Assignment 1 – Problem Set 1

ENPM 667-0101

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1. Find values of α & β that make the eqn

$$dF_{(x,y)} = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right)dx + (xy^{\beta}+1)\,dy$$ an exact differential. For those values solve $dF(x,y) = 0$

Soln

If the given eqn needs to be an exact differential

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

Rewrite the eqn as $A(x,y)\,dx + B(x,y)\,dy$ — (1)

$$\frac{\partial A}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right)$$

$$= 0 - \frac{\alpha}{y^2} = -\frac{\alpha}{y^2}$$

$$\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}(xy^{\beta}+1) = y^{\beta}$$

Now $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \Rightarrow -\frac{\alpha}{y^2} = y^{\beta} \Rightarrow -\alpha = y^{\beta+2}$

Comparing the LHS & RHS of it, we can deduce that

$$-\alpha = 1 \Rightarrow \alpha = -1$$

$$\beta + 2 = 0 \Rightarrow \beta = -2$$

So now sub. α & β in eqn (1) $\Rightarrow \left(\frac{1}{x^2+2} - \frac{1}{y}\right)dx + (xy^{-2}+1)\,dy$

Now to solve for $dF(x,y) = 0$

lets say the integral is $U(x,y)$ which is given by

$$U = \int A(x,y)\,dx + F(y) = c_1$$

$$= \int \left(\frac{1}{x^2+2} - \frac{1}{y}\right) dx + F(y) = c_1$$

$$\frac{1}{\sqrt{2}} \int \left(\frac{\sqrt{2}}{x^2+(\sqrt{2})^2} - \frac{1}{y}\right) dx + F(y) = c_1$$

$$\therefore U = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + F(y) = c_1. \quad -(2)$$

Now we need to solve for $F(y)$ so we differentiate U wrt y

$$\frac{dU}{dy} = 0 + \frac{x}{y^2} + F'(y) = \frac{x}{y^2} + F'(y)$$

equate $\frac{dU}{dy}$ to $B(x,y)$

$$\frac{x}{y^2} + F'(y) = xy^{-2} + 1$$

$$F'(y) = \cancel{xy^{-2}} + 1 - \cancel{xy^{-2}}$$

$F'(y) = 1$; Integrate it to find $F(y)$

$F(y) = y + c_2 \Rightarrow$ sub $F(y)$ in eqn (2).

$$\boxed{\therefore U(x,y) = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + y = c}$$

where $c = c_1 - c_2$

(2.) A series electric circuit contains a resistance R, a capacitance C and a battery supplying a time-varying electromotive force $V(t)$. The charge q on the capacitor therefore obeys the equation $R\frac{dq}{dt} + \frac{q}{C} = V(t)$. Assuming that initially there is no charge on the capacitor, and given that $V(t) = V_0 \sin wt$, find the charge on the capacitor as a function of time.

Soln

Given eqn of circuit $\Rightarrow R\frac{dq}{dt} + \frac{q}{C} = V(t)$

Rearranging the eqn $\Rightarrow \frac{dq}{dt} + \frac{1}{RC}\cdot q = \frac{V(t)}{R}$

Its of the form $\Rightarrow \frac{dy}{dx} + P.y = Q(x)$ a linear ODE in 1st order & 1st degree.

To solve this eqn we first need to find the Integrating Factor

$$I.F = e^{\int P.dx}$$

$$= e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

The form of the solution to this equation is

$$y \times I.F = \int (Q \times I.F)\,dx + C$$

$$\therefore\ q \times e^{t/RC} = \int \left(\frac{V(t)}{R} \times e^{t/RC}\right) dt + \text{Constant}$$

Sub $V(t) = V_0 \sin wt$

$$q\, e^{t/RC} = \frac{1}{R}\int (V_0 \sin wt \times e^{t/RC})\, dt + \text{const} \quad -(1)$$

let $I_1 = \int V_0 \sin wt \times e^{t/RC}$

Solve this by using integrating by parts

$$\int f(x)\cdot g(x) = f(x)\cdot \int g(x) - \int f'(x)\cdot \int g(x)\, dx$$

$$I_1 = V_0 \sin wt \int e^{t/RC} dt - \int \left(\frac{d}{dx}(V_0 \sin wt)\cdot \int e^{t/RC} dt\right) dt$$

$$= V_0 \sin wt\, e^{t/RC} \times RC - \int (V_0 w \cos wt \cdot e^{t/RC} \cdot RC)\, dt$$

$$I_1 = V_0 \sin wt\, e^{t/RC} \times RC - wRC \int (V_0 \cos wt \times e^{t/RC})\, dt \quad -(2)$$

let $I_2 = \int (V_0 \cos wt \times e^{t/RC})\, dt$

Integrate by parts

$$= V_0 \cos wt \int e^{t/RC} dt - \int \left(\frac{d}{dx}(V_0 \cos wt) \times \int e^{t/RC} dt\right) dt$$

$$= V_0 \cos wt\, e^{t/RC} \cdot RC - \int (-w V_0 \sin wt \times e^{t/RC} \times RC)\, dt$$

$$I_2 = V_0 \cos wt\, e^{t/RC} \cdot RC + wRC \underbrace{\int (V_0 \sin wt \cdot e^{t/RC})\, dt}_{= I_1}$$

$$\therefore I_2 = V_0 \cos wt\, e^{t/RC} \cdot RC + wRC\, I_1$$

sub I_2 in eqn (2)

$$I_1 = V_0 \sin wt\, e^{t/RC} \cdot RC - wRC\,(RC V_0 \cos wt\, e^{t/RC} + wRC I_1)$$

$$I_1 = RC V_0 \sin wt\, e^{t/RC} - wR^2C^2 V_0 \cos wt\, e^{t/RC} - w^2R^2C^2 I_1$$

$$I_1(1+\omega^2R^2C^2) = RC.V_0 \sin\omega t\, e^{t/RC} - \omega R^2C^2 V_0 \cos\omega t . e^{t/RC}$$ (5)

$$I_1 = \frac{RC V_0 \sin\omega t . e^{t/RC} - \omega R^2C^2 V_0 \cos\omega t . e^{t/RC}}{1+\omega^2R^2C^2}$$

$$I_1 = \frac{V_0 RC e^{t/RC}(\sin\omega t - \omega RC\cos\omega t)}{1+\omega^2R^2C^2}$$

sub I_1 in eqn (1)

$$q . e^{t/RC} = \frac{1}{\cancel{R}} \times \frac{V_0 \cancel{R} C \cancel{e^{t/RC}}(\sin\omega t - \omega RC\cos\omega t)}{1+\omega^2R^2C^2} + \underline{\text{constant}}$$

$$q = \frac{V_0 C(\sin\omega t - \omega RC\cos\omega t)}{1+\omega^2R^2C^2} + \text{constant} \quad (3)$$

Now using the initial conditions; at $t=0$; $q=0$ to solve for value of constant

$$0 = \frac{V_0 C(0-\omega RC)}{1+\omega^2R^2C^2} + \text{constant}$$

$$\therefore \text{constant} = \frac{V_0 RC^2\omega}{1+\omega^2R^2C^2} \Rightarrow \text{sub back in eqn (3)}$$

$$q = \frac{V_0 C(\sin\omega t - \omega RC\cos\omega t)}{1+\omega^2R^2C^2} + \frac{V_0 RC^2\omega}{1+\omega^2R^2C^2}$$

$$\boxed{\therefore\ q = \frac{V_0 C(\sin\omega t - \omega RC\cos\omega t + RC\omega)}{1+\omega^2R^2C^2}}$$

(3) By finding an appropriate Integrating Factor solve the equation $\frac{dy}{dx} = \frac{-2x^2+y^2+x}{xy}$

Soln

Rewriting the given equation, we get

$$(2x^2+y^2+x)\,dx + (xy)\,dy = 0 \quad -(1)$$

Its of the form $A(x,y)\,dx + B(x,y)\,dy = 0$

$$A(x,y) = 2x^2+y^2+x \; ; \; B(x,y) = xy$$

$$\frac{\partial A}{\partial y} = 2y \; ; \; \frac{\partial B}{\partial x} = y \;\; \& \;\; \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$$

They aren't exact, so we need to find an Integrating Factor and multiply both sides of eqn (1) with it.

$$F(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = \frac{1}{xy}\cdot(2y-y) = \frac{1}{x}$$

$$I.F = e^{\int F(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

Multiply on both sides of eqn (1) with $I.F = x$.

$$x(2x^2+y^2+x)\,dx + (x y)\,dy \cdot x = 0$$

$$(2x^3+xy^2+x^2)\,dx + (x^2y)\,dy = 0$$

Now we get $A_1(x,y) = 2x^3+xy^2+x^2$ & $B_1(x,y) = x^2y$.

Its solution can be written as $V(x,y)$

$$U(x,y) = \int A(x,y)\,dx + F(y) = C_1$$

$$= \int (2x^3 + xy^2 + x^2)\,dx + F(y) = C_1$$

$$U(x,y) = \frac{x^4}{2} + \frac{x^2y^2}{2} + \frac{x^3}{3} + F(y) = C_1 \quad -2$$

Now differentiate with respect to y.

$$\frac{dU}{dy} = x^2y + F'(y)$$

Equate it to $B(x,y)$

$$\cancel{x^2y} + F'(y) = \cancel{x^2y}$$

$F'(y) = 0$; Integrate to find value of $F(y)$

$$F(y) = C_2$$

Sub $F(y)$ back in eqn 2

$$\frac{x^4}{2} + \frac{x^2y^2}{2} + \frac{x^3}{3} + C_2 = C_1$$

$\therefore$ The solution is

$$\boxed{\therefore \frac{x^4}{2} + \frac{(xy)^2}{2} + \frac{x^3}{3} = C \quad \text{where } C = C_1 - C_2}$$

(4) Solve the following equations using the Method of Undetermined coefficients for the stated boundary conditions. (8)

(a.) $\frac{d^2f}{dt^2} + 2\frac{df}{dt} + 5f = 0$; $f(0) = 1$ and $f'(0) = 0$

Soln

From the eqn we can see that $f(x) = 0$; ~~There~~

The eqn can be written as $f(t) = f_c(t) + f_p(t)$

Since $f(x)$ component is 0; we need only the complementary function $f_c(t)$ & no need for particular integral $f_p(t)$

Finding the complementary function $f_c(t)$.

Since RHS of ODE is already 0, express the ODE in its auxillary form $\Rightarrow$ $\lambda^2 + 2\lambda + 5 = 0$.

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$\lambda = -1 \pm 2i \,;\; \lambda_1 = -1+2i \;\&\; -1-2i = \lambda_2$$

Here we can see that the roots are complex; so the complementary function will be of the form; $f_c(t) = e^{\alpha t}(d_1 \cos\beta t + d_2 \sin\beta t)$

From the roots; $\alpha = -1$; $\beta = 2$

$$\therefore f_c(t) = e^{-t}(d_1 \cos 2t + d_2 \sin 2t)$$

$\therefore$ The solution is $f(t) = e^{-t}(d_1 \cos 2t + d_2 \sin 2t)$ — ①

Apply the boundary conditions to solve for constants d_1 & d_2

(i.) $f(0) = 1$

$$f(0) = e^{-0}(d_1 \cos 2(0) + d_2 \sin 2(0)) = 1$$

$$d_1 = 1$$

(ii.) $f'(0) = 0$

$$f'(t) = e^{-t}(-2 d_1 \sin 2t + 2 d_2 \cos 2t) + e^{-t}(-1)(d_1 \cos 2t + d_2 \sin 2t)$$

Sub $d_1 = 1$ & apply $f'(0) = 0$

$$e^{-0}(-2 \cancel{d_1} 1 \sin 2(0) + 2 d_2 \cos 2(0)) - e^{-0}(1 \cos 2(0) + d_2 \sin 2(0)) = 0$$

$$2d_2 - 1 = 0$$

$$d_2 = \tfrac{1}{2}$$

Sub d_1 & d_2 in eqn ①

$$\boxed{\therefore f(t) = e^{-t}\left(\cos 2t + \tfrac{1}{2} \sin 2t\right)}$$

(4.b) $\dfrac{d^2 f}{dt^2} + 2\dfrac{df}{dt} + 5f = e^{-t}\cos(3t)$

The solution for this ODE is of the form $f(t) = f_c(t) + f_p(t)$

Finding the complementary function $f_c(t)$

Equate the RHS to 0 & express the ODE in its auxillary form

$$\cancel{\lambda^2 + \lambda}: \lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$\lambda = -1 \pm 2i; \quad \lambda_1 = -1+2i; \quad \lambda_2 = -1-2i$$

Here we can see that the roots are complex; so the complementary function will be of the form $f_c(t) = e^{\alpha t}(d_1 \cos\beta t + d_2 \sin\beta t)$

From the roots; $\alpha = -1$; $\beta = 2$

$$\therefore f_c(t) = e^{-t}(d_1 \cos 2t + d_2 \sin 2t)$$

Solving for the particular integral $f_p(t)$

From the ODE $\Rightarrow \underset{(RHS)}{f(t)} = e^{-t}\cos(3t)$

$\therefore$ The trial function will be of form $f_p(t) = e^{-t}(b_1 \sin 3t + b_2 \cos 3t)$

Find 1st derivative wrt 't'.

$$\frac{df_p}{dt} = e^{-t}(b_1 3\cos 3t - b_2 3\sin 3t) + (-e^{-t})(b_1 \sin 3t + b_2 \cos 3t)$$

$$\frac{df_p}{dt} = e^{-t}(3b_1 \cos 3t - 3b_2 \sin 3t - b_1 \sin 3t - b_2 \cos 3t)$$

Find 2nd derivative wrt t

$$\frac{d^2 f_p}{dt^2} = e^{-t}(-9b_1 \sin 3t - 9b_2 \cos 3t - 3b_1 \cos 3t + 3b_2 \sin 3t) + (3b_1 \cos 3t - 3b_2 \sin 3t - b_1 \sin 3t - b_2 \cos 3t)(e^{-t}(-1))$$

$$= e^{-t}(-9b_1\sin 3t - 9b_2\cos 3t - 3b_1\cos 3t + 3b_2\sin 3t$$
$$-3b_1\cos 3t + 3b_2\sin 3t + b_1\sin 3t + b_2\cos 3t)$$

$$\frac{d^2 f_p}{dt^2} = e^{-t}(-8b_1\sin 3t - 6b_1\cos 3t + 6b_2\sin 3t - 8b_2\cos 3t)$$

Sub the f, $\frac{df}{dt}$ & $\frac{d^2f}{dt^2}$ in the ODE eqn

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + 5f = e^{-t}\cos 3t$$

$$\cancel{e^{-t}}(-8b_1\sin 3t - 6b_1\cos 3t + 6b_2\sin 3t - 8b_2\cos 3t)$$
$$+2\cancel{e^{-t}}(3b_1\cos 3t - 3b_2\sin 3t - b_1\sin 3t - b_2\cos 3t) = \cancel{e^{-t}}\cos 3t$$
$$+5\cancel{e^{-t}}(b_1\sin 3t + b_2\cos 3t)$$

$$-8b_1\sin 3t \cancel{- 6b_1\cos 3t} + \cancel{6b_2\sin 3t} - 8b_2\cos 3t + \cancel{6b_1\cos 3t}$$
$$\cancel{-6b_2\sin 3t} - 2b_1\sin 3t - 2b_2\cos 3t + 5b_1\sin 3t + 5b_2\cos 3t = \cos 3t$$

$$-5b_1\sin 3t - 5b_2\cos 3t = \cos 3t$$

equating the eqn; we see $b_1 = 0$ & $b_2 = -\frac{1}{5}$

sub b_1 & b_2 in $f_p(t)$

$$\therefore f_p(t) = e^{-t}\left(-\frac{1}{5}\cos 3t\right)$$

$$f_p(t) = \frac{-e^{-t}}{5}\cos 3t$$

$\therefore$ The solution to the ODE is

$$f(t) = f_c(t) + f_p(t)$$

$$= e^{-t}(d_1 \cos 2t + d_2 \sin 2t) - \frac{e^{-t}}{5}\cos 3t$$

$$f(t) = e^{-t}(d_1 \cos 2t + d_2 \sin 2t - \frac{1}{5}\cos 3t) \quad -(1)$$

Apply necessary boundary conditions

(i) $f(0) = 0$

$$f(0) = e^{-0}(d_1 \cos 2(0) + d_2 \sin 2(0) - \frac{1}{5}\cos 3(0)) = 0$$

$$\cancel{d_1 = \frac{1}{5}} \quad d_1 = \frac{1}{5}$$

(ii) $f'(0) = 0$

$$f'(t) = e^{-t}(-2d_1 \sin 2t + 2d_2 \cos 2t + \frac{3}{5}\sin 3t) + e^{-t}(-1)$$
$$(d_1 \cos 2t + d_2 \sin 2t - \frac{1}{5}\cos 3t)$$

$$= e^{-t}(-2d_1 \sin 2t + 2d_2 \cos 2t + \frac{3}{5}\sin 3t - d_1 \cos 2t$$
$$- d_2 \sin 2t + \frac{1}{5}\cos 3t)$$

$$f'(0) = e^{-0}(2(\frac{1}{5})\sin 2(0) + 2d_2 \cos 2(0) + \frac{3}{5}\sin 3(0)$$
$$- \frac{1}{5}\cos 2(0) - d_2 \sin 2(0) + \frac{1}{5}\cos 3(0)) = 0$$

$$= (0 + 2d_2 + 0 - \cancel{\frac{1}{5}} - 0 + \cancel{\frac{1}{5}}) = 0$$

$$2d_2 = 0$$

$$d_2 = 0.$$

Sub d_1 & d_2 in eqn ①

$$f(t) = e^{-t}\left(\frac{1}{5}\cos 2t + 0 - \frac{1}{5}\cos 3t\right)$$

$$\boxed{\therefore f(t) = \frac{e^{-t}}{5}(\cos 2t - \cos 3t)}$$

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⑤ Using the method of Laplace Transform solve the following differential equation: $y''(t) + y(t) = \sin(2t)$ satisfying the initial conditions $y(0) = 2$; $y'(0) = 1$

Soln

From Laplace transforms we know that the laplace transform of the n^{th} derivative of $f(x)$ is given by

$$\bar{f^n}(s) = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{n-2}(0) - f^{n-1}(0)$$

$$ODE = y''(t) + y(t) = \sin(2t) \quad -①$$

apply Laplace Transform to both sides of eqn ①

$$\mathcal{L}(y''(t) + y(t)) = \mathcal{L}(\sin 2t)$$

$$s^2 \bar{y}(s) - s y(0) - y'(0) + \bar{y}(s) = \frac{2}{s^2+2^2} \quad -②$$

$$\left(\text{WKT Laplace Transform of } \sin 2x = \frac{a}{s^2+a^2}\right)$$

Apply initial conditions to eqn ①

$$s^2\bar{y}(s) - s(2) - 1 + \bar{y}(s) = \frac{2}{s^2+4}$$

rearranging the eqn we get

$$\bar{y}(s)(1+s^2) - 2s - 1 = \frac{2}{s^2+4}$$

$$\bar{y}(s)(1+s^2) = \frac{2}{s^2+4} + 2s + 1$$

$$\bar{y}(s) = \frac{2s^3+s^2+8s+6}{(s^4+4)(s^2+1)} \quad -③$$

Break down the RHS of eqn ③ into partial fractions

$$\cancel{\frac{2s^3+s^2+8s+6}{s^4+}}$$

$$\frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} \quad -④$$

$$\frac{2s^3+s^2+8s+6}{\cancel{(s^2+4)(s^2+1)}} = \frac{(As+B)(s^2+1) + (Cs+D)s^2+4}{\cancel{(s^2+4)(s^2+1)}}$$

$$2s^3+s^2+8s+6 = As^3 + As + Bs^2 + B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$2s^3+s^2+8s+6 = (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D)$$

equate respective terms on LHS & RHS

$$A+C = 2 \Rightarrow A = 2-C$$

$A + 4C = 8$ | $2 + 3C = 8$ | $C = 2$
$(2-C) + 4C = 8$ | $3C = 6$ | $\therefore A = 2 - 2 = 0$

$B + D = 1$ | $B + 4D = 6$ | $3D = 5$ | $B = 1 - \frac{5}{3}$
$B = 1 - D$ | $1 - D + 4D = 6$ | $D = \frac{5}{3}$ | $B = -\frac{2}{3}$
| $1 + 3D = 6$ | |

Sub A, B, C, D in eqn 4 we get

$$\frac{2S^3 + S^2 + 8S + 6}{(S^2+4)(S^2+1)} = \frac{-2}{3(S^2+4)} + \frac{2S + \frac{5}{3}}{S^2+1} = \frac{-2}{3(S^2+4)} + \frac{2S}{S^2+1} + \frac{5}{3(S^2+1)}$$

Sub in eqn 3

$$\bar{y}(s) = -\frac{1}{3} \cdot \frac{2}{S^2+2^2} + 2 \cdot \frac{S}{S^2+1^2} + \frac{5}{3} \cdot \frac{1}{S^2+1^2}$$

Applying the inverse laplace transform on B.S we get

$$y(t) = -\frac{1}{3} \sin 2t + 2 \cos t + \frac{5}{3} \sin t$$

Since inverse transform of $\frac{a}{S^2+a^2} = \sin ax$; $\frac{S}{S^2+a^2} = \cos ax$

$\therefore$ The solution for ODE is

$$\boxed{y(t) = -\frac{1}{3} \sin 2t + 2 \cos t + \frac{5}{3} \sin t}$$

6. Find the general solution of the following third-order linear differential equation by using the Method of Undetermined Coefficients. (16)

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 30e^{-x}$$

Soln

The soln for this ~~eqn~~ ODE is $y(x) = y_c(x) + y_p(x)$

Solve for the complementary function $y_c(x)$

Equate the RHS to 0 & write the eqn in auxillary form

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

By trial & error of long division we find that $(\lambda+1)$ is a factor of the cubic equation

$(\lambda+1)(\lambda^2+2\lambda+1) = 0$; Now factoring the quadratic equation

$$(\lambda+1)(\lambda^2+\lambda+\lambda+1) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda+1) = 0$$

$$\therefore \lambda_1 = \lambda_2 = \lambda_3 = -1$$

Here we see that the root, λ is repeated three times.

Since it has repeatable roots, we can write the complementary function as

$$y_c(x) = (C_1 + C_2x + C_3x^2)e^{-x}$$

Now we need to find the particular Integral $y_p(x)$

Since $f(x) = 30e^{-x}$; the best fit trial eqn for $y_p(x)$ is

$y_p(x) = b e^{-x}$

But since this term already exists in the complementary function, the next available term with integral power of x is

$$y_p(x) = bx^3e^{-x} \quad -(1)$$

1^{st} Differential wrt x

$$\frac{dy_p}{dx} = b(-x^3e^{-x} + e^{-x}3x^2) = b(3e^{-x}x^2 - x^3e^{-x})$$

2^{nd} Differential wrt x

$$\frac{d^2y_p}{dx^2} = b(3(2xe^{-x} - x^2e^{-x}) - (3e^{-x}x^2 - e^{-x}x^3)$$

$$= b(6xe^{-x} - 6x^2e^{-x} + x^3e^{-x})$$

3^{rd} differential wrt x

$$\frac{d^3y_p}{dx^3} = b(6(e^{-x} - xe^{-x}) - 6(2xe^{-x} - x^2e^{-x}) + (3x^2e^{-x} - x^3e^{-x})$$

$$= b(6e^{-x} - 6xe^{-x} - 12xe^{-x} + 6x^2e^{-x} + 3x^2e^{-x} - x^3e^{-x})$$

$$\frac{d^3y_p}{dx^3} = b(6e^{-x} - 18xe^{-x} + 9x^2e^{-x} - x^3e^{-x})$$

Sub $y_p(x)$, $\frac{dy_p}{dx}$, $\frac{d^2y_p}{dx^2}$, $\frac{d^3y_p}{dx^3}$ in the original ODE

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 30e^{-x}$$

$$b\cancel{e^{-x}}(6-18x+9x^2-x^3)+3b\cancel{e^{-x}}(6x-6x^2+x^3) + 3b\cancel{e^{-x}}(3x^2-x^3)+b\cancel{e^{-x}}x^3 = 30\cancel{e^{-x}}$$

$$b(6-\cancel{18x}+9x^2-x^3+\cancel{18x}-18x^2+3x^2+9x^2-3x^3+x^3) = 30$$

$$6b = 30$$

$$b = 5$$

Sub. $b=5$ in eqn ① $\quad y_p(x) = b\,x^3e^{-x}$

$$\therefore y_p(x) = 5x^3e^{-x}$$

$\therefore$ The solution to the ODE is $y(x) = y_c(x) + y_p(x)$

$$y(x) = (c_1+c_2x+c_3x^2)e^{-x} + 5x^3e^{-x}$$

$$\boxed{\therefore y(x) = e^{-x}(c_1+c_2x+c_3x^2+5x^3)}$$

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⑦ Use the Method of Variation of Parameters to find the solution of the following equations $\quad$ Hint $\int x^n e^{-x}dx = -e^{-x}n!\sum_{m=0}^{n}\frac{x^m}{m!}$

ⓐ $y'' - y = x^n$

Soln

Solution for this equation is $y(x) = y_c(x) + y_p(x)$

Find the complementary function $y_c(x)$

Equate the RHS of the eqn to 0 & write the auxillary eqn.

$$\lambda^2 - 1 = 0$$
$$\lambda^2 = 1$$
$$\lambda = \pm 1 \quad \text{roots are real \& distinct}$$

$$\therefore \; y_c(x) = c_1 e^{x} + c_2 e^{-x}$$

Seeing the complementary function we assume a particular integral of the form $y_p(x) = K_1(x) . e^{x} + K_2(x) e^{-x}$

lets impose additional constrains

$$K_1'(x) e^{x} + K_2'(x) e^{-x} = 0 \quad -①$$

$$K_1'(x) e^{x} - K_2'(x) e^{-x} = \frac{f(x)}{a_n(x)} = \frac{x^n}{1} \quad -②$$

Adding eqns ① & ②

$$2K_1'(x) e^{x} = x^n$$

$$K_1'(x) e^{x} = \frac{x^n}{2}$$

$$K_1'(x) = \frac{x^n}{2} e^{-x}$$

Sub eqn ② from ① ; we get

$$2K_2'(x) e^{-x} = -x^n$$
$$K_2'(x) = -\frac{x^n}{2} e^{+x}$$

Integrate the terms $K_1'(x)$ & $K_2'(x)$ & ignore constant of integration

$$K_1'(x) = \frac{x^n}{2} e^{-x}$$

$$K_1(x) = \frac{1}{2} \int x^n e^{-x} = \frac{1}{2} \left(-e^{-x} n! \sum_{m=0}^{n} \frac{x^m}{m!} \right)$$

$$K_1(x) = -\frac{e^{-x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!}$$

$$K_2'(x) = -\frac{1}{2} \int x^n e^{x} = -\frac{1}{2} \left(+e^{x} n! \sum_{m=0}^{n} \frac{x^m}{m!} (-1)^{n-m} \right)$$

$$K_2(x) = -\frac{e^{+x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!} (-1)^{n-m}$$

Sub $K_1(x)$ & $K_2(x)$ back in $y_p(x)$

$$\therefore y_p(x) = -\frac{e^{-x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!} \times e^{x} - \frac{e^{+x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!} (-1)^{n-m} \times e^{-x}$$

$\therefore$ The general solution is given by

$$y(x) = (C_1 + K_1(x)) e^{x} + (C_2 + K_2(x)) e^{-x}$$

$$\therefore y(x) = \left(C_1 - \frac{e^{-x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!} \right) e^{x} + \left(C_2 - \frac{e^{x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!} (-1)^{n-m} \right) e^{-x}$$

$$\boxed{\therefore y(x) = \left(C_1 - \frac{e^{-x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!} \right) e^{x} + \left(C_2 - \frac{e^{x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!} (-1)^{n-m} \right) e^{-x}}$$

7.6 $y'' + y = \tan(x)$; $0 < x < \pi/2$

Soln The solution is given by $y(x) = y_c(x) + y_p(x)$

Find the complementary function $y_c(x)$

Equate the RHS of the ODE to 0 & write the auxillary eqn

$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i \;;\; \lambda_1 = +i;\; \lambda_2 = -i \quad \Big| \quad \alpha = 0,\; \beta = 1$$

The roots are complex, so the form of the complementary function is $f_c(x) = e^{\alpha x}(d_1 \cos\beta x + d_2 \sin\beta x)$

$$\therefore f_c(x) = d_1 \cos x + d_2 \sin x$$

Keeping the complementary function in mind lets assume the particular integral is of the form $f_p(x) = K_1(x)\cos x + K_2(x)\sin x$

lets impose additional constraints

$$K_1'(x)\cos x + K_2'(x)\sin x = 0 \quad - (1)$$

$$-K_1'(x)\sin x + K_2'(x)\cos x = \tan x \quad - (2)$$

Eqn (1) $\times \sin x$ & eqn (2) $\times \cos x$ we get

$$K_1'(x)\cos x\sin x + K_2'(x)\sin^2 x = 0 \quad - (3)$$
$$-K_1'(x)\sin x\cos x + K_2'(x)\cos^2 x = \sin x \quad - (4)$$

Adding ③ & ④, we get

$$K_2'(x)(\cos^2 x + \sin^2 x) = \sin x$$

$$K_2'(x) = \sin x$$

Sub $K_2'(x)$ in eqn ①

$$K_1'(x)\cos x + \sin^2 x = 0$$

$$K_1'(x) = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

Now integrate $K_1'(x)$ & $K_2'(x)$ & ignore constants of integration.

$$K_2(x) = \int \sin x = -\cos x$$

$$K_1(x) = \int \cos x - \sec x = \sin x - \ln(\sec x + \tan x)$$

∴ The solution $y(x)$ is $((d_1 + K_1(x))\cos x + (d_2 + K_2(x))\sin x)$

$$\boxed{\therefore y(x) = (d_1 + \sin x - \ln(\sec x + \tan x))\cos x + (d_2 - \cos x)\sin x}$$