question could be: how many tangles have length k in  $\mathcal{B}_N$ ? Using the results presented, we enumerated all tangles up to  $\mathcal{B}_{10}$  and obtained Table 3, let's call it T(N,k). Some clear pattern emerge, for example T(N,1)=2(N-1) (as expected), or  $T(N,1)=T(N-2,\frac{N(N-1)}{2}-1)$  for N > 5, but we were unable to find a general formula.

We don't have a proof for any of the above statements, nor we have a candidate formula for the odd entries. We leave these questions open as a further research direction.

<sup>3</sup>https://oeis.org/A132911

$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\frac{1}{2}$
1     2     4     6     8     10     12     14     16     18       2     8     20     36     56     80     108     140     176       3     2     36     102     208     362     572     846     1192       4     30     196     562     1224     2294     3900     6186       5     10     228     1110     3192     7266     14380     25870       6     2     212     1650     6620     18746     43764     90034       7     106     1966     11090     40166     112250     26646       8     42     1914     15890     73278     247494     68277	$\frac{1}{2}$
2     8     20     36     56     80     108     140     176       3     2     36     102     208     362     572     846     1192       4     30     196     562     1224     2294     3900     6186       5     10     228     1110     3192     7266     14380     25870       6     2     212     1650     6620     18746     43764     90034       7     106     1966     11090     40166     112250     26646       8     42     1914     15890     73278     247494     68277	$\frac{1}{2}$
3     2     36     102     208     362     572     846     1192       4     30     196     562     1224     2294     3900     6186       5     10     228     1110     3192     7266     14380     25870       6     2     212     1650     6620     18746     43764     90034       7     106     1966     11090     40166     112250     26646       8     42     1914     15890     73278     247494     68277	$\frac{1}{2}$
4     30     196     562     1224     2294     3900     6186       5     10     228     1110     3192     7266     14380     25870       6     2     212     1650     6620     18746     43764     90034       7     106     1966     11090     40166     112250     26646       8     42     1914     15890     73278     247494     68277	$\frac{1}{2}$
5     10     228     1110     3192     7266     14380     25870       6     2     212     1650     6620     18746     43764     90034       7     106     1966     11090     40166     112250     26646       8     42     1914     15890     73278     247494     68277	$\frac{1}{2}$
6     2     212     1650     6620     18746     43764     90034       7     106     1966     11090     40166     112250     26646       8     42     1914     15890     73278     247494     68277	$\frac{1}{2}$
7     106     1966     11090     40166     112250     26646       8     42     1914     15890     73278     247494     68277	52
8 42 1914 15890 73278 247494 68277	
Y	
	-
10 2 830 20910 166400 825422 31001	
11 414 18798 212250 1291638 56670	
12 162 15402 244730 1853554 95146	
13 56 10174 255188 2448214 14804	
14 14 6154 240828 3003652 21502	
15 2 3282 207968 3411904 29298 1500 161044 2027000 277000	
1530 161844 3627806 37604	
17 648 113490 3585522 45596	
18 234 73978 3325568 52372	
19 72 44336 2856302 57069	
20 16 24354 2325126 59057	
2 12462 1741684 58153	
22 5848 1238988 54397	
23 2502 830378 48420	
972 523782 41150	
25 324 312886 33243	
26 90 176806 25585	
18 94362 18883	
28 2 47280 13337	
29 22294 90284	
30 9756 58569	
31 3908 36537	
32 1406 21860	
33 434 12537	
34 110 68844	
35 20 36137	
36 2 18048	
37   85298	
37930	
39 15636	)
40 5880	
41 1972	
42 566	
43 132	
44 22	
45   2	

Table 3: The number of tangles in  $\mathcal{B}_N$  with length k.

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### References

- [1] Anders Björner and Francesco Brenti. Combinatorics of Coxeter groups, volume 231. Springer, 2005.
- [2] Richard Brauer. On algebras which are connected with the semisimple continuous groups. *Annals of Mathematics*, pages 857–872, 1937.
- [3] Manuel Clavel, Francisco Durán, Steven Eker, Santiago Escobar, Patrick Lincoln, Narciso Marti-Oliet, José Meseguer, Rubén Rubio, and Carolyn Talcott. *Maude Manual (Version 3.2.1)*. The Maude System (https://maude.cs.illinois.edu/), 2022.
- [4] Igor Dolinka and James East. Twisted brauer monoids. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 148(4):731–750, 2018.
- [5] Igor Dolinka, James East, and Robert D Gray. Motzkin monoids and partial brauer monoids. Journal of Algebra, 471:251–298, 2017.
- [6] James East, Attila Egri-Nagy, James D Mitchell, and Yann Péresse. Computing finite semi-groups. *Journal of Symbolic Computation*, 92:110–155, 2019.
- [7] Jeff Erickson. Algorithms. Independently published (https://jeffe.cs.illinois.edu/teaching/algorithms/), 2023.
- [8] Dana C Ernst, Michael G Hastings, and Sarah K Salmon. Factorization of temperley-lieb diagrams. *Involve*, a Journal of Mathematics, 10(1):89–108, 2016.
- [9] Véronique Froidure and Jean-Eric Pin. Algorithms for computing finite semigroups. In Foundations of Computational Mathematics: Selected Papers of a Conference Held at Rio de Janeiro, January 1997, pages 112–126. Springer, 1997.
- [10] Louis Kauffman and Yuri Magarshak. Vassiliev knot invariants and the structure of rna folding. Knots and Applications, 03 1995.
- [11] Jon Kleinberg and Eva Tardos. Algorithm design. Pearson Education India, 2006.
- [12] Donald Knuth. The art of Computer Programming: Volume 3: Sorting and Searching. Addison-Wesley Professional, 1998.
- [13] Daniele Marchei and Emanuela Merelli. Rna secondary structure factorization in prime tangles. BMC bioinformatics, 23(6):1–18, 2022.
- [14] Edward F Moore. The shortest path through a maze. In *Proc. of the International Symposium* on the Theory of Switching Part II, pages 285–292. Harvard University Press, 1959.

- [15] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2024. Published electronically at http://oeis.org.
- [16] Arun Ram. Characters of brauer's centralizer algebras. *Pacific journal of Mathematics*, 169(1):173–200, 1995.
- [17] Richard P Stanley. On the number of reduced decompositions of elements of coxeter groups. European Journal of Combinatorics, 5(4):359–372, 1984.
- [18] Harold NV Temperley and Elliott H Lieb. Relations between the 'percolation'and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 322(1549):251–280, 1971.
- [19] Konrad Zuse. Plankalkül. Konrad Zuse Internet Archive (http://zuse.zib.de), 1946.

# A Factorization algorithms for submonoids of $\mathcal{B}_N$

#### A.1 Factorization of $S_N$

Every element in  $S_N$  can be uniquely described as a permutation of the string  $1, 2, \dots, N$ . A simple isomorphism from a string permutation  $s_1, s_2, \dots, s_N$  and a tangle in  $S_N$  is to connect the upper node i to the lower node  $s_i'$ . The identity permutation string is therefore the one in which every element is in ascending order. In this context, factorizing a tangle in  $S_N$  is the same as finding the shortest sequence of adjacent transpositions (i.e.  $s_i, s_{i+1} \to s_{i+1}, s_i$ ) such that the original string is reduced to the identity string. In other words, we have to sort the string. This is a well-known problem in Computer Science and there are numerous fast algorithms for solving it. However, due to the constraint of using only adjacent transpositions, we are bound to a quadratic time complexity since the longest minimal factorization in  $S_N$  has size  $\frac{N(N-1)}{2}$ . One such algorithm is the Bubblesort [12], which specifically sorts strings using adjacent transpositions, yielding the shortest possible sequence (see Algorithm 4).

#### **Algorithm 4** Factorization algorithm for any tangle in $S_N$ .

```
Require: X \in S_N

function Factorize S_N(X)

s \leftarrow string permutation for X

F \leftarrow empty list

for j \in [N \dots 1] do

for i \in [1 \dots j-1] do

if s_i > s_{i+1} then

Swap s_i and s_{i+1}

Append T_i to F

return F
```

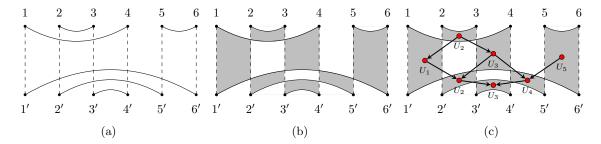


Figure 16: (a) A tangle in  $\mathcal{TL}_N$ . Drawing N imaginary edges (i,i') we see that the tangle is now divided into N-1 columns and each column is divided into regions, delimited by edges. (b) Select the regions with odd depth. (c) If two regions are diagonally adjacent, connect the top one with the bottom one, thus obtaining a Directed Acyclic Graph. Label each node as  $U_i$ , where i is the index of the columns it is in. If we read this DAG from top to bottom and from left to right we obtain the minimal factorization  $U_2 \circ U_5 \circ U_1 \circ U_3 \circ U_2 \circ U_4 \circ U_3$ .

#### A.2 Factorization of $\mathcal{TL}_N$

The Temperley-Lieb monoid  $\mathcal{TL}_N$  [18] is a submonoid of  $\mathcal{B}_N$  in which only *U*-primes are taken as generators. Informally speaking,  $\mathcal{TL}_N$  contains all tangles with no crossings. A  $\Omega(N^2)$  factorization algorithm was first proposed by Ernst et. al [8] and in this Section we will give a surface-level explanation of how it works (see Figure 16).

Given a tangle X in  $\mathcal{TL}_N$ , divide it into N-1 columns by adding a vertical line for each pair (i,i'). Each column now will be further divided into different regions delimited by the edges it contains. Each region has a depth value indicated by how many other regions there are above it. We will call regions with even depth 0-regions and regions with odd depth 1-regions. Two regions,  $R_1$  and  $R_2$ , in the same column are vertically adjacent if  $|depth(R_1) - depth(R_2)| = 1$ . Given two regions  $R_1$  and  $R_2$  in adjacent columns and two points  $p_1 \in R_1$  and  $p_2 \in R_2$ , if we can draw a straight line between them without crossing an edge in X, then we say that  $R_1$  and  $R_2$  are horizontally adjacent. If there is a region R' vertically adjacent to  $R_1$  and horizontally adjacent to  $R_2$ , then  $R_1$  and  $R_2$  are diagonally adjacent. We will  $R_1 \to R_2$  in the special case where R' is below  $R_1$ .

Let  $\mathfrak{R}$  be the set of all 1-regions. From here we construct a Directed Acyclic Graph (DAG) G such that every region  $R \in \mathfrak{R}$  is a vertex of G and given two regions  $R_1$  and  $R_2$ , if  $R_1 \to R_2$ , then  $(R_1, R_2)$  is an edge of G. If a vertex does not have incoming edges, then we call it a root of G. Finally, to obtain the factorization for X we traverse each vertex from top to bottom and from left to right, i.e. we list all roots  $r_i, r_j, \ldots, r_k$  of G, store  $U_i \circ U_j \circ \cdots \circ U_k$  and delete the roots, now other nodes will not have incoming edges, list them as the new roots  $r'_i, r'_j, \ldots, r'_k$ , and so on until G is empty (Algorithm 5).

The time complexity of this algorithm is bounded below by  $\Omega(N^2)$  because the number of 1-regions is the same as the number of nodes in G, which is equal to the length of the factorization for the input tangle and, by Corollary 3, we know it is quadratic. Assuming that retrieving the roots of G can be done in constant time, the while loop basically enumerates the nodes of G, therefore it does not influence the time complexity. If we assume that the number of total regions is still quadratic and they could be enumerated in quadratic time as well, we have that Algorithm 5 has a time complexity of  $\mathcal{O}(N^2)$ .

#### **Algorithm 5** Factorizes tangle in $\mathcal{TL}_N$

```
procedure Factorize \mathcal{TL}(X)
Divide X in columns R \leftarrow \text{regions of } X
\mathfrak{R} \leftarrow \text{1-regions of } R
G \leftarrow \text{DAG from } \mathfrak{R} \text{ s.t. if } R_1 \rightarrow R_2, \text{ then } (R_1, R_2) \in G
F \leftarrow \text{empty list}
while G \neq \emptyset do
S \leftarrow \text{roots of } G
for s \in S do
i \leftarrow \text{column where } s \text{ lies in}
Append U_i to F
Remove nodes in S from G
return F
```

# B Testing the Assumptions

Here we present how methodology for empirically testing Assumption 1 and Assumption 2. The interested reader can find the Python code in the GitHub repository https://github.com/DanieleMarchei/BrauerMonoidFactorization.

#### **B.1** Preparation

To test the two assumptions we need a database of minimal factorizations. We cannot use Algorithm 3, as it would be circular reasoning. This is because its correctness relies on them being true (it is used in Section 5 to partially apply  $\tau$  and in Section 7 to argue that  $\ell_P$  is independent of the factorization). Thus we decided to create this database by exploring the right Cayley Graph of  $\mathcal{B}_N$  using a Breadth First Search (BFS)<sup>4</sup> starting from the identity tangle. The BFS algorithm has the nice property of always returning the shortest path in a graph (which will be a minimal factorization in our case) and by instructing it to first explore the edges labelled with a U-prime, and then all edges labelled with a T-prime, every time we reach an unexplored tangle we know we have obtained a minimal factorization that uses the least amount of T-primes. We ran this procedure up to  $\mathcal{B}_8$ .

### B.2 Assumption 1

Assumption 1 stated that every factorization in the Brauer monoid can be reduced to a minimal one by a sequence of "delete", "braid" and "swap" rules. In other words, we do not need to increase the factorization length in order to find a shorter one.

To test this empirically, we implemented the axioms for  $\mathcal{B}_N$  (Table 1) as a Term Rewriting System (TRS) using the Maude System [3]. The approach is similar to what described in [13], but to have this paper self-contained, we will give a high-level illustration of the procedure.

The TRS accepts in input a factorization and iteratively tries to apply as many "delete" rules it can. Then, it tries to apply "braid" and "swap" rules non-deterministically until a new delete rule

<sup>&</sup>lt;sup>4</sup>According to [7], the BFS algorithm was first proposed in [19] and not in [14], as it is often attributed.

is applicable, and at this point the TRS starts again with the newly shortened factorization. This reduction procedure stops when no delete rule is applicable after all move rules have been applied. The actual test for the assumption is performed as follows:

- 1. generate a random factorization F with length  $|F| \in [2, s \frac{N(N-1)}{2}]$  and compute the corresponding tangle  $X \in \mathcal{B}_N$ ;
  - if the factorization for X stored in the database has the same length of F, then repeat step 1. We only want to test non-minimal factorizations
- 2. get the minimal factorization  $F^*$  for X in the database;
- 3. reduce F using the TRS, obtaining  $\hat{F}$ ;
- 4. if  $|\hat{F}| \neq |F^*|$  we have found a counterexample, otherwise repeat from step 1.

where s is a scale factor that parametrizes the maximum length possible that can be generated. To avoid spending too much time on this procedure, we keep track of the tangles generated, effectively testing the assumption once for each tangle. Since the probability of generating a tangle we haven't already generated decreases each time we find a new one, we implemented a "patience" counter that is decreased each time we do not generate a new tangle. The procedure stops when the patience reaches to zero or all tangles are tested. In our test we set s=2 and the patience counter to 2000000. Because this procedure takes a long time to terminate, we ran it up to N=6 with the results shown in Table 4. We never found a counterexample.

N	2	3	4	5	6
$ \mathcal{B}_N $	3	15	105	945	10395
n. tangles tested	3	15	105	942	10043

Table 4: Test table for Assumption 1. We ran the test up to  $\mathcal{B}_6$  and, among all tangles tested, we never found a counterexample.

#### B.3 Assumption 2

Assumption 2 stated that if a tangle X has k crossings, then there exists a minimal factorization F with exactly k T-primes and no other factorization with fewer T-primes exists.

This assumption is easier and faster to check:

- 1. pick a tangle X with minimal factorization F from the database;
  - by construction, F will have the minimal amount of T-primes.
- 2. count the number of crossings c in X;
- 3. if  $c \neq \#T(F)$  then we have found a counterexample, otherwise repeat from step 1.

We ran this procedure up to N=8 and never found a counterexample.