EMPM - 667. Section 0101

Problem Set 5 (Assignment 5)

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1. Let $A \in R^{n\times n}$ & $B \in R^{m\times m}$, Show that $X(t) = e^{At} X(0) e^{Bt}$ is the solution to the following eqn.

$$\dot{X}(t) = AX(t) + X(t)B.$$

Soln. The given eqn $X(t) = e^{At} X(0) e^{Bt}$ — (1)

By differentiating w.r.t t on both sides,

we get

$$\dot{X}(t) = X(0)\left[e^{At} \frac{d}{dt}(e^{Bt}) + e^{Bt} \frac{d}{dt}(e^{At}) \right]$$

$$= X(0)\left[e^{At} \cdot B \times e^{Bt} + e^{Bt} \cdot A \times e^{At} \right]$$

$$= \underbrace{e^{At} \cdot X(0) \cdot e^{Bt}}_{} [B + A]$$

= this is nothing but the main eqn (1)

$$= X(t)[B+A] = A(X(t)) + B(X(t))$$

$$\therefore \dot{X}(t) = AX(t) + X(t) \cdot B.$$

Hence proved.

5. Consider the following state-space representation

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

<u>Soln</u> (a) Standard form of Uncontrollable Systems using Similarity Transformations.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$n = 3$

$\therefore \quad \{B \quad AB \quad A^2B\}$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$\therefore [B \;\; AB \;\; A^2B] =$

Reduce to Row Echelon form to find rank.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is reduced to its echelon form & we can see that it has only two non-zero rows, hence rank $= n_2 = 2$.

$\therefore$ There are $\{v_1, \dots v_{n_2}\}$ vectors, linearly independent columns of $[B \;\; AB \;\; A^{n-1}B]$.

So the first non-zero row element act as pivot points and thus those are indicative of the columns which are linearly independent of the rest

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot points

linearly independent vectors.

$\therefore$ From the original $[B \;\; AB \;\; A^2B]$ we get the following linearly independent set of vectors.

$$\therefore \{V_1, V_2\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We know that the similarity transformation matrix S is a $n \times n$ matrix given by.

$$S = \left[V_1 \,\vdots\, V_2 \cdots \vdots\, V_{n_r} \,\vdots\, S_{n-n_r} \right]$$

Here S_{n-n_r} is a $n \times n-n_r$ matrix which contains $n-n_r$ linearly independent vectors chosen so that S is a non-singular matrix (Invertible).

$$\therefore S = \left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & S_{n-n_r} \\ 0 & 1 & \end{array}\right] \qquad \text{let } S_{n-n_r} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow S = \left[\begin{array}{cc|c} 0 & 1 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{array}\right] \quad R_3 \to R_3 - R_1$$

Reduce to row echelon form & do Gaussian-Elemination so that the set of linear eqns. have no solutions.

$$\rightarrow \left[\begin{array}{cc|c} 0 & 1 & a \\ 1 & 0 & b \\ 0 & 0 & c-a \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & c-a \end{array}\right]$$

$\therefore$ We just need to show that $c-a \neq 0$
$\Rightarrow a \neq c$
So if $a = 1$ & we can choose the other two to be 0, 0.
$\therefore a, b, c = 1, 0, 0$

$$\therefore S_{n-n_r} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

we can see that the three vectors are linearly independent of each other.

now we need to find the pair $(\hat{A}, \hat{B})$ which is the standard form of uncontrollable systems
$\therefore \hat{A} = S^{-1} A S$; $\hat{B} = S^{-1} B$.
$\Rightarrow$

$$S^{-1} = \frac{Adj(S)}{\det S}. \quad |S| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 0(0-0) - 1(0-1) + 0(0-1) = 0 + 1 + 0 = 1$$

$\therefore \det S = 1$

To find the Adjoint of a matrix first we find its co-factor matrices of each element.

$C_{11} = (-1)^{1+1}(0) = 0$ | $C_{21} = (-1)^{2+1}(-1) = 1$

$C_{12} = (-1)^{1+2}(0) = 0$ | $C_{22} = (-1)^{2+2}(0) = 0$

$C_{13} = (-1)^{1+3}(1) = 1$ | $C_{23} = (-1)^{2+3}(0) = 0$

$C_{31} = (-1)^{3+1}(0) = 0$; $C_{32} = (-1)^{3+2}(-1) = 1$

$C_{33} = (-1)^{3+3}(-1) = -1$

∴ The co-factor matrix is as follows

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$; Next we take the transpose of the co-factors matrix to get the adjoint of 'S'.

$$Adj.(S) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow S^{-1} = \frac{Adj\,S}{\det S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

(∵ since $\det S = 1$).

$$\hat{A} = S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+1 & 0+0+1 \\ 1+0+0 & 1+0-1 & 0+0-1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+1+0 & 0+0+1 & 0+0+0 \\ 0+0+0 & 1+0-1 & 1+0+0 \end{bmatrix}$$

$$\therefore \hat{A} = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{c:c} A_1 & A_{12} \\ \hdashline 0 & A_2 \end{array}\right]$$

Here $A_1 \in R^{n_2 \times n_2} \Rightarrow A_1 \in R^{2\times 2}$

$$\therefore A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

now lets find $\hat{B} \Rightarrow S^{-1}B$

$$\hat{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+1 \\ 0+0+0 & 1+0-1 \end{bmatrix}$$

$$= \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{c} B_1 \\ \hdashline 0 \end{array}\right]$$

Here $B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$B_1 \in R^{n_1 \times m} \Rightarrow B_1 \in R^{2\times 2}$

Here the standard form uncontrollable system is the pair $(\hat{A}, \hat{B})$

where $\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $\hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) Write the controllable part of the system as state eqn.

∴ From the above system the pair (A_1, B_1) is controllable part.

$\Rightarrow A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$; $B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

∴ If we write this in state space representation, we get

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

(6) Consider the system

$$\dot{x} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} x$$

Investigate the stability of this system using the the following Lyapunov eqn.

$$A^T P + PA = -Q \quad , \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Soln

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$$

Since we know that P must be a symmetric matrix, we assume P as follows.

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\therefore A^T P + PA \Rightarrow \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -3a-b & -3b-c \\ 2a-b & 2b-c \end{bmatrix} + \begin{bmatrix} -3a-b & 2a-b \\ -3b-c & 2b-c \end{bmatrix} = Q$$

$$\begin{bmatrix} -6a-2b & 2a-4b-c \\ 2a-4b-c & 4b-2c \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

lets equate the corresponding terms, we get:

$-6a-2b=-1$ — ①

$2a-4b-c=0$
$c=2a-4b$ — ②

$4b-2c=-1$ — ③

sub ② in ③, we get

$4b-2(2a-4b)=-1$
$4b-4a+8b=-1$

$12b-4a=-1$ — ④

④ $\rightarrow$ $\cancel{12b}-4a=-1$
$+6\times$ ① $\rightarrow$ $\cancel{-12b}-36a=-6$
$-40a=-7$

$a=\frac{7}{40}$

sub a in ①; we get
$-6\left(\frac{7}{40}\right)-2b=-1$

$$-2b = \frac{42}{40} - 1$$

$$2b = 1 - \frac{42}{40}$$

$$b = -\frac{1}{40}$$

sub b in 3, we get

$$\cancel{4}\left(-\frac{1}{\cancel{40}\,10}\right) - 2c = -1$$

$$-2c = \frac{1}{10} - 1$$

$$2c = 1 - \frac{1}{10}$$

$$c = \frac{9}{20}$$

∴ The P matrix will be as follows.

$$P = \begin{bmatrix} \frac{7}{40} & -\frac{1}{40} \\ -\frac{1}{40} & \frac{9}{20} \end{bmatrix}$$

Now, lets check if P is a ~~[illegible]~~ positive definite matrix also.

A 2x2 symmetric matrix is said to be a ~~pos~~ positive definite matrix if both its eigen values are positive

(i.e) $\lambda_1 > 0$ & $\lambda_2 > 0$.

$\therefore$ Assume a vector x such that

$Px = \lambda x$ where λ is any scalar

$\Rightarrow |P - \lambda I| = 0$

$$\begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{40} & \frac{9}{20} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left| \begin{bmatrix} \frac{7}{40} - \lambda & -\frac{1}{40} \\ -\frac{1}{40} & \frac{9}{20} - \lambda \end{bmatrix} \right|$$

$\Rightarrow$

$$\left(\left(\frac{7}{40} - \lambda\right)\left(\frac{9}{20} - \lambda\right)\right) - \left(\left(-\frac{1}{40}\right)\left(-\frac{1}{40}\right)\right) = 0$$

$$\left(\frac{7 - 40\lambda}{40}\right)\left(\frac{18 - 40\lambda}{40}\right) - \left(\frac{1}{40 \times 40}\right) = 0$$

$$(7-40\lambda)(18-40\lambda)-1=0$$

$$126-280\lambda-720\lambda+1600\lambda^2-1=0$$

$$1600\lambda^2-1000\lambda+125=0$$

$$64\lambda^2-40\lambda+5=0$$

The root of this eqn is

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Here $a=64$; $b=-40$; $c=5$

$$\therefore \lambda = \frac{40 \pm \sqrt{(-40)^2-4(64)(5)}}{2(64)}$$

$$= \frac{40 \pm \sqrt{1600-1280}}{128}$$

$$\lambda = \frac{40 \pm \sqrt{320}}{128} = \frac{40 \pm 8\sqrt{5}}{128}$$

$\therefore$ Both λ values are +ve (>0)

Hence we can say that P is a symmetric positive definite matrix.

Hence the given system is <u>STABLE</u>.

(14)

(4) Consider the state-space eqn

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad \text{—} (1)$$

$$J = \int_0^\infty \left(x^T Q x + u^2 \right) dt.$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} ; \quad \gamma > 0.$$

Design LQR controller that gives state feedback $u = Kx$

Soln If for a pair (A, B_K) to be stabilizable, we look for K that minimizes the following cost function

$$J(K, X(0)) = \int_0^\infty X^T(t) Q X(t) + U_K^T(t) R U_K(t) dt.$$

Since we know that $u = kx$ is a single (i.e) 1X1 matrix,

$$u^2 \Rightarrow u \times u \text{ or } u^T \times u$$

now compare this with the cost function as given in the question, we get.

$$J = \int_0^\infty (x^T Q x + u^2) dt$$

we arrive at the conclusion that the value of $R = 1$

the optimal soln is given by the following controller $K = -R^{-1} B_K^T P$.

with the input being $u(t) = Kx$
Here P is the symmetric positive definite soln

of the following stationary Riccati eqn

$$A^T P + PA - PB_K R^{-1} B_K^T P = -Q \quad (1)$$

Since we know that P must be symmetric lets consider it as the following matrix

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

lets sub all known values into eqn (1) we get

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(1)\begin{bmatrix} 0 & 1 \end{bmatrix} \times$$

$$\times \begin{bmatrix} a & b \\ b & c \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} + \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} - \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 & a \\ a & 2b \end{bmatrix} - \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ b & c \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 & a \\ a & 2b \end{bmatrix} - \begin{bmatrix} b^2 & bc \\ bc & c^2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -b^2 & a-bc \\ a-bc & 2b-c^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -\gamma \end{bmatrix}$$

comparing the corresponding elements, we get

$$b^2 = 1 \;; \quad a = bc \;; \quad 2b - c^2 = -\gamma$$

Here $b^2 = 1 \Rightarrow b = \pm 1$

lets first consider $b = -1$

so, we get

$$b = -1 \;;$$

$$a = (-1)c \Rightarrow a = -c$$

$$\cancel{2b}\; 2(-1) - c^2 = -\gamma$$

$$-c^2 = -\gamma + 2$$

$$c^2 = \gamma - 2$$

$$\therefore c = \sqrt{\gamma - 2}$$

$$\therefore a = -\sqrt{\gamma - 2}$$

Then $P = \begin{bmatrix} -\sqrt{\gamma-2} & -1 \\ -1 & \sqrt{\gamma-2} \end{bmatrix}$

lets check if the matrix P is positive definite.

The leading principle minor must be >0.

But $a = -\sqrt{\gamma - 2} < 0$

$\therefore$ The matrix isn't positive definite.

Lets now consider the second case, $b = 1$

$$\Rightarrow b = 1 \; ; \; a = c$$

$$2(1) - c^2 = -\gamma$$

$$-c^2 = -\gamma - 2$$

$$c^2 = \gamma + 2$$

$$c = \sqrt{\gamma + 2}$$

$$\therefore a = \sqrt{\gamma + 2}$$

$\therefore$ Now the P. matrix is given as

$$P = \begin{bmatrix} \sqrt{\gamma + 2} & 1 \\ 1 & \sqrt{\gamma + 2} \end{bmatrix}$$

Another test for a positive definite for a 2×2 matrix is that for $a > 0$, then $ac - b^2 > 0$ (determinants test)

$\therefore ac - b^2 \Rightarrow (\sqrt{\gamma+2})(\sqrt{\gamma+2}) - (1)^2$

$$\gamma + 2 - 1$$
$$= \gamma + 1$$

From the question we know that $\gamma > 0$

$\therefore \gamma + 1 > 0$ it is positive & satisfies the determinants test.

Hence matrix P is positive definite matrix. & thus proves that its a soln for the Riccati eqn.

Going back to the controller eqn & sub P in it, we get.

$$K = -R^T B_k^T P$$

$$= (-1) \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\gamma+2} & 1 \\ 1 & \sqrt{\gamma+2} \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & \sqrt{\gamma+2} \end{bmatrix}$$

$\therefore K = -\begin{bmatrix} 1 & \sqrt{\gamma+2} \end{bmatrix}$ is the controller for the provided system.

2. A (Euclidean) ball, $B(x_c, r)$ in R^n is given by

$B(x_c, r) = \{x \in R^n \mid \|x - x_c\| \leq r\}$ where $r > 0$ and $\|\cdot\|$ denots Euclidean norm. $\|x\| = \sqrt{x^T x}$.
Prove that $B(x_c, r)$ is a convex set.

Soln Any set, say C is a convex set, if the segment b/w any two points in C, lies in C i.e, if for any x_1 & $x_2 \in C$ & for any θ with $0 \leq \theta \leq 1$

$$\theta x_1 + (1-\theta) x_2 \in C.$$

lets take $y, z \in B(x_c, r)$ & $x, y \in R^n$

$\therefore \|y - x_c\| \leq r$ & $\|z - x_c\| \leq r$
(according to definition above)

Here we take $t \in [0, 1]$, so the point lies on the line segment joining the two points y & z. $\therefore P = yt + (1-t) z$.

$\Rightarrow \|yt + (1-t) z - x_c\|$
$= \|yt - x_c t + x_c t + (1-t) z - x_c\|$

Above ~~there~~ we added & subtracted $x_c t$ to get the following.

$$= \| t(y - x_c) + (1-t)(z - x_c) \|$$

$$\leq \| t(y - x_c \| + \| (1-t)(z - x_c) \|$$

(from the triangular inequalities property of Euclidean norms)

$$= |t| \, \| y - x_c \| + |(1-t)| \, \| z - x_c \|$$

(from the positive homogenity property of Euclidean norms).

$$\leq |t| r + |(1-t)| r$$

$$\cancel{|t|r} + r - \cancel{|t|r}$$

$$\leq r$$

$\therefore \| yt + (1-t)z - x_c \| \leq r.$

Hence $yt + (1-t)z \in B(x_c, r)$

Hence $B(x_c, r)$ is a convex set.

Hence proved.

(3) Let $X(t)$ be a zero-mean Gaussian random process with covariance function given by

$$C_X(t_1, t_2) = \sigma^2 e^{-|t_1 - t_2|}$$

Find the probability ~~distribution~~ density function of $X(t)$ & $X(t+s)$.

Soln Its given that $C_X(t_1, t_2) = \sigma^2 e^{-|t_1 - t_2|}$ — (1)

W.K.T. from the definition

$$\Sigma = \begin{bmatrix} C_X(t, t) & C_X(t, t+s) \\ C_X(t+s, t) & C_X(t+s, t+s) \end{bmatrix}$$

(covariance matrix)

From (1), we can write the elements of Σ as follows.

$$C_X(t, t) = \sigma^2 e^{-|t-t|} = \sigma^2$$
$$C_X(t, t+s) = \sigma^2 e^{-|t-(t+s)|} = \sigma^2 e^{-|s|}$$
$$C_X(t+s, t) = \sigma^2 e^{-|t+s-t|} = \sigma^2 e^{-|s|}$$
$$C_X(t+s, t+s) = \sigma^2 e^{-|t+s-t-s|} = \sigma^2$$

$$\therefore \Sigma = \begin{bmatrix} \sigma^2 & \sigma^2 e^{-|s|} \\ \sigma^2 e^{-|s|} & \sigma^2 \end{bmatrix}$$

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & e^{-|\Delta|} \\ e^{-|\Delta|} & 1 \end{bmatrix} \qquad \Sigma^{-1} = \frac{\text{Adj}\,\Sigma}{\det \Sigma}$$

$$|\Sigma| = \sigma^2 (1 - e^{-2|\Delta|})$$

~~$\therefore$ Σ^{-1} Adj Σ =~~

Adj of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

~~$\therefore$ Adj Σ = (σ^2 ($-\sigma^2 e^{-|\Delta|}$~~

$$\text{Adj}\,\Sigma = \sigma^2 \begin{bmatrix} 1 & -e^{-|\Delta|} \\ -e^{-|\Delta|} & 1 \end{bmatrix}$$

$$\therefore \Sigma^{-1} = \frac{\cancel{\sigma^2} \begin{bmatrix} 1 & -e^{-|\Delta|} \\ -e^{-|\Delta|} & 1 \end{bmatrix}}{\cancel{\sigma^2}(1 - e^{-2|\Delta|})}$$

$$\Sigma^{-1} = \frac{1}{(1 - e^{-2|\Delta|})} \begin{bmatrix} 1 & -e^{-|\Delta|} \\ -e^{-|\Delta|} & 1 \end{bmatrix}$$

The joint probability density function of $X(t)$ & $X(t+s)$ is given as

$F_{X(t) \cdot X(t+s)}$

$$= \left[\frac{1}{(2\pi)^{k/2} \left(\sigma^2 (1-e^{-2|s|})\right)^{1/2}} \times \exp\left(\frac{1}{2} x^T \left(\frac{1}{(1-e^{-2|s|})} \begin{bmatrix} 1 & -e^{-|s|} \\ -e^{-|s|} & 1 \end{bmatrix} \right) x \right) \right]$$

here $K=2$ & since its zero mean, $m=0$.

& $x = \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix}$ & $x^T = \begin{bmatrix} x(t) & x(t+s) \end{bmatrix}$