ENPM-667 Section 0101

Problem Set 3 (Assignment -3)

Hemanth Joseph Raj

UID: 117518955

Dir ID: hemanth1

(1.) For an $m \times n$ matrix A, prove using the definition provided above that the spectral norm is given by

(a.) $$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Soln For any vector x, such that its not $\vec{0}$. ($\vec{x} \neq \vec{0}$)

$\therefore$ $\|Ax\|$ can be reewritten as.

$$\|Ax\| = \|x\| \cdot \left\| A\left(\frac{x}{\|x\|}\right) \right\|$$

We know that $\left\| \frac{1}{\|x\|} \cdot x \right\| = 1$

at max value for $\|A\|$

$$\therefore \|Ax\| \leq \|x\| \cdot \|A\|$$

$$\frac{\|Ax\|}{\|x\|} \leq \|A\|$$

($x \neq 0$ otherwise LHS $\to \infty$).

The above eqn is nothing but.

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Hence proved.

(b.) For any $n \times 1$ vector x, conclude that
$$\|Ax\| \leq \|A\| \, \|x\|$$

$\Rightarrow$ From previous proof of spectral radius, we can say that

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

It can be re-written as-

$$\|A\| \geq \frac{\|Ax\|}{\|x\|}$$ (since we already know that x ~~is~~ a $n \times 1$ vector $\neq 0$)

$$\Rightarrow \|A\| \cdot \|x\| \geq \|Ax\|$$

$$\therefore \|Ax\| \leq \|A\| \cdot \|x\|$$

Hence proved.

(c) Using the conclusion above, prove for conformable matrices A & B (AB is well defined).

$$\|AB\| \leq \|A\| \, \|B\|$$

$\Rightarrow$ Using definition of spectral norm, we know that

$$\|AB\| = \max_{\|x\|=1} \|ABx\|$$

using the submultiplicity axiom, we can write.

$$\max_{\|x\|=1} \|ABx\| \leq \max_{\|x\|=1} \left(\|A\| \cdot \|Bx\|\right)$$

Since we know that $\max_{||x||=1} ||A||$ is a number

the max only applies to $||Bx||$.

$$\Rightarrow \max_{||x||=1} ||ABx|| \leq ||A|| \cdot \max_{||x||} ||Bx||$$

using spectral norm definition, we get

$$\Rightarrow \max_{||x||=1} ||ABx|| \leq ||A|| \cdot ||B||$$

using spectral norm definition on LHS, we get

$$||AB|| \leq ||A|| \cdot ||B||$$

Hence proved.

(2.) Show that for the $n \times n$ matrix A

$$\sigma(A) \leq ||A||$$

<u>Soln</u> For any non zero vector x with eigen value λ, we can write A as.

$$Ax = \lambda x$$

apply spectral norm to it, we get

$$||Ax|| = ||\lambda x||$$

using the Homogeneous axiom, we can get

$$||Ax|| = |\lambda| \cdot ||x||$$

Since x is a non-zero vector, we divide both sides by $||x||$, we get.

$$\frac{||Ax||}{||x||} = \frac{|\lambda| \cdot \cancel{||x||}}{\cancel{||x||}} \quad (\because x \neq 0)$$

$$|\lambda| = \frac{||Ax||}{||x||} \quad (x \neq 0)$$

using the spectral norm statement, we get

$$|\lambda| \leq ||A||$$

where $||A||$ - spectral norm of A.

$\therefore$ This proves that for any λ, $|\lambda|$ will alway be less than or equal to $||A||$, ~~take~~ the same applies to the max of all $|\lambda|$ values,

Since we know that $\sigma(A) = \max\{|\lambda| : \lambda \text{ is eigen value of } A\}$

Hence we can say that.

$$\sigma(A) \leq ||A||$$

Hence proved.

3 If $A(t)$ is a continuously-differentiable $n \times n$ matrix function that is invertible at each t. Show that

$$\frac{d}{dt} A^{-1}(t) = -A^{-1}(t)\, \dot{A}(t) . A^{-1}(t)$$

Soln From the product rule for differentiation of matrices, we know that.

$$\frac{d}{dt}\left[A(t) . B(t)\right] = \dot{A}(t) . B(t) + A(t) . \dot{B}(t)$$

Since we know that $A(t)$ is continuously-differentiable we can apply this rule; moreover we know that A is invertible at each t. So we can replace $B(t)$ with $A^{-1}(t)$ in the above eqn

$$\Rightarrow \frac{d}{dt}[\underbrace{A(t) . A^{-1}(t)}_{I}] = \dot{A}(t) . A^{-1}(t) + A(t) . \dot{A}^{-1}(t)$$

$$\Rightarrow \frac{d}{dt}(I) = 0$$

$$0 = \dot{A}(t) . A^{-1}(t) + A(t) . \dot{A}^{-1}(t)$$

$$-\dot{A}(t) . A^{-1}(t) = A(t) . \dot{A}^{-1}(t)$$

Forward multiply on BS wit $A^{-1}(t)$

$$-A^{-1}(t) \times \dot{A}(t) . A^{-1}(t) = \underbrace{A^{-1}(t) . A(t)}_{I} . \dot{A}^{-1}(t)$$

$$-A^{-1}(t).\dot{A}(t).A^{-1}(t) = \underbrace{I.\dot{A}^{-1}(t)}_{\dot{A}^{-1}(t)}$$

$$\Rightarrow -A^{-1}(t).\dot{A}(t).A^{-1}(t) = \underbrace{\dot{A}^{-1}(t)}_{\frac{d}{dt}(A^{-1}(t))}$$

$\Rightarrow$

$$\Rightarrow -A^{-1}(t).\dot{A}(t).A^{-1}(t) = \frac{d}{dt}A^{-1}(t)$$

$$\therefore \frac{d}{dt}A^{-1}(t) = -A^{-1}(t).\dot{A}(t).A^{-1}(t)$$

Hence proved.

6. Prove that

$$\frac{\partial}{\partial \tau} \Phi(t, \tau) = -\Phi(t, \tau) \cdot A(\tau)$$

Soln For a constant matrix, $A(t) = A$, the transition matrix is given by

$$\Phi(t, t_0) = e^{A(t-t_0)}$$

We know that $\frac{\partial}{\partial t} \Phi(t, t_0) = A(t) \cdot \Phi(t, t_0)$ — ①

We also know that the transition matrix for $A(t)$ is invertible for every t & t_0 & is given by

$$\Phi^{-1}(t, t_0) = \Phi(t_0, t) \text{ — ②}$$

Now we have our question

LHS ⇒

$\frac{\partial}{\partial \tau} \Phi(t, \tau)$; using eqn ② we can write it as

$$= \frac{\partial}{\partial \tau} \Phi^{-1}(\tau, t)$$

From problem ③, we have proved for continuously differentiable matrix invertible at each t.

$$\frac{d}{dt} A^{-1}(t) = -A^{-1}(t) \cdot \dot{A}(t) \cdot A^{-1}(t) \text{ — ③}$$

Applying eqn ③, we get

$$\Rightarrow \frac{\partial}{\partial \tau} \Phi^{-1}(\tau, t) = -\Phi^{-1}(\tau, t) \cdot \frac{\partial}{\partial \tau} \Phi(\tau, t) \cdot \Phi^{-1}(\tau, t)$$

Use eqn 1 to expand $\frac{\partial}{\partial \tau}\Phi(\tau,t)$ we get

$$= -\Phi^{-1}(\tau,t)\cdot A(\tau)\cdot\underbrace{\Phi(\tau,t)\cdot\Phi^{-1}(\tau,t)}_{I}$$

$$= -\Phi^{-1}(\tau,t)\cdot A(\tau)$$

Applying eqn 2, we get

$$= -\Phi(t,\tau)\cdot A(\tau).$$

LHS = RHS.

Hence proved.

8. Compute the matrix exponential e^{At} for the following 3X3 matrix A

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix}$$

Soln To find the matrix exponential, we need to do Eigen values decomposition of the matrix. First the matrix needs to be diagonalizable.

Lets see if matrix A is diagonalizable or not

Let $\vec{x}$ be denoted as eigen vectors of A with λ as its corresponding eign value.

$$\Rightarrow A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} = (\lambda I).\vec{x}$$
$$(A-\lambda I).\vec{x} = 0$$

For it to have a non-trivial soln $|A-\lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 6 & 5 & -2 \end{vmatrix} = 0$$

$$-\lambda(2\lambda+\lambda^2-5) - 1(0-6) + 0 = 0$$
$$-\lambda^3 - 2\lambda^2 + 5\lambda + 6 = 0$$
$$\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$$

$$(\lambda+1)(\lambda^2+\lambda-6)=0$$
$$(\lambda+1)(\lambda(\lambda+3)-2(\lambda+3))=0$$
$$(\lambda+1)(\lambda-2)(\lambda+3)=0$$

$\therefore$ The eigen values are $\lambda_1=-1$; $\lambda_2=2$; $\lambda_3=-3$

Now lets calculate eigen vector corresponding to

$\lambda_1=-1$

$$\Rightarrow (A-\lambda_1 I).\vec{x}_1=\vec{0}$$

$$\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 6 & 5 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$\Rightarrow \cancel{x_1+x_2=0}$ $\quad\Rightarrow \cancel{x_1=-x_2} \Rightarrow \cancel{x_1=x_3}$

$\cancel{x_2+x_3=0}$ $\quad \cancel{x_2=-x_3}$

$\cancel{6x_1+5x_2+x_3=0}$

$\Rightarrow \cancel{6x_3-5x_3+x_3=0}$

Perform row reduction to Echelon form

Write the augmented matrix

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 6 & 5 & 1 & 0\end{array}\right)$$

$R_3 \rightarrow R_3-6R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$R_3 \to R_2 + R_3$

Writing eqn back in form

$$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$
$$x_2 + x_3 = 0 \qquad x_2 = -x_3$$

If we say $x_3 = K$; we get
$x_2 = -K$; $x_1 = K$
$\therefore \vec{x}_1 = (K \ \ -K \ \ K)^T$
normalizing it, we get

$$\sqrt{K^2 + (-K)^2 + K^2} = 1$$
$$K = 1/\sqrt{3}$$

$$\therefore \vec{x}_1 = \left(\frac{1}{\sqrt{3}} \ \ \frac{-1}{\sqrt{3}} \ \ \frac{1}{\sqrt{3}}\right)^T$$

Now lets calculate eign vectors for $\lambda_2 = 2$

$$(A - \lambda_2 I) \cdot \vec{x}_2 = \vec{0}$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & 5 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the augmental matrix

$$\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 6 & 5 & -4 & 0 \end{array}\right) R_1 \to -\frac{1}{2} \times R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 6 & 5 & -4 & 0 \end{array}\right) R_3 \to R_3 - 6R_1$$

$$\begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 8 & -4 & 0 \end{pmatrix} R_3 \to R_3 + 4R_2$$

$$\begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_2 \to -\frac{1}{2} R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Write it back in eqn form

$$x_1 - 1/2\, x_2 = 0 \Rightarrow x_1 = 1/2\, x_2$$
$$x_2 - 1/2\, x_3 = 0 \qquad x_2 = 1/2\, x_3$$

$\Rightarrow$ lets say $x_3 = K$; $x_2 = K/2$; $x_1 = K/4$

$\therefore \vec{x_2} = \begin{pmatrix} K/4 & K/2 & K \end{pmatrix}^T$

normalizing it, we get

$$\sqrt{\left(\frac{K}{4}\right)^2 + \left(\frac{K}{2}\right)^2 + K^2} = 1$$

$$\Rightarrow \frac{K^2}{16} + \frac{K^2}{4} + K^2 = 1$$

$$\Rightarrow K^2 + 4K^2 + 16K^2 = 16$$

$$21K^2 = 16$$

$$\Rightarrow K^2 = \frac{16}{21} \Rightarrow K = \sqrt{\frac{16}{21}} = \frac{4}{21}$$

$$\therefore \vec{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{21}} & \frac{2}{\sqrt{21}} & \frac{4}{\sqrt{21}} \end{pmatrix}^T$$

Now lets calculate eigen vectors for $\lambda_3 = -3$

$$\Rightarrow \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 6 & 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write in augmented form

$$\left(\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right) \begin{array}{l} R_1 \to R_1 \times \frac{1}{3} \\ R_2 \to R_2 \times \frac{1}{3} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right) R_3 \to R_3 - 6R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \xrightarrow{} \qquad R_3 \to R_3 - 3R_2 \qquad \Rightarrow \left(\begin{array}{ccc|c} 1 & 1/3 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Writing it back in eqn form, we get

$$\Rightarrow x_1 = -1/3\, x_2$$
$$x_2 = -1/3\, x_3$$

lets say $x_3 = K$ $\therefore x_2 = -K/3$; $x_1 = K/9$

$$\therefore \vec{x}_3 = \begin{pmatrix} K/9 & -K/3 & K \end{pmatrix}^T$$

normalizing it, we get

$$\sqrt{\left(\frac{K}{9}\right)^2 + \left(\frac{-K}{3}\right)^2 + K^2} = 1$$

$$\frac{K^2}{81} + \frac{K^2}{9} + K^2 = 1$$

$$K^2 + 9K^2 + 81K^2 = 81$$

$$91K^2 = 81$$

$$K^2 = 81/91$$

$$K = \sqrt{\frac{81}{91}} = \frac{9}{\sqrt{91}}$$

$$\cancel{K^2}$$

$$\therefore \vec{x}_3 = \begin{pmatrix} \frac{1}{\sqrt{91}} & -\frac{3}{\sqrt{91}} & \frac{9}{\sqrt{91}} \end{pmatrix}^T$$

$\therefore$ The three eigen vectors of A are as follows

$$\vec{x}_1 = \left(1/\sqrt{3} \quad -1/\sqrt{3} \quad 1/\sqrt{3}\right)^T$$

$$\vec{x}_2 = \left(1/\sqrt{21} \quad 2/\sqrt{21} \quad 4/\sqrt{21}\right)^T$$

$$\vec{x}_3 = \left(1/\sqrt{91} \quad -3/\sqrt{91} \quad 9/\sqrt{91}\right)^T$$

As we can see that the three vectors can't be written as linear combinations of each other, hence they are all linearly independent. Therefore matrix A is diagonalizable

Assume a vector matrix $V = \left[\vec{x}_1 \,\vdots\, \vec{x}_2 \,\vdots\, \vec{x}_3\right]$

$\therefore$ We can rewrite matrix A as $A = V \Lambda V^{-1}$ where Λ is the eigen value diagonal matrix.

$$\therefore V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{91}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{21}} & \frac{-3}{\sqrt{91}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{21}} & \frac{9}{\sqrt{91}} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

lets find V^{-1} ; $V^{-1} = (\text{Adjoint } V / \det V)$

$$\Rightarrow \det(V) = \frac{1}{\sqrt{3}}\left(\frac{18}{\sqrt{21}\sqrt{91}} + \frac{12}{\sqrt{21}\sqrt{91}}\right) - \frac{1}{\sqrt{21}}\left(\frac{-9}{\sqrt{3}\sqrt{91}} + \frac{3}{\sqrt{3}\sqrt{91}}\right) + \frac{1}{\sqrt{91}}\left(\frac{-4}{\sqrt{3}\sqrt{21}} - \frac{2}{\sqrt{3}\sqrt{21}}\right)$$

$$= \frac{30}{\sqrt{3}\sqrt{21}\sqrt{91}} + \cancel{\frac{6}{\sqrt{3}\sqrt{21}\sqrt{91}}} - \cancel{\frac{6}{\sqrt{3}\sqrt{21}\sqrt{91}}}$$

$$\therefore \det(V) = \frac{30}{\sqrt{3}\sqrt{21}\sqrt{91}}$$

lets find Adjoint V $\Rightarrow$ rewriting & replacing the elements of matrix V, with its co-factors & then take its transpose, we get the adjoint.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} \frac{2}{\sqrt{21}} & \frac{-3}{\sqrt{91}} \\ \frac{4}{\sqrt{21}} & \frac{9}{\sqrt{91}} \end{vmatrix} = \frac{18}{\sqrt{21}\sqrt{91}} + \frac{12}{\sqrt{21}\sqrt{91}} = \frac{30}{\sqrt{21}\sqrt{91}}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} \frac{-1}{\sqrt{3}} & \frac{-3}{\sqrt{91}} \\ \frac{1}{\sqrt{3}} & \frac{9}{\sqrt{91}} \end{vmatrix} = (-1)\left(\frac{-9}{\sqrt{3}\sqrt{91}} + \frac{3}{\sqrt{3}\sqrt{91}}\right) = \frac{6}{\sqrt{3}\sqrt{91}}$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{21}} \end{vmatrix} = \left(\frac{-4}{\sqrt{3}\sqrt{21}} - \frac{2}{\sqrt{3}\sqrt{21}}\right) = \frac{-6}{\sqrt{3}\sqrt{21}}$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{91}} \\ \frac{4}{\sqrt{21}} & \frac{9}{\sqrt{91}} \end{vmatrix} = (-1)\left(\frac{9}{\sqrt{21}\sqrt{91}} - \frac{4}{\sqrt{21}\sqrt{91}}\right) = \frac{-5}{\sqrt{21}\sqrt{91}}$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{91}} \\ \frac{1}{\sqrt{3}} & \frac{9}{\sqrt{91}} \end{vmatrix} = \left(\frac{9}{\sqrt{3}\sqrt{91}} - \frac{1}{\sqrt{3}\sqrt{91}}\right) = \frac{8}{\sqrt{3}\sqrt{91}}$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{21}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{21}} \end{vmatrix} = (-1)\left(\frac{4}{\sqrt{3}\sqrt{21}} - \frac{1}{\sqrt{3}\sqrt{21}}\right) = \frac{-3}{\sqrt{3}\sqrt{21}}$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{91}} \\ \frac{2}{\sqrt{21}} & \frac{-3}{\sqrt{91}} \end{vmatrix} = \frac{-3}{\sqrt{21}\sqrt{91}} - \frac{2}{\sqrt{21}\sqrt{91}} = \frac{-5}{\sqrt{21}\sqrt{91}}$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{91}} \\ \frac{-1}{\sqrt{3}} & \frac{-3}{\sqrt{91}} \end{vmatrix} = (-1)\left(\frac{-3}{\sqrt{3}\sqrt{91}} + \frac{1}{\sqrt{3}\sqrt{91}}\right) = \frac{2}{\sqrt{3}\sqrt{91}}$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{21}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{21}} \end{vmatrix} = \frac{2}{\sqrt{3}\sqrt{21}} + \frac{1}{\sqrt{3}\sqrt{21}} = \frac{3}{\sqrt{3}\sqrt{21}}$$

The Co-factor matrix is

$$\begin{bmatrix} \frac{30}{\sqrt{21}\sqrt{91}} & \frac{6}{\sqrt{3}\sqrt{91}} & \frac{-6}{\sqrt{3}\sqrt{21}} \\ \frac{-5}{\sqrt{21}\sqrt{91}} & \frac{8}{\sqrt{3}\sqrt{91}} & \frac{-3}{\sqrt{3}\sqrt{21}} \\ \frac{-5}{\sqrt{21}\sqrt{91}} & \frac{2}{\sqrt{3}\sqrt{91}} & \frac{3}{\sqrt{3}\sqrt{21}} \end{bmatrix}$$

Take transpose of this matrix to get the adjoint of V

$$\therefore \mathrm{Adj}(V) = \begin{bmatrix} \frac{30}{\sqrt{21}\sqrt{91}} & \frac{-5}{\sqrt{21}\sqrt{91}} & \frac{-5}{\sqrt{21}\sqrt{91}} \\ \frac{6}{\sqrt{3}\sqrt{91}} & \frac{8}{\sqrt{3}\sqrt{91}} & \frac{2}{\sqrt{3}\sqrt{91}} \\ \frac{-6}{\sqrt{3}\sqrt{21}} & \frac{-3}{\sqrt{3}\sqrt{21}} & \frac{3}{\sqrt{3}\sqrt{21}} \end{bmatrix}$$

$$V^{-1} = \mathrm{Adj}(V) / \det(V)$$

$$\therefore V^{-1} = \frac{1}{\frac{30}{\sqrt{3}\sqrt{21}\sqrt{91}}} \begin{bmatrix} \frac{30}{\sqrt{21}\sqrt{91}} & \frac{-5}{\sqrt{21}\sqrt{91}} & \frac{-5}{\sqrt{21}\sqrt{91}} \\ \frac{6}{\sqrt{3}\sqrt{91}} & \frac{8}{\sqrt{3}\sqrt{91}} & \frac{2}{\sqrt{3}\sqrt{91}} \\ \frac{-6}{\sqrt{3}\sqrt{21}} & \frac{-3}{\sqrt{3}\sqrt{21}} & \frac{3}{\sqrt{3}\sqrt{21}} \end{bmatrix}$$

$$V^{-1} = \frac{1}{30} \begin{bmatrix} 30\sqrt{3} & -5\sqrt{3} & -5\sqrt{3} \\ 6\sqrt{21} & 8\sqrt{21} & 2\sqrt{21} \\ -6\sqrt{91} & -3\sqrt{91} & 3\sqrt{91} \end{bmatrix}$$

$$\Rightarrow A = V \Lambda V^{-1}$$

$$\therefore A = \frac{1}{30} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{91}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{21}} & \frac{-3}{\sqrt{91}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{21}} & \frac{9}{\sqrt{91}} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 30\sqrt{3} & -5\sqrt{3} & -5\sqrt{3} \\ 6\sqrt{21} & 8\sqrt{21} & 2\sqrt{21} \\ -6\sqrt{91} & -3\sqrt{91} & 3\sqrt{91} \end{bmatrix}$$

Now that we have written A as $V\Lambda V^{-1}$, we can find the value of e^{At} as follows.

W.K.T $\rightarrow$ e^{x} expansion $\Rightarrow \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$

if A is a square matrix, then

$$e^{At} = \frac{I}{0!} + \frac{At}{1!} + \frac{A^2t^2}{2!} + \dots + \frac{A^Kt^K}{K!} + \dots\dots + \frac{A^nt^n}{n!}$$

sub $A = V\Lambda V^{-1}$

$$\therefore e^{At} = e^{V\Lambda V^{-1}t}$$

$$= \underbrace{I}_{VV^{-1}} + V\Lambda V^{-1}t + \frac{1}{2!}(V\Lambda V^{-1})^2 t^2 + \dots + \frac{1}{K!}(V\Lambda V^{-1})^K t^K + \dots$$

Suppose $\rightarrow$ $V\Lambda \underbrace{V^{-1} \times V}_{I}\Lambda \underbrace{V^{-1} \times V}_{I}\Lambda V^{-1} = V\Lambda^3 V^{-1}$

apply this to e^{At}, we get

$$e^{At} = VV^{-1} + V\Lambda V^{-1}t + \frac{V\Lambda^2 V^{-1}t^2}{2!} + \dots + \frac{V\Lambda^K V^{-1}t^K}{K!} + \dots\dots$$

$$= V\left[I + \Lambda t + \frac{\Lambda^2 t^2}{2!} + \cdots \frac{\Lambda^K t^K}{K!} + \cdots\right] V^{-1}$$

We know that $\Lambda^K = \begin{bmatrix} \lambda_1^K & 0 & \cdots & & 0 \\ 0 & \lambda_2^K & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_{n-1}^K & 0 \\ 0 & \cdots & & 0 & \lambda_n^K \end{bmatrix}$

$$\therefore e^{At} = V \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix} V^{-1}$$

we know our eigen values are $\lambda_1 = -1; \lambda_2 = 2; \lambda_3 = -3$

$$\therefore e^{At} = V \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} V^{-1}$$

we know the values of V & V^{-1}, [illegible] sub them here

$$\therefore e^{At} = \frac{1}{30} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{91}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{21}} & \frac{-3}{\sqrt{91}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{21}} & \frac{9}{\sqrt{91}} \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 30\sqrt{3} & -5\sqrt{3} & -5\sqrt{3} \\ 6\sqrt{21} & 8\sqrt{21} & 2\sqrt{21} \\ -6\sqrt{91} & -3\sqrt{91} & 3\sqrt{91} \end{bmatrix}$$

$$= \frac{1}{30}\begin{bmatrix} \frac{e^{-t}}{\sqrt{3}} & \frac{e^{2t}}{\sqrt{21}} & \frac{e^{-3t}}{\sqrt{91}} \\ \frac{-e^{-t}}{\sqrt{3}} & \frac{2e^{2t}}{\sqrt{21}} & \frac{-3e^{-3t}}{\sqrt{91}} \\ \frac{e^{-t}}{\sqrt{3}} & \frac{4e^{2t}}{\sqrt{21}} & \frac{9e^{-3t}}{\sqrt{91}} \end{bmatrix}\begin{bmatrix} 30\sqrt{3} & -5\sqrt{3} & -5\sqrt{3} \\ 6\sqrt{21} & 8\sqrt{21} & 2\sqrt{21} \\ -6\sqrt{91} & -3\sqrt{91} & 3\sqrt{91} \end{bmatrix}$$

$$= \frac{1}{30}\begin{bmatrix} 30e^{-t}+6e^{2t}-6e^{-3t} & -5e^{-t}+8e^{2t}-3e^{-3t} & -5e^{-t}+2e^{2t}+3e^{-3t} \\ -30e^{-t}+12e^{2t}+18e^{-3t} & 5e^{-t}+16e^{2t}+9e^{-3t} & 5e^{-t}+4e^{2t}-9e^{-3t} \\ 30e^{-t}+24e^{2t}-54e^{-3t} & -5e^{-t}+32e^{2t}-27e^{-3t} & -5e^{-t}+8e^{2t}+27e^{-3t} \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} e^{-t}+\frac{e^{2t}}{5}-\frac{e^{-3t}}{5} & -\frac{e^{-t}}{6}+\frac{4e^{2t}}{15}-\frac{e^{-3t}}{10} & -\frac{e^{-t}}{6}+\frac{e^{2t}}{15}+\frac{e^{-3t}}{10} \\ -e^{-t}+\frac{2e^{2t}}{5}+\frac{3e^{-3t}}{5} & \frac{e^{-t}}{6}+\frac{8e^{2t}}{15}+\frac{3e^{-3t}}{10} & \frac{e^{-t}}{6}+\frac{2e^{2t}}{15}-\frac{3e^{-3t}}{10} \\ e^{-t}+\frac{4e^{2t}}{5}-\frac{9e^{-3t}}{5} & -\frac{e^{-t}}{6}+\frac{16e^{2t}}{15}-\frac{9e^{-3t}}{10} & -\frac{e^{-t}}{6}+\frac{4e^{2t}}{15}+\frac{9e^{-3t}}{10} \end{bmatrix}$$

(4) Use Laplace Transforms to solve $\dot{x} = ax(t) + b(t).u(t)$ with initial condition $x(0)$. $a \to$ constant, $x(t)$, $b(t)$ & $u(t)$ are real valued functions.

Soln Applying Laplace Transform, we get

$$\Rightarrow s\bar{x}(s) - x(0) = a\bar{x}(s) + \mathcal{L}[b(t).u(t)]$$

$$\Rightarrow s\bar{x}(s) - x(0) = a\bar{x}(s) + b(s) * u(s)$$

$* \to$ convolution operation

$$\Rightarrow \bar{x}(s)[s-a] = x(0) + b(s) * u(s)$$

$$\Rightarrow \bar{x}(s) = \frac{x(0)}{s-a} + \frac{b(s) * u(s)}{s-a}$$

Taking inverse Laplace, for calculating value of $x(t)$, we get

$$x(t) = e^{at}.x(0) + \mathcal{L}^{-1}\left[\frac{b(s) * u(s)}{(s-a)}\right]$$

$$\Rightarrow x(t) = e^{at}.x(0) + \mathcal{L}^{-1}[b(s) * u(s)] * \mathcal{L}^{-1}\left[\frac{1}{s-a}\right]$$

$$\Rightarrow x(t) = e^{at}.x(0) + b(t).u(t) * e^{at}$$

From convolution theorem, we know that

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau$$

Apply this for the last term, we get

$$x(t) = e^{at} \cdot x(0) + \int_{-\infty}^{\infty} b(t-\tau) \cdot u(t-\tau) e^{a\tau} d\tau.$$

(5) Define state variables such that the n^{th} order D.E

$$y^{(n)}(t)+a_{n-1}t^{-1}y^{(n-1)}(t)+\dots+a_1 t^{-n+1}y^{(1)}(t)+a_0 t^{-n}y(t)=0$$

where $y^{(n)}(t)=\dfrac{d^n y(t)}{dt^n}$, can be written as the linear state eqn. $\dot{x}(t)=t^{-1}Ax(t)$ where A is a constant matrix $n\times n$

Soln $\Rightarrow y^{(n)}(t)=-a_0 t^{-n}y(t)-a_1 t^{-n+1}y^{(1)}(t)-\dots-a_{n-2}t^{-2}y^{(n-2)}(t)$
$-a_{n-1}t^{-1}y^{(n-1)}(t)$

— (1)

Thus the state variables are taken as.

$$\left.\begin{aligned} x_1(t)&=t^{-n+1}y(t)\\ x_2(t)&=t^{-n+2}y'(t)\\ &\vdots\\ x_{n-1}(t)&=t^{-1}y^{(n-2)}(t)\\ x_n(t)&=t^{0}y^{(n-1)}(t)\end{aligned}\right\} \quad (2)$$

Taking derivatives of the state variables, we get

$$\dot{x}_1=(1-n)t^{-n}y(t)+t^{-n+1}\cdot y^{(1)}(t)$$

$$=t^{-1}\Big((-n+1)\underbrace{t^{-n+1}y(t)}_{x_1}+\underbrace{t^{-n+2}y^{(1)}(t)}_{x_2}\Big)$$

$\therefore \dot{x}_1 = t^{-1}((-n+1)x_1 + x_2))$ from eqn ②

$\Rightarrow \dot{x}_2 = (-n+2)t^{-n+1}y^{(1)}(t) + t^{-n+2}y^{(2)}(t)$

$= t^{-1}((-n+2)\underbrace{t^{-n+2}y^{(1)}(t)}_{x_2} + \underbrace{t^{-n+3}y^{(2)}(t)}_{x_3})$

$\therefore \dot{x}_2 = t^{-1}((-n+2)x_2 + x_3)$ from eqn ②

Similarly we can find the values for the remaining state variables.

Lets find the values for last two variables

$\dot{x}_{n-1} = -1 \times t^{-2}y^{(n-2)}(t) + t^{-1}y^{(n-1)}(t)$

$= t^{-1}(\underbrace{-t^{-1}y^{(n-2)}(t)}_{x_{n-1}} + \underbrace{y^{(n-1)}(t)}_{x_n})$

$\therefore \dot{x}_{n-1} = t^{-1}(-x_{n-1} + x_n)$ from eqn ②

$\Rightarrow \dot{x}_n = y^{n}(t)$

$= t^{-1}(t\,y^{(n)}(t))$

Sub value for $y^{(n)}(t)$ here from eqn ①, we get

$\dot{x}_n = t^{-1}(t(-a_0 t^{-n}y(t) - a_1 t^{-n+1}y^{(1)}(t) - \ldots - a_{n-2}t^{-2}y^{(n-2)}(t)$
$- a_{n-1}t^{-1}y^{(n-1)}(t)))$

So, from our eqn ② replace the $ty(t)$ terms with x_i terms.

$$\therefore \dot{x}_n = t^{-1}(-a_0 x_1 - a_1 x_2 - \dots - a_{n-2} x_{n-1} - a_{n-1} x_n)$$

$\therefore$ The state variables are.

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

differentiating it, we get

$$\Rightarrow \dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix}$$

replace the $\dot{x}_i(t)$ terms with our calculated values, we get

$$\dot{x}(t) = \begin{bmatrix} t^{-1}((-n+1)\, x_1(t) + x_2(t)) \\ t^{-1}((-n+2)\, x_2(t) + x_3(t)) \\ \vdots \\ t^{-1}(-x_{n-1}(t) + x_n(t)) \\ t^{-1}(-a_0 x_1 - a_1 x_2 - \dots - a_{n-2} x_{n-1} - a_{n-1} x_n \end{bmatrix}$$

Breaking the above ~~two~~ vector as a two different matrices, we get.

$$\dot{x}(t) = t^{-1} \underbrace{\begin{bmatrix} (-n+1) & 1 & 0 & \cdots & 0 & 0 \\ 0 & (-n+2) & 1 & & \vdots & \vdots \\ 0 & 0 & (-n+3) & & \vdots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & & -1 & 1 \\ -a_0 & -a_1 & -a_3 & & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}}_{x(t)}$$

$$\therefore \dot{x}(t) = t^{-1} A x(t)$$

~~The final form of y(t) is~~

~~y(t) = Cx~~

The final form of $y(t)$ is

$$y(t) = \begin{bmatrix} t^{n-1} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

(7) Compute the state-transition matrix $\Phi(t, t_0)$ for the following matrix $A(t) = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix}$

Soln The state-transition matrix $\Phi(t, t_0)$ which is a function of two variables, is given by the Peano-Baker series.

$$\therefore \Phi(t, t_0) = I + \int_{t_0}^{t} A(\sigma_1)\, d\sigma_1 + \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2)\, d\sigma_2\, d\sigma_1 +$$

$$+ \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) \int_{t_0}^{\sigma_2} A(\sigma_3) \cdot d\sigma_3 \cdot d\sigma_2\, d\sigma_1 + \cdots\cdots +$$

$$\cdots + \int_{t_0}^{t} \int_{t_0}^{\sigma_1} \cdots \int_{t_0}^{\sigma_K} A(\sigma_1) A(\sigma_2) \cdots A(\sigma_K)\, d\sigma_K\, d\sigma_{K-1} \cdots d\sigma_1$$

We apply this to our matrix.

1st term $= I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2nd term $= \int_{t_0}^{t} A(\sigma_1)\, d\sigma_1 = \int_{t_0}^{t} \begin{bmatrix} 1 & 0 \\ 1 & \eta(\sigma_1) \end{bmatrix} d\sigma_1$

$$2^{nd}\ \underline{\underline{\text{term}}} = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_1 & \int_{t_0}^{t} \eta(\sigma_1)\,d\sigma_1 \end{bmatrix}\Bigg|_{t_0}^{t} = \begin{bmatrix} t-t_0 & 0 \\ t-t_0 & \int_{t_0}^{t} \eta(\sigma_1)\,d\sigma_1 \end{bmatrix}$$

$$3^{rd}\ \text{term} = \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2)\,d\sigma_2\,d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} \begin{bmatrix} 1 & 0 \\ 1 & \eta(\sigma_2) \end{bmatrix} d\sigma_2\,d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \begin{bmatrix} \sigma_2 & 0 \\ \sigma_2 & \int_{t_0}^{\sigma_1} \eta(\sigma_2)\,d\sigma_2 \end{bmatrix}\Bigg|_{t_0}^{\sigma_1} d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \begin{bmatrix} \sigma_1 - t_0 & 0 \\ \sigma_1 - t_0 & \int_{t_0}^{\sigma_1} \eta(\sigma_2)\,d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \int_{t_0}^{t} \begin{bmatrix} \sigma_1 & 0 \\ \sigma_1 & \eta(\sigma_1) \end{bmatrix} \begin{bmatrix} \sigma_1 - t_0 & 0 \\ \sigma_1 - t_0 & \int_{t_0}^{\sigma_1} \eta(\sigma_2)\,d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \int_{t_0}^{t} \begin{bmatrix} \sigma_1^2 - \sigma_1 t_0 & 0 \\ \sigma_1^2 - \sigma_1 t_0 + \eta(\sigma_1)(\sigma_1 - t_0) & \eta(\sigma_1) \int_{t_0}^{\sigma_1} \eta(\sigma_2)\,d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \int_{t_0}^{t} \begin{bmatrix} 1 & 0 \\ 1 & \eta(\sigma_1) \end{bmatrix} \begin{bmatrix} \sigma_1 - t_0 & 0 \\ \sigma_1 - t_0 & \int_{t_0}^{\sigma_1} \eta(\sigma_2) d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \int_{t_0}^{t} \begin{bmatrix} \sigma_1 - t_0 & 0 \\ \sigma_1 - t_0 + \eta(\sigma_1)(\sigma_1 - t_0) & \eta(\sigma_1) \int_{t_0}^{\sigma_1} \eta(\sigma_2) d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \begin{bmatrix} \frac{\sigma_1^2}{2} - \sigma_1 t_0 & 0 \\ \frac{\sigma_1^2}{2} - \sigma_1 t_0 + \int_{t_0}^{t} \eta(\sigma_1)(\sigma_1 - t_0) d\sigma_1 & \int_{t_0}^{t} \eta(\sigma_1) \int_{t_0}^{\sigma_1} \eta(\sigma_2) d\sigma_2 \, d\sigma_1 \end{bmatrix}_{t_0}^{t}$$

$\underline{\underline{3^{rd}\ \text{term}}} = \begin{bmatrix} \frac{(t-t_0)^2}{2} - (t-t_0) t_0 & 0 \\ \frac{(t-t_0)^2}{2} - (t-t_0) t_0 + \int_{t_0}^{t} \eta(\sigma_1)(\sigma_1 - t_0) d\sigma_1 & \int_{t_0}^{t} \eta(\sigma_1) \int_{t_0}^{\sigma_1} \eta(\sigma_2) d\sigma_2 \, d\sigma_1 \end{bmatrix}$

4^{th} term $= \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) \int_{t_0}^{\sigma_2} A(\sigma_3) \, d\sigma_3 \, d\sigma_2 \, d\sigma_1$

$$= \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) \int_{t_0}^{\sigma_2} \begin{bmatrix} 1 & 0 \\ 1 & \eta(\sigma_3) \end{bmatrix} d\sigma_3 \, d\sigma_2 \, d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) \begin{bmatrix} \sigma_3 & 0 \\ \sigma_3 & \int_{t_0}^{\sigma_2} \eta(\sigma_3) d\sigma_3 \end{bmatrix}_{t_0}^{\sigma_2} d\sigma_2 \, d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) \begin{bmatrix} \sigma_2 - t_0 & 0 \\ \sigma_2 - t_0 & \int_{t_0}^{\sigma_2} \eta(\sigma_3)\, d\sigma_3 \end{bmatrix} d\sigma_2\, d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} \begin{bmatrix} 1 & 0 \\ 1 & \eta(\sigma_2) \end{bmatrix} \begin{bmatrix} \sigma_2 - t_0 & 0 \\ \sigma_2 - t_0 & \int_{t_0}^{\sigma_2} \eta(\sigma_3)\, d\sigma_3 \end{bmatrix} d\sigma_2\, d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} \begin{bmatrix} \sigma_2 - t_0 & 0 \\ \sigma_2 - t_0 + \eta(\sigma_2)(\sigma_2 - t_0) & \eta(\sigma_2) \int_{t_0}^{\sigma_2} \eta(\sigma_3)\, d\sigma_3 \end{bmatrix} d\sigma_2\, d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \begin{bmatrix} \frac{\sigma_2^2}{2} - \sigma_2 t_0 & 0 \\ \frac{\sigma_2^2}{2} - \sigma_2 t_0 + \int_{t_0}^{\sigma_1} \eta(\sigma_2)(\sigma_2 - t_0)\, d\sigma_2 & \int_{t_0}^{\sigma_1} \eta(\sigma_2) \int_{t_0}^{\sigma_2} \eta(\sigma_3)\, d\sigma_3\, d\sigma_2 \end{bmatrix}_{t_0}^{\sigma_1} d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \begin{bmatrix} \frac{(\sigma_1 - t_0)^2}{2} - (\sigma_1 - t_0) t_0 & 0 \\ \frac{(\sigma_1 - t_0)^2}{2} - (\sigma_1 - t_0) t_0 + \int_{t_0}^{\sigma_1} \eta(\sigma_2)(\sigma_2 - t_0)\, d\sigma_2 & \int_{t_0}^{\sigma_1} \eta(\sigma_2) \int_{t_0}^{\sigma_2} \eta(\sigma_3)\, d\sigma_3\, d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \int_{t_0}^{t} A(\sigma_1) \begin{bmatrix} \frac{\sigma_1^2 - 2\sigma_1 t_0 + t_0^2}{2} - \sigma_1 t_0 - t_0^2 & 0 \\ \frac{\sigma_1^2 - 2\sigma_1 t_0 + t_0^2}{2} - \sigma_1 t_0 - t_0^2 + \int_{t_0}^{\sigma_1} \eta(\sigma_2)(\sigma_2 - t_0)\, d\sigma_2 & \int_{t_0}^{\sigma_1} \eta(\sigma_2) \int_{t_0}^{\sigma_2} \eta(\sigma_3)\, d\sigma_3\, d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \int_{t_0}^{t} \begin{bmatrix} 1 & 0 \\ 1 & \eta(\sigma_1) \end{bmatrix} \begin{bmatrix} \frac{\sigma_1^2 - 2\sigma_1 t_0 + t_0^2}{2} - \sigma_1 t_0 - t_0^2 & 0 \\ \frac{\sigma_1^2 - 2\sigma_1 t_0 + t_0^2}{2} - \sigma_1 t_0 - t_0^2 + \int_{t_0}^{\sigma_1} \eta(\sigma_2)(\sigma_2 - t_0)\, d\sigma_2 & \int_{t_0}^{\sigma_1} \eta(\sigma_2) \cdot \int_{t_0}^{\sigma_2} \eta(\sigma_3)\, d\sigma_3\, d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \int_{t_0}^{t} \begin{bmatrix} \frac{\sigma_1^2 - 2\sigma_1 t_0 + t_0^2}{2} - \sigma_1 t_0 - t_0^2 & 0 \\ \frac{\sigma_1^2 - 2\sigma_1 t_0 + t_0^2}{2} - \sigma_1 t_0 - t_0^2 + \eta(\sigma_1)(\sigma_1 t_0 - t_0^2) + \int_{t_0}^{\sigma_1} \eta(\sigma_2)(\sigma_2 - t_0)\, d\sigma_2 & \eta(\sigma_1) \int_{t_0}^{\sigma_1} \eta(\sigma_2) \cdot \int_{t_0}^{\sigma_2} \eta(\sigma_3)\, d\sigma_3\, d\sigma_2 \end{bmatrix} d\sigma_1$$

$$= \begin{bmatrix} \frac{\sigma_1^3}{6} - \frac{\sigma_1^2}{2} t_0 + \frac{\sigma_1 t_0^2}{2} - \frac{\sigma_1^2 t_0}{2} - \sigma_1 t_0^2 & 0 \\ \frac{\sigma_1^3}{6} - \frac{\sigma_1^2}{2} t_0 + \frac{\sigma_1 t_0^2}{2} - \frac{\sigma_1^2 t_0}{2} - \sigma_1 t_0^2 + \int_{t_0} \eta(\sigma_1)(\sigma_1 t_0 - t_0^2) + \int_{t_0}^{\sigma_1} \eta(\sigma_2)(\sigma_2 - t_0)\, d\sigma_2 & \int_{t_0}^{t} \eta(\sigma_1) \int_{t_0}^{\sigma_1} \eta(\sigma_2) \int_{t_0}^{\sigma_2} \eta(\sigma_3) \cdot d\sigma_3\, d\sigma_2\, d\sigma_1 \end{bmatrix}_{t_0}^{t}$$

$$4^{th}\ \text{term} = \begin{bmatrix} \dfrac{(t-t_0)^3}{6} - (t-t_0)^2 t_0 - \dfrac{(t-t_0)t_0^2}{2} \\ \dfrac{(t-t_0)^3}{6} - (t-t_0)^2 t_0 - \dfrac{(t-t_0)t_0^2}{2} + \int_{t_0}^{t} \eta(\sigma_1)(\sigma_1 t_0 - t_0^2) + \int_{t_0}^{\sigma_1} \eta(\sigma_2)(\sigma_2 - t_0)\, d\sigma_2 \\ \int_{t_0}^{t} \eta(\sigma_1) \int_{t_0}^{\sigma_1} \eta(\sigma_2) \int_{t_0}^{\sigma_2} \eta(\sigma_3) \cdot d\sigma_3 d\sigma_2 d\sigma_1 \end{bmatrix}$$

0

Calculate so on for all terms

∴ The final state transition matrix will be

$$\Phi(t, t_0) = I + \underset{\text{term}}{\text{second}} + \underset{\text{term}}{\text{third}} + \underset{\text{term}}{\text{fourth}} + \ldots\ldots$$