ENPM667 - Control for Robotics Systems

Final Project - Technical Report

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1 Part 1

1.1 A. Equations of motion for the system and the corresponding nonlinear state-space representation

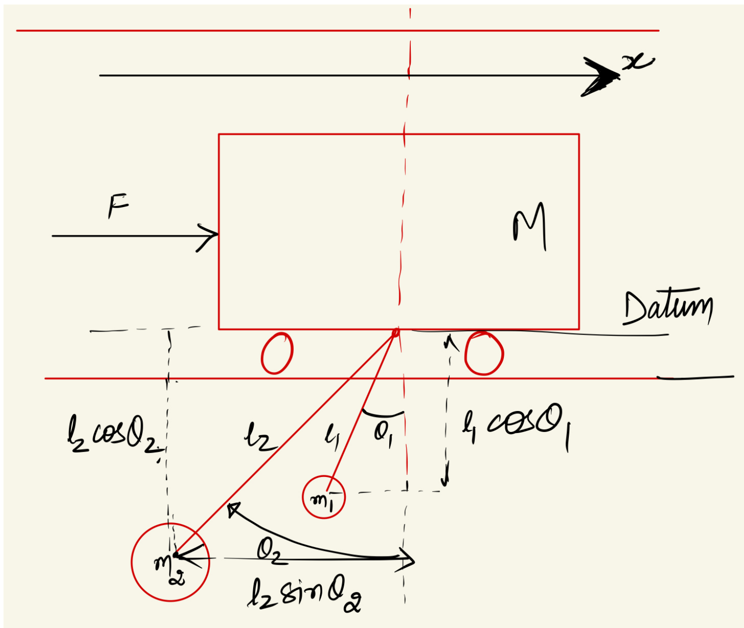

Figure 1: Force Diagram of the System

From the above diagram, the Kinetic Energy equation can be derived as

$$K = \frac{1}{2}m_2(\dot{x}-\dot{\theta}_2 l_2 cos\theta_2)^2+\frac{1}{2}m_1(\dot{x}-\dot{\theta}_1 l_2 cos\theta_1)^2+\frac{1}{2}M(\dot{x})^2+\frac{1}{2}m_2(\dot{\theta}_2)^2(l_2 sin\theta_2)^2+\frac{1}{2}m_1(\dot{\theta}_1)^2(l_1 sin\theta_1)^2$$

and the Potential Energy can be derived as

$$P = -m_2 g l_2 cos\theta_2 - m_1 g l_1 cos\theta_1$$

The Lagrangian is given as the difference between the Kinetic and Potential Energies of a system

$$L = K - P$$

Therefore the Lagrangian can be written as follows

$$L = K - P$$

$$= \frac{1}{2}m_2\{(\dot{x}-\dot{\theta}_2 l_2 cos\theta_2)^2 + (\dot{\theta}_2 l_2 sin\theta_2)^2 + \frac{1}{2}M(\dot{x})^2 + \frac{1}{2}m_1\{(\dot{x}-\dot{\theta}_1 l_1 cos\theta_1)^2 + (\dot{\theta}_1 l_1 sin\theta_1)^2\}$$

$$+\{m_2 g l_2 cos\theta_2 + m_1 g l_1 cos\theta_1\}$$

Take the Partial derivative of L with respect to $\dot{x}$

$$\frac{\partial L}{\partial \dot{x}} = m_2(\dot{x}-\dot{\theta}_2 l_2 cos\theta_2) + M(\dot{x}) + m_1(\dot{x}-\dot{\theta}_1 l_1 cos\theta_1)$$

Now differentiate the partial derivative computed above with respect to t

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m_2(\ddot{x} - \ddot{\theta_2}l_2cos\theta_2) + \dot{\theta_2}^2 l_2sin\theta_2) + M(\ddot{x}) + m_1(\ddot{x} - \ddot{\theta_1}l_1cos\theta_1 + \dot{\theta_1}^2 l_1sin\theta_1)$$

Take the partial derivative of L with respect to x

$$\frac{\partial L}{\partial x} = 0$$

We know that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

Therefore we can write the force on the equation as follows,

$$F = m_2(\ddot{x} - \ddot{\theta_2}l_2cos\theta_2) + \dot{\theta_2}^2 l_2sin\theta_2) + M(\ddot{x}) + m_1(\ddot{x} - \ddot{\theta_1}l_1cos\theta_1 + \dot{\theta_1}^2 l_1sin\theta_1)$$

Similarly, we do the above derivations for the other two variables θ_1 and θ_2

First we consider the variable θ_2

Take the partial derivative of L with respect to $\dot{\theta_2}$

$$\frac{\partial L}{\partial \dot{\theta_2}} = m_2\{(-l_2cos\theta_2(\dot{x} - \dot{\theta_2}l_2cos\theta_2) + l_2^2\dot{\theta_2}(sin^2\theta_2)\}$$

$$= m_2\{-l_2cos\theta_2\dot{x} + l_2^2\dot{\theta_2}(cos^2\theta_2) + l_2^2\dot{\theta_2}(sin^2\theta_2)\}$$

$$\implies \frac{\partial L}{\partial \dot{\theta_2}} = m_2\{l_2^2\dot{\theta_2} - l_2cos\theta_2\dot{x}\}$$

Now differentiate the partial derivative computed above with respect to t

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_2}} = m_2\{l_2^2\ddot{\theta_2} - l_2cos\theta_2\ddot{x} + l_2\dot{\theta_2}\dot{x}sin\theta_2\}$$

Now take the partial derivative of L with respect to θ_2

$$\frac{\partial L}{\partial \theta_2} = m_2\{(\dot{x} - \dot{\theta_2}l_2cos\theta_2)\dot{\theta_2}l_2sin\theta_2 + (l_2\dot{\theta_2})^2sin\theta_2cos\theta_2\} - m_2gl_2sin\theta_2$$

$$= m_2\{\dot{x}\dot{\theta_2}l_2sin\theta_2 - gl_2sin\theta_2\}$$

We know that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_2}} - \frac{\partial L}{\partial \theta_2} = 0$$

$$\implies m_2\{l_2^2\dot{\theta_2} - l_2cos\theta_2\ddot{x} + l_2\dot{\theta_2}\dot{x}sin\theta_2\} - m_2\{\dot{x}\dot{\theta_2}l_2sin\theta_2 - gl_2sin\theta_2\} = 0$$

$$\implies m_2\{l_2^2\ddot{\theta_2} - l_2cos\theta_2\ddot{x} + gl_2sin\theta_2\} = 0$$

Next we consider the variable θ_1

Take the partial derivative of L with respect to $\dot{\theta_1}$

$$\frac{\partial L}{\partial \dot{\theta_1}} = m_1\{(-l_1cos\theta_1(\dot{x} - \dot{\theta_1}l_1cos\theta_1) + l_1^2\dot{\theta_1}(sin^2\theta_1)\}$$

$$= m_1\{-l_1cos\theta_1\dot{x} + l_1^2\dot{\theta_1}(cos^2\theta_1) + l_1^2\dot{\theta_1}(sin^2\theta_1)\}$$

$$\implies \frac{\partial L}{\partial \dot{\theta_1}} = m_1\{l_1^2\dot{\theta_1} - l_1 cos\theta_1 \dot{x}\}$$

Now differentiate the partial derivative computed above with respect to t

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_1}} = m_1\{l_1^2\ddot{\theta_1} - l_1 cos\theta_1 \ddot{x} + l_1\dot{\theta_1}\dot{x}sin\theta_1\}$$

Now take the partial derivative of L with respect to θ_1

$$\frac{\partial L}{\partial \theta_1} = m_1\{(\dot{x} - \dot{\theta_1}l_1 cos\theta_1)\dot{\theta_1}l_1 sin\theta_1 + (l_1\dot{\theta_1})^2 sin\theta_1 cos\theta_1\} - m_1 g l_1 sin\theta_1$$

$$= m_1\{\dot{x}\dot{\theta_1}l_1 sin\theta_1 - g l_1 sin\theta_1\}$$

We know that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_1}} - \frac{\partial L}{\partial \theta_1} = 0$$

$$\implies m_1\{l_1^2\dot{\theta_1} - l_1 cos\theta_1 \ddot{x} + l_1\dot{\theta_1}\dot{x}sin\theta_1\} - m_1\{\dot{x}\dot{\theta_1}l_1 sin\theta_1 - g l_1 sin\theta_1\} = 0$$

$$\implies m_1\{l_1^2\ddot{\theta_1} - l_1 cos\theta_1 \ddot{x} + g l_1 sin\theta_1\} = 0$$

Therefore the three equations of motions can be expressed as follows

$$F = m_2(\ddot{x} - \ddot{\theta_2}l_2 cos\theta_2) + \dot{\theta_2}^2 l_2 sin\theta_2) + M(\ddot{x}) + m_1(\ddot{x} - \ddot{\theta_1}l_1 cos\theta_1 + \dot{\theta_1}^2 l_1 sin\theta_1) \quad (1)$$

$$m_1\{l_1^2\ddot{\theta_1} - l_1 cos\theta_1 \ddot{x} + g l_1 sin\theta_1\} = 0 \quad (2)$$

$$m_2\{l_2^2\ddot{\theta_2} - l_2 cos\theta_2 \ddot{x} + g l_2 sin\theta_2\} = 0 \quad (3)$$

By rewriting equation 1 we get,

$$F = (m_2 + M + m_1)\ddot{x} - \ddot{\theta_2}(m_2 l_2 cos\theta_2) - \ddot{\theta_1}(m_1 l_1 cos\theta_1) + m_1 l_1(\dot{\theta_1})^2 sin\theta_1 + m_2 l_2(\dot{\theta_2})^2 sin\theta_2$$

$$\ddot{x}(m_2 + M + m_1) - \ddot{\theta_2}(m_2 l_2 cos\theta_2) - \ddot{\theta_1}(m_1 l_1 cos\theta_1) = F - m_1 l_1(\dot{\theta_1})^2 sin\theta_1 + m_2 l_2(\dot{\theta_2})^2 sin\theta_2 \quad (4)$$

Equation 2 can be rewritten as

$$\ddot{x}(-l_1 cos\theta_1) + \ddot{\theta_1}l_1^2 = -g l_1 sin\theta_1$$

$$\ddot{\theta_1} = \frac{-g sin\theta_1 + \ddot{x}cos\theta_1}{l_1} \quad (5)$$

Similarly, equation 3 can be rewritten as

$$\ddot{x}(-l_2 cos\theta_2) + \ddot{\theta_2}l_2^2 = -g l_2 sin\theta_2$$

$$\ddot{\theta_2} = \frac{-g sin\theta_2 + \ddot{x}cos\theta_2}{l_2} \quad (6)$$

Now by substituting equation 5 and 6 in equation 4 we get

$$\ddot{x}(m_2 + M + m_1) - (\frac{-g sin\theta_2 + \ddot{x}cos\theta_2}{l_2})(m_2 l_2 cos\theta_2) - (\frac{-g sin\theta_1 + \ddot{x}cos\theta_1}{l_1})(m_1 l_1 cos\theta_1)$$

$$= F - m_1 l_1(\dot{\theta_1})^2 sin\theta_1 + m_2 l_2(\dot{\theta_2})^2 sin\theta_2$$

By isolating the $\ddot{x}$ term to the LHS and the rest of the equation to the RHS we get the following equation

$$\ddot{x} = \frac{F - m_2 g cos\theta_2 sin\theta_2 - m_1 g cos\theta_1 sin\theta_1 - m_1 l_1 \dot{\theta_1}^2 sin\theta_1 - m_2 l_2 \dot{\theta_2}^2 sin\theta_2}{m_2 + M + m_1 - m_2(cos\theta_2)^2 - m_1(cos\theta_1)^2} \tag{7}$$

Using equation 7 and equations 5 and 6 from above we proceed to write our state space representation.

The Nonlinear Matrix Form can be written as:

$$\left(\begin{array}{c} -\frac{\sigma_2}{\sigma_1} - \frac{l_1\, m_1\, \dot{\theta}_1^2\, \sin(\theta_1)}{\sigma_1} - \frac{l_2\, m_2\, \dot{\theta}_2^2\, \sin(\theta_2)}{\sigma_1} \\ \dot{x} \\ -\frac{g\, \sin(\theta_1)}{l_1} - \frac{\cos(\theta_1)\,\sigma_2}{l_1\,\sigma_1} - \frac{m_1\,\dot{\theta}_1\, \cos(\theta_1)\, \sin(\theta_1)}{\sigma_1} - \frac{l_2\, m_2\, \dot{\theta}_2\, \cos(\theta_1)\, \sin(\theta_2)}{l_1\,\sigma_1} \\ \dot{\theta}_1 \\ -\frac{g\, \sin(\theta_2)}{l_2} - \frac{\cos(\theta_2)\,\sigma_2}{l_2\,\sigma_1} - \frac{m_2\,\dot{\theta}_2\, \cos(\theta_2)\, \sin(\theta_2)}{\sigma_1} - \frac{l_1\, m_1\, \dot{\theta}_1\, \cos(\theta_2)\, \sin(\theta_1)}{l_2\,\sigma_1} \\ \dot{\theta}_2 \end{array}\right)$$

And the State space form can be expressed as:

$$\begin{bmatrix} \ddot{x} \\ \dot{x} \\ \ddot{\theta}_1 \\ \dot{\theta}_1 \\ \ddot{\theta}_2 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{l_1\, m_1\, \dot{\theta}_1\, \sin(\theta_1)}{\sigma_1} & -\frac{\sigma_2}{\theta_1\,\sigma_1} & -\frac{l_2\, m_2\, \dot{\theta}_2\, \sin(\theta_2)}{\sigma_1} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1\, \cos(\theta_1)\, \sin(\theta_1)}{\sigma_1} & -\frac{\frac{g\, \sin(\theta_1)}{l_1} + \frac{\cos(\theta_1)\,\sigma_2}{l_1\,\sigma_1}}{\theta_1} & -\frac{l_2\, m_2\, \cos(\theta_1)\, \sin(\theta_2)}{l_1\,\sigma_1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{l_1\, m_1\, \cos(\theta_2)\, \sin(\theta_1)}{l_2\,\sigma_1} & -\frac{\frac{g\, \sin(\theta_2)}{l_2} + \frac{\cos(\theta_2)\,\sigma_2}{l_2\,\sigma_1}}{\theta_1} & -\frac{m_2\, \cos(\theta_2)\, \sin(\theta_2)}{\sigma_1} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \\ \dot{\theta}_1 \\ \theta_1 \\ \dot{\theta}_2 \\ \theta_2 \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{\sigma_1} \\ 0 \\ \frac{cos(\theta_1)}{l_1(\sigma_1))} \\ 0 \\ \frac{cos(\theta_2)}{l_2(\sigma_1)} \\ 0 \end{bmatrix} F$$

where

$$\sigma_1 = -m_1 \cos(\theta_1)^2 - m_2 \cos(\theta_2)^2 + M + m_1 + m_2$$

$$\sigma_2 = g\, m_1\, \cos(\theta_1)\, \sin(\theta_1) + g\, m_2\, \cos(\theta_2)\, \sin(\theta_2)$$

1.2 B. Obtaining the linearized system around equilibrium points (x = 0, $\theta_1 = \theta_2 = 0$) and writing the state space representation of the linearized system

Now We need to find the Jacobian of the given state space in order to linearize the system.

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \cdots & \frac{\partial F_2}{\partial X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial X_1} & \frac{\partial F_n}{\partial X_2} & \cdots & \frac{\partial F_n}{\partial X_n} \end{bmatrix}$$

And B can be computed as:

$$B = \begin{bmatrix} \frac{\partial F_1}{\partial U_1} \\ \frac{\partial F_2}{\partial U_1} \\ \vdots \\ \frac{\partial F_n}{\partial U_1} \end{bmatrix}$$

After Putting Values of Equilibrium Points A and B are

$$A = \begin{pmatrix} 0 & 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{g+\frac{g\,m_1}{M}}{l_1} & 0 & -\frac{g\,m_2}{M\,l_1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g+\frac{g\,m_2}{M}}{l_2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} \frac{1}{M} \\ 0 \\ \frac{1}{M\,l_1} \\ 0 \\ \frac{1}{M\,l_2} \\ 0 \end{pmatrix}$$

Here A and B are Matrices for our linear state equation

$$\dot{X} = AX + BU$$

1.3 C. Conditions on M, m_1, m_2, l_1, l_2 for which the linearized system is controllable

In order to find controllability of the system, following condition must hold

$$rank\left[B \quad AB \quad A^2B \ldots A^{n-1}B\right] = n$$

And Hence the determinant of Matrix $\left[B \quad AB \quad A^2B \ldots A^{n-1}B\right]$ should not be equal to zero. The determinant is computed with $Mgm_1m_2l_1l_2$

$$det\left[B \quad AB \quad A^2B \ldots A^{n-1}B\right] = \frac{g^6\,{l_1}^2 - 2\,g^6\,l_1\,l_2 + g^6\,{l_2}^2}{M^6\,{l_1}^6\,{l_2}^6} \neq 0$$

The condition will be $l_1 \neq l_2$ and M can not be very large value.

1.4 D. Substituting given parameter values and obtaining an LQR Controller

After Substituting the given parameters as:

$$\begin{array}{cc} m_1 & 100\ Kg \\ m_2 & 100\ Kg \\ M & 1000\ Kg \\ l_1 & 20\ m \\ l_2 & 10\ m \end{array}$$

State looks as follows:

$$\begin{bmatrix} \ddot{x} \\ \dot{x} \\ \ddot{\theta}_1 \\ \dot{\theta}_1 \\ \ddot{\theta}_2 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{11}{20} & 0 & -\frac{1}{20} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{10} & 0 & -\frac{11}{10} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \\ \dot{\theta}_1 \\ \theta_1 \\ \dot{\theta}_2 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{1000} \\ 0 \\ \frac{1}{20000} \\ 0 \\ \frac{1}{10000} \\ 0 \end{bmatrix} F$$

For LQR Design, Q is selected as:

$$Q = diag \begin{bmatrix} 5 & 50 & 50 & 500 & 50 & 500 \end{bmatrix}$$

and $R = 0.02$

The LQR then Modelled as follows:

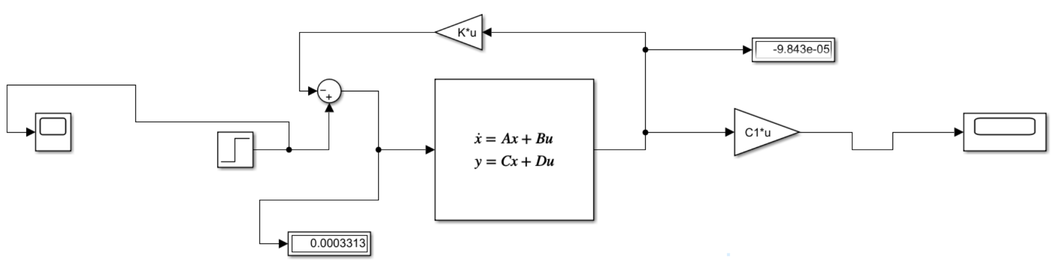

Figure 2: Simulink Model for the LQR

The plot of vector x can be seen as:

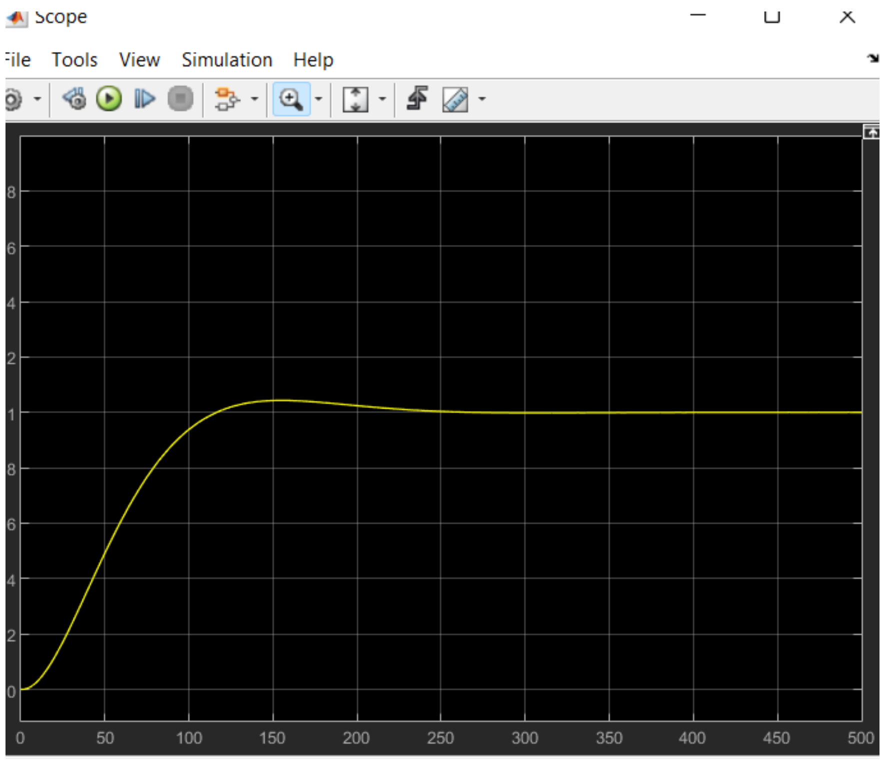

Figure 3: Plot of LQR

For stability we checked the eigen values of the Control Matrix $A - KB$ and found out that all 6 were lying in left half plane

Thus *Stability* proved.

2 Part 2

2.1 E. Selecting output vectors and determine for which of them the linearized system is observable

For checking Observability for vectors rank of $\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$ for different vectors ranks were as follows:

x	6
$x \; and \; \theta_2$	4
$\theta_1 \; and \; \theta_2$	6
$x \; and \; \theta_1 \; and \; \theta_2$	6

Thus, for the output vectors x and θ_2 the linearized system is not observable.

2.2 F. Obtaining the "best" Luenberger observer for each one of the output vectors for which the system is observable

The LeunBerger Observer is Modelled as: And Plots were:

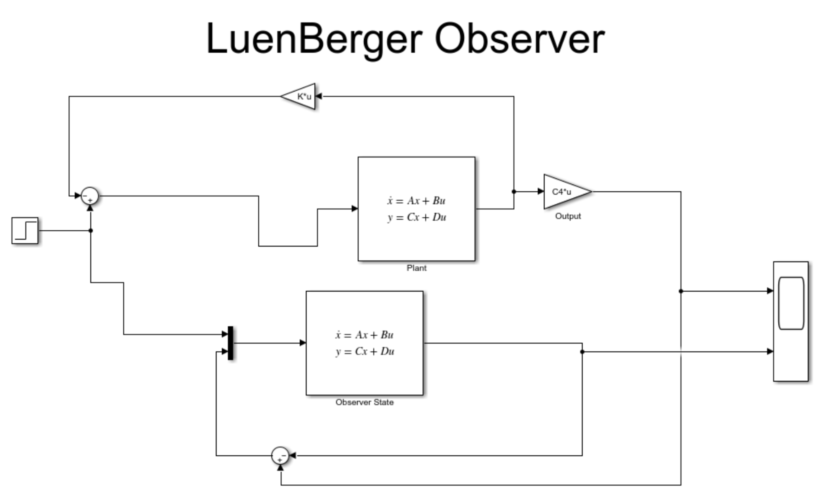

Figure 4: LeunBerger Observer

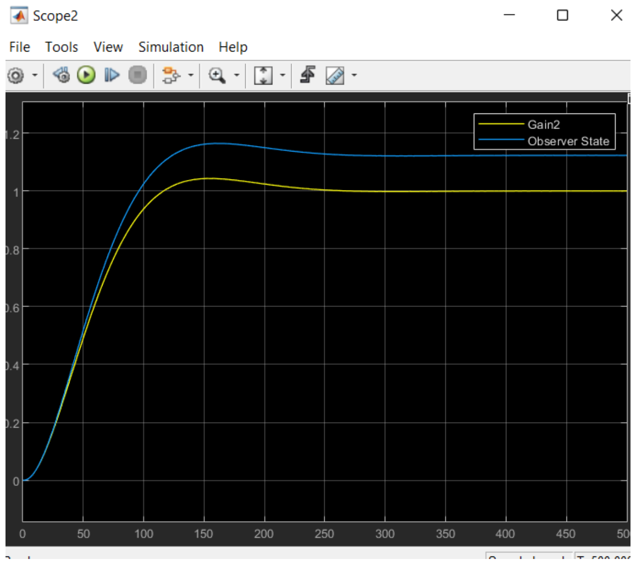

Figure 5: LeunBerger Estimation

2.3 G. LQG Controller

Now the LQG Controllers is designed as

$$\dot{X} = A - LC + [B\ L]U$$

And Modelled as:

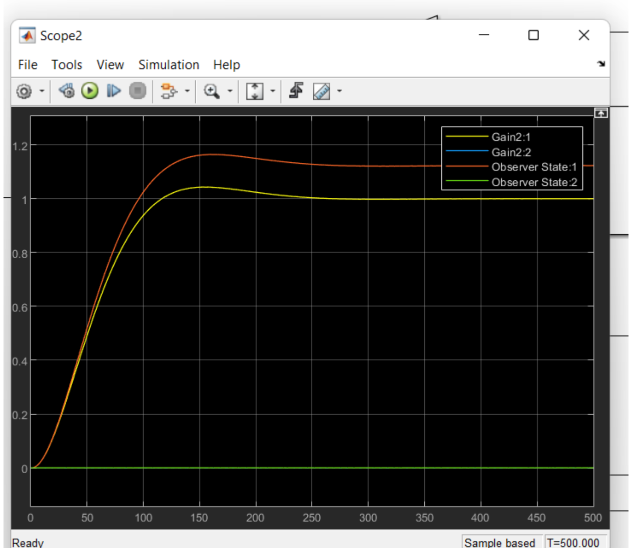

Figure 6: Plot of LQR

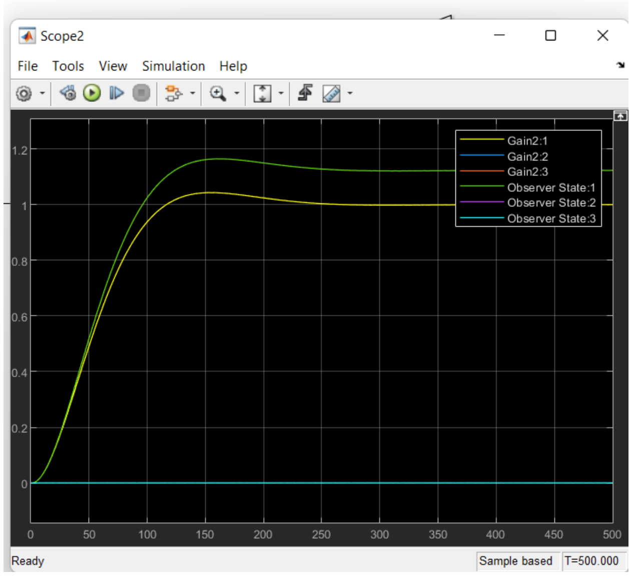

Figure 7: Plot of LQR

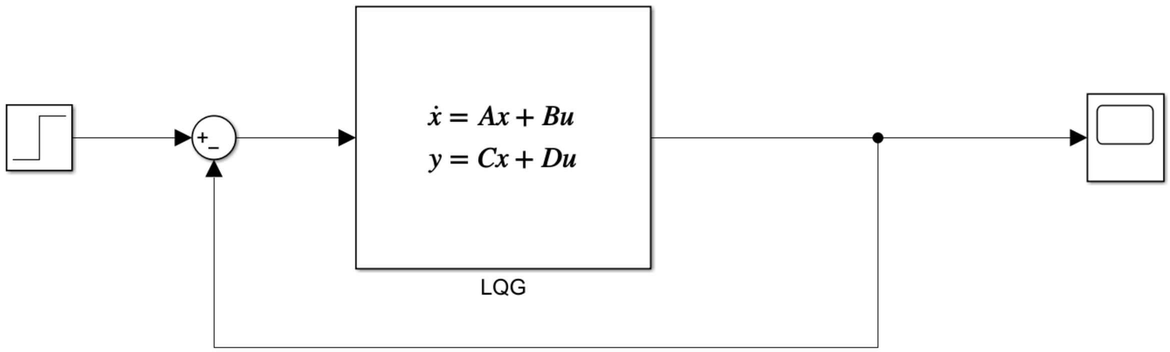

Figure 8: Model of LQG

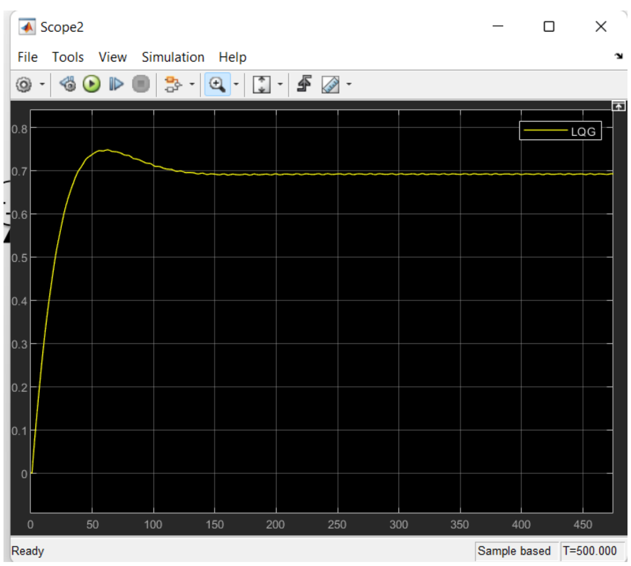

Figure 9: Plot of LQG