SYLLABUS

UNIT- I

Matrices: Types of Matrices, Symmetric; Hermitian; Skew-symmetric; Skew-Hermitian; orthogonal matrices; Unitary Matrices; rank of a matrix by Echelon form and Normal form, Inverse of Non-singular matrices by Gauss-Jordan method; System of linear equations; solving system of Homogeneous and Non-Homogeneous equations. Gauss elimination method; Gauss Seidel Iteration Method.

UNIT-II

Eigen values and Eigen vectors: Linear Transformation and Orthogonal Transformation: Eigen values and Eigenvectors and their properties: Diagonalization of a matrix; Cayley-Hamilton Theorem (without proof); finding inverse and power of a matrix by Cayley-Hamilton Theorem; Quadratic forms and Nature of the Quadratic Forms; Reduction of Quadratic form to canonical forms by Orthogonal Transformation

UNIT-III

Sequences & Series

Sequence: Definition of a Sequence, limit; Convergent, Divergent and Oscillatory sequences.

Series: Convergent, Divergent and Oscillatory Series; Series of positive terms; Comparison test, p-test, D-Alembert's ratio test; Raabe's test; Cauchy's Integral test; Cauchy's root test; logarithmic test. Alternating series: Leibnitz test; Alternating Convergent series: Absolute and Conditionally Convergence.

UNIT-IV

Calculus Mean value theorems: Rolle's theorem, Lagrange's Mean value theorem with their Geometrical Interpretation and applications, Cauchy's Mean value Theorem. Taylor's Series. Applications of definite integrals to evaluate surface areas and volumes of revolutions of curves (Only in Cartesian coordinates), Definition of Improper Integral: Beta and Gamma functions and their applications.

UNIT-V

Multivariable calculus (Partial Differentiation and applications):
Definitions of Limit and continuity. Partial Differentiation; Euler's Theorem; Total derivative; Jacobian; Functional dependence & independence, Maxima and minima of functions of two variables and three variables using method of Lagrange multipliers.

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Theory

Unit - I

Matrices

1) Define matrix. List and Explain the types of matrices with Examples.

Ans:- Matrix:- The contribution of Elements into rows and columns is called a matrix.

Example:- (i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ order = 2x3 (ii) $\begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix}$ order = 3x2 (iii) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$

Types of matrix:-

(i) square matrix :- The number of rows is Equal to the number of columns.

Example:- $\begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix}_{2\times2}$ $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}_{2\times2}$.

(ii) Rectangular Matrix:- A matrix which is not a square matrix is called a rectangular matrix.

Example:- $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}_{2\times3}$

(iii) Row matrix:- A matrix of order 1 x m is called a row matrix.

Example:- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times3}$

(iv) Column matrix:- A matrix of order $n \times 1$

Example $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3\times 1}$

(v) unit matrix:- If $A = [a_{ij}]_{n\times n}$ such that $a_{ij} = 1$ for $i=j$ and $a_{ij} = 0$ for $i \neq j$, Then 'A' is called as a unit matrix. It is denoted by I_n.

Example:- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(vi) Null matrix (or) zero matrix:- If $A = [a_{ij}]_{m\times n}$ such that $a_{ij} = 0$, $\forall$ i and j, Then 'A' is called a zero matrix (or) a null matrix. It is denoted by 'O' (or more clearly $O_{m\times n}$)

Example:- $O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

(vii) Diagonal matrix:- A square matrix is said to be diagonal if at least on element of principle diagonal is non-zero and all are zero.

Example:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(viii) Scalar matrix:- A diagonal matrix whose leading diagonal Elements are all Equal is called a scalar matrix.

Example:-

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(ix) Equal matrices :- Two matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ are said to be Equal if and only if (i) A and B are of the same type or (order) and (ii) $a_{ij} = b_{ij}$ for Every i and j.

(x) Symmetric Matrix :- A square matrix $A=[a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for Every i and j. Examples:-

(i) $A = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 5 & 4 \\ 5 & 4 & 7 \end{bmatrix}$ Then $A' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 5 & 4 \\ 5 & 4 & 7 \end{bmatrix}$

$\Rightarrow$ A is a symmetric matrix.

$A = A'$ (or) $A' = A$.

(xi) skew-symmetric matrix:- A square matrix $A=[a_{ij}]$ is said to be skew-symmetric if $a_{ij} = -a_{ji}$ for Every i and j.

$\Rightarrow$ A is a skew-symmetric matrix

$A = -A'$ (or) $A' = -A$.

Example:-

$$A' = \begin{pmatrix} 0 & -5 & -3 \\ 5 & 0 & 8 \\ 3 & -8 & 0 \end{pmatrix} = -A \begin{pmatrix} 0 & 5 & 3 \\ -5 & 0 & -8 \\ -3 & 8 & 0 \end{pmatrix}$$

$A' = -A$

(xii) Hermitian matrix:- A square matrix A is said to be Hermitian if $\boxed{A = (\bar{A})^T}$ (or) $\boxed{A^\theta = A}$ $\boxed{A^T = \bar{A}}$ (or) $\boxed{(\bar{A})^T = A}$

Example:-

$$A=\begin{bmatrix} 1 & 1+2i \\ 1-2i & 2 \end{bmatrix}, \bar{A}=\begin{bmatrix} 1 & 1-2i \\ 1+2i & 2 \end{bmatrix}$$

$$(\bar{A})^T=\begin{pmatrix} 1 & 1+2i \\ 1-2i & 2 \end{pmatrix}$$

$\therefore A$ is a Hermitian matrix

(xiii) Skew-Hermitian matrix:- A square matrix 'A' is said to be Skew-Hermitian

$\boxed{A = -(\bar{A})^T}$ (or) $\boxed{A^\theta = -A}$ $\boxed{\bar{A}^T = -\bar{A}}$ (or) $\boxed{(\bar{A})^T = -A}$

Example:-

$$A=\begin{bmatrix} i & 1+i \\ 1-i & 2i \end{bmatrix}, \bar{A}=\begin{bmatrix} -i & -1-i \\ 1+i & -2i \end{bmatrix}$$

$$(\bar{A})^T=\begin{pmatrix} -i & 1+i \\ 1-i & -2i \end{pmatrix} = -(\bar{A})^T=\begin{pmatrix} i & 1+i \\ 1-i & 2i \end{pmatrix}.$$

(xiv) Orthogonal matrices:- A square matrix 'A' is said to be orthogonal if $AA' = A'A = I$ That is $A^T = A^{-1}$.

Example:- $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.\ A^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(xv) Unitary matrices:- A square matrix 'A' such that $(\bar{A})^T = A^{-1}$ i.e., $(\bar{A})^T A =$ $A(\bar{A})^T = I$ (or) $A^\theta A = I$ is called a unitary matrix

Example:- $\begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$

(xvi) Triangular Matrix :- A square matrix all of whose elements below the leading diagonal are zero is called an upper Triangular matrix.

→ A square matrix all of whose elements above the leading diagonal are zero is called a Lower Triangular matrix.

Example:- Upper triangular matrix :-

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Lower triangular matrix :-

$$\begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 & 0 \\ 2 & 1 & -8 & 5 & 0 \\ 2 & 0 & 4 & 1 & 6 \end{bmatrix}$$

(xvii) Idempotent Matrix :- If 'A' is a square matrix such that $A^2 = A$ then 'A' is called Idempotent.

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2) Define Rank of matrix with an Example

Ans):- Rank of matrix is defined as the order of the largest square submatrix whose determinant is not zero.

→ Rank is denoted by 'ρ'(row).

Example:-

$$A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}_{3\times3}$$

$-1(18-1)-0(9+5)+6(3+30)$

$-1(17)-0(13)+6(33)$

$\Rightarrow -17-0+198$

$\Rightarrow 198-17$

$\Rightarrow 181.$ (Ans).

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3) Write short note on Echelon form and normal form

Ans) → The following are the three methods used for finding the rank of a matrix

(i) Echelon form (or) Triangular form

(ii) Normal form (or) Canonical form

(iii) Normal form of the type 'PAQ'.

(i) Echelon form (or) Triangular form of a matrix:-

→ A matrix is said to be in Echelon form if it has the following properties

(i) Zero rows, if any, are below any non-zero row.

(ii) The first non-zero element in each non-zero row is equal to 1.

(iii) The number of zero rows before the first non-zero element of a row is less than the number of such zeros in the next row.

Example:- $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is an Echelon form.

(ii) Normal form:- A matrix is also a form $[I_r]$ (or) $[I_r\ 0]$ (or) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ (or canonical form):-

when 'r' is the rank of a matrix (or) Else the rank of a matrix.

Example:- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

→ In normal we will use row and column operations.

→ In a row and column if they is a single element, It's Better to divide that row (or) column by the same element.

(iii) Normal form of the type 'PAQ':-

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

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4) Explain working rule for finding the inverse of matrix (or) non singular matrices by gauss-Jordan method.

Ans) Gauss-Jordan method:- By Gauss-Jordan method we can find the inverse of a non-singular square matrix using Elementary row operation only. This method is known as Gauss-Jordan method.

The inverse of a matrix by elementary transformations:- (Inverse by gauss-Jordan method):-

→ We can find the inverse of a non-singular square matrix using Elementary row operation only. This method is known as Gauss-Jordan method.

Working Rule for finding the inverse of a matrix:-

Suppose A is a non-singular square matrix of order n.

We write $A = I_n A$.

Now, we apply elementary row operations only to the matrix 'A' and the prefactor 'I_n' of the R.H.S. we will do this till we get an equation of the form

$$I_n = BA$$

Then obviously 'B' is the inverse of 'A'

Alternatively, consider [A/I] and apply elementary row operations on both 'A' and 'I' until 'A' gets transformed to 'I'.

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5) Define system of linear Equation. Explain

Ans):- System of Linear Simultaneous Equations

Defination:- An equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$ — (1)

where, $x_1, x_2, \ldots x_n$ are unknowns

$a_1, a_2, a_3 \ldots a_n$, b are constants is called a linear equation in 'n' unknowns

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \text{— (2)}$$

$$AX = B \text{ — (3)}$$

$$A = [a_{ij}], X = (x_1, x_2, \ldots x_n)^T \text{ — (4)}$$

and $B = (b_1, b_2, \ldots, b_m)^T$ — (5)

6) Give the procedure for :-

a) Gauss (or) Gaussian Elimination method.

b) Gauss seidel iteration method.

Ans):-

Gaussian Elimination Method

→ This method of solving a system of 'n' linear equations in 'n' unknowns consists of eliminating the coefficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution.

→ consider the system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{——} (1)$$

The augmented matrix of this system is

$$[A,B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \text{——} (2)$$

By performing row operation R_2 & R_3

$$[A,B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_2 \\ 0 & \alpha_{32} & \alpha_{33} & \beta_3 \end{bmatrix} \text{——} (3)$$

Now, By performing R_3'

$$[A,B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_2 \\ 0 & 0 & \gamma_{33} & \delta_3 \end{bmatrix} \quad \text{---} (4)$$

The augmented matrix (4) corresponds to an upper triangular system which can be solved by Backward substitution. The solution obtained is Exact.

Gauss seidel iteration method

→ This is a modification of Gauss-Jacobi's method. ~~we consider the system of equations.~~

→ Consider

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{---} (1)$$

The system of Equation (1) may be written as

$$\left.\begin{aligned} x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \\ x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \\ x_3 &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \end{aligned}\right\} (2)$$

let the initial approximate solution be $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$. substituting $x_2^{(0)}, x_3^{(0)}$ in the first Equation of (2), we get

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right] \quad \text{--- } 3(a)$$

This is taken as the first approximation of x_1.

substituting $x_1^{(1)}$ for x_1 and $x_3^{(0)}$ for $x_3^{(0)}$ in the second approxn. Equation of (2), we get.

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right] \quad \text{--- } 3(b)$$

This is taken as the first approximation of x_2

Next, substituting $x_1^{(1)}$ for x_1 and $x_2^{(1)}$ for x_2 in the last Equation of (2), we get

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right] \quad \text{--- } 3(c).$$

→ The values obtained in 3(a), 3(b), 3(c) constitute the first iterates of the solution.

THE END

Unit - II

Eigen Values and Eigen vectors

1) Define Eigen values and Eigen vectors (or) Define Latent roots and vectors. ~~Give their properties~~

Ans):- **Eigen values** :- we can find the eigen values by determining the roots of the characteristic Equation.

$$\boxed{|A-\lambda I|=0}$$

→ In the above Equation 'λ' is a scalar known as the eigen value (or) characteristic value associated with Eigen vector 'v'

Eigen vectors :- If 'v' is a vector that is not zero, then it is an eigen vector of a square matrix 'A' If Av is a scalar multiple of 'v'.

→ this condition should be written as the Equation

$$\boxed{Av=\lambda v}$$

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2) Define Diagonalization of a matrix

Ans):- A matrix 'A' is a diagonalizable if there Exists an invertible matrix 'p' such that $p^{-1}AP = D$ is a diagonal matrix. Also the matrix 'p' is then said to diagonalize 'A' (or) transform 'A' to diagonal form.

3) Define Quadratic form. Explain the nature of Quadratic form.

Ans)

Quadratic form

Defination :- A homogeneous expression of the second degree in any number of variables is called a quadratic form.

Nature of Quadratic form :- the quadratic form X^TAX in 'n' variables is said to be

(i) positive definite :- if $r=n$ and $s=n$ (or) if all the Eigen values of 'A' are positive.

(ii) Negative definite :- If $r=n$ and $s=0$ (or) if all the Eigen values of 'A' are negative.

(iii) positive semidefinite :- if $r<n$ and $s=r$ (or) if all the Eigen values of $A \geq 0$ and at least one Eigen value is zero.

(iv) negative semidefinite :- if $r<n$ and $s=0$ (or) if all the Eigen values of $A \leq 0$ and at least one Eigen value is zero.

(v) indefinite :- in all other cases

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4) List the various methods of reduction of quadratic form to canonical form (or) sum of squares form. Explain reduction of quadratic form to canonical forms by orthogonal transformation

Ans) :-

Ans):- Methods of Reduction of Quadratic form to canonical form (or sum of squares form).

→ Any quadratic form may be reduced to canonical form by means of the following methods:-

1. Diagonalisation (Reduction to canonical form using Linear transformation (or) Linear transformation of quadratic form).

2. orthogonalisation (Reduction to canonical form using orthogonal transformation).

3. Lagrange's reduction.

* Reduction of quadratic form to canonical forms by orthogonal transformation:-

→ Procedure to reduce quadratic form to canonical form by orthogonal transformation:-

1) write the coefficient matrix 'A' associated with the given quadratic form.

2) find the Eigen values of 'A'.

3) write the canonical form using $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$.

4) form the matrix 'P' containing the normalized Eigen vectors of 'A' as column vectors. Then $X = PY$ gives the required orthogonal transformation which reduces quadratic form to canonical form.

5) Define the following.

a) Linear transformation :-

If , $x' = a_1x + a_2y$

$y' = b_1x + b_2y$

(or) $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

let, $X = AY$

where,

$$X = \begin{bmatrix} x' \\ y' \end{bmatrix}, \; A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} x \\ y \end{bmatrix}$$

so, such a transformation is called Linear transformation in two dimensions.

b) Orthogonal transformation:- If 'A' is an orthogonal matrix and X, Y are two column vectors, then the transformation $\boxed{Y = AX}$ is called an orthogonal transformation.

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6) State Cayley - Hamilton theorem and use it to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

Sol):- Cayley - Hamilton theorem :- Every square matrix satisfies its own characteristic equation.

given, $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

The characteristic equation of A is $|A-\lambda I|=0$.

i.e; $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix}=0.$

$\Rightarrow (1-\lambda)\left[(1+\lambda)^2-4\right]-2\left[-2(1+\lambda)-12\right]+3\left[2+3(1+\lambda)\right]=0.$

$\Rightarrow (1-\lambda)\left[(1+\lambda)^2-4\right]-2\left[-2-\lambda-12\right]+3\left[2+3+3\lambda\right]=0.$

$\Rightarrow (1-\lambda)\left[1^2+\lambda^2+2\lambda-4\right]-2\left[-2-\lambda-12\right]+3\left[2+3+3\lambda\right]=0.$

$\Rightarrow (1-\lambda)\left[\lambda^2+2\lambda-3\right]-2\left[-\lambda-14\right]+3\left[3\lambda+5\right]=0.$

$\Rightarrow \lambda^2+2\lambda-3-\lambda^3-2\lambda^2+3\lambda+2\lambda+28+9\lambda+15=0$

$\Rightarrow -\lambda^3-2\lambda^2+\lambda^2+3\lambda+4\lambda+9\lambda+28+15-3=0$

$\Rightarrow -\lambda^3-\lambda^2+18\lambda+40=0.$

$\Rightarrow -(\lambda^3+\lambda^2-18\lambda-40)=0$

$\lambda^3+\lambda^2-18\lambda-40=0.$

By Cayley-Hamilton theorem

$A^3+A^2-18A-40I=0.$

Now multiplying with A^{-1} on b.s.

$A^{-1}(A^3+A^2-18A-40I)=0.$

$A^2+A-18I-40A^{-1}=0$

$A^2+A-18I=40A^{-1}$

$A^{-1}=\frac{1}{40}\left[A^2+A-18I\right]$

Now

$$A^2 = A \times A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 14 & 14 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{40}\left[\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}\right]$$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

(Ans)

Unit - III

Sequences and series

1) Define Sequence. Give Convergent, Divergent and oscillatory sequences

Ans):- Limit of a sequence:- Let $\{s_n\}$ be a sequence and $l \in R$. l is said to be limit of the sequence $\{s_n\}$, if to each $\varepsilon > 0$, there exists $m \in Z^+$ such that $|s_n - l| < \varepsilon, \forall n \geq m$.

$$\lim_{n\to\infty} s_n = l \text{ (or) } \lim s_n = l.$$

Sequence:- set of numbers is arranged in a order is called a sequence and it is denoted by $\{a_n\}$ (or) $\langle a_n \rangle$

Examples

(i) $1, 2, 3, 4 \dots$

(ii) $3^2, 3^4, 3^6, 3^8 \dots$

Convergent sequence:- If $\lim\limits_{n\to\infty} s_n = l$, then we say that the sequence $\{s_n\}$ converges to 'l' (or) $\{s_n\}$ is convergent to limit l.

Divergent sequence:- A sequence which is not convergent is called a divergent sequence.

i.e., if $\lim\limits_{n\to\infty} s_n = \pm\infty$

Oscillatory sequences:- If $\lim_{n\to\infty} s_n$ is not unique (oscillates finitely) (or) $\pm\infty$ (oscillates infinitely).

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2) Define series, give convergent, divergent and oscillatory series.

Ans):- Series (or) Infinite series:- An expression of the form $u_1+u_2+u_3+\dots+u_n+\dots$ where $\{u_n\}$ is a sequence is called a series.

→ If the series is like $u_1+u_2+\dots+u_n$ it is called a finite series.

Convergence, Divergent and oscillation of a series:-

→ Consider the sequence $\{s_n\}$

1) if $\lim_{n\to\infty} s_n = l$ and 'l' is finite, then $\sum u_n$ converges.

2) If $\lim s_n = \infty$ (or) $-\infty$, then $\sum u_n$ diverges.

3) If $\{s_n\}$ does not tend to a unique limit as $n\to\infty$, then the series $\sum u_n$ is said to be oscillatory (or) non-convergent.

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3) Define series of positive terms (or) series of negative terms

Ans):- Series of Non-negative terms (or) positive terms:-

We say that $\sum u_n$ is a series of non-negative terms if $u_n \geq 0$, $\forall n \in Z^+$, we say that $\sum u_n$ is a series of positive terms (or) series of non-negative terms.

4) Define Comparison Test

Ans):- Comparison Test:- If $\sum u_n$ and $\sum v_n$ are two series of positive terms and $\lim_{n\to\infty} \frac{u_n}{v_n} = l \neq 0$, then the series $\sum u_n$

and Σv_n both converge (or) both diverge.

5) Explain theorem on as Auxiliary series (or) p-Test (or) p-Series Test

Ans Theorem:-

The series $\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots, p \in R$

(a) Converges if $p > 1$

(b) diverges if $p \leq 1$

Proof:- let 'S_n' be the n^{th} partial sum of $\Sigma \frac{1}{n^p}$

$$\therefore S_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p}$$

(a) let $p > 1$

$\frac{1}{1^p} = 1$ (one term)

$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}$ (two terms)

$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{2^{2(p-1)}}$ ($2^2 = 4$ term)

- - - - -

$$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \dots$$

The series in the R.H.S is a geometric series with $r = \frac{1}{2^{p-1}} < 1$

Since, $r < 1$ the series in the R.H.S is convergent

$\therefore \Sigma \frac{1}{n^p}$ is also convergent.

(b) Case (i) let $p=1$, then

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots$$

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{2}{4} \Rightarrow \frac{1}{2}.$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{4}{8} \Rightarrow \frac{1}{2}.$$

$$\therefore \sum \frac{1}{n^p} > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\ldots$$

$$\therefore \sum \frac{1}{n^p} > \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$$

Thus, the series in R.H.S is divergent

$\therefore$ It is a geometric series with $r=1$

$\therefore$ The series in L.H.S is divergent

Case (ii) let $p<1$, then $n>n^p$

$$\therefore \frac{1}{n} < \frac{1}{n^p}$$

$$\Rightarrow \sum \frac{1}{n} < \sum \frac{1}{n^p}$$

The series in L.H.S is divergent (by case (i)).

$\therefore$ The series in R.H.S is also divergent.

---

6) Explain Cauchy's integral test

Theorem :-

let, $\sum u_n = u_1+u_2+u_3+\ldots+u_n+\ldots$ be series with positive and non-increasing terms i.e., $u_1 \geq u_2 \geq u_3 \geq \ldots$

let $f(x)$ be a non-negative decreasing function such that $f(1)=u_1$, $f(2)=u_2, \ldots f(n)=u_n$ on $[1,\infty)$. Then the series

$\sum_{n=1}^{\infty} f(n)$ Converges (or) diverges according to the improper integral

$\int_1^{\infty} f(x)dx$ is finite (or) infinite.

Note:- we must understand $\int_1^{\infty} f(x)\,dx$ as $\lim_{l \to \infty} \int_1^{l} f(x)dx$.

---

7) List out alternating Convergence series and define them

Ans):- a) There are two alternating Convergence series, They are :-

1) Absolute Convergence
2) Conditionally Convergence.

1) Absolute Convergence :- The series Σu_n is said to be absolutely convergent if $\Sigma |u_n|$ is convergent.

2) Conditional Convergence:- If Σu_n Converges and $\Sigma |u_n|$ diverges, then we say that Σu_n Converges Conditionally (or) Converges non-absolutely or semi-Convergent).

---

8) What are the various tests of Convergence

Ans):- The following are the Various tests of Convergence

1) p-test (or) p-series test (or)
2) D'Alembert's ratio test.
3) Cauchy's n^{th} root test
4) Rabee's test
5) Logarithmic test
6) D'morgan's and Bertrand's test after addn

Q) Proof that

Unit - IV

Calculus

1) What are the various mean value theorems, Define them with Geometrical Interpretation

Mean Value theorems

Ans):-

→ The following are the various mean value theorems :-

1) Rolle's Theorem
2) Lagrange's mean value Theorem
3) Cauchy's mean value Theorem
4) Taylor's theorem (or) series.

Rolle's Theorem

Statement:- let $f(x)$ be the function.

(i) It is continuous in closed interval $[a,b]$

(ii) It is differentiable in open interval (a,b) and

(iii) $f(a) = f(b)$

Then there exists at least one point 'c' in open (a,b), such that $f'(c) = 0$.

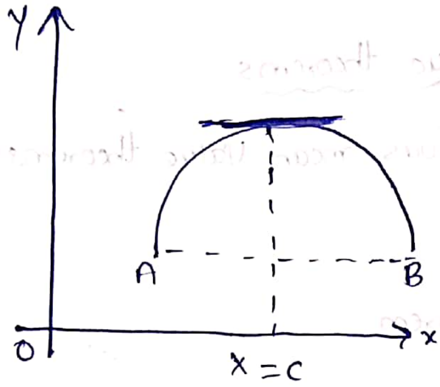
Geometrical interpretation

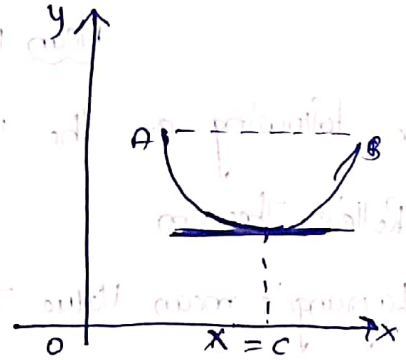
geometrically, the three assumptions in Rolle's theorem mean the following:

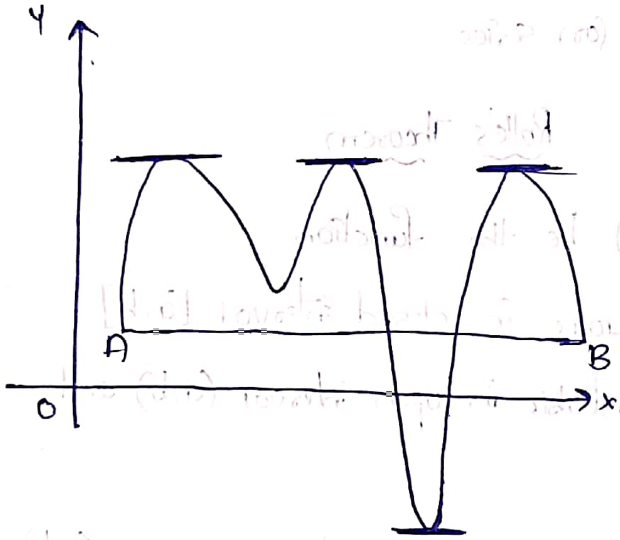
(i) The curve $y = f(x)$ is continuous in the closed interval $[a,b]$.

(ii) At every point $x = c$ where $a < c < b$, at the point $(c, f(c))$ on the curve $y = f(x)$, there is a unique tangent to the curve.

(iii) $f(a) = f(b)$ (i.e) the two end points of the curve $y = f(x)$ corresponding to $x = a$, $x = b$ have the same ordinate. (i.e) they are at the same height (or) depth with reference to the axis.







2) Lagrange's mean Value theorem

statement: let $f(x)$ be a function such that

(i) It is continuous in closed interval $[a, b]$ and

(ii) Differentiable in open interval (a, b).

Then there exists at least one point 'c' in open interval (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

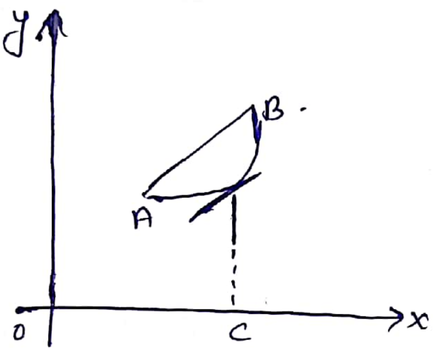
Geometrical Interpretation:-

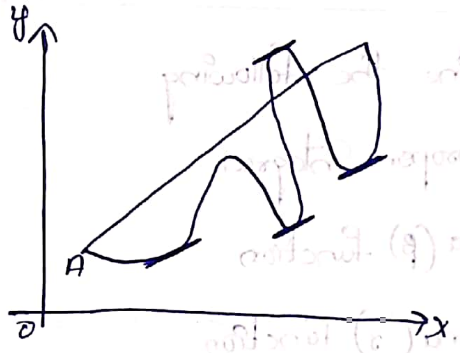
Geometrically, the two assumptions of Lagrange's mean value theorem mean the following.

(i) The curve $y = f(x)$ is continuous in closed interval $[a,b]$.

(ii) At every point $x = c$, where $a<c<b$, at the point $(c, f(c))$ on the curve $y = f(x)$, there is a unique tangent to the curve.

→ $f'(c)$ is equal to the slope of the chord $AB = \frac{f(b)-f(a)}{b-a}$





3) Cauchy's mean value theorem

statement:- If $f:[a,b] \to R$, $g:[a,b] \to R$ are such that

(i) f, g are continuous on $[a,b]$

(ii) f, g are differentiable on (a,b) and

(iii) $g'(x) \neq 0 \ \forall \ x \in (a,b)$

then, there exists a point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(b)-f(a)}{b-a}$$

4) Taylor's theorem (or) series

statement:- If $f:[a,b] \to R$ is such that

(a) $f^{(n-1)}$ is continuous on $[a,b]$

(b) $f^{(n-1)}$ is derivable on (a,b) (or) $f^{(n)}$ exists on (a,b) and $p \in Z^+$ then there exists a point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where,

$$R_n = \frac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\, p}$$

Most important

2) Define the following

a) Improper Integral

b) Beta (β) function

c) Gamma (γ) Function

Ans) a) Improper Integral:-

statement:-

Consider the integral $\int_a^b f(x)\,dx$. Such an integral, for which

(i) Either the interval of integration is not finite, i.e; $a = -\infty$ (or) $b = \infty$ (or) both.

(or)

(ii) The function $f(x)$ is unbounded at one (or) more points in $[a,b]$ is called an improper integral.

Examples:-

1) $\int_0^\infty \frac{dx}{1+x^4}$ and $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ are improper integrals of the first kind.

2) $\int_0^1 \frac{dx}{1-x^2}$ is are Improper Integrals of second kind

3) $\int_0^{-\infty} e^{-x} x^{n-1} dx$ when $n > 0$ is are Improper Integral of third kind.

b) Beta Function:-

Beta function The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is called the Beta function.

→ It is denoted by $B(m,n)$ (or) Beta m,n.

c) Gamma Function:-

The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the gamma function

→ It is denoted by $\Gamma(n)$ (or) Gamma 'n'.

→ It is called as Eulerian integral of the 2nd kind.

---

3) Explain the applications of definite integrals to evaluate surface areas and volumes of revolutions of curves.

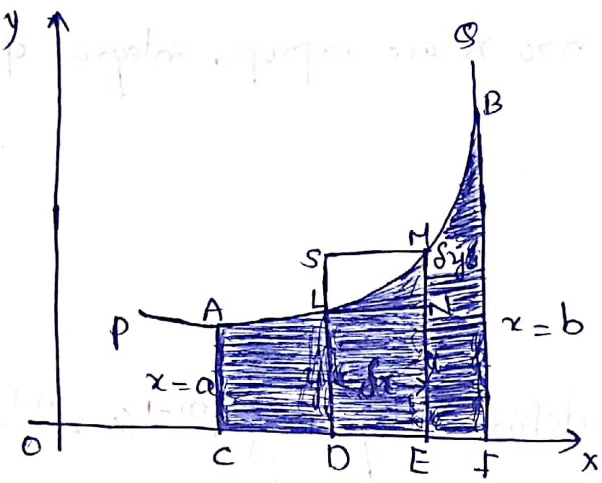
Ans):- Surface Areas of revolution:- (Cartesian form (or) Coordinates).

(i) Revolution about the x-axis

Theorem:- The surface area of the solid generated by the revolution about the x-axis of the area bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$, $x=b$ is

$$\int_a^b 2\pi y\, ds \quad (\text{or}) \int_a^b 2\pi y \frac{ds}{dx} . dx$$

Proof:-



Thus, the surface area of the solid generated by the revolution about x-axis of the arc of the curve $y = f(x)$, between two ordinates $x = a$ and $x = b$ is given by

$$S = \int_a^b 2\pi y \, ds = \int_a^b 2\pi y \frac{ds}{dx} dx$$

$$= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

(ii) Revolution about the y-axis

$$S = \int_c^d 2\pi x \, ds$$

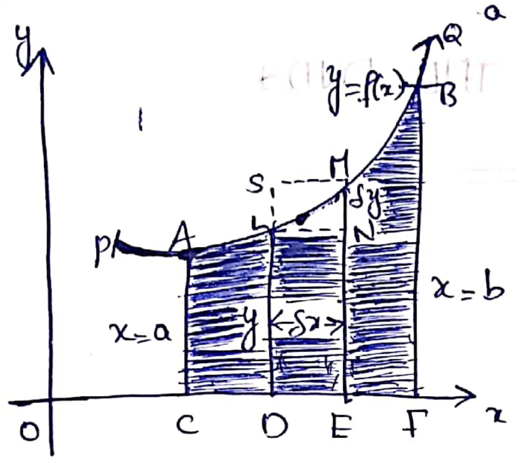
$$= \int_c^d 2\pi x \cdot \frac{ds}{dy} \cdot dy$$

$$= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy .$$

Volumes of Solids of revolution (Cartesian form (or) Co-ordinates)

(i) Volume of the solid about x-axis :-

The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a, x = b$ is given by $\int_a^b \pi y^2 dx$.



The volume of the solid generated by revolving the area bounded by the curve $y = f(x)$, x-axis and the co-ordinates $x=a$ and $x=b$ is $\int_a^b \pi y^2 dx$

(ii) Volume of the solid about y-axis :- let $x = f(y)$ be the curve. The volume of the solid generated by the revolution about y-axis of the area bounded by the curve $x = f(y)$. the y-axis and the abscissae $y = a, y = b$ is given by $\int_a^b \pi x^2 dy$.

(iii) Volume Bounded by two Curves :-

(iii)(a) Volume of the solid generated by the revolution of the area bounded by the curves $y = y_1(x), y = y_2(x)$ and the ordinates $x = a$, $x = b$ about x-axis is $\int_b^a \pi(y_1^2 - y_2^2)dx$.

where, y_1 and y_2 are the ordinates of the upper and lower curves.

$$= \int_{x=a}^{b} \pi y_1^2 dx - \int_a^b \pi y_2^2 dx = \int_a^b \pi (y_1^2 - y_2^2) dx.$$

(b) Volume of the solid generated by the curves $x = x_1(y)$, $x = x_2(y)$ and the abscissae $y = a$, $y = b$ about y-axis is $\int_b^a \pi (x_1^2 - x_2^2) dy$.

THE END

Multivariable Calculus (partial Differentiation and applications):

1) Define Limit and Continuity

Ans) a) * Limit of a function of two Variable :-

A function $f(x,y)$ is said to be a limit at a point (a,b), if Corresponding to any given positive number 'ε' there exists a positive number 'δ' such that $|f(x,y)-l|<\varepsilon$ for all points (x,y) whenever $|x-a|\le\delta, |y-b|\le\delta$.

i.e., $\lim_{\substack{x\to a\\ y\to b}} (x,y) = l$ (or) $\lim_{(x,y)\to(a,b)} f(x,y)=l$.

it may be noted that

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \Rightarrow \lim_{x\to a} (x,b)=l = \lim_{y\to b} f(a,y).$$

Continuity of a function of two variables at a point:-

A function $f(x,y)$ is said to be a continuous at a point (a,b), if Corresponding to any given positive number 'ε' there Exists a positive number 'δ' such that $|f(x,y)-f(a,b)|<\varepsilon$

i.e., $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

→ A function which is not Continuous is said to be discontinuous.

---

2) state and prove:-

a) Euler's Theorem

b) Jacobian

c) Functional Dependence.

d)

Ans):-

a) Euler's Theorem

statement:- The Euler's theorem on homogeneous function of degree 'n' is given by

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

Proof:

If $z=f(x,y)$ is homogeneous function of degree 'n' in x and y then

$$z=x^n g\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

Now, ~~differentiate~~ Differentiate (1) partially w.r.t x and y

$$\Rightarrow \frac{\partial z}{\partial x}=nx^{n-1} g\left(\frac{y}{x}\right)+x^n\cdot g'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) \quad \text{--- (2)}$$

and $$\frac{\partial z}{\partial y}=x^n g'\left(\frac{y}{x}\right)\cdot\frac{1}{x}=x^{n-1} g'\left(\frac{y}{x}\right) \quad \text{--- (3)}.$$

Now, multiplying eq^n (2) & (3); w.r.t x and y

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nx^n g\left(\frac{y}{x}\right)=nz$$

Similarly, multiplying eq^n (2) & (3) w.r.t x, y and z

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu.$$

b) Jacobian

statement:- If $u=u(x,y)$ and $v=v(x,y)$ then, the determinant are

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ (or) } \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

is called as Jacobian of u,v w.r.t x,y (or) the Jacobian of the transformation.

The above determinant value is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

Proof:-

Consider, the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ (or) } \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

The determinant value is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\left(\frac{u,v}{x,y}\right).$$

→ The Jacobian is a 2nd order functional determinant is given by

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ (or) } \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

If $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$, then the Jacobian of u, v, w with respect to x, y, z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \text{ (or) } \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

→ In the same way we define Jacobian of a 'n' independent variables.

c) Functional Dependence

Statement:- If the functions 'u' and 'v' of the independent variables x and y are functionally dependent then the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ vanishes.

Proof:- Consider $F(u,v) = 0$.

Differentiating $F(u,v) = 0$ partially w.r.t x and y, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0.$$

A non-trivial solution $F_u \neq 0$, $F_v \neq 0$ to this system exists if the coefficient determinant is zero.

$$\Rightarrow \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0 \text{ (i.e) } \frac{\partial(u,v)}{\partial(x,y)} = 0. \text{ (for } u, v \text{ functional)}$$

Similary, for u,v,w functions the Equation is given by

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_w & w_z \end{vmatrix} = 0$$

If the Jacobian is not Equal to zero, then the functions, u,v,w are said to be functionally independent.

---

3) Define maximum and minimum of functions of two variables with Extreme value

Ans):- Defination:- let $f(x,y)$ be a function of two variables x and y.

At $x=a$; $y=b$, $f(x,y)$ is said to have maximum (or) minimum value, if $f(a,b) > f(a+h, b+k)$ or $f(a,b) < f(a+h, b+k)$ respectively where 'h' and 'k' are small values.

Extreme value:- $f(a,b)$ is said to be an Extreme value of f, if it is a maximum (or) minimum value.

(1) The necessary Conditions for (x,y) to have a maximum (or) minimum at (a,b) are

$$f_x(a,b) = 0; \; f_y(a,b) = 0.$$

(2) sufficient conditions: Suppose that $f_x(a,b)=0$; $f_y(a,b)=0$ and let

$$\frac{\partial^2}{\partial x^2} f(a,b) = l; \; \frac{\partial^2}{\partial x \partial y} f(a,b) = m; \; \frac{\partial^2}{\partial y^2} f(a,b) = n.$$

Then

(i) $f(a,b)$ is a maximum value if $ln - m^2 > 0$ and $l < 0$.

(ii) $f(a,b)$ is a minimum value if $ln-m^2>0$ and $l>0$

(iii) $f(a,b)$ is not an Extreme value if $ln-m^2<0$.

(iv) If $ln-m^2=0$ then $f(x,y)$ fails to have maximum (or) minimum value and it needs further investigation.

4) Explain Lagrange multipliers method for 3 variables

Ans):- Lagrange's method of undetermined multipliers:

→ we are required to find the extremum of a function subject to some other conditions involving the variables.

→ such type of problems can be solving using the method of Lagrange's undetermined multipliers.

→ Let $f(x,y,z)$ be a function of three Independent variables x,y,z where x,y,z are connected by the relation:-

$$\phi(x,y,z)=0 \quad \text{—— (1)}$$

Since the stationary values occur when $\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0$, we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \quad \text{—— (2)}$$

By total differentiation of (1), we get

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \quad \text{—— (3)}$$

multiplying (3) by 'λ', and adding to (2), we get

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)dz=0 \quad \text{--- (4)}$$

The Equation (4) will hold good if

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \quad \text{--- (5)}$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0. \quad \text{--- (6)}$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \quad \text{--- (7)}$$

solving the four Equations (1), (5), (6), (7) we can find the values of x, y, z and λ for which $f(x,y,z)$ has stationary value.

* THE END *

Problems

+

Theorems

Unit - I

Matrices (problems)

1) Evaluate $A^2-3A+9I$ where $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$ and I is a unit matrix.

Sol):- given, $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$

let, $I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Now, $A^2 = A\times A = \begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}\times\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$

$$=\begin{bmatrix}1\times1-(2\times2)+3(-3) & 1(-2)-2(3)+3(1) & 1\times3+(-2)(-1)+3(2)\\ 2\times1+3\times2+(-1)(-3) & 2(-2)+3(3)+(-1)(1) & 2\times3+3(-1)+(-1)(2)\\ (-3)\times1+1\times2+2(-3) & (-3)(-2)+1(3)+2(1) & (-3)(3)+1(-1)+2\times2\end{bmatrix}$$

$$=\begin{bmatrix}1-4-9 & -2-6+3 & 3+2+6\\ 2+6+3 & -4+9-1 & 6-3-2\\ -3+2-6 & 6+3+2 & -9-1+4\end{bmatrix}$$

$$= \begin{bmatrix} 1-13 & 3-8 & 11 \\ 11 & 9-5 & 6-5 \\ 2-9 & 11 & 4-10 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$\therefore A^2 - 3A + 9I$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - 3\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -5 & 11 \\ 11 & 13 & 1 \\ -7 & 11 & 3 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$$ Ans.

2) Express the matrix 'A' as sum of symmetric and skew-symmetric matrices, where $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

Sol:- Given $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$$

Now $A + A^T = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix}$$

Let symmetric matrix $(P) = \frac{1}{2}(A + A^T)$

$$= \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{6/2}\ 3 & 0/2 & 11/2 \\ 0/2 & \cancel{14/2}\ 7 & 3/2 \\ 11/2 & 3/2 & 0/2 \end{bmatrix}$$

$$P = \begin{bmatrix} 3 & 0 & \frac{11}{2} \\ 0 & 7 & 3/2 \\ \frac{11}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

~~lly, skew-symmetric matrix (Q) = A - A~~

Now, $A - A^T = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$

$$A - A^T = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & 5 \\ -1 & 5 & 0 \end{bmatrix}$$

Let skew-symmetric matrix $(Q) = \frac{1}{2}(A - A^T)$

$$= \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & 5 \\ -1 & 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0}{2} & \frac{-4}{2} & \frac{1}{2} \\ \frac{4}{2} & \frac{0}{2} & \frac{5}{2} \\ \frac{-1}{2} & \frac{5}{2} & \frac{0}{2} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 2 & 0 & \frac{5}{2} \\ \frac{-1}{2} & \frac{5}{2} & 0 \end{bmatrix}$$

Hence $A = P + Q$, which is sum of a symmetric & skew-symmetric matrix.

= (Ans)

3). Define adjoint of a matrix & hence find A^{-1} by using adjoint of A where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (42)

Sol:-

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$\text{Det } A = 1(3(-4) - (-4)(-3)) - 1(1(-4) - (-2)(-3)) + 3(1(-4) - (-2)(3))$

$= 1(-12-12) - 1(-4 - 6) + 3(-4+6)$

$= 1(-24) - 1(-10) + 3(2)$

$= -24 + 10 + 6$

$= -24 + 16$

$= -8 \neq 0$

$\therefore$ A is non-singular, so, A^{-1} exists.

Calculation of cofactors:-

For Row-1:-

Cofactor of $1 = (-1)^{1+1} \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} = 3(-4) - (-4)(-3)$

$= -12 - 12$

$= -24$

Cofactor of $1 = (-1)^{1+2} \begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} = -[-4 - 6]$

$= 4 + 6 = 10$

Cofactor of $3 = (-1)^{1+3} \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -(-4+6) = 2$

Row - (2) :-

Cofactor of $1 = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ -4 & -4 \end{vmatrix} = -(-4+12)$
$= -8.$

Cofactor of $3 = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -(-4+6)$
$= 2.$

Cofactor of $-3 = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = -1(-4+2)$
$= 2.$

Row - (3) :-

Cofactor of $-2 = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 3 & -3 \end{vmatrix} = 1(-3-9)$
$= -12.$

Cofactor of $-4 = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = 1(-3-3)$
$= 6.$

Cofactor of $-4 = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 1(3-1)$
$= 2.$

The matrix formed by the cofactors of elements of A is $B = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}$

$$\therefore \text{Adj } A = B^T = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Hence; $A^{-1} = \dfrac{\text{adj} A}{\det A}$

$$= \begin{bmatrix} \dfrac{\cancel{+24}^{3}}{\cancel{+8}_{1}} & \dfrac{\cancel{+8}^{1}}{\cancel{+8}} & \dfrac{\cancel{+12}^{3}}{\cancel{+8}_{2}} \\ \dfrac{10}{-8} & \dfrac{2}{-8} & \dfrac{6}{-8} \\ \dfrac{2}{-8} & \dfrac{2}{-8} & \dfrac{2}{-8} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 1 & 3 \\ \dfrac{-5}{4} & \dfrac{-1}{4} & \dfrac{-3}{4} \\ -\dfrac{1}{4} & -\dfrac{1}{4} & -\dfrac{1}{4} \end{bmatrix}$$

= (Ans)

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(4) Prove that the following matrix are orthogonal.

(i) $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$ (iii) $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

Sol:- (i) given that, $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$

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then, $A^T = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$.

$$A \cdot A^T = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2+(-3)(-3)+1(1) & 2\times4+(-3)3+1\times1 & 2(-3)+-3(1)+1(9) \\ 4\times2+3(-3)+1\times1 & 4\times4+3\times3+1\times1 & 4\times(-3)+3(1)+1\times9 \\ -3\times(2)+1(-3)+9\times1 & -3\times4+1(3)+9\times1 & -3\times(-3)+1\times1+9\times9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9+1 & 8-9+1 & -6+3+9 \\ 8+9+1 & 16+9+1 & -12+3+9 \\ -6+3+9 & -12+3+9 & 9+1+81 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 1+8-9 & 12-6 \\ 0 & 17+9 & 12-12 \\ 12-6 & 12-12 & 91 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 & 6 \\ 0 & 26 & 0 \\ 6 & 0 & 91 \end{bmatrix} \neq I_3$$

Hence, the matrix 'A' is not orthogonal $\Rightarrow$ (Ans)

(iii)
$$\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}.$$

Sol:-

Given that, $A = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$.

then, $A^T = \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \times \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 \times -2/3 + 1/3 \times 1/3 + 2/3 \times 2/3 & \frac{-2}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} & \frac{-2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{-2}{3} + \frac{2}{3} \times \frac{2}{3} \\ \frac{2}{3} \times \frac{-2}{3} + \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} & \frac{2}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{3} & \frac{2}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{-2}{3} + \frac{1}{3} \times 2/3 \\ \frac{1}{3} \times \frac{-2}{3} + \left(\frac{-2}{3}\right) \times \left(\frac{1}{3}\right) + \frac{2}{3} \times \frac{2}{3} & \frac{1}{3} \times \frac{2}{3} + \left(\frac{-2}{3}\right) \times \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right) \times \left(\frac{1}{3}\right) & \frac{1}{3} \times \frac{1}{3} + \left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right) + \frac{2}{3} \times \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{9}+\frac{1}{9}+\frac{4}{9} & \frac{-4}{9}+\frac{2}{9}+\frac{2}{9} & \frac{-2}{9}-\frac{2}{9}+\frac{4}{9} \\ \frac{-4}{9}+\frac{2}{9}+\frac{2}{9} & \frac{4}{9}+\frac{4}{9}+\frac{1}{9} & \frac{2}{9}-\frac{4}{9}+\frac{2}{9} \\ \frac{-2}{9}-\frac{2}{9}+\frac{4}{9} & \frac{2}{9}-\frac{4}{9}+\frac{2}{9} & \frac{1}{9}+\frac{4}{9}+\frac{4}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{9} & \frac{4}{9}-\frac{4}{9} & \frac{4}{9}-\frac{4}{9} \\ \frac{4}{9}-\frac{4}{9} & \frac{8}{9}+\frac{1}{9} & \frac{2}{9}-\frac{2}{9} \\ \frac{4}{9}-\frac{4}{9} & \frac{4}{9}-\frac{4}{9} & \frac{8}{9}+\frac{1}{9} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence, the matrix 'A' is orthogonal. (Ans)

(iii) given that,

$$A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Then, $A^T = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

$$A\cdot A^T=\frac{1}{2}\begin{bmatrix}-1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{bmatrix}\times\frac{1}{2}\begin{bmatrix}-1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}-1\times-1+1\times1+1\times1+1\times1 & -1\times1+1(-1)+1\times1+1\times1 & -1\times1+1\times1+1(-1)+1\times1 & -1\times1+1\times1+1\times1+1\times-1\\ 1\times-1+(-1)\times(1)+1\times1+1\times1 & 1\times1+(-1)\times(-1)+1\times1+1\times1 & 1\times1+(-1)\times1+1\times(-1)+1\times1 & 1\times1+(-1)\times1+1\times1+1(-1)\\ 1\times(-1)+1\times1+(-1)\times1\times1\times1 & 1\times1+(1)\times(-1)+(-1)\times(1)+1\times1 & 1\times1+1\times1+(-1)\times(-1)+1\times1 & 1\times1+1\times1+(-1)\times1+1\times(-1)\\ 1\times(-1)+1\times1+1\times1+(-1)\times1 & 1\times1+(-1)\times(-1)+1\times1+(-1)\times1 & 1\times1+1\times1+1\times(-1)+(-1)\times1 & 1\times1+1\times1+1\times1+1(-1)\times(-1)\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}1+1+1+1 & \cancel{-1-1+1+1} & -1+1\cancel{-1}+1 & -1+1+1-1\\ -1-1+1+1 & 1+1+1+1 & 1-1-1+1 & 1-1+1-1\\ -1+1-1+1 & 1-1-1+1 & 1+1+1+1 & 1+1-1-1\\ -1+1+1-1 & 1-1+1-1 & 1+1-1-1 & 1+1+1+1\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}4&0&0&0\\0&4&0&0\\0&0&4&0\\0&0&0&4\end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{4} & \frac{0}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{4}{4} & \frac{0}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{4}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{0}{4} & \frac{0}{4} & \frac{4}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4.$$

∴ Hence, the matrix 'A' is orthogonal. (Ans).

Imp

(5) Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix

Sol:-

Given, $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

then, $A^T = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}$

$\therefore A^{\theta} = (\bar{A}^T) = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$A \times A^{\theta} = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \times \begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 1+i \times 1-i \; -1+i \times -1-i & 1+i \times 1-i \; -1+i \times 1+i \\ 1+i \times 1-i + 1-i \times -1-i & 1+i \times 1-i + 1-i \times 1+i \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} \frac{4}{4} & \frac{0}{4} \\ \frac{0}{4} & \frac{4}{4} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore AA^{\theta} = I$, Hence A is unitary matrix

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6. show that $A = \begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}$ is unitary if $a^2+b^2+c^2+d^2=1$

Sol:-

given that, $A = \begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}$

then, $A^T = \begin{pmatrix} a+ic & b+id \\ -b+id & a-ic \end{pmatrix}$

$\therefore A^\theta = (\overline{A^T}) = \begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix}$

Now, $A \cdot A^\theta = \begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix} \begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix}$

$= \left(\cancel{a+ic \quad a-ic - b+id \quad -b-id} \right.$

$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$

$\therefore AA^\theta = I$, if & only if $a^2+b^2+c^2+d^2 = 1$.

i.e A is unitary if & only if $a^2+b^2+c^2+d^2 = 1$.

= (Ans)

---

(7). Prove that every Hermitian matrix can be written as $A+iB$, where real and symmetric and 'B' is real and skew-symmetric

Sol:- Let 'c' be a Hermitian matrix.

Then $c^\theta = c \longrightarrow (1)$

Consider $A = \frac{1}{2}(c+\bar{c})$

$B = \frac{1}{2i}(C - \bar{C})$.

$C = A + iB$ $\left[C = \frac{1}{2}(C+\bar{C}) + i\left(\frac{1}{2}(C-\bar{C})\right) \right]$

NOW we have to P.T

A is symmetric & B is skew symmetric

$A^T = \frac{1}{2}(C+\bar{C})^T$

$= \frac{1}{2}\left[C^T + (\bar{C})^T\right]$

$= \frac{1}{2}\left[C^T + C^\theta\right]$ $\left[\because (\bar{C})^T = C^\theta\right]$

$= \frac{1}{2}\left[(C^\theta)^T + C\right]$ $(\because$ from eq (1) $C^\theta = C)$

$= \frac{1}{2}\left[\{(\bar{C}^T)\}^T + C\right]$

$= \frac{1}{2}\left[\bar{C} + C\right]$ $\left[\because \{(\bar{C}^T)\}^T = \bar{C}\right]$

$\boxed{A^T = A}$

$\therefore$ A is symmetric.

$B^T = \frac{1}{2i}(C - \bar{C})^T$

$= \frac{1}{2i}\left[C^T - (\bar{C})^T\right]$

$= \frac{1}{2i}\left[C^T - C^\theta\right]$

$$= \frac{1}{2i}\left[(C^{\theta})^T - C\right]$$

$$= \frac{1}{2i}\left[\{(\bar{C})^T\}^T - C\right]$$

$$= \frac{1}{2i}(\bar{C} - C)$$

$$= -\frac{1}{2i}(C - \bar{C})$$

$$\boxed{B^T = -B}$$

$\therefore$ B is skew-symmetric

(Ans)

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8) find the rank of the following matrices

a) $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}_{3\times3}$, b) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Sol:-)

a) given, $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

$$\det A = -1(6\times3 - 1\times1) - 0(3\times3 - (-5)\times1) + 6(3\times1 ...$$

$$= -1(18-1) - 0(9+5) + 6(3+30)$$

$$= -1(17) - 0(14) + 6(33)$$

$$= -17 - 0 \cancel{+84} + 198$$

$= 198 - 17$

$= 181 \neq 0$

$\therefore$ Since the order $3 \neq 0$.

$\therefore$ Rank $\rho(A) = 3$. (Ans).

b) given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$\det A = 1(4\times6 - 5\times5) - 2(3\times6 - 4\times5) + 3(3\times5 - 4\times5)$.

$= 1(24 - 25) - 2(18 - 20) + 3(15 - 16)$

$= 1(-1) - 2(-2) + 3(-1)$

$= -1 + 4 - 3$

$= 4 - 4$

$= 0 = 0$

$\therefore$ Rank$(\rho)A \neq 3$

$\therefore$ so it must be less than 3.

Consider 2×2 matrix $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ ~~$= 1(1\times4) - 2($~~

$[\because ad - bc]$

$= (1\times4 - 3\times2)$

$= 4 - 6$

$= -2 \neq 0$

$\therefore$ since the order $2 \neq 0$

$\therefore$ Rank $\rho(A) = 2$ (Ans)

c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \end{bmatrix}$

Sol:- Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \end{bmatrix}$

Consider 3x3 matrix $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 6 \end{vmatrix}$

$\det A = 1(6\times6-7\times7)-2(5\times6-7\times8)+3(5\times7-6\times8)$

$= 1(36-49) - 2(30-56) + 3(35-48)$

$= 1(-13) - 2(-26) + 3(-13)$

$= -13+52-39$

$= 52-39-13$

$= 52-52$

$= 0.$ $\therefore$ Rank $\rho(A) \neq 3.$

Consider, 2x2 matrix $\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix}$

$(\because ad-bc)$

$= (1\times6 - 5\times2)$

$= (6-10)$

$= (-4)$

$= -4 \neq 0.$

$\therefore$ Rank $(P) = 2$.

D) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol:-

let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

consider 3x3 matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{vmatrix}$$

$$\det A = 1(6\times0-7\times7)-2(5\times0-7\times8)+3(5\times7-6\times8)$$

$$= 1(0-49) -2(0-56)+3(35-48)$$

$$= 1(-49)-2(-56)+3(-13)$$

$$= -49+112-39$$

$$= 112-39-49$$

$$= 112-88.$$

$$= 24 \neq 0.$$

$\therefore$ Rank $(A) = 3$.
(Ans).

9) Reduce the matrices

(i) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ into Echelon form and Hence find its rank

Sol):-

(i) given, $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form.

$\therefore$ no. of non-zero rows is 3.

$\therefore$ Rank (A) = 3.

=(Ans).

(ii). Given $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form.

$\therefore \text{Rank}(A) = 2$

=(Ans).

(10). Find the rank of

(i). $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ (ii). $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 1 & 2 \\ -3 & -4 & 5 & 8 \\ 1 & 3 & 10 & 14 \end{bmatrix}$

(iii). $\begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

Solution

(i). Given $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 4R_1$

$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - 3R_2$

$$\approx \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in Echelon form

$\therefore$ The no. of non-zero rows $= 2$.

$\therefore$ Rank $(A) = 2$. (Ans).

(ii)

$$\text{Given, } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 1 & 2 \\ -3 & -4 & 5 & 8 \\ 1 & 3 & 10 & 14 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + 3R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 10 \\ 0 & 2 & 14 & 20 \\ 0 & 1 & 7 & 10 \end{bmatrix}$$

$R_3 \to \dfrac{R_3}{2}$.

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 10 \\ 0 & 1 & 7 & 10 \\ 0 & 1 & 7 & 10 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_2$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is an Echelon form

$\therefore$ The no. of non-zero rows $= 2$.

$\therefore$ Rank $(A) = 2$. (Ans)

(iii)

$$\text{given, } A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$.

$$\sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 2 & -4 & 3 & -1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$$

$R_4 \to 9R_4 + 5R_3$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 0 & 18 & -16 \end{bmatrix}$$

This is an Echelon form

$\therefore$ The no. of non zero rows is '4'.

$\therefore$ Rank (A) = 4. // (Ans).

(iv) Given, $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + R_1$

$R_4 \to R_4 + 3R_1$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

$R_2 \to \frac{R_2}{5}$

$R_3 \to \frac{R_3}{10}$

$R_4 \to \frac{R_4}{15}$.

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is an Echelon form

∴ The no. of non-zero rows is '2'

$\therefore$ Rank(A) = 2 // (Ans).

(11) Find the value of 'k' if the Rank of matrix is 2, where (64)

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

Sol:-

(i) Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{vmatrix}$$

$$= 1(10k - 6\times7) - 2(2\times10 - 7\times3) + 3(2\times6 - 3k)$$

$$= 1(10k - 42) - 2(20 - 21) + 3(12 - 3k)$$

$$= 1(10k - 42) - 2(-1) + (36 - 9k)$$

$$= (10k - 42) + 2 + (36 - 9k)$$

$$= 10k - 42 + 2 + 36 - 9k$$

$$= 10k - 9k + 38 - 42$$

$$= k - 4$$

$$\boxed{k = 4}$$

= (Ans).

(ii) given, $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$R_4 \rightarrow R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & k-1 & -1 \end{bmatrix}$$

let, $k - 1 = -3$.

$k = -3 + 1$

$= \cancel{-4}\ -2$

$\boxed{k = -2}$ //.(Ans).

---

(12) Determine the Rank of the matrices

(i) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ By reducing

it to normal form (or) Canonical form

Solutions

(i) given, $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

$C_1 \leftrightarrow C_2$.

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$\longrightarrow R$

$| \longrightarrow C$

$R_2 \rightarrow \frac{R_2}{2}$.

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ C_1 & C_2 & C_3 & C_4 \\ C_1 & C_2 & C_3 & C_4 \\ C_1 & C_2 & C_3 & C_4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 - 2C_1$

$0 - 2(0)$

$C_4 \rightarrow C_4 + 2C_1$

$0 - 0.$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to \frac{C_2}{4}$

$C_4 \to \frac{C_4}{3}.$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2.$

$C_4 \to C_4 - C_2.$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ This is in the form of normal form

Hence Rank $\rho(A) = 2$ (Ans).

(iii) given, $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

$R_1 \longleftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$C_2 \to C_2 + C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_1 \to R_1 + \frac{R_2}{6}$

$\cancel{R_1 \to R_1 + \frac{R_2}{6}}$ $\cancel{C_2 \to C_2 + C_3}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\cancel{R_1 \to R_1 + \frac{R_2}{6}}$

$C_4 \to C_4 - \frac{4}{3} C_1$

$C_4 \to 6C_4 + 10C_2$

$C_4 \to C_4 + 6C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_2 \to \frac{C_2}{6}$, ~~and~~ $R_3 \to R_3 - R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\cong \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ This is in the form of normal form.

Hence, Rank $(A) = 3$ // (Ans).

(iii)

given, $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_1$

$R_4 \rightarrow R_4 - R_1$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 2R_3$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_3$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 2C_3 - 3C_1$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_1 \to \frac{C_1}{2}$

$C_3 \to \frac{C_3}{8}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 4C_1$

$C_4 \to C_4 - C_3$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ This is in the form of normal form.

Hence, Rank(A) = 3 // (Ans).

(iv) given, $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 5 & 3 & 7 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$.

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 8C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to \frac{R_3}{11}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 + 7C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 6C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to \frac{C_3}{3}$

$C_4 \to \frac{C_4}{2}.$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\simeq \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ This is in the form of normal form

Hence, Rank (A) = 3. (Ans).

(v)

given, $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

$R_2 \to 2R_2 - 3R_1$

$R_3 \to 2R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & -13 & -13 \\ 0 & -9 & -9 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{-13}$$

$$R_3 \to \frac{R_3}{-9}$$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - 3R_2$$

$$\sim \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to C_3 - 2C_1$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to C_3 - C_2$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{2}$.

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\simeq \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ This is in the form of normal form

Hence Rank(A) = 2.
(Ans).

---

(13). Obtain non-singular matrices P & Q such that PAQ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Sol:- Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

$$\boxed{A = I_3 A I_3}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \rightarrow C_2 - C_1$

$C_3 \rightarrow C_3 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underset{(P)}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}} \underset{(A)}{A} \underset{(Q)}{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

This is of the form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \;\&\; Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\det P = 1 \neq 0$; $\det Q = 1 \neq 0$

$\therefore$ P & Q are non-singular matrix such that

$$PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Rank(A) = 2

(Ans).

(4). Find the non-singular matrices P & Q such that PAQ is in normal form where $A=\begin{bmatrix}3 & 2 & -1 & 5\\5 & 1 & 4 & -2\\1 & -4 & 11 & -19\end{bmatrix}$ (77)

Sol:- Given $A=\begin{bmatrix}3 & 2 & -1 & 5\\5 & 1 & 4 & -2\\1 & -4 & 11 & -19\end{bmatrix}$

$R \to$ —

$C \to |$

$$\boxed{A = I_3 A I_4}$$

$$\begin{bmatrix}3 & 2 & -1 & 5\\5 & 1 & 4 & -2\\1 & -4 & 11 & -19\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix}1 & -4 & 11 & -19\\5 & 1 & 4 & -2\\3 & 2 & -1 & 5\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$C_2 \to C_2 + 4C_1$

$C_3 \to C_3 - 11C_1$

$C_4 \to C_4 + 19C_1$

$$\begin{bmatrix}1 & 0 & 0 & 0\\5 & 21 & -51 & 93\\3 & 14 & -34 & 62\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix} A \begin{bmatrix}1 & 4 & -11 & 19\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$R_2 \to R_2 - 5R_1$

$R_3 \to R_3 - 3R_1$

$C_2 \to \frac{C_2}{7}, \frac{C_3}{-17}, \frac{C_4}{31}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 \\ 0 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 4/7 & 11/17 & 19/31 \\ 0 & \frac{1}{7} & 0 & 0 \\ 0 & 0 & \frac{-1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$

$R_2 \to \frac{R_2}{3}$

$R_3 \to \frac{R_3}{2}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1/2 & 0 & -3/2 \end{bmatrix} A \begin{bmatrix} 1 & 4/7 & 11/17 & 19/31 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1/31 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$C_4 \to C_4 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1/2 & 0 & -3/2 \end{bmatrix} A \begin{bmatrix} 1 & 4/7 & 9/119 & 9/217 \\ 0 & 1/7 & -1/7 & -1/7 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1/31 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1/2 & -1/3 & 1/6 \end{bmatrix} A \begin{bmatrix} 1 & 4/17 & 9/119 & 2/217 \\ 0 & 1/7 & -1/7 & -1/7 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1/31 \end{bmatrix}$$

This is of the form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

where $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & -5/3 \\ 1/2 & -1/3 & 1/6 \end{bmatrix}$ & $Q = \begin{bmatrix} 1 & 4/17 & 9/119 & 9/217 \\ 0 & 1/7 & -1/7 & -1/7 \\ 0 & 0 & -1/17 & 0 \\ 0 & 0 & 0 & 1/31 \end{bmatrix}$

$\therefore$ P & Q are non-singular such that

$$PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank(A) = 2

=(Ans)

(15). Find the inverse of (i). $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

(ii). $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ By using elementary row operations (or) Gauss-Jordan method.

Soln's

(i). Given $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

$$\boxed{A = I_3 A}$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \longleftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} A$$

$R_3 \rightarrow R_3 + 5R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_1 \rightarrow R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} A$$

$R_3 \rightarrow \frac{R_3}{8}$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix} A$$

$R_1 \rightarrow R_1 - 2R_3$

$R_2 \rightarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{8} & \frac{6}{8} & \frac{4}{8} \\ \frac{-1}{8} & \frac{3}{8} & \frac{-2}{8} \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{bmatrix} A$$

$R_2 \rightarrow \frac{R_2}{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2/8 & 6/8 & 4/8 \\ 1/8 & -3/8 & 2/8 \\ 1/8 & 5/8 & 2/8 \end{bmatrix} A$$

$$= \frac{1}{8}\begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix} A$$

This is of the form $I_3 = BA$

where $A^{-1} = B = \frac{1}{8}\begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$.

= (Ans).

(ii). Given $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$.

$$\boxed{A = I_4 A}$$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + 2R_1$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to 2R_1 - 3R_2$

$R_3 \to 2R_3 - 11R_2$

$R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_4$

$R_3 \to R_3 + 6R_4$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -1 & -3 & 0 & -2 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 & -6 \\ -2 & -2 & 2 & 4 \\ -1 & 1 & 2 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to \frac{R_1}{-2}$

$R_2 \to \frac{R_2}{-2}$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

This is of the form $I_4 = BA$

$$A^{-1} = B = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$ (Ans).

(16). Write the following equ in matrix form $AX=B$ (83)

& solve for X by finding A^{-1}

$x+y-2z=3, 2x-y+z=0, 3x+y-z=8.$

Sol:- Given equ

$$x+y-2z=3$$
$$2x-y+z=0$$
$$3x+y-z=8$$

Let $\boxed{AX=B}$

$$A=\begin{bmatrix}1 & 1 & -2\\ 2 & -1 & 1\\ 3 & 1 & -1\end{bmatrix};\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix};\ B=\begin{bmatrix}3\\ 0\\ 8\end{bmatrix}$$

Let $\boxed{A=I_3 A}$

$$\begin{bmatrix}1 & 1 & -2\\ 2 & -1 & 1\\ 3 & 1 & -1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & -2\\ 2 & -1 & 1\\ 3 & 1 & -1\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix}1 & 1 & -2\\ 0 & -3 & 5\\ 0 & -2 & 5\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}A$$

$R_1 \to 3R_1 + R_2$

$R_3 \to 3R_3 - 2R_2$

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & -3 & 5 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & -2 & 3 \end{bmatrix} A$$

$R_1 \to 5R_1 + R_3$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 15 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 3 & -3 \\ -5 & -2 & 3 \end{bmatrix} A$$

$\frac{R_1}{15}, \frac{R_2}{-3}, \frac{R_3}{5}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/5 & 1/5 \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix} A$$

This is of the form $I_3 = CA$

$$A^{-1} = C = \begin{bmatrix} 0 & 1/5 & 1/5 \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix}$$

Hence $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1/5 & 1/5 \\ -1 & -1 & 1 \\ -1 & -2/5 & 3/5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 5 \\ 9/5 \end{bmatrix}$$

$\therefore$ The sol^n is $x = \frac{8}{5}, y = 5, z = \frac{9}{5}$.

= (Ans)

(17). Solve the system of linear equ by matrix method.

$x+y+z=6$, $2x+3y-2z=2$, $5x+y+2z=13$.

Sol:- Let $A=\begin{bmatrix}1&1&1\\2&3&-2\\5&1&2\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}6\\2\\13\end{bmatrix}$

$$AX=B$$

$$\begin{bmatrix}1&1&1\\2&3&-2\\5&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\2\\13\end{bmatrix}$$

$R_2 \to R_2-2R_1$

$R_3 \to R_3-5R_1$

$$\begin{bmatrix}1&1&1\\0&1&-4\\0&-4&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-10\\-17\end{bmatrix}$$

$R_1 \to R_1-R_2$

$R_3 \to R_3+4R_2$

$$\begin{bmatrix}1&0&5\\0&1&-4\\0&0&-19\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-10\\-57\end{bmatrix}$$

$R_3 \to \frac{R_3}{-19}$

$$\begin{bmatrix}1&0&5\\0&1&-4\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-10\\3\end{bmatrix}$$

$R_1 \to R_1-5R_3$

$R_2 \to R_2+4R_3$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Hence the solⁿ is $x=1, y=2, z=3$ =(Ans)

(18). Find whether the following equ's are consistent, if so solve them:-

(i). $x+y+2z=4,\ 2x-y+3z=9,\ 3x-y-z=2$

(ii). $2x-y+z=5,\ 3x+y-2z=-2,\ x-3y-z=2$.

(iii). $x+2y+2z=2,\ 3x-2y-z=5,\ 2x-5y+3z=-4,\ x+4y+6z=0$

(iv). $3x+3y+2z=1,\ x+2y=4,\ 10y+3z=-2,\ 2x-3y-z=5$.

solⁿ's

(i).

sol:- Let given equ's are:-

$x+y+2z=4$

$2x-y+3z=9$

$3x-y-z=2$

let $AX=B$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

Now Augmented Matrix $[A/B]=\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

$R_2 \to R_2-2R_1$

$R_3 \to R_3-3R_1$

$$[A/B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

$R_3 \to 3R_3 - 4R_2$

$$[A/B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

No. of non-zero rows = 3

Since Rank of $A = 3$

Rank of $[A/B] = 3$

$\therefore$ Rank of A = Rank of $[A/B]$

The given system is consistent, so it has a solution

Since Rank of A = Rank of $[A/B]$ = no. of unknowns

$\therefore$ The given system has a unique sol^n.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

So, $x + y + 2z = 4 \to (1)$

$-3y - z = 1 \to (2)$

$\cancel{+}17z = \cancel{+}34$

$17z = 34$

$z = \frac{34}{17}$

$\boxed{z = 2} \to (3)$

Now sub z in eq (2)

(2) $\Rightarrow -3y - 2 = 1$

$-3y = 1 + 2$

~~$+3y = +1$~~ $-\cancel{3}y = \cancel{3}$

$\boxed{y = -1}$

Now sub z, y in eq (1)

$$(1) \Rightarrow x - 1 + 2(2) = 4$$

$$x - 1 + 4 = 4$$

$$x + 3 = 4$$

$$x = 4 - 3$$

$$\boxed{x = 1}$$

$\therefore x = 1,\ y = -1,\ z = 2$ is the sol^n.

⇐ (Ans).

(ii). Given equ's.

$$2x - y + z = 5$$

$$3x + y - 2z = -2$$

$$x - 3y - z = 2$$

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -2 \\ 1 & -3 & -1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$

Let $AX = B$

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -2 \\ 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

Now $[A|B] = \begin{bmatrix} 2 & -1 & 1 & 5 \\ 3 & 1 & -2 & -2 \\ 1 & -3 & -1 & 2 \end{bmatrix}$

$R_2 \rightarrow 2R_2 - 3R_1$

$R_3 \rightarrow 2R_3 - R_1$

$$\sim \begin{bmatrix} 2 & -1 & 1 & 5 \\ 0 & 5 & -7 & -19 \\ 0 & -5 & -3 & -1 \end{bmatrix}$$

$R_1 \to 5R_1 + R_2$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 10 & 0 & -2 & 6 \\ 0 & 5 & -7 & -19 \\ 0 & 0 & -10 & -20 \end{bmatrix}$$

No. of non-zero rows = 3.

then, Rank(A) = 3

Rank[A|B] = 3

$\therefore$ Rank(A) = Rank [A|B]

The given system is consistent, so it has a solⁿ

$\therefore$ The given system has a unique solⁿ.

$$\begin{bmatrix} 10 & 0 & -2 \\ 0 & 5 & -7 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -19 \\ -20 \end{bmatrix}$$

so, $10x - 2z = 6 \longrightarrow (1)$

$5y - 7z = -19 \longrightarrow (2)$

$\cancel{-}10z = \cancel{-}20$

$\boxed{z = 2} \longrightarrow (3)$

sub z in eq (2)

② $\Rightarrow 5y - 7(2) = -19$

$5y - 14 = -19$

$5y = -19 + 14$

$5y = -5$

$\boxed{y = -1}$

Now, sub z, y in eq(1)

$$① \Rightarrow 10x - 2(2) = 6$$

$$10x - 4 = 6$$

$$10x = 6 + 4$$

$$\cancel{10}x = \cancel{10}$$

$$\boxed{x = 1}$$

$\therefore x = 1, y = -1, z = 2$ is the unique soln. =(Ans).

(iii). Let given equ's are:-

$$x + 2y + 2z = 2$$

$$3x - 2y - z = 5$$

$$2x - 5y + 3z = -4$$

$$x + 4y + 6z = 0.$$

Consider $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$$

Now $[A|B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - R_1$

$$[A/B] \sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$R_3 \to 8R_3 - 9R_2$

$R_4 \to 4R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

$\frac{R_3}{55}, \frac{R_4}{9}$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of non-zero rows $= 3$.

Rank $(A) = 3$

Rank $[A/B] = 3$

$\therefore$ Rank$(A) =$ Rank $[A/B]$

The given system is consistent, so it has a sol^n

$\therefore$ The given system has a unique sol^n.

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$x + 2y + 2z = 2 \longrightarrow (1)$

$-8y - 7z = -1 \longrightarrow (2)$

$z = -1 \longrightarrow (3)$.

Now, sub z in eq (2)

$(2) \Rightarrow -8y - 7(-1) = -1$

$-8y + 7 = -1$

$-8y = -1 - 7$

$\cancel{+}8y = \cancel{+8}$

$\boxed{y = 1}$

Sub y & z in eq (1)

$(1) \Rightarrow x + 2(1) + 2(-1) = 2$

$x + \cancel{2} - \cancel{2} = 2$

$\boxed{x = 2}$

$\therefore x = 2,\ y = 1,\ z = -1$ is the unique sol^n

(Ans).

(iv).

Sol:- Given equ's are:-

$$3x+3y+2z=1$$

$$x+2y=4$$

$$10y+3z=-2$$

$$2x-3y-z=5$$

Consider $A=\begin{bmatrix}3&3&2\\1&2&0\\0&10&3\\2&-3&-1\end{bmatrix}$; $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$; $Z=\begin{bmatrix}1\\4\\-2\\5\end{bmatrix}$

$$AX=B$$

$$\begin{bmatrix}3&3&2\\1&2&0\\0&10&3\\2&-3&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\4\\-2\\5\end{bmatrix}$$

$$[A/B]\sim\begin{bmatrix}3&3&2&1\\1&2&0&4\\0&10&3&-2\\2&-3&-1&5\end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim\begin{bmatrix}1&2&0&4\\3&3&2&1\\0&10&3&-2\\2&-3&-1&5\end{bmatrix}$$

$$R_2 \rightarrow R_2-3R_1$$

$$R_4 \rightarrow R_4-2R_1$$

$$\sim\begin{bmatrix}1&2&0&4\\0&-3&2&-11\\0&10&3&-2\\0&-7&-1&-3\end{bmatrix}$$

$$\frac{R_2}{-3}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

$R_3 \to R_3 - 10R_2$

$R_4 \to R_4 + 7R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}$$

$\cancel{R_2 \to}\ \frac{3}{29} R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}$$

$R_4 + \frac{17}{3} R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of non-zero rows = 3

Rank (A) = 3

Rank [A/B] = 3

∴ Rank (A) = Rank (A/B).

The given system is consistent, so it has a soln

∴ The given system has a unique soln

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/3 \\ -4 \\ 0 \end{bmatrix}$$

$x + 2y = 4 \longrightarrow (1)$

$y - 2/3 z = 11/3 \longrightarrow (2)$

$z = -4 \longrightarrow (3)$

Sub z in eq (2)

(2) $\Rightarrow y - \frac{2}{3}(-4) = 11/3$

$y + \frac{8}{3} = \frac{11}{3}$

$y = \frac{11}{3} - \frac{8}{3}$

$y = \frac{3}{3}$

$\boxed{y = 1}$

Sub y & z in eq (1)

(1) $\Rightarrow x + 2(1) = 4$

$x + 2 = 4$

$x = 4 - 2$

$\boxed{x = 2}$

$\therefore x = 2, y = 1, z = -4$ is unique sol

(19). S.T the equ's $x-4y+7z=14$, $3x+8y-2z=13$ & $7x-8y+26z=5$ are not Consistent. (96)

Sol:-

Given equ's

$$x-4y+7z=14$$
$$3x+8y-2z=13$$
$$7x-8y+26z=5$$

Let $A=\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B=\begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}$

$$AX=B$$

$$\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}$$

$$[A|B]\sim\begin{bmatrix} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{bmatrix}$$

$R_2 \to R_2-3R_1$

$R_3 \to R_3-7R_1$

$$\sim\begin{bmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{bmatrix}$$

$R_3 \to R_3-R_2$

$$\sim \begin{bmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & -64 \end{bmatrix}$$

$\therefore$ No of non-zero rows is 3.

Rank (A) = 2

Rank [A|B] = 3

Rank (A) $\neq$ Rank [A|B]

$\therefore$ The system is inconsistent

(Ans)

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20. Find the values of 'p' and 'q' so that the equations $2x+3y+5z=9$, $7x+3y+2z=8$, $2x+3y+pz=q$ have

(i) No solution.

(ii) unique solution.

(iii) Infinite Number of solutions.

[OR]

Find the values of a and b such that the system $2x+3y+5z=9$, $7x+3y+2z=8$, $2x+3y+az=b$.

(i) No solution.

(ii) unique solution.

(iii) Infinite number of solutions.

Sol:-

given equations are

$2x+3y+5z=9$.

$7x+3y+2z=8$

$2x+3y+pz=q$

$$A=\begin{bmatrix}2 & 3 & 5\\ 7 & 3 & 2\\ 2 & 3 & p\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\ B=\begin{bmatrix}9\\ 8\\ q\end{bmatrix}$$

The Equations (or) system can be written as written ~~in~~ matrix form $AX=B$

$$\begin{bmatrix}2 & 3 & 5\\ 7 & 3 & 2\\ 2 & 3 & p\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 8\\ q\end{bmatrix}$$

Now, Augumented matrix

$[A|B]$

$$\sim\left[\begin{array}{ccc|c}2 & 3 & 5 & 9\\ 7 & 3 & 2 & 8\\ 2 & 3 & p & q\end{array}\right]$$

$R_2 \to 2R_2 - 7R_1$

$R_3 \to R_3 - R_1$

$$\sim\begin{bmatrix}2 & 3 & 5 & 9\\ 0 & -15 & -31 & -47\\ 0 & 0 & p-5 & q-9\end{bmatrix}$$

~~$R_2 \to 2R_2 - 7R_1$~~

~~$R_3 \to R_3 - R_1$~~

$\cancel{2x\ 3\ 5}\ 9$

$\cancel{17}$

det A =
Let,

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & p \end{bmatrix}$$

Let, det A = $\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & p \end{vmatrix}$

$= 2(3\times p - 2\times 3) - 3(7\times p - 2\times 2) + 5(7\times 3 - 2\times 3)$

$= 2(3p-6) - 3(7p-4) + 5(21-6)$

$= 6p - 12 - 21p + 12 + 105 - 30.$

$= -21p + 6p + 105 + 12 - 30 - 12$

$= -15p + 117 - 42$

$= 117 - 42 - 15p$

$= 75 - 15p$

$= p = \frac{75}{15}$

$\boxed{p = 5}$

$\therefore$ det A $= 0. \Rightarrow p = 5.$

Case(ii).- ~~No solution~~
No solution

when, $p=5$, $q \neq 9$

The rank(A) = 2 and

rank (A|B) = 3

Since, rank(A) $\neq$ Rank(A|B).

$\therefore$ The system is inconsistent

$\therefore$ The system will not have any solution.

Case (ii)

~~Unique solution~~ unique solution

when $p \neq 5$, det $A \neq 0$

$\therefore$ ~~The system~~ The system is unique solution.

Case (iii) :-

~~Infinite No of solutions.~~ Infinite No. of solutions.

when $p=5$, $q=9$

The rank (A) = 2. and

rank (A|B) = 2

Since, rank (A) = rank(A|B).

$\therefore$ The system is consistent

$\therefore$ The system is having infinite No. of solutions.

=(Ans).

Q1) Solve the system of Equations

(i) $x+2y+3z=1$

$2x+3y+8z=2$

$x+y+z=3$

(ii) $x+y+2z=4$

$3x+y-3z=-4$

$2x-3y-5z=-5$

Sol:-

(i) The given Equations are

$$x+2y+3z=1$$
$$2x+3y+8z=2$$
$$x+y+z=3.$$

let, $A=\begin{bmatrix}1&2&3\\2&3&8\\1&1&1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}1\\2\\3\end{bmatrix}$

The Equations (or) system can be written as in matrix form, $AX=B$.

$$\begin{bmatrix}1&2&3\\2&3&8\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Now, Augmented matrix

$[A|B]$

$$\sim\left[\begin{array}{ccc|c}1&2&3&1\\2&3&8&2\\1&1&1&3\end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\sim\begin{bmatrix}1&2&3&1\\0&-1&2&0\\0&-1&-2&2\end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \end{array}\right]$$

This is in Echelon form

$\therefore$ The no. of non-zero rows $= 3$.

$\therefore$ Rank $(A) = 3$

$\therefore$ Rank $(A|B) = 3$

Rank (A) = Rank $(A|B)$

$\therefore$ The above system is an unique solution.

Consider,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Then, ~~convert~~ solve it into $AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$x + 2y + 3z = 1$ —— ①

$-y + 2z = 0$ —— ②

$-\overset{2}{\cancel{4}}z = \underset{1}{\cancel{2}}$.

$\boxed{\cancel{z = -2}}$ —— ③

$\boxed{z = -1/2}$ —— ③

Now, substitute eqⁿ ③ in ②

② ⇒ $\cancel{-y+2z=0}$ ⇒ $-y+2\left(-\frac{1}{2}\right)=0$

⇒ $\cancel{-y+2(-2)}=0$ ⇒ $-y-1=0$

⇒ $\cancel{-4+y=0}$ $\cancel{-y-4=0}$ ⇒ $-y=1$

⇒ $\cancel{y=4}$ ⇒ $\cancel{-y=4}$ = $\boxed{y=-1}$

$\cancel{\boxed{y=-4}}$ $\cancel{\Rightarrow y=}$

Now, substitute y, z in eqⁿ ①

① ⇒ $x+2y+3z=1$

⇒ $x+2(-1)+3\left(-\frac{1}{2}\right)=1$

⇒ $2x-2+6(-1)=1$

⇒ $2x-2-6=1$

⇒ $2x-8=1$

⇒ $2x=8+1$

⇒ $2x=9$

⇒ $\boxed{x=9/2}$

∴ $x=9/2,\ y=-1,\ z=-1/2$

Take, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$X=\begin{bmatrix} 9/2 \\ -1 \\ -1/2 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 9/2 \\ -1 \\ -1/2 \end{bmatrix}$ is the unique solution //(Ans).

(ii) The given Equations are

$x+y+2z=4$

$3x+y-3z=-4$

$2x-3y-5z=-5$

let, $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & -3 \\ 2 & -3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ -4 \\ -5 \end{bmatrix}$

The equations (or) system can be written as in matrix form, $AX = B$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & -3 \\ 2 & -3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -5 \end{bmatrix}$$

Now, Augumented matrix $[A|B]$.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 3 & 1 & -3 & -4 \\ 2 & -3 & -5 & -5 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -2 & -9 & -16 \\ 0 & -5 & -9 & -13 \end{bmatrix}$$

$R_1 \rightarrow 2R_1 + R_2$

$R_3 \rightarrow 2R_3 - 5R_2$

$$\sim \begin{bmatrix} 2 & 0 & -5 & -8 \\ 0 & -2 & -9 & -16 \\ 0 & 0 & 27 & 54 \end{bmatrix}$$

$R_3 \rightarrow R_3/27.$

$$\sim \begin{bmatrix} 2 & 0 & -5 & -8 \\ 0 & -2 & -9 & -16 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 5R_3$

$R_2 \rightarrow R_2 + 9R_3$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This in the Echelon form

$\therefore$ The no. of non-zero rows = 3.

$\therefore$ Rank(A) = 3

$\therefore$ Rank(A|B) = 3

Rank(A) = Rank(A|B)

$\therefore$ The above system is an unique solution.

(ii) consider,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Then solve it into $AX = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$2x = 2$ —— (1) $\quad x = 2/2 = 1 \quad \boxed{x = 1}$

$-2y = 2$ —— (2) $\Rightarrow y = \frac{-\cancel{2}}{\cancel{2}} = -1 \quad \boxed{y = -1}$

$1z = 2$ —— (3)

$z = 2/1 \Rightarrow \boxed{z = 2}$

$x = 1, \; y = -1, \; z = 2.$

Take, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is the unique solution. //

(Ans)

22) Solve the Equations

(i) $x + 2y + 3z = 1$

$2x + 3y + 8z = 2$

$x + y + z = 3$

(ii) $3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4$

(iii) $2x_1 + x_2 + x_3 = 10$

$3x_1 + 2x_2 + 3x_3 = 18$

$x_1 + 4x_2 + 9x_3 = 16.$

(iv) $2x_1 + x_2 + 2x_3 + x_4 = 6.$

$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$

using gauss-Elimination method (or) partial ~~Partting~~ pivoting gauss-Elimination method

Solutions

(i) given equations,

$$x+2y+3z=1$$
$$2x+3y+8z=2$$
$$x+y+z=3$$

let, $A=\begin{bmatrix}1&2&3\\2&3&8\\1&1&1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}1\\2\\3\end{bmatrix}$

The equations (or) system can be written as in matrix

form, $AX=B$

$$\begin{bmatrix}1&2&3\\2&3&8\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Now, Augumented matrix

$[A|B]$

$$\sim\left[\begin{array}{ccc|c}1&2&3&1\\2&3&8&2\\1&1&1&3\end{array}\right]$$

$R_2 \to R_2-2R_1$

$R_3 \to R_3-R_1$

$$\sim\begin{bmatrix}1&2&3&1\\0&-1&2&0\\0&-1&-2&2\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \end{bmatrix}$$

No. of non-zero rows = (3)

Rank (A) = 3

Rank (A|B) = 3.

Rank (A) = Rank (A|B).

∴ The above system is consistent.

Consider,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$x + 2y + 3z = 1 \longrightarrow (1)$

$-y + 2z = 0 \longrightarrow (2)$

$-4z = 2.$

$z = \frac{\cancel{2}}{\cancel{-4}_{2}}$

$$\boxed{z = \frac{-1}{2}} \longrightarrow (3)$$

sub. z in eq^n (2).

(2) $\Rightarrow -y + 2z = 0.$

$\Rightarrow -y + \cancel{2}\left(\frac{-1}{\cancel{2}}\right) = 0$

$\Rightarrow -y - 1 = 0$

~~$\Rightarrow y + 1 = 0$~~ $\quad -y - 1 = 0$

$\boxed{y=-1}$

Sub, 'y' and "z" in eqn ①

$① \Rightarrow x+2y+3z=1$

$\Rightarrow x+2(-1)+3\left(\frac{-1}{2}\right)=1$

$\Rightarrow \cancel{x-2} \quad \cancel{\Rightarrow 2x-4-6=1} \qquad x-2-\frac{3}{2}=1$

$\cancel{\Rightarrow 2x-4-6=2} \qquad 2x-4-3=2$

$\cancel{\Rightarrow 2x-10=1} \qquad 2x-7=2$

$\cancel{\Rightarrow 2x=10+1} \qquad 2x=7+2$

$\cancel{\Rightarrow 2x=11} \qquad \boxed{x=9/2}$

$\cancel{\Rightarrow x=11/2}$

$\therefore$ The solution $x=\frac{9}{2}, y=-1, z=\frac{-1}{2}$

=(Ans)

(ii). Given equ's

$3x+y+2z=3$

$2x-3y-z=-3$

$x+2y+z=4$

let $A=\begin{bmatrix}3&1&2\\2&-3&-1\\1&2&1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}3\\-3\\4\end{bmatrix}$

$AX=B$

$$\begin{bmatrix}3&1&2\\2&-3&-1\\1&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-3\\4\end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix}$$

$R_3 \to 7R_3 - 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix}$$

No. of non-zero rows = 3

Rank (A) = 3,

Rank [A|B] = 3.

$\therefore$ Rank(A) = Rank [A|B]

$\therefore$ The above system is consistent

Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 8 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ -11 \\ -8 \end{bmatrix}$

$$x + 2y + z = 4 \rightarrow (1)$$

$$-7y - 3z = -11 \rightarrow (2)$$

$8z = -8$

$\boxed{z = -1} \rightarrow (1)$

Now sub $z = -1$ in eq (2)

(2) $\Rightarrow -7y - 3(-1) = -11$

$-7y + 3 = -11$

$-7y = -11 - 3$

$+7y = +14$

$7y = 14$

$y = \frac{14}{7}$

$\boxed{y = 2}$

Now sub z & y in eq (1)

(1) $\Rightarrow x + 2y + z = 4$

$x + 2(2) - 1 = 4$

$x + 4 - 1 = 4$

$x + 3 = 4$

$x = 4 - 3$

$\boxed{x = 1}$

$\therefore$ The soln is $x = 1$, $y = 2$ & $z = -1$

= (Ans),

(i (iii). Given eq's

$$2x_1+x_2+x_3=10$$

$$3x_1+2x_2+3x_3=18$$

$$x_1+4x_2+9x_3=16$$

Let $A=\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B=\begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$

$$AX=B$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

$$[A|B]=\begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$$

$R_2 \to 2R_2-3R_1$

$R_3 \to 2R_3-R_1$

$$\sim\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{bmatrix}$$

$R_3 \to R_3-7R_2$

$$\sim\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{bmatrix}$$

No. of non-zero rows $=3$

Rank$(A)=3$

Rank$[A|B]=3$

$\therefore$ Rank(A) = Rank[A|B]

Consistent system.

Consider

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 6 \\ -20 \end{bmatrix}$$

$$2x_1 + x_2 + x_3 = 10 \rightarrow (1)$$

$$x_2 + 3x_3 = 6 \rightarrow (2)$$

$$-4x_3 = -20$$

$$x_3 = \frac{\cancel{20}^{5}}{\cancel{4}_{1}}$$

$$\boxed{x_3 = 5} \rightarrow (3)$$

Sub x_3 in eq (2)

$(2) \Rightarrow x_2 + 3(5) = 6$

$$x_2 + 15 = 6$$

$$x_2 = 6 - 15$$

$$\boxed{x_2 = -9}$$

Sub x_2 & x_3 values in eq (1)

$(1) \Rightarrow 2x_1 - 9 + 5 = 10$

$$2x_1 - 4 = 10$$

$$2x_1 = 10 + 4$$

$$2x_1 = 14$$

$$\boxed{x_1 = 7}$$

$\therefore$ The solution is $x_1 = 7, x_2 = -9, x_3 = 5$ // (Ans) (iv).

Sol:- Given equ

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$

$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$

$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$$

$$2x_1 + 2x_2 - x_3 + x_4 = 10.$$

Let $A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$

$$AX = B$$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{6}$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 2 & 1 & 2 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 4 & -3 & -3 & -2 \end{bmatrix}$$

$R_3 \to 3R_3 - 7R_2$

$R_4 \to 3R_4 - 4R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & -9 & 3 & 18 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & 0 & 39 & 117 \end{bmatrix}$$

No. of Non zero rows = 4

Rank(A) = 4

Rank[A|B] = 4

Rank(A) = Rank(AB)

Consistent

Consider

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & -3 & -12 \\ 0 & 0 & 0 & 39 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 6 \\ -6 \\ -33 \\ 117 \end{bmatrix}$$

$$x_1 - x_2 + x_3 + 2x_4 = 6 \rightarrow (1)$$

$$3x_2 - 3x_4 = -6 \rightarrow (2)$$

$$-3x_3 - 12x_4 = -33 \rightarrow (3)$$

$$39x_4 = 117$$

$$\boxed{x_4 = 3} \rightarrow (4)$$

sub eq (4) in eq (3) & (2)

(3) $\Rightarrow -3x_3 - 12(3) = -33$

$$-3x_3 - 36 = -33$$

$$-3x_3 = -33 + 36$$

$$-\cancel{3}x_3 = \cancel{3}^{1}$$

$$\boxed{x_3 = -1}$$

(2) $\Rightarrow 3x_2 - 3(3) = -6$

$$3x_2 - 9 = -6$$

$$3x_2 = -6 + 9$$

$$3x_2 = 3$$

$$\boxed{x_2 = 1}$$

sub x_2, x_3 & x_4 in eq (1)

(1) $\Rightarrow x_1 - 1 - 1 + 2(3) = 6$

$$x_1 - 1 - 1 + 6 = 6$$

$$x_1 + 4 = 6$$

$$x_1 = 6 - 4$$

$$\boxed{x_1 = 2}$$

$\therefore$ The solution is $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$ = (Ans)

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23. Solve the system of eq's using Gauss-seidel iteration method (or) successive Displacement method (or) Gauss-seidal method.

(i). $10x + y + z = 12$

$2x + 10y + z = 13$

$2x + 2y + 10z = 14$

(ii). $x_1 + 10x_2 + x_3 = 6$

$10x_1 + x_2 + x_3 = 6$

$x_1 + x_2 + 10x_3 = 6$

(iii). $10 - 2y - z - u = 3$

$-2x + 10y - z - u = 15$

$-x - y + 10z - 2u = 27$

$-x - y - 2z + 10u = -9$

(iv). $8x_1 - 3x_2 + 2x_3 = 20$

$4x_1 + 11x_2 - x_3 = 33$

$6x_1 + 3x_2 + 12x_3 = 36$

(v). $10x_1 - 2x_2 - x_3 - x_4 = 3$

$-2x_1 + 10x_2 - x_3 - x_4 = 15$

$-x_1 - x_2 + 10x_3 - 2x_4 = 15$

$-x_1 - x_2 - 2x_3 + 10x_4 = -9$ (Correct to 3 decimal places) (i). Given system

$$\left.\begin{aligned} 10x+y+z&=12 \\ 2x+10y+z&=13 \\ 2x+2y+10z&=14 \end{aligned}\right\} \rightarrow (I)$$

The given system is diagonally dominant & we write it as

$\Rightarrow 10x+y+z=12$

$\Rightarrow 10x=12-y-z$

$\Rightarrow x=\frac{1}{10}[12-y-z] \rightarrow (1)$

$y=\frac{1}{10}[13-2x-z] \rightarrow (2)$

$z=\frac{1}{10}[14-2x-2y] \rightarrow (3).$

First Appx:—

We start iteration by taking $y=0, z=0$ in eq(1).

(1) $\Rightarrow x^{(1)}=\frac{1}{10}(12-y-z]$

$x^{(1)}=\frac{1}{10}[12-0-0]$

$x^{(1)}=\frac{1}{10}[12]$

$\boxed{x^{(1)}=1.2}$

Now, putting $x^{(1)}=x=1.2$ & $z=0$ in eq(2)

(2) $\Rightarrow y^{(1)}=\frac{1}{10}[13-2x-z]$

$y^{(1)}=\frac{1}{10}[13-2(1.2)-0]$

$y^{(1)}=\frac{1}{10}[13-2.4]$

$y^{(1)}=\frac{1}{10}[10.6]$

$\boxed{y^{(1)}=1.06}$

Now putting $x^{(1)}=1.2$, $y^{(1)}=1.06$ in (3)

$$③ \Rightarrow z^{(1)} = \frac{1}{10}\left[14-2x-2y\right]$$

$$z^{(1)} = \frac{1}{10}\left[14-2(1.2)-2(1.06)\right]$$

$$z^{(1)} = \frac{1}{10}\left[14-2.4-2.12\right]$$

$$z^{(1)} = \frac{1}{10}\left[14-4.52\right]$$

$$z^{(1)} = \frac{1}{10}\left[9.48\right]$$

$$\boxed{z^{(1)} = 0.948 \simeq 0.95}$$

Second Appr

Now taking $y^{(1)}, z^{(1)}$ as the initial values in (1).

$$① \Rightarrow x^{(2)} = \frac{1}{10}\left[12-1.06-0.95\right]$$

$$\boxed{x^{(2)} = 0.999}$$

Now, putting $x=x^{(2)}$ & $z=z^{(1)}$ in eq(2)

$$② \Rightarrow y^{(2)} = \frac{1}{10}\left[13-1.998-0.95\right]$$

$$\boxed{y^{(2)} = 1.005}$$

Now putting $x=x^{(2)}$ & $y=y^{(2)}$ in eq(3)

$$③ \Rightarrow z^{(2)} = \frac{1}{10}\left[14-1.998-2.010\right]$$

Third Appr

$$\boxed{z^{(2)} = 0.999}$$

Now taking $y^{(2)}, x^{(2)}$ & $z^{(2)}$ in eq(1) & (2) & (3)

$$① \Rightarrow x^{(3)} = \frac{1}{10}\left(12-1.005-0.999\right)$$

$$= 0.9996$$

$$\boxed{x^{(3)} = 1.00}$$

(2) $\Rightarrow y^{(3)} = \frac{1}{10}(13 - 2.0 - 0.999)$

$y^3 = 1.0001$

$\boxed{y^3 = 1.00}$

(3) $\Rightarrow z^{(3)} = \frac{1}{10}(14 - 2.0 - 2.0)$

$\boxed{z^{(3)} = 1.00}$

Fourth App:-

Uy, we find the fourth approximations of x, y, z rep let them as

$x^{(4)} = 1.00$

$y^{(4)} = 1.00$

$z^{(4)} = 1.00.$

Tabular format:-

Variable	Ist approx.	IInd app	IIIrd app	IVth approx.
x	1.20	0.999	1.00	1.00
y	1.06	1.005	1.005	1.00
z	0.95	0.999	1.00	1.00.

Thus, the soln of the given system of eqn is

$x \simeq 1,\ y \simeq 1,\ z \simeq 1$

(Ans)

(ii). given System

$$\left.\begin{array}{l} x_1 + 10x_2 + x_3 = 6 \\ 10x_1 + x_2 + x_3 = 6 \\ x_1 + x_2 + 10x_3 = 6 \end{array}\right\} \text{—(I)}$$

The given system is diagonally dominant & we can write as

$$\Rightarrow x_1 + 10x_2 + x_3 = 6$$

$$\Rightarrow x_1 = 6 - 10x_2 - x_3$$

$$\cancel{\Rightarrow x_1 = \frac{1}{6}\left[-10x_2 - x_3\right]}$$

$$\Rightarrow x_1 = \frac{6 - x_2 - x_3}{10} \rightarrow (1)$$

lly

$$\cancel{x_2 = 10x_1 + x_2 + x_3}$$

$$x_2 = \frac{6 - x_3 - x_1}{10} \rightarrow (2)$$

$$x_3 = \frac{6 - x_1 - x_2}{10} \rightarrow (3).$$

Ist Approximation :-

$x_2 = 0,\ x_3 = 0$ in eq(1).

$$(1) \Rightarrow x_1^{(1)} = \frac{6 - 0 - 0}{10}$$

$$x_1^{(1)} = \frac{6}{10}$$

$$\boxed{x_1^{(1)} = 0.6}$$

Take $x_1^{(1)} = 0.6,\ x_3 = 0$ in eq(2)

$$(2) \Rightarrow x_2^{(1)} = \frac{6 - 0 - 0.6}{10}$$

$$\boxed{x_2^{(1)} = 0.54}$$

Take. $x_1^{(1)} = 0.6$, $x_2^{(1)} = 0.54$

$$③ \Rightarrow x_3^{(1)} = \frac{6 - 0.6 - 0.54}{10}$$

$$\boxed{x_3^{(1)} = 0.486}$$

IInd Approximation :-

Taking, $\cancel{x_1^{(1)} = x_1 = 0.6}$, $x_2^{(1)} = x_2 = 0.54$

$x_3^{(1)} = x_3 = 0.486$. eq(1).

$$① \Rightarrow x_1^{(2)} = \frac{6 - x_2 - x_3}{10}$$

$$x_1^{(2)} = \frac{6 - 0.54 - 0.486}{10}$$

$$\boxed{x_1^{(2)} = 0.4974.}$$

$\cancel{② \Rightarrow x_2^{(2)} = 6 - 0.486 -}$

Take $x_1^{(2)} = 0.4974$, $x_3^{(1)} = 0.486$ in eq(2)

$$② \Rightarrow x_2^{(2)} = \frac{6 - 0.486 - 0.4974}{10}$$

$$\boxed{x_2^{(2)} = 0.5017}$$

Take, $x_1^{(2)} = 0.4974$, $x_2^{(2)} = 0.5017$. in Eq^n(3)

$$③ \Rightarrow x_3^{(2)} = \frac{6 - 0.4974 - 0.5017}{10}$$

$$\boxed{x_3^{(2)} = 0.4992}$$

IIIrd Approximation

Taking $x_3^{(2)} = 0.4992$, $x_2^{(2)} = 0.5107$ in eq(1)

$$① \Rightarrow x_1^{(3)} = \frac{10 - 0.5017 - 0.4992}{10}$$

$$\boxed{x_1^{(3)} = 0.4999}$$

Taking $x_1^{(3)} = 0.4999$, $x_3 = x_3^{(2)} = 0.4992$

$$② \Rightarrow x_2^{(3)} = \frac{10 - 0.4999 - 0.4992}{10}$$

$$\boxed{x_2^{(3)} = 0.5000}$$

Taking $x_1^{(3)} = 0.4999$, $x_2^{(3)} = 0.5000$

$$③ \Rightarrow x_3^{(3)} = \frac{10 - 0.4999 - 0.5000}{10}$$

$$\boxed{x_3^{(3)} = 0.5000}$$

Tabular format

	Ist app	IInd app	IIIrd app
x_1	0.6	0.4974	0.4999 = 0.5000
x_2	0.54	0.5017	0.5000
x_3	0.486	0.4992	0.5000

Thus, the solution is $x_1 = 0.5$, $x_2 = 0.5$ & $x_3 = 0.5$

— (Ans)

Theorems

(1). If A & B are square symmetric matrices of same order then P.T (i). AB+BA is symmetric (ii). AB−BA is skew-symmetric.

Proof:- Given that A & B are symmetric matrices

so, $A^T = A$ & $B^T = B \rightarrow (1)$.

(i). AB+BA is symmetric:-

Consider $(AB+BA)^T = (AB)^T + (BA)^T$ $[\because (A+B)^T = A^T + B^T]$

$= B^T A^T + A^T B^T$ $[\because (AB)^T = B^T A^T]$

$= BA + AB$ [from eq(1)]

$$\boxed{(AB+BA)^T = AB+BA}$$

$\therefore$ AB+BA is a symmetric matrix //

(ii). AB−BA is skew-symmetric:-

Consider $(AB-BA)^T = (AB)^T - (BA)^T$ $[\because (A-B)^T = A^T - B^T]$

$= B^T A^T - A^T B^T$ $[\because (AB)^T = B^T A^T]$

$= BA - AB$

$$\boxed{(AB-BA)^T = -(AB-BA)}$$

$\therefore$ AB−BA is skew-symmetric

=(Ans).

(2) Every square matrix can be expressed as the sum of a symmetric & skew-symmetric matrices in one & only way (uniquely).

(or)

S.T any square matrix $A = B + C$ where B is symmetric & C is skew-symmetric matrices.

Proof:- Let A be any square matrix.

Consider $P = \frac{1}{2}(A + A')$ & $Q = \frac{1}{2}(A - A')$

So, $P' = \left\{\frac{1}{2}(A + A')\right\}'$ & $Q' = \left\{\frac{1}{2}(A - A')\right\}'$

Take $P' = \left\{\frac{1}{2}(A + A')\right\}'$

$\Rightarrow P' = \frac{1}{2}\left(A' + (A')'\right)$.

$= \frac{1}{2}(A' + A)$

$\therefore \boxed{P' = P}$

$\therefore$ P is a symmetric matrix.

Similarly, $Q' = \left\{\frac{1}{2}(A - A')\right\}'$

$= \frac{1}{2}\left(A' - (A')'\right)$

$= \frac{1}{2}(A' - A)$

$\therefore \boxed{Q' = -Q}$

$\therefore$ Q is a skew-symmetric matrix.

Thus square matrix = symmetric matrix + skew-symmetric matrix

Hence the matrix A is the sum of a symmetric matrix & a skew-symmetric matrix.

TO P.T the sum is unique!- (126)

Consider $R' = R$ & $S' = -S$ [$R \rightarrow$ symmetric matrix, $-S \rightarrow$ skew – 〃 〃]

Now $A' = (R+S)' \Rightarrow R' + S' \Rightarrow R - S$

$= \frac{1}{2}(A + A')$

$= \frac{1}{2}(R + S + R - S)$

$= R$

Ily, $= \frac{1}{2}(A - A')$

$= \frac{1}{2}(R + S - R + S)$

$= Q = S$.

$\Rightarrow R = P$ & $S = Q$

Thus, the representation is unique =(Ans).

---

(3). P.T inverse of a non-singular symmetric matrix A is symmetric.

Proof!- Since A is non-singular symmetric matrix.

$\therefore A^{-1}$ exists & $A^T = A \rightarrow (1)$

Now, we have to prove that A^{-1} is symmetric

$\Rightarrow (A^{-1})^T$

$= (A^T)^{-1}$

$\Rightarrow A^{-1}$ (from eq(1))

Since $(A^{-1})^T = A^{-1}$

$\therefore A^{-1}$ is symmetric =(Ans)

④. If A is a symmetric matrix, then P.T adj A is also symmetric.

Proof:— Since A is symmetric, we have

$$A^T = A \rightarrow (1)$$

Consider $(\text{adj} A)^T = \text{adj} A^T$

$$\boxed{(\text{adj} A)^T = \text{adj} A} \text{ (from eq(1))}$$

Since $(\text{adj} A)^T = \text{adj} A$, therefore, adj A is a symmetric matrix.

=(Ans)

⑤. If A, B are orthogonal matrices, each of order n then AB & BA are orthogonal matrices.

Proof:— Since A & B are both orthogonal matrices,

$$\therefore AA^T = A^TA = I \rightarrow (1)$$

$$\& \ BB^T = B^TB = I \longrightarrow (2)$$

Now $(AB)^T = B^TA^T$

$$\therefore (AB)^T(AB) = (B^TA^T)(AB)$$

$$= B^T(A^TA)B$$

$$= B^TIB \text{ (from (1))}$$

$$= B^TB$$

$$= I \text{ (from (2))}$$

$\therefore$ AB is orthogonal.

Similarly, we can P.T BA is also orthogonal.

=(Ans).

6) P.T the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

Proof:-

let 'A' be an orthogonal matrix

Then $A^T A = A A^T = I$

Consider, $A^T . A = I$

Taking inverse on both sides

$$(A^T . A)^{-1} = I^{-1}$$

$$\Rightarrow A^{-1} (A^T)^{-1} = I$$

$$\Rightarrow A^{-1} (A^{-1})^T = I$$

$\therefore A^{-1}$ is orthogonal

Again $A^T . A = I$

Taking Transpose on both sides.

we get, $(A^T . A)^T = I^T$

$$\Rightarrow A^T . (A^T)^T = I$$

$$(\because (AB)^T = B^T A^T)$$

Hence A^T is orthogonal. (Ans).

---

7) If A&B are Hermitian matrices, P.T AB-BA is a skew-Hermitian

Proof:-

given A and B are Hermitian

$$\therefore (\bar{A})^T = A \text{ and } (\bar{B})^T = B \quad \text{—— (1)}$$

Now, $(\overline{AB - BA})^T$

$$= (\overline{AB} - \overline{BA})^T$$

$$= (\bar{A}\bar{B} - \bar{B}\bar{A})^T$$

$= (\bar{A}\bar{B})^T - (\bar{B}\bar{A})^T$

$= (\bar{B})^T (\bar{A})^T - (\bar{A})^T (\bar{B})^T$

$= BA - AB$, By Eqⁿ (1)

$= -(AB - BA)$

Thus, $AB - BA$ is a skew-Hermitian matrix. (Ans).

8) Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew-Hermitian matrix

Proof:

let, A be any square matrix

Now, $(A+A^\theta)^\theta$

$= A^\theta + (A^\theta)^\theta$

$= A^\theta + A$.

Since, $(A+A^\theta)^\theta$

$= A + A^\theta$

$\therefore$ $A + A^\theta$ is a Hermitian matrix

$\therefore \frac{1}{2}(A+A^\theta)$ is also a Hermitian matrix

Now, $(A - A^\theta)^\theta$

$= A^\theta - (A^\theta)^\theta$

$= A^\theta - A$

$= -(A - A^\theta)$.

Hence $(A - A^\theta)^\theta$ is a skew-Hermitian matrix

$\therefore \frac{1}{2}(A - A^\theta)$ is also a skew-Hermitian matrix.

Now,

we have $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$

$= P + Q$ (say).

where, 'P' is a Hermitian matrix

and 'Q' is a skew-Hermitian matrix

~~Thus~~, Every square matrix can be expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

To prove that the representation is unique:-

let, $A = R + S$ be another representation of A.

where, 'R' is Hermitian and

'S' is a skew-Hermitian

Then to P.T $R = P$ and

$S = Q$.

then, $A^{\theta} = (R+S)^{\theta}$

$= R^{\theta} + S^{\theta}$

$= R - S$ ($\because$ 'R' is Hermitian and 'S' is skew-Hermitian).

$\therefore R = \frac{1}{2}(A + A^{\theta}) = P$

and $S = \frac{1}{2}(A - A^{\theta}) = Q$.

Thus the representation is unique. (Ans).

---

9) Prove that Every Hermitian matrix can be written as $A + iB$, where 'A' is real and symmetric and 'B' is real and skew-symmetric.

~~Proof~~:-

~~let, 'e' den~~ a

Proof

let 'c' be a Hermitian matrix.

Then $c^\theta = c$ — ①

let us take,

$A = \frac{1}{2}(c+\bar{c})$ and

$B = \frac{1}{2i}(c-\bar{c})$

Then obviously both 'A' and 'B' are real matrices

Now, we have to prove 'A' is symmetric and 'B' is skew-symmetric

$$A^T = \frac{1}{2}(c+\bar{c})^T$$

$$= \frac{1}{2}\left[c^T + (\bar{c})^T\right]$$

$$= \frac{1}{2}\left[c^T + c^\theta\right]$$

$$= \frac{1}{2}\left[(c^\theta)^T + c\right] \text{ (By ①)}$$

$$= \frac{1}{2}\left[\{(\bar{c})^T\}^T + c\right]$$

$$= \frac{1}{2}(\bar{c}+c) \Rightarrow A$$

$= A.$ is symmetric

Now, $B^T = \frac{1}{2}\left[\frac{1}{2i}(c-\bar{c})^\theta\right]^T$

$$= \frac{1}{2i}(c-\bar{c})^T$$

$$= \frac{1}{2i}\left[c^T - (\bar{c})^T\right]$$

$$= \frac{1}{2i}\left[c^T - c^\theta\right]$$

$$= \frac{1}{2i}\left[(c^\theta)^T - c\right] \text{ (By ①)}$$

$$= \frac{1}{2}\left[\{(\bar{C})^T\}^T - C\right]$$

$$= \frac{1}{2i}(\bar{C} - C)$$

$$= \frac{-1}{2i}(C - \bar{C}) = -B.$$

$\therefore$ 'B' is skew-symmetric // (Ans)

(10) Show that the inverse of a unitary matrix is unitary.

Proof

Let 'A' be an unitary matrix

Then, $AA^{\theta} = I$ —— ①

$\Rightarrow (AA^{\theta})^{-1} = I^{-1}$ (Taking inverse on Both sides)

$\Rightarrow (A^{\theta})^{-1} A^{-1} = I$

$\Rightarrow (A^{-1})^{\theta} A^{-1} = I.$

Thus A^{-1} is unitary matrix. // (Ans).

11) Prove that the product of two unitary matrices is unitary

proof: Let 'A' and 'B' are the two unitary matrices

$$\left.\begin{array}{l} \therefore A.A^{\theta} = A^{\theta}.A = I \text{ and} \\ B.B^{\theta} = B^{\theta}.B = I \end{array}\right\} \text{—— ①}$$

Consider,

$$(AB).(AB)^{\theta} = (AB)(B^{\theta}A^{\theta})$$

$$= A(BB^{\theta})A^{\theta}$$

$$= AIA^{\theta} \quad (\because \text{from ①})$$

$$= A.A^{\theta} = I \quad (\because \text{from ①})$$

Again Consider,

$(AB)^{\theta} \cdot (AB)$

$= (B^{\theta} A^{\theta}) \cdot (AB)$

$= B^{\theta}(A^{\theta} A) B$

$= B^{\theta} I B$

$= B^{\theta} B = I$ ($\because$ from ①)

$\therefore (AB) \cdot (AB)^{\theta}$

$= (AB)^{\theta} \cdot (AB) = I$

Thus AB is a unitary matrix// (Ans).

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12) prove that the transpose of a unitary matrix is unitary

Proof:

let 'A' be a unitary matrix

Then, $(\bar{A}^{T}) = (\bar{A})^{T} = \bar{A}^{1}$ —— ①

Taking transpose, we get

$$\left((\bar{A}^{T})\right)^{T} = (\bar{A}^{-1})^{T} = (\bar{A}^{T})^{-1}$$

let, $A^{T} = B$.

Now (i) becomes $(\bar{B})^{T} = \bar{B}^{-1}$

Thus 'B' is unitary

i.e., A^{T} is unitary//
(Ans)

✱ THE END ✱

Unit-II (Problems)

Eigen Values and Eigen Vectors

1) Find the characteristic roots of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Sol: Given, matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic Eqn of A is $|A-\lambda I|=0$.

i.e., $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0.$

$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda)-(2)(1)]-2[1(2-\lambda)-1(1)]+1[1(2)-1(3-\lambda)]=0$

$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda)-2]-2[2-\lambda-1]+1[2-3+\lambda]=0$

$\Rightarrow (2-\lambda)[6-3\lambda-2\lambda+\lambda^2-2]-2[1-\lambda]+1[\lambda-1]=0$

$\Rightarrow (2-\lambda)[\lambda^2-5\lambda+4]-2[1-\lambda]+1[\lambda-1]=0$

$\Rightarrow 2\lambda^2-10\lambda+8-\lambda^3+5\lambda^2-4\lambda-2+2\lambda+\lambda-1=0.$

$\Rightarrow -\lambda^3+2\lambda^2+5\lambda^2-10\lambda-4\lambda+2\lambda+\lambda+8-2-1=0.$

$\Rightarrow -\lambda^3+7\lambda^2-14\lambda+3\lambda+5=0$

$\Rightarrow -\lambda^3+7\lambda^2-11\lambda+5=0$

$\Rightarrow -(\lambda^3-7\lambda^2+11\lambda-5)=0$

$\lambda^3-7\lambda^2+11\lambda-5=0$

$(\lambda-1)(\lambda^2 - 7\lambda + 11$ ~~$\lambda^2-(7)\lambda^2+(7)\lambda$~~

$$\Rightarrow (\lambda-1)(\lambda^2-6\lambda+5)=0.$$

$$\Rightarrow (\lambda-1)(\lambda^2-\lambda-5\lambda+5)=0$$

$$\Rightarrow (\lambda-1)[\lambda(\lambda-1)-5(\lambda-5)]=0$$

$$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-5)=0$$

$\lambda-1=0$ | $\lambda-1=0$ | $\lambda-5=0$

$\boxed{\lambda=1}$ | $\boxed{\lambda=1}$ | $\boxed{\lambda=5}$

$$\therefore \lambda=1,1,5$$

Hence, the characteristic roots of A are 1,1,5 (Ans)

2) Find the Eigen values and the corresponding Eigen vectors of

(i) $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ (ii) $A=\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solutions

(i) given, $A=\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

The characteristic Equation $|A-\lambda I|=0$.

$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}=0 \rightarrow (1)$$

$\Rightarrow (5-\lambda)(2-\lambda) - 4\times 1 = 0$

$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0.$

$\Rightarrow 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0.$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0.$

$\Rightarrow \lambda^2 - \lambda - 6\lambda + 6 = 0$

$\Rightarrow \lambda(\lambda-1) - 6(\lambda-1) = 0$

$\Rightarrow \lambda - 1 = 0 \ ; \ \lambda - 6 = 0.$

$\Rightarrow \boxed{\lambda = 1} \ ; \ \boxed{\lambda = 6}$

$\therefore \lambda = 1, 6 \rightarrow (2)$

The roots of the equ are $\lambda = 1, 6$.

Hence the eigen values of the matrix A are 1, 6.

Consider the system $\begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow (3)$

To get the eigen vector X corresponding to each eigen value λ, we have to solve the above system.

Eigen vector corresponding to $\lambda = 1$

Put, $\boxed{\lambda = 1}$ in Eqn 3

$(3) \Rightarrow \begin{pmatrix} 5-1 & 4 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

$\Rightarrow \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow 4x_1 + 4x_2 = 0$

$\Rightarrow x_1 + x_2 = 0.$

$\Rightarrow x_2 = -x_1$

Take, $x_1 = \alpha$ we get $x_2 = -\alpha$

$\therefore X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

where, $\alpha \neq 0$ is a scalar.

Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an Eigen vector of A corresponding to the Eigen value $\boxed{\lambda = 1.}$

Eigen vector corresponding to $\lambda = 6$

Put, $\boxed{\lambda = 6}$ in Eqn 3

3 $\Rightarrow \begin{pmatrix} 5-6 & 4 \\ 1 & 2-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

$\Rightarrow -x_1 + 4x_2 = 0.$

$\Rightarrow x_1 - 4x_2 = 0.$

$\Rightarrow x_1 = 4x_2$

Take, $x_1 = 4\alpha$ we get $x_2 = \alpha$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

where, $\alpha \neq 0$ is a scalar.

Hence $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an Eigen vector of A corresponding to the Eigen value $\boxed{\lambda = 6}$ // (Ans).

(ii) given, $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{pmatrix}$.

The characteristic Eqn $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & 6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0.$$

(ii). Given $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{pmatrix}$

Consider $(A - \lambda I)X = O$

i.e $$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & 6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (1)$$

The characteristic eq $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & 6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$-2-\lambda\left[(1-\lambda)(-\lambda)-(-6)(-2)\right]-2\left[2(-\lambda)-(-6)(-1)\right]$$

$$+3\left[2(-2)-(1-\lambda)(-1)\right]=0$$

$$\Rightarrow -(2+\lambda)\left[(1-\lambda)(-\lambda)-12\right]-2\left[-2\lambda-6\right]-3\left[-4+1-\lambda\right]=0$$

$$\cancel{\Rightarrow -(2+\lambda)\left[-\lambda+\lambda^2-12\right]+4\lambda+12+12+3+3\lambda=0}$$

$$\cancel{\Rightarrow -(2+\lambda)\left[\lambda^2-\lambda-12\right]+4\lambda+12+12+3+3\lambda=0}$$

$$\cancel{\Rightarrow -\left[2\lambda^2-2\lambda-24+\lambda^3-\lambda^2-12\lambda\right]+4\lambda+12+12+3+3\lambda=0}$$

$$\cancel{\Rightarrow -\left[\lambda^3+2\lambda^2-\lambda^2-14\lambda-24+4\lambda+24+3+3\lambda\right]=0}$$

$$\cancel{\left[\lambda^3+\lambda^2-7\lambda+3\right]=0}$$

$$\Rightarrow -(2+\lambda)\left[\lambda^2-\lambda-12\right]-2\left[-2(\lambda+3)\right]-3\left[-\lambda-3\right]=0$$

$$\Rightarrow -(2+\lambda)\left[\lambda^2-\lambda-12\right]-2\left[-2(\lambda+3)\right]-3\left[-(\lambda+3)\right]=0$$

$$\Rightarrow -(2+\lambda)\left[\lambda^2-\lambda-12\right]-2\left[-2(\lambda+3)\right]+3(\lambda+3)=0$$

$$\Rightarrow -(2+\lambda)\left[\lambda^2-\lambda-12\right]+4(\lambda+3)+3(\lambda+3)=0$$

$$\Rightarrow -(2+\lambda)\left[\lambda^2-\lambda-12\right]+7(\lambda+3)=0$$

$$\Rightarrow -(\lambda+2)(\lambda-4)(\lambda+3)+7(\lambda+3)=0$$

$$\Rightarrow (\lambda+3)\left[-(\lambda+2)(\lambda-4)+7\right]=0$$

$$\Rightarrow -(\lambda+3)\left[(\lambda+2)(\lambda-4)-7\right]=0$$

$$\Rightarrow -(\lambda+3)\left[\lambda^2-4\lambda+2\lambda-8-7\right]=0$$

$$\Rightarrow -(\lambda+3)\left[\lambda^2-2\lambda-15\right]=0.$$

$\Rightarrow -(\lambda+3)\,[\lambda^2+\lambda-3\lambda-15]=0$

~~$\Rightarrow (\lambda+3)\,(\lambda($~~

$(\lambda+3)\,[\lambda(\lambda+1)-3(\lambda+5)]=0$

$\Rightarrow -(\lambda+3)\,(\lambda^2-2\lambda-15)=0.$

$\Rightarrow (\lambda+3)\,(\lambda+3)\,(\lambda-5)=0$

$\lambda+3=0$	$\lambda+3=0$	$\lambda-5=0$
$\lambda=-3$	$\lambda=-3$	$\lambda=5$

$\therefore \lambda=-3,-3,5$

$\therefore$ The eigen values of A are $-3,-3,5$

Eigen vector of A Corresponding to $\lambda=-3$:-

sub, $\lambda=-3$ in eq (1)

$$(1)\Rightarrow \begin{pmatrix} -2-3 & 2 & -3 \\ 2 & 1+3 & 6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[AB]\sim\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{bmatrix}$$

$R_2 \to 2R_2-2R_1$

$R_3 \to R_3+R_1$

$$\sim\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0 \longrightarrow (1)$$

$$\Rightarrow x_1 = -2x_2 + 3x_3 \longrightarrow (2)$$

$$0 = 0$$

$$0 = 0$$

$\cancel{x_1 = -2\alpha + 3\beta}$

Take, $x_2 = \alpha$ and $x_3 = \beta$

~~we get~~ $\cancel{x_1 + 2(\alpha) - 3(\beta) = 0}$ $x_1 = -2\alpha + 3\beta;\ x_2 = \alpha,\ x_3 = \beta$

$\cancel{\Rightarrow x_2 \ldots 2}$

$$(1) \Rightarrow x_1 + 2(\alpha) - 3(\beta) = 0$$

$$x_1 + 2\alpha - 3\beta = 0.$$

$$x_1 = -2\alpha + 3\beta$$

Hence

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2\alpha \\ \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 3\beta \\ 0 \\ \beta \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

It is the Eigen vector corresponding to $\lambda = -3$.

Eigen vectors Correspondng to $\lambda = 5$:—

Sub, $\lambda = 5$ in eqn (1)

$$(1) \Rightarrow \begin{pmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & 6 \\ -1 & -2 & 0-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & 6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$[A|B] \sim \left(\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & 6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right)$$

$R_1 \leftrightarrow R_3$

$R_1 \to (-)R_1$

$$\sim \begin{pmatrix} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 7R_1$

$$\sim \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{pmatrix}$$

$R_2 \to R_2/-8$.

$$\sim \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 16 & 32 & 0 \end{pmatrix}$$

$R_1 \to R_1 - 2R_2$

$R_3 \to R_3 - 16R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_1 + x_3 = 0$ —— (1)

$\Rightarrow x_2 + 2x_3 = 0$ —— (2)

$0 = 0.$

Take, $x_3 = \alpha_1$, sub in Eq^n (1)

(1) $\Rightarrow x_1 + \alpha_1 = 0.$

$\Rightarrow$ ~~$x_1 = \alpha_1$~~, $x_1 = -\alpha_1$

sub $x_1 = -\alpha_1$, $x_3 = \alpha_1$ in Eq^n (2)

(2) $\Rightarrow x_2 + 2(\alpha_1) = 0.$

$\Rightarrow x_2 + 2\alpha_1 = 0$

$\Rightarrow x_2 = -2\alpha_1$

Thus,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\alpha_1 \\ -2\alpha_1 \\ \alpha_1 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha_1 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore$ The eigen vectors of 'A' corresponding to $\boxed{\lambda = 5}$ is

$(-1 \;\; -2 \;\; 1)^T$ //.

(Ans).

(iii) given, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

~~the~~ character Consider $(A - \lambda I)X = 0$

i.e.; $\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ — (1)

The characteristic eqn $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0.$$

$\Rightarrow C_1 \to C_1 - C_2$

$C_2 \to C_2 - C_3$.

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ \lambda & -\lambda & 1 \\ 0 & \lambda & 1-\lambda \end{vmatrix} = 0.$$

$R_2 \to R_2 + R_1$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & \lambda & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda[-\lambda(1-\lambda) - 2\lambda] = 0$

~~$\Rightarrow \lambda^2 [1-\lambda+2) = 0$~~ $-\lambda[-\lambda + \lambda^2 - 2\lambda] = 0$

$-\lambda[\lambda^2 - 2\lambda - \lambda] = 0$

$$-\lambda(\lambda^2-3\lambda)=0$$

$$\lambda^2(3-\lambda)=0$$

$$\lambda^2=0 \quad \Big| \quad 3-\lambda=0$$

$$+\lambda=+3$$

$$\boxed{\lambda=3}$$

$$\lambda=0, \lambda=0, \lambda=3.$$

$$\therefore \boxed{\lambda=0,0,3}$$

Eigen vector of A corresponding to $\lambda=0$:-

Put $\lambda=0$ in eq (1)

$$(1) \Rightarrow \begin{bmatrix} 1-0 & 1 & 1 \\ 1 & 1-0 & 1 \\ 1 & 1 & 1-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1+x_2+x_3=0$$

Consider $x_2=\alpha$ & $x_3=\beta$

$$x_1+\alpha+\beta=0$$

$$x_1=-\alpha-\beta$$

$$x_1=-(\alpha+\beta)$$

$$\therefore X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(\alpha+\beta) \\ \alpha \\ \beta \end{bmatrix} = \alpha\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the eigen vectors of A corresponding to eigen value

$\lambda=0$ are $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\0\\1\end{bmatrix}$. (146)

Eigen vector of A corresponding to $\lambda=3$:-

put $\lambda=3$ in eq (1)

$$(1) \Rightarrow \begin{bmatrix}1-3 & 1 & 1\\1 & 1-3 & 1\\1 & 1 & 1-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}-2 & 1 & 1\\1 & -2 & 1\\1 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-2x_1+x_2+x_3=0$$

$$x_1-2x_2+x_3=0$$

$$x_1+x_2-2x_3=0$$

Take $x_1=x_2=x_3=k$.

$$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}k\\k\\k\end{bmatrix}=k\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Hence the eigen vectors of A corresponding to eigen value $\lambda=3$ are $\begin{bmatrix}1\\1\\1\end{bmatrix}$

Thus, the eigen values & eigen vectors of A are 0,0,3 &

$\begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ $\begin{bmatrix}1\\1\\1\end{bmatrix}$ // (Ans) (iv). $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (1)$$

The characteristic equ of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow 8-\lambda\left(\left(7-\lambda\right)\left(3-\lambda\right) - (-4)(-4)\right) + 6\left((-6)(3-\lambda) - 2(-4)\right) + 2\left((-6)(-4) - 2(7-\lambda)\right) = 0$

$\Rightarrow 8-\lambda\left(21 - 7\lambda - 3\lambda + \lambda^2 + 8\right) + 6\left(-18 + 6\lambda + 8\right) + 2\left(24 - 14 + 2\lambda\right) = 0$

$\cancel{\Rightarrow 168 - 56\lambda - 24\lambda + 8\lambda^2 + 64 - 21\lambda + 7\lambda^2 + 3\lambda^2 - \lambda^3 - 8\lambda - 108 + 36\lambda + 48 + 48 - 28 + 4\lambda}$

$\cancel{\Rightarrow -\lambda^3 + 18\lambda^2 + 40\lambda - 109\lambda +}$

$\Rightarrow 8-\lambda\left(\lambda^2 - 10\lambda + 29\right) + 6\left(6\lambda - 10\right) + 2\left(2\lambda + 10\right) = 0$

$\Rightarrow 8\lambda^2 - 80\lambda + 232 - \lambda^3 + 10\lambda^2 - 29\lambda + 36\lambda - 60 + 4\lambda + 20 =$

$\Rightarrow -\lambda^3 + 18\lambda^2 + \cancel{14} \; 40\lambda - 109\lambda +$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda(\lambda-3)(\lambda-15) = 0$

$\therefore \boxed{\lambda = 0, 3, 15}$

Eigen vector corresponding to eigen value $\lambda = 0$:-

Put $\lambda=0$ in eq(1)

$$\begin{bmatrix} 8-\cancel{0} & -6 & 2 \\ -6 & 7-\cancel{0} & -4 \\ 2 & -4 & 3-\cancel{0} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

Solving (1)st two equ's

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -6 & & 2 & & 8 & & -6 \\ 7 & & -4 & & -6 & & 7 \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56+36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

i.e $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k$

$\therefore x_1 = k,\ x_2 = 2k,\ x_3 = 2k$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 0$ is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Eigen vector corresponding to $\lambda = 3$

put $\lambda = 3$ in eq (1)

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 + 4x_3 = 0$

$2x_1 - 4x_2 + 0x_3 = 0$.

Now solving first 2 equ

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{array}$$

$$\frac{x_1}{+24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{+16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore x_1 = -2,\ x_2 = -1,\ x_3 = +2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ +2 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 3$ is $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 15$

put $\lambda = 5$ in eq(1)

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 - 6x_2 - 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

$2x_1 - 4x_2 - 12x_3 = 0.$

Solving 1st two eq's.

$$\begin{array}{ccccc} & x_1 & & x_2 & x_3 \\ -6 & & -2 & -7 & -6 \\ -8 & & -4 & -6 & -8 \end{array}$$

$$\frac{x_1}{24-16} = \frac{x_2}{12-28} = \frac{x_3}{56-36}.$$

$$\frac{x_1}{8} = \frac{x_2}{-16} = \frac{x_3}{20}$$

$x_1 = 2, x_2 = -2, x_3 = 1.$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 15$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Hence the eigen value of A are 0, 3, 15 & the corresponding eigen vectors of A are $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

= Ans.

(V). $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

The characteristic equ of 'A' is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0.$$

$$(2-\lambda)\left[(5-\lambda)(3-\lambda) - 0\right] - 2\left[2(3-\lambda) - 0\right] + 0$$

$$(2-\lambda)(15 - 5\lambda - 3\lambda + \lambda^2) - 4(3-\lambda) = 0.$$

$$30 - 10\lambda - 6\lambda + 2\lambda^2 - 15\lambda + 5\lambda^2 + 3\lambda^2 - \lambda^3 - 12 + 4\lambda = 0$$

$$\Rightarrow -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0.$$

$-(\lambda^3-10\lambda^2+27\lambda-18)=0$

$\lambda^3-10\lambda^2+27\lambda-18=0$.

Dividing with $(\lambda-1)$

S.D

$\lambda=1$	1	-10	27	-18
	0	1	-9	18
	1	-9	18	$\boxed{0}$

$\Rightarrow (\lambda-1)(\lambda^2-9\lambda+18)=0$

$(\lambda-1)(\lambda^2-6\lambda-3\lambda+18)=0$

$(\lambda-1)[\lambda(\lambda-6)-3(\lambda-6)]=0$

$\lambda-1=0,\ \lambda-6=0,\ \lambda-3=0$

$\lambda=1,\ \lambda=6,\ \lambda=3$

$\therefore \boxed{\lambda=1,6,3}$

$\therefore$ Thus, the eigen values of 'A' are 1, 6, 3.

Eigen vector corresponding to $\lambda=1$:-

$$\begin{bmatrix}1&2&0\\2&4&0\\0&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1+2x_2=0$

$2x_1+4x_2=0$

$2x_3=0$.

$\boxed{x_3=0}$

Consider $x_2=k$

$\Rightarrow x_1+2(k)=0$

$x_1=-2k$

$$\therefore \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2k\\k\\0\end{bmatrix}=k\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

Eigen vector corresponding to $\lambda=3$:—

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1+2x_2=0 \Rightarrow 2x_2=x_1 \rightarrow (1)$

$2x_1+2x_2=0 \Rightarrow \cancel{2x_2=2x_1 \Rightarrow x_1}$

$\Rightarrow x_1+x_2=0 \rightarrow (2)$

From eq (1) & (2) we get

$2x_2+\cancel{x_1}+x_2-\cancel{x_1}=0$

$3x_2=0$

$\boxed{x_2=0}$

(1) $\Rightarrow 2(0)=x_1$

$\Rightarrow \boxed{x_1=0.}$

Consider $x_3=k$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad [k=1]$$

$\therefore$ The eigen vector corresponding to $\lambda=3$ is $X_2=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 6$:-

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x_1 + 2x_2 = 0$

$2x_1 - x_2 = 0 \Rightarrow 2x_1 = x_2$

$-3x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

Let $\boxed{x_1 = k}$

Then $2x_1 = x_2 \Rightarrow 2(k) = x_2 \Rightarrow \boxed{x_2 = 2k}$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} k \\ 2k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ The eigen vector corresponding to $\lambda = 6$ is $X_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\therefore$ The eigen vectors corresponding to $\lambda = 1, 3, 6$ is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

= Ans)

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3) Find the Eigen values and eigen vectors of

(i) $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 7 & 0 \\ 2 & 6 & 1 \end{bmatrix}$

Sol:-

(i) given, $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

Consider, $(A-\lambda I)X = O$

i.e., $\begin{pmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cancel{x_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ —(1)

The characteristic equation is $|A-\lambda I| = 0$.

$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(5-\lambda) - (1)(4) = 0.$

$\Rightarrow 10 - 2\lambda - 5\lambda + \lambda^2 - 4 = 0.$

$\Rightarrow \lambda^2 - 5\lambda - 2\lambda + 10 - 4 = 0.$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0.$

$\Rightarrow \lambda^2 - \cancel{\lambda}6\lambda + 6 = 0.$

~~$\Rightarrow \lambda(\lambda-6)+6=0$~~ $\Rightarrow \lambda(\lambda-1) - 6(\lambda-1) = 0$

~~$\Rightarrow \lambda - 6 \lambda + 6$~~ $\Rightarrow \lambda - 1 = 0, \lambda - 6 = 0$

$\lambda = 1, \lambda = 6.$

$\therefore$ The Eigen values are $\lambda = 1, 6.$

Case (i):- The corresponding eigen vectors ~~values~~ of $\boxed{1, \lambda = 1}$:-

(1) $\Rightarrow \begin{pmatrix} 2-1 & 1 \\ 4 & 5-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 + x_2 = 0$

$\Rightarrow 4x_1 + 4x_2 = 0.$

Take, $x_1 = k$, $x_2 = -k$.

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ -k \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a Eigen vector corresponding to $\boxed{\lambda = 1}$

Case (ii) :- The corresponding Eigen vectors of $\boxed{\lambda = 6}$:-

$$① \Rightarrow \begin{pmatrix} 2-6 & 1 \\ 4 & 5-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ — } ③$$

$$\Rightarrow \begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -4x_1 + x_2 = 0.$

$\Rightarrow 4x_1 - x_2 = 0.$

Take, $x_1 = k$, $x_2 = 4k$.

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ 4k \end{pmatrix} = k\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is a Eigen vector corresponding to $\boxed{\lambda = 6}$

$\therefore$ The Eigen vectors for Eigen values 1, 6 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ //(Ans).

(ii) given, $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 7 & 0 \\ 2 & 6 & 1 \end{bmatrix}$

Consider $(A-\lambda I)X = 0$

i.e.; $$\begin{pmatrix} 3-\lambda & 0 & 0 \\ 5 & 7-\lambda & 0 \\ 2 & 6 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{--- (1)}$$

The characteristic equation is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 5 & 7-\lambda & 0 \\ 2 & 6 & 1-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (3-\lambda)\left[(7-\lambda)(1-\lambda)-(6)(0)\right] - 0\left[5(1-\lambda)-0(2)\right] + \underbrace{0\left[5(6)-(7-\lambda)(2)\right]}_{=0}$$

$$\Rightarrow (3-\lambda)\left[(7-\lambda)(1-\lambda)\cancel{-6}\right] \underbrace{\cancel{-0[5}}_{=0}$$

$$\Rightarrow (3-\lambda)\left[(7-7\lambda-\lambda+\lambda^2)\cancel{-6}\right] = 0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-7\lambda-\lambda+7\right] = 0.$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-8\lambda+7\right] = 0.$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-\lambda-7\lambda+7\right] = 0.$$

$$\Rightarrow (3-\lambda)\left[\lambda(\lambda-1)-7(\lambda-1)\right] = 0$$

$$\Rightarrow (3-\lambda)(\lambda-1)(\lambda-7) = 0$$

$\Rightarrow (3-\lambda) = 0,$ $3 = \lambda$ $\boxed{\lambda = 3}$ | $\lambda - 1 = 0$ $\boxed{\lambda = 1}$ | $\lambda - 7 = 0$ $\boxed{\lambda = 7}$

$\therefore$ The Eigen values are $\lambda = 3, 1, 7$

case(i) :- The Eigen vectors are corresponding to $\boxed{\lambda=1}$

$$① \Rightarrow \begin{pmatrix} 3-1 & 0 & 0 \\ 5 & 7-1 & 0 \\ 2 & 6 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 5 & 6 & 0 \\ 2 & 6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \boxed{x_1 = 0}$

$\Rightarrow 5x_1 + 6x_2 = 0. \Rightarrow 5(0) + 6x_2 = 0$

$\Rightarrow 6x_2 = 0$

$\Rightarrow 2x_1 + 6x_2 = 0.$ $\Rightarrow \boxed{x_2 = 0}$

$\Rightarrow \cancel{2(0) + 6(0)} = 0$

Take, ~~k=1~~ $\boxed{x_3 = k}$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$$

Take, $K = 1$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is The Eigen vectors Corresponding to $\boxed{\lambda = 1}$

case(ii):- The Eigen vectors are corresponding to $\boxed{\lambda=3}$

$$① \Rightarrow \begin{pmatrix} 3-3 & 0 & 0 \\ 5 & 7-3 & 0 \\ 2 & 6 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

~~The characteristic~~ $\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \cancel{x_1 = 0}$

$\Rightarrow 5x_1 + 4x_2 = 0 \Rightarrow 5x_1 = -4x_2 \Rightarrow x_1 = \frac{-4}{5}x_2$

$\Rightarrow x_1 = -4/5\, x_2$

$\Rightarrow 2x_1 + 6x_2 - 2x_3 = 0.$

$\Rightarrow 2\left(\frac{-4}{5}x_2\right) + 6x_2 - 2x_3 = 0$

$\Rightarrow \frac{-8}{5}x_2 + 6x_2 - 2x_3 = 0$

$\Rightarrow \cancel{\frac{-8x_2 + 6x_2 - 2x_3}{5} = 0}$ $\qquad \frac{22x_2}{5} - 2x_3 = 0$

$\Rightarrow \cancel{-40x_2 + 30x_2 - 10x_3 = 0}$ $\qquad 22x_2 - 10x_3 = 0$

$22x_2 = 10x_3.$

$\Rightarrow \cancel{\frac{-4}{5}\,=0} \Rightarrow \cancel{-2x_2}\; 2x_3$

$\Rightarrow x_3 = \frac{22}{10}$

$\Rightarrow x_3 = \frac{\cancel{22}\,11}{\cancel{10}\,5}x_2$

$\Rightarrow x_3 = \frac{11}{5}x_2.$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-4}{5}x_2 \\ x_2 \\ \frac{11}{5}x_2 \end{bmatrix} = \frac{-x_2}{5}\begin{bmatrix} 4 \\ -5 \\ -11 \end{bmatrix}$$

$\begin{pmatrix} 4 \\ -5 \\ -11 \end{pmatrix}$ is The Eigen vectors of the corresponding vectors to $\boxed{\lambda = 3}$

Case (iii) The Eigen vectors Corresponding to $\boxed{\lambda = 7}$:-

$$① \Rightarrow \begin{pmatrix} 3-7 & 0 & 0 \\ 5 & 7-7 & 0 \\ 2 & 6 & 1-7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -4x_1 = 0 \quad \Rightarrow \boxed{x_1 = 0}$

$\Rightarrow 5x_1 = 0 \quad \Rightarrow \boxed{x_1 = 0}$

$\Rightarrow 2x_1 + 6x_2 - 6x_3 = 0$

$\Rightarrow 2(0) + 6x_2 - 6x_3 = 0$

$\Rightarrow 6x_2 - 6x_3 = 0$

$\Rightarrow \cancel{6x_2 - 6x_3 = 0} \quad 6x_2 = 6x_3$

$\Rightarrow \boxed{x_2 = x_3}$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is the Eigen vector of the corresponding vectors to $\lambda = 6$

$\therefore$ The Eigen vectors for Eigen values $= 1, 3, 7$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

= (ANS)

---

4) Find the characteristic ~~equation~~ roots and characteristic vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Sol:-

given, $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

~~The characteristic equation $|A-\lambda I|$~~

Consider, $(A-\lambda I)X=0$

i.e., $\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ —(1)

The characteristic equation $|A-\lambda I|=0$

$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$C_2 \to C_2 + C_3$

$\Rightarrow \begin{vmatrix} 6-\lambda & 0 & 2 \\ -2 & 2-\lambda & -1 \\ 2 & 2-\lambda & 3-\lambda \end{vmatrix} = 0.$

Taking, $(2-\lambda)$ common from column c_2

$\Rightarrow (2-\lambda)\begin{vmatrix} 6-\lambda & 0 & 2 \\ -2 & 1 & -1 \\ 2 & 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)\left[(6-\lambda)\{(3-\lambda)(1)+1\}\right] - 0\left[-2(3-\lambda)-2(-1)\right] + 2\left[(-2)(1)-2(1)\right] = 0$

$\Rightarrow (2-\lambda)\left[(6-\lambda)(3-\lambda+1)\right] - 0 \{+2(-2-2)\} = 0$

$\Rightarrow (2-\lambda)\left[(6-\lambda)(4-\lambda)\right] + 2(-4)\} = 0$

$\Rightarrow (2-\lambda)\left[(6-\lambda)(4-\lambda)-8\right] = 0$

$\Rightarrow (2-\lambda)\left[24-6\lambda-4\lambda+\lambda^2-8\right]=0.$

$\Rightarrow (2-\lambda)\left[\lambda^2-10\lambda+16\right]=0$

$\Rightarrow (2-\lambda)\left[\lambda^2-2\lambda-8\lambda+16\right]=0$

$\Rightarrow (2-\lambda)\left[\lambda(\lambda-2)-8(\lambda-2)\right]=0$

$\Rightarrow (2-\lambda)(\lambda-2)(\lambda-8)=0$

$2-\lambda=0$	$\lambda-2=0$	$\lambda-8=0$
$\lambda=+2$	$\lambda=2$	$\lambda=8.$

$\therefore$ The characteristic ~~eigen values~~ roots are $\lambda=2,2,8.$

Case(i). The characteristic vectors ~~roots~~ are corresponding to $\boxed{\lambda=2}$:

$$① \Rightarrow \begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

~~$R_2 \to R_2 + 2R_1$~~

~~$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0.$~~ $R_3 \to R_3 + R_2$

~~$\Rightarrow -2x_1 + x_2 - x_3 = 0$~~

~~$\Rightarrow 2x_1 - x_2 + x_3 = 0.$~~

$$\Rightarrow \begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to 2R_2 + R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$

$\Rightarrow 2(2x_1 - x_2 + x_3) = 0$

$\Rightarrow 2x_1 - x_2 + x_3 = 0 \quad \text{—(1)}.$

let, $x_3 = k_1$ and $x_2 = k_2$

(1) $\Rightarrow 2x_1 - k_2 + k_1 = 0.$

$\Rightarrow x_1 = \dfrac{k_2 - k_1}{2}.$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k_2 - k_1}{2} \\ k_2 \\ k_1 \end{bmatrix} = \frac{k_2}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{k_1}{2}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$\therefore$ The characteristic vectors relating to $\lambda = 2$ are given by

$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ //

case (ii): The characteristic vector corresponding to $\boxed{\lambda = 8}$:-

$$(1) \Rightarrow \begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to \frac{R_1}{2}$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to \frac{R_2}{-3}$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 - x_2 + x_3 = 0 \longrightarrow (1)$

$\Rightarrow x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$ ~~(1) ⇒ 2x₁ − x₂ + x₃ = 0.~~

~~From (1) we have~~

~~(1) ⇒ 2x₁ − x₂ + x₃ = 0.~~

~~$x_1 = -x_2 + x_3$~~

~~$x_3 =$~~

(1) $\Rightarrow -x_1 - x_2 + x_3 = 0.$

$+x_1 = -x_2 + x_3.$

$\Rightarrow x_1 = x_3 + x_3.$

$$\boxed{x_1 = 2x_3}$$

Take $\boxed{x_3 = k}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ The characteristic vector corresponding to $\lambda = 8$ is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ = (Ans)

---

(5) Find the eigen values of the following system $8x - 4y = \lambda x$, $2x + 2y = \lambda y$.

Sol:-

Given system

$$8x - 4y = \lambda x$$

$$8x - 4y - \lambda x = 0$$

$$(8 - \lambda)x - 4y = 0$$

$$2x + 2y = \lambda y$$

$$2x + 2y - \lambda y = 0$$

$$(2 - \lambda)y + 2x = 0$$

Take $A = \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

The eigen value of A is $|A - kI| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda-k & -4 \\ 2 & 2-\lambda-k \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda-k)(2-\lambda-k) - 2(-4) = 0$$

$$\Rightarrow (8-\lambda-k)(2-\lambda-k) + 8 = 0$$

$$\Rightarrow 16 - 8\lambda - 8k - 2\lambda + \lambda^2 + \lambda k - 2k + \lambda k + k^2 + 8 = 0.$$

$$\Rightarrow k^2 - 10k + 2\lambda k + (24 - 10\lambda) = 0$$

$$\Rightarrow \underset{a}{k^2} + k\underset{b}{(2\lambda - 10)} + \underset{c}{(24 - 10\lambda)} = 0.$$

This is a quadratic in k.

$$\therefore k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(2\lambda - 10) \pm \sqrt{(2\lambda-10)^2 - 4(1)(24-10\lambda)}}{2(1)}$$

$$= \frac{(10-2\lambda) \pm \sqrt{(2\lambda-10)^2 - 4(24-10\lambda)}}{2}$$

$$= \frac{2(5-\lambda) \pm \sqrt{4\lambda^2 + 100 - 2(2\lambda)(10) - 96 + 40\lambda}}{2}$$

$$= \frac{2(5-\lambda) \pm \sqrt{4\lambda^2 + 100 - 4\lambda - 96 + 40\lambda}}{2}$$

$$= \frac{2(5-\lambda) \pm \sqrt{4\lambda^2 + 4}}{2}$$

$$= (5-\lambda) \pm \sqrt{\lambda^2 + 1}$$

Hence $(5-\lambda) + \sqrt{\lambda^2+1}$ & $(5-\lambda) - \sqrt{\lambda^2+1}$ are eigen values

(Ans)

(6). Find the sum & product of the eigen values of the matrix:-

(i). $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{bmatrix}$ (ii). $A = \begin{bmatrix} 2 & 5 & 7 \\ 1 & 4 & 6 \\ 2 & -2 & 3 \end{bmatrix}$

Sol^n's

(i)

Sol:- Given $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

(a). sum of the eigen values = trace of the matrix = sum of the diagonal elements

$= 2+4+2$

$= 8$

(b). product of the eigen values = $\det A = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{vmatrix}$

$= 2(4\times2 - 0\times2) - 1(3\times2 - 1\times2) - 1(3\times0 - 4\times1)$

$= 2(8) - 1(6-2) - 1(-4)$

$= 16 - 1(4) - 1(-4)$

$= 16 - \cancel{4} + \cancel{4}$

$= 16$

(Ans).

(ii)

Sol:- $A = \begin{bmatrix} 2 & 5 & 7 \\ 1 & 4 & 6 \\ 2 & -2 & 3 \end{bmatrix}$

(a). sum of the eigen values = Trace of the matrice

= sum of the diagonal elements

$= 2+4+3 = 9$

(b). product of the eigen values $\Rightarrow \det A = \begin{vmatrix} 2 & 5 & 7 \\ 1 & 4 & 6 \\ 2 & -2 & 3 \end{vmatrix}$

$= 2(4\times 3-(-2)(6)) - 5(1\times 3 - 2\times 6) + 7(1(-2)-2(4))$

$= 2(12+12) - 5(3-12) + 7(-2-8)$

$= 2(24) - 5(-9) + 7(-10)$

$= 48 + 45 - 70$

$= 93 - 70$

$= 23$

(Ans)

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⑦ Find the eigen values & eigen vectors of the matrix A & its inverse, where $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

Sol:- Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

$(A - I)X = 0$

$$\begin{bmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

The characteristic equ of 'A' is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda) - 0\times 5] - 3[0(3-\lambda) - 0\times 5] + 4(0\times 0 - 0(2-\lambda)] = 0$

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)] = 0$

$(1-\lambda) = 0$ | $2-\lambda = 0$ | $3-\lambda = 0$

$\boxed{\lambda = 1}$ | $\boxed{\lambda = 2}$ | $\boxed{\lambda = 3}$

$\therefore \lambda = 1, 2, 3.$

$\therefore$ The eigen values are $\lambda = 1, 2, 3.$

(i). To Find characteristic vector of $\lambda = 1$:-

sub $\lambda = 1$ in eq (1)

$$\begin{bmatrix} 1-1 & 3 & 4 \\ 0 & 2-1 & 5 \\ 0 & 0 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_2 + 4x_3 = 0 \rightarrow (i)$

$x_2 + 5x_3 = 0 \longrightarrow (ii)$

$2x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

since 'x_1' is not there in any of the equ, take $\boxed{x_1 = \alpha}$

$(i) \Rightarrow 3x_2 + 4(0) = 0$

$3x_2 = 0$

$\boxed{x_2 = 0}$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Characteristic vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(ii). To Find characteristic vector of '2' :-

sub. $\lambda = 2$ in eq (2)

$$\begin{bmatrix} 1-2 & 3 & 4 \\ 0 & 2-2 & 5 \\ 0 & 0 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1+3x_2+4x_3=0$

$5x_2=0 \Rightarrow x_2=0$

$x_3=0$

Take $x_1=k$

$$\therefore X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}k\\0\\0\end{bmatrix}=k\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$\therefore$ Hence, characteristic vector is $\begin{bmatrix}1\\0\\0\end{bmatrix}$

(iii). To find characteristic vector of '3' :-

$$\begin{bmatrix}1-3 & 3 & 4\\0 & 2-3 & 5\\0 & 0 & 3-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2 & 3 & 4\\0 & -1 & 5\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-2x_1+3x_2+4x_3=0 \rightarrow (i)$

$-x_2+5x_3=0 \rightarrow (ii)$

$5x_3=x_2$

Take $\boxed{x_3=k}$ } sub in (i)

$\Rightarrow \boxed{5k=x_2}$

$-2x_1+3(5k)+4(k)=0$

$-2x_1+15k+4k=0$

$-2x_1+19k=0$

$2x_1=19k$

$\boxed{x_1=\frac{19}{2}k}$

$$\therefore X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\frac{19}{2}\\5k\\k\end{bmatrix}=\frac{k}{2}\begin{bmatrix}19\\10\\2\end{bmatrix}$$

$\therefore$ Characteristic vector is $\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$

Finding eigen values & eigen vectors of A^{-1}:-

W.K.T the eigen values of A^{-1} are the reciprocals of the eigen values of A & eigen vectors of A^{-1} are same as eigen vectors of matrix "A".

Hence, eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ i.e

The eigen values of A are : 1, 2, 3
$(\lambda_1)(\lambda_2)(\lambda_3)$

$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}$

$1, \frac{1}{2}, \frac{1}{3}$

and eigen vectors of A^{-1} are same as eigen vectors of the matrix A.

=(Ans)

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(8). Find the eigen values of A^{-1}, if $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Sol:- Given

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

The given matrix is an upper Δ^{lar} matrix.

so, the eigen values of A are its diagonal elements 2, 4, 3

If λ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

$\therefore$ The eigen values of A are 2, 4, 3

Consider $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 4$

$\therefore$ The eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

=(Ans)

(9). Find the eigen values of adj A, if $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol:-

Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic equ of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$

$1-\lambda=0$	$2-\lambda=0$	$3-\lambda=0$
$\boxed{\lambda=1}$	$\boxed{\lambda=2}$	$\boxed{\lambda=3}$

$\therefore \lambda = 1, 2, 3.$

Now det (A) (or) $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$

$= 1(6-2) + 0(3-2) - 1(2-4)$

$= 1(4) + 0 - 1(-2)$

$= 4 + 2$

$= 6.$

$\therefore$ The eigen values of adj A are $\frac{|A|}{\lambda}$

$\Rightarrow \frac{6}{1}, \frac{6}{2}, \frac{6}{3}$

$\Rightarrow 6, 3, 2$

= (Ans).

(10). For the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3 + 5A^2 - 6A + 2I$.

Sol:- Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

The characteristic equ of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(3-\lambda)(-2-\lambda) - 0(2)\right] - 2\left[0(-2-\lambda) - 0\times 2\right] - 3\left[0\times 0 - (3-\lambda)(0)\right] = 0$

$\Rightarrow (1-\lambda)\left[(3-\lambda)(-2-\lambda)\right] = 0$

$\Rightarrow (1-\lambda)(3-\lambda)(-2-\lambda) = 0$

$\Rightarrow (1-\lambda)(3-\lambda)(-(2+\lambda)) = 0$

$\Rightarrow (1-\lambda)(3-\lambda)(2+\lambda) = 0$

$1-\lambda = 0$	$3-\lambda = 0$	$2+\lambda = 0$
$\lambda = 1$	$\lambda = 3$	$\lambda = -2$

$\therefore \lambda = 1, 3, -2$

$\therefore$ Eigen values of A are $\lambda = 1, 3, -2$

Let $f(A) = 3A^3 + 5A^2 - 6A + 2I$. [$\because$ Put $I = 1$]

Then the eigen values of $f(A)$ are $f(1), f(3), f(-2)$.

$\therefore f(1) = 3(1)^3 + 5(1)^2 - 6(1) + 2(1)$

$= 3 + 5 - 6 + 2$

$= 10 - 6$

$\therefore \boxed{f(1) = 4}$

$\therefore f(3) = 3(3)^3 + 5(3)^2 - 6(3) + 2(1)$

$\boxed{f(3) = 110}$

$\therefore f(-2) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2(1)$

$\boxed{f(-2) = 10}$

$\therefore$ Thus the eigen values of $3A^3 + 5A^2 - 6A + 2I$ are

4, 110, 10

= (Ans)

---

(11). If 2, 3, 5 are the eigen values of a matrix A, then find the eigen values of $2A^3 + 3A^2 + 5A + 3I$.

Sol:- The eigen values are $\lambda = 2, 3, 5$ (given)

Let $f(A) = 2A^3 + 3A^2 + 5A + 3I$

Take $I = 1$

$\therefore f(2) = 2(2)^3 + 3(2)^2 + 5(2) + 3(1) = 41$

$\therefore f(3) = 2(3)^3 + 3(3)^2 + 5(3) + 3(1) = 99$

$\therefore f(5) = 2(5)^3 + 3(5)^2 + 5(5) + 3(1) = 353$.

Hence, the eigen values of $2A^3 + 3A^2 + 5A + 3I$ are

41, 99 & 353 = (Ans)

(12). Find the eigen values of the following matrices:-

(i). $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ (ii). $B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ (iii). $C = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

Sol's

(i). Given $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

Characteristic equ of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 4-\lambda & 1-3i \\ 1+3i & 7-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)(7-\lambda) - (1+3i)(1-3i) = 0$

$\Rightarrow (4-\lambda)(7-\lambda) - (1-9\lambda^2) = 0$

$\Rightarrow 28 - 11\lambda + \lambda^2 - 10 = 0$

$\Rightarrow \lambda^2 - 11\lambda + 18 = 0$

$\Rightarrow (\lambda-9)(\lambda-2) = 0$

$\therefore \lambda = 9, 2.$

Also, $\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix} = A^T.$

$\therefore$ A is Hermitian.

This verifies that the eigen values of a Hermitian matrix are real.

= (Ans).

(ii). Given $B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

so, $\bar{B} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$

$B^T = \begin{bmatrix} 3i & 2+i \\ 2-i & -i \end{bmatrix}$

$\therefore \bar{B} = -B^T.$

Thus, B is a skew-Hermitian matrix.

The characteristic equ of B is

$|B-\lambda I|=0$

$$\begin{vmatrix} 3i-\lambda & 2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3i-\lambda)(-i-\lambda) - (-2+i)(2+i) = 0$

$\Rightarrow -3i^2 - 3i\lambda + \lambda i + \lambda^2 - [-4-2i + 2i + i^2] = 0$

$\Rightarrow -3i^2 - 3i\lambda + \lambda i + \lambda^2 + 4 + \cancel{2i} - \cancel{2i} + i^2 = 0.$ (176)

$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$ $\quad [\because i^2 = -1]$

The roots are $4i, -2i$.

This verifies that the roots of skew-Hermitian matrices will be purely imaginary (or) zero.

(iii). $C = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$.

$\bar{C} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$

$\bar{C}^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$

We can see that $(\bar{C})^T C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Thus 'C' is a unitary matrix.

The characteristic equ of C is $|C - \lambda I| = 0$

$$\begin{vmatrix} \frac{1}{2}i - \lambda & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i - \lambda \end{vmatrix} = 0.$$

$\therefore \lambda = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ & $\frac{-\sqrt{3}}{2} + \frac{1}{2}i$ are eigen values.

$$\left|\frac{\pm\sqrt{3}}{2} + \frac{1}{2}i\right|^2 = 1.$$

Thus, the characteristic roots of unitary matrix have absolute value 1.

(Ans)

(13). Diagonalize the matrix :-

(i). $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (ii). $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Sol's

(i)

Sol:- Given that $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (8-\lambda)\,[(-3-\lambda)(1-\lambda)-8] + 8[4(1-\lambda)+6] - 2[-16-3(-3-\lambda)] = 0.$

$\Rightarrow (8-\lambda)\,[(-3-\lambda)(1-\lambda)-8] + 8[4(1-\lambda)+6] - 2[-19(-3-\lambda)] = 0.$

$\Rightarrow (8-\lambda)[-3+3\lambda-\lambda+\lambda^2-8] + 8[4-4\lambda+6] - 2[57+19\lambda] = 0.$

$\Rightarrow (8-\lambda)\,[\lambda^2-8-3+3\lambda-\lambda] + 8[6+4-4\lambda] - 2[57+19\lambda] = 0.$

$\Rightarrow (8-\lambda)\,[\lambda^2+2\lambda-11] + 8[-4\lambda+10] - 2[19\lambda+57] = 0.$

$\Rightarrow 8\lambda^2+16\lambda-88-\lambda^3-2\lambda^2+11\lambda-32\lambda+80-38\lambda-114 = 0.$

$\Rightarrow -\lambda^3+8\lambda^2-2\lambda^2+16\lambda+11\lambda-32\lambda-38\lambda-88+80-114 = 0.$

$\Rightarrow -\lambda^3+8\lambda^2-2\lambda^2-38\lambda-32\lambda+27\lambda+80-114-88 = 0.$

$\Rightarrow$ ~~$-\lambda^3+6\lambda^2-43\lambda-122=0$~~ $-\lambda^3+6\lambda^2-11\lambda+6=0.$

$\Rightarrow -(\lambda^3-6\lambda^2+11\lambda-6) = 0.$

$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6 = 0.$

$\lambda^3 \quad \lambda^2 \quad \lambda \quad I.$

Take, $\boxed{\lambda=1}$ is a root

$(\lambda-1)(\lambda^2-5\lambda+6)=0$

Synthetic Division

1	1	−6	11	−6
	0	1	−5	6
	1	−5	6	$\boxed{0}$

$\cancel{(\lambda-1)(\lambda^2+\lambda-6\lambda+6)=0}$

$\cancel{(\lambda-1)(\lambda(\lambda+1)-6(\lambda+1))}$

$(\lambda-1)(\lambda^2-2\lambda-3\lambda+6)=0$

$(\lambda-1)(\lambda(\lambda-2)-3(\lambda-2))=0.$

$(\lambda-1)(\lambda-2)(\lambda-3)=0.$

$\therefore \boxed{\lambda=3,1,2}$

$\therefore$ Diagonal matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

To find the Eigen vector for $\boxed{\lambda=3}$

The Eigen vector corresponding to $\boxed{\lambda=3}$

$\Rightarrow (A-\lambda I)X=0.$

$$\Rightarrow \begin{pmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 8-3 & -8 & -2 \\ 4 & -3-3 & -2 \\ 3 & -4 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$5x_1 - 8x_2 - 2x_3 = 0$ —— ①

$4x_1 - 6x_2 - 2x_3 = 0$ —— ②

$3x_1 - 4x_2 - 2x_3 = 0$. —— ③.

Solving, eqn ① & ②.

we get,

x_1		x_2	x_3
-8	-2	5	-8
-6	-2	4	-6

$$\frac{x_1}{16-12} = \frac{x_2}{-8+10} = \frac{x_3}{-30+32}$$

$$\frac{x_1}{\cancel{4}\,2} = \frac{x_2}{\cancel{2}\,1} = \frac{x_3}{\cancel{2}\,1}$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vector corresponding to $\boxed{\lambda=3}$ is $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

To find the Eigen vector for $\boxed{\lambda=1}$

The Eigen vector corresponding to $\boxed{\lambda=1}$

$\Rightarrow (A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} 8-1 & -8 & -2 \\ 4 & -3-1 & -2 \\ 3 & -4 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$7x_1 - 8x_2 - 2x_3 = 0. \quad \text{—} (1)$

$4x_1 - 4x_2 - 2x_3 = 0. \quad \text{—} (2)$

$3x_1 - 4x_2 = 0. \quad \text{—} (3).$

Solving, Eq^n (1) & (2).

we get,

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ \hline -8 & -2 & 7 & -8 \\ -4 & -2 & 4 & -4 \end{array}$$

$$\frac{x_1}{16-(8)} = \frac{x_2}{-8-(-14)} = \frac{x_3}{-28-(-32)}$$

$-8+14. \qquad -28+32$

$$\frac{x_1}{\cancel{8}\ 4} = \frac{x_2}{\cancel{6}\ 3} = \frac{x_3}{\cancel{4}\ 2}$$

$$\cancel{X_2} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

$\therefore$ Eigen vectors corresponding to $\boxed{\lambda = 1}$ is $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

To find the eigen vector for $\boxed{\lambda = 2}$

The eigen vector corresponding to $\boxed{\lambda = 2}$

$\Rightarrow (A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} \cancel{8-2}\ 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 6x_1 - 8x_2 - 2x_3 = 0$ — (1)

$\Rightarrow 4x_1 - 5x_2 - 2x_3 = 0$ — (2)

$\Rightarrow 3x_1 - 4x_2 - 1x_3 = 0$ — (3).

solving eqn (1) & (2).

we get

x_1	x_2	x_3	
-8	6	-2	-8
-5	4	-2	-5

ad-bc.

$$\frac{x_1}{-32-(-30)} = \frac{x_2}{-12-(-8)} = \frac{x_3}{10+(16)}$$

$32+30 \qquad -12+8 \qquad 26.$

$\cancel{62} \qquad -\cancel{4} \qquad \cancel{26}.$

$31 \qquad -2 \qquad 13$

$R_3 \to R_3 - \frac{1}{2}R_1$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 6 & -8 & -2 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow 6x_1 - 8x_2 - 2x_3 = 0$ —— (1)

$\Rightarrow -2x_1 + 3x_2 = 0$

$\Rightarrow 3x_2 = 2x_1$ —— (2).

solving ~~from~~ (1) & (2) we get. let $x_2 = k_3$

then

x_1	x_2	x_3	
-8	6	-2	-8.
3	0	-2	3

$x_1 = \frac{3}{2}k_3$

$x_3 = -\frac{1}{2}k_3$

$$\frac{x_1}{-18} = \frac{x_2}{-12} = \frac{x_3}{-6-(16)}$$

$$\frac{x_1}{-18} = \frac{x_2}{-12} = \frac{x_3}{-6-16}$$

$6 \quad 4 \quad -22$

$2 \quad$

$3.$

$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{2}k_3 \\ k_3 \\ -\frac{1}{2}k_3 \end{bmatrix}$

$X_3 = \frac{k_3}{2}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Thus $P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ which is called the modal matrix of 'A'.

Hence, $P^{-1}AP = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (Ans)

(ii)

given, $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Characteristic Equation of A is $|A - \lambda I| = 0$.

$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0.$

$C_3 \to C_3 + C_2$

$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & 0 \\ 1 & 2-\lambda & 3-\lambda \\ -1 & -1 & -\lambda-1 \end{vmatrix} = 0.$

$\Rightarrow (-1-\lambda)[(2-\lambda)(-\lambda-1) + 1(3-\lambda)] - 2[(-\lambda-1) + 1(3-\lambda)] = 0.$

$\Rightarrow \cancel{\lambda^3}\lambda$

$-2[(\cancel{-\lambda-1})]$

$\Rightarrow \cancel{(-1-\lambda)[(2-\lambda)(-\lambda-1)+(3-\lambda)]} \cancel{+[2\lambda+2+3-3\lambda]} = 0.$

$\Rightarrow (-1-\lambda)[(2-\lambda)(-\lambda-1) + (3-\lambda)] - 2[(-\lambda-1) + (3-\lambda)] = 0$

$\Rightarrow (-1-\lambda)[(-2\lambda - 2 + \lambda^2 + \lambda) + (3-\lambda)] - 2[(-\lambda-1) + (3-\lambda)] = 0$

$\Rightarrow -(\lambda-1)[\lambda^2 - \lambda - 2 + 3 - \lambda] - 2[-(\lambda+1) + (3-\lambda)] = 0.$

$\Rightarrow -(\lambda-1)$

(ii). Given $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

characteristic equ of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

$+2-2 = 0$

$2+1-\lambda$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$C_3 \to C_3 + C_2$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & 0 \\ 1 & 2-\lambda & 3-\lambda \\ -1 & -1 & -\lambda-1 \end{vmatrix} = 0$$

$\Rightarrow (-1-\lambda)\left[(2-\lambda)(-\lambda-1) + 1(3-\lambda)\right] - 2\left[(-\lambda-1) + 1(3-\lambda)\right] = 0$

$\Rightarrow (-1-\lambda)\left[-2\lambda - 2 + \lambda^2 + \cancel{\lambda} + 3 - \cancel{\lambda}\right] - 2\left[(-\lambda-1) + (3-\lambda)\right] = 0.$

$\Rightarrow (-1-\lambda)\left[\lambda^2 - 2\lambda + 1\right] - 2\left[-\lambda - 1 + 3 - \lambda\right] = 0.$

$\Rightarrow (-1-\lambda)\left[\lambda^2 - 2\lambda + 1\right] - 2\left[-2\lambda + 2\right] = 0.$

$\Rightarrow (-1-\lambda)\left(\lambda^2 - 2\lambda + 1\right) - 2\left(2(-\lambda + 1)\right) = 0.$

$\Rightarrow (-1-\lambda)\left[\lambda^2 - 2\lambda + 1\right] - 4(-\lambda + 1) = 0.$

$\Rightarrow (-1-\lambda)\left[\lambda^2 - 2\lambda + 1\right] + 4\lambda - 4 = 0.$

$\Rightarrow -\lambda^2 + 2\lambda - 1 - \lambda^3 + 2\lambda^2 - \lambda + 4\lambda - 4 = 0$

$\cancel{\Rightarrow -\lambda^3 + \lambda^2 + 6\lambda}$

$\Rightarrow -\lambda^3 + \lambda^2 + 6\lambda - \lambda - 5 = 0$

$\Rightarrow -\lambda^3 + \lambda^2 + 5\lambda - 5 = 0.$

$\Rightarrow -(\lambda^3 - \lambda^2 - 5\lambda + 5) = 0$

$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda + 5 = 0.$

$\Rightarrow (\lambda - 1)(\lambda^2 - 5) = 0.$

$\lambda - 1 = 0$ | $\lambda^2 - 5 = 0$

$\boxed{\lambda = 1}$ | $\lambda^2 = 5 \Rightarrow \boxed{\lambda = \pm\sqrt{5}}$

$\lambda = 1, \pm\sqrt{5}$

$\therefore$ A is diagonalisable.

<u>TO Find the eigen vector of $\lambda = 1$:-</u>

The eigen vector corresponding to $\boxed{\lambda = 1}$

$\Rightarrow (A - \lambda I)X = 0.$

$$\Rightarrow \begin{pmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} -1-1 & 2 & -2 \\ 1 & 2-1 & 1 \\ -1 & -1 & 0-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -2x_1 + 2x_2 - 2x_3 = 0.$ —— (1)

$\Rightarrow x_1 + x_2 + x_3 = 0.$ —— (2)

$\Rightarrow -x_1 - x_2 - x_3 = 0.$ —— (3).

Solving, Eq^n (1) & (2).

we get,

x_1	x_2	x_3	
2	-2	-2	2
1	1	1	1

$$\frac{x_1}{2-(-2)} = \frac{x_2}{-2-(-2)} = \frac{x_3}{-2-(-2)}$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{-2}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

To find the Eigen vector corresponding to $\lambda = \sqrt{5}$:

The Eigen vec

$\Rightarrow (A - \lambda I) X = 0.$

$$\Rightarrow \begin{pmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & 0-\sqrt{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ The required diagonalised matrix is $P^{-1}AP =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

= (Ans)

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(14) If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ find (a) A^8 (b) A^4

Sol:-

Given, $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

The characteristic equ $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\cancel{\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-4] - 1[0-(-4)(1)] + 1[0-(-4)(2-\lambda)] = 0}$$

$$\cancel{\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-4] - 1[4] + 1[4(2-\lambda)] = 0}$$

$$\cancel{\Rightarrow (1-\lambda)[6-2\lambda-3\lambda+\lambda^2-4] - 4 + 8 - 4\lambda = 0}$$

$$\cancel{\Rightarrow (1-\lambda)[\lambda^2-5\lambda+2] - 4\lambda + 4 = 0}$$

$$\cancel{\Rightarrow \lambda^2 - 5\lambda + 2 - \lambda^3 + 6\lambda^2 - 2\lambda - 4\lambda + 4 = 0}$$

$\Rightarrow \cancel{\lambda^3 + 7\lambda^2 - 11\lambda + 4 = 0}$

(or)

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$

$1-\lambda=0$	$2-\lambda=0$	$3-\lambda=0$
$\lambda=1$	$\lambda=2$	$\lambda=3$

$\therefore$ Characteristic values of A $\lambda = 1, 2, 3$.

Characteristic vector corresponding to $\lambda=1$:-

sub $\lambda=1$ in below equ (1)

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

$$\begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 2-1 & 1 \\ -4 & 4 & 3-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y + z = 0 \longrightarrow (i)$

$y + z = 0 \longrightarrow (ii)$

$-4x + 4y + 2z = 0 \longrightarrow (iii)$

solve eq (ii) & (iii).

	x		y		z
1		1		0	1
4		2		-4	4

$$\frac{x}{2-4} = \frac{y}{-4-0} = \frac{z}{0+4}$$

$$\frac{x}{-2} = \frac{y}{-4} = \frac{z}{4}$$

$$X_1 = \begin{bmatrix} x_1 \\ y_2 \\ z_3 \end{bmatrix} = \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} = -2\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=1$.

Characteristic vector corresponding to $\lambda=2$:—

Put $\lambda=2$

$$(1) \Rightarrow \begin{bmatrix} 1-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ -4 & 4 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x+y+z=0 \rightarrow (i)$$

$$\boxed{z=0} \rightarrow (ii)$$

$$-4x+4y+z=0 \rightarrow (iii)$$

Sub $z=0$ in eq (i)

$$(i) \Rightarrow -x+y+0=0.$$

$$-x+y=0$$

$$\boxed{y=x}$$

Let $x=K$

$y = K$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} K \\ K \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$ //

<u>characteristic vector corresponding to $\lambda = 3$:-</u>

put $\lambda = 3$.

$$(1) \Rightarrow \begin{bmatrix} 1-3 & 1 & 1 \\ 0 & 2-3 & 1 \\ -4 & 4 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + z = 0 \rightarrow (i)$

$-y + z = 0 \rightarrow (ii) \Rightarrow \boxed{z = y}$

$-4x + 4y = 0 \rightarrow (iii)$.

let $y = k$

$z = k$

$x = k$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

Consider $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

We have $|P| = -1$ & $P^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$

We have $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{diag}(1,2,3) = D$ (say)

(a).

$$D^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 3^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix}$$

We have $A^8 = P D^8 P^{-1}$.

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$$

(b). $D^4 = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$

Now $A^4 = P D^4 P^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 16 & 81 \\ 2 & 16 & 81 \\ -2 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & \end{bmatrix}$$

$$= \begin{bmatrix} -1+64-162 & 1-48+162 & 0-16+81 \\ -2+64-162 & 2-48+162 & 0-16+81 \\ 2+0-162 & -2-0+162 & 0-0+81 \end{bmatrix}$$

$$= \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & -160 & 81 \end{bmatrix}$$

= (Ans)

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(15). Diagonalize $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ & hence find A^8.

Sol:-

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

~~The chara~~ $(A-\lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

The characteristic equ of A is $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-0\right] - 6\left[1(3-\lambda)-0\right] + 1\left[0-0\right] = 0$

$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)\right] - 6\left[(3-\lambda)\right] = 0.$

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) - 6(3-\lambda) = 0$

Take $(3-\lambda)$ common

$\Rightarrow (3-\lambda)\left[(1-\lambda)(2-\lambda) - 6\right] = 0$

$\Rightarrow (3-\lambda)\left[(2-\lambda-2\lambda+\lambda^2)-6\right]=0$

$\Rightarrow (3-\lambda)\left[\lambda^2-3\lambda-4\right]=0.$

$\Rightarrow (3-\lambda)\left[\lambda^2+\lambda-4\lambda-4\right]=0$

$\Rightarrow (3-\lambda)\left[\lambda(\lambda+1)-4(\lambda+1)\right]=0$

$\Rightarrow (3-\lambda)(\lambda+1)(\lambda-4)=0.$

$3-\lambda=0$	$\lambda+1=0$	$\lambda-4=0$
$\lambda=3$	$\lambda=-1$	$\lambda=4$

$\therefore \lambda = 3, -1, 4$ are the Eigen values.

~~(Ans)~~.

The Eigen vector corresponding to $\lambda=3$:-

put $\lambda=3$ in eq (1)

$$(1)\Rightarrow \begin{bmatrix} 1-3 & 6 & 1 \\ 1 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1+6x_2+x_3=0$

$x_1-x_2=0$ ~~$\Rightarrow x_1=x_2$~~

Sub above two equ's

x_1		x_2		x_3	
6	1		-2		6
-1	0		1		-1

$$\frac{x_1}{0+1} = \frac{x_2}{1+0} = \frac{x_3}{2-6}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-4}$$

$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$ is the eigen vectors corresponding to $\lambda = 3$.

The Eigen vectors corresponding to $\lambda = 4$:–

put $\lambda = 4$ in eq(1)

$$(1) \Rightarrow \begin{bmatrix} 1-4 & 6 & 1 \\ 1 & 2-4 & 0 \\ 0 & 0 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 6x_2 + x_3 = 0 \rightarrow (i)$

$x_1 - 2x_2 = 0 \rightarrow (ii)$

$-x_3 = 0 \rightarrow (iii)$

solve (i) & (ii).

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 6 & & 1 & & -3 & & 6 \\ -2 & & 0 & & 1 & & -2 \end{array}$$

$$\frac{x_1}{0+2} = \frac{x_2}{1+0} = \frac{x_3}{6-6}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 4$

Eigen vector corresponding to $\lambda = -1$:-

Put $\lambda = -1$ in eq (1)

$$① \Rightarrow \begin{bmatrix} 1+1 & 6 & 1 \\ 1 & 2+1 & 0 \\ 0 & 0 & 3+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 6x_2 + x_3 = 0 \rightarrow (i)$$

$$x_1 + 3x_2 = 0 \rightarrow (ii)$$

$$4x_3 = 0 \rightarrow (iii)$$

Solve (i) & (ii)

$$\begin{array}{cccccccc} & x_1 & & & x_2 & & x_3 & \\ 6 & & 1 & & 2 & & 6 & \\ 3 & & 0 & & 1 & & 3 & \end{array}$$

$$\frac{x_1}{0-3} = \frac{x_2}{1-0} = \frac{x_3}{6-6}$$

$$\frac{x_1}{-3} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -1$.

Consider $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$

$$\det P = 1(0-0) - 2(0+4) - 3(0+4)$$

$$= -2(4) - 3(4)$$

$$= -8 - 12$$

$$= -20 \neq 0$$

$$\therefore P^{-1} = \frac{-1}{20} \begin{bmatrix} 0 & 0 & -5 \\ -4 & 12 & 4 \\ 4 & 8 & 1 \end{bmatrix}$$

Now $P^{-1}AP = \frac{-1}{20} \begin{bmatrix} 0 & 0 & -5 \\ -4 & 12 & 4 \\ 4 & 8 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D = \text{diag}(3, 4, -1).$$

Now $D^8 = \begin{bmatrix} 3^8 & 0 & 0 \\ 0 & 4^8 & 0 \\ 0 & 0 & (-1)^8 \end{bmatrix} = \begin{bmatrix} 6561 & 0 & 0 \\ 0 & 65536 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^8 = PD^8P^{-1}$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 1 \\ -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6561 & 0 & 0 \\ 0 & 65536 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{-1}{20} \begin{bmatrix} 0 & 0 & -5 \\ -4 & 12 & 4 \\ 4 & 8 & 1 \end{bmatrix}$$ (Ans)

(16). S.T the matrix $A=\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ cannot be diagonalized. (M.T)

Sol:- $A=\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$$(A-\lambda I)X=0$$

$$\begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

The characteristics equ of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix}=0.$$

$\Rightarrow (2-\lambda)\left[(2-\lambda)(1-\lambda)-0\right]-3\left[0+0\right]+4\left[0-0\right]=0.$

$\Rightarrow (2-\lambda)\left[(2-\lambda)(1-\lambda)\right]-3\left[0\right]=0$

$\Rightarrow (2-\lambda)\left[(2-\lambda)(1-\lambda)\right]=0$

$\cancel{\Rightarrow (2-\lambda)\left[2-2\lambda-\lambda+\lambda^2\right]=0.}$

$\cancel{\Rightarrow (2-\lambda)\left[\lambda^2-3\lambda+2\right]=0.}$

$\cancel{\Rightarrow 2\lambda^2-6\lambda+4-\lambda^3+3\lambda^2-2\lambda=0.}$

$\cancel{\Rightarrow -\lambda^3+5\lambda^2-8\lambda+4=0.}$

$\cancel{\Rightarrow -(\lambda^3-5\lambda^2+8\lambda-4)=0}$

$\cancel{\lambda^3-5\lambda^2+8\lambda-4=0}$

$\Rightarrow (2-\lambda)=0 \quad | \quad (2-\lambda)=0 \quad | \quad (1-\lambda)=0$

$\lambda=2 \quad | \quad \lambda=2 \quad | \quad \lambda=1.$

$\therefore \lambda = 2,2,1$ are the characteristic value of A.

Since the eigen values of A are not distinct,

$\therefore$ the eigen vectors of A are not linear independent.

Hence, the matrix A is not diagonalized.

Now, put $\lambda=2$ in eq(1)

$$(1) \Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix is 2. so, these equ's will have $3-2=1$ independent soln. Thus, the geometric multiplication of these eigen value 2 is 1 & algebraic multiplicatiog is 2. Since, the algebraic multiplicity is not equal to geometric multiplicity. A is not similar to a diagonal matrix. Thus, the matrix cannot be diagonalized.

(17). Is the matrix $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ diagonalizable? (199)

Sol:-

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$(A-\lambda I)x = 0$$

$$\begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (1)$$

Characteristic equ of A is $|A-\lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (3-\lambda)\left[(-3-\lambda)(7-\lambda) - 5(-4)\right] - 10\left[(-2)(7-\lambda) - 3(-4)\right]$
$+ 5\left[(-2)(5) - 3(-3-\lambda)\right]$

$\Rightarrow (3-\lambda)\left[(-3-\lambda)(7-\lambda) + 20\right] - 10\left[-14 + 2\lambda + 12\right] + 5\left[-10 + 9 + 3\lambda\right] =$

$\Rightarrow (3-\lambda)\left[-21 + 3\lambda - 7\lambda + \lambda^2 + 20\right] - 10\left(2\lambda - 2\right] + 5\left[3\lambda - 1\right] = 0$

$\Rightarrow (3-\lambda)\left[\lambda^2 - 4\lambda - 1\right] - 20\lambda + 20 + 15\lambda - 5 = 0.$

$\Rightarrow 3\lambda^2 - 12\lambda - 3 - \lambda^3 + 4\lambda^2 + \lambda - 15 - 5\lambda = 0$

$\Rightarrow -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0.$

$\Rightarrow -(\lambda^3 - 7\lambda^2 + 16\lambda - 12) = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0.$

$\Rightarrow (\lambda-2)(\lambda^2 - 5\lambda + 6) = 0$

Consider $\lambda=2$ is a root

$$\begin{array}{r|rrrr} \lambda=2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\Rightarrow (\lambda-2)(\lambda^2-2\lambda-3\lambda+6)=0$

$\Rightarrow (\lambda-2)(\lambda(\lambda-2)-3(\lambda-2))=0$

$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-3)=0.$

Thus $\lambda=2,2,3$ are the eigen values of A.

Since, the eigen values of A are not distinct.

$\therefore$ the eigen vectors of A are not linearly independent.

Hence, the given matrix is not diagonalizable.

<u>Eigen vector corresponding to $\lambda=2$;-</u>

Put $\lambda=2$ in eq(1)

$$① \Rightarrow \begin{bmatrix} 3-2 & 10 & 5 \\ -2 & -3-2 & -4 \\ 3 & 5 & 7-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2+2R_1$

$R_3 \to R_3-3R_1$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{R_2}{3}, \frac{R_3}{-5}$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 10x_2 + 5x_3 = 0 \rightarrow (1)$$

$$5x_2 + 2x_3 = 0 \rightarrow (2)$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 10 & 5 & 1 & 10 \\ 5 & 2 & 0 & 5 \end{array}$$

$$\frac{x_1}{20-25} = \frac{x_2}{0-2} = \frac{x_3}{5-0}$$

$$\frac{x_1}{-5} = \frac{x_2}{-2} = \frac{x_3}{5}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 5 \end{bmatrix} = -\begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$$ //

The rank of the coefficient matrix is 2. so, these equ will have 3-2=1 independent sol^n. Thus, the geometric multiplication of the eigen value 2 is 1 & algebraic multiplicaty is 2. since, the algebric multiplicaty is not equal to geometric multiplicity, A is not similar to a diagonal matrix. Thus, the matrix cannot be diagonalized

(Ans)

(18). Diagonalize the matrix where $A=\begin{bmatrix}1&0&0\\0&3&-1\\0&-1&3\end{bmatrix}$ by orthogonal reduction. (2021)

Sol:- Given $A=\begin{bmatrix}1&0&0\\0&3&-1\\0&-1&3\end{bmatrix}$

$$(A-\lambda I)X=0$$

$$\begin{bmatrix}1-\lambda&0&0\\0&3-\lambda&-1\\0&-1&3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\longrightarrow(1)$$

The characteristic equ $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda&0&0\\0&3-\lambda&-1\\0&-1&3-\lambda\end{vmatrix}=0.$$

$\Rightarrow(1-\lambda)\left[(3-\lambda)^2-(-1)(-1)\right]=0$

$\Rightarrow(1-\lambda)\left[(3-\lambda)^2-1\right]=0.$

$\Rightarrow(1-\lambda)(\lambda^2-6\lambda+8)=0.$

$\Rightarrow(1-\lambda)\left[\lambda^2-2\lambda-4\lambda+8\right]=0$

$\Rightarrow(1-\lambda)\left[\lambda(\lambda-2)-4(\lambda-2)\right]=0$

$\Rightarrow(1-\lambda)(\lambda-2)(\lambda-4)=0.$

$\lambda=1,2,4$ are the eigen values of 'A'.

<u>Eigen vector corresponding to $\lambda=1$:-</u>

Put $\lambda=1$ in eq(1)

$$(1) \Rightarrow \begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 3-1 & -1 \\ 0 & -1 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$2x_2 - x_3 = 0$$

$$-x_2 + 2x_3 = 0.$$

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 2 & & -1 & & 0 & & 2 \\ -1 & & 2 & & 0 & & -1 \end{array}$$

$$\frac{x_1}{4-(-1)(-1)} = \frac{x_2}{0-0} = \frac{x_3}{0-0}.$$

$$\frac{x_1}{4-1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 2$:-

$$(1) \Rightarrow \begin{bmatrix} 1-2 & 0 & 0 \\ 0 & 3-2 & -1 \\ 0 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 = 0 \rightarrow (i)$

$x_2 - x_3 = 0 \rightarrow (ii)$

$-x_2 + x_3 = 0$

equ (i) & (ii)

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ 0 & & 0 & & -1 & & 0 & \\ 1 & & -1 & & 0 & & 1 & \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 4$:—

(1) $\Rightarrow \begin{bmatrix} 1-4 & 0 & 0 \\ 0 & 3-4 & -1 \\ 0 & -1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 = 0$

$-x_2 - x_3 = 0$

$-x_2 - x_3 = 0 \quad \nearrow$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ 0 & & 0 & & -1 & & 0 & \\ -1 & & -1 & & 0 & & -1 & \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0-1} = \frac{x_3}{+1-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} //$$

We notice that these vectors are pairwise orthogonal.

Normalizing these vectors.

$$\frac{X_1}{\|X_1\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad \frac{X_2}{\|X_2\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} ; \quad \frac{X_3}{\|X_3\|} = \begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Consider $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Then $P^T = P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

We can verify that $PAP^{-1} = D$ where $D = diag(1,2,4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus A is reduced to diagonal form by orthogonal reduction // (Ans)

(19) Find the diagonal matrix orthogonally similar to the following real symmetric matrix. Also obtain the transforming matrix. $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$ (206)

Sol:- Given $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The characteristic equ of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0.$$

$$C_3 \to C_3 + C_2$$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 4 & 0 \\ 4 & -8-\lambda & -9-\lambda \\ -4 & -1 & -9-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ Take $-9-\lambda$ common

$$(-9-\lambda) \begin{vmatrix} 7-\lambda & 4 & 0 \\ 4 & -8-\lambda & 1 \\ -4 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow (-9-\lambda)[(7-\lambda)(-8-\lambda)+1]-4[4+4]=0.$

$\Rightarrow (-9-\lambda)\left[(7-\lambda)(-8-\lambda+1)\right]-32=0.$

$\Rightarrow (-9-\lambda)\left[(7-\lambda)(-7-\lambda)-32\right]=0.$

$\Rightarrow (-9-\lambda)\left[49-\cancel{7\lambda}+\cancel{7\lambda}+\lambda^2-32\right]=0.$

$\Rightarrow (-9-\lambda)\left[\lambda^2-81\right]=0$

$\therefore \lambda = 9, -9, -9$ are the eigen values.

Eigen vectors corresponding to $\lambda = 9$:-

(1) $\Rightarrow \begin{bmatrix} 7-9 & 4 & -4 \\ 4 & -8-9 & -1 \\ -4 & -1 & -8-9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\Rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ 0 & -18 & -18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 + 4x_2 - 4x_3 = 0$

$4x_1 - 17x_2 - x_3 = 0$

$-18x_2 - 18x_3 = 0$

solve Ist two equ's

we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 4 & -4 & -2 & 4 \\ -17 & -1 & 4 & -17 \end{array}$$

$$\frac{x_1}{-4-68} = \frac{x_2}{-16-2} = \frac{x_3}{34-16}$$

$$\frac{x_1}{\cancel{72}\;4} = \frac{x_2}{-\cancel{18}\;1} = \frac{x_3}{\cancel{18}\;1}$$

$$\therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$ corresponding to $\lambda = 9$ //

Eigen vector corresponding to $\lambda = -9$:-

$$(1) \Rightarrow \begin{bmatrix} 7+9 & 4 & -4 \\ 4 & -8+9 & -1 \\ -4 & -1 & -8+9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$16x_1 + 4x_2 - 4x_3 = 0$$

$$4x_1 + x_2 - x_3 = 0$$

$$-4x_1 - x_2 + x_3 = 0$$

Solving Ist two equ's

$$\begin{array}{ccccc} \cancel{x_1} & x_1 & x_2 & x_3 & \\ & 4 & -4 & 16 & 4 \\ & 1 & -1 & 4 & 1 \end{array}$$

$$\frac{x_1}{-4+4} = \frac{x_2}{-16+16} = \frac{x_3}{16-16}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{0}$$

$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -9$

(Ans)

(20) Verify cayley - Hamilton theorem for the matrix

$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

Sol:-

given ~~matrix~~ $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

$\therefore$ The characteristic equation of A is $|A - \lambda I| = 0$.

i.e; $\begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0.$

$\Rightarrow (8-\lambda)[(-3-\lambda)(1-\lambda)-8] - 4[-8(1-\lambda)+8] + 3[16-2(-3-\lambda)] = 0$

$\Rightarrow (8-\lambda)[-3+3\lambda-\lambda+\lambda^2-8] - 4[\cancel{-8}+8\lambda\cancel{+8}] + 3[16+6+2\lambda] = 0$

$\Rightarrow (8-\lambda)[\lambda^2+2\lambda-11] - 4[8\lambda] + 3[2\lambda+22] = 0.$

$\Rightarrow (8-\lambda)[\lambda^2+2\lambda-11] - 32\lambda + 6\lambda + 66 = 0.$

$\Rightarrow 8\lambda^2 + 16\lambda - 88 - \lambda^3 - 2\lambda^2 + 11\lambda - 32\lambda + 6\lambda + 66 = 0$

$\Rightarrow \cancel{-\lambda^3 + 6\lambda^2 + 27\lambda + 6\lambda - 5\lambda - 22 = 0}$

$\Rightarrow \cancel{-\lambda^3 + 6\lambda^2 +}$

$\Rightarrow -\lambda^3 + 6\lambda^2 + 33\lambda - 32\lambda - 22 = 0.$

$\Rightarrow -\lambda^3 + 6\lambda^2 + \lambda - 22 = 0$

$\Rightarrow -(\lambda^3 - 6\lambda^2 - \lambda + 22) = 0$

$\lambda^3 - 6\lambda^2 - \lambda + 22 = 0.$

Now $\boxed{\lambda = A}$

To verify Cayley-Hamilton Theorem, we have to P.T

$A^3 - 6A^2 - A + 22I = 0.$

Now, $A^2 = A \cdot A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}.$

And, $A^3 = A \cdot A^2 = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$

$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$

Now, $A^3 - 6A^2 - A + 22I$.

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix} - 6\begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + 22\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$

Hence, Cayley-Hamilton Theorem is verified. (Ans)

(21) Find the inverse of the following matrices by using Cayley-Hamilton theorem

(or)

Verify Cayley-Hamilton theorem for the matrices, Hence find A^{-1}

(i) $\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Sol:-

given, $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

$\therefore$ The characteristic equation of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (2-\lambda)[-6-3\lambda+2\lambda+\lambda^2]-5[-2-\lambda]-1(3-6+2\lambda)=0$. 212

$\Rightarrow (2-\lambda)(-6-\lambda+\lambda^2)+10+5\lambda+3-2\lambda=0$.

$\Rightarrow -12-2\lambda+2\lambda^2+6\lambda+\lambda^2-\lambda^3+13+3\lambda=0$

$\Rightarrow -\lambda^3+3\lambda^2+7\lambda+1=0$

$\Rightarrow -(\lambda^3-3\lambda^2-7\lambda-1)=0$

$\Rightarrow \lambda^3-3\lambda^2-7\lambda-1=0$.

To verify cayley - Hamilton theorem, we have to prove that

$$A^3-3A^2-7A-I=0.$$

Now, $A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$

Now, $A^3 = A^2 \cdot A = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

Now, A^3-3A^2-7A-I

$$\Rightarrow \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} + \begin{bmatrix} -21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix} + \begin{bmatrix} -14 & -7 & -14 \\ -35 & -21 & -21 \\ 7 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

∴ ~~To verify~~ Hence, cayley-Hamilton theorem is verified.

To find A^{-1}

$$\Rightarrow A^3 - 3A^2 - 7A - I = 0$$

$$\Rightarrow A^{-1}(A^3 - 3A^2 - 7A - I) = 0$$

$$\Rightarrow A^2 - 3A - 7I - A^{-1} = 0$$

$$\Rightarrow A^{-1} = A^2 - 3A - 7I$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -6 & -3 & -6 \\ 15 & 9 & -9 \\ 3 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

check :- $AA^{-1} = I$ (Identity matrix).

(ii)

given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

$\therefore$ The characteristic equation of A is $|A-\lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (1-\lambda)[(1+\lambda)^2-4] - 2[-2(1+\lambda)-12] + 3[2+3(1+\lambda)] = 0.$

$\Rightarrow (1-\lambda)[(1+\lambda)^2-4] - 2[-2-2\lambda-12] + 3[2+3+3\lambda] = 0$

$\Rightarrow (1-\lambda)[(1+\lambda)^2-4] - 2[-2\lambda-14] + 3[3\lambda+5] = 0.$

$\Rightarrow (1-\lambda)[(1+\lambda)(1+\lambda)-4] - 2[-2\lambda-14] + 3[3\lambda+5] = 0.$

$\Rightarrow (1-\lambda)[1+\lambda+\lambda+\lambda^2-4] - 2[-2\lambda-14] + 3[3\lambda+5] = 0$

$\Rightarrow (1-\lambda)[\lambda^2+2\lambda-3] - 2[-2\lambda-14] + 3[3\lambda+5] = 0.$

$\Rightarrow \lambda^2+2\lambda-3-\lambda^3-2\lambda^2+3\lambda+4\lambda+28+9\lambda+15 = 0.$

$\Rightarrow -\lambda^3+\lambda^2+18\lambda+43-3 = 0$

$\Rightarrow -\lambda^3+\lambda^2+18\lambda+40 = 0.$

$\Rightarrow -(\lambda^3-\lambda^2-18\lambda-40) = 0$

$\lambda^3-\lambda^2-18\lambda-40 = 0.$

To ~~Now Consider~~ :)

Verify cayley - Hamilton Theorem, we have to prove that multiply, A^{-1} on Both sides. $\Rightarrow A^{-1}(A^3+A^2-18A-40I)=0$.

$\Rightarrow A^2+A-18I=40A^{-1}$

$$\Rightarrow A^{-1}=\frac{1}{40}\left[A^2+A-18I\right]$$

then, $A^2=A.A$ $= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

$$A^2=\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$\therefore A^{-1}=\frac{1}{40}\left[A^2+A-18I\right]$

$$\therefore A^{-1}=\frac{1}{40}\left\{\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}+\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}-18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$A^{-1}=\frac{1}{40}\left\{\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}+\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}-\begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1}=\frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$ //(Ans).

(vii)

given, $A=\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$\therefore$ The characteristic eq^n is $|A-\lambda I|=0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & -1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix}=0$$

$\Rightarrow (1-\lambda)(\lambda^2-1-1)+3(-2+1+\lambda)=0$

$\Rightarrow (1-\lambda)(\lambda^2-2)+3(\lambda-1)=0$

$\Rightarrow (1-\lambda)(\lambda^2-2)+3\lambda-3=0$

$\Rightarrow (1-\lambda)(\lambda^2\underbrace{-2)+3\lambda-3}_{-5}=0$

$\Rightarrow (1-\lambda)(\lambda^2-5)=0$

$\Rightarrow \lambda^2-5-\lambda^3+5\lambda=0$

$\Rightarrow -\lambda^3+\lambda^2+5\lambda-5=0$

$\Rightarrow -(\lambda^3-\lambda^2-5\lambda+5)=0$

$\lambda^3-\lambda^2-5\lambda+5=0.$

To verify Cayley - Hamilton Theorem, we have to prove that.

$$A^3-A^2-5A+5I=0.$$

Now, $A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$

$= \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Now, $A^3 = A^2 \cdot A.$

$= \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -3 & 21 \\ 9 & -8 & 1 \\ 5 & -5 & 5 \end{bmatrix}$$

Now, $A^3 - A^2 - 5A + 5I = 0$

$$= \begin{bmatrix} 4 & -3 & 21 \\ 9 & -8 & 1 \\ 5 & -5 & 5 \end{bmatrix} - \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 15 \\ 10 & -5 & -5 \\ 5 & -5 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\therefore$ Cayley - Hamilton theorem is verified.

To find A^{-1}

$\Rightarrow \cancel{A^{-1}(A^3) - A^{-1}(A^2) - A^{-1}(5A) = A^{-1}}$

$\Rightarrow A^{-1} = A^3 - A^2 - 5A + 5I = 0.$

$\Rightarrow A^3 - A^2 - 5A + 5I = 0$

$\Rightarrow \cancel{A^{-1}(A^2) - A^{-1}(A) - 5}$

$\Rightarrow$ multiply with A^{-1}, we get

$A^2 - A - 5 + 5A^{-1} = 0.$

$\Rightarrow 5A^{-1} = -A^2 + A + 5I.$

$\Rightarrow A^{-1} = \frac{1}{5}\left(-A^2 + A + 5I\right).$

$$\therefore A^{-1} = \frac{1}{5}\left\{ -\begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.$$

$$= \frac{1}{5}\left\{ \begin{bmatrix} -4 & 3 & -6 \\ 1 & -2 & -6 \\ 0 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right.$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 3 & -3 \\ 3 & 2 & -7 \\ 1 & -1 & 1 \end{bmatrix}$$ // (Ans)

22. Using cayley-Hamilton theorem find the inverse & A^4 of the matrix 218

$$A=\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

Sol:- Let $A=\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation of A is given by $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix}=0$$

$R_1 \rightarrow R_1-R_3$

$R_2 \rightarrow R_2+R_3$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & \lambda-1 \\ 0 & 1-\lambda & 1-\lambda \\ 6 & 2 & -1-\lambda \end{vmatrix}=0$$

Now Take $(1-\lambda)$ common

$$\Rightarrow (1-\lambda)(1-\lambda)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)^2\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix}=0$$

$\Rightarrow (1-\lambda)^2\left[1(-(1+\lambda)-2\right]-1\left[0-6\right]=0$

$\Rightarrow (1-\lambda)^2\left[1(-1-\lambda)-2\right]-1\left[-6\right]=0$

$\Rightarrow (1-\lambda)^2\left[-1-\lambda-2\right]-1\left[-6\right]=0$

$\Rightarrow (1-\lambda)^2 [-\lambda-3]=0$

$\Rightarrow (1-\lambda)^2 [3-\lambda]=0$

~~$\Rightarrow 3-\lambda=$~~ $\Rightarrow (1+\lambda^2-2(1)(\lambda))(3-\lambda)=0$.

$\Rightarrow (1+\lambda^2-2\lambda)(3-\lambda)=0$

$\Rightarrow 3-\lambda+3\lambda^2-\lambda^3-6\lambda+2\lambda^2=0$

$\Rightarrow -\lambda^3+5\lambda^2-7\lambda+3=0$

$\Rightarrow -(\lambda^3-5\lambda^2+7\lambda-3)=0$

$\Rightarrow \lambda^3-5\lambda^2+7\lambda-3=0.$

By cayley-Hamilton theorem, Consider $\lambda=A$

$A^3-5A^2+5A-3I=0 \longrightarrow (1)$

Let $A^2=A\times A=\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}\times\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

$A^2=\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$

Let $A^3=A^2\times A=\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}\times\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

$A^3=\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ +78 & 26 & 25 \end{bmatrix}$

To find A^{-1}, multiply with A^{-1} on b.s of eq(1)

$$(1) \Rightarrow A^{-1}[A^3 - 5A^2 + 7A - 3I] = 0$$

$$\Rightarrow A^2 - 5A + 7I - 3A^{-1} = 0$$

$$\Rightarrow 3A^{-1} = A^2 - 5A + 7I$$

$$\Rightarrow A^{-1} = \frac{1}{3}[A^2 - 5A + 7I] \longrightarrow (2)$$

NOW
$$A^2 - 5A + 7I = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$A^2 - 5A + 7I = \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

$$(2) \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

Multiplying A' for eq (1)

$$(1) \Rightarrow A^4 - 5A^3 + 7A^2 - 3A = 0.$$

$$A^4 = 5A^3 - 7A^2 + 3A.$$

$$A^4 = 5\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & 25 \end{bmatrix} - 7\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 3\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

= (Ans)

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(23) Reduce the following quadratic form to the Canonical (or) normal form:—

(i). $3x^2+2y^2+3z^2-2xy-2yz$.

(ii). $2x^2+5y^2+3z^2+4xy$.

Sol's

(i). Given Q.F $3x^2+2y^2+3z^2-2xy-2yz$.

The matrix of the QF is given by $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

We write $A = I_3 A I_3$

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Same $\leftarrow$ $R_2 \to 3R_2+R_1$ ↳ Row operation ↳ Column operation

$C_2 \to 3C_2 + C_1$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 15 & -3 \\ 0 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 5R_3 + R_2$

$5C_3 \to 5C_3 + C_2$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 60 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 5 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\frac{R_1}{\sqrt{3}}, \frac{R_2}{\sqrt{15}}, \frac{R_3}{\sqrt{60}}, \frac{C_1}{\sqrt{3}}, \frac{C_2}{\sqrt{15}}, \frac{C_3}{\sqrt{60}}$$

same

$$\begin{bmatrix} \frac{3}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{15}{\sqrt{15}} & 0 \\ 0 & 0 & \frac{60}{\sqrt{60}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{15}} & \frac{3}{\sqrt{15}} & 0 \\ \frac{1}{\sqrt{60}} & \frac{3}{\sqrt{60}} & \frac{5}{\sqrt{60}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{15}} & \frac{3}{\sqrt{60}} \\ 0 & \frac{3}{\sqrt{15}} & \frac{3}{\sqrt{60}} \\ 0 & 0 & \frac{5}{\sqrt{60}} \end{bmatrix}$$

This is of the form P^TAP. The normal form is $y_1^2+y_2^2+y_3^2$, where this is done using the linear transformation

$$X = PY$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{15}} & \frac{3}{\sqrt{60}} \\ 0 & \frac{3}{\sqrt{15}} & \frac{3}{\sqrt{60}} \\ 0 & 0 & \frac{5}{\sqrt{60}} \end{bmatrix}$$

= (Ans)

(ii). Given Q.F $2x^2+5y^2+3z^2+4xy$.

$$A=\begin{bmatrix}2&2&0\\2&5&0\\0&0&3\end{bmatrix}$$

$$A=I_3AI_3.$$

$$\begin{bmatrix}2&2&0\\2&5&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$R_2\to R_2-R_1$

$C_2\to C_2-C_1$

$$\begin{bmatrix}2&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\frac{R_1}{\sqrt2},\ \frac{R_2}{\sqrt3},\ \frac{R_3}{\sqrt3},\ \frac{C_1}{\sqrt2},\ \frac{C_2}{\sqrt3},\ \frac{C_3}{\sqrt3}$$

$$\begin{bmatrix}\frac{2}{\sqrt2}&0&0\\0&\frac{3}{\sqrt3}&0\\0&0&\frac{3}{\sqrt3}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt2}&0&0\\\frac{-1}{\sqrt3}&\frac{1}{\sqrt3}&0\\0&0&\frac{1}{\sqrt3}\end{bmatrix}A\begin{bmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt3}&0\\0&\frac{1}{\sqrt3}&0\\0&0&\frac{1}{\sqrt3}\end{bmatrix}$$

$$X=PY$$

$$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix};\ Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix};\ P=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt3}&0\\0&\frac{1}{\sqrt3}&0\\0&0&\frac{1}{\sqrt3}\end{bmatrix}$$

(24). Find the symmetric matrix corresponding to the quadratic form. (224)

(i). $x_1^2+6x_1x_2+5x_2^2$ (ii). $x^2+2y^2+3z^2+4xy+5yz+6zx$

(iii). $x_1^2+2x_2^2+4x_2x_3+x_3x_4$

Sol^n

(i). Given Q.F $x_1^2+6x_1x_2+5x_2^2$

The above Q.F can be written as

$x_1.x_1+3x_1.x_2+3x_2.x_1+5x_1.x_2$

Let A be the symmetric matrix of this Q.F.

Then $A=\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

Let $X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; $X^T=[x_1 \quad x_2]$

We have X^TAX

$=[x_1 \quad x_2]\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\boxed{x_1^2+6x_1x_2+5x_2^2}$

∴ The above equ is a symmetric matrix corresponding to the Q.F

=(Ans).

(ii). $x^2+2y^2+3z^2+4xy+5yz+6zx$.

The above Q.F can be written as

~~x_1x_1+~~ $x.x+2xy+3xz+2yx+2y.y+\frac{5}{2}yz+3zx+\frac{5}{2}yz+3z.z$

Let A be the symmetric matrix of the Q.F.

Then $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $X^T = [x \; y \; z]$

let $X^T A X$

$$[x \; y \; z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

= (Ans).

(iii). $x_1^2 + 2x_2^2 + 4x_2x_3 + x_3x_4$.

The above Q.F can be written as:-

$x_1 \cdot x_1 + 0 \cdot x_1x_2 + 0 \cdot x_1x_3 + 0 \cdot x_1x_4 + 0 \cdot x_2 \cdot x_4 + 2x_2 \cdot x_2 +$
$2x_2x_3 + 0x_3x_4 + 0x_3x_1 + 2x_3x_2 + 0x_3x_3 + \frac{1}{2}x_3x_4 +$
$0 \cdot x_4x_1 + 0 \cdot x_4x_2 + \frac{1}{2}x_3x_4 + 0 \cdot x_4x_4$.

Then $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; $X^T = [x_1 \; x_2 \; x_3 \; x_4]$

We have $X^T A X$

$$[x_1 \; x_2 \; x_3 \; x_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 1/2 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

= (Ans).

(25). write the matrix relating to the Q.F $ax^2+2hxy+by^2$ 226

Sol:- Given Q.F $ax^2+2hxy+by^2$

The above Q.F can be written as:-

$X=[x \;\; y]$; $A=\begin{bmatrix} a & h \\ h & b \end{bmatrix}$; $X^T=\begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore$ The Corresponding matrix is $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$

=(Ans).

---

(26). Find the Q.F corresponding to the symmetric matrix (or) matrix:-

(i)
$$A=\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$$

(ii).
$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Soln

(i). Given $A=\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

Let $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $X^T=[x \;\; y \;\; z]$

The Q.F Corresponding to the symmetric matrix A is given by:-

X^TAX.

$$=[x \;\; y \;\; z]\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0\times x + 5\times y - 1z \\ 5x + 1\times y + 6\times z \\ -1x + 6y + 2\times z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 5y - z \\ 5x + y + 6z \\ -x + 6y + 2z \end{bmatrix}$$

$\Rightarrow x(5y-z) + y(5x+y+6z) + z(-x+6y+2z)$

$\Rightarrow 5xy - xz + 5xy + y^2 + 6yz - zx + 6zy + 12z^2$

$\therefore X^TAX = y^2 + 12z^2 + 10xy + 12yz - 2zx$

=(Ans).

(ii). given, $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

Let $X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$; $X^T = \begin{bmatrix} x & y & z & t \end{bmatrix}$

The Quadratic form corresponding to the symmetric matrix 'A' is given by

$$X^TAX.$$

$$= \begin{bmatrix} x & y & z & t \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z & t \end{bmatrix} \begin{bmatrix} y+2z+3t \\ x+2y+3z+4t \\ 2x+3y+4z+5t \\ 3x+4y+5z+6t \end{bmatrix}$$

$$= x(y+2z+3t) + y(x+2y+3z+4t) + z(2x+3y+4z+5t) + t(3x+4y+5z+6t).$$

$$= (xy+2xz+3xt)+(xy+2y^2+3yz+4yt)+(2zx+3zy+4z^2+5zt) + (3xt+4ty+5zt+6t^2)$$

$$= 2y^2+4z^2+6t^2+2xy+6yz+4zx+6xt+8yt+10zt$$

$\therefore$ It is the required quadratic form. (Ans).

---

(27) Find the inverse transformation of

$$y_1 = 2x_1+x_2+x_3,\ y_2 = x_1+x_2+x_3,\ y_3 = x_1-2x_3$$

Sol:- The given transformation can be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e., $Y = AX$

Now, $|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$

$$= 2(-2-0) - 1(-2-2) + 1(0-1)$$

$= \cancel{2-1+2+1-1} \Rightarrow 2(-2) - 1(-4) + 1(-1)$

$= \cancel{4+1+2-1} \Rightarrow -4+4-1$

$= \cancel{7-1}$ $= -1 \neq 0$

$\therefore$ The matrix 'A' is non-singular

Hence, the transformation is regular.

$\therefore$ The inverse transformation is given by $X = A^{-1}Y$

i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\therefore \Rightarrow x_1 = 2y_1 - 2y_2 - y_3$

$\Rightarrow x_2 = -4y_1 + 5y_2 + 3y_3$

$\Rightarrow x_3 = y_1 - y_2 - y_3$ (Ans)

29) Show that the transformation $y_1 = x_1\cos\theta + x_2\sin\theta$, $y_2 = -x_1\sin\theta + x_2\cos\theta$ is orthogonal.

Sol:-

The given transformation can be written as

$Y = AX$, where

$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

The matrix transformation is

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Now, $A^{-1} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A^T$.

$\therefore$ The transformation is orthogonal.

(Ans).

30). Identify / Discuss / Find the nature of the

Quadratic form:-

(i). $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$

(ii). $x^2+4xy+6xz-y^2+2yz+4z^2$.

(iii). $2x^2+2y^2+2z^2+2yz$.

Sol^n

(i). Given Q.F $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$.

The matrix of the given Q.F is $A = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \end{array}$

The characteristic equ $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 - R_3$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & \lambda \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow \lambda \begin{vmatrix} -1 & 0 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda\left[-1(4-\lambda)(1-\lambda)-(-2)(-2)\right]+1\left[(-2)(-2)-1(4-\lambda)\right]$$

$$\Rightarrow \lambda\left[-1(4-\lambda)(1-\lambda)-4\right]+1\left[+\cancel{4}-\cancel{4}+\lambda\right]$$

$$\Rightarrow \lambda\left[-(\cancel{4}-4\lambda-\lambda+\lambda^2-\cancel{4}\right]+\lambda=0.$$

$$\Rightarrow \lambda\left(-(-5\lambda+\lambda^2)+\lambda\right)=0$$

$\Rightarrow \lambda[5\lambda-\lambda^2+\lambda]=0.$

$\Rightarrow \lambda[6\lambda-\lambda^2]=0$

$\Rightarrow \lambda[\lambda(6-\lambda)]=0$

$\Rightarrow \lambda^2(6-\lambda)=0.$

$\Rightarrow \lambda^2(-(\lambda-6))=0$

$\Rightarrow \lambda^2(\lambda-6)=0$

$\Rightarrow \lambda^2=0, \quad \lambda-6=0$

$\lambda=6$

$\therefore$ Eigen values are $\lambda=0,0,6$ which are +ve & 2 values are zero.

Hence the QF is positive semi-definite. =(Ans).

(ii). Given QF $x^2+4xy+6xz-y^2+2yz+4z^2$.

The matrix of the given Q.F is $A=\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \end{array}$

characteristic equ of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix}=0.$$

~~$\Rightarrow (1-\lambda)[-4-1]-2(8-3)+3[2+3]$~~

~~$\Rightarrow (1-\lambda)[-5]-2[5]+3[5]$~~

~~$\Rightarrow (1-\lambda)(-5)-10+15$~~

~~$\Rightarrow -5+5\lambda-10+15$~~

$\Rightarrow (1-\lambda)[(-1-\lambda)(4-\lambda)-1]-2[2(4-\lambda)-3]+3[2-3(-1-\lambda)]=0$

$\Rightarrow (1-\lambda)[(-1-\lambda)(4-\lambda)-1]-2(8-2\lambda-3)+3(2+3+3\lambda)=0.$

$\Rightarrow (1-\lambda)[-4+\lambda-4\lambda+\lambda^2-1]-2[-2\lambda+5]+3[3\lambda+5]=0$

$\Rightarrow (1-\lambda)[\lambda^2-3\lambda-5]-2[-2\lambda+5]+3[3\lambda+5]=0$

$\Rightarrow \lambda^2-3\lambda-\cancel{5} -\lambda^3+3\lambda^2+5\lambda+4\lambda \cancel{+10} +9\lambda +\cancel{15} = 0.$ 232

$\Rightarrow -\lambda^3+4\lambda^2+18\lambda-3\lambda \cancel{+120} = 0.$

$\Rightarrow -\lambda^3+4\lambda^2+15\lambda \cancel{+120} = 0.$

$\Rightarrow -(\lambda^3-4\lambda^2-15\lambda) = 0$

$\Rightarrow -\lambda(\lambda^2-4\lambda-15) = 0.$

$\Rightarrow +\lambda = 0$ | $\lambda^2-4\lambda-15=0.$

$\lambda = 2 \pm \sqrt{19}$

$\lambda = 2+\sqrt{19},\ 2-\sqrt{19}$

$= \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{4 \pm \sqrt{(-4)^2-4(1)(-15)}}{2(1)}$

$= \dfrac{4 \pm \sqrt{16+60}}{2}$

$= \dfrac{4 \pm \sqrt{76}}{2}$

$= \dfrac{4+\sqrt{76}}{2};\ \dfrac{4-\sqrt{76}}{2}$

$= 2+\sqrt{19};\ 2-\sqrt{19}$

$\therefore \lambda = 0,\ 2+\sqrt{19},\ 2-\sqrt{19}.$

Thus, the Q.F is indefinite.

= (Ans).

(iii). Given QF $= 2x^2+2y^2+2z^2+2yz$

The matrix of the given Q.F is

$$A = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

Characteristic equ of A is $|A-\lambda I| = 0.$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (2-\lambda)\left[(2-\lambda)(2-\lambda)-1\right]$

$\Rightarrow (2-\lambda)\left[4-2\lambda-2\lambda+\lambda^2-1\right]$

$\Rightarrow (2-\lambda)\left[\lambda^2-4\lambda+3\right] = 0.$

$\Rightarrow (2-\lambda)\left[\lambda^2-\lambda-3\lambda+3\right] = 0$

$\Rightarrow (2-\lambda)\left[\lambda(\lambda-1)-3(\lambda-1)\right] = 0$

$\Rightarrow (2-\lambda)(\lambda-3)(\lambda-1)=0.$

$2-\lambda=0$	$\lambda-3=0$	$\lambda-1=0$
$-\lambda=-2$	$\lambda=3$	$\lambda=1$
$\lambda=2$		

$\therefore \lambda = 2, 3, 1$

$\therefore$ The roots of the characteristic equ are 1, 2, 3.

All roots are positive.

$\therefore$ The Q.F is +ve Definite.

= (Ans).

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(31). Find the matrix, rank, index & signature of the quadratic form $x_1x_2 - 4x_1x_4 - 2x_2x_3 + 12x_3x_4$.

Sol:- Given Q.F $x_1x_2 - 4x_1x_4 - 2x_2x_3 + 12x_3x_4$

(i). The matrix of the Q.F is

$$A = \begin{array}{c|c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & -2 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & -1 & 0 & 6 \\ -2 & 0 & 6 & 0 \end{bmatrix} \end{array}$$

We write $A = I_4 A I_4$

$$\begin{bmatrix} 0 & 1/2 & 0 & -2 \\ 1/2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 6 \\ -2 & 0 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R.O $\quad R_1 \to R_1 \leftrightarrow R_2 \quad$ R.O $\quad$ C.O

$C_1 \to C_1 \leftrightarrow C_2$

$$\begin{bmatrix} 0 & 1/2 & -1 & 0 \\ 1/2 & 0 & 0 & -2 \\ -1 & 0 & 0 & 6 \\ 0 & -2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$, $C_3 \to C_3 + 2C_2$

$$\begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_3$, $C_2 \to C_2 + C_3$

$$\begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 & 3 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sqrt{2}R_1$, $\sqrt{2}R_2$, $\sqrt{2}C_1$, $\sqrt{2}C_2$, $\frac{R_3}{\sqrt{2}}$, $\frac{R_4}{\sqrt{2}}$, $\frac{C_3}{\sqrt{2}}$, $\frac{C_4}{\sqrt{2}}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 \\ 3\sqrt{2} & 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} A \begin{bmatrix} 0 & 3\sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$R_1 \to R_1 \longleftrightarrow R_2$, $R_3 \longleftrightarrow R_4$

$$\underset{(I_4)}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} = \underset{(P^T)}{\begin{bmatrix} 3\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \\ \sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix}} \underset{(A)}{A} \underset{(P)}{\begin{bmatrix} 3\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}}$$

This is of the form $I_4 = P^TAP$.

Where $P = \begin{bmatrix} 3\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 \end{bmatrix}$; $P^T = \begin{bmatrix} 3\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \\ \sqrt{2} & 0 & 1/\sqrt{2} & 0 \end{bmatrix}$

$\therefore$ Rank of Q.F (r) = no. of non-zero term in normal form

$= 4.$

$\therefore$ Index of Q.F (s) = no. of +ve terms in the normal form

$= 4$

$\therefore$ Signature of Q.F $= 2s - r = 2(4) - 4 = 8 - 4 = 4$

= Ans

(32). Reduce the Q.F $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical (or) normal form by orthogonal transformation (or) reduction.

(835)

(or)

Reduce the Q.F $3x^2+5y^2+3z^2-2yz+2zx-2xy$ (or) $3x_1^2+5x_2^2+3x_3^2-2x_1x_2+2x_1x_3-2x_2x_3$ to canonical (or) normal form & hence find/state nature, rank, index & signature of the Q.F.

Sol:- Given Q.F $3x^2+5y^2+3z^2-2yz+2zx-2xy$.

The matrix of the given Q.F is

$$A = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \end{array}$$

The characteristic equ of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (3-\lambda)\left[(5-\lambda)(3-\lambda)-(-1)(-1)\right]+1\left[(-1)(3-\lambda)-1(-1)\right]+1\left[(-1)(-1)-1(5-\lambda)\right]=0.$

$\Rightarrow (3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right]+1\left[-3+\lambda+1\right]+1\left[1-5+\lambda\right]=0$

$\Rightarrow (3-\lambda)\left[15-5\lambda-3\lambda+\lambda^2-1\right]+1\left[\lambda-2\right]+1\left[\lambda-4\right]=0$

$\Rightarrow (3-\lambda)\left[\lambda^2-8\lambda+14\right]+1\left[\lambda-2\right]+1\left[\lambda-4\right]=0$

$\Rightarrow 3\lambda^2-24\lambda+42-\lambda^3+8\lambda^2-14\lambda+\lambda-2+\lambda-4=0$

$\Rightarrow -\lambda^3+11\lambda^2+2\lambda-38\lambda+36=0$

$\Rightarrow -\lambda^3+11\lambda^2 - 36\lambda+36=0$

$\Rightarrow -(\lambda^3-11\lambda^2+36\lambda-36)=0$

$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$

$\Rightarrow (\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$

$\lambda = 2$

$$\begin{array}{r|rrrr} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\Rightarrow (\lambda-2)(\lambda^2 - 3\lambda - 6\lambda + 18) = 0$

$\Rightarrow (\lambda-2)(\lambda(\lambda-3) - 6(\lambda-3)) = 0$

$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-6) = 0.$

$\lambda - 2 = 0$ | $\lambda - 3 = 0$ | $\lambda - 6 = 0$

$\lambda = 2$ | $\lambda = 3$ | $\lambda = 6.$

$\therefore \lambda = 2, 3, 6.$ are the eigen values.

The Corresponding eigen vectors are given by $(A-\lambda I)X = 0$.

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1).$$

<u>Case-1</u>:- Let $\lambda = 2$.

$$(1) \Rightarrow \begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - y + z = 0$

$-x + 3y - z = 0$

$x - y + z = 0$

Solve I^{st} two equ's.

$$\begin{array}{cccc} & x & y & z \\ -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{array}$$

$$\frac{x}{1-3} = \frac{y}{-1+1} = \frac{z}{3-1}$$

$$\frac{x}{-\cancel{2}_{1}} = \frac{y}{0} = \frac{z}{\cancel{2}_{1}}$$

$$\therefore X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus $\hat{X}_1$ (or) $\frac{X_1}{\sqrt{\lambda}} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Case-2:- Let $\lambda = 3$

$$① \Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-y + z = 0$$

$$-x + 2y - z = 0$$

$$x - y = 0$$

Solve Ist two eq's.

$$\begin{array}{cccc} & x & y & z \\ -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 2 \end{array}$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1}$$

$\therefore X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Thus $\hat{X}_2$ (or) $\dfrac{X_2}{\sqrt{\lambda}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Case-3:- Let $\lambda = 6$:-

(1) $\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x - y + z = 0$

$-x - y - z = 0$

$x - y \cancel{+} 3z = 0.$

Solve Ist ~~two~~ equ

$$\begin{array}{cccccc} & x & & y & & z \\ -1 & & 1 & & -3 & & -1 \\ -1 & & -1 & & -1 & & -1 \end{array}$$

$\dfrac{x}{\cancel{2}\ 1} = \dfrac{y}{-\cancel{4}\ 2} = \dfrac{z}{\cancel{2}\ 1}$

$\therefore X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\hat{X}_3$ (or) $\dfrac{X_3}{\lambda} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

* The modal matrix is $[X_1 \quad X_2 \quad X_3] = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

* The normalised modal matrix $P = [\hat{X}_1 \quad \hat{X}_2 \quad \hat{X}_3]$

$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

This is an orthogonal matrix.

$\therefore$ Diagonalised matrix $(D) = P^{-1}AP$.

$$\therefore D = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Canonical (or) normal Form $= Y^T D Y$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

which is the required canonical form.

$\therefore$ The orthogonal transformation which reduces the Q.F X^TAX to canonical form is given by $X = PY$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\therefore$ Rank $(r) = 3$

$\therefore$ Index $(s) = 3$

$\therefore$ Signature $= 2s - r$

$= 2(3) - 3$

$= 6 - 3$

$= 3$

(Ans)

THE END

Sequences & Series

(or)

Infinite Series

Definition of a sequence:– A sequence is a succession of no's (or) terms formed according to some definite rule.

For example, if $u_n = 2n+1$

By giving different values of n in u_n, we get different terms of the sequence.

Thus, $u_1 = 3, u_2 = 5, u_3 = 7 \ldots$

A sequence having unlimited no. of terms is known as an infinite sequence.

Definition of a limit:– If a sequence tends to a limit l, then we write

$$\lim_{n \to \infty} (u_n) = l.$$

Convergent sequence:– If the limit of a sequence is finite, the sequence is convergent.

Eg:- $1, 1/4, 1/9, 1/16, \ldots, 1/n^2 + \ldots$ is a convergent sequence.

Divergent sequence:– If the limit of a sequence does not tend to a finite number, the sequence is said to be divergent.

Eg:- $3, 5, 7 \ldots (2n+1) \ldots$ is a divergent sequence.

Bounded sequence:- $u_1, u_2, u_3 \ldots u_n$ is a bounded sequence if $u_n < k$ for every n.

Monotonic sequence:- The sequence is either increasing or decreasing, such sequences are called monotonic.

Eg:- $1, 4, 7, 10 \ldots$ is a monotonic sequence.

$1, 1/2, 1/3, 1/4 \ldots$ is also a monotonic sequence.

$1, -1, 1, -1, 1, \ldots$ is not a monotonic sequence.

- A sequence which is monotonic & bounded is a Convergent sequence.

Series:- A series is the sum of a sequence.

Let $u_1, u_2, u_3, \ldots u_n, \ldots$ be a given sequence.

Then, the expression ~~u_1+u_2~~ $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ is called the series associated with the given sequence.

Eg:- $1+3+5+7+\ldots$ is a series.

If the no. of terms of a series is limited, the series is called finite. When the no. of terms of a series are unlimited, it is called an infinite series.

$$u_1 + u_2 + u_3 + u_4 + \ldots + u_n + \ldots \infty$$

is called an infinite series & it is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$. The sum of the first 'n' terms of a series is denoted by S_n.

Convergent, Divergent & Oscillatory series:-

Consider the infinite series

$$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots \infty$$

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

Three cases arise :-

(i). If 'S_n' tends to a finite number as $n \to \infty$, the series $\sum u_n$ is said to be "Convergent".

(ii). If 'S_n' tends to infinity as $n \to \infty$, the series $\sum u_n$ is said to be "divergent".

(iii). If 'S_n' does not tend to a unique limit, finite or infinite, the series $\sum u_n$ is called "oscillatory".

Properties of Infinite series:-

①. The nature of an infinite series does not change!

(i). By multiplication of all terms by a constant k.

(ii). By addition or deletion of a finite no. of terms.

②. If two series $\sum u_n$ & $\sum v_n$ are convergent, then $\sum (u_n + v_n)$ is also convergent.

Ex:- ①

Examine the nature of the series $1+2+3+4+\cdots+n+\cdots\infty$

Sol:- $S_n = 1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$ [series in AP]

since, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} \Rightarrow \infty$

Hence, this series is divergent.

=(Ans)

Ex-②:- Test the Convergence of the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\infty$$

Sol:- $S_n = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\infty$ [series in G.P]

$$= \frac{1}{1-\frac{1}{2}} = 2 \qquad \left[S_n = \frac{a}{1-r}\right]$$

$$\lim_{n\to\infty} S_n = 2.$$

Hence, the series is Convergent.

(Ans)

Ex-③:- Discuss the nature of the series

$$2-2+2-2+2-\cdots\infty$$

Sol:- Let $S_n = 2-2+2-2+2-\cdots\infty$

$= 0$ if 'n' is even.

$= 2$ if 'n' is odd.

Hence, S_n does not tend to a unique limit so therefore, the given series is oscillatory.

(Ans)

Series of positive terms / Positive term series:- If all terms after few negative terms in an infinite series are positive, such a series is a positive term series.

Eg:- $-10-6-1+5+12+20+\cdots$ is a positive term series.

By omitting the $-$ve terms, the nature of a $+$ve term series remains unchanged.

Comparison test:- If two positive terms Σu_n & Σv_n be such that

$\lim_{n\to\infty} \frac{u_n}{v_n} = k$ (finite no.), then both series converge or diverge together.

Proof:- By definition of limit there exists a +ve no. ε, however small, such that

$$\left|\frac{u_n}{v_n} - k\right| < \varepsilon \text{ for } n>m \quad \text{i.e. } -\varepsilon < \frac{u_n}{v_n} - k < +\varepsilon$$

$$k - \varepsilon < \frac{u_n}{v_n} < k + \varepsilon \text{ for } n>m$$

Ignoring the first m terms of both series, we have

$$k - \varepsilon < \frac{u_n}{v_n} < k + \varepsilon \text{ for all } n. \longrightarrow (1)$$

Case-①:- Σv_n is convergent, then

$\lim_{n\to\infty} (v_1+v_2+\dots+v_n) = h$ (say) where h is a finite no.

from eq (1), $u_n < (k+\varepsilon)v_n$ for all n.

$$\lim_{n\to\infty} (u_1+u_2+\dots+u_n) < (k+\varepsilon) \lim_{n\to\infty} (v_1+v_2+\dots+v_n) = (k+\varepsilon)h.$$

Hence, Σu_n is also convergent.

Case-②:- Σv_n is divergent then

$$\lim_{n\to\infty} (v_1+v_2+v_3+\dots+v_n) \to \infty \longrightarrow (2)$$

from eq (2) $\Rightarrow \lim_{n\to\infty} (u_1+u_2+\dots+u_n) \to \infty$

Hence, Σu_n is also divergent.

Ex-①:- Test the series $\sum_{n=1}^{\infty} \frac{1}{n+10}$ for Convergence or divergence.

Sol:- Given $u_n = \frac{1}{n+10}$

$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+10} = \lim_{n\to\infty} \frac{1}{1+\frac{10}{n}} = 1 = \text{finite no.}$$

Acc. to Comparison test both series converges or diverge together, but Σv_n is divergent as $p=1$.

$\therefore \Sigma u_n$ is also divergent.

(Ans)

Ex-②:- Test for Convergence the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \dots \infty$$

Sol:- $u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{1}{n^2} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

let $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2 = \text{finite no.}$$

Acc. to Comparision test both series Converge or diverse together, but Σv_n is Convergenent as $p=2$.

$\therefore \Sigma u_n$ is also convergent.

(Ans)

Ex-③:- Test the following for Convergence:-

$$\sum_{1}^{\infty} \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7}}$$

Sol:- $u_n = \dfrac{(3n^2+1)^{1/3}}{(4n^3+2n+7)^{1/4}} = \dfrac{3^{1/3} n^{2/3}\left[1+\frac{1}{3n^2}\right]^{1/3}}{n^{3/4}\left[4+\frac{2}{n^2}+\frac{7}{n^3}\right]^{1/4}}$

$$= \frac{3^{1/3}\left[1+\frac{1}{3n^2}\right]^{1/3}}{n^{1/12}\left[4+\frac{2}{n^2}+\frac{7}{n^3}\right]^{1/4}}$$

Let $V_n = \dfrac{1}{n^{1/12}}$

$$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{3^{1/3}\left[1+\frac{1}{3n^2}\right]^{1/3}}{\left[4+\frac{2}{n^2}+\frac{7}{n^3}\right]^{1/4}}$$

$$= \frac{3^{1/3}}{\sqrt{2}} = \text{finite no.}$$

According to Comparison test both series converge or diverge together; but ΣV_n is divergent as $P = \frac{1}{12} < 1$.

$\therefore \Sigma u_n$ is also divergent.

=(Ans)

Ex-④:- Test for Convergence the series whose nth term is $n^{\log x}$.

Sol:- The series is $1^{\log x} + 2^{\log x} + 3^{\log x} + 4^{\log x} + \dots \infty$

$$= \frac{1}{1^{-\log x}} + \frac{1}{2^{-\log x}} + \frac{1}{3^{-\log x}} + \frac{1}{4^{-\log x}} + \dots \infty$$

Put $-\log x = p$

$$= \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \infty$$

(a). which is Convergent if $p > 1$ i.e $-\log x > 1$

$$\Rightarrow \log_e \frac{1}{x} > \log_e e \Rightarrow \frac{1}{x} > e \text{ (or) } x < \frac{1}{e}$$

(b). which is divergent if $p \leq 1 \Rightarrow -\log x \leq 1$

$$\Rightarrow \log_e \frac{1}{x} \leq \log_e e \Rightarrow \frac{1}{x} \leq e \Rightarrow \frac{1}{e} \leq x.$$

=(Ans)

Ex-⑤:- Test the series for convergence

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \cdots$$

Sol:- Given series is $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \cdots$

Here $u_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$

let $v_n = \frac{1}{n^{p-1}}$

$$\therefore \frac{u_n}{v_n} = \frac{1+\frac{1}{n}}{n^{p-1}} \times \frac{n^{p-1}}{1}$$

$$= 1 + \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1.$$

$\therefore$ both the series are either convergent or divergent.

But Σv_n is convergent if $p-1 > 1$, i.e if $p > 2$

and is divergent if $p-1 \leq 1$ i.e if $p \leq 2$

$\therefore$ The given series is convergent if $p > 2$ & divergent if $p \leq 2$.

=(Ans)

Ex-⑥:- Test the series for convergence

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\dots$$

Sol:- Let us consider $\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\dots$

Here $u_n=\frac{n^n}{(n+1)^{n+1}}=\frac{n^n}{(n+1)(n+1)^n}=\frac{n^n}{n(1+\frac{1}{n})\times n^n(1+\frac{1}{n})^n}$

$$=\frac{1}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}$$

$$\frac{u_n}{v_n}=\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}\cdot \text{ Let } v_n=\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}=\text{finite quantity.}$$

Hence, either both the series are convergent or both the series are divergent.

But Σv_n is divergent as $p=1$.

$\therefore$ given series is divergent.

=(Ans).

D'-Alembert's Ratio Test:-

Statement:- If Σu_n is a +ve term series such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k \text{ then}$$

(i). The series is Convergent if $k<1$

(ii). " " " Divergent " $k>1$.

Proof:-

Case-①:- When $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k < 1$

By definition of a limit, we can find a no. $r(<1)$ such that

$$\frac{u_{n+1}}{u_n} < r \text{ for all } n \geq m \quad \left[\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r \dots\right]$$

Omitting the first m terms, let the series be

$$u_1 + u_2 + u_3 + u_4 + \dots \infty$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \dots\right) = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \dots\infty\right)$$

$$< u_1(1 + r + r^2 + r^3 + \dots \infty) \quad (r<1)$$

$$= \frac{u_1}{1-r}, \text{ which is a finite quantity.}$$

Hence, Σu_n is Convergent.

Case-②:- When $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k > 1$

By definition of limit, we can find a no. m ~~to~~ such that

$$\frac{u_{n+1}}{u_n} \geq 1 \text{ for all } n \geq m$$

$$\frac{u_2}{u_1} > 1, \quad \frac{u_3}{u_2} > 1, \quad \frac{u_4}{u_3} > 1$$

Ignoring the first m terms, let the series be

$$u_1 + u_2 + u_3 + u_4 + \cdots \infty$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \cdots\right)$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \cdots \infty\right)$$

$$\geq u_1(1+1+1+1 \ldots \text{ to } n \text{ terms}) = n u_1$$

$$\lim_{n\to\infty}(u_1 + u_2 + \cdots + u_n) = u_r$$

$$\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} n u_1 = \infty$$

Hence, Σu_n is divergent.

Note:- When $\frac{u_{n+1}}{u_n} = 1$ $(k=1)$.

The Ratio test fails.

Ex-(1):- Test for Convergence the series whose nth term is:-

(a). $\frac{n^2}{2^n}$

(b). $\frac{2^n}{n^3}$

Sol:-

(a). $u_n = \dfrac{n^2}{2^n}$, $u_{n+1} = \dfrac{(n+1)^2}{2^{n+1}}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2$$

$$= \frac{1}{2} < 1$$

Hence, the series is Convergent by D-Alembert's Ratio Test.

=(Ans)

(b). $u_n = \dfrac{2^n}{n^3}$; $u_{n+1} = \dfrac{2^{n+1}}{(n+1)^3}$

By D-Alembert's Ratio Test

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} = \frac{2}{\left(1+\frac{1}{n}\right)^3}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^3} = 2 > 1$$

Hence, the series is divergent.

=(Ans)

Raabe's Test (Higher Ratio Test) :-

Statement :- If $\sum u_n$ is a +ve term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k, \text{ then}$$

(i). The series is Convergent if $k > 1$

(ii). " " " divergent " $k < 1$.

Case-I ($k > 1$)

Proof :- Let 'p' be such that $k > p > 1$ & Compare the given series. $\sum u_n$ with $\sum \frac{1}{n^p}$ which is Convergent as $p > 1$.

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \text{ (or) } \left(\frac{u_n}{u_{n+1}}\right) > \left(1 + \frac{1}{n}\right)^p > 1 + \frac{p}{n} + \frac{p(p-1)}{\lfloor 2} \frac{1}{n^2} + \dots$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{n!} \frac{1}{n^2} + \dots$$

If $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$ &

$k > p$ which is true as $k > p > 1$;

$\sum u_n$ is Convergent when $k > 1$.

Case - II ($k < 1$) :- Same steps as in Case - I.

Note :-

~~①. Rab~~ ①. Raabe's Test fails if $k = 1$.

②. Raabe's Test is applied only when D-Alembert's Ratio Test fails.

Ex-(1):- Test the Convergence of the series

$$\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \dots$$

Sol:- $u_n = \dfrac{x^n}{(2n-1)2n}$

$$u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(2n+1)(2n+2)} \times \frac{(2n-1)2n}{x^n}$$

$$= \frac{x\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$$

(i). If $x<1$, Σu_n is Convergent.

(ii). If $x>1$, Σu_n is divergent.

(iii). If $x=1$, Test fails

Let us apply Rabee's Test when $x=1$.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{2n(2n-1)} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)-2n(2n-1)}{2n(2n-1)}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(8n+2)}{(2n)(2n-1)}\right]$$

$$= \lim_{n\to\infty} \frac{\left(2+\frac{2}{4n}\right)}{1\left(1-\frac{1}{2n}\right)};$$

$$= 2.$$

So, the series is Convergent.

Hence, we can say that the given series is Convergent if $x \le 1$ & divergent. if $x > 1$

=(Ans).

Ex-②:- Test the following series for Convergence

$$\sum \frac{1}{\sqrt{n+1}-1}$$

Sol:- $u_n = \frac{1}{\sqrt{n+1}-1}$; $u_{n+1} = \frac{1}{\sqrt{n+2}-1}$

$$\frac{u_n}{u_{n+1}} = \frac{\sqrt{n+2}-1}{\sqrt{n+1}-1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n}-\frac{1}{n^2}}}{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}} = 1$$

D-Alembert's test fails.

Let us apply ~~Rabe~~ Raabe's test.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{\sqrt{n+2}-1}{\sqrt{n+1}-1}-1\right)$$

$$= \lim_{n\to\infty} n\left[\frac{\sqrt{n+2}\cancel{-1}-\sqrt{n+1}\cancel{+1}}{\sqrt{n+1}-1}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-1}\right]$$

$$= \lim_{n\to\infty} n \left[\frac{\sqrt{1+\frac{2}{n}} - \sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{1}{n} - \frac{1}{n^2}}} \right] = 0 < 1$$

Hence, Σu_n is divergent
=(Ans)

Ex-③ :- Test for convergence the series for +ve values of x :-

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \dots + \frac{2^n-2}{2^n+1}x^{n-1} + \dots$$

Sol:- $u_n = \dfrac{2^n-2}{2^n+1}x^{n-1}$

$u_{n+1} = \dfrac{2^{n+1}-2}{2^{n+1}+1}x^n$

By D-Alembert's Ratio Test

$$\frac{u_{n+1}}{u_n} = \left(\frac{2^{n+1}-2}{2^{n+1}+1}x^n\right)\left(\frac{2^n+1}{2^n-2} \cdot \frac{1}{x^{n-1}}\right)$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{n+1}\left(1-\frac{1}{2^n}\right)x}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)} \left[\frac{2^n\left(1+\frac{1}{2^n}\right)}{2^n\left(1-\frac{1}{2^{n-1}}\right)}\right]$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x$$

(i). If $x<1$, Σu_n is convergent.

(ii). If $x>1$, Σu_n is divergent.

(iii). If $x=1$, test fails.

Let us apply Raabe's Test when $x=1$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\left(\frac{2^n-2}{2^n+1}\right)\left(\frac{2^{n+1}+1}{2^{n+1}-2}\right)-1\right]$$

$$= \lim_{n\to\infty} n\left[1-1\right]$$

$$=0<1$$

So, the series is divergent.

Hence, the given series is divergent if $x \geq 1$ & Convergent if $x<1$.

=(Ans)

Ex-④:- Show that the series $\frac{1}{x}+\frac{\lfloor 2}{x(x+1)}+\frac{\lfloor 3}{x(x+1)(x+2)}+\ldots$ Converges if $x>2$ & diverges if $x\leq 2$.

Sol:-

$$u_n = \frac{\lfloor n}{x(x+1)(x+2)\cdots(x+n-1)}$$

$$u_{n+1} = \frac{\lfloor n+1}{x(x+1)(x+2)\cdots(x+n-1)(x+n)}$$

By D-Alembert's Test.

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{(x+n)} ; \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1+\frac{1}{n}}{1+\frac{x}{n}} = 1$$

Test fails.

Let us apply Raabe's Test...

$$\lim_{n\to\infty} \left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{x+n}{n+1}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{x-1}{n+1}\right)$$

$$= \lim_{n\to\infty} \frac{x-1}{1+\frac{1}{n}}$$

$$= x-1$$

If $x-1>1$ (or) $x>2$, Σu_n is Convergent.

If $x-1<1$ (or) $x<2$, Σu_n is divergent.

=End.

Ex-⑤:- Discuss the Convergence of the series:-

$$\frac{x^2}{2\log 2} + \frac{x^3}{3\log 3} + \frac{x^4}{4\log 4} + \cdots$$

Sol:- Given $\frac{x^2}{2\log 2} + \frac{x^3}{3\log 3} + \frac{x^4}{4\log 4} + \cdots$

$$u_n = \frac{x^{n+1}}{(n+1)\log(n+1)}$$

$$u_{n+1} = \frac{x^{n+2}}{(n+2)\log(n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)\log(n+1)} \times \frac{(n+2)\log(n+2)}{x^{n+2}}$$

$$= \frac{1}{x}\left(\frac{n+2}{n+1}\right)\frac{\log(n+2)}{\log(n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{x}\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)\frac{\log\left(1+\frac{2}{n}\right)}{\log\left(1+\frac{1}{n}\right)} = \frac{1}{x}$$

(i). When $\frac{1}{x} > 1$ i.e $x < 1$, the series is convergent.

(ii). $x > 1$, the series is divergent.

(iii). when $x = 1$, the test fails.

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+2}{n+1}\right) \frac{\log(n+2)}{\log(n+1)}$$

$$= \left(\frac{n+2}{n+1}\right) \frac{\log n + \log\left(1+\frac{2}{n}\right)}{\log n + \log\left(1+\frac{1}{n}\right)}$$

$$= \left(\frac{n+2}{n+1}\right) \frac{\log n + \frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \cdots}{\log n + \frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} + \cdots}$$

$$= \left(\frac{n+2}{n+1}\right) \frac{1 + \frac{2}{n\log n} + \cdots}{1 + \frac{1}{n\log n} + \cdots}$$

$$= \left(\frac{n+2}{n+1}\right)\left[1 + \frac{1}{n\log n}\right]$$

$$= \left[1 + \frac{1}{n+1}\right]\left[1 + \frac{1}{n\log n}\right]$$

$$= 1 + \frac{1}{n\log n}$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[1 + \frac{1}{n\log n} - 1\right]$$

$$= \frac{1}{\log n}$$

$$= 0 < 1$$

Ans

Cauchy's Integral Test:-

Statement :- A positive term series

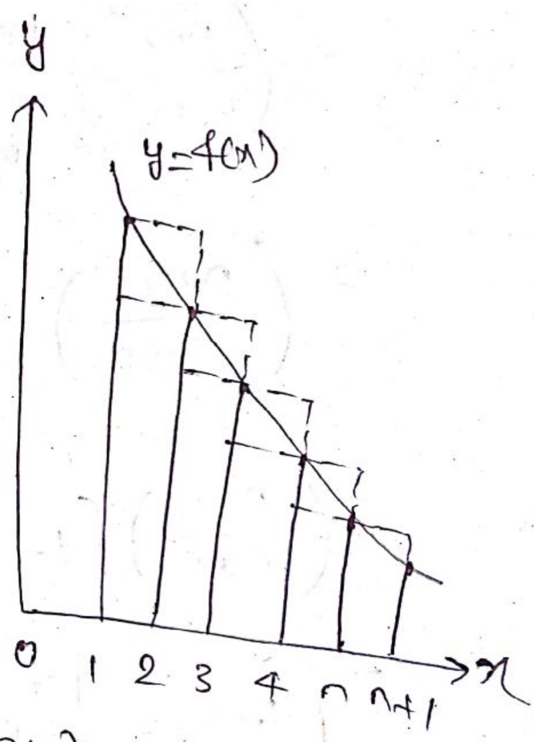
$$f(1)+f(2)+f(3)+\dots+f(n)+\dots$$

where $f(n)$ decreases as n increases, converges or diverges according to the integral.

$$\int_1^\infty f(x)\,dx.$$

is finite (or) infinite.

Proof:- In the fig, the area under the curve from $x=1$ to $x=n+1$ lies b/w the sum of the areas of small rectangles & sum of the areas of large rectangles.



$$\Rightarrow f(1)+f(2)+\dots+f(n) \geq \int_1^{n+1} f(x)dx$$

$$\geq f(2)+f(3)+\dots+f(n+1)$$

$$S_n \geq \int_1^{n+1} f(x)\,dx \geq S_{n+1}-f(1)$$

As, $n\to\infty$, from the second inequality that if the integral has a finite value then $\lim_{n\to\infty} S_{n+1}$ is also finite, so $\sum f(n)$ is convergent.

Similarly, if the integral is infinite, then from the first inequality that $\lim_{n\to\infty} S_n \to \infty$, so the series is divergent.

Ex-①:- Apply the integral test to determine the convergence of the p-series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots+\frac{1}{n^p}+\dots\infty$$

(i). When $p>1$

$$f(x)=\frac{1}{x^p}$$

$$\int_1^\infty f(x)\,dx = \lim_{m\to\infty}\int_1^m \frac{1}{x^p}\,dx = \lim_{m\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^m$$

$$=\lim_{m\to\infty}\frac{1}{1-p}\left[m^{1-p}-1\right]$$

$$=\lim_{m\to\infty}\frac{1}{1-p}\left[\frac{1}{m^{p-1}}-1\right]$$

$$=\frac{1}{p-1}, \text{ which is finite.}$$

By Cauchy's Integral Test, the series is convergent for $p>1$.

(ii). When $p<1$

$$\int_1^\infty f(x)\,dx=\frac{1}{1-p}\left[\lim_{m\to\infty}(m^{1-p}-1)\right]\to\infty$$

Thus, the series is divergent, if $p<1$.

(iii). When $p=1$.

$$\int_1^\infty \frac{1}{x}dx = [\log x]_1^\infty \to \infty$$

Thus, the series is divergent

Thus, $\sum \frac{1}{n^p}$ is Convergent if $p>1$ & divergent if $p\leq 1$.

Ex-②:- Examine the Convergence of:-

(a). $\sum_{n=2}^{\infty} \frac{1}{n\log n}$ (b). $\sum_{n=1}^{\infty} ne^{-n^2}$

Sol:-

(a). Here $f(x) = \frac{1}{x\log x}$

$$\int_2^\infty \frac{1}{x\log x}dx = \lim_{m\to\infty} [\log\log x]_2^m$$

$$= \lim_{m\to\infty} [\log\log m - \log\log 2]$$

By Cauchy's Integral Test the series is divergent.

=Ans

(b). Here, $f(x) = xe^{-x^2}$

Now $\int_1^\infty xe^{-x^2}dx = \lim_{m\to\infty}\left[\frac{e^{-x^2}}{-2}\right]_1^m$

$$= \lim_{m\to\infty}\left[\frac{e^{-m^2}}{-2} + \frac{e^{-1}}{2}\right]$$

$$= \frac{e^{-1}}{2} = \frac{1}{2e}$$, which is finite.

Hence, the given series is Convergent.

=Ans

Cauchy's root test:-

Statement:- If Σu_n is a +ve term series such that

$\lim_{n\to\infty} (u_n)^{1/n} = k$, then:-

(i). If $k<1$; the series Converges.

(ii). If $k>1$, the series diverges.

Proof:- By definition of limit

$$|(u_n)^{1/n} - k| < \varepsilon \text{ for } n>m$$

$$k-\varepsilon < (u_n)^{1/n} < k+\varepsilon \text{ for } n>m$$

(i).

$$k<1$$

$$k+\varepsilon < r < 1$$

$$(u_n)^{1/n} < k$$

$$u_n < k^n$$

$$u_1+u_2+\dots\infty < k+k^2+\dots+k^n+\dots\infty$$

$$< \frac{1}{1-k} \text{ (a finite quantity)}$$

$\therefore$ The series is Convergent.

(ii). $k>1$

$$k-\varepsilon>1$$

$$(u_n)^{1/n} > k-\varepsilon > 1$$

$$u_n>1$$

$$s_n = u_1+u_2+\dots+u_n > n$$

$$\lim_{n\to\infty} s_n \to \infty$$

$\therefore$ The series is divergent.

(iii). $k=1$

If $\lim_{n\to\infty} (u_n)^{1/n} = 1$, the test fails.

For Exp — $\sum u_n = \sum \frac{1}{n^p}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^p}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^{-p}$$

$$= 1 \text{ for all } p,\ k=1$$

But $\sum \frac{1}{n^p}$ is Convergent for $p>1$ & divergent for $p \le 1$.

Thus, we cannot say whether $\sum u_n$ is Convergent or divergent for $k=1$.

Ex (1):- Examine the Convergence of the series $\sum \frac{1}{(1+\frac{1}{n})^{n^2}}$

Sol:- $u_n = \frac{1}{(1+\frac{1}{n})^{n^2}}$, $(u_n)^{1/n} = \left[\frac{1}{(1+\frac{1}{n})^{n^2}}\right]^{1/n} = \frac{1}{(1+\frac{1}{n})^n}$.

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n}$$

$$= \frac{1}{e} < 1$$

Hence, the given series is Convergent.

= (Ans)

Ex-②:- Discuss the Convergence of the following series:-

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots\infty$$

Sol:- $u_n=\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{(n+1)}{n}\right]^{-n}$

$$[u_n]^{1/n}=\left[\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{(n+1)}{n}\right]^{-n}\right]^{1/n}$$

$$=\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right]^{-1}$$

$$\lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)\right]^{-1}$$

$$=(e-1)^{-1}$$

$$=\frac{1}{e-1}<1$$

Hence, the given series is convergent.

=Ans.

Ex-③:- Discuss the convergence of the series

$$\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\cdots\infty$$

Sol:- Ignoring the ~~term~~ first term, we have:-

$$u_n=\left(\frac{n+1}{n+2}\right)^n x^n$$

$$(u_n)^{1/n}=\left(\frac{n+1}{n+2}\right)x \qquad \text{[Cauchy Root Test]}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) x = x$$

Hence, the series Converges if $x<1$ & diverges if $x>1$.

If $x=1$.

$$u_n = \left(\frac{n+1}{n+2}\right)^n = \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2}$$

$$= \frac{e}{e^2}$$

$$= \frac{1}{e} \neq 0.$$

∴ The series is divergent by Cauchy Fundamental Theorem.

$\Rightarrow$ Ans

Logarithmic (Log) test:-

Statement:- If Σu_n is a +ve term series such that

$$\lim_{n\to\infty} \left(n \log \frac{u_n}{u_{n+1}}\right) = k$$

(i). If $k>1$, then the series is Convergent.

(ii). If $k<1$, then the series is divergent.

proof:-

(i). If $k>1$.

Compare Σu_n with $\Sigma \frac{1}{n^p}$, if $k>p>1$, Σu_n Converges.

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p \rightarrow (1)$$

Taking logarithm (log) of b.s of (1):

$$\log \frac{u_n}{u_{n+1}} > p \log\left(1+\frac{1}{n}\right)$$

$$\log \frac{u_n}{u_{n+1}} > p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right)$$

$$n \log \frac{u_n}{u_{n+1}} > p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \cdots\right)$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > p$$

i.e $k>p$ which is true as $k>p>1$ $\left[\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = k\right]$

Hence, Σu_n is Convergent.

when $p<1$

Similarly, when $p<1$, Σu_n is divergent.

when $p=1$, the test fails.

Ex:- Test the Convergence of the series

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\cdots$$

Sol:- The given series is

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\cdots$$

Here $u_n = \frac{n^{n-1}x^{n-1}}{n!}$ $\therefore u_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^n x^n}{(n+1)!} \times \frac{n!}{n^{n-1} x^{n-1}} = \frac{(n+1)^n}{(n+1)n^{n-1}} x$$

$$= \frac{(n+1)^{n-1}}{n^{n-1}} x = \left[\frac{n+1}{n}\right]^{n-1}$$

$$x = \left(1+\frac{1}{n}\right)^n \times \left(1+\frac{1}{n}\right)^{-1} x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = ex \quad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

By D-Alembert's Ratio Test, Σu_n is Convergent if $|ex|<1$, i.e $|x| < \frac{1}{e}$ & ~~its~~ is divergent if $|ex| > 1$ i.e $|x| > \frac{1}{e}$.

If $ex = 1$, i.e $x = \frac{1}{e}$, this test fails.

If $x = 1/e$

$$\frac{u_{n+1}}{u_n} = \left(1+\frac{1}{n}\right)^{n-1} \times \frac{1}{e}$$

$$\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^{n-1}}$$

$$\log\left[\frac{u_n}{u_{n+1}}\right] = \log\left[\frac{e}{\left(1+\frac{1}{n}\right)^{n-1}}\right] \quad [\because \log \text{ on b.s}]$$

$$= \log e - (n-1)\log\left(1+\frac{1}{n}\right) = \log e - (n-1)\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right]$$

$$= 1 - \left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \dots - \frac{1}{n} + \frac{1}{2n^2} + \dots\right]$$

$$= 1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{n} - \frac{1}{2n^2} - \dots$$

$$= \frac{3}{2n} - \frac{5}{6n^2}$$

By Logarithmic Test

$\therefore n\log \frac{u_n}{u_{n+1}} = \frac{3}{2} - \frac{5}{6n}$

$\therefore \lim_{n\to\infty}\left[n\log \frac{u_n}{u_{n+1}}\right] = \frac{3}{2}$ i.e >1

$\therefore$ for $x=\frac{1}{e}$, Σu_n is Convergent.

Hence, if $x \le \frac{1}{e}$, Σu_n is Convergent & if $x > \frac{1}{e}$, Σu_n is divergent.

=(Ans)

P-test/P-series :- The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots\infty$ is:-

(i). Convergent if $p>1$ (ii). Divergent if $p\le 1$.

Sol:-

Case 1: ($p>1$):- The given series can be grouped as:-

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\frac{1}{11^p}+\frac{1}{12^p}+\frac{1}{13^p}+\frac{1}{14^p}+\frac{1}{15^p}\right)+\dots$$

Now $\frac{1}{1^p}=1 \longrightarrow (1)$

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p} \longrightarrow (2)$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p} \longrightarrow (3)$$

$$\frac{1}{8^p}+\frac{1}{8^p}+\dots+\frac{1}{15^p}<\frac{1}{8^p}+\frac{1}{8^p}+\dots+\frac{1}{8^p}=\frac{8}{8^p} \rightarrow (4)$$

On adding (1), (2), (3) & (4) we get:

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\frac{1}{8^p}+\dots+\frac{1}{15^p}\right)+\dots<\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\dots$$

$$<1+\left(\frac{1}{2}\right)^{p-1}+\left(\frac{1}{2}\right)^{2p-2}+\left(\frac{1}{2}\right)^{3p-3}+\dots$$

$$<\frac{1}{1-\left(\frac{1}{2}\right)^{p-1}} \quad \left[\begin{array}{l} G.P \\ r=(1/2)^{p-1} \\ S=\frac{1}{1-r}\end{array}\right]$$

< Finite no. if $p>1$.

Hence, the given series is convergent when $p>1$.

<u>Case 2:-</u> $p=1$

When $p=1$, the given series becomes.

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{16}\right)+\dots$$

$$1+\frac{1}{2}=1+\frac{1}{2} \rightarrow (1)$$

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2} \rightarrow (2)$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2} \rightarrow (3)$$

$$\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{16}>\frac{1}{16}+\frac{1}{16}+\dots+\frac{1}{16}=\frac{8}{16}=\frac{1}{2} \rightarrow (4)$$

on adding (1), (2), (3) & (4) we get

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{16}\right)+\dots$$

$$>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots>1+\frac{n}{2} \quad (n\rightarrow\infty)$$

$>\infty$

Hence, the given series is divergent when $p=1$.

Case ③ :- $p<1$

$$\frac{1}{2^p} > \frac{1}{2}, \quad \frac{1}{3^p} > \frac{1}{3}, \quad \frac{1}{4^p} > \frac{1}{4} \text{ \& so on.}$$

$$\therefore \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$> \text{divergent series } (p=1)$

Hence, the given series is divergent when $p<1$.

Alternating series :- A series in which the terms are alternately negative is called the "alternating series".

Eg :- $u_1 - u_2 + u_3 - u_4 + \cdots \infty$

Leibnitz's Rule for Convergence of an Alternating series :-

(i). Each term is numerically less than its preceeding term.

(ii). $\lim\limits_{n\to\infty} u_n = 0$.

Ex - ① :- Test the following series for Convergence

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \infty$$

Sol :- The terms of the given series are alternately +ve & -ve.

$$|u_n| = \frac{1}{n} \; ; \; |u_{n+1}| = \frac{1}{n+1}$$

(i). $|u_{n+1}| < u_n$ as $\frac{1}{n+1} < \frac{1}{n}$

(ii). $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

As, both the Conditions for Convergence are satisfied, the given series is Convergent by Leibnitz's rule.

=(Ans).

Ex-② :- Test the Convergence of the series:-

$$\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - \cdots \infty$$

Sol:- The terms of the given series are alternately +ve & -ve

$$u_n = (-1)^{n-1} \frac{n}{5n+1}$$

$$|u_n| - |u_{n+1}| = \frac{n}{5n+1} - \frac{n+1}{5(n+1)+1}$$

$$= \frac{\cancel{5n^2} + 6n - \cancel{5n^2} - 5n - n - 1}{(5n+1)(5n+6)}$$

$$= \frac{-1}{(5n+1)(5n+6)}$$

$$\therefore |u_n| > |u_{n+1}|$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{1}{5+\frac{1}{n}} = \frac{1}{5} \neq 0$$

Hence, the series is not Convergent. It is oscillatory.

=(Ans)

Alternating Convergent series:- There are two types of alternating convergent series:-

(1). Absolutely Convergent series.

(2). Conditionally Convergent series.

(1). Absolutely Convergent Series:- If $u_1+u_2+u_3+\dots$ be such that $|u_1|+|u_2|+|u_3|+\dots$ be convergent then $u_1+u_2+u_3+\dots\infty$ is called absolutely convergent.

(2). Conditionally Convergent Series:- If $|u_1|+|u_2|+|u_3|+\dots$ be divergent & $u_1+u_2+u_3+\dots$ be convergent then $u_1+u_2+u_3+\dots\infty$ is called conditionally convergent.

Ex-(1):- S.T the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$ is convergent but not absolutely convergent.

Sol:- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$

The terms of the series are alternately +ve & -ve.

(i). $|u_{n+1}|<|u_n|$ as $\frac{1}{n+1}<\frac{1}{n}$

(ii). $\lim\limits_{n\to\infty} u_n=0$.

Thus, the given series is convergent.

But $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\infty$ is divergent. (P-series, $P=1$)

Hence, the given series is conditionally convergent.

=(Ans)

Ex-②:- What can you say about the series

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\infty\ ?$$

Sol:- Given $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

$$|u_n|=\frac{1}{n^2}\ ;\ |u_{n+1}|=\frac{1}{(n+1)^2}$$

(i). $|u_{n+1}|<|u_n|$ (ii). $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n^2}=0.$

Thus, the given series is Convergent by Leibnitz's rule.

And $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$ is also Convergent (P-series $P=2$)

Thus, the given series is absolutely convergent.

=(Ans).

Ex-③:- Discuss the series for Convergence

$$1-\frac{1}{3}+\frac{1}{2}-\frac{1}{3^3}+\frac{1}{2^2}-\frac{1}{3^5}+\frac{1}{2^3}-\frac{1}{3^7}+\cdots$$

Sol:- The given series is rewritten as:-

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots-\left(\frac{1}{3}+\frac{1}{3^3}+\frac{1}{3^5}+\frac{1}{3^7}+\cdots\right)$$

$$\lim_{n\to\infty}S_n=\frac{1}{1-\frac{1}{2}}-\frac{\frac{1}{3}}{1-\frac{1}{3^2}}=2-\frac{3}{8}$$

$$=1\frac{5}{8}$$

The given series is Convergent.

Again, $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{3}+\frac{1}{3^3}+\frac{1}{3^5}+\cdots$

$$\lim_{n\to\infty}S_n=2+\frac{3}{8}=\frac{16+3}{8}=\frac{19}{8}$$

This series is also convergent.

Hence, the given series is absolutely convergent.

=(Ans)

List of the Tests for Convergence:-

①. Cauchy's Fundamental Test:-

$$\lim_{n\to\infty} u_n \neq 0, \ \Sigma u_n \text{ is divergent.}$$

②. Comparison Test:-

$$\lim_{n\to\infty} \frac{u_n}{v_n} = k, \ \Sigma u_n \ \& \ \Sigma v_n \text{ are of the same nature.}$$

$$\Sigma \frac{1}{n^p}, \ p>1, \text{ Convergent}; \ p\leq 1, \text{ divergent.}$$

③. D-Alembert's Test:-

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k, \text{ if } k<1, \ \Sigma u_n \text{ is Convergent};$$

$$\text{if } k>1, \ \Sigma u_n \text{ is divergent.}$$

④. Raabe's Test:-

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k, \text{ if } k>1, \ \Sigma u_n \text{ is Convergent}; \text{ if } k<1,$$

Σu_n is divergent.

⑤. Cauchy's Root Test:-

$\lim_{n\to\infty} (u_n)^{1/n} = k$, if $k<1$, Σu_n is Convergent; if $k>1$, Σu_n is divergent.

⑥. Logarithmic (or) Log Test:-

$$\lim_{n\to\infty} n\log \frac{u_n}{u_{n+1}} = k, \text{ if } k>1, \ \Sigma u_n \text{ is Convergent}; \text{ if } k<1,$$

Σu_n is divergent.

(7). Leibnitz's Test :-

Alternately +ve & -ve is Convergent if :-

(i). $|u_{n+1}| > |u_n|$

(ii). $\lim_{n \to \infty} u_n = 0.$

THE END

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Unit - IV

Calculus

(1). verify Rolle's theorem for :- (or) Find the value of 'c' By Using RT.

(i). $f(x) = (x+2)^3 (x-3)^4$ in $[-2,3]$.

(ii). $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{2}, \sqrt{2}]$.

(iii). $f(x) = (x-a)^m (x-b)^n$ where m, n are +ve integers in $[a,b]$.

(iv). $f(x) = \dfrac{\sin x}{e^x}$ (or) $e^{-x} \sin x$ in $[0, \pi]$.

(v). $\log \left[\dfrac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$, $a>0, b>0$.

(vi). $g(x) = 8x^3 - 6x^2 - 2x + 1$, has a zero b/w 0 & 1.

(vii). $f(x) = x^2 - 2x - 3$ in the interval $(1,-3)$.

(viii). $f(x) = e^x \sin x$ in $[0, \pi]$.

Sol^n's

(i). Given $f(x) = (x+2)^3 (x-3)^4$

→ f(x) is a polynomial in x & hence is continuous in closed interval $[-2,3]$.

Now $f'(x) = 3(x+2)^2 (x-3)^4 + (x+2)^3 \cdot 4(x-3)^3$

$= (x+2)^2 (x-3)^3 \left[3(x-3) + 4(x-3)\right]$

$\cancel{= (x+2)^2 (x-3)^2 \left(3(x^2+9-2(9)(x)) + 4x - 12\right)}$

$\cancel{= (x+2)^2 (x-3)^2 \left[3x^2 + 27 - 18x\right] + 4x - 12}$

$\cancel{= (x+2)^2 (x-3)^2 \left[3x^2 - 18x + 4x + 27 - 12\right]}$

$= (x+2)^2 (x-3)^3 \left[3x - 9 + 4x - 12\right]$

$= (x+2)^2(x-3)^3(7x-1)$

and thus $f'(x)$ exists on $(-2,3)$.

$\therefore f(-2)=0 \; ; \; f(3)=0$

$\therefore \boxed{f(-2)=f(3)}$

Thus, all the 3 conditions of Rolle's theorem are satisfied.

Hence there exists a point 'c' in $(-2,3)$ such that $f'(c)=0$.

Verification:– Consider $f'(c)=0$.

$\Rightarrow (c+2)^2(c-3)^3(7c-1)=0$.

$c=-2$ (or) 3 (or) $\frac{1}{7}$

$\therefore \boxed{c=\frac{1}{7}\in(-2,3)}$ i.e $-2<\frac{1}{7}<3$.

Hence Rolle's theorem is verified.

=(Ans).

(ii). $f(x)=2x^3+x^2-4x-2$ in $[-\sqrt{2},\sqrt{2}]$. $\rightarrow (1)$

Sol:–

$\rightarrow f(x)$ is a polynominal & hence is continuous & differentiate for all x. Hence the Ist two conditions of Rolle's theorm are satisfied.

Now, $f(\sqrt{2})=2(\sqrt{2})^3+(\sqrt{2})^2-4\sqrt{2}-2$ (from eq(1))

~~$=4\sqrt{2}+2-\sqrt{4}$~~, $=\cancel{4\sqrt{2}}+\cancel{2}-\cancel{4\sqrt{2}}-\cancel{2}$

$\boxed{f(\sqrt{2})=0}$

and, $f(-\sqrt{2})=2(-\sqrt{2})^3+(-\sqrt{2})^2-4\sqrt{2}-2$ (from eq(1))

$=-\cancel{4\sqrt{2}}+\cancel{2}+\cancel{4\sqrt{2}}-\cancel{2}$

$\boxed{f(-\sqrt{2})=0}$

$\therefore \boxed{f(\sqrt{2})=f(-\sqrt{2})=0}$

Thus conditions of three Rolle's theorem is satisfied.

Hence by Rolle's theorem, there exists a point c in $(-\sqrt{2}, \sqrt{2})$ such that $f'(c) = 0$.

verification :— Consider $f'(c) = 0$.

i.e $6c^2 + 2c - 4 = 0 \Rightarrow 3c^2 + c - 2 = 0$.

$\cancel{= 3c^2 - c + 2c - 2 = 0}$

$\cancel{= c(3c-1) + 2(c-1) = 0}$.

$= (3c-2)(c+1) = 0$.

$\Rightarrow 3c = 2 \quad | \quad c + 1 = 0$

$\boxed{c = 2/3} \quad | \quad \boxed{c = -1}$

Both $c = \frac{2}{3}$ & -1 are in b/w $(-\sqrt{2}, \sqrt{2})$.

Hence Rolle's theorem is verified (Ans)

(iii). $f(x) = (x-a)^m (x-b)^n$ where m, n are +ve integers in $[a,b]$.

Sol:- $f(x)$ is a polynomial, since every polynomial is continuous & differentiate for all x. Hence the Ist two conditions of Rolle's theorem are satisfied.

Now $f(a) = 0$ & $f(b) = 0$

$\therefore f(a) = f(b)$.

Thus, three conditions of Rolle's theorem are satisfied.

$\therefore$ There exists $c \in (a,b)$ s.T $f'(c) = 0$.

$(c-a)^{m-1} (c-b)^{n-1} [(m+n)c - (mb+na)] = 0$.

$\Rightarrow (m+n)c - (mb+na) = 0 \Rightarrow (m+n)c = mb+na$.

$\Rightarrow c = \frac{mb+na}{m+n} \in (a,b)$

since the point $c \in (a,b)$ divides a,b internally in the ratio $m:n$

$\therefore$ Rolle's theorem is verified

=(Ans).

(iv). $f(x) = \dfrac{\sin x}{e^x}$ (or) $e^{-x}\sin x$ in $[0,\pi]$

Sol:

(i). since $\sin x$ & e^x are both continuous funⁿ in $[0,\pi]$, $\dfrac{\sin x}{e^x}$ is also continuous in $[0,\pi]$.

(ii). Since $\sin x$ & e^x be derivable in $(0,\pi)$, $\dfrac{\sin x}{e^x}$ is also derivable in $(0,\pi)$.

(iii). $f(0) = \dfrac{\sin(0)}{e^0} = 0$ & $f(\pi) = \dfrac{\sin\pi}{e^\pi} = 0$.

$$\therefore f(0) = f(\pi).$$

Thus all three conditions of Rolle's theorem are satisfied.

$\therefore$ There exists $c \in (0,\pi)$ such that $f'(c) = 0$.

Now $f'(x) = \dfrac{e^x \cos x - \sin x \cdot e^x}{(e^x)^2} = \dfrac{\cancel{e^x}(\cos x - \sin x)}{(e^x)^{\cancel{2}}}$

$$= \frac{\cos x - \sin x}{e^x}$$

$\therefore f'(c) = 0 \Rightarrow \dfrac{\cos c - \sin c}{e^x} = 0.$

$\Rightarrow \cos c - \sin c = 0 \Rightarrow \cos c = \sin c \Rightarrow \tan c = 1.$

$\therefore \boxed{c = \pi/4 \in (0,\pi)}$

Hence Rolle's theorem is verified //(Ans).

(v)

sol:- Let $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$, $a>0, b>0$.

$= \log(x^2+ab) - \log x - \log(a+b)$.

(i). since $f(x)$ is a composite fun^n of continuous fun's in $[a,b]$, it is continuous in $[a,b]$.

(ii). $f'(x) = \frac{1}{x^2+ab} 2x - \frac{1}{x}$

$f'(x) = \frac{x^2-ab}{x(x^2+ab)}$

$\therefore f'(x)$ exists for all $x \in (a,b)$.

(iii). $f(a) = \log\left[\frac{a^2+ab}{a^2+ab}\right]$

$= \log(1)$

$\boxed{f(a) = 0.}$

lly, $f(b) = \log\left(\frac{a^2+ab}{a^2+ab}\right]$

$= \log(1)$

$\boxed{f(b) = 0.}$

$\therefore f(a) = f(b)$.

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem

$\therefore$ There exists $c \in (a,b)$ s.t $f'(c) = 0$.

$\Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 = ab \Rightarrow c = \pm\sqrt{ab}$

$\therefore c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem is verified. // (Ans) (vi). $f(x) = x^2 - 2x - 3$ in the interval $(1, -3)$.

Sol:- The specified interval $(1,-3)$ in the given problem is not Correct. It may be $(-1, 3)$.

(i). Let $f(x) = x^2 - 2x - 3$ is a polynomial in x. so it is contineous & derivable in $(-1, 3)$.

Now $f(-1) = (-1)^2 - 2(-1) - 3$
$= 1 + 2 - 3$
$= 3 - 3$
$\boxed{f(-1) = 0}$

$f(3) = 3^2 - 2(3) - 3$
$= 9 - 6 - 3$
$= 0$
$\boxed{f(3) = 0}$

$$\boxed{f(-1) = f(3)}$$

$\therefore f(x)$ satisfies all the three conditions of Rolle's theorem.

$\rightarrow$ There must exist at least one $c \in (-1, 3)$ s.T $f'(c) = 0$.

$\Rightarrow 2c - 2 = 0 \Rightarrow \cancel{2}c = \cancel{2}^{1} \Rightarrow \boxed{c = 1}$

$\boxed{c = 1 \in (-1, 3)}$

Hence, Rolle's Theorem is verified (Ans).

(vii). $f(x) = e^x \sin x$ in $[0, \pi]$

(i). $f(x)$ is continuous for every value of x since both fun's e^x & $\sin x$ are continuous for every value of x.

$\therefore f(x)$ is continuous in the closed interval $[0, \pi]$.

(ii). $f'(x) = e^x \cos x + \sin x . e^x = e^x(\cos x + \sin x)$ which exists for any x in $(0, \pi)$.

$\therefore f(x)$ is derivable in the open interval $(0, \pi)$.

(iii). Now $f(0) = e^0 \sin 0 = 1(0) = 0$ & $f(\pi) = e^{\pi}\sin\pi = e^{\pi}(0) = 0$.

$\therefore f(0) = f(\pi)$

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[0,\pi]$.

$\therefore$ There exists at least one value of 'c' of x in $(0,\pi)$ such that $f'(c) = 0$.

i.e $e^c(\cos c + \sin c) = 0 \Rightarrow \cos c + \sin c = 0$ [$\because e^c \neq 0$ for any c]

$\Rightarrow \sin c = -\cos c$ (or) $\tan c = -1$ (or) $\tan c = -\tan\frac{\pi}{4}$

$= \tan\left(\pi - \frac{\pi}{4}\right)$

$= \tan\frac{3\pi}{4}$

$\therefore \boxed{c = \frac{3\pi}{4} \in (0,\pi)}$ (Ans).

(viii). $g(x) = 8x^3 - 6x^2 - 2x + 1$ in b/w 0 & 1

The above equ is a polynomial, it is continuous on $[0,1]$ & differentiable on $(0,1)$.

Now $g(0) = 1$ & $g(1) = 8(1)^3 - 6(1)^2 - 2(1) + 1$

$= 8 - 6 - 2 + 1$

$= 1$

$\therefore \boxed{g(0) = g(1)}$

Hence, all the three conditions of RT are verified.

$\therefore$ there exists a no. $c \in (0,1)$ s.T $g'(c) = 0$.

$g'(x) = 24x^2 - 12x - 2$

$\therefore g'(c) = 0 \Rightarrow 24c^2 - 12c - 2 = 0$ (or) $12c^2 - 6c - 1 = 0$.

$\Rightarrow c = \frac{3 \pm \sqrt{21}}{12}$ i.e $c = 0.63$ (or) -0.132

only the value $c = 0.63$ lies in $(0,1)$.

Thus, there exists at least one root b/w 0 & 1.

= (Ans).

Q). verify whether rolle's theorem can be applied to the following fun's in the interval cited:-

(i). $f(x) = |x|$ in $[-1,1]$.

(ii). $f(x) = \tan x$ in $[0,\pi]$.

(iii). $f(x) = \frac{1}{x^2}$ in $[-1,1]$

(iv). $f(x) = x^3$ in $[1,3]$

Sol's

(i). $f(x) = |x|$ in $[-1,1]$.

(a). $f(x)$ is continuous for every value of x.

$\therefore f(x)$ is continuous in the closed interval $[-1,1]$. (lies)

(b). $f(x)$ is not derivable at $x=0$. i.e $f(0) = |0| = 0$.

$\therefore$ Rolle's theorem is not applicable (Ans)

(ii). $f(x) = \tan x$ in $[0,\pi]$

$f(x)$ is discontinuous at $x = \pi/2$ as it is not defined.

$\therefore$ The condition (1) of Rolle's Theorem is not satisfied

Hence ~~with~~ we cannot apply rolle's Theorem. = (Ans)

(iii) $f(x) = \frac{1}{x^2}$ in $[-1,1]$.

~~condition~~ $f(x)$ is discontinuous at $x=0$

Hence Rolle's theorem cannot be applied.

(iv) $f(x) = x^3$ in $[1,3]$

conditions (1) and (2) are satisfied by $f(x)$

But $f(1) \neq f(3)$

Hence, Rolle's theorem is not applicable to $f(x) = x^3$ in $[1,3]$ = (Ans)

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(5) Verify Lagrange's mean value theorem (or) find the value of c by using Lagranges mean value theorem. (LMVT).

(i) $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$.

(ii) $f(x) = \log_e x$ in $[1,e]$

(iii) $f(x) = \cos x$ in $[0, \frac{\pi}{2}]$

(iv) $f(x) = x(x-2)(x-3)$ in $(0,4)$.

Solutions

(i) given, $f(x) = x^3 - x^2 - 5x + 3$ is a polynomial in x:

$\therefore$ It is continuous and derivable for every value of x.

Now, $f(x)$ is continuous in closed interval $[0,4]$.

$f(x)$ is derivable in open interval $(0,4)$.

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b-a}}$$

$$f'(c) = \frac{f(4) - f(0)}{4-0}.$$

$$3c^2 - 2c - 5 = \frac{f(4) - f(0)}{4} \quad \text{--- (1).}$$

Now, $f(4) = 4^3 - 4^2 - 5 \cdot 4 + 3$

$= 64 - 16 - 20 + 3$

$= 67 - 36$

$= 31$

and $f(0) = 3$.

$$\therefore \frac{[f(4) - f(0)]}{4}$$

$$= \frac{31 - 3}{4} \Rightarrow 7 \quad \text{—— (2)}$$

from (1) & (2), we have.

$\Rightarrow 3c^2 - 2c - 5 = 7$

$\Rightarrow 3c^2 - 2c - 12 = 0$ (is quadratic eq^n in 'c').

$$\boxed{c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\therefore c = \frac{2 \pm \sqrt{4 + 144}}{6}$$

$$= \frac{2 \pm \sqrt{148}}{6}$$

$$= \frac{1 \pm \sqrt{37}}{3}$$

$\frac{1 \pm \sqrt{37}}{3}$ lies in open interval $(0, 4)$.

$\therefore$ Lagrange's mean value Theorem is verified.

(AN)

(ii) $f(x) = \log_e^x$ in $[1, e]$.

Sol:-

given, $f(x) = \log_e^x$

lies in $[1, e]$

The function is continuous in closed interval $[1, e]$

The function is derivable in open interval $(1, e)$

$\therefore$ The Lagrange's mean value Theorem is applicable.

By this theorem, Exists a point 'c' in open interval

such that, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

$$= \frac{f(e)-f(1)}{e-1}$$

$$= \frac{1-0}{e-1} \Rightarrow \boxed{\frac{1}{e-1}}$$

But, $f'(c) = \frac{1}{c} \Rightarrow \frac{1}{c}$

$$\Rightarrow \frac{1}{e-1} \qquad (\because c = e-1)$$

$(e,-1)$ lies in the interval $(1,e)$.

$\therefore$ Lagrange's mean value Theorem is verified.
= (Ans).

(iii) given, $f(x) = \cos x$ lies in $[0, \frac{\pi}{2}]$.

$f(x) = \cos x$ is continuous on $\left[0, \frac{\pi}{2}\right]$

$f(x) = \cos x$ is derivable on $\left(0, \frac{\pi}{2}\right)$.

$\therefore$ Lagrange's mean value theorem is applicable.

$\therefore f'(x) = -\sin x$.

By Lagrange's mean value theorem.

$$\cancel{f'(c)} = \frac{f(b)-f(a)}{b-a} = f'(c).$$

$$= \frac{f(\pi/2) - f(0)}{\frac{\pi}{2} - 0} \quad -\sin c$$

$$\Rightarrow \frac{\cos \pi/2 - \cos 0}{\pi/2} = -\sin c$$

$\Rightarrow \frac{1}{\pi/2} - \sin c$

$\Rightarrow \sin c = \frac{2}{\pi}$

$\Rightarrow c = \sin^{-1}\frac{2}{\pi} \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ Lagrange's mean value theorem is verified //(Ans).

given,

(iv) $f(x) = x(x-2)(x-3)$ in $(0,4)$.

$f(c)$. x_1 x_2

$\therefore f(x) = x(x-2)(x-3)$ is a polynomial function in x.

$\therefore$ It is continuous and derivable for every value of x.

Since, $f(x)$ is continuous in $[0,4]$

and $f(x)$ is derivable in $(0,4)$.

By Lagrange's mean value theorem; There exists a point 'c' in $(0,4)$, such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

$$= \frac{f(4)-f(0)}{4-0} \quad \text{—— (1)}$$

Now, ~~$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$~~

$f'(x) = f(x_1) = (x_2 - x_1) f'(c)$.

$f'(x) = (x-2)(x-3) + x(x-2) + x(x-3)$

$= x^2 - 5x + 6 + x^2 - 2x + x^2 - 3x$.

$= 3x^2 - 10x + 6$.

Also, $f(0) = 0$

and $f(4) = 4(4-2)(4-3)$

$= (16-8)(4-3)$

$= 8(1) = 8$.

from ①, we have.

$$\Rightarrow 3c^2 - 10c + 6 = \frac{8^2 - 0}{4}$$

$$= 2.$$

$$\Rightarrow 3c^2 - 10c + 6 - 2$$

$$\Rightarrow 3c^2 - 10c + 4 = 0.$$

$$\therefore \boxed{c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\therefore c = \frac{10 \pm \sqrt{100 - 4(3)(4)}}{6}$$

$$= \frac{10 \pm \sqrt{100 - 12(4)}}{6}.$$

$$= \frac{10 \pm \sqrt{100 - 48}}{6}$$

$$= \frac{10 \pm \sqrt{52}}{6}.$$

$$= \frac{\cancel{10}^{5} \pm \cancel{2}\sqrt{13}}{\cancel{6}_{3}}$$

$$\boxed{\therefore c = \frac{5 \pm \sqrt{13}}{3}}$$

i.e., $c = \frac{5 + \sqrt{13}}{3}$ and $c = \frac{5 - \sqrt{13}}{3}$

$\Rightarrow 2.87$ $\Rightarrow 0.46.$

$\therefore$ Both values of 'c' lies in $[0, 4]$.

$\therefore$ Lagrange's mean value theorem is verified

//(Ans)

Q) Verify cauchy's mean value theorem (or) find the value of 'c' by using Cauchy's mean value theorem (CMVT).

(i) $f(x) = x^2, g(x) = x^3$ in $[1,2]$.

(ii) $f(x) = \sin x, g(x) = \cos x$ on $[0, \frac{\pi}{2}]$

(iii) $f(x)$ and $f'(x)$ in $(1, e)$ given $f(x) = \log x$

(iv) $f(x) = e^x$; $g(x) = e^{-x}$ in $[3, 7)$ and find the value of c.

(v) $f(x) = \log x$ and $g(x) = x^2$ in $[a,b]$ with $b > a > 1$, prove that

$$\frac{\log b - \log a}{b - a} = \frac{a+b}{2c^2}$$

Solutions

(i) given, $f(x) = x^2$ and

$g(x) = x^3$.

They are in form of $x^n, n \in Z$

$\therefore f, g$ are continuous on $[1,2]$.

f, g are differentiable on $(1,2)$.

Now, $g'(x) = 3x^2 \neq 0 \;\forall\; x \in (1,2)$

$\therefore f, g$ satisfies the conditions of Cauchy's mean value theorem

There, exists $c \in (a, b)$.

Such that, $\boxed{\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}}$

$$\Rightarrow \frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{(2)^2-(1)^2}{(2)^3-(1)^3} = \frac{4-1}{8-1} = \frac{2c}{3c^2}$$

$$\Rightarrow \quad \Rightarrow \frac{4-1}{8-1} = \frac{2c}{3c^2}$$

$$\frac{3}{7} = \frac{2}{3c} \quad \Rightarrow \frac{3}{7} = \frac{2}{3}$$

$$c \Rightarrow \frac{14}{9}$$

$$\Rightarrow \in (1,2)$$

∴ Cauchy's mean value theorem is verified. //(Ans).

(ii) given, $f(x) = \sin x$, $g(x) = \cos x$ on $\left[0, \frac{\pi}{2}\right]$ lies

$f(x) = \sin x$, $g(x) = \cos x$ are continuous on $[0, \pi/2]$

$f(x) = \sin x$, $g(x) = \cos x$ are differentiable on $\left(0, \frac{\pi}{2}\right)$.

Now, $g'(x) = -\sin x \neq 0$ on $\left(0, \frac{\pi}{2}\right)$.

∴ function $f(x)$ and $g(x)$ satisfies conditions of Cauchy's mean value theorem

∴ Cauchy's mean value theorem

$\exists\ c \in \left(0, \frac{\pi}{2}\right)$

$$\Rightarrow \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{f\left(\frac{\pi}{2}\right)-f(0)}{g\left(\frac{\pi}{2}\right)-g(0)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{1-0}{0-1} = -\cot c$$

$\Rightarrow c = \pi/4 \in (0, \frac{\pi}{2})$.

$\therefore$ Cauchy's mean value theorem is verified. = (Ans)

(iii) given, f(x) and f'(x) in (1, e) and also given $f(x) = \log x$.

Now, $f(x) = \log x, \; x \in (1, e)$

and $g(x) = f'(x) = \frac{1}{x}$.

f(x) and g(x) are continuous in (1, e)

f(x) and g(x) are differentiable in (1, e)

$g'(x) = -\frac{1}{x^2} \neq 0, \; \forall x \in (1, e)$.

$\therefore$ f(x) and g(x) satisfies the cond^n's of cauchy's mean value Theorem

$\therefore c \in (1, e)$,

Such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(e) - f(1)}{g(e) - g(1)}$$

$$\Rightarrow \frac{\frac{1}{c}}{-\frac{1}{c^2}} = \frac{\log(e) - \log(1)}{\frac{1}{e} - \frac{1}{1}}$$

$$\Rightarrow -c = \frac{1}{\frac{1}{e} - 1} = \frac{e}{1 - e}$$

$$\Rightarrow c = \frac{e}{e - 1} \in (1, e)$$

$\therefore$ Cauchy's mean value Theorem is verified. (Ans)

(iv) given, $f(x) = e^x$, $g(x) = e^{-x}$ in $[3, 7]$ ~~and find the value of c.~~ lies.

$f(x)$ and $g(x)$ are continuous and derivable for all values of x.

f and g are continuous in $[3, 7]$

f and g are differentiable in $(3, 7)$

Now, $g'(x) = e^{-x} \neq 0$ for any $x \in (3, 7)$

f and g satisfies the conditions of Cauchy's mean value Theorem.

There exists a point $c \in (3, 7)$,

Such that. $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$

$\Rightarrow \dfrac{f(7) - f(3)}{g(7) - g(3)} = \dfrac{f'(c)}{g'(c)}$

$\Rightarrow \dfrac{e^7 - e^3}{e^{-7} - e^{-3}} = \dfrac{e^c}{-e^{-c}}$

$\Rightarrow \dfrac{e^7 - e^3}{\frac{1}{e^7} - \frac{1}{e^3}} = -e^{2c}$

$\Rightarrow -e^{7+3} = -e^{2c}$

$\Rightarrow 2c = 10$ (or) $c = 5 \in (3, 7)$

$\therefore$ Cauchy's mean value theorem is verified. //(Ans)

(v) given, $f(x) = \log x$ and $g(x) = x^2$ in $[a, b]$.

$f(a) = \log x$

$\Rightarrow f(a) = \log a$, $f(b) = \log b$.

and $g(x) = x^2$

$\Rightarrow g(a) = a^2$, $g(b)$

$\Rightarrow b^2$

Also, $f'(x) = \frac{1}{x}$ and

$g'(x) = 2x$.

By cauchy's mean value theorem,

$$\Rightarrow \boxed{\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}}$$

$$= \frac{\log b - \log a}{b^2 - a^2} = \frac{1/c}{2c}$$

$$\Rightarrow \frac{\log b - \log a}{(b-a)(b+a)} = \frac{1}{2c^2}.$$

$$\therefore \frac{\log b - \log a}{b - a} = \frac{a+b}{2c^2}.$$

$\therefore$ cauchy's mean value Theorem is verified. (Ans)

Q. obtain the Taylor's series expansion of:-

(i). $\sin x$ in power of $x - \pi/4$.

(ii). $\sin 2x$ about $x = \frac{\pi}{4}$ (or) $\sin 2x$ in power of $x - \pi/4$.

(iii). $f(x) = \log \sin x$ about $x = 3$.

(iv). $f(x) = e^x$ in power (or) about $x = -1$ (or) $x+1$.

Sol

(i). Given $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$\Rightarrow f''(x) = -\sin x$

$\Rightarrow f'''(x) = -\cos x$

$\Rightarrow f^4(x) = \sin x.$

Take $a = \pi/4$; $x - \pi/4 \Rightarrow x = \frac{\pi}{4}$

Now, $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$; $f''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$; $f^4(\pi/4) = f^5(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f'(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$; $f'''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$; $f^6(\frac{\pi}{4}) = f^7(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

The Taylor's series Expansion is given by:-

$$\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots}$$

$$\Rightarrow \sin x = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right) \cdot \frac{1}{\sqrt{2}} - \frac{\left(x - \frac{\pi}{4}\right)^2}{\sqrt{2!}} \frac{1}{2} - \frac{\left(x - \frac{\pi}{4}\right)^3}{3!} \frac{1}{\sqrt{2}} + \frac{\left(x - \frac{\pi}{4}\right)^4}{4!} \frac{1}{\sqrt{2}} + \cdots$$

Now Take $\frac{1}{\sqrt{2}}$ common.

$$= \frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{(x-\pi/4)^2}{2!}-\frac{(x-\pi/4)^3}{3!}+\frac{(x-\pi/4)^4}{4!}+\frac{(x-\pi/4)^5}{5!}\cdots\right]$$

= (Ans).

(ii). Given $f(x) = \sin 2x$; $x = \frac{\pi}{4}$.

Now take $t = x - \frac{\pi}{4}$

$$\frac{\pi}{4} + t = x$$

$$\therefore \sin 2x = \sin 2\left(\frac{\pi}{4}+t\right)$$

$$= \sin\left(\frac{\pi}{2}+2t\right)$$

$$= \cos 2t$$

$$= 1 - \frac{2^2 t^2}{2!} + \frac{2^4 t^4}{4!} - \cdots \text{ for all values of } 't'.$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \cdots \text{ for all values of } 'x'.$$

Now put $x - \pi/4 = t$

$$= 1 - \frac{2^2}{2!}(t^2)$$

(ii). Given $f(x) = \sin 2x$

Take $x - \pi/4 = t \Rightarrow \boxed{x = \pi/4 + t}$

$$\therefore \sin 2x = \sin 2\left(\frac{\pi}{4} + t\right)$$

$$= \sin\left(\frac{\pi}{4} \times 2 + t \times 2\right)$$

$$= \sin\left(\frac{\pi}{2} + 2t\right)$$

$$\boxed{\sin 2x = \cos 2t}$$

Now Take $1 - \frac{2^2 t^2}{2!} + \frac{2^4 t^4}{4!} - \cdots\cdots$ (for all values of 't').

$$\Rightarrow 1 - \frac{2^2}{2!}(t)^2 + \frac{2^4}{4!}(t)^4 - \cdots$$

$$\Rightarrow 1 - \frac{2^2}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{2^4}{4!}\left(x - \frac{\pi}{4}\right)^4 - \cdots$$

(for all values of 'x').

= (Ans).

(iii). $f(x) = \log \sin x$; $x = 3 \Rightarrow a = 3$.

$$f'(x) = \frac{1}{\sin x}(\cos x) = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f'''(x) = -2\operatorname{cosec}^2 x \cot x$$

By, Taylor's series expansion we have.

$$\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots}$$

$\therefore \log \sin x = \log \sin 3 + (x-3)\cot 3 + \frac{(x-3)^2}{2!}(-\operatorname{cosec}^2 3) + \frac{(x-3)^3}{3!}$

$+ \times - = -$ $\quad 2\operatorname{cosec}^2 3 \cot 3$.

$= \log \sin 3 + (x-3)\cot 3 - \frac{(x-3)^2}{2}\operatorname{cosec}^2 3 + \frac{2}{3}(x-3)^3 \operatorname{cosec}^2 3 \cot 3$.

= (Ans).

(iv). let $f(x) = e^x$; put $x+1=t$

$\therefore x = t-1$

Now $f(x) = e^x = e^{t-1} = \frac{1}{e}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]$ for all values of 't'.

$= \frac{1}{e}\left[1 + (x+1) + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \cdots\right]$

(for all values of x)

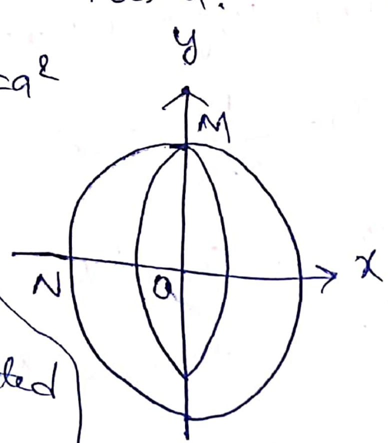
= (Ans).

---

⑥. Find the vol. of a sphere of radius a.

Sol:- The equ of the circle is $x^2+y^2=a^2$

$\Rightarrow y^2 = a^2 - x^2$



$\therefore$ The required volume of the sphere = 2(volume of the solid generated by the revolution about x-axis).

i.e $= 2\int_0^a \pi y^2\,dx$

Now Take π outside ($\because \pi$ - constant)

$2\pi \int_0^a y^2\,dx$

Now put $(y^2 = a^2 - x^2)$

$$= 2\pi \int_0^a (a^2 - x^2)\, dx$$

Diff WRT 'x'

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

[∵UL−LL]

$$= 2\pi \times \left[a^2 \times a - \frac{a^3}{3} \right] - [0]$$

$$= 2\pi \times \left[a^3 - \frac{a^3}{3} \right]$$

$$= 2\pi \times \frac{3a^3 - a^3}{3}$$

$$= 2\pi \times \frac{2a^3}{3}$$

$$= \frac{4\pi a^3}{3}$$

∴ $\boxed{\text{The vol. of sphere} = \frac{4\pi a^3}{3}}$ —(Ans)

---

(7). Find the volume of the solid that result when the region enclosed by the curve $y = x^3$, $x = 0$, $y = 1$ is revolved about the y-axis.

Sol:- Given curve is $y = x^3$

∴ $\boxed{\text{Required vol} = \int_0^1 \pi x^2\, dy}$ $\boxed{\text{For (y-axis)}}$

$$= \pi \int_0^1 y^{2/3}\, dy$$

$$= \pi\left(\frac{3}{5}\right)\left(y^{5/3}\right)_0^1$$

$$\boxed{\text{Vol. of the solid} = \frac{3\pi}{5} \text{ cu. units}}$$

=(Ans).

⑧. Find the volume of the solid when ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(0<b<a)$ rotates about minor axis.

Sol:- equ. of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\Rightarrow \frac{x^2}{a^2} = 1-\frac{y^2}{b^2}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{b^2-y^2}{b^2}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{1}{b^2} \times b^2-y^2$$

$$\Rightarrow x^2 = \frac{a^2}{b^2}(b^2-y^2)$$

$$\therefore \boxed{\text{Required vol.} = \int_{-b}^{b} \pi x^2\,dy}$$

$$= \pi \int_{-b}^{b} x^2\,dy$$

$$= \pi \int_{-b}^{b} \frac{a^2}{b^2}(b^2-y^2)\,dy$$

$$= \frac{\pi a^2}{b^2} \int_{-b}^{b} (b^2 - y^2)\, dy$$

$$= \frac{2\pi a^2}{b^2} \int_{0}^{b} (b^2 - y^2)\, dy$$

Diff WRT 'y'

$$= \frac{2\pi a^2}{b^2} \left(b^2 y - \frac{y^3}{3} \right)_0^b$$

UL-LL

$$= \frac{2\pi a^2}{b^2} \left(b^2 \times b - \frac{b^3}{3} \right)$$

$$= \frac{2\pi a^2}{b^2} \left(b^3 - \frac{b^3}{3} \right)$$

$$= \frac{2\pi a^2}{b^2} \left(\frac{3b^3 - b^3}{3} \right)$$

$$= \frac{2\pi a^2}{\cancel{b^2}} \times \frac{2b^{\cancel{3}}}{3}$$

$$\boxed{\text{Vol. of solid} = \frac{4\pi a^2 b}{3} \text{ cubic units.}}$$ (Ans)

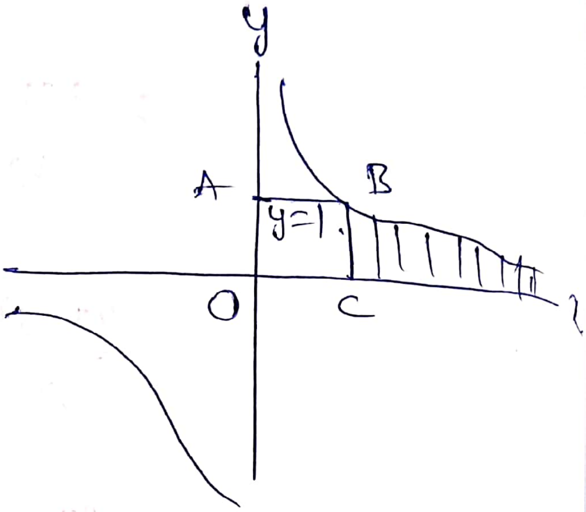
9). Find the vol. of the solid that results when the region enclosed by the curves $xy=1$, x-axis & $x=1$ rotated about x-axis.

Sol:- Given curve $xy = 1 \Rightarrow y = \frac{1}{x}$

required volume $= \int_{x=1}^{\infty} \pi y^2 dx$

$$= \int_{x=1}^{\infty} \frac{\pi}{x^2}\, dx$$

$$= \pi \int_{1}^{\infty} \frac{1}{x^2}\, dx$$



$$= \pi \left[\frac{-1}{x} \right]_1^{\infty}$$

$$= \pi\, [0+1]$$

$$= \pi .$$

$\therefore$ The required volume $= \pi + \pi = 2\pi$ cubic units.

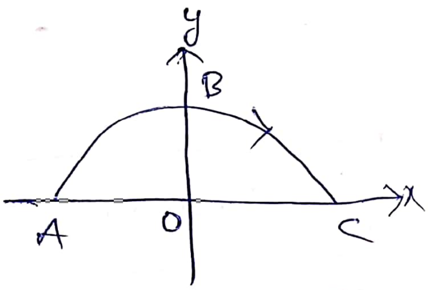
(Ans)

---

(10). Find the surface area of a sphere generated by the circle $x^2+y^2=16$ about its diameter.

Sol:- Given circle $x^2+y^2=16$

$$y^2 = 16 - x^2 \rightarrow (1)$$



Diff (1) WRT 'x'.

$$\cancel{2}y \frac{dy}{dx} = -\cancel{2}x$$

$$\Rightarrow y \frac{dy}{dx} = -x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$\therefore$ Required surface area $= 2\pi \int_{-4}^{4} y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= 2\pi \int_{-4}^{4} y \sqrt{1+\frac{x^2}{y^2}}\, dx$$

$$= 2\pi \int_{-4}^{4} \sqrt{x^2+y^2}\, dx$$

$$= 2\pi \int_{-4}^{4} \sqrt{16}\, dx$$

$$= 8\pi \int_{-4}^{4} dx$$

$$= 8\pi (x)_{-4}^{4}$$

$$= 8\pi (UL-LL)$$

$$= 8\pi (4+4)$$

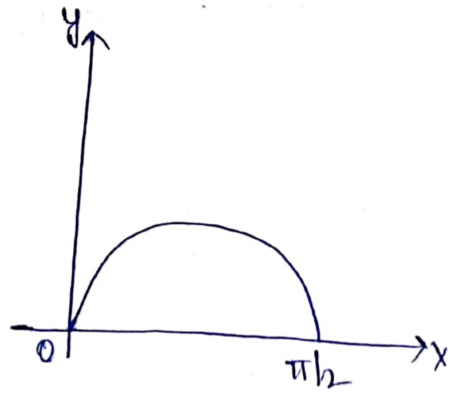
$$= 8\pi (8)$$

$$= 64\pi$$

Ans

(11). Find the area of the surface of revolution generated by revolving one arch of the curve $y=\sin x$ about x-axis.

Sol:



given, curve is $y = \sin x$

'x' varies from 0 to $\pi/2$

$$\boxed{S = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.}$$

$$\therefore \frac{dy}{dx} = \cos x$$

required surface area

$$\Rightarrow 2\pi \int_0^{\pi/2} y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\Rightarrow 2\pi \int_0^{\pi/2} \sin x \cdot \sqrt{1+\cos^2 x}\, dx$$

$$\Rightarrow 2\pi \int_0^{1} \sqrt{1+t^2}\, dt \quad [\text{put } \cos x = t]$$

$$\Rightarrow 2\pi \left[\frac{1}{2}\sqrt{1+t^2} + \frac{1}{2}\sinh^{-1} t \right]_0^1$$

$$\Rightarrow 2\pi \left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\sinh^{-1}(1) - 0 - 0 \right]$$

$$\Rightarrow \pi \left[\sqrt{2} + \sinh^{-1}(1) \right]$$

$$\Rightarrow \pi \left[\sqrt{2} + \log(1+\sqrt{2}) \right].$$

//(ans).

---

(12) The arc ~~gence~~ of the curve $x = y^3$ between $y=0$ and $y=2$ is revolved about y-axis. Find the area of surface so generated

Sol:-

given curve is $x = y^3$

$$\therefore \frac{dx}{dy} = 3y^2$$

Hence required surface area

$$\Rightarrow S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\cancel{\Rightarrow 2\pi \int_0^{\pi/2} \sin x \cdot \sqrt{1+\cos^2 x}\,dx}$$

$$\Rightarrow 2\pi \int_0^2 x\sqrt{1+(3y^2)^2}\;dy$$

$$\Rightarrow 2\pi \int_0^2 y^3\sqrt{1+9y^4}\;dy$$

$$\Rightarrow 2\pi \int_1^{145} \sqrt{t}\;\frac{dt}{36} \quad (\text{put } 1+9y^4 = t)$$

$$\Rightarrow \frac{\pi}{18}\left(\frac{2}{3}t^{3/2}\right)_1^{145}$$

$$\Rightarrow \frac{\pi}{27}\left[(145)^{3/2} - 1\right].$$

// (Ans).

Formulas.

Axis of revolution	Volume of area	Surface area
X-axis	$V = \pi\int_a^b y^2 dx$	$S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$
Y-axis	$V = \pi\int_c^d x^2 dy$	$S = 2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$

Properties of Beta Function.

1) To prove that (or) to show that :-

(i) Symmetry of Beta function i.e., $B(m,n) = B(n,m)$.

(ii) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

(iii) $B(m,n) = B(m+1,n) + B(m,n+1)$

(or)

$B(p,q) = B(p+1,q) + B(p,q+1)$.

(iv) If m and n are positive integers, then $B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$

(v) $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$ (or) $\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$

(or)

$B(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy$

Solutions

Proofs

* Symmetry of Beta function i.e $B(m,n) = B(n,m)$.

(i) By definition, we have

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx}$$

Put $1-x = y$

$dx = -dy$

$$\therefore B(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\, dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$$

$$\therefore \boxed{B(m,n) = B(n,m)}$$ = (Hence proved)

(ii). $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$ = CH.

(ii). By definition, we have

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx}$$

Put $x = \sin^2\theta$

$\Rightarrow dx = \sin 2\theta\, d\theta$

$$\therefore B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \sin 2\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta\,(2\sin\theta\cos\theta)\, d\theta$$

$$\boxed{B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.}$$

(Hence proved).

(iii). $B(m,n) = B(m+1,n) + B(m,n+1)$

(or)

$B(p,q) = B(p+1,q) + B(p,q+1)$.

Proof:- By, definition we have.

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.}$$

$$B(m+1,n)+B(m,n+1) = \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 \left[x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\right]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\left[x+(1-x)\right]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$\boxed{B(m+1,n)+B(m,n+1) = B(m,n)}$$

Hence proved.

(iv). If m & n are +ve integers, then $B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$

P:- By definition, we have

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx}$$

$$= \left[x^{m-1}\frac{(1-x)^n}{n(-1)}\right]_0^1 - \int_0^1 \frac{(1-x)^n}{n(-1)}(m-1)x^{m-2}dx$$

$$= \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n dx$$

$$B(m,n) = \frac{m-1}{n}B(m-1,n+1) \longrightarrow (1)$$

Now we have to find $B(m-1,n+1)$. To obtain this put $m=m-1$ & $n=n+1$ in eq (1). Then.

$$B(m-1,n+1) = \frac{m-2}{n+1}B(m-2,n+2)$$

Putting this value of $B(m-1,n+1)$ in eq (1)

$$B(m,n) = \frac{m-1}{n}\cdot\frac{m-2}{n+1}B(m-2,n+2) \longrightarrow (2)$$

Changing $m \to m-2$; $n \to n+2$. ~~from eq (1)~~

$$B(m-2, n+2) = \frac{m-3}{n+2} B(m-3, n+3).$$

from eq (2)

$$B(m,n) = \frac{m-1}{n} \cdot \frac{m-2}{n+1} \cdot \frac{m-3}{n+2} B(m-3, n+3) \longrightarrow (3).$$

proceeding like this, we get

$$B(m,n) = \frac{(m-1)(m-2)(m-3)\cdots[m-(m-1)]}{n(n+1)(n+2)\cdots[n+(m-2)]} B[m-(m-1), n+(m-1)]$$

$$= \frac{(m-1)(m-2)(m-3)\cdots 1}{n(n+1)(n+2)\cdots(n+m-2)} B(1, n+m-1) \longrightarrow (4)$$

But $B(1, n+m-1) = \int_0^1 x^0 (1-x)^{n+m-2} dx$

$$= \int_0^1 (1-x)^{n+m-2} dx$$

$$= \left[\frac{(1-x)^{n+m-1}}{(n+m-1)(-1)} \right]_0^1$$

$$= \frac{-1}{n+m-1}(0-1)$$

$$= \frac{1}{n+m-1}$$

(4) $\Rightarrow$ $B(m,n) = \dfrac{(m-1)(m-2)(m-3)\cdots 1}{n(n+1)(n+2)\cdots(n+m-2)} \cdot \dfrac{1}{n+m-1}.$

$$= \frac{(m-1)!}{(n+m-1)(n+m-2)\cdots(n+2)(n+1)n}.$$

Multiplying the N&D by $(n-1)!$ we have

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(n+m-1)(n+m-2)\ldots(n+2)(n+1)\,n(n-1)!}$$

$$\boxed{B(m,n) = \frac{(m-1)!\,(n-1)!}{(n+m-1)!}}$$

// (Hence proved)

(V). To show

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \ (\text{or}) \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy.$$

(or)

$$B(P,Q) = \int_0^\infty \frac{y^{Q-1}}{(1+y)^{P+Q}}\,dy.$$

Proof:- From Defination, we have

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx} \longrightarrow (1)$$

Put $x = \frac{1}{1+y}$

$$dx = -\frac{dy}{(1+y)^2}$$

From (1)

$$B(m,n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1} \cdot \frac{-dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+1}\,(1+y)^{n-1}}\,dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\therefore B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \longrightarrow (2).$$

Again, since Beta funⁿ is symmetrical in m & n.

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \longrightarrow (3).$$

Hence $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \rightarrow (4).$

(or) $B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dx = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$

(Hence proved)

Properties of Gamma Function

~~312~~ 313

①. To prove / To show that:-

(i). $\Gamma(1) = 1$

(ii). $\Gamma(n) = (n-1)\Gamma(n-1)$ where $n > 1$.

(iii). If n is a non-negative integer, then $\Gamma(n+1) = n!$

(iv).

Proof's

(i). $\Gamma(1) = 1$.

Proof:-

By defination of Gamma function, we have.

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx.}$$

$$\therefore \Gamma(1) = \int_0^\infty e^{-x} x^0\,dx.$$

$$= \int_0^\infty e^{-x}\,dx = (-e^{-x})_0^\infty$$

$$= -(0-1) \Rightarrow 1.$$

$$\therefore \boxed{\Gamma(1) = 1.}$$

=(Ans).

(ii) Proof

By defination, we have.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx = \left[x^{n-1}\frac{e^{-x}}{(-1)}\right]_0^\infty - \int_0^\infty (n-1)x^{n-2}\cdot\left(\frac{e^{-x}}{-1}\right)dx$$

(Integrate by parts)

$$= -\lim_{x\to\infty} \frac{x^{n-1}}{e^x} + 0 + (n-1)\int_0^\infty e^{-x} x^{n-2}\, dx.$$

$$= (n-1)\int_0^\infty e^{-x} x^{n-2}\, dx \qquad \left(\because \lim_{x\to\infty} \frac{x^{n-1}}{e^x} = 0 \text{ for } n>1\right)$$

$$\therefore \boxed{\Gamma(n) = (n-1)\,\Gamma(n-1).} \text{ (Ans)}$$

(iii) Proof :-

~~From property~~

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1).$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3)\Gamma(n-3)$$

$$= n(n-1)(n-2)(n-3)\ldots 3.2.1\ \Gamma(1) = n!$$

$$= \cancel{\Gamma}\ \cancel{\Gamma(1)}\ n! \qquad [\because \Gamma(1) = 1].$$

$$\Gamma(n+1) = n!\ (n = 0, 1, 2 \ldots) \text{ // (Ans).}$$

THE END

Unit-V

Multivariable calculus

(Partial Differentiation & applications)

1. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ S.T $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

(or) find $J\left(\frac{u,v,w}{x,y,z}\right)$.

Sol:- G.T $u = \frac{yz}{x}$; $v = \frac{zx}{y}$; $w = \frac{xy}{z}$

$$\frac{\partial u}{\partial x} = \frac{-yz}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{z}{y} \qquad \frac{\partial w}{\partial x} = \frac{y}{z}$$

$$\frac{\partial u}{\partial y} = \frac{z}{x} \qquad \frac{\partial v}{\partial y} = \frac{-xz}{y^2} \qquad \frac{\partial w}{\partial y} = \frac{x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{y}{x} \qquad \frac{\partial v}{\partial z} = \frac{x}{y} \qquad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

x x x

x x x

y y y

z z

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

Multiplying $C_1 \to x,\ C_2 \to y,\ C_3 \to z$

$$= \frac{1}{xyz}\begin{vmatrix} \frac{-xyz}{x^2} & \frac{zy}{x} & \frac{yz}{x} \\ \frac{xz}{y} & \frac{-xyz}{y^2} & \frac{xz}{y} \\ \frac{xy}{z} & \frac{xy}{z} & \frac{-xyz}{z^2} \end{vmatrix}$$

Now, Taking common $\frac{yz}{x}$ from R_1, $\frac{xz}{y}$ from R_2 & $\frac{xy}{z}$ from R_3.

$$= \frac{1}{xyz}\cdot\frac{yz}{x}\cdot\frac{xz}{y}\cdot\frac{xy}{z}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{x^2y^2z^2}{(xyz)^2}\left[-1(1-1)-1(-1-1)+1(1+1)\right]$$

$$= -1(0)-1(-2)+1(2)$$

$$= 0+2+2$$

$$= 4.$$

$\therefore$ $$\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4}$$

= (Ans).

(2) If $x=u(1+v)$, $y=v(1+u)$ then P.T $\frac{\partial(x,y)}{\partial(u,v)}=1+u+v$.

Sol:- Given that $x=u(1+v)$; $y=v(1+u)$.

$$\frac{\partial x}{\partial u}=1+v$$

$$\frac{\partial y}{\partial u}=v$$

$$\frac{\partial x}{\partial v}=u$$

$$\frac{\partial y}{\partial v}=1+u$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1+v & u\\ v & 1+u\end{vmatrix}=(1+v)(1+u)-uv$$

$$=1+u+v+\cancel{uv}-\cancel{uv}$$

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)}=1+u+v}$$

= Ans.

---

(3) If $u=x^2-2y$; $v=x+y+z$; $w=x-2y+3z$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Sol:- G.T

$u=x^2-2y$; $v=x+y+z$; $w=x-2y+3z$

$\frac{\partial u}{\partial x}=2x$; $\frac{\partial v}{\partial x}=1$; $\frac{\partial w}{\partial x}=1$.

$\frac{\partial u}{\partial y}=-2$; $\frac{\partial v}{\partial y}=1$; $\frac{\partial w}{\partial y}=-2$

$\frac{\partial u}{\partial z}=0$; $\frac{\partial v}{\partial z}=1$; $\frac{\partial w}{\partial z}=3$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3+2) + 2(3-1)$$

$$= 2x(5) + 2(2)$$

$$\therefore \boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = 10x + 4}$$

$\Rightarrow$ (Ans).

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4. If $x+y+z=u$, $y+z=uv$, $z=uvw$, then evaluate

(i). $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ (ii). $J\left(\dfrac{u,v,w}{x,y,z}\right)$

(or)

If $u=x+y+z$, $y+z=uv$, $z=uvw$, S.T $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

(1) Sol:- Given $x+y+z=u \longrightarrow (1)$

$y+z=uv \longrightarrow (2)$

$z=uvw \longrightarrow (3)$.

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$(2) \Rightarrow uv=y+z \Rightarrow y=uv-z \Rightarrow \boxed{y=uv-uvw}$ (By (3))

$(1) \Rightarrow u=x+y+z \Rightarrow x=u-(y-z) \Rightarrow x=u-(y+z)$

$\Rightarrow \boxed{x=u-uv}$ (By (2))

$\therefore \frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(x) = \frac{\partial}{\partial u}(u-uv) = 1-v.$

$\therefore \frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(x) = \frac{\partial}{\partial v}(u-uv) = -u.$

$\therefore \frac{\partial x}{\partial w} = \frac{\partial}{\partial w}(x) = \frac{\partial}{\partial w}(u-uv) = 0$

$\therefore \frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(y) = \frac{\partial}{\partial u}(uv-uvw) = v-vw$

$\therefore \frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(y) = \frac{\partial}{\partial v}(uv-uvw) = u-uw$

$\therefore \frac{\partial y}{\partial w} = \frac{\partial}{\partial w}(y) = \frac{\partial}{\partial w}(uv-uvw) = -uv.$

$\therefore \frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(z) = \frac{\partial}{\partial u}(uvw) = vw.$

$\therefore \frac{\partial z}{\partial v} = \frac{\partial}{\partial v}(z) = \frac{\partial}{\partial v}(uvw) = uw.$

$\therefore \frac{\partial z}{\partial w} = \frac{\partial}{\partial w}(z) = \frac{\partial}{\partial w}(uvw) = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= 1-v\left[(u-uw)(uv)-(uw)(-uv)\right]+u\left[(v-vw)(uv)-(-uv)(vw)\right]$$

$$= 1-v\left[(u-uw)(uv)+(uw)(uv)\right]+u\left[(v-vw)(uv)+uv(vw)\right]$$

$$= 1-v\left[u^2v-\cancel{u^2wv}+\cancel{u^2wv}\right]+u\left[v^2u-\cancel{v^2uw}+\cancel{uv^2w}\right]$$

$$= 1-v\left[u^2v\right]+u\left[v^2u\right]$$

$$= u^2v-\cancel{u^2v^2}\cancel{+v^2u^2}$$

$$= u^2v$$

$$\therefore \boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v}$$

(ii).

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{u^2v}$$

— (Ans)

5. If $x = r\cos\theta, y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$ AISO

S.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1.$ (246)

Sol:- Given $x = r\cos\theta, y = r\sin\theta \longrightarrow (1)$

W.K.T $r^2 = x^2 + y^2, \theta = \tan^{-1}\left(\frac{y}{x}\right) \rightarrow (2)$

$(1) \Rightarrow \frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(x) = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$

$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(y) = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta$

$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(x) = \frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta$

$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(y) = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta.$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & +r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r(1)$$

$$\boxed{\frac{\partial(x,y)}{\partial(r,\theta)} = r}$$

$$\text{Ily, } \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix}$$

$$= \frac{1}{r}\left(\cos^2\theta + \sin^2\theta\right)$$ 247

$$= \frac{1}{r}(1)$$

$$\therefore \boxed{\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)}$$

$$\Rightarrow \cancel{r} \times \frac{1}{\cancel{r}}$$

$$= \underline{1}.$$

$$\therefore \boxed{\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1}$$

=(Ans)

⑥ If $u = \frac{x+y}{1-xy}$ & $v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$.

Sol:- Given $u = \frac{x+y}{1-xy}$; $v = \tan^{-1}x + \tan^{-1}y$.

$$\frac{\partial u}{\partial x} = \frac{(1-xy).1-(x+y)(-y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}}$$

Illy, $\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$

$\&\ \frac{\partial v}{\partial x} = \frac{1}{1+x^2}\ ;\ \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \Rightarrow \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$\Rightarrow \frac{\cancel{1+y^2}}{(1-xy)^2} \times \frac{1}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{(1-xy)^2} \times \frac{1}{\cancel{1+x^2}}$$

$$\Rightarrow \cancel{\frac{1}{(1-xy)^2}} - \cancel{\frac{1}{(1-xy)^2}}$$

$$\Rightarrow 0$$

$$\therefore \boxed{\frac{\partial(u,v)}{\partial(x,y)} = 0}$$

(Ans)

7. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, S.T $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ & find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$.

Sol:- Given $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(x) = \frac{\partial}{\partial r}(r\sin\theta\cos\phi) = \sin\theta\cos\phi$$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(x) = \frac{\partial}{\partial \theta}(r\sin\theta\cos\phi) = r\cos\theta\cos\phi$$

$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}(x) = \frac{\partial}{\partial \phi}(r\sin\theta\cos\phi) = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(y) = \frac{\partial}{\partial r}(r\sin\theta\sin\phi) = \sin\theta\sin\phi$$

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(y) = \frac{\partial}{\partial \theta}(r\sin\theta\sin\phi) = r\cos\theta\sin\phi$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}(y) = \frac{\partial}{\partial \phi}(r\sin\theta\sin\phi) = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}(z) = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}(z) = \frac{\partial}{\partial \theta}(r\cos\theta) = -r\sin\theta$$

$$\frac{\partial z}{\partial \phi} = \frac{\partial}{\partial \phi}(z) = \frac{\partial}{\partial \phi}(r\cos\theta) = 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Now Expand "R_3". 250

$$= \cos\theta\left[(r\cos\theta\cos\phi)(r\sin\theta\cos\phi) + (r\cos\theta\sin\phi)(r\sin\theta\sin\phi)\right]$$

$$+ r\sin\theta\left[(\sin\theta\cos\phi)(r\sin\theta\cos\phi) + (\sin\theta\sin\phi)(r\sin\theta\sin\phi)\right]$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi)\right] + r\sin\theta\left[r^2\sin^2\theta(\cos^2\phi + \sin^2\phi)\right]$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta(1)\right] + r\sin\theta\left[r^2\sin^2\theta(1)\right]$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta\right] + r\sin\theta\left[r^2\sin^2\theta\right]$$

$$= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta)$$

$$= r^2\sin\theta(1)$$

$$= r^2\sin\theta$$

$$\therefore \boxed{\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta}$$

$$\therefore \boxed{\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{r^2\sin\theta}}$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \times \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$$

$$r^2\cancel{\sin\theta} \times \frac{1}{r^2\cancel{\sin\theta}}$$

$$= 1.$$

$$\therefore \boxed{\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \times \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = 1}$$ // (Ans)

8. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

given that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$. (251)

[OR]

find the Jacobian transformation from 3D-cartesian co-ordinates to spherical polar coordinates.

Sol:-

Given, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

refer Q.no-7 ~~sol~~ (pg no: 250). _(AM)

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9. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ & $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Sol:-

given, $x = \sqrt{vw}$,

$y = \sqrt{wu}$,

$z = \sqrt{uv}$ and

$u = r\sin\theta\cos\phi$

$v = r\sin\theta\sin\phi$

$w = r\cos\theta$.

since x, y, z are fun's of u, v, w which are in turn fun's of r, θ, ϕ

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} \rightarrow (1)$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(x) = \frac{\partial}{\partial u}(\sqrt{vw}) = 0.$$ 252

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(x) = \frac{\partial}{\partial v}(\sqrt{vw}) = \frac{1}{2}\sqrt{\frac{w}{v}}$$

$$\frac{\partial x}{\partial w} = \frac{\partial}{\partial w}(x) = \frac{\partial}{\partial w}(\sqrt{vw}) = \frac{1}{2}\sqrt{\frac{v}{w}}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(y) = \frac{\partial}{\partial u}(\sqrt{wu}) = \frac{1}{2}\sqrt{\frac{w}{u}}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(y) = \frac{\partial}{\partial v}(\sqrt{wu}) = 0$$

$$\frac{\partial y}{\partial w} = \frac{\partial}{\partial w}(y) = \frac{\partial}{\partial w}(\sqrt{wu}) = \frac{1}{2}\sqrt{\frac{u}{w}}$$

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(z) = \frac{\partial}{\partial u}(\sqrt{uv}) = \frac{1}{2}\sqrt{\frac{v}{u}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}(z) = \frac{\partial}{\partial v}(\sqrt{uv}) = \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}(z) = \frac{\partial}{\partial w}(\sqrt{uv}) = 0$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

Expand c_3'

$$= \frac{1}{2}\sqrt{\frac{v}{w}}\left[\frac{1}{4}\sqrt{\frac{w}{u}}\cdot\sqrt{\frac{u}{v}} - 0\right] - \frac{1}{2}\sqrt{\frac{u}{w}}\left[0 - \frac{1}{4}\sqrt{\frac{w}{v}}\cdot\sqrt{\frac{v}{u}}\right] + 0$$

$$= \frac{1}{8}\sqrt{\cancel{\frac{v}{w}\cdot\frac{w}{u}\cdot\frac{u}{v}}} + \frac{1}{8}\sqrt{\cancel{\frac{u}{w}\cdot\frac{w}{v}\cdot\frac{v}{u}}}$$

$$= \frac{1}{8} + \frac{1}{8}$$

$$= \frac{\cancel{2}^{1}}{\cancel{8}_{4}}$$

$$= \frac{1}{4}$$

$$\therefore \boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{4}}$$

~~We are also~~ Given that

$u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$.

Refer QNO-7, PNO-250

(2).

$$① \Rightarrow \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)}\cdot\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)}$$

$$= \frac{1}{4} \times r^2\sin\theta$$

= (Ans)

(10). If $x = r\cos\theta,\ y = r\sin\theta,\ z = z$. find $\frac{\partial(r,\theta,z)}{\partial(x,y,z)}$ G.T (254)

$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r.$

sol:- Given $x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r \Rightarrow \frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{r}$$

(or)

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(x) = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta.$$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(x) = \frac{\partial}{\partial \theta}(r\cos\theta) = r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}(x) = \frac{\partial}{\partial \phi}(r\cos\theta) = 0$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(y) = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta$$

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(y) = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}(y) = \frac{\partial}{\partial \phi}(r\sin\theta) = 0$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}(z) = \frac{\partial}{\partial r}(z) = 0$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}(z) = \frac{\partial}{\partial \theta}(z) = 0$$

$$\frac{\partial z}{\partial \phi} = \frac{\partial}{\partial \phi}(z) = \frac{\partial}{\partial \phi}(z) = \phi$$

$$\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\boxed{\frac{\partial(r,\theta,z)}{\partial(x,y,z)} = \frac{1}{r}}$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,z)} \times \frac{\partial(r,\theta,z)}{\partial(x,y,z)}$$

$$\cancel{r} \times \frac{1}{\cancel{r}}$$

$$= 1$$

= (Ans).

(11). If $u = x^2 - y^2$, $v = 2xy$, where $x = r\cos\theta$, $y = r\sin\theta$,

S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$. (256)

(or)

If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$ & $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Sol:-

given, $u = x^2 - y^2$ —— (1)

$v = 2xy$ and —— (2)

$x = r\cos\theta$ —— (3)

$y = r\sin\theta$. —— (4)

Sub Eq^n (3) & (4) in Eq^n (1)

(1) $\Rightarrow u = (r\cos\theta)^2 - (r\sin\theta)^2$

$u = r^2\cos^2\theta - r^2\sin^2\theta$

$u = r^2(\cos^2\theta - \sin^2\theta)$

$u = r^2(\cos 2\theta)$ —— (5)

Sub, Eq^n (3) & (4) in Eq^n (2).

(2) $\Rightarrow v = 2(r\cos\theta).(r\sin\theta)$

$v = 2r\cos\theta \cdot r\sin\theta$

$v = r^2(2\cos\theta\sin\theta)$

$v = r^2 \sin 2\theta$ —— (6)

Now, differentiate Eq^n (5) & (6) partially w.r.t r & θ

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(u) = \frac{\partial}{\partial r}\left(r^2(\cos 2\theta)\right) = 2r\cos 2\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}(u) = \frac{\partial}{\partial \theta}\left(r^2\cos 2\theta\right) = -2r^2\sin 2\theta$$

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}(v) = \frac{\partial}{\partial r}\left(r^2\sin 2\theta\right) = 2r\sin 2\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}(v) = \frac{\partial}{\partial \theta}\left(r^2\sin 2\theta\right) = 2r^2\cos 2\theta.$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$$= (2r\cos 2\theta)(2r^2\cos 2\theta) - (2r\sin 2\theta)(-2r^2\sin 2\theta)$$

$$= (2r\cos 2\theta)(2r^2\cos 2\theta) + (2r\sin 2\theta)(2r^2\sin 2\theta)$$

$$= 4r^3\left(\cos^2 2\theta + \sin^2 2\theta\right)$$

$$= 4r^3(1)$$

$$\boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$

Ans.

(1) If $x = e^r \sec\theta$, $y = e^r \tan\theta$ prove that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$.

(253)

Sol:-

given, $\left.\begin{aligned} x &= e^r \sec\theta \\ y &= e^r \tan\theta. \end{aligned}\right\}$ —— (1)

$$\therefore \frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(x) = \frac{\partial}{\partial r}(e^r \sec\theta) = e^r \sec\theta.$$

$$\therefore \frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(x) = \frac{\partial}{\partial \theta}(e^r \sec\theta) = e^r \sec\theta \tan\theta$$

$$\therefore \frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(y) = \frac{\partial}{\partial r}(e^r \tan\theta) = e^r \tan\theta$$

$$\therefore \frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(y) = \frac{\partial}{\partial \theta}(e^r \tan\theta) = e^r \sec^2\theta.$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} e^r \sec\theta & e^r \sec\theta \tan\theta \\ e^r \tan\theta & e^r \sec^2\theta \end{vmatrix}$$

$$= \cancel{e^{2r}(\sec\theta)} =$$

$$= (e^r \sec\theta)(e^r \sec^2\theta) - (e^r \sec\theta \tan\theta)(e^r \tan\theta)$$

$$= e^{2r}(\sec^3\theta - \sec\theta \tan^2\theta)$$

$$= e^{2r} \sec\theta(\sec^2\theta - \tan^2\theta) \Rightarrow e^{2r} \sec\theta \text{ —— (2)}$$

$(1) \Rightarrow \frac{y}{x} = \frac{\tan\theta}{\sec\theta} = \sin\theta$

$$x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta) = e^{2r}.$$

Thus $\boxed{r = \frac{1}{2}\log(x^2 - y^2)}$

$\boxed{\theta = \sin^{-1}\left(\frac{y}{x}\right)}$

These five

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(r) = \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2 - y^2)\right) = \frac{1}{2}\cdot\frac{1}{x^2 - y^2}(2x) = \frac{x}{x^2 - y^2}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(r) = \frac{\partial}{\partial y}\left(\frac{1}{2}\log(x^2 - y^2)\right) = \frac{1}{2}\cdot\frac{1}{x^2 - y^2}(-2y) = \frac{-y}{x^2 - y^2}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}(\theta) = \frac{\partial}{\partial x}\left(\sin^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{\sqrt{1 - (y^2/x^2)}}\cdot\left(\frac{-y}{x^2}\right) = -\frac{y}{x\sqrt{x^2 - y^2}}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}(\theta) = \frac{\partial}{\partial y}\left(\sin^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{\sqrt{1 - (y^2/x^2)}}\cdot\frac{1}{(x)} = \frac{1}{\sqrt{x^2 - y^2}}$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2 - y^2} & \frac{-y}{x^2 - y^2} \\ \frac{-y}{x\sqrt{x^2 - y^2}} & \frac{1}{\sqrt{x^2 - y^2}} \end{vmatrix}$$

$$= \frac{x}{(x^2 - y^2)^{3/2}} - \frac{y^2}{x(x^2 - y^2)^{3/2}}$$

$$= \frac{1}{(x^2 - y^2)^{3/2}}\left[x - \frac{y^2}{x}\right].$$

$$= \frac{1}{x\sqrt{x^2-y^2}} \quad \text{---} (4)$$

Eq^n (2) & (4) are the required Jacobians

Now, sub x from (1) and

for $\sqrt{x^2-y^2}$ from (3) in (4), we get

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{e^r \sec\theta} \cdot \frac{1}{e^r}$$

$$= \frac{1}{e^{2r}\sec\theta} \quad \text{---} (5)$$

Hence, $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$ ($\because$ from (2) & (5)) // (Ans).

---

(13) If $x = u(1-v)$, $y = uv$ prove that $JJ' = 1$

(or)

If $x = u(1-v)$ and $y = uv$ prove that $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$.

Sol:-

Given, $\left.\begin{array}{l} x = u(1-v) \\ y = uv. \end{array}\right\} \to (1)$

$\frac{\partial}{\partial u}(x) = \frac{\partial}{\partial u}(u(1-v)) = 1-v$

$\frac{\partial}{\partial v}(x) = \frac{\partial}{\partial v}(u(1-v)) = -u$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$\frac{\partial}{\partial u}(y) = \frac{\partial}{\partial u}(uv) = v$

$\frac{\partial}{\partial v}(y) = \frac{\partial}{\partial v}(uv) = u$

$$= (1-v)(u) - v(-u)$$

$$= u - \cancel{uv} + \cancel{vu}$$

$$\boxed{J = \frac{\partial(x,y)}{\partial(u,v)} = u} \to (2)$$

$$(1) \Rightarrow x = u - uv = u - y \Rightarrow \boxed{u = x + y} \;\; \& \;\; \boxed{v = \frac{y}{u} = \frac{y}{x+y}}$$

(261)

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(u) = \frac{\partial}{\partial x}(x+y) = 1$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(u) = \frac{\partial}{\partial y}(x+y) = 1$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(v) = \frac{\partial}{\partial x}\left(\frac{y}{x+y}\right) = \frac{-y}{(x+y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(v) = \frac{\partial}{\partial x}\left(\frac{y}{x+y}\right) = \frac{(x+y).1 - y(0+1)}{(x+y)^2}$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{(x+y).1 - y(0+1)}{(x+y)^2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= 1\left(\frac{x}{(x+y)^2}\right) - \left(\frac{-y}{(x+y)^2}\right)\cdot(1)$$

$$= \left(\frac{x}{(x+y)^2}\right) + \left(\frac{y}{(x+y)^2}\right)$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{\cancel{x+y}}{(x+y)^{\cancel{2}}}$$

$$= \frac{1}{x+y}$$

$$J' = \frac{1}{u}$$

$$\therefore JJ' = \cancel{u} \times \frac{1}{\cancel{u}}$$

$$\boxed{JJ' = 1}$$

(Ans)

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(14). (i) If $x = \frac{u^2}{v}$, $y = \frac{v^2}{u}$. find $\frac{\partial(u,v)}{\partial(x,y)}$

(ii) If $x = uv$, $y = u/v$ then find $\frac{\partial(x,y)}{\partial(u,v)}$

(iii) If $x = uv$, $y = \frac{u}{v}$ verify that $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$.

Sol:-

(i) given, $x = \frac{u^2}{v}$

$y = \frac{v^2}{u}$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(x) = \frac{\partial}{\partial u}\left(\frac{u^2}{v}\right) = \frac{2u}{v}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(x) = \frac{\partial}{\partial v}\left(\frac{u^2}{v}\right) = \frac{-u^2}{v^2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(y) = \frac{\partial}{\partial u}\left(\frac{v^2}{u}\right) = \frac{-v^2}{u^2}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(y) = \frac{\partial}{\partial v}\left(\frac{v^2}{u}\right) = \frac{2v}{u}.$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v^2}{u^2} & \frac{2v}{u} \end{vmatrix}$$

$$= \left(\frac{2\cancel{u}}{\cancel{v}}\right)\left(\frac{2\cancel{v}}{\cancel{u}}\right) - \left(\frac{-\cancel{v^2}}{\cancel{u^2}}\right)\left(\frac{-\cancel{u^2}}{\cancel{v^2}}\right)$$

$$= 4 - 1$$

$$\therefore \boxed{\frac{\partial(x,y)}{\partial(u,v)} = 3}$$

$$\therefore \boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{3}.}$$ // (Ans).

(ii). given,

$x = uv$

$y = u/v.$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(x) = \frac{\partial}{\partial u}(uv) = v$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(x) = \frac{\partial}{\partial v}(uv) = u$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(y) = \frac{\partial}{\partial u}\left(u/v\right) = 1/v$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(y) = \frac{\partial}{\partial v}\left(u/v\right) = -u/v^2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$= \cancel{v}\left(\frac{-u}{v^{\cancel{2}}}\right) - \frac{1}{v}(u)$$

$$= \frac{-u}{v} - \frac{u}{v}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{-2u}{v} \quad \text{//(Ans)}$$

(iii) Given,

$$x = uv$$

$$y = \frac{u}{v}$$

Same as (14)(ii)

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = \frac{-2u}{\cancel{v}} \cdot \frac{\cancel{v}}{-2u}$$

$$= 1 \text{ (Ans)}$$

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(15) Evaluate

(i) $\lim\limits_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2y}{x^2+y^2+1}$ (ii) $\lim\limits_{\substack{x \to 1 \\ y \to 1}} \frac{x(y-1)}{y(x-1)}$

(iii) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2}$

Solutions

(ii) given, $\lim\limits_{\substack{x \to 1 \\ y \to 1}} \frac{x(y-1)}{y(x-1)}$

∴ The given limit is does not exist (Ans)

(iii) given, $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y^2}$

∴ The given limit is does not exist (Ans).

(i) given, $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{2x^2y}{x^2+y^2+1}$

for, $\lim\limits_{x\to 1}$ $\Rightarrow \lim\limits_{x\to 1}\left\{\lim\limits_{y\to 2}\left(\dfrac{2x^2y}{x^2+y^2+1}\right)\right\}$

$$\Rightarrow \lim_{x\to 1}\frac{4x^2}{x^2+5} = \frac{4}{6} = \frac{2}{3}.$$

for, $\lim\limits_{y\to 2}$ $\Rightarrow \lim\limits_{y\to 2}\left\{\lim\limits_{x\to 1}\dfrac{2x^2y}{x^2+y^2+1}\right\}$

$$\Rightarrow \lim_{y\to 2}\frac{2y}{y^2+2}$$

$$\Rightarrow \frac{\cancel{4}\,2}{\cancel{6}\,3}$$

$$\Rightarrow 2/3 \quad \text{(Ans)}.$$

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16 (2) If $f(x,y) = \dfrac{x-y}{2x+y}$ show that $\lim\limits_{x\to 0}\left\{\lim\limits_{y\to 0} f(x,y)\right\} \neq \lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} f(x,y)\right\}$.

Sol:-

given, $f(x,y) = \dfrac{x-y}{2x+y}$

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\}$$

$$= \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x-y}{2x+y}\right\}$$

$$= \lim_{x \to 0} \frac{x}{2x} = \frac{1}{2} \text{ (cancel 'x')}.$$

Now,

$$\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x,y) \right\}$$

$$= \lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{x-y}{2x+y} \right\}$$

$$= \lim_{y \to 0} \frac{-y}{y} = -1 \text{ (cancel 'y')}.$$

---

(17)

(i) Discuss the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(ii) Examine for continuity at the origin of the function defined by

$$f(x,y) = \frac{x^2}{\sqrt{x^2+y^2}} \quad \text{for } x \neq 0, y \neq 0$$

$$\text{for } x = 0, y = 0.$$

Redefine the function to make it continuous.

Solution:

(i) given, $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Consider, the limit of the function for testing the continuity along the line, $y = mx$

Nao, $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$

$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{2xy}{x^2+y^2}$

$= \lim_{x\to 0} \frac{2m^2}{x^2+m^2x^2}$

$= \frac{2m}{1+m^2}$

which is different for the different m selected

$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ doesnot Exist

Consider,

$\lim_{x\to 0} f(x,0)$

$= \lim_{x\to 0} \frac{2x(0)}{x^2+0}$

$= \lim_{x\to 0} 0 = 0$

$= f(0,0).$

Nao,

$\lim_{y\to 0} f(0,y)$

$= \lim_{y\to 0} \frac{2.0y}{0+y^2}$

$= \lim_{y\to 0} 0 = 0$

$\Rightarrow f(0,0).$

$\therefore f(x,y)$ is Continuous for the values x and y, but it is not continuous at $(0,0)$ (Ans)

(ii) given

$$f(x,y) = \frac{x^2}{\sqrt{x^2+y^2}} \quad \text{for } x \neq 0, y \neq 0$$
$$\text{for } x=0, y=0$$

∴ The continuity of the given function at (0,0).

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\}$$

$$= \lim_{x\to 0}\left\{\lim_{y\to 0} \frac{x^2}{\sqrt{x^2+y^2}}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{x^2}{x}\right\}$$

$$= \lim_{x\to 0} x = 0.$$

Also, $\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}$

$$= \lim_{y\to 0}\left\{\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+y^2}}\right\}$$

$$= \lim_{y\to 0}\left\{\frac{0}{\sqrt{0+y^2}}\right\}$$

$$= \lim_{y\to 0} (0) = 0.$$

$$\therefore \lim_{x\to 0}\left\{\lim_{y\to 0} (f(x,y)\right\}$$

$$= \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}$$

Also, $y = mx$.

$$\Rightarrow \lim_{x \to 0} f(x,y)$$

$$\Rightarrow \lim_{x \to 0} \frac{x^2}{\sqrt{x^2+m^2x^2}}$$

$$\Rightarrow \lim_{x \to 0} \frac{x}{\sqrt{1+m^2}} = 0.$$

Similarly, $y = mx^2$

$$\lim_{x \to 0} f(x,y) = 0.$$

$\therefore$ The function $f(x,y)$ is continuous at the origin if $f(x,y) = 0$. for $x=0, y=0$. if not $f(x,y)$ is not continuous at the origin.

If $f(x,y)$ is not continuous at $(0,0)$ then $f(x,y) = 0$ for $x=0, y=0$, so that $f(x,y)$ is continuous at origin // (Ans).

---

(18) Verify Euler's theorem for.

(i) $z = ax^2 + 2hxy + by^2$

(ii) $xy + yz + zx$

(iii) $z = \dfrac{1}{x^2+y^2+xy} = (x^2+xy+y^2)^{-1}$

(iv) $u = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$ and prove that $\dfrac{\partial^2 u}{\partial x \cdot \partial y} = \dfrac{x^2-y^2}{x^2+y^2}$

(v) $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$

Solutions

(i) given,

$$z = ax^2 + 2hxy + by^2 \quad \text{--- (1)}$$

∴ This is a homogeneous function of second degree
To verify Euler's Theorem, we have to show that

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z}$$

Now,

$$\frac{\partial z}{\partial x} = 2ax + 2hy \quad \text{--- (2)}.$$

and $$\frac{\partial z}{\partial y} = 2hx + 2by \quad \text{--- (3)}$$

Now, multiply Eqn (2) by 'x'.

$$(2) \Rightarrow x \times \frac{\partial z}{\partial x} = (2ax + 2hy) \cdot x$$

$$\Rightarrow x \times \frac{\partial z}{\partial x} = 2ax^2 + 2hxy \quad \text{--- (4)}$$

Now, multiply Eqn (3) by 'y'.

$$(3) \Rightarrow y \times \frac{\partial z}{\partial y} = (2hx + 2by) \cdot y$$

$$\Rightarrow y \times \frac{\partial z}{\partial z} = 2hxy + 2by^2 \quad \text{--- (5)}.$$

Now, add Eqn (4) & (5).

$$(4) \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial z} = 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$= 2(ax^2 + hxy + hxy + by^2)$$

$$= 2(ax^2 + 2hxy + by^2)$$

$$= 2(z) \quad \Rightarrow 2z \ //(\text{Ans}).$$

Hence, Euler's Theorem is verified.

(ii) given, $xy + yz + zx$.

let, $f(x,y,z) = xy + yz + zx$

Now,
$f(kx, ky, kz) = k^2 f(x,y,z)$ —— ①.

∴ This is a homogeneous function of second degree

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}} \text{ —— ②}$$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f) = \frac{\partial}{\partial x}(xy + yz + zx) = y + z$ —— ③

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f) = \frac{\partial}{\partial y}(xy + yz + zx) = x + z$ —— ④

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(f) = \frac{\partial}{\partial z}(xy + yz + zx) = x + y$. —— ⑤.

Now, sub Eqn ③, ④, ⑤ in Eqn ②.

$\Rightarrow$ ② $\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$

$\Rightarrow x(y+z) + y(x+z) + z(x+y)$

$\Rightarrow xy + zx + yx + yz + xz + yz$

$\Rightarrow 2xy + 2zx + 2yz$

$\Rightarrow 2(xy + zx + yz)$

$\Rightarrow \cancel{2xy + 2zx + 2yz}$

$\Rightarrow 2 \cdot f(x,y,z)$

∴ Euler's Theorem is verified // (Ans).

(iii) given, $z = \dfrac{1}{x^2+xy+y^2} = (x^2+xy+y^2)^{-1}$

'z' is a homogeneous function in x, y of degree '-2'.

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -(2)z}$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(z) = \frac{\partial}{\partial x}(x^2+xy+y^2)^{-1} = (-1)(x^2+xy+y^2)^{-2}(2x+y) \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(z) = \frac{\partial}{\partial y}(x^2+xy+y^2)^{-1} = (-1)(x^2+xy+y^2)^{-2}(x+2y) \quad \text{--- (2)}$$

multiply, Eqn (1) by 'x'.

$$(1) \Rightarrow x \cdot \frac{\partial z}{\partial x} = (-1)(x^2+xy+y^2)^{-2}(2x+y).$$

$$x \cdot \frac{\partial z}{\partial x} = -(x^2+xy+y^2)^{-2}(2x+y) \quad \text{--- (3)}$$

Now, multiply Eqn (2) by 'y'.

$$(2) \Rightarrow y \cdot \frac{\partial z}{\partial y} = (-1)(x^2+xy+y^2)^{-2}(x+2y).$$

$$\Rightarrow y \cdot \frac{\partial z}{\partial y} = -(x^2+xy+y^2)^{-2}(x+2y). \quad \text{--- (4)}.$$

Now, Add Eqn (3) & (4).

$$\Rightarrow x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = -(x^2+xy+y^2)^{-2}(2x+y) - (x^2+xy+y^2)^{-2}(x+2y)$$

$$= -(x^2+xy+y^2)^{-2}(2x^2+2xy+2y^2)$$

$$= -2(x^2+xy+y^2)^{-2}(x^2+xy+y^2)$$ (2-4)

$$= -2(x^2+xy+y^2)^{-1}$$

$$= -2z.$$

Hence, Euler's Theorem is verified. = (Ans)

(iv) given, $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ — (1).

'u' is a homogeneous function of degree '2'.

By, Euler's Theorem.

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u} \text{ — (2).}$$

Consider,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(u) = \frac{\partial}{\partial x}\left(x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y \text{ — (3).}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(u) = \frac{\partial}{\partial y}\left(x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$\frac{\partial u}{\partial y} = 2y\tan^{-1}\left(\frac{x}{y}\right) + x \text{ — (4).}$$

Now, multiply Eqn (3) by 'x':

$$(3) \Rightarrow \quad x \times \frac{\partial u}{\partial x} = \left(2x\tan^{-1}\left(\frac{y}{x}\right) - y\right) \cdot x$$

$$x \times \frac{\partial u}{\partial x} = 2x^2\tan^{-1}\left(\frac{y}{x}\right) - yx \text{ — (5)}$$

Now, multiply Eqⁿ (4) by y.

$$(4) \Rightarrow y \times \frac{\partial u}{\partial y} = \left(-2y \tan^{-1}\left(\frac{x}{y}\right) + x\right) y$$

$$\Rightarrow y \times \frac{\partial u}{\partial y} = \left(-2y^2 \tan^{-1}\left(\frac{x}{y}\right) + xy\right. \quad \text{—— (6).}$$

Now, add Eqⁿ (5) & (6).

$$\Rightarrow x \times \frac{\partial u}{\partial x} + y \times \frac{\partial u}{\partial y} = 2x^2 \tan^{-1}\left(\frac{y}{x}\right) - \cancel{yx} - 2y^2 \tan^{-1}\left(\frac{x}{y}\right) + \cancel{xy}$$

$$= 2x^2 \tan^{-1}\left(\frac{y}{x}\right) - 2y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

$$= 2\left(x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= 2(u)$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \quad \text{—— (7)}$$

from Eqⁿ (2) & (7) we observe that Euler's Theorem is verified.

$$\text{Let, } \frac{\partial^2 u}{\partial x \cdot \partial y} = -2y \frac{1}{1+\frac{x^2}{y^2}} \left(\frac{1}{y}\right) + 1$$

$$= \frac{-2y^2}{x^2+y^2} + 1$$

$$= \frac{x^2-y^2}{x^2+y^2} \quad //(\text{Ans}).$$

(1) given, $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ —(1).

u is a homogeneous function of degree '0'.

By Euler's theorem.

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0} \quad \text{—(2)}.$$

Consider, $x\dfrac{\partial u}{\partial x} = \dfrac{xy}{\sqrt{y^2-x^2}} - \dfrac{xy}{x^2+y^2}$ —(3)

Consider,

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{—(4)}.$$

Now, substitute Eqn (3) & (4) in (2).

$\Rightarrow$ (2) $\Rightarrow x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

$$\Rightarrow \cancel{\frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}}$$

$$\Rightarrow \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$\Rightarrow 0$.

$\therefore x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ Euler's Theorem is verified

// (Ans).

Maximum and minimum values

* for $f(x,y)$ to be maximum (or) minimum, $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.
* for $f(x)$ to be maximum (or) minimum, $f'(x)=0$.
* Consider the Lagrangean function

$$F(x,y,z) = u(x,y,z) + \lambda \cdot \phi(x,y,z).$$

THE END

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