ENPM - 667. Section 0101

Problem Set 4 (Assignment 4)

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① Consider the following state equation

$$\dot{x}(t) = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} x(t) + e^{t/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

Determine if its controllable or not.

Soln The equation is a linear time varying system

B is dependent on time.

where $A = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ $B = e^{t/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For this to be controllable, the grammian of controllability should be invertible on $[t_0, t_f]$

$$\therefore W(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, \tau) B(\tau) B^T(\tau) \Phi^T(t_0, \tau) d\tau$$

should be invertible.

Since $A(t) = A$ is a constant matrix.

$$\Phi(t_0, \tau) = e^{A(t_0 - \tau)}$$

($\because$ From property of transition matrices).

From cayley-Hamilton theorem, we know that for matrix exponentials, there exists analytic functions such that (for n expansion)

$$e^{At} = \sum_{K=0}^{n-1} \alpha_K A^K \quad - \quad (*)$$

where α_K is determined from equations given by eigen values of A.

$$\Rightarrow e^{\lambda_i t} = \sum_{K=0}^{n-1} \alpha_K \lambda_i^K \quad - (1)$$ where λ_i eigen values of A.

Lets find eigen values of A.

$$A = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

Lets say x is an eigen vector of A where

$$Ax = \lambda x \quad (\lambda \text{ is a scalar})$$
$$Ax - \lambda I x = 0$$
$$(A - \lambda I)x = 0.$$
$$\therefore |A - \lambda I| = 0$$

$$A - \lambda I = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \frac{5}{12} - \lambda & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} - \lambda \end{bmatrix}$$

$$\therefore A - \lambda I = \frac{1}{12}\begin{bmatrix} 5 - 12\lambda & 1 \\ 1 & 5 - 12\lambda \end{bmatrix}$$

$$\therefore |A-\lambda I| = \frac{1}{12}\left((5-12\lambda)^2 - 1^2\right) = 0$$

$$(5-12\lambda)^2 = 1$$

$$5-12\lambda = 1 \quad \text{or} \quad 5-12\lambda = -1$$

$$\therefore \lambda_1 = \frac{1}{3} \quad \& \quad \lambda_2 = \frac{1}{2}$$

Sub in eqn (1) we get

$$e^{\frac{1}{3}(t)} = \sum_{k=0}^{n-1} \alpha_k \frac{1}{3}^k \text{ here sub } n=2 \ \& \ t = t_0 - \tau$$

$$\Rightarrow e^{\frac{1}{3}(t_0-\tau)} = \sum_{k=0}^{1} \alpha_k \frac{1}{3}^k$$

$$\Rightarrow e^{\frac{1}{3}(t_0-\tau)} = \alpha_0(1) + \alpha_1\left(\frac{1}{3}\right)$$

$$\Rightarrow \alpha_0 + \frac{1}{3}\alpha_1 = e^{\frac{1}{3}(t_0-\tau)} \quad \text{— (2)}$$

sub $\lambda = \frac{1}{2}$

$$e^{\frac{1}{2}(t_0-\tau)} = \sum_{k=0}^{1} \alpha_k \frac{1}{2}^k$$

$$\rightarrow e^{\frac{1}{2}(t_0-\tau)} = \alpha_0(1) + \alpha_1\left(\frac{1}{2}\right)$$

$$\Rightarrow \alpha_0 + \frac{1}{2}\alpha_1 = e^{\frac{1}{2}(t_0-\tau)} \quad \text{— (3)}$$

$\Rightarrow$ ③ − ② we get

$$\alpha_1\left(\frac{1}{2}-\frac{1}{3}\right) = e^{\frac{1}{2}(t_0-\tau)} - e^{\frac{1}{3}(t_0-\tau)}$$

$$\alpha_1 = 6\left(e^{\frac{1}{2}(t_0-\tau)} - e^{\frac{1}{3}(t_0-\tau)}\right)$$

Sub α_1 in eqn ③, we get

$$\Rightarrow \alpha_0 + \frac{1}{\cancel{2}}\left(\overset{3}{\cancel{6}}\left(e^{\frac{1}{2}(t_0-\tau)} - e^{\frac{1}{3}(t_0-\tau)}\right)\right) = e^{\frac{1}{2}(t_0-\tau)}$$

$$\Rightarrow \alpha_0 + 3e^{\frac{1}{2}(t_0-\tau)} - 3e^{\frac{1}{3}(t_0-\tau)} = e^{\frac{1}{2}(t_0-\tau)}$$

$$\alpha_0 = 3e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)}$$

Now from the main eqn (✗), we know

$$e^{At} = \sum_{K=0}^{n-1} \alpha_K A^K$$

$\therefore$

$$\Phi(t_0, \tau) = e^{A(t_0-\tau)}$$

$$\therefore e^{A(t_0-\tau)} = \sum_{K=0}^{n-1} \alpha_K A^K$$

expanding the RHS, we get (since $n=2$)

$$= \alpha_0 A^0 + \alpha_1 A^1$$

Sub α_0 & α_1 values obtained above

$$\Rightarrow A^0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad A^1 = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$e^{A(t_0-\tau)} = 3e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} +$$

$$\frac{1}{\cancel{12}\,2}\left(\cancel{12}^{1}\left(e^{\frac{1}{2}(t_0-\tau)} - e^{\frac{1}{3}(t_0-\tau)}\right)\right) \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)} & 3e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)} \\ 3e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)} & 3e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)} \end{bmatrix} +$$

$$\begin{bmatrix} \frac{5}{2}\left(e^{\frac{1}{2}(t_0-\tau)} - e^{\frac{1}{3}(t_0-\tau)}\right) & \frac{1}{2}\left(e^{\frac{1}{2}(t_0-\tau)} - e^{\frac{1}{3}(t_0-\tau)}\right) \\ \frac{1}{2}\left(e^{\frac{1}{2}(t_0-\tau)} - e^{\frac{1}{3}(t_0-\tau)}\right) & \frac{5}{2}\left(e^{\frac{1}{2}(t_0-\tau)} - e^{\frac{1}{3}(t_0-\tau)}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left(e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)}\right) & \frac{5}{2}e^{\frac{1}{3}(t_0-\tau)} - \frac{3}{2}e^{\frac{1}{2}(t_0-\tau)} \\ \frac{5}{2}e^{\frac{1}{3}(t_0-\tau)} - \frac{3}{2}e^{\frac{1}{2}(t_0-\tau)} & \frac{1}{2}\left(e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)}\right) \end{bmatrix}$$

$$\therefore e^{A(t_0-\tau)} = \Phi(t_0,\tau) = \frac{1}{2}\begin{bmatrix} e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} \\ 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} & e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} \end{bmatrix}$$

Sub $\Phi(t_0, \tau)$ into the grammian eqn, we get.

$$\Rightarrow W(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, \tau) B(\tau) B^T(\tau) \Phi^T(t_0, \tau)\, d\tau$$

$$\Rightarrow \int_{t_0}^{t_f} \frac{1}{2}\begin{bmatrix} e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} \\ 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} & e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} \end{bmatrix} \times$$

$$\begin{bmatrix} e^{\tau/2} \\ e^{\tau/2} \end{bmatrix} \times \begin{bmatrix} e^{\tau/2} & e^{\tau/2} \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} \\ 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} & e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} \end{bmatrix} d\tau$$

$$= \frac{1}{4}\int_{t_0}^{t_f} \begin{bmatrix} e^{\tau/2}\left(e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} + 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)}\right) \\ e^{\tau/2}\left(5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} + e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)}\right) \end{bmatrix} \times \begin{bmatrix} e^{\tau/2} & e^{\tau/2} \end{bmatrix}$$

$$\times \begin{bmatrix} e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} \\ 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} & e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} \end{bmatrix} d\tau$$

$$= \frac{1}{4}\int_{t_0}^{t_f} \begin{bmatrix} e^{\tau/2}\left(6e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)}\right) \\ e^{\tau/2}\left(6e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)}\right) \end{bmatrix} \begin{bmatrix} e^{\tau/2} & e^{\tau/2} \end{bmatrix}$$

$$\begin{bmatrix} e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} \\ 5e^{\frac{1}{3}(t_0-\tau)} - 3e^{\frac{1}{2}(t_0-\tau)} & e^{\frac{1}{3}(t_0-\tau)} + e^{\frac{1}{2}(t_0-\tau)} \end{bmatrix} d\tau$$

taking terms common, we get

$$= \frac{1}{4}\int_{t_0}^{t_f} e^{\tau^2/4}\left(6e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)}\right)\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix} e^{\frac{1}{3}(t_0-\tau)}+e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)}-3e^{\frac{1}{2}(t_0-\tau)} \\ 5e^{\frac{1}{3}(t_0-\tau)}-3e^{\frac{1}{2}(t_0-\tau)} & e^{\frac{1}{3}(t_0-\tau)}+e^{\frac{1}{2}(t_0-\tau)}\end{bmatrix} d\tau$$

$$= \frac{1}{4}\int_{t_0}^{t_f} e^{\tau^2/4}\left(6e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)}\right)\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix} e^{\frac{1}{3}(t_0-\tau)}+e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)}-3e^{\frac{1}{2}(t_0-\tau)} \\ 5e^{\frac{1}{3}(t_0-\tau)}-3e^{\frac{1}{2}(t_0-\tau)} & e^{\frac{1}{3}(t_0-\tau)}+e^{\frac{1}{2}(t_0-\tau)}\end{bmatrix} d\tau$$

$$= \frac{1}{4}\int_{t_0}^{t_f} e^{\tau^2/4}\left(6e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)}\right)\begin{bmatrix} e^{\frac{1}{3}(t_0-\tau)}+e^{\frac{1}{2}(t_0-\tau)}+5e^{\frac{1}{3}(t_0-\tau)}-3e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)}-3e^{\frac{1}{2}(t_0-\tau)}+e^{\frac{1}{3}(t_0-\tau)}+e^{\frac{1}{2}(t_0-\tau)} \\ e^{\frac{1}{3}(t_0-\tau)}+e^{\frac{1}{2}(t_0-\tau)}+5e^{\frac{1}{3}(t_0-\tau)}-3e^{\frac{1}{2}(t_0-\tau)} & 5e^{\frac{1}{3}(t_0-\tau)}-3e^{\frac{1}{2}(t_0-\tau)}+e^{\frac{1}{3}(t_0-\tau)}+e^{\frac{1}{2}(t_0-\tau)}\end{bmatrix} d\tau$$

$$\therefore W(t_0,t_f) = \frac{1}{4}\int_{t_0}^{t_f} e^{\tau/4}\left(6e^{\frac{1}{3}(t_0-\tau)} - 2e^{\frac{1}{2}(t_0-\tau)}\right)^2\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} d\tau$$

Here we notice that the matrix has all same elements, if we integrate under the range of $[t_0, t_f]$ also, we will get a matrix with same elements & then ~~of~~ the determinant of such a matrix is 0.

$\therefore$ The matrix $W(t_0, t_f)$ is not invertible

Hence the system is NOT CONTROLLABLE.

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(4) Given the linear second-order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u.$$

find a linear state feedback control, $u \Rightarrow$
$u = k_1 x_1 + k_2 x_2$ so that closed loop system has poles at $s = -2, 2$

Soln Given $u = k_1 x_1 + k_2 x_2$

Consider the term $\begin{bmatrix} 1 \\ -2 \end{bmatrix} u$

$$\Rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} k_1 x_1 + k_2 x_2 = \begin{bmatrix} k_1 x_1 + k_2 x_2 \\ -2(k_1 x_1 + k_2 x_2) \end{bmatrix}$$

plugging it back into the closed loop, we get.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_1 x_1 + k_2 x_2 \\ -2(k_1 x_1 + k_2 x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ -2k_1 & -2k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \cancel{\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}}$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1+K_1 & -3+K_2 \\ 1-2K_1 & -2-2K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We have the poles of the system, $s = -2, 2$

Which is nothing but the eigenvalues of the A_c matrix

$\therefore$ we know that $|A_c - \lambda I| = 0$

Sub $\lambda_1 = -2$; we get

$$\begin{vmatrix} 1+K_1+2 & -3+K_2 \\ 1-2K_1 & \cancel{-2}-2K_2\cancel{+2} \end{vmatrix} = 0$$

$$\begin{vmatrix} K_1+3 & K_2-3 \\ 1-2K_1 & -2K_2 \end{vmatrix} = 0$$

$$(k_1+3)(-2k_2) - (k_2-3)(1-2k_1) = 0$$

$$\Rightarrow -2k_1k_2 - 6k_2 - (k_2 - 2k_1k_2 - 3 + 6k_1) = 0$$

$$\Rightarrow \cancel{-2k_1k_2} - 6k_2 - k_2 + \cancel{2k_1k_2} + 3 - 6k_1 = 0$$

$$-7k_2 + 3 - 6k_1 = 0 \quad - 1$$

Sub $\lambda_2 = 2$; we get

$$\begin{vmatrix} 1+k_1-2 & -3+k_2 \\ 1-2k_1 & -2-2k_2-2 \end{vmatrix} = 0$$

$$\begin{vmatrix} k_1-1 & k_2-3 \\ 1-2k_1 & -4-2k_2 \end{vmatrix} = 0$$

$$-(k_1-1)(4+2k_2) - (k_2-3)(1-2k_1) = 0$$

$$-(4k_1 + 2k_1k_2 - 4 - 2k_2) - (k_2 - 2k_1k_2 - 3 + 6k_1) = 0$$

$$-4k_1 \cancel{-2k_1k_2} + 4 + 2k_2 - k_2 + \cancel{2k_1k_2} + 3 - 6k_1 = 0$$

$$-10k_1 + k_2 + 7 = 0 \quad - 2$$

$$\therefore \quad k_2 = 10k_1 - 7$$

Sub in 1

$$-7(10k_1 - 7) + 3 - 6k_1 = 0$$

$$-70k_1 + 49 + 3 - 6k_1 = 0$$

$$-76k_1 + 52 = 0$$

$$\cancel{-}76k_1 = +52 \Rightarrow k_1 = \frac{52}{76}$$

$$\therefore \quad k_1 = \frac{13}{19}$$

Sub K_1 in (2)

$$K_2 = 10\left(\frac{13}{19}\right) - 7$$

$$K_2 = -\frac{3}{19}$$

$\therefore$ The stateequation becomes.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1+\frac{13}{19} & -3+\left(\frac{-3}{19}\right) \\ 1-2\left(\frac{13}{19}\right) & -2-2\left(\frac{-3}{19}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{32}{19} & \frac{-60}{19} \\ \frac{-7}{19} & \frac{-32}{19} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\therefore$ The control law becomes.

$$u = \frac{13}{19} x_1 - \frac{3}{19} x_2$$

(5) Repeat the above problem if possible for the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u.$$

Soln W.K.T. $u = K_1 x_1 + K_2 x_2$

consider $\begin{bmatrix}0\\ 1\end{bmatrix} u = \begin{bmatrix}0\\ 1\end{bmatrix} K_1 x_1 + K_2 x_2$

$$= \begin{bmatrix}0\\ K_1 x_1 + K_2 x_2\end{bmatrix}$$

plug it back to the eqn, we get

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ K_1 & K_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\therefore \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ K_1 & K_2+2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

We have the poles of the system, $s = -2, 2$ which is nothing but the eigen values of the A_c - matrix

W.K.T $\Rightarrow |A_c - \lambda I| = 0$

Sub $\lambda_1 = -2$; we get

$$\begin{vmatrix} -1+2 & 0 \\ K_1 & K_2+2+2 \end{vmatrix} = 0$$

$$(K_2+4) - 0 = 0$$
$$K_2 = -4$$

Sub $\lambda_2 = 2$; we get

$$\begin{vmatrix} -1-2 & 0 \\ K_1 & K_2+2-2 \end{vmatrix} = 0 \quad \Rightarrow K_2 = 0$$

Here we totally lose the K_1 term.
& we get two values for K_2 which is not possible.
Hence the poles cannot be placed at $s=-2$

Lets check for controllability.

$n = 2$

rank $[B_K \quad AB_K]$

$\rightarrow$ rank $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 \times -1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$ reduced to echelon form,

Here we can see that it has two pivot points

$\therefore$ rank $= 2 = n$

Hence the

We use the P-B-H test to check for stabilizability

rank $[(\lambda I - A) \vdots B] = n$ for all λ, $\mathrm{Re}\{\lambda\} \geq 0$; then its stabilizable.

Lets find the poles for the A matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad |A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow -(\lambda+1)(2-\lambda) = 0$$

$$\cancel{-(2\lambda - \lambda^2 + 2 - \lambda) = 0}$$

$$\cancel{\lambda^2 - \lambda - 2 = 0}$$

$\lambda = -1$ or $\lambda = 2$

$\cancel{\lambda^2 - 2\lambda + \lambda - 2 = 0}$

$\cancel{\lambda\lambda}$

we get $\lambda = -1$ or $\lambda = 2$

Lets check for stabilizability first.

$Re\{\lambda\} \geq 0 \Rightarrow \lambda = 2$

rank $[(\lambda I - A) \vdots B]$

$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \to \frac{R_1}{3}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

no. of non. zero rows $= 2$

$\therefore$ rank $= 2 = n$

$\therefore$ It is stabilizable.

Lets check if its controllable.

rank $[(\lambda I - A) \vdots B] = n$ for all λ then it is controllable.

$\because \lambda = -1, 2$ we already checked the condition for $\lambda = 2$

lets check for $\lambda = -1$

rank $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

There is a non-zero row, therefore rank = 1

rank = 1 $\neq n$

$\therefore$ The system is not controllable.

Hence we can show that the rank is not equal to n (2) for all values of λ hence it is not controllable.

Therefore we show that the given system is stabilizable but not controllable.

(6.) Repeat for the below system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Soln W.K.T, $u = K_1 x_1 + K_2 x_2$

Consider the term $\begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} K_1 x_1 + K_2 x_2$

$= \begin{bmatrix} 0 \\ K_1 x_1 + K_2 x_2 \end{bmatrix}$ plug it back to the eqn, we get.

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u$$

$$\Rightarrow\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\K_1x_1+K_2x_2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0&0\\K_1&K_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}1&0\\K_1&K_2+2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

we have the poles of the system, $s=-2,2$
which is the eigen values of A_C matrix

W.K.T. $\Rightarrow$ $|A_C-\lambda I|=0$

sub $\lambda_1=-2$; we get.

$$\begin{vmatrix}1+2&0\\K_1&K_2+2+2\end{vmatrix}=0$$

$$3(K_2+4)=0;\quad K_2=-4$$

sub $\lambda_1=2$; we get.

$$\begin{vmatrix}1-2&0\\K_1&K_2+2-2\end{vmatrix}=0$$

$$\Rightarrow K_2=0$$

Here too, we totally lose the K_1 term & we get

two values for k_2 which is not possible.
Hence the poles cannot be placed at $s=-2$.

So lets find the poles of the system. which are nothing but eigen values of A matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \therefore \quad |A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0$$

$$\lambda = 1 \text{ or } \lambda = 2$$

Since all the eigen values are +ve; we can check for both stability & controllability for all λ values.

$\lambda = 1$

$$\text{rank}\,[(\lambda I - A) \,\vdots\, B]$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{no. of non-zero rows} = 1 \\ \therefore \text{rank} = 1 \neq n \end{array}$$

$\therefore$ It is not controllable or stabilizable (since it hasn't satisfied the condition of one λ value).

8. Find the solution for the following linear-state eqn.

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix} x(t) \; ; \quad x(t_0) = x_0$$

Soln Since it is a linear-time invariant system, the solution to this is

$$\theta(t) = e^{A(t-t_0)} \times x(t_0)$$

Here I assume time starts at 0 : $t_0 = 0$.

$$e^{At} \Rightarrow e^{At} \times x(t_0)$$

find e^{At} we the Inverse laplace transform to find the eigen decomposition e^{At}

$$e^{At} = \mathcal{L}^{-1}\{(SI-A)^{-1}\}$$

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+4 & -4 \\ 0 & 1 & s \end{bmatrix}$$

now lets find $(sI-A)^{-1}$

$$\det(sI-A) = s+1\,(s^2+4s+4) \neq 0$$

Co-factors.

$C_{11} = (-1)^{1+1} \times s^2+4s+4 = s^2+4s+4$

$C_{12} = (-1)^{1+2} \times 0 = 0$

$C_{13} = (-1)^{1+3} \times 0 = 0$

$C_{21} = (-1)^{2+1} \times \cancel{0(s^2+s-0)} = 0$

$C_{22} = (-1)^{2+2} \times (s^2+s-0) = s^2+s$

$C_{23} = (-1)^{2+3} \times (s+1) = -s-1$

$C_{31} = (-1)^{3+1} \times 0 = 0$

$C_{32} = (-1)^{3+2} \times (-4s-4) = 4s+4$

$C_{33} = (-1)^{3+3} \times s^2+5s+4 = s^2+5s+4$

$\therefore$ The co-factor matrix is as follows.

$$\begin{bmatrix} s^2+4s+4 & 0 & 0 \\ 0 & s^2+s & -s-1 \\ 0 & 4s+4 & s^2+5s+4 \end{bmatrix}$$

For the adjoint, take the transpose of above matrix

$$\therefore \operatorname{Adj}(sI-A) = \begin{bmatrix} s^2+4s+4 & 0 & 0 \\ 0 & s^2+s & 4s+4 \\ 0 & -s-1 & s^2+5s+4 \end{bmatrix}$$

$$\therefore (sI-A)^{-1} = \frac{1}{(s+1)(s^2+4s+4)} \begin{bmatrix} s^2+4s+4 & 0 & 0 \\ 0 & s(s+1) & 4(s+1) \\ 0 & -1(s+1) & s^2+5s+4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s}{s^2+4s+4} & \frac{4}{s^2+4s+4} \\ 0 & \frac{-1}{(s^2+4s+4)} & \frac{s^2+5s+4}{(s+1)(s^2+4s+4)} \end{bmatrix}$$

now, by using laws of partial fractions, for the ~~abov~~ elements of the matrix we can write the following matrix.

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$$\begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} - \frac{2}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & \frac{-1}{(s+2)^2} & \frac{1}{s+2} + \frac{2}{(s+2)^2} \end{bmatrix} = (sI - A)^{-1}$$

now we perform the inverse laplace tranform on the above matrix.

$$\mathcal{L}^{-1}\{(sI-A)^{-1}\} = e^{At}$$

$$= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} - e^{-2t} \times 2t & e^{-2t} \times 4t \\ 0 & -e^{-2t} \times t & e^{-2t} + e^{-2t} \times 2t \end{bmatrix}$$

Solution is $e^{At} \times x(t_0)$

$$\Rightarrow \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} - e^{-2t} \times 2t & e^{-2t} \times 4t \\ 0 & -e^{-2t} \times t & e^{-2t} + e^{-2t} \times 2t \end{bmatrix} x_0$$

(7) Show that if $y(t)$ is the output of a linear time-invariant system corresponding to input $u(t)$ then the output of the system corresponding to the input $\dot{u}(t)$ is given by $\dot{y}(t)$. Assume that the initial state is zero, $x(t_0) = 0$.

Soln According to the Convolution Theorem, the o/p of the Linear Time Invariant System can be given as follows.

$$y(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$

let it be eqn (1)

$y(t)$ = output & $h(t)$ is the impulse response
$u(t)$ = input

Lets take the o/p of system at t_0 ; $\dot{u}(t) = y_1(t)$

$$\therefore y_1(t) = \int_{-\infty}^{\infty} h(\tau) \frac{d}{dt} u(t-\tau)\, d\tau$$

$$= \frac{d}{dt} \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$

$$= \frac{d}{dt}(y(t)) \quad \text{(as per eqn (1))}$$

$= \dot{y}(t)$

$\therefore$ The output for $\dot{u}(t)$ is $= \dot{y}(t)$

Hence we obtain that for any LTI system corresponding to input $u(t)$ the output of the system for the input $\dot{u}(t)$ is equal to $\dot{y}(t)$.

Hence proved.