ENPM - 667 Section 0101

Problem Set 2 (Assignment-2)

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1. Evaluate the determinants

(a.)
$$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix}$$

Soln Let given matrix be denoted by A.
The determinant of any matrix A, order $N \times N$ is given by $\Rightarrow A_{11}C_{11} + A_{12}C_{12} + \dots A_{1n}C_{1n}$
where ~~C_{1n}~~ C_{1i} $(i \to 1-n)$ is the cofactor of the particular element.

$$C_{ij} = (-1)^{i+j} M_{ij}$$ where M_{ij} is the minor of the element

M_{ij} is obtained by det of $N-1 \times N-1$ matrix obtained by removing all elements of i^{th} row & j^{th} column

$\therefore$ For the matrix A, its determinant is calculated as

$$\det A = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$$

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 1 & -2 & 1 \\ -3 & 4 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 1((4\times1) - (-2\times-2)) - (-2)((-3\times1) - (-2\times1)) + 1((-3\times-2) - (4\times1))$$

$$= 1(4-4) + 2(-3+2) + 1(6-4)$$

$$= 0 - 2 + 2 = 0$$

$$\therefore C_{11} = 0$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 0 & -2 & 1 \\ 3 & 4 & -2 \\ -2 & -2 & 1 \end{vmatrix}$$

$$= (-1)\Big(0((4\times1) - (-2\times-2)) - (-2)((3\times1) - (-2\times-2))$$
$$+ 1((3\times-2) - (4\times-2))\Big)$$

$$= -1(0 + 2(3-4) + 1(-6+8))$$

$$= -1(-2+2) = 0$$

$$\therefore C_{12} = 0$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 0 & 1 & 1 \\ 3 & -3 & -2 \\ -2 & 1 & 1 \end{vmatrix}$$

$$= 1\Big(0((-3\times 1)-(-2\times 1)) - (1)\big((3\times 1)-(-2\times -2)\big) + 1((3\times 1)-(-3\times -2))\Big)$$

$$= (0 - 1(3-4) + 1(3-6))$$

$$= 1 - 3 = -2$$

$$\therefore C_{13} = -2$$

$$C_{14} = (-1)^{1+4} M_{14} = (-1)^5 \begin{vmatrix} 0 & 1 & -2 \\ 3 & -3 & 4 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= (-1)\Big(0((-3\times -2)-(4\times 1)) - (1)\big((3\times -2)-(4\times -2)\big) - 2((3\times 1)-(-3\times -2))\Big)$$

$$= -1(0 - 1(-6+8) - 2(3-6))$$

$$= -1(-2+6) = -1(4) = -4$$

$$\therefore C_{14} = -4$$

$\therefore \det A = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$

$= (1)(0) + (0)(0) + (2)(-2) + (3)(-4)$

$= 0 + 0 - 4 - 12 = -16$

$\boxed{\therefore \det A = -16}$

(1.b) Evaluate determinants

$$\begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}$$

Soln Let the given matrix be denoted as A.

The determinant of matrix A is

$\det A = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$

Find all the abovementioned co-factors.

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} b & b & b \\ e & e & b+e \\ b & b+f & b+d \end{vmatrix}$$

$$=1\Big(b\big((e\times(b+d))-((b+e)\times(b+f))\big)-b\big((e\times(b+d))$$
$$-(b\times(b+e))\big)+b\big((e\times(b+f))-(e\times b)\big)\Big)$$

$$=\Big(b\big((eb+ed)-(b^2+bf+eb+ef)\big)-b\big((eb+ed)-(b^2+be)\big)$$
$$+b\big((eb+ef)-(eb)\big)\Big)$$

$$=b(\cancel{eb}+ed-b^2-bf-\cancel{eb}-ef)-b(\cancel{eb}+ed-b^2$$
$$-\cancel{be})+b(\cancel{eb}+ef-\cancel{eb})$$

$$=\cancel{bed}-\cancel{b^3}-b^2f-\cancel{bef}-\cancel{bed}+\cancel{b^3}+\cancel{bef}$$

$$\therefore C_{11}=-b^2f$$

$$C_{12}=(-1)^{1+2}M_{12}=(-1)^3\begin{vmatrix}0 & b & b\\ c & e & b+e\\ a & b+f & b+d\end{vmatrix}$$

$$=-1\Big(0\big((e\times(b+d))-((b+e)(b+f))\big)-b\big((c\times(b+d))-(a\times(b+e))\big)$$
$$+b\big((c\times(b+f))-(a\times e)\big)\Big)$$

$$=-1\big(0-b(cb+cd-ab-ae)+b(cb+cf-ae)\big)$$

$$=-1(-\cancel{b^2c}-bcd+ab^2+\cancel{abe}+\cancel{b^2c}+bcf-\cancel{abe})$$

$$=-1(-bcd+ab^2+bcf)$$

$$\therefore C_{12}=bcd-ab^2-bcf$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & b & b+d \end{vmatrix}$$

$$= 1\Big(0\big((e\times(b+d)) - (b\times(b+e))\big) - b\big((c\times(b+d)) - (a\times(b+e))\big)$$
$$+ b\big((b\times c) - (a\times e)\big)\Big)$$

$$= (0 - b((bc+cd) - (ab+ae)) + b(bc - ae))$$

$$= -b(bc + cd - ab - ae) + b^2c - abe$$

$$= -\cancel{b^2c} - bcd + ab^2 + \cancel{abe} + \cancel{b^2c} - \cancel{abe}$$

$$\therefore C_{13} = ab^2 - bcd$$

$$C_{14} = (-1)^{1+4} M_{14} = (-1)^5 \begin{vmatrix} 0 & b & b \\ c & e & e \\ a & b & b+f \end{vmatrix}$$

$$= -1\Big(0\big((e\times(b+f)) - (b\times e)\big) - b\big((c\times(b+f)) - (a\times e)\big)$$
$$+ b\big((c\times b) - (a\times e)\big)\Big)$$

$$= -1(0 - b((bc+cf) - (ae)) + b((bc) - (ae)))$$

$$= -1(-b(bc + cf - ae) + b(bc - ae))$$

$$= -1(-\cancel{b^2c} - bcf + \cancel{abe} + \cancel{b^2c} - \cancel{abe}) = bcf$$

$$\therefore C_{14} = bcf$$

$$\therefore \det A = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$$

$$= (gc)(-b^2 f) + (ge)(bcd - ab^2 - bcf)$$

$$+ (a+ge)(ab^2 - bcd) + (gb+ge)(bcf)$$

$$= \cancel{-b^2cfg} + \cancel{bcdeg} - \cancel{ab^2ge} - \cancel{bcfge} + a^2b^2 - abcd$$

$$+ \cancel{ab^2ge} - \cancel{bcdge} + \cancel{b^2cfg} + \cancel{bcefg}$$

$$= a^2b^2 - abcd = \cancel{ab(ab - bd)}$$

$$= ab(ab - cd)$$

$$\boxed{\therefore \det A = ab(ab - cd)}$$

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2. Using the properties of determinants, solve with a minimum of calculations ~~for~~ the following equations for x:

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

Soln: let the above matrix be denoted by A.

W.K.T. $\det A = 0$

$$\Rightarrow A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14} = 0 \quad \text{—} (1)$$

Lets find the above mentioned co-factors.

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = (1) \begin{vmatrix} x & b & 1 \\ b & x & 1 \\ b & c & 1 \end{vmatrix}$

$= x(x-c) - b(b-b) + 1(bc - bx)$

$= x^2 - cx - 0 + bc - bx$

$\therefore C_{11} = x^2 - x(b+c) + bc$

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = (-1) \begin{vmatrix} a & b & 1 \\ a & x & 1 \\ a & c & 1 \end{vmatrix}$

$= -1(a(x-c) - b(a-a) + 1(ac - ax))$

$\therefore \cancel{C_{12}} = -1(\cancel{ax} - \cancel{ac} - 0 + \cancel{ac} - \cancel{ax}) = 0$

$\therefore C_{12} = 0$

$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13} = (1) \begin{vmatrix} a & x & 1 \\ a & b & 1 \\ a & b & 1 \end{vmatrix}$

$= a(b-b) - x(a-a) + 1(ab - ab) = 0$

$\therefore C_{13} = 0$

$C_{14} = (-1)^{1+4} M_{14} = (-1)^5 M_{14} = (-1) \begin{vmatrix} a & x & b \\ a & b & x \\ a & b & c \end{vmatrix}$

$= -1(a(bc - bx) - x(ac - ax) + b(ab - ab))$

$= -1(abc - abx - acx + ax^2 + \cancel{bc} 0)$

$\therefore C_{14} = -abc + abx + acx - ax^2$

Sub $C_{11}, C_{12}, C_{13}, C_{14}$ in eqn ①

$\Rightarrow (x)(x^2 - x(b+c) + bc) + (a)(0) + (a)(0)$
$+ 1(-abc + abx + acx - ax^2) = 0$

$\Rightarrow x^3 - x^2(b+c) + bcx - abc + abx + acx - ax^2 = 0$

$x^3 - x^2(a+b+c) + x(ab+bc+ca) - abc = 0$

rewriting the eqn we get

$x^3 - x^2(b+c) + xbc - ax^2 + (b+c)ax - abc = 0$

$x(x^2 - (b+c)x + bc) - a(x^2 - (b+c)x + bc) = 0$

$(x-a)(x^2 - (b+c)x + bc) = 0$

$x = a$ or $x^2 - (b+c)x + bc = 0$

rewriting this eqn, we get

$x^2 - bx - cx + bc = 0$

$x(x-b) - c(x-b) = 0$

$(x-b)(x-c) = 0$

$\therefore x = b$ or $x = c$

$$\boxed{\therefore\ x = a \text{ or } x = b \text{ or } x = c}$$

(3.) Given a matrix $A = \begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where γ_1 & γ_2 are non-zero complex numbers

(a.) Find the Eigen values & Eigen vectors of A

(b.) Find the conditions for the eigen values to be real.

(c.) Find the conditions for the eigenvectors to be orthogonal.

Soln

If γ_1 & γ_2 are non-zero complex numbers then we can say $\gamma_1 = \alpha_1 + i\beta_1$ & $\gamma_2 = \alpha_2 + i\beta_2$

∴ The matrix A will be

$$A = \begin{bmatrix} 1 & \alpha_1 + i\beta_1 & 0 \\ \alpha_2 + i\beta_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigen values & eigen vectors are defined as follows.

A linear operator, A transforms a vector x in any vector space to another vector Ax in the same vector space & it the resulting vector is a multiple of the same vector

we can say that $Ax = \lambda x$ where λ is a scalar

Then x is the eigen vector & λ is its corresponding eigen value

$Ax = \lambda x$ can be rewritten as

$$Ax = \lambda I x$$

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0$$

The above eqn has a non-trivial soln if

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 1 & \alpha_1 + i\beta_1 & 0 \\ \alpha_2 + i\beta_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & \alpha_1 + i\beta_1 & 0 \\ \alpha_2 + i\beta_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & \alpha_1+i\beta_1 & 0 \\ \alpha_2+i\beta_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left((1-\lambda)^2-0\right)-(\alpha_1+i\beta_1)\left((\alpha_2+i\beta_2)(1-\lambda)-0\right)$$
$$+0\left((0\times(\alpha_2+i\beta_2))-(0\times(1-\lambda))\right)=0$$

$$(1-\lambda)^3-(\alpha_1+i\beta_1)(\alpha_2+i\beta_2)(1-\lambda)=0$$

$$1-\lambda\left((1-\lambda)^2-(\alpha_1+i\beta_1)(\alpha_2+i\beta_2)\right)=0$$

$\lambda=1$ or $(1-\lambda)^2-(\alpha_1+i\beta_1)(\alpha_2+i\beta_2)=0$

$$(1-\lambda)^2=(\alpha_1+i\beta_1)(\alpha_2+i\beta_2)$$

$$1-\lambda=\pm\sqrt{(\alpha_1+i\beta_1)(\alpha_2+i\beta_2)}$$

$$\lambda=1\pm\sqrt{(\alpha_1+i\beta_1)(\alpha_2+i\beta_2)}$$

$\therefore \lambda_1 = 1$

$\lambda_2 = 1 + \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}$

$\lambda_3 = 1 - \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}$

For the first eigen value ~~λ~~ $\lambda_1 = 1$; the suitable eigen vector is x^1 with elements x_1, x_2, x_3 which satisfies $Ax^1 = 1x^1$

$$\Rightarrow \quad x_1 + (\alpha_1 + i\beta_1)x_2 + 0 \times x_3 = x_1,$$

$$(\alpha_2 + i\beta_2)x_1 + (1)x_2 + 0 \times x_3 = x_2,$$

$$0 \times x_1 + 0 \times x_2 + 1 \times x_3 = x_3.$$

For the above equations we can see & deduce that the values of $x_3 = K$ & that of x_1 & x_2 are zero

So the corresponding eigen vector for λ_1 is

$x^1 = [0, 0, K]^T$ normalizing it

we get $x^1 = [0, 0, 1]^T$

Solve for eigen value x^2 for λ_2 with elements x_1, x_2, x_3 is $Ax^2 = \left(1 + \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}\right) x^2$

Writing the eqns we get

$$x_1 + (\alpha_1 + i\beta_1) x_2 + 0 \times x_3 = x_1 + \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}\, x_1$$

$$(\alpha_2 + i\beta_2) x_1 + x_2 + 0 \times x_3 = x_2 + \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}\, x_2$$

$$0 x_1 + 0 x_2 + 1 x_3 = x_3 + \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}\, x_3$$

For ease of calculation lets use $\alpha_1 + i\beta_1 = \gamma_1$ & $\alpha_2 + i\beta_2 = \gamma_2$ as originally stated in the problem

$$x_1 + \gamma_1 x_2 + 0 \times x_3 = x_1 + \sqrt{\gamma_1 \gamma_2}\, x_1 \quad -(1)$$

$$\gamma_2 x_1 + x_2 + 0 x_3 = x_2 + \sqrt{\gamma_1 \gamma_2}\, x_2 \quad -(2)$$

$$0 x_1 + 0 x_2 + 1 x_3 = x_3 + \sqrt{\gamma_1 \gamma_2}\, x_3 \quad -(3)$$

Eqn (3) only holds true with (1) (2) & (3) if $x_3 = 0$

$\Rightarrow$ (1) − (2) we get

$$\cancel{x_1} + \gamma_1 x_2 - \gamma_2 x_1 - \cancel{x_2} = \cancel{x_1} + \sqrt{\gamma_1 \gamma_2}\, x_1 - \cancel{x_2} - \sqrt{\gamma_1 \gamma_2}\, x_2$$

$$\gamma_1 x_2 - \gamma_2 x_1 = \sqrt{\gamma_1\gamma_2}\,(x_1 - x_2)$$

$$\frac{\gamma_1 x_2}{\sqrt{\gamma_1\gamma_2}} - \frac{\gamma_2 x_1}{\sqrt{\gamma_1\gamma_2}} = x_1 - x_2$$

$$\sqrt{\frac{\gamma_1}{\gamma_2}}\, x_2 - \sqrt{\frac{\gamma_2}{\gamma_1}}\, x_1 = x_1 - x_2$$

lets assume $x_1 = \sqrt{\gamma_1}$ to eliminate x_1 term.

$$\sqrt{\frac{\gamma_1}{\gamma_2}}\, x_2 - \sqrt{\gamma_2} = \sqrt{\gamma_1} - x_2$$

$$\sqrt{\frac{\gamma_1}{\gamma_2}}\, x_2 + x_2 = \sqrt{\gamma_1} + \sqrt{\gamma_2}$$

$$x_2\left(\frac{\cancel{\sqrt{\gamma_1} + \sqrt{\gamma_2}}}{\sqrt{\gamma_2}}\right) = \cancel{\sqrt{\gamma_1} + \sqrt{\gamma_2}}$$

$$x_2 = \sqrt{\gamma_2}$$

$\therefore$ Eigen vector x^2 is given as $(\sqrt{\gamma_1}, \sqrt{\gamma_2}, 0)^T$

or $x^2 = \left(\sqrt{(\alpha_1 + i\beta_1)}, \sqrt{(\alpha_2 + i\beta_2)}, 0\right)^T$

Solve for eigen value x^3 for λ_3 with elements x_1, x_2, x_3 is $Ax^3 = \left(1 - \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}\right) x^3$

For ease of calculations replace $\alpha_1 + i\beta_1$ with γ_1 & $\alpha_2 + i\beta_2$ with γ_2 as originally stated in the problem.

Writing the eqns, we get

$$x_1 + r_1 x_2 + 0 \times x_3 = x_1 - \sqrt{r_1 r_2}\, x_1 \quad -(1)$$

$$r_2 x_1 + x_2 + 0 \times x_3 = x_2 - \sqrt{r_1 r_2}\, x_2 \quad -(2)$$

$$0 \times x_1 + 0 \times x_2 + 1 \times x_3 = x_3 - \sqrt{r_1 r_2}\, x_3 \quad -(3)$$

Similarly we can say that eqn (3) holds with (1), (2) & (3) if $x_3 = 0$

$\Rightarrow$ (1) $-$ (2) we get

$$\cancel{x_1} + r_1 x_2 - r_2 x_1 - \cancel{x_2} = \cancel{x_1} - \sqrt{r_1 r_2}\, x_1 - \cancel{x_2} + \sqrt{r_1 r_2}\, x_2$$

$$r_1 x_2 - r_2 x_1 = \sqrt{r_1 r_2}\,(x_2 - x_1)$$

$$\frac{r_1 x_2}{\sqrt{r_1 r_2}} - \frac{r_2 x_1}{\sqrt{r_1 r_2}} = (x_2 - x_1)$$

$$\sqrt{\frac{r_1}{r_2}}\, x_2 - \sqrt{\frac{r_2}{r_1}}\, x_1 = x_2 - x_1$$

assume $x_1 = \sqrt{r_1}$

$$\sqrt{\frac{r_1}{r_2}}\, x_2 - \sqrt{r_2} = x_2 - \sqrt{r_1}$$

$$\sqrt{\frac{r_1}{r_2}}\, x_2 - x_2 = \sqrt{r_2} - \sqrt{r_1}$$

$$x_2 \left(\frac{\sqrt{r_1} - \sqrt{r_2}}{\sqrt{r_2}} \right) = -(\sqrt{r_1} - \sqrt{r_2})$$

$x_2 = -\sqrt{r_2}$

$\therefore$ eigen vector x^3 is given as $(\sqrt{r_1}, -\sqrt{r_2}, 0)^T$

or $x^3 = \left(\sqrt{(\alpha_1 + i\beta_1)}, -\sqrt{(\alpha_2 + i\beta_2)}, 0\right)^T$

Hence the eigen values are

$\lambda_1 = 1$; $\lambda_2 = 1 + \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}$ &

$\lambda_3 = 1 - \sqrt{(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)}$

and their corresponding eigen vectors are

$x^1 = [0, 0, 1]^T$

$x^2 = \left[\sqrt{(\alpha_1 + i\beta_1)}, \sqrt{(\alpha_2 + i\beta_2)}, 0\right]^T$

$x^3 = \left[\sqrt{(\alpha_1 + i\beta_1)}, -\sqrt{(\alpha_2 + i\beta_2)}, 0\right]^T$

(b.) For the eigen values to be real

$\lambda_1 = 1$ is anyway real

for $\lambda_{2,3}$ to be real $(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2)$ must be real & positive which means

β_1 & $\beta_2 = 0$ & $\alpha_1 \alpha_2 \geq 0$

(c.) To find the condition for the vector to be mutually orthogonal, we need to check if the eigen vectors are orthogonal between themselves.

(i.) x^1 & x^2 is orthogonal if

$$(x^1)^\dagger x^2 = 0$$

$$(x^1)^\dagger = (x^{1*})^T$$

$$= \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^*\right]^T$$ ~~Since there are~~

$$= [0 \quad 0 \quad 1]$$

$(x^1)^\dagger x^2$

$$\cancel{[0 \quad 0 \quad 1] \begin{bmatrix} \sqrt{\alpha_1 + i\beta_1} \\ \sqrt{\alpha_2 + i\beta_2} \\ \sqrt{\alpha_3 + i\gamma} \end{bmatrix}}$$

$$[0 \quad 0 \quad 1] \begin{bmatrix} \sqrt{\alpha_1 + i\beta_1} \\ \sqrt{\alpha_2 + i\beta_2} \\ 0 \end{bmatrix} = 0.$$

$$0 \times \sqrt{\alpha_1 + i\beta_1} + 0 \times \sqrt{\alpha_2 + i\beta_2} + 1 \times 0 = 0$$

inconclusive condition

(ii) x^1 & x^3 is orthogonal if $(x^1)^\dagger x^3 = 0$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\alpha_1 + i\beta_1} \\ -\sqrt{\alpha_2 + i\beta_2} \\ 0 \end{bmatrix} = 0$$

$$0 \times \sqrt{\alpha_1 + i\beta_1} + 0 \times -\sqrt{\alpha_2 + i\beta_2} + 1 \times 0 = 0$$

inconclusive condition.

(iii) x^2 & x^3 is orthogonal if $(x^2)^\dagger x^3 = 0$

$$(x^2)^\dagger = (x^{2*})^T$$

$$= \begin{bmatrix} \sqrt{\alpha_1 - i\beta_1} \\ \sqrt{\alpha_2 - i\beta_2} \\ 0 \end{bmatrix}^T = \begin{bmatrix} \sqrt{\alpha_1 - i\beta_1} & \sqrt{\alpha_2 - i\beta_2} & 0 \end{bmatrix}$$

$\therefore (x^2)^\dagger \cdot x^3 = 0$

$$\begin{bmatrix} \sqrt{\alpha_1 - i\beta_1} & \sqrt{\alpha_2 - i\beta_2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\alpha_1 + i\beta_1} \\ -\sqrt{\alpha_2 + i\beta_2} \\ 0 \end{bmatrix} = 0$$

$$\sqrt{(\alpha_1 - i\beta_1)(\alpha_1 + i\beta_1)} - \sqrt{(\alpha_2 - i\beta_2)(\alpha_2 + i\beta_2)} = 0$$

we know that the product of a complex number & its complex conjugate is equal to the square of the numbers modulus; apply that to the above eqn, we get

$$\sqrt{(|\alpha_1 + i\beta_1|)^2} - \sqrt{(|\alpha_2 + i\beta_2|)^2} = 0$$

$$|\alpha_1 + i\beta_1| - |\alpha_2 + i\beta_2| = 0$$

$$|\alpha_1 + i\beta_1| = |\alpha_2 + i\beta_2|$$

$$|\gamma_1| = |\gamma_2|$$

④ Make an LU decomposition of the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix}.$$

Hence solve $Ax = b$ for (i) $b = [4 \ 1 \ 8 \ -5]^T$

(ii) $b = [-10 \ 0 \ -3 \ -24]^T$

Deduce that $\det(A) = -160$ & confirm this by direct calculation.

Soln

Decompose the matrix A into the product of a square lower triangular matrix & square upper triangular matrix.

$$A = LU$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}$$

$$A = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} & L_{21}U_{14}+U_{24} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} & L_{31}U_{14}+L_{32}U_{24}+U_{34} \\ L_{41}U_{11} & L_{41}U_{12}+L_{42}U_{22} & L_{41}U_{13}+L_{42}U_{23}+L_{43}U_{33} & L_{41}U_{14}+L_{42}U_{24}+L_{43}U_{34}+U_{44} \end{bmatrix}$$

Equate the above eqn with A & solve for the 16 unknown elements.

$U_{11} = 2$; $U_{12} = -3$; $U_{13} = 1$; $U_{14} = 3$

$L_{21} \cdot U_{11} = 1$	$L_{31} \cdot U_{11} = 5$	$L_{41} \cdot U_{11} = 3$
$L_{21} \times 2 = 1$	$L_{31} \times 2 = 5$	$L_{41} \times 2 = 3$
$L_{21} = \frac{1}{2}$	$L_{31} = \frac{5}{2}$	$L_{41} = \frac{3}{2}$

$$L_{21}U_{12}+U_{22}=4$$
$$\left(\frac{1}{2}\right)(-3)+U_{22}=4$$
$$U_{22}=4+\frac{3}{2}$$
$$U_{22}=\frac{11}{2}$$

$$L_{21}U_{13}+U_{23}=-3$$
$$\left(\frac{1}{2}\right)(1)+U_{23}=-3$$
$$U_{23}=-3-\frac{1}{2}$$
$$U_{23}=-\frac{7}{2}$$

$$L_{21}U_{14}+U_{24}=-3$$
$$\left(\frac{1}{2}\right)(3)+U_{24}=-3$$
$$U_{24}=-3-\frac{3}{2}$$
$$U_{24}=-\frac{9}{2}$$

$$L_{31}U_{12}+L_{32}U_{22}=3$$
$$\left(\frac{5}{2}\right)(-3)+L_{32}\left(\frac{11}{2}\right)=3$$
$$-15+11L_{32}=6$$
$$11L_{32}=15+6$$
$$L_{32}=\frac{21}{11}$$

$$L_{41}U_{12}+L_{42}U_{22}=-6$$
$$\left(\frac{3}{2}\right)(-3)+L_{42}\left(\frac{11}{2}\right)=-6$$
$$-9+11L_{42}=-12$$
$$11L_{42}=-3$$
$$L_{42}=-\frac{3}{11}$$

$$L_{31}U_{13}+L_{32}U_{23}+U_{33}=-1$$
$$\left(\frac{5}{2}\right)(1)+\left(\frac{21}{11}\right)\left(-\frac{7}{2}\right)+U_{33}=-1$$
$$U_{33}=\frac{1}{2}\left(\frac{21\times 7}{11}-5\right)-1$$
$$U_{33}=\frac{35}{11}$$

$$L_{41}U_{13}+L_{42}U_{23}+L_{43}U_{33}=-3$$
$$\left(\frac{3}{2}\right)(1)+\left(-\frac{3}{11}\right)\left(-\frac{7}{2}\right)+L_{43}\left(\frac{35}{11}\right)=-3$$
$$\frac{35}{11}L_{43}=-\frac{21}{11\times 2}-\frac{3}{2}-3$$

$$\frac{35}{\cancel{11}}L_{43}=-\frac{60}{\cancel{11}}$$
$$L_{43}=\frac{-60}{35}$$

$$L_{31}U_{14} + L_{32}U_{24} + U_{34} = -1$$

$$\left(\frac{5}{2}\right)(3) + \left(\frac{21}{11}\right)\left(-\frac{9}{2}\right) + U_{34} = -1$$

$$\frac{15}{2} - \frac{189}{22} + U_{34} = -1$$

$$U_{34} = \frac{189}{22} - \frac{15}{2} - 1$$

$$U_{34} = \frac{1}{11}$$

$$L_{41}U_{14} + L_{42}U_{24} + L_{43}U_{34} + U_{44} = 1$$

$$\left(\frac{3}{2}\right)(3) + \left(-\frac{3}{11}\right)\left(-\frac{9}{2}\right) + \left(\frac{-60}{35}\right)\left(\frac{1}{11}\right) + U_{44} = 1$$

$$\frac{9}{2} + \frac{27}{22} - \frac{60}{35 \times 11} + U_{44} = 1$$

$$U_{44} = 1 - \frac{9}{2} - \frac{27}{22} + \frac{60}{385} = -\frac{32}{7}$$

$$U_{44} = -\frac{32}{7}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{5}{2} & \frac{21}{11} & 1 & 0 \\ \frac{3}{2} & \frac{-3}{11} & \frac{-60}{35} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{bmatrix}$$

We need to solve $Ax = b$

$$LUx = b$$

(1) $b = [-4 \quad 1 \quad 8 \quad -5]^T = \begin{bmatrix} -4 \\ 1 \\ 8 \\ -5 \end{bmatrix}$

W.K.T. $LUx = b$ can be written as.

$$Ly = b \, ; \; Ux = y$$

Solve for y using $Ly = b$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ +\frac{1}{2} & 1 & 0 & 0 \\ \frac{5}{2} & \frac{21}{11} & 1 & 0 \\ \frac{3}{2} & \frac{-3}{11} & \frac{-60}{35} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 8 \\ -5 \end{bmatrix}$$

$y_1 = -4$

$+\frac{1}{2} y_1 + y_2 = 1 \Rightarrow +\frac{1}{2}(-4) + y_2 = 1 \Rightarrow y_2 = 3$

$\frac{5}{2} y_1 + \frac{21}{11} y_2 + y_3 = 8 \Rightarrow \frac{5}{2}(-4) + \frac{21}{11}(3) + y_3 = 8 \Rightarrow y_3 = \frac{135}{11}$

$\frac{3}{2} y_1 - \frac{3}{11} y_2 - \frac{60}{35} y_3 + y_4 = -5 \Rightarrow \frac{3}{2}(-4) - \frac{3}{11}(3) - \frac{60}{35}\left(\frac{135}{11}\right) + y_4 = -5$

$y_4 = \frac{160}{7}$

$\therefore y_1 = -4 \, ; \, y_2 = 3 \, ; \, y_3 = \frac{135}{11} \, ; \, y_4 = \frac{160}{7}$

sub y_1, y_2, y_3, y_4 into $Ux = y$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ \frac{135}{11} \\ \frac{160}{7} \end{bmatrix}$$

$-\frac{32}{\cancel{7}} x_4 = \frac{160}{\cancel{7}} \Rightarrow x_4 = \frac{-160}{32} = -5$

$\frac{35}{\cancel{11}} x_3 + \frac{x_4}{\cancel{11}} = \frac{135}{\cancel{11}}$

$35 x_3 - 5 = 135$

$35 x_3 = 140$

$x_3 = \frac{140}{35} = 4$

$\frac{11}{2} x_2 - \frac{7}{2} x_3 - \frac{9}{2} x_4 = 3$

$\frac{11}{2} x_2 - \frac{7}{2}(4) - \frac{9}{2}(-5) = 3$

$11 x_2 - 28 + 45 = 6$

$11 x_2 = 6 + 28 - 45$

$11 x_2 = -11$

$x_2 = -1$

$$2x_1 - 3x_2 + x_3 + 3x_4 = -4$$

$$2x_1 - 3(-1) + 4 + 3(-5) = -4$$

$$2x_1 + 3 + 4 - 15 = -4$$

$$2x_1 - 8 = -4$$

$$2x_1 = 4$$

$$x_1 = 2$$

$\therefore x_1 = 2;\ x_2 = -1;\ x_3 = 4;\ x_4 = -5$

$$\therefore x = \begin{bmatrix} 2 & -1 & 4 & -5 \end{bmatrix}^T$$

(ii) Solve for $Ax = b$ for $b = (-10 \quad 0 \quad -3 \quad -24)^T$

W.K.T $\rightarrow LUx = b \rightarrow$ rewritten as

$$Ly = b \ \& \ Ux = y$$

Solve for y using $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{5}{2} & \frac{4}{11} & 1 & 0 \\ \frac{3}{2} & \frac{-3}{11} & \frac{-60}{35} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ -3 \\ -24 \end{bmatrix}$$

$y_1 = -10$

$\frac{y_1}{2} + y_2 = 0$

$\frac{-\cancel{10}^{5}}{\cancel{2}} + y_2 = 0$

$y_2 = 5$

$\frac{5y_1}{2} + \frac{21y_2}{11} + y_3 = -3$

$\frac{5}{\cancel{2}}(-\cancel{10}^{5}) + \frac{21}{11}(5) + y_3 = -3$

$-25 + \frac{105}{11} + y_3 = -3$

$y_3 = 25 - \frac{105}{11} - 3$

$y_3 = \frac{137}{11}$

$\frac{3}{2}y_1 - \frac{3}{11}y_2 - \frac{60}{35}y_3 + y_4 = -24$

$\frac{3}{\cancel{2}}(-\cancel{10}^{5}) - \frac{3}{11}(5) - \frac{60}{35}\left(\frac{137}{11}\right) + y_4 = -24$

$-15 - \frac{15}{11} - \frac{1644}{77} + y_4 = -24$

$y_4 = 15 + \frac{15}{11} + \frac{1644}{77} - 24$

$y_4 = \frac{96}{7}$

$\therefore y_1 = -10;\ y_2 = 5;\ y_3 = \frac{137}{11};\ y_4 = \frac{96}{7}$

Sub y_1, y_2, y_3, y_4 in $Ux = y$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{35}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \\ \frac{137}{11} \\ \frac{96}{7} \end{bmatrix}$$

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$-\frac{32}{\cancel{7}}x_4 = \frac{96}{\cancel{7}}$

$x_4 = -\frac{96}{32}$

$x_4 = -3$

$\frac{35}{\cancel{11}}x_3 + \frac{1}{\cancel{11}}x_4 = \frac{137}{\cancel{11}}$

$35x_3 - 3 = 137$

$x_3 = \frac{140}{35}$

$x_3 = 4$

$\frac{11}{2}x_2 - \frac{7}{2}x_3 - \frac{9}{2}x_4 = 5$

$11x_2 - 7(4) - 9(-3) = 10$

$11x_2 = 28 - 27 + 10$

$x_2 = \frac{11}{11}$; $x_2 = 1$

$2x_1 - 3x_2 + x_3 + 3x_4 = -10$

$2x_1 - 3 + 4 - 9 = -10$

$2x_1 = -2$; $x_1 = -1$

$\therefore x_1 = -1$; $x_2 = 1$; $x_3 = 4$; $x_4 = -3$

$\therefore x = [-1 \quad 1 \quad 4 \quad -3]^T$

To find determinant of A

$|A| = |L||U|$

Since L & U are triangular matrices, their determinants are equal to the product of their diagonal elements

$$|L| = 1 \times 1 \times 1 \times 1 = 1$$

$$|U| = \cancel{2} \times \frac{\cancel{11}}{\cancel{2}} \times \frac{\cancel{35}^{5}}{\cancel{11}} \times -\frac{32}{\cancel{7}_{1}} = -160$$

$$\therefore |A| = 1 \times -160 = -160$$

Hence we've deduced that $|A| = -160$

<u>Direct calculation:-</u>

$$|A| = \begin{vmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{vmatrix}$$

$$\det(A) = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & -3 & -3 \\ 3 & -1 & -1 \\ -6 & -3 & 1 \end{vmatrix} = 4(-1-3) + 3(3-6) - 3(-9-6)$$

$$= 4(-4) + 3(-3) - 3(-15)$$

$$C_{11} = 20$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -3 & -3 \\ 5 & -1 & -1 \\ 3 & -3 & 1 \end{vmatrix} = (-1)\big(1(-1-3) + 3(5+3) - 3(-15+3)\big)$$

$$= (-1)(-4 + 24 + 36) = -56$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 4 & -3 \\ 5 & 3 & -1 \\ 3 & -6 & 1 \end{vmatrix} = 1(3-6) - 4(5+3) - 3(-30-9)$$

$$-3 - 32 + 117 = 82$$

$$C_{14} = (-1)^{1+4} \begin{vmatrix} 1 & 4 & -3 \\ 5 & 3 & -1 \\ 3 & -6 & -3 \end{vmatrix} = (-1)\big(1(-9-6) - 4(-15+3) - 3(-30-9)\big)$$

$$= (-1)(-15 + 48 + 117) = -150$$

$$\det A = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} + A_{14}C_{14}$$

$$(2)(20) + (-3)(-56) + (1)(82) + (3)(-150)$$

$$= -160$$

$$\therefore \; |A| = -160$$

Hence proved (confirmed) by direct calculation.

5. Consider the following 3x3 matrix

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

(i.) Is this matrix diagonalizable

(ii.) If yes, provide a diagonalization of this matrix

Soln

(i.) let $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$$A^T = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

We see that $A = A^T$

$\therefore$ The matrix is symmetric & hence normal

$\therefore$ The matrix is diagonalizable.

(ii.) Let x be denoted as the eigenvector of A with λ as its eigenvalue.

$$(A - \lambda I)\, x = 0$$

For non-trivial soln, $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$-(1+\lambda)\left(+(1+\lambda)^2-4\right)-2\left(-2(1+\lambda)-4\right)$$
$$+2\left(4+2(1+\lambda)\right)=0$$

$$-(\lambda+1)\left((1+\lambda)^2-4\right)+4(1+\lambda)+8+8+4(1+\lambda)=0$$

$$-(1+\lambda)^3+4(1+\lambda)+8(1+\lambda)+16=0$$

$$-(1+\lambda)^3+12(1+\lambda)+16=0$$

$$(\lambda+1)^3-12(\lambda+1)-16=0$$

$$\lambda^3+3\lambda^2+3\lambda+1-12\lambda-12-16=0$$

$$\lambda^3+3\lambda^2-9\lambda-27=0$$

$$\cancel{(\lambda+5)}\ (\lambda+3)(\lambda^2-9)=0$$

$$(\lambda+3)(\lambda+3)(\lambda-3)=0$$

$$\lambda_1=-3;\ \lambda_2=-3;\ \lambda_3=3$$

Arriving at an eigen vector for $\lambda_1 = -3 = \lambda_2$

$Ax^{1,2} = \lambda x^{1,2}$

$x^{1,2} \Rightarrow$

$$-x_1 + 2x_2 + 2x_3 = -3x_1$$
$$2x_1 - x_2 + 2x_3 = -3x_2$$
$$2x_1 + 2x_2 - x_3 = -3x_3$$

$$\Rightarrow \quad 2x_1 + 2x_2 + 2x_3 = 0$$
$$2x_1 + 2x_2 + 2x_3 = 0$$
$$2x_1 + 2x_2 + 2x_3 = 0$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 0$$

let $x_1 = -A$ & $x_2 = -B$

$\therefore x_3 = A + B$

$$\therefore x^{1,2} = \begin{bmatrix} -A \\ -B \\ A+B \end{bmatrix} = \begin{bmatrix} -A + 0B \\ 0A - B \\ A + B \end{bmatrix} = A\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + B\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore x^1 = [-1 \ 0 \ 1]^T$ & $x^2 = [0 \ -1 \ 1]^T$

normalizing them we get

$x^1 = \frac{1}{\sqrt{2}}[-1 \ 0 \ 1]^T$ & $x^2 = \frac{1}{\sqrt{2}}[0 \ -1 \ 1]^T$

Obtaining ation eigen vector for $\lambda_3 = 3$

$x^3 \Rightarrow$

$$-x_1 + 2x_2 + 2x_3 = 3x_1$$
$$2x_1 - x_2 + 2x_3 = 3x_2$$
$$2x_1 + 2x_2 - x_3 = 3x_3$$

$$\Rightarrow -4x_1 + 2x_2 + 2x_3 = 0$$
$$2x_1 - 4x_2 + 2x_3 = 0$$
$$2x_1 + 2x_2 - 4x_3 = 0$$

$$\Rightarrow 2x_1 - \cancel{2}x_2 - x_3 = 0$$
$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 + x_2 - 2x_3 = 0$$

we can assume that $x_1 = x_2 = x_3 = K$ (lets say)

$$x^3 = (K, K, K)^T$$

normalizing the vector we get

$$K^2 + K^2 + K^2 = 1$$

$$3K^2 = 1 \Rightarrow K = \frac{1}{\sqrt{3}}$$

$$x^3 = \left(\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}\right)^T$$

∴ The matrix S is constructed out of the eigen vectors.

$$S = (x_1^1 \quad x_2^2 \quad x_3^3)$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

We can see that the eigen vectors are linearly independent so it is diagonalizable

The diagonalized matrix is given by

$$A' = S^{-1} A S$$

Finding the inverse of S.

$$\det(S) \Rightarrow -\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}\right) - 0 + \frac{1}{\sqrt{3}}\left(0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{3}} + \frac{1}{2\sqrt{3}}$$

$$= \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \neq 0$$

Hence S is invertible

$$S^{-1} = \frac{\text{Adjoint}(S)}{\det S}$$

Adjoint $\cancel{A}$ = Replacing / rewriting the elements of a matrix with its co-factors & take its transpose.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{vmatrix} = \left(\frac{-1}{\sqrt{2}\sqrt{3}} - \frac{1}{\sqrt{2}\sqrt{3}}\right) = -\frac{\sqrt{2}}{\sqrt{3}}$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{vmatrix} = (-1)\left(0 - \frac{1}{\sqrt{2}\sqrt{3}}\right) = \frac{1}{\sqrt{2}\sqrt{3}}$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \left(0 + \frac{1}{2}\right) = \frac{1}{2}$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{vmatrix} = (-1)\left(0 - \frac{1}{\sqrt{2}\sqrt{3}}\right) = \frac{1}{\sqrt{2}\sqrt{3}}$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{vmatrix} = \left(-\frac{1}{\sqrt{2}\sqrt{3}} - \frac{1}{\sqrt{2}\sqrt{3}}\right) = \frac{-\sqrt{2}}{\sqrt{3}}$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = (-1)\left(-\frac{1}{2} - 0\right) = \frac{1}{2}$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{vmatrix} = \left(0 + \frac{1}{\sqrt{2}\sqrt{3}}\right) = \frac{1}{\sqrt{2}\sqrt{3}}$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{vmatrix} = (-1)\left(\frac{-1}{\sqrt{2}\sqrt{3}} - 0\right) = \frac{1}{\sqrt{2}\sqrt{3}}$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} \end{vmatrix} = \left(\frac{1}{2} - 0\right) = \frac{1}{2}$$

co-factors of S $\Rightarrow$

$$\begin{bmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} & \frac{1}{2} \end{bmatrix}$$

now transpose it to get the Adjoint of S..

$$\therefore \text{Adj } S = \begin{bmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} \\ \frac{1}{\sqrt{2}\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$S^{-1} = \frac{\text{Adj } S}{\det S} = \frac{1}{\frac{\sqrt{3}}{2}} \begin{bmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} \\ \frac{1}{\sqrt{2}\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\therefore A' = S^{-1} A S$$

$$= \frac{2}{\sqrt{3}} \begin{bmatrix} \frac{-\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} \\ \frac{1}{\sqrt{2}\sqrt{3}} & \frac{-\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{2}\sqrt{3}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$S^{-1} * A \Rightarrow$

$$\frac{2}{\sqrt{3}} \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{2}\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{2}\sqrt{3}} \\ \frac{-1}{\sqrt{2}\sqrt{3}} - \frac{2\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} - \frac{2\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{2}\sqrt{3}} \\ -\frac{1}{2} + 1 + 1 & 1 - \frac{1}{2} + 1 & 1 + 1 - \frac{1}{2} \end{bmatrix}$$

$$= \frac{2}{\sqrt{3}} \begin{bmatrix} \sqrt{2}\sqrt{3} & -\frac{\sqrt{3}}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} \\ \frac{-\sqrt{3}}{\sqrt{2}} & \sqrt{2}\sqrt{3} & -\frac{\sqrt{3}}{\sqrt{2}} \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

$S^{-1} * A * S \Rightarrow$

$$= \frac{2}{\sqrt{3}} \begin{bmatrix} \sqrt{2}\sqrt{3} & -\frac{\sqrt{3}}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} \\ -\frac{\sqrt{3}}{\sqrt{2}} & \sqrt{2}\sqrt{3} & -\frac{\sqrt{3}}{\sqrt{2}} \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\cancel{\frac{2}{\sqrt{3}}\left(\sqrt{2}\sqrt{3}\times\frac{-1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}\times 0 - \frac{\sqrt{3}}{\sqrt{2}}\times\frac{1}{\sqrt{2}} - 0 +\right.}$$

$$= \frac{2}{\sqrt{3}} \begin{bmatrix} -\sqrt{3}+0-\frac{\sqrt{3}}{2} & 0+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2} & \sqrt{2}-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{2}+0-\frac{\sqrt{3}}{2} & 0-\sqrt{3}-\frac{\sqrt{3}}{2} & \frac{-1}{\sqrt{2}}+\sqrt{2}-\frac{1}{\sqrt{2}} \\ \frac{-3}{2\sqrt{2}}+0+\frac{3}{2\sqrt{2}} & 0-\frac{3}{2\sqrt{2}}+\frac{3}{2\sqrt{2}} & \frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \frac{2}{\sqrt{3}} \begin{bmatrix} -\frac{3\sqrt{3}}{2} & 0 & 0 \\ 0 & -\frac{3\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{3\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3\sqrt{3}}{2} \times \frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{3\sqrt{3}}{2} \times \frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{3\sqrt{3}}{2} \times \frac{2}{\sqrt{3}} \end{bmatrix}$$

$$\therefore A' = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Hence the matrix A has been diagonalized.

⑥ Let $A \in R^{n\times n}$, $U \in R^{n\times k}$ & $V \in R^{n\times k}$ be given matrices. Suppose that A, $A + UV^T$ & $I + V^T A^{-1} U$ are non-singular matrices.

Prove that

$$(A + UV^T)^{-1} = A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}$$

Soln

Take LHR

$(A + UV^T)^{-1}$

since A is invertible

$$(A + UV^T)^{-1} = (A.I + A.A^{-1} UV^T)^{-1}$$

$$= (A(I + A^{-1} UV^T))^{-1}$$

since we know that $(A.B)^{-1} = B^{-1}.A^{-1}$

$$= (I + A^{-1} UV^T)^{-1} A^{-1}$$

Now, using the matrix inverse identity below

- $(I + P)^{-1} = I - (I + P)^{-1} P$

we get

$$= [I - (I + A^{-1} UV^T)^{-1} (A^{-1} UV^T)] A^{-1}$$

$$= A^{-1} - (I + A^{-1}UV^T)^{-1} A^{-1} U V^T A^{-1}$$

Now applying the next matrix inverse identity as below,

- $(I + PQ)^{-1} P = P (I + QP)^{-1}$

we get

$$= A^{-1} - (I + \underbrace{A^{-1}}_{P} \underbrace{UV^T}_{Q})^{-1} \underbrace{A^{-1}}_{P} U V^T A^{-1}$$

$$= A^{-1} - A^{-1} (I + \underbrace{U}_{P} \underbrace{V^T A^{-1}}_{Q})^{-1} \underbrace{U}_{P} V^T A^{-1}$$

now applying the above identity again

we get

$$= A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}$$

$$= \text{RHS}$$

Hence proved that

$$(A + UV^T)^{-1} = A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}$$