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UNIT - I

First order ODE

Topics

→ Exact, Linear and Bernouli's equations, Applications

→ Newton's law of cooling, Law of natural growth and decay

→ Equations not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and ~~Clairaut's~~ Clairaut's method

Exact D.E:- Let $M(x,y)dx + N(x,y)dy = 0$ be a first order & first degree D.E.

where M, N are functions in terms of x, y.

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $Mdx + Ndy = 0$ is said to be exact D.E.

Working rule to solve exact D.E.

* The given equation is of the form $Mdx + Ndy = 0$
* Find $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$
* If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then given equation is an exact D.E.
* The solution can be obtained as

$$\underset{(y \text{ as constant})}{\int M dx} + \underset{(\text{the terms not containing } x)}{\int N dy} = C$$

Problems

(1) Solve $(e^y+1)\cos x\, dx + e^y \sin x\, dy = 0$

Sol:- The equation is of the form $Mdx + Ndy = 0$

where $M = (e^y+1)\cos x$ $\qquad N = e^y \sin x$

$\frac{\partial M}{\partial y} = e^y \cos x$ $\qquad \frac{\partial N}{\partial x} = e^y \cos x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so the given eqn is an exact D.E.

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The general soln is

$$\underset{(y \text{ as constant})}{\int M\,dx} + \underset{(\text{terms not containing } x)}{\int N\,dy} = c$$

$$\int (e^y+1)\cos x\,dx + \int 0\,dy = c$$

$$(e^y+1)\sin x = c$$

(2) Solve $2xy\,dy - (x^2-y^2+1)\,dx = 0$

Sol:- The given eqn is of the form $M\,dx + N\,dy = 0$

where $M = -(x^2-y^2+1)$ $\quad N = 2xy$

$M = -x^2+y^2-1$

$\frac{\partial M}{\partial y} = 2y$ $\quad \frac{\partial N}{\partial x} = 2y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the given eqn is an exact D.E.

The general soln is

$$\underset{(y \text{ as constant})}{\int M\,dx} + \underset{(\text{terms not containing } x)}{\int N\,dy} = c$$

$$\int (y^2-x^2-1)\,dx + \int 0\,dy = c$$

$$y^2x - \frac{x^3}{3} - x = c$$

$$3xy^2 - x^3 - 3x = c$$

(3) Solve $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol:- The given eqn is of the form $M\,dx + N\,dy = 0$

where $M = 1+e^{x/y}$ $\quad N = e^{x/y}\left(1-\frac{x}{y}\right)$

$\frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right)$ $\quad \frac{\partial N}{\partial x} = e^{x/y}\left(0-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\left(\frac{1}{y}\right)$

$= \frac{-xe^{x/y}}{y^2}$ $\quad = \frac{-xe^{x/y}}{y^2}$

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Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the given equ is an exact DiE.

The general soln is

$$\underset{(y \text{ as constant})}{\int M\,dx} + \underset{(\text{terms not containing } x)}{\int N\,dy} = c$$

$$\int (1 + e^{x/y})\,dx + \int 0\,dy = c$$

$$x + \frac{e^{x/y}}{\frac{1}{y}} = c$$

$$x + y e^{x/y} = c$$

(4) Solve $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

The given equ is of the form $M\,dx + N\,dy = 0$

where $M = 5x^4 + 3x^2y^2 - 2xy^3$ $\qquad N = 2x^3y - 3x^2y^2 - 5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$ $\qquad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the given equ is an exact DiE.

The general soln is

$$\underset{(y \text{ as constant})}{\int M\,dx} + \underset{(\text{terms independent of } x)}{\int N\,dy} = c$$

$$\int (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int -5y^4\,dy = c$$

$$\frac{\cancel{5}x^5}{\cancel{5}} + \frac{\cancel{3}x^3}{\cancel{3}}y^2 - \frac{\cancel{2}x^2}{\cancel{2}}y^3 - \frac{\cancel{5}y^5}{\cancel{5}} = c$$

$$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = c$$

H.W

(5) Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

(6) Solve $(r + \sin\theta - \cos\theta)\,dr + r(\sin\theta + \cos\theta)\,d\theta = 0$

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Non exact D.E.

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then $Mdx + Ndy = 0$ is said to be non exact D.E.

Note

We can convert non exact D.E. into exact D.E. by multiplying with Integrating Factors.

To find an Integrating Factors (I.F.) of $Mdx + Ndy = 0$

Method (I)

If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous D.E. and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an I.F. of $Mdx + Ndy = 0$

(1) Solve $x^2y\,dx - (x^3+y^3)dy = 0$ — (1)

Sol:- Here, given eqn is an homogeneous D.E. and comparing $Mdx + Ndy = 0$, $M = x^2y$ $N = -x^3-y^3$

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Given eqn is not an exact D.E.

In this case,

$$I.F. = \frac{1}{Mx+Ny}$$

$$= \frac{1}{\cancel{x^3y} - \cancel{x^3y} - y^4}$$

$$= -\frac{1}{y^4}$$

multiplying eq (1) with I.F., then

$$-\frac{1}{y^4}(x^2y)dx + \frac{1}{y^4}(x^3+y^3)dy = 0$$

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$$\frac{x^2}{y^3}dx - \frac{(x^3+y^3)}{y^4}dy = 0$$

Comparing with $M_1 dx + N_1 dy = 0$ then

$$M_1 = \frac{x^2}{y^3} \qquad N_1 = -\frac{x^3-y^3}{+y^4} = \frac{x^3}{y^4} + \frac{1}{y}$$

$$\frac{\partial M_1}{\partial y} = \frac{-3x^2}{y^4} \qquad \frac{\partial N_1}{\partial x} = -\frac{3x^2}{y^4}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ so the ~~gi~~ equ is in exact.

The general soln is

$$\int M_1 dx + \int N_1 dy = c$$

(y as constant) (terms independent of x)

$$\int \frac{x^2}{y^3}dx + \int \frac{1}{y}dy = c$$

$$\boxed{\frac{x^3}{3y^3} + \log y = c}$$

② Solve $xydx - (x^2+2y^2)dy = 0$ ①

$$M = xy \qquad N = -x^2 - 2y^2$$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = -2x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so it is not an exact.

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{\cancel{x^2y} - \cancel{x^2y} - 2y^3} = -\frac{1}{2y^3}$$

multiplying with I.F. to eq①

$$\frac{xy}{-2y^3}dx - \frac{(x^2+2y^2)}{-2y^3}dy = 0$$

$$-\frac{x}{2y^2}dx + \left(\frac{x^2}{2y^3} + \frac{1}{y}\right)dy = 0$$

$$M_1 dx + N_1 dy = 0$$

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$M_1 = \frac{-x}{2y^2}$ $\qquad$ $N_1 = \frac{x^2}{2y^3} + \frac{1}{y}$

$\frac{\partial M_1}{\partial y} = \frac{-x}{\cancel{2y^2}}\left(\frac{-\cancel{2}}{y^3}\right)$ $\qquad$ $\frac{\partial N_1}{\partial x} = \frac{2x}{2y^3} + 0$

$= \frac{x}{y^3}$ $\qquad$ $= \frac{x}{y^3}$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ so the eqn is in exact.

The general soln is

$$\underset{(y \text{ as constant})}{\int M_1\,dx} + \underset{(\text{terms independent of } x)}{\int N_1\,dy} = c$$

$$\int\left(\frac{-x}{2y^2}\right)dx + \int\left(\frac{x^2}{2y^3} + \frac{1}{y}\right)dy = c$$

$$\Rightarrow -\frac{x^2}{4y^2} + \frac{x^2}{2}\left(\frac{-3}{y^2}\right) + \log y = c$$

$$\Rightarrow \frac{-x^2}{4y^2} - \frac{3x^2}{2y^2} + \log y = c$$

<u>HW</u>

3) Solve $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$

4) Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

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Method (II)

If the equation $Mdx + Ndy = 0$ is of the form $yf(xy)dx + xg(xy)dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an I.F. of $Mdx + Ndy = 0$

① Solve $y(xy\sin xy + \cos xy)dx + (xy\sin xy - \cos xy)xdy = 0$ — ①

Sol:- eq ① is of the form $y f(xy)dx + x g(xy)dy = 0$

eq ① comparing with $Mdx + Ndy = 0$

$M = xy^2\sin xy + y\cos xy$ $\qquad$ $N = x^2y\sin xy - x\cos xy$

Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ It is ~~an~~ not an exact D.E.

In this case

$$I.F = \frac{1}{Mx-Ny} = \frac{1}{\cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy}$$

$$= \frac{1}{2xy\cos xy}$$

multiplying eq ① with I.F

$$\frac{1}{2\cancel{xy}\cos xy}\cancel{y}(xy\sin xy + \cos xy)dx + \frac{1}{2xy\cos xy}x(xy\sin xy - \cos xy)dy = 0$$

$$\frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)dx + \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)dy = 0$$

which is of the form $M_1dx + N_1dy = 0$

$M_1 = \frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)$ $\qquad$ $N_1 = \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)$

$\frac{\partial M_1}{\partial y} = \frac{1}{2}(y\sec^2 xy \cdot x + \tan xy)$ $\qquad$ $\frac{\partial N_1}{\partial x} = \frac{1}{2}(x\sec^2 xy \cdot y + \tan y)$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ so the equation is an exact.

The general soln is

$$\underset{(y \text{ as constant})}{\int M_1 dx} + \underset{(\text{terms independent of } x)}{\int N_1 dy} = c$$

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$$\frac{1}{2}\int\left(y\tan xy + \frac{1}{x}\right)dx + \frac{1}{2}\int\left(\cancel{x\tan xy} - \frac{1}{y}\right)dy = c$$

$$\frac{1}{2}\,\cancel{y}\log\left(\frac{\sec xy}{\cancel{y}}\right) + \frac{1}{2}\log x + \frac{1}{2}(-\log y) = c$$

$$\Rightarrow \frac{1}{2}\left[\log\sec xy + \log x - \log y\right] = \log c$$

$$\Rightarrow \log|\sec xy| + \log\frac{x}{y} = \log c$$

$$\Rightarrow \boxed{\frac{x\sec xy}{y} = c}$$

HW ② Solve $(2xy+1)y\,dx + (1+2xy-x^3y^3)x\,dy = 0$

HW ③ Solve $y(1+xy)dx + x(1-xy)dy = 0$

Method III

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ & a continuous single variable function $f(x)$ such that $f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ then $e^{\int f(x)dx}$ is an I.F. of $Mdx + Ndy = 0$

① Solve $2xy\,dy - (x^2+y^2+1)dx = 0$ — ①

eq ① is of the form $Mdx + Ndy = 0$

$M = -x^2 - y^2 - 1$ $\qquad$ $N = 2xy$

$\frac{\partial M}{\partial y} = -2y$ $\qquad$ $\frac{\partial N}{\partial x} = 2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ not an exact.

$$f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{2xy}\left[-2y - 2y\right] = \frac{-\cancel{4y}^{\,2}}{\cancel{2}x\cancel{y}} = \frac{-2}{x}$$

$$I.F = e^{\int f(x)dx} = e^{-2\int\frac{1}{x}dx} = e^{-2\log x} = e^{\log\frac{1}{x^2}} = \frac{1}{x^2}$$

$$I.F = \frac{1}{x^2}$$

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multiplying eq ① with I.F.

$$\frac{2xy}{x^2}dy - \frac{(x^2+y^2+1)}{x^2}dx = 0$$

$$\frac{2y}{x}dy - \left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right]dx = 0$$

It is in the form of $M_1 dx + N_1 dy = 0$

$$M_1 = -1 - \frac{y^2}{x^2} - \frac{1}{x^2} \qquad N_1 = \frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = -\frac{2y}{x^2} \qquad \frac{\partial N_1}{\partial x} = -\frac{2y}{x^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow$ is an exact

The general soln is

$$\underset{(y \text{ as constant})}{\int M_1 dx} + \underset{(\text{terms not containing } x)}{\int N_1 dy} = c$$

$$-\int \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx + 0 = c$$

$$\Rightarrow -\left[x + \frac{y^2}{x} - \frac{1}{x}\right] = c$$

$$\Rightarrow x^2 - (y^2+1) = cx$$

HW

② Solve $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$

HW

③ Solve $(x^3 - 2y^2)dx + 2xy\,dy = 0$

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Method (IV)

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ & a Continuous single variable function $g(y)$ such that $\cancel{\frac{f(x)}{g(y)}} = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$ then $e^{\int g(y)dy}$ is an I.F. of $Mdx + Ndy = 0$

① Solve the D.E. $y(xy + e^x)dx - e^x dy = 0$ — ①

Sol:- $M = y(xy + e^x) = xy^2 + e^x y$ $\qquad N = -e^x$

$\frac{\partial M}{\partial y} = 2xy + e^x$ $\qquad \frac{\partial N}{\partial x} = -e^x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now $f(x) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{xy^2 + e^x y}\left[-e^x - 2xy - e^x\right]$

$= \frac{-2\cancel{[e^x + xy]}}{y\cancel{[e^x + xy]}} = -\frac{2}{y} = g(y)$

$I.F = e^{\int g(y)dy}$

$= e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = e^{\log \frac{1}{y^2}} = \frac{1}{y^2}$

multplying eq ① with I.F.

$\frac{1}{y^2}\left[y(xy + e^x)\right]dx - \frac{e^x}{y^2}dy = 0$

$\frac{xy + e^x}{y}dx - \frac{e^x}{y^2}dy = 0$

It is in the form of $M_1 dx + N_1 dy = 0$

$M_1 = x + \frac{e^x}{y}$ $\qquad N_1 = -\frac{e^x}{y^2}$

$\frac{\partial M_1}{\partial y} = -\frac{e^x}{y^2}$ $\qquad \frac{\partial N_1}{\partial x} = -\frac{e^x}{y^2}$

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Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow$ which is exact.

The general soln is

$$\underset{(y \text{ as constant})}{\int M_1 dx} + \underset{(\text{terms independent of } x)}{\int N_1 dy} = c$$

$$\int \left(x + \frac{e^x}{y}\right) dx + \int 0 \, dy = c$$

$$\boxed{\frac{x^2}{2} + \frac{e^x}{y} = c}$$

HW

(2) Solve $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3)dy = 0$

HW

(3) Solve $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3)dy = 0$

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Linear Differential equations of first order

The general form of first order Linear D.E. is

$$\frac{dy}{dx} + P(x)y = Q(x)$$

procedure

* The Linear D.E. is $\frac{dy}{dx} + P(x)y = Q(x)$
* Write down $I.F = e^{\int P(x)dx}$
* The general soln is given by

$$y \times I.F = \int (Q(x) \times I.F)\, dx + c$$

(OR)

* The Linear D.E. is of the type $\frac{dx}{dy} + P(y)x = Q(y)$
* Write down $I.F = e^{\int P(y)dy}$
* The general soln is given by

$$x \times I.F = \int (Q(y) \times I.F)\, dy + c$$

problems

① Solve $x\frac{dy}{dx} + y = \log x$

Given eqn can be written as

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\log x}{x} \quad \text{—— ①}$$

eq ① is a linear form ie; $\frac{dy}{dx} + P(x)y = Q(x)$

$$P(x) = \frac{1}{x}, \quad Q(x) = \frac{\log x}{x}$$

$$I.F = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

The general soln is

$$y \times I.F = \int (Q(x) \times I.F)\, dx$$

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$$y \times x = \int \left(\frac{\log x}{\cancel{x}} \times \cancel{x}\right) dx$$

$$yx = x\log x - x + c$$

$$\boxed{xy = x(\log x - 1) + c}$$

② Solve $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

Given equ can be written as

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x\sqrt{1-x^2}}{1-x^2} = \frac{x}{\sqrt{1-x^2}}$$

Comparing with Linear equation

$$P(x) = \frac{2x}{1-x^2} \qquad Q(x) = \frac{x}{\sqrt{1-x^2}}$$

$$I.F = e^{\int P(x)dx} = e^{\int \frac{2x}{1-x^2}dx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

The general soln is

$$y \times I.F = \int (Q(x) \times I.F)dx$$

$$y\frac{1}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{1-x^2} dx$$

$$= \int \frac{x}{(1-x^2)^{3/2}} dx$$

put $1-x^2 = t$

$-2x\,dx = dt$

$x\,dx = -\frac{dt}{2}$

$$= -\frac{1}{2}\int \frac{1}{t^{3/2}} dt$$

$$= -\frac{1}{2}\left[\frac{t^{-3/2+1}}{-\frac{3}{2}+1}\right] + c$$

$$= \frac{1}{\sqrt{t}} + c = \frac{1}{\sqrt{1-x^2}} + c \Rightarrow \boxed{\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c}$$

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③ Solve $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

Here $P(x) = \frac{1}{x\log x}$ $\quad Q(x) = \frac{\sin 2x}{\log x}$

$$I.F = e^{\int P(x)dx} = e^{\int \frac{1}{x\log x}dx}$$

$$= e^{\log(\log x)} = \log x$$

The general soln is

$$y \times I.F = \int (Q(x) \times I.F)dx$$

$$y\log x = \int \left(\frac{\sin 2x}{\cancel{\log x}} \cancel{\log x}\right)dx$$

$$\boxed{y\log x = -\frac{\cos 2x}{2} + c}$$

④ Solve $(x + 2y^3)\frac{dy}{dx} = y$

The given eqn can be written as

$$\frac{dy}{dx} = \frac{y}{x+2y^3}$$

$$\frac{dx}{dy} = \frac{x+2y^3}{y} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$

Comparing with $\cancel{\frac{dy}{dx} + P(x)y = Q}$ $\quad \frac{dx}{dy} + P(y)x = Q(y)$

$P(y) = -\frac{1}{y}$ $\quad Q(y) = 2y^2$

$$I.F = e^{\int P(y)dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = e^{\log \frac{1}{y}} = \frac{1}{y}$$

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The general soln is

$$x \times I.F = \int (Q(y) \times I.F)\,dy$$

$$x \times \frac{1}{y} = \int 2y^{\cancel{2}} \times \frac{1}{\cancel{y}}\,dy$$

$$= \cancel{2}\frac{y^2}{\cancel{2}} + c$$

$$\boxed{\frac{x}{y} = y^2 + c}$$

(5) Solve $(1+y^2)dx = (\tan^{-1}y - x)dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

Here $P(y) = \frac{1}{1+y^2}$, $Q(y) = \frac{\tan^{-1}y}{1+y^2}$

$$I.F = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

The general soln is

$$x \times I.F = \int (Q(y) \times I.F)\,dy$$

$$x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy$$

put $\tan^{-1}y = t$

$\frac{1}{1+y^2}dy = dt$

$$= \int e^t . t\,dt$$

$$= e^t(t-1) + c$$

$$\boxed{x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c}$$

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H.W

(6) Solve $(x+1)\frac{dy}{dx} - xy = e^{x}(x+1)^{n+1}$

H.W

(7) Solve $\sin 2x \frac{dy}{dx} - y = \tan x$

H.W

(8) Solve $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$ given $y=0$ when $x=1$

Bernouli's equation

The general form of Bernouli's equation is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Procedure

The given eqn can be written as

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^n}P(x)y = Q(x)$$

$$y^{-n}\frac{dy}{dx} + y^{1-n}P(x) = Q(x)$$

put $y^{1-n} = t$

$$(1-n)\,y^{\cancel{1}-n-\cancel{1}}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}$$

$$\frac{1}{1-n}\frac{dt}{dx} + t\,P(x) = Q(x)$$

$$\frac{dt}{dx} + (1-n)t\,P(x) = Q(x)(1-n)$$

which is in Linear equation of first order.

problems

(1) Solve $x\frac{dy}{dx} + y = x^3y^6$

Given eqn can be written as

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{y}{xy^6} = \frac{x^3y^6}{xy^6}$$

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$$y^{-6}\frac{dy}{dx} + y^{-5}\frac{1}{x} = x^2$$

It is a bernouli's equation.

put $y^{-5} = t$

$$-5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

$$-\frac{1}{5}\frac{dt}{dx} + t.\frac{1}{x} = x^2$$

$$\Rightarrow \frac{dt}{dx} - \frac{5}{x}t = -5x^2$$

It is a Linear equation

$$I.F = e^{-\int \frac{5}{x}dx} = e^{-5\log x} = e^{\log\frac{1}{x^5}} = \frac{1}{x^5}$$

The general soln is

$$t \times I.F = \int (Q(x) \times I.F)\,dx$$

$$t.\frac{1}{x^5} = \int (-5x^2)\left(\frac{1}{x^5}\right)dx$$

$$\frac{1}{x^5y^5} = -5\int x^{-3}dx$$

$$= -5\frac{x^{-2}}{-2} + c$$

$$\frac{1}{x^5y^5} = \frac{5}{2}.\frac{1}{x^2} + c$$

$$\boxed{\frac{1}{y^5} = \frac{5}{2}x^3 + cx^5}$$

(2) Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x\,2\sin y\cancel{\cos y}}{\cos y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2\tan y\, x = x^3$$

put $\tan y = t$

$$\sec^2 y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + 2tx = x^3$$

$$I.F = e^{2\int x\,dx} = e^{\frac{2x^2}{2}} = e^{x^2}$$

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The general soln is

$$t \times I.F = \int (Q(x) \times I.F)\,dx$$

$$t e^{x^2} = \int x^3 e^{x^2} dx$$

$$= \int x^2 . x e^{x^2} dx$$

$x^2 = u$

$2x\,dx = du$

$x\,dx = \frac{du}{2}$

$$= \int u e^{u} \frac{du}{2}$$

$$t e^{x^2} = \frac{1}{2} e^{u}(u-1) + c$$

$$\boxed{t \tan y\, e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c}$$

Hw

③ Solve $3\frac{dy}{dx} - y\cos x = y^4(\sin 2x - \cos x)$

Hw

④ Solve $e^x \frac{dy}{dx} = 2xy^2 + ye^x$

Hw

⑤ Solve $\frac{dy}{dx} + y\cot x = y^2 \sin^2 x \cos^2 x$

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Applications of DE

Newton's law of cooling

The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let θ be the temperature of the body at time 't' & θ_0 be the temperature of surrounding medium, By the Newton's law of cooling, we have

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$$\int \frac{d\theta}{\theta - \theta_0} = -k \int dt$$

$$\Rightarrow \log(\theta - \theta_0) = -kt + \log c$$

$$\log\left(\frac{\theta - \theta_0}{c}\right) = -kt$$

$$\Rightarrow \boxed{\theta - \theta_0 = c e^{-kt}}$$

Problems

(1) The temperature of the body drops from 100°c to 75°c in 10mins when the surrounding air is at 20°c temperature. What will be it's temperature after an half an hour? When will the temperature be 25°c.

Sol:- Given $\theta_0 = 20°c$

We have $\theta - \theta_0 = ce^{-kt}$

$$\theta - 20 = ce^{-kt} \quad \text{—— (1)}$$

The temperature drops from 100°c to 75°c in 10mins.

i.e; When $t = 0$ $\quad \theta = 100°c \Rightarrow (1) \Rightarrow 100 - 20 = ce^{0(-k)} \Rightarrow c = 80$

When $t = 10$ $\quad \theta = 75°c \Rightarrow (1) \Rightarrow 75 - 20 = 80e^{-K(10)}$

$$\Rightarrow k = -\frac{1}{10}\log\frac{55}{80}$$

~~K = 0.016~~

After half an hour (30mins) the temperature becomes (from (1))

$$\theta - 20 = 80 \times e^{-30K}$$

$$= 80 \times e^{-30\left(-\frac{1}{10}\log\frac{55}{80}\right)}$$

$$\boxed{\theta = 46°c}$$

When $\theta = 25°c \Rightarrow t = ?$

eq (1) becomes $25 - 20 = 80e^{-\left(-\frac{1}{10}\log\frac{55}{80}\right)t} \Rightarrow t = 74.86$ mins

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② If the air is maintained at 15°c and the temperature of the body drops from 70°c to 40°c in 10mins. What will be it's temperature after 30mins.

Sol:- Given $\theta_0 = 15°c$

We have $\theta - \theta_0 = ce^{-kt}$ —— ①

When $t = 0$ $\theta = 70°c$

$70 - 15 = ce^{-k(0)} \Rightarrow c = 55$

eq ① becomes

$\theta - \theta_0 = 50e^{-kt}$ —— ②

When $t = 10$ mins $\theta = 40°c$

② becomes

$40 - 15 = 55e^{-k(10)}$

$\Rightarrow e^{-10k} = \frac{25}{55} = \frac{5}{11}$

When $t = 30$mins, the temperature becomes

$\theta = 15 + 55(e^{-10k})^3$

$= 15 + 55\left(\frac{5}{11}\right)^3$

$\boxed{\theta = 20.16°c}$

③ A body kept in air with temperature 25°c cools from 140°c to 80°c in 20mins. Find when the body cools down to 35°c.

Sol:- Given $\theta_0 = 25°c$ we have $\theta - \theta_0 = ce^{-kt}$

When $t = 0$ $\theta = 140°c$ $\theta - 25 = ce^{-kt}$ —— ①

① becomes $140 - 25 = c(1) \Rightarrow c = 115$

① becomes

$\theta - 25 = 115e^{-kt}$ —— ②

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When $t = 20$, $\theta = 80^\circ c$

$$80 - 25 = 115 e^{-k(20)}$$

$$\frac{\cancel{55}^{\,11}}{\cancel{115}_{\,23}} = e^{-20k} \Rightarrow e^{-20k} = \frac{11}{23}$$

When $\theta = 35^\circ c$ $\quad t = ?$

$$35 - 25 = 115 e^{-kt}$$

$$10 = 115 \left(e^{-20k}\right)^{\frac{t}{20}}$$

$$10 = 115 \left(\frac{11}{23}\right)^{t/20} \Rightarrow t = 66.2$$

After 66.2 mins, the ~~temper~~ body cools down to $35^\circ c$

Hw

(4) An object cools from $120^\circ c$ to $95^\circ c$ in half an hour when surrounded by air whose temperature is $70^\circ F$. Find its temperature at the end of another half an hour.

Hw

(5) A copper ball is heated to a temperature of $80^\circ c$. Then at time $t = 0$ it is placed in water which is maintained at $30^\circ C$. If at $t = 3$ min, the temperature of the ball is reduced to $50^\circ c$. Find the time at which the temperature of the ball is $40^\circ C$

Hw

(6) Suppose that an object is heated to $300^\circ F$ and allowed to cool in a room whose air temperature is $80^\circ F$, if after 10 mins the temperature of the object is $250^\circ F$, what will be it's temperature after 20 mins.

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Law of natural growth and decay

Let $x(t)$ be the amount of substance at time 't'. A law of chemical conversion states that the rate of change of amount $x(t)$ of a chemically changing substance is proportional to the amount of substance available at that time.

i.e: $\frac{dx}{dt} \propto x$

$$\frac{dx}{dt} = kx \text{ (growth)}$$

$$\frac{1}{x}dx = k\,dt$$

$$\int \frac{1}{x}dx = k\int dt$$

$$\log x = kt + \log c$$

$$\log \frac{x}{c} = kt$$

$$\frac{x}{c} = e^{kt}$$

$$\boxed{x = ce^{kt}}$$

$$\frac{dx}{dt} = -kx \text{ (decay)}$$

$$\frac{1}{x}dx = -k\,dt$$

$$\int \frac{1}{x}dx = -k\int dt$$

$$\log x = -kt + \log c$$

$$\log \frac{x}{c} = -kt$$

$$\frac{x}{c} = e^{-kt}$$

$$\boxed{x = ce^{-kt}}$$

Note:- * If t increases and x increases we can take $\frac{dx}{dt} = kx$ $(k>0)$

* If t increases and x decreases we can take $\frac{dx}{dt} = -kx$ $(k>0)$

Problems

① A bacterial culture, growing exponentially increases from 200 to 500 gm in the period of 6am to 9am. How many grams will be present at noon (12pm)

Sol:- Given $x = 200$ $t = 0$

$x = 500$ $t = 3$hrs (6am to 9am)

$x = ?$ $t = 6$hrs (12pm (noon))

By law of natural growth

$$\frac{dx}{dt} \propto x \Rightarrow x = ce^{kt} \quad \text{—— (1)}$$

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$t=0$, $x=200$

$200 = ce^{k(0)} \Rightarrow c = 200$

① becomes

$x = 200e^{kt}$ — ②

When $x = 500$, $t = 3$ hrs

$500 = 200e^{3k}$

$\Rightarrow e^{3k} = \frac{500}{200} = \frac{5}{2}$

$\Rightarrow 3k = \log\frac{5}{2}$

$\Rightarrow k = \frac{1}{3}\log\frac{5}{2}$

When $t = 6$ hrs $x = ?$

From ②

$x = 200e^{\frac{1}{3}\log\frac{5}{2}\cdot 6}$

$x = 1250$ gms

② A bacterial culture, growing exponentially, increases from 100 to 400 gms in 10 hrs. How much was present after 3 hrs, from the initial instant?

Sol: Given $x = 100$ $t = 0$

$x = 400$ $t = 10$ hrs

$x = ?$ $t = 13$ hrs

By law of natural growth

$\frac{dx}{dt} \propto x \Rightarrow x = ce^{kt}$ — ①

When $t=0$, $x=100$

$\Rightarrow 100 = ce^{k(0)} \Rightarrow c = 100$

① becomes $x = 100e^{kt}$ — ②

When $x = 400$ $t = 10$ hrs

$400 = 100e^{10k}$

$\Rightarrow e^{10k} = \frac{400}{100} = 4$

$\Rightarrow 10k = \log 4$

$\Rightarrow k = \frac{1}{10}\log 4$

When $t = 13$ hrs $x = ?$

From ②

$x = 100e^{(\frac{1}{10}\log 4)13}$

$x = 606.2$ gms

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Equations of first order But not of the first degree

The general form of the first order D.E. of degree $n>1$ is

$$P_0\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots + P_{n-1}\left(\frac{dy}{dx}\right) + P_n = 0 \quad \text{--- (1)}$$

where $P_0\ P_1\ P_2 \cdots P_{n-1}\ P_n$ are functions of x and y.

If we denotes $\frac{dy}{dx}$ as p then eq (1) becomes

$$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots P_{n-1} p + P_n = 0$$

Equations solvable for p

Let $P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0$ — (1)

By taking into n linear factors, eq (1) can be written as

$$[p - f_1(x,y)][p - f_2(x,y)] \cdots [p - f_n(x,y)] = 0 \quad \text{— (2)}$$

equating each factor equal to zero, we obtain n equations of the first order and first degree, then

$$p = \frac{dy}{dx} = f_1(x,y),\ p = \frac{dy}{dx} = f_2(x,y) \cdots, p = \frac{dy}{dx} = f_n(x,y)$$

The solutions can be

$$F_1(x,y,c_1) = 0,\ F_2(x,y,c_2) = 0, \cdots F_n(x,y,c_n) = 0$$

where $c_1\ c_2\ c_3 \cdots c_n$ are arbitrary constants,

If we replace all the constants $c_1\ c_2\ c_3 \cdots c_n$ with a single constant 'c' then 'n' solutions becomes

$$F_1(x,y,c) = 0,\ F_2(x,y,c) = 0 \cdots F_n(x,y,c) = 0$$

Combining the above equations, the solution of (1) is

$$F_1(x,y,c) \cdot F_2(x,y,c) \cdots F_n(x,y,c) = 0$$

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problems

① Solve the following D.E's

i) $p^2 = ax^3$ where $p = \frac{dy}{dx}$

Sol:- Given $p^2 = ax^3$

$$p = \pm (ax^3)^{1/2}$$

$$p = \pm a^{1/2} x^{3/2}$$

$$\frac{dy}{dx} = \pm \sqrt{a}\, x^{3/2}$$

$$\Rightarrow dy = \pm \sqrt{a}\, x^{3/2} dx$$

$$\Rightarrow \int dy = \pm \sqrt{a} \int x^{3/2} dx$$

$$\Rightarrow y = \pm \sqrt{a} \left[\frac{x^{3/2+1}}{3/2+1} \right] + c$$

$$\Rightarrow y = \pm \sqrt{a} \frac{2}{5} x^{5/2} + c$$

$$\Rightarrow 5y = \pm 2\sqrt{a}\, x^{5/2} + 5c$$

$$\Rightarrow 5y - 5c = \pm 2\sqrt{a}\, x^{5/2}$$

$\Rightarrow$ squaring on both sides

$$25(y-c)^2 = \left(\pm 2\sqrt{a}\, x^{5/2}\right)^2 = 4ax^5.$$

$\boxed{25(y-c)^2 = 4ax^5}$ which is the required soln.

ii) $\left(\frac{dy}{dx}\right)^3 - ax^4 = 0$

$$\left(\frac{dy}{dx}\right)^3 = ax^4$$

$$\frac{dy}{dx} = a^{1/3} x^{4/3}$$

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$$dy = a^{1/3} x^{4/3} dx$$

$$\int dy = a^{1/3} \int x^{4/3} dx$$

$$y = a^{1/3} \left[\frac{x^{4/3+1}}{\frac{4}{3}+1} \right] + c$$

$$y = \frac{3}{7} a^{1/3} x^{7/3} + c$$

$$y - c = \frac{3}{7} a^{1/3} x^{7/3}$$

$$7(y-c) = 3 a^{1/3} x^{7/3}$$

cubing on bothsides

$$343(y-c)^3 = (3a^{1/3} x^{7/3})^3 = 27 a x^7.$$

(iii) $p^2 - 5p + 6 = 0$

Sol:- $(p-2)(p-3) = 0$

$p - 2 = 0$, $p - 3 = 0$

$p = 2$ $\qquad$ $p = 3$

$\frac{dy}{dx} = 2$ $\qquad$ $\frac{dy}{dx} = 3$

$dy = 2dx$ $\qquad$ $dy = 3dx$

$\int dy = 2\int dx$ $\qquad$ $\int dy = 3\int dx$

$\Rightarrow y = 2x + c$ $\qquad$ $y = 3x + c$

$y - 2x - c = 0$, $y - 3x - c = 0$

$\therefore$ The required soln is

$$\boxed{(y - 2x - c)(y - 3x - c) = 0}$$

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iv) $p^2 - 7p + 12 = 0$

$(p-3)(p-4) = 0$

$p = 3 \qquad p = 4$

$\frac{dy}{dx} = 3 \qquad \frac{dy}{dx} = 4$

$\int dy = 3\int dx \qquad \int dy = 4\int dx$

$y = 3x + c \qquad y = 4x + c$

$\Rightarrow y - 3x - c = 0, \quad y - 4x - c = 0$

$\therefore$ The combined soln is

$(y - 3x - c)(y - 4x - c) = 0$

Hw

v) $p^2 - 2p\cosh x + 1 = 0$

② Solve

i) $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$

Given eqn can be written as

$yp^2 + (x-y)p - x = 0$

$yp^2 + xp - yp - x = 0$

$yp^2 - yp + xp - x = 0$

$yp(p-1) + x(p-1) = 0$

$(yp + x)(p-1) = 0$

$p - 1 = 0 \qquad\qquad yp + x = 0$

$p = 1 \qquad\qquad y\frac{dy}{dx} + x = 0$

$\frac{dy}{dx} = 1 \qquad\qquad ydy + xdx = 0$

$dy = 1dx \qquad\qquad ydy = -xdx$

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$\int dy = \int 1 dx$

$y = x + c$

$y - x - c = 0$

$\int y dy = -\int x dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + c$

$\frac{x^2}{2} + \frac{y^2}{2} - c = 0$

$\Rightarrow x^2 + y^2 - 2c = 0$

$\therefore$ The Combined soln is

$(y - x - c)(x^2 + y^2 - 2c) = 0$

ii) Solve $x^2p^2 + xyp - 6y^2 = 0$

which is a quadratic equ in 'p'.

$x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

$\Rightarrow xp(xp + 3y) - 2y(xp + 3y) = 0$

$\Rightarrow (xp + 3y)(xp - 2y) = 0$

$\Rightarrow xp + 3y = 0$

$x\frac{dy}{dx} = -3y$

$\int \frac{1}{y} dy = \int \frac{-3}{x} dx$

$\Rightarrow \log y = -3\log x + \log c$

$\Rightarrow \log y + 3\log x = \log c$

$\Rightarrow yx^3 = c$

$\Rightarrow x^3y - c = 0$

$xp - 2y = 0$

$x\frac{dy}{dx} - 2y = 0$

$x\frac{dy}{dx} = 2y$

$\Rightarrow \int \frac{1}{y} dy = \int \frac{2}{x} dx$

$\Rightarrow \log y = 2\log x + \log c$

$\Rightarrow \log y - \log x^2 = \log c$

$\Rightarrow \log \frac{y}{x^2} = \log c$

$\Rightarrow \frac{y}{x^2} = c$

$\frac{y}{x^2} = c$

$\therefore$ The Combined soln is,

$(x^3y - c)\left(\frac{y}{x^2} - c\right) = 0$

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iii) $p(p+y) = x(x+y)$

$p^2 + py = x^2 + xy$

$\Rightarrow p^2 + py - x^2 - xy = 0$

$\Rightarrow p^2 - x^2 + y(p-x) = 0$

$\Rightarrow (p-x)(p+x) + y(p-x) = 0$

$\Rightarrow (p-x)(p+x+y) = 0$

$\Rightarrow p - x = 0$

$\Rightarrow \frac{dy}{dx} - x = 0$

$\Rightarrow dy = x\,dx$

$\Rightarrow y = \frac{x^2}{2} + c$

$\Rightarrow y - \frac{x^2}{2} - c = 0$

$p + y = -x$

$\frac{dy}{dx} + y = -x$

$I.F = e^{\int 1\,dx} = e^x$

$ye^x = -\int xe^x\,dx$

$ye^x = -e^x(x-1) + c$

$y = -(x-1) + ce^{-x}$

$y + x - 1 - ce^{-x} = 0$

$\therefore$ The combined soln is

$$\left(y - \frac{x^2}{2} - c\right)\left(y + x - 1 - ce^{-x}\right) = 0$$

(3) Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

Given eqn can be written as

$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$$

$$\frac{p^2 - 1}{p} = \frac{x}{y} - \frac{y}{x}$$

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$$p^2 - 1 = p\left(\frac{x}{y} - \frac{y}{x}\right)$$

$$p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$$

$$p = \frac{x}{y} \quad \text{and} \quad p = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y} \qquad \frac{dy}{dx} = \frac{-y}{x}$$

$$\int y\,dy = \int x\,dx \qquad \int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c \qquad \log y = -\log x + \log c$$

$$\log y + \log x = \log c$$

$$\frac{y^2}{2} - \frac{x^2}{2} - c = 0 \qquad xy = c$$

$\therefore$ The combined soln is

$$\left(\frac{y^2}{2} - \frac{x^2}{2} - c\right)(xy - c) = 0$$

$$p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$$

$$\Rightarrow p = \frac{-\left(\frac{y}{x} - \frac{x}{y}\right) \pm \sqrt{\left(\frac{x}{y} - \frac{y}{x}\right)^2 + 4\frac{y}{x}}}{2}$$

$$= \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \left(\frac{x}{y} + \frac{y}{x}\right)}{2}$$

$$= \frac{x}{y} \text{ (or) } -\frac{y}{x}$$

Hw

(4) i) Solve $p^2 + 2py\cot x = y^2$

ii) If the curve whose D.E. is $p^2 + 2py\cot x = y^2$ passes through $\left(\frac{\pi}{2}, 1\right)$

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Equations solvable for y

The general form of the solvable for y is given by

$$y = f(x, p) \quad \text{——} \quad (1)$$

Diff (1) w.r.to x

$$\frac{dy}{dx} = F\left(x, p, \frac{dp}{dx}\right)$$

$$p = F\left(x, p, \frac{dp}{dx}\right) \quad \text{——} \quad (2)$$

It is a D.E. of first order in two variables.

The soln of (2) is

$$\phi(x, p, c) = 0 \quad \text{——} \quad (3)$$

eliminating p from (1) & (3), we will get a relation between x, y and c which is the required soln of the given system.

Note: (1) In some cases, the elimination of p between (1) & (3) is not possible, then (1) and (3) together constitute the solution,

i.e., $x = F_1(p, c)$, $y = F_2(p, c)$

Note:- (2) In some cases, eq (2) can be expressed as $f_1(x, p) \cdot f_2\left(x, p, \frac{dp}{dx}\right) = 0$

In such cases, we ignore the first factor $f_1(x, p)$ which doesnot involve $\frac{dp}{dx}$ and proceed with $f_2\left(x, p, \frac{dp}{dx}\right) = 0$

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Problems

① Solve the following D.E.'s

(i) $y = (x-a)p - p^2$

Sol:- Given eqn is $y = (x-a)p - p^2$ —— ①

Diff ① w.r.to x

$$\frac{dy}{dx} = (x-a)\frac{dp}{dx} + p - 2p\frac{dp}{dx}$$

$$\cancel{p} = ((x-a) - 2p)\frac{dp}{dx} + \cancel{p}$$

$$\frac{dp}{dx}(x-a-2p) = 0$$

$$\Rightarrow \frac{dp}{dx} = 0$$

Integrating we get

$$p = c \quad \text{—— ②}$$

Eliminating p from ① and ②

$$\boxed{y = (x-a)c - c^2}$$

which is the required soln.

(ii) $y = (1+p)x + p^2$ —— ①

Diff ① w.r.to x

$$\frac{dy}{dx} = (1+p)\cdot 1 + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$\cancel{p} = 1 + \cancel{p} + (x+2p)\frac{dp}{dx}$$

$$\Rightarrow (x+2p)\frac{dp}{dx} = -1$$

$$\frac{dp}{dx} = \frac{-1}{x+2p}$$

$$\frac{dx}{dp} = -x - 2p$$

$$\frac{dx}{dp} + x = -2p$$

which is a linear eqn in x.

Here $P = 1$, $Q = -2p$

$$I.F = e^{\int dp} = e^p$$

The soln is

$$x \times e^p = \int (-2p)e^p\,dp$$

$$xe^p = -2e^p(p-1) + c$$

$$x = 2(1-p) + ce^{-p} \quad \text{—— ②}$$

sub ② in ①

$$\boxed{y = (1+p)(2(1-p) + ce^{-p}) + p^2}$$

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iii) $y = x + a\tan^{-1}p$ —— ①

Diff ① w.r.to x

$$\frac{dy}{dx} = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{a}{1+p^2}\frac{dp}{dx} = p-1$$

$$\Rightarrow \frac{dp}{dx} = \frac{(p-1)(1+p^2)}{a}$$

$$\Rightarrow \frac{dx}{a} = \frac{1}{(p-1)(p^2+1)}dp$$

$$\frac{1}{(p-1)(p^2+1)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+1}$$

$A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}$

$$\frac{1}{(p-1)(p^2+1)} = \frac{1}{2}\left[\frac{1}{p-1} - \frac{p}{p^2+1} - \frac{1}{p^2+1}\right]$$

$$\Rightarrow dx = \frac{a}{2}\left[\frac{1}{p-1} - \frac{p}{p^2+1} - \frac{1}{p^2+1}\right]dp$$

$$\Rightarrow \int dx = \frac{a}{2}\int\left(\frac{1}{p-1} - \frac{p}{p^2+1} - \frac{1}{p^2+1}\right)dp$$

$$x = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1}p + \log c\right]$$

$$x = \frac{a}{2}\left[\log\left[\frac{c(p-1)}{\sqrt{p^2+1}}\right] - \tan^{-1}p\right] \quad \text{—— ②}$$

sub ② in ①

$$y = \frac{a}{2}\left[\log\left[\frac{c(p-1)}{\sqrt{p^2+1}}\right] - \tan^{-1}p\right] + a\tan^{-1}p$$

iv) $y = 2px - p^2$ —— ①

Diff ① w.r.to x

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$p - 2p = (2x - 2p)\frac{dp}{dx}$$

$$\cancel{2(x-p)\frac{dp}{dx} + p = 0}$$

$$\frac{-p}{2(x-p)} = \frac{dp}{dx}$$

$$\frac{dx}{dp} = \frac{-2(x-p)}{p}$$

$$= -\frac{2}{p}x + \frac{2\cancel{p}}{\cancel{p}}$$

$$\frac{dx}{dp} + \frac{2}{p}x = 2 \quad \text{—— ②}$$

which is a linear D.E.

$P(x) = \frac{2}{p}$, $Q(x) = 2$

$$I.F = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$

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The soln is

$$x \cdot p^2 = \int 2p^2 dp$$

$$xp^2 = \frac{2p^3}{3} + c$$

$$x = \frac{2p^3}{3p^2} + \frac{c}{p^2}$$

$$x = \frac{2p}{3} + \frac{c}{p^2} \quad \text{—— (3)}$$

sub (3) in (1)

$$y = 2p\left[\frac{2p}{3} + \frac{c}{p^2}\right] - p^2$$

$$\boxed{y = \frac{4}{3}p^2 + \frac{2c}{p} - p^2}$$

(2) Solve the following D.E's

i) $y + px = x^4 p^2$

Sol:- Given eqn can be written as

$$y = -px + x^4 p^2 \quad \text{—— (1)}$$

Diff (1) w.r.to x

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + x^4 2p\frac{dp}{dx} + p^2 4x^3$$

$$p + p + x\frac{dp}{dx} = 2px^3\left(x\frac{dp}{dx} + 2p\right)$$

$$\left(2p + x\frac{dp}{dx}\right) - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\left(2p + x\frac{dp}{dx}\right)\left(1 - 2px^3\right) = 0$$

Neglecting the factor, which does not involve $\frac{dp}{dx}$, we have

$$2p + x\frac{dp}{dx} = 0$$

$$x\frac{dp}{dx} = -2p$$

$$-\frac{1}{2p}dp = \frac{1}{x}dx$$

$$\cancel{-\frac{1}{2}\log p = \log x + \log c = \log cx}$$

$$\frac{1}{p}dp = -\frac{2}{x}dx$$

$$\log p = -2\log x + \log c = \log\frac{c}{x^2}$$

$$\left(p = c/x^2\right) \quad \text{—— (2)}$$

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sub ② in ①

$$y + \frac{c}{x^{\cancel{2}}}\cancel{x} = \cancel{x^4}\cdot\frac{c^2}{\cancel{x^4}}$$

$$\boxed{xy + c = c^2 x}$$

ii) $y = 2px + p^4x^2$ — ①

Diff ① w.r.to x

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + p^4\cdot 2x + x^2 4p^3\frac{dp}{dx}$$

$$p - 2p - 2xp^4 = (2x + 4x^2p^3)\frac{dp}{dx}$$

$$-p - 2xp^4 = 2x(1 + 2xp^3)\frac{dp}{dx}$$

$$-p(1 + 2xp^3) = 2x(1 + 2xp^3)\frac{dp}{dx}$$

$$(1 + 2p^3x)\left(p + 2x\frac{dp}{dx}\right) = 0$$

$$p + 2x\frac{dp}{dx} = 0$$

$$\Rightarrow 2x\frac{dp}{dx} = -p$$

$$\Rightarrow \int \frac{1}{p}\,dp = -\int \frac{1}{2x}\,dx$$

$$\log p = -\frac{1}{2}\log x + \log c$$

$$\cancel{2\log p} =$$

$$p = \frac{c}{\sqrt{x}}$$

$$p = \frac{c}{\sqrt{x}} \quad \text{— ②}$$

sub ② in ①

$$\boxed{y = 2\frac{c}{\sqrt{x}}x + \left(\frac{c}{\sqrt{x}}\right)^4 x^2}$$

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③ i) $y = p\tan p + \log\cos p$ where $p = \frac{dy}{dx}$

Sol:- Given. $y = p\tan p + \log\cos p$ —— ①

Diff ① w.r.to x

$$\frac{dy}{dx} = p\sec^2 p\frac{dp}{dx} + \tan p\frac{dp}{dx} + \frac{1}{\cos p}(-\sin p)\frac{dp}{dx}$$

$$\cancel{p} = \cancel{p}\sec^2 p\frac{dp}{dx} + \cancel{\tan p\frac{dp}{dx}} - \cancel{\tan p\frac{dp}{dx}}$$

$$\sec^2 p\frac{dp}{dx} = 1$$

$$\Rightarrow dx = \sec^2 p\,dp$$

$$\int dx = \int \sec^2 p\,dp$$

$$x = \tan p + c \quad \text{—— ②}$$

eq ① & ② together form the required soln in parametric form.

ii) $y = p\sin p + \cos p$ —— ①

Diff ① w.r.to x

$$\frac{dy}{dx} = \cancel{p\cos p + \sin p\frac{dp}{dx}}\ p\cos p\frac{dp}{dx} + \cancel{\sin p\frac{dp}{dx}} - \cancel{\sin p\frac{dp}{dx}}$$

$$\cancel{p} = \cancel{p}\cos p\frac{dp}{dx}$$

$$\cos p\frac{dp}{dx} = 1$$

$$\Rightarrow \int \cos p\,dp = \int 1\,dx$$

$$\sin p = x + c$$

$$x = \sin p + c \quad \text{—— ②}$$

eq ① & ② together form the required soln in parametric form.

H.W

① $y = \sin p - p\cos p$

② $4y = x^2 + p^2$

③ $yp^2 - 2xp + y = 0$

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Equations solvable for x

The general form of the solvable for x is given by

$$x = f(y, p) \quad \text{--- (1)}$$

Diff w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$\frac{dx}{dy} = F\left(y, p, \frac{dp}{dy}\right)$$

$$\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right)$$

The soln is

$$\phi(y, p, c) = 0 \quad \text{--- (2)}$$

Eliminating p between (1) & (2), we get the required soln of (1)

i.e; $\phi(x, y, c) = 0$

Note: (1) If the elimination of p between (1) & (2) is not possible, then we solve (1) & (2) to express x and y in terms of p and c.

i.e; $x = f_1(p, c)$, $y = f_2(p, c)$ --- (3)

These two equations together will be the soln of (1)

Note: (2):- Sometime the soln (3) is also not possible, in this case (1) and (2) together constitute the soln giving x and y in terms of p.

Note: (3):- In some problems eq $F\left(y, p, \frac{dp}{dy}\right)$ can be expressed as

$$F_1(y, p)\, F_2\left(y, p, \frac{dp}{dy}\right) = 0$$

In such case we ignore the first factor $F_1(y, p)$ which does not involve $\frac{dp}{dy}$ and proceed with $F_2\left(y, p, \frac{dp}{dy}\right) = 0$

Note: (4):- If instead of ignoring the factor $F_1(y, p)$, we eliminate p between (1) and $F_1(y, p) = 0$ we obtain an equation involving no const 'c'. This is known as singular soln of (1)

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problems

(1) Solve the following D.E's

i) $x = 3y - \log p$

Sol:- Given $x = 3y - \log p$ — (1)

Diff (1) w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$\frac{dx}{dy} = 3 - \frac{1}{p}\frac{dp}{dy}$$

$$\frac{1}{p} - 3 = -\frac{1}{p}\frac{dp}{dy}$$

$$\frac{1-3p}{\cancel{p}} = \frac{-1}{\cancel{p}}\frac{dp}{dy}$$

$$\Rightarrow dy = \frac{1}{3p-1}dp$$

$$\int dy = \int \frac{1}{3p-1}dp$$

$$y = \frac{1}{3}\log(3p-1) + \log c$$

$$y = \log(3p-1)^{1/3}c.$$

$$e^y = c(3p-1)^{1/3}$$

$$(3p-1)^{1/3} = ce^{y}$$

$$3p-1 = ce^{3y}$$

$$p = \frac{1+ce^{3y}}{3} \quad \text{—— (2)}$$

eliminating p from (1) & (2)

$$\boxed{x = 3y - \log\left(\frac{1+ce^{3y}}{3}\right)}$$

which is the required soln.

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iii) $x = 4(p + p^3)$ —— (1)

Diff (1) w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$\frac{dx}{dy} = 4\left(\frac{dp}{dy} + 3p^2\frac{dp}{dy}\right)$$

$$\frac{1}{p} = 4\left(\frac{dp}{dy} + 3p^2\frac{dp}{dy}\right)$$

$$\frac{1}{p} = 4(1+3p^2)\frac{dp}{dy}$$

$$dy = 4p(1+3p^2)\,dp$$

$$\int dy = \int 4p\,dp + 12\int p^3\,dp$$

$$y = \frac{\cancel{4}^{2}p^2}{\cancel{2}} + \frac{\cancel{12}^{3}p^4}{\cancel{4}} + c$$

$$y = 2p^2 + 3p^4 + c \quad \text{—— (2)}$$

Both (1) & (2) constitutes the soln of (1).

Ho

(4) $p^3 - 4xyp + 8y^2 = 0$

(2) Solve the following D.E's

i) $y = 2px + p^3y^2$

Given eqn can be written as

$$2px = y - p^3y^2$$

$$2x = \frac{y}{p} - p^2y^2 \quad \text{—— (1)}$$

Diff (1) w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$2\frac{dx}{dy} = \frac{p.1 - y\frac{dp}{dy}}{p^2} - \left(p^2 2y + y^2 2p\frac{dp}{dy}\right)$$

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2p^2y - 2y^2p\frac{dp}{dy}$$

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$$\frac{2}{p} - \frac{1}{p} + 2p^2y = -\frac{dp}{dy}\left(\frac{y}{p^2} + 2y^2p\right)$$

$$\left(\frac{1}{p} + 2p^2y\right) + \frac{y}{p}\left(\frac{1}{p} + 2p^2y\right)\frac{dp}{dy} = 0$$

$$\left(\frac{1}{p} + 2p^2y\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

We can neglect first factor, because it is not involving $\frac{dp}{dy}$, so

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

$$\frac{y}{p}\frac{dp}{dy} = -1$$

$$\frac{dp}{dy} = \frac{-p}{y}$$

$$\int \frac{dp}{p} = -\int \frac{1}{y}\,dy$$

$$\log p = -\log y + \log c$$

$$\log p = \log \frac{c}{y}$$

$$p = \frac{c}{y} \quad \text{—} \quad (2)$$

Sub (2) in given eqn

$$y = 2\left(\frac{c}{y}\right)x + \left(\frac{c^3}{y^3}\right)y^2$$

$$y = \frac{2cx}{y} + \frac{c^3}{y}$$

$$\boxed{y^2 = 2cx + c^3}$$

ii) $x = \frac{y\log y}{p} - \frac{p}{y}$ —— (1)

Diff (1) w.r.to y and write $\frac{dx}{dy}$ with $\frac{1}{p}$

$$\frac{dx}{dy} = \frac{p\left(y\cdot\frac{1}{y} + \log y\cdot 1\right) - y\log y\frac{dp}{dy}}{p^2} - \frac{y\frac{dp}{dy} - p\cdot 1}{y^2}$$

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$$= \frac{p(1+\log y) - y\log y \frac{dp}{dy}}{p^2} - \frac{y\frac{dp}{dy} - p}{y^2}$$

$$\frac{1}{p} = \frac{1}{p}(1+\log y) - \frac{y}{p^2}\log y \frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\cancel{\frac{1}{p}} = \cancel{\frac{1}{p}} + \frac{1}{p}\log y - \frac{y}{p^2}\log y \frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\Rightarrow \frac{1}{p}\log y + \frac{p}{y^2} = \left(\frac{y}{p^2}\log y + \frac{1}{y}\right)\frac{dp}{dy}$$

$$\Rightarrow \frac{p}{y^2}\left(\frac{y^2}{p^2}\log y + 1\right) = \frac{1}{y}\left(\frac{y^2}{p^2}\log y + 1\right)\frac{dp}{dy}$$

$$\left(\frac{y^2}{p^2}\log y + 1\right)\left(\frac{p}{y} - (1)\frac{dp}{dy}\right) = 0$$

Neglecting first factor

$$\frac{p}{y} - \frac{dp}{dy} = 0$$

$$\Rightarrow \frac{p}{y} = \frac{dp}{dy}$$

$$\Rightarrow \int \frac{dp}{p} = \int \frac{1}{y}\,dy$$

$$\Rightarrow \log p = \log y + \log c$$

$$p = cy \quad \text{—— (2)}$$

sub (2) in (1)

$$x = \frac{\cancel{y}\log y}{c\cancel{y}} - \frac{c\cancel{y}}{\cancel{y}}$$

$$cx = \log y - c^2$$

$$\boxed{\log y = cx + c^2}$$

iii) $p = \tan\left(x - \frac{p}{1+p^2}\right)$

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Clairaut's equation

An equation of the form $y = px + f(p)$ is known as Clairaut's form. The general soln is obtained by replacing "p" with "c"

procedure for singular soln

→ Given eqn is of the form $y = px + f(p)$ —— (1)

→ The general soln of clairaut's equation is obtained by replacing p with c

ie; $y = cx + f(c)$ —— (1)

Diff (1) w.r.to x $\frac{dy}{dx} = p.1 + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$p = p + \frac{dp}{dx}(x + f'(p))$

$\Rightarrow \frac{dp}{dx}(x + f'(p)) = 0$

$\Rightarrow \frac{dp}{dx} = 0 \Rightarrow x = c$ $\quad x + f'(p) = 0$

→ ~~Diff (1) w.r.to c~~

$x + f'(c) = 0$ —— (2)

Eliminate 'c' from (1) & (2) which will gives the singular soln.

problems

(1) Solve the following D.E's

i) $y = px + p^n$ —— (1)

~~The general~~ eq (1) is the clairaut's equation.

The generical soln is obtained replacing 'p' with c

$y = cx + c^n$

ii) $xp^2 - yp + 2 = 0$

Given eqn can be written as

$yp = xp^2 + 2$

$y = \frac{xp^2}{p} + \frac{2}{p}$

$y = xp + \frac{2}{p}$

soln is

$\boxed{y = xc + \frac{2}{c}}$

iii) $y = px + \log p$

$y = cx + \log c$

iv) $p = \log(px - y)$

$e^p = px - y$

$y = px - e^p$

$\boxed{y = cx - e^c}$

v) $p = \tan(px - y)$

$px - y = \tan^{-1}p$

$y = px - \tan^{-1}p$

$\boxed{y = cx - \tan^{-1}c}$

vi) $(y - px)(p - 1) = p$

$y - px = \frac{p}{1-p}$

$y = px + \frac{p}{1-p}$

$\boxed{y = cx + \frac{c}{1-c}}$

vii) $\cos y \cos px + \sin y \sin px = p$

$\cos(y - px) = p$

$y - px = \cos^{-1}p$

$y = px + \cos^{-1}p$

$\boxed{y = cx + \cos^{-1}c}$

viii) $\sin px \cos y = \cos px \sin y + p$ (Hw)

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② Solve $y = px + p^2$ — ① and obtain singular soln

eq ① is a clairaut's form

soln of ① is

$y = cx + c^2$ — ②

Diff ② w.r.to x

$\frac{dy}{dx} = p \cdot 1 + x\frac{dp}{dx} + 2p\frac{dp}{dx}$

Taking $\frac{dy}{dc} = 0$

$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$

$\Rightarrow x\frac{dp}{dx} + 2p\frac{dp}{dx} = 0$

$\Rightarrow (x + 2p)\frac{dp}{dx} = 0$

Taking $x + 2p = 0$ & $\frac{dp}{dx} = 0$

$p = -\frac{x}{2}$ $\quad$ $p = c$

substituting $p = -\frac{x}{2}$ in ①

$y = -\frac{x}{2} \cdot x + \left(-\frac{x}{2}\right)^2$

$y = -\frac{x^2}{2} + \frac{x^2}{4} = -\frac{x^2}{4}$

$\boxed{x^2 + 4y = 0}$

which is the required singular soln.

③ Find the general and singular of

i) $\sin(px - y) = p$

$px - y = \sin^{-1} p$

$y = px - \sin^{-1} p$ — ①

Diff ① w.r.to x

$\frac{dy}{dx} = p \cdot 1 + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$

$p = p + \left(x - \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$

$\Rightarrow \left(x - \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$

$\Rightarrow x - \frac{1}{\sqrt{1-p^2}} = 0$ & $\frac{dp}{dx} = 0$

$x = \frac{1}{\sqrt{1-p^2}}$ $\quad$ $\frac{dp}{dx} = 0$

$p = c$

sub $p = c$ in ①

$y = cx + c^2$

which is general soln

$\sqrt{1-p^2} = \frac{1}{x}$

$1 - p^2 = \frac{1}{x^2}$

$p^2 = 1 - \frac{1}{x^2}$

$p = \sqrt{1 - \frac{1}{x^2}}$

substituting p in ①

$y = \sqrt{1 - \frac{1}{x^2}} \cdot x - \sin^{-1}\left(\sqrt{1 - \frac{1}{x^2}}\right)$

$= \frac{\sqrt{x^2-1}}{x} \cdot x - \sin^{-1}\left(\sqrt{1 - \frac{1}{x^2}}\right)$

$y = \sqrt{x^2 - 1} - \sin^{-1}\left(\sqrt{1 - \frac{1}{x^2}}\right)$

which is required singular soln.

HW

ii) $y = px - \sqrt{1 + p^2}$

HW

iii) $y = px + p - p^2$

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UNIT-II

ODE of Higher order

* Second order Linear D.E. with constant coefficients:
  Non homogeneous terms of the type e^{ax}, $\sin ax$, $\cos ax$, polynomials in x, $e^{ax}V(x)$ and $xV(x)$,
* Method of variation of parameters.
* Equations reducible to linear ODE with constant coefficients: Legendre's equation, Cauchy-Euler equation,

Second order Linear D.E. with constant coefficients

The general form of second order linear D.E. is

$$f(D)y = Q(x) \quad \text{—— (1)}$$

where $f(D)$ be the D.E. of second order

If $Q(x) = 0$ then $f(D)y = 0$ is called homogeneous Linear D.E.

If $Q(x) \neq 0$ then $f(D)y = Q(x)$ is called non homogeneous Linear D.E.

The complete solution of eq (1) is

$$y = C.F. + P.I.$$

where C.F = Complementary Function
P.I. = Particular Integral.

(OR)

$$y = y_c + y_p$$

y_c = complementary soln
y_p = particular soln

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To find the Complementary Function

$f(D)y = 0$ be the given equation — (1)

Auxillary Equation of (1) is

$f(m) = 0$

Case (i)

If the roots are real and distinct

ie; $m = a, b, c, d$ then

Complementary function is

$$y_c = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx} + c_4 e^{dx}$$

Case (ii)

If the roots are real and repeated.

ie; $m = a, a, b, c$ then

Complementary function is

$$y_c = (c_1 + c_2 x) e^{ax} + c_3 e^{bx} + c_4 e^{cx}$$

Case (iii)

If the roots are real and repeated

ie; $m = a, a, a, b$ then

Complementary function is

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{ax} + c_4 e^{bx}$$

Case (iv)

If the roots are imaginary

ie; $m = \pm bi, \pm di$ then

$$y_c = c_1 \cos bx + c_2 \sin bx + c_3 \cos dx + c_4 \sin dx$$

Case (v) If the roots are complex

ie; $m = a \pm bi \quad c \pm di$ then

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$$y_c = e^{ax}\left(c_1\cos bx + c_2\sin bx\right) + e^{cx}\left(c_3\cos dx + c_4\sin dx\right)$$

Case (vi)

If the roots are complex & repeated

$m = a \pm bi,\ a \pm bi$ then

$$y_c = e^{ax}\left((c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx\right)$$

Problems

1. Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

The given eqn can be written as

$$(D^2 + 5D + 6)y = 0$$

which is of the form $f(D)y = 0$

A.E. is $f(m) = 0$

$\Rightarrow m^2 + 5m + 6 = 0$

$\Rightarrow (m+2)(m+3) = 0$

$\Rightarrow m = -2, -3$

Here the roots are real and distinct.

The solution is

$$\boxed{y_c = c_1 e^{-2x} + c_2 e^{-3x}}$$

2. Solve $(D^2 + 6D + 9)y = 0$

A.E is $f(m) = 0$

$\Rightarrow m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)(m+3) = 0$

$\Rightarrow m = -3, -3$

Here the roots are repeated

The solution is $\boxed{y_c = (c_1 + c_2 x)e^{-3x}}$

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③ Solve $(D^2+4)y=0$

A.E. is $f(m)=0$

$\Rightarrow m^2+4=0$

$\Rightarrow m=\pm 2i$

Roots are Imaginary

The soln is

$$\boxed{y_c = c_1\cos 2x + c_2 \sin 2x}$$

④ Solve $(D^2+2D+2)y=0$

A.E. is $f(m)=0$

$\Rightarrow m^2+2m+2=0$

$\Rightarrow m = \frac{-2\pm 2i}{2}$

$\Rightarrow m = -1\pm i$

roots are imaginary & complex

$$\boxed{y_c = e^{-x}(c_1\cos x + c_2 \sin x)}$$

⑤ Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

Sol:- Given eqn can be written as

$(D^2+D+1)y=0$

A.E. is $f(m)=0$

$\Rightarrow m^2+m+1=0$

$\Rightarrow m = \frac{-1\pm\sqrt{1-4}}{2}$

$m = \frac{-1\pm i\sqrt{3}}{2}$

$\Rightarrow m = -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

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The roots are complex and conjugate.

The general soln is

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

(6) Solve $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = 1$

Sol:- Given eqn can be written as $(D^2 + D - 2)y = 0$

A.E is $f(m) = 0$

$\Rightarrow m^2 + m - 2 = 0$

$\Rightarrow (m-1)(m+2) = 0$

$\Rightarrow m = 1, -2$

The soln is $y = c_1 e^{x} + c_2 e^{-2x}$ —— (1)

Given $y(0) = 4$

$\Rightarrow y(0) = c_1 e^{0} + c_2 e^{-2(0)} = 4$

$\Rightarrow c_1 + c_2 = 4$ —— (2)

Diff (1) w.r.to x

$y' = c_1 e^{x} - 2c_2 e^{-2x}$

$y'(0) = c_1 e^{0} - 2c_2 e^{-2(0)} = 1$

$\Rightarrow c_1 - 2c_2 = 1$ —— (3)

From (2) & (3)

$\Rightarrow c_1 = 3,\ c_2 = 1$

The general soln is

$$y = 3e^{x} + 1.e^{-2x}$$

HW (7) Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$ $m = -1, -1, 2, 2$

$y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}$

HW (8) Solve $(D^4 + 8D^2 + 16)y = 0$ $m = 2i, 2i, -2i, -2i$

$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

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To find the soln of $f(D)y = Q(x)$

Inverse operator

The operator D^{-1} is called the inverse of the differential operator D.

$f(D)y = Q(x)$

$$P.I. \text{ of } y = \frac{1}{f(D)} Q(x)$$

If $f(D) = D - \alpha$

$$y_p = \frac{1}{D-\alpha} Q(x) = e^{\alpha x} \int Q(x)\, e^{-\alpha x} dx$$

If $f(D) = D + \alpha$

$$y_p = \frac{1}{D+\alpha} Q(x) = e^{-\alpha x} \int Q(x)\, e^{\alpha x} dx$$

problems

① Find i) $\frac{1}{D} x^2$ ii) $\frac{1}{D^3} \cos x$

i) $\frac{1}{D} x^2 = \frac{1}{(D-0)} x^2 = e^{-0.x} \int x^2 e^{0.x} dx = \int x^2 dx = \frac{x^3}{3}$

ii) $\frac{1}{D^3} \cos x = \frac{1}{D^2}\left(\frac{1}{D} \cos x\right) = \frac{1}{D^2}(\sin x) = \frac{1}{D}\left(\frac{1}{D} \sin x\right) = \frac{1}{D}(-\cos x)$

$= -\frac{1}{D}(\cos x)$

$= -\sin x$

② Solve $(D^2 - 5D + 6)y = xe^{4x}$

Given eqn is of the form $f(D)y = Q(x)$

A.E. is $f(m) = 0$

$\Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3$

$y_c = c_1 e^{2x} + c_2 e^{3x}$

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$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D^2-5D+6)} xe^{4x}$$

D

$$= \frac{1}{(D-2)(D-3)} xe^{4x}$$

$$= \left[\frac{1}{D-3} - \frac{1}{D-2}\right] xe^{4x}$$

$$= \frac{1}{D-3} xe^{4x} - \frac{1}{D-2} xe^{4x}$$

$$= e^{3x}\int xe^{4x} \cdot e^{-3x} dx - e^{2x}\int xe^{4x} \cdot e^{-2x} dx$$

$$= e^{3x}\int xe^{x} dx - e^{2x}\int xe^{2x} dx$$

$$= e^{3x}\left[xe^{x} - e^{x}\right] - e^{2x}\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]$$

$$y_p = e^{4x}\frac{(2x-3)}{4}$$

The general soln is $y = y_c + y_p$

$$\boxed{y = c_1 e^{2x} + c_2 e^{3x} + e^{4x}\left(\frac{2x-3}{4}\right)}$$

(3) Hw Solve $(D^2+4)y = \tan 2x$

(4) Hw Solve $(D^2+5D+6)y = e^{x}$

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I. of $f(D)y = Q(x)$ when $Q(x) = e^{ax}$

$$f(D)y = e^{ax}$$

$$y_p = \frac{1}{f(D)} e^{ax}$$

put $D = a$

$$y_p = \frac{1}{f(a)} e^{ax} \quad \text{if } f(a) \neq 0$$

If $f(D) = (D-a)^k$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^k} e^{ax} = \frac{e^{ax} x^k}{k!} \quad \text{if } f(a) = 0$$

problems

① Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$

Sol:- Given eqn can be written as

$$(D^2 + 4D + 3)y = e^{2x} \quad \text{—— ①}$$

A.E. of ① is $f(m) = 0$

$$\Rightarrow m^2 + 4m + 3 = 0$$

$$\Rightarrow (m+1)(m+3) = 0$$

$$\Rightarrow m = -1, -3$$

Complementary soln is

$$\boxed{y_c = c_1 e^{-x} + c_2 e^{-3x}}$$

To find y_p

$$y_p = \frac{1}{f(D)} Q(x)$$

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$$= \frac{1}{(D^2+4D+3)} e^{2x}$$

put $D = 2$

$$= \frac{1}{4+8+3} e^{2x}$$

$$= \frac{e^{2x}}{15}$$

$\therefore$ The general soln is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-3x} + \frac{e^{2x}}{15}}$$

② Solve $(D^2+2D+1)y = e^{-x}$ —— ①

Sol:- A.E. of ① is $f(m) = 0$

$\Rightarrow m^2+2m+1 = 0$

$\Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$

$$y_c = (c_1 + c_2 x) e^{-x}$$

To find y_p

$$y_p = \frac{1}{D^2+2D+1} e^{-x}$$

$$= \frac{1}{(D+1)^2} e^{-x}$$

$f(D) = (D+1)^2$

$f(-1) = (-1+1)^2 = 0$

Here $f(D) = 0$

Since $f(a) = 0$

$$y_p = e^{-x} \cdot \frac{x^2}{2!}$$

$\therefore$ The general soln is, $y = y_c + y_p$

$$\boxed{y = (c_1 + c_2 x) e^{-x} + \frac{x^2 e^{-x}}{2}}$$

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③ Solve $(D^2+6D+9)y = 2e^{-3x}$ — ①

Sol: A.E. of ① is $f(m)=0$

$\Rightarrow m^2+6m+9=0$

$\Rightarrow (m+3)^2=0$

$\Rightarrow m=-3,-3.$

$y_c = (c_1+c_2x)e^{-3x}$

To find y_p

$y_p = \frac{1}{D^2+6D+9} 2e^{-3x}$

$= 2\left[\frac{e^{-3x}}{(D+3)^2}\right]$

Since $f(-3)=0$

$y_p = \cancel{2}\, e^{-3x} \cdot \frac{x^2}{\cancel{2!}}$

$y_p = x^2 e^{-3x}$

∴ The general soln is

$y = y_c + y_p$

$$\boxed{y = (c_1+c_2x)e^{-3x} + x^2e^{-3x}}$$

④ Solve $(D^3-1)y = (e^x+1)^2$ — ①

A.E. of ① is $f(m)=0$

$\Rightarrow m^3-1=0$

$\Rightarrow m = 1, \frac{-1\pm i\sqrt{3}}{2}$

$y_c = c_1e^x + e^{-\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$

$y_p = \frac{1}{D^3-1}(e^x+1)^2$

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$$= \frac{1}{D^3-1}\left(e^{2x}+2e^{x}+1\right)$$

$$= \frac{e^{2x}}{D^3-1} + \frac{2\,e^{x}}{D^3-1} + \frac{e^{0\cdot x}}{D^3-1}$$

$$= \frac{e^{2x}}{8-1} + \frac{2\,e^{x}}{(D-1)(D^2+D+1)} + \frac{1}{-1}$$

$$= \frac{e^{2x}}{7} - 1 + \frac{2}{3}\cdot\frac{1}{D-1}e^{x}$$

$$y_p = \frac{e^{2x}}{7} - 1 + \frac{2}{3}\frac{x\,e^{x}}{1!}$$

The general soln is

$$y = y_c + y_p$$

$$y = c_1 e^{x} + e^{-\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] + \frac{e^{2x}}{7} - 1 + \frac{2}{3}xe^{x}$$

HW

(5) Solve $(D^3-3D^2+4)y = (1+e^{-x})^3$

HW

(6) Solve $(D^3+3D^2-4)y = \sinh 2x + 7$

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To find $f(D)y = Q(x)$ when $Q(x) = \sin ax$ (or) $\cos ax$

$$P.I. = y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} \sin ax$$

$$= \frac{1}{\phi(D^2)} \sin ax \quad (\because \text{Convert } f(D) \text{ into } \phi(D^2))$$

put $D^2 = -a^2$

$$= \frac{1}{\phi(-a^2)} \sin ax \quad (\because \phi(-a^2) \neq 0)$$

If $\phi(-a^2) = 0$ then

$$y_p = \frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$$

(or)

$$y_p = \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$$

problems

(1) Solve $(D^2+3D+2)y = \sin 3x$ (1)

Sol:- Comparing with $f(D)y = Q(x)$

$f(D) = D^2+3D+2 \qquad Q(x) = \sin 3x$

A.E. of (1) is

$f(m) = 0$

$\Rightarrow m^2+3m+2 = 0$

$\Rightarrow m = -1, -2$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_p = \frac{1}{f(D)} \sin 3x$$

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$$= \frac{1}{D^2+3D+2} \sin 3x$$

put $D^2 = -3^2 = -9$

$$= \frac{1}{-9+3D+2} \sin 3x$$

$$= \frac{1}{3D-7} \sin 3x$$

$$= \frac{(3D+7)}{9D^2-49} \sin 3x$$

$$= \frac{(3D+7)}{-81-49} \sin 3x$$

$$y_p = \frac{-1}{130}\left[9\cos 3x + 7\sin 3x\right]$$

The general soln is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{130}\left[9\cos 3x + 7\sin 3x\right]}$$

② Solve $(D^2-4)y = 2\cos^2 x$ — ①

Sol:- $f(D)y = Q(x)$

A.E. of ① is $f(m) = 0 \Rightarrow m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

$$y_c = c_1 e^{-2x} + c_2 e^{2x}$$

To find y_p

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2-4}(1+\cos 2x)$$

$$= \frac{e^{0x}}{D^2-4} + \frac{\cos 2x}{D^2-4}$$

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$$= \frac{e^{0.x}}{0-4} + \frac{\cos 2x}{-4-4}$$

$$= -\frac{1}{4} - \frac{1}{8}\cos 2x$$

$\therefore$ The general soln is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{4} - \frac{\cos 2x}{8}}$$

③ Solve $(D^2+1)y = \sin x \sin 2x$ — ①

A.E of ① is $f(m) = 0$

$$m^2+1 = 0 \Rightarrow m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

<u>To find y_p</u>

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2+1} \sin x \sin 2x$$

$$= \frac{1}{2} \frac{1}{D^2+1} 2\sin x \sin 2x$$

$$= \frac{1}{2} \frac{\cos x - \cos 3x}{D^2+1}$$

$$= \frac{1}{2} \frac{x}{2} \sin x - \frac{1}{2} \frac{\cos 3x}{-9+1}$$

$$y_p = \frac{x}{4} \sin x + \frac{1}{16} \cos 3x$$

The general soln is

$$y = y_c + y_p$$

$$\boxed{y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x + \frac{1}{16} \cos 3x}$$

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Solve $(D^2+9)y = \cos 3x + \sin 2x$ — (1)

A.E. of (1) is $f(m) = 0$

$\Rightarrow m^2+9 = 0$

$\Rightarrow m = \pm 3i$

$y_c = c_1 \cos 3x + c_2 \sin 3x$

To find y_p

$$y_p = \frac{1}{D^2+9}(\cos 3x + \sin 2x)$$

$$= \frac{\cos 3x}{D^2+9} + \frac{\sin 2x}{D^2+9}$$

$$= \frac{x}{2(3)} \sin 3x + \frac{\sin 2x}{-4+9}$$

$$y_p = \frac{x}{6} \sin 3x + \frac{1}{5} \sin 2x$$

The general soln is

$$y = y_c + y_p$$

$$= (c_1 \cos 3x + c_2 \sin 3x) + \frac{x}{6} \sin 3x + \frac{1}{5} \sin 2x$$

HW

(5) Solve $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = \cos x$

HW

(6) Solve $y'' + 4y' + 20y = 23 \sin t - 15 \cos t$, $y(0) = 0$, $y'(0) = -1$.

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To find P.I. of $f(D)y = Q(x)$ where $Q(x) = x^K$

$$y_p = \frac{1}{f(D)} x^K$$

Now write $\frac{1}{f(D)}$ as $[1 \pm \phi(D)]^{-1}$ and expand its ascending powers of D using Binomial theorem upto the term containing D^K.

Important formulae's

$$\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$$

$$\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$$

$$\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$$

$$\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$$

$$\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots$$

$$\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots$$

problems

① Solve $(D^3 - 3D - 2)y = x^2$ —— ①

Sol:- A.E. of ① is $f(m) = 0$

$\Rightarrow m^3 - 3m - 2 = 0$

$\Rightarrow m = 1, 1, 2$

$$y_c = (c_1 + c_2 x)e^x + c_3 e^{2x}$$

$$y_p = \frac{1}{D^3 - 3D - 2} x^2$$

$$= \frac{1}{-2\left[1 - \left(\frac{D^3 - 3D}{2}\right)\right]} x^2$$

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$$= -\frac{1}{2}\left[1 + \frac{D^3-3D}{2} + \left(\frac{D^3-3D}{2}\right)^2 + \cdots\right]x^2$$

power

$$y_p = -\frac{1}{2}\left[x^2 - 3x + 18\right]$$

$$\therefore y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^x + c_3 e^{2x} - \frac{1}{2}(x^2 - 3x + 18)$$

2) Solve $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x + x$ — (1)

Sol:- A.E. of (1) is $f(m) = 0$

$$\Rightarrow m^3 - 3m^2 + 4m - 2 = 0$$

$$\Rightarrow m = 1, 1 \pm i$$

$$y_c = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x)$$

$$y_p = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x + x)$$

$$= \frac{e^x}{D^3 - 3D^2 + 4D - 2} + \frac{\cos x}{D^3 - 3D^2 + 4D - 2} + \frac{x}{D^3 - 3D^2 + 4D - 2}$$

$$= \frac{1}{(D-1)(D^2 - 2D + 2)}e^x + \frac{\cos x}{-D + 3 + 4D - 2} + \frac{1}{-2\left(1 - \frac{D^3 - 3D^2 + 4D}{2}\right)}$$

$$= \frac{e^x}{(1-2+2)(D-1)} + \frac{\cos x}{3D+1} - \frac{1}{2}\left[1 - \left(\frac{D^3 - 3D^2 + 4D}{2}\right)\right]^{-1}$$

$$= xe^x + \frac{(3D-1)}{9D^2 - 1}\cos x - \frac{1}{2}\left[1 + \frac{D^3 - 3D^2 + 4D}{2} + \cdots\right]x$$

$$= xe^x - \frac{1}{10}(-3\sin x - \cos x) - \frac{1}{2}(x+2)$$

$$= xe^x + \frac{1}{10}(3\sin x + \cos x) - \frac{1}{2}(x+2)$$

$$\therefore y = y_c + y_p$$

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Hw ③ Solve $(D^3+3D^2+2)y = 2\cos(2x+3) + 2e^x + x^2$

Hw ④ Solve $(D^2-4D+4)y = 8x^2 + e^{2x}$

Hw ⑤ Solve $(D^3-3D^2-10D+24)y = x+3$

To find P.I. of $f(D)y = Q(x)$ when $Q(x) = e^{ax}v(x)$

$f(D)y = Q(x)$

P.I. of y is $y_p = \dfrac{1}{f(D)} Q(x)$

$= \dfrac{1}{f(D)} e^{ax} v(x)$

$= e^{ax} \dfrac{1}{f(D+a)} v(x)$

problems

① Solve $(D^2+2)y = e^x \cos x$ — ①

A.E. of ① is $f(m) = 0$

$\Rightarrow m^2+2 = 0$

$\Rightarrow m = \pm\sqrt{2}i$

$y_c = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

To find y_p

$y_p = \dfrac{1}{D^2+2} e^x \cos x$

$= e^x \dfrac{1}{(D+1)^2+2} \cos x$

$= e^x \dfrac{1}{D^2+2D+3} \cos x$

$= e^x \dfrac{1}{-1+2D+3} \cos x$

$= e^x \dfrac{1}{2D+2} \cos x$

$= \dfrac{e^x}{2} \dfrac{(D-1)}{(D+1)(D-1)} \cos x$

$= \dfrac{e^x}{2} \dfrac{(D-1)}{D^2-1} \cos x$

$= \dfrac{e^x}{2} \dfrac{(D-1)\cos x}{-2}$

$= -\dfrac{e^x}{4} [-\sin x - \cos x]$

$= \dfrac{e^x}{4} [\sin x + \cos x]$

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$\therefore y = y_c + y_p$

$$y = (c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{e^x}{4}[\sin x + \cos x]$$

(2) Solve $(D^3 - 3D^2 + 3D - 1)y = x^2 e^x$ — (1)

A.E. of (1) is $f(m) = 0$

$$\Rightarrow m^3 - 3m^2 + 3m - 1 = 0$$

$$\Rightarrow m = 1, 1, 1$$

$$y_c = (c_1 + c_2 x + c_3 x^2)e^x$$

<u>To find y_p</u>

$$y_p = \frac{1}{(D^3 - 3D^2 + 3D - 1)} x^2 e^x$$

$$= \frac{1}{(D-1)^3} x^2 e^x$$

$$= e^x \frac{1}{(D+\cancel{1}-\cancel{1})^3} x^2$$

$$= e^x \frac{1}{D^2} x^2$$

$$= e^x \frac{1}{D^2}\left[\frac{1}{D} x^2\right]$$

$$= e^x \cdot \frac{1}{D^2} \frac{x^3}{3}$$

$$= \frac{e^x}{3} \frac{1}{D}\left(\frac{1}{D} x^3\right)$$

$$= \frac{e^x}{3} \frac{1}{D}\left(\frac{x^4}{4}\right)$$

$$= \frac{e^x}{12} \frac{x^5}{5}$$

$$= \frac{e^x x^5}{60}$$

$\therefore y = y_c + y_p$

$$= (c_1 + c_2 x + c_3 x^2)e^x + \frac{e^x x^5}{60}$$

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③ Solve $\frac{d^2y}{dx^2} + y = e^{-x} + x^3 + e^x \sin x$

Sol:-

$$(D^2+1)y = e^{-x} + x^3 + e^x \sin x \quad \leftarrow (1)$$

A.E. of (1) is $f(m) = 0$

$$\Rightarrow m^2+1 = 0$$

$$\Rightarrow m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

To find y_p

$$y_p = \frac{1}{D^2+1}\left[e^{-x} + x^3 + e^x \sin x\right]$$

$$= \frac{e^{-x}}{D^2+1} + \frac{x^3}{D^2+1} + \frac{e^x \sin x}{D^2+1}$$

$$= \frac{e^{-x}}{1^2+1} + (1+D^2)^{-1} x^3 + e^x \left[\frac{1}{(D+1)^2+1}\right] \sin x$$

$$= \frac{e^{-x}}{2} + (1 - D^2 + D^4 - D^6 + \cdots) x^3 + e^x \left[\frac{1}{D^2+2D+2}\right] \sin x$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^x \left[\frac{1}{-1+2D+2}\right] \sin x$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^x \left(\frac{1}{2D+1} \sin x\right)$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^x \left[\frac{(2D-1)}{4D^2-1} \sin x\right]$$

$$= \frac{e^{-x}}{2} + x^3 - 6x + e^x \left[\frac{(2D-1)\sin x}{-4-1}\right]$$

$$= \frac{e^{-x}}{2} + x^3 - 6x - \frac{e^x}{5}\left[2\cos x - \sin x\right]$$

$$\therefore y = y_c + y_p$$

$$y = c_1 \cos x + c_2 \sin x + \frac{e^{-x}}{2} + x^3 - 6x - \frac{e^x}{5}\left[2\cos x - \sin x\right]$$

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(4) Solve $(D^3-4D-D+4)y = e^{3x}\cos 2x$

Hw

(5) Solve $(D^3-7D^2+14D-8)y = e^{x}\cos 2x$

Hw

(6) Solve $(D^2+1)y = x^2\cosh x$

Hw

(7) Solve $(D^2+1)y = \sin x \sin 2x + e^{x}x^2$

Hw

(8) Solve $(D^2-2D+2)y = e^{x}\tan x$

Hw

(9) Solve $(D^2+9)y = (x^2+1)e^{3x}$

To find P.I. of $f(D)y = Q(x)$ when $Q(x) = xV(x)$

P.I. is $y_p = \dfrac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} xV(x)$$

$$y_p = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} V(x)$$

Problems

(1) Solve $(D^2+2D+1)y = x\cos x$ — (1)

Sol:- A.E of (1) is $f(m) = 0$

$$\Rightarrow m^2+2m+1=0$$

$$\Rightarrow m = -1, -1$$

$$y_c = (c_1+c_2x)e^{-x}$$

To find y_p

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D^2+2D+1)} x\cos x$$

$$= \left[x - \frac{2D+2}{D^2+2D+1}\right] \frac{1}{D^2+2D+1} \cos x$$

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$$= \left[x - \frac{2D+2}{D^2+2D+1}\right] \frac{\cos x}{-1+2D+1}$$

$$= \left[x - \frac{2D+2}{D^2+2D+1}\right] \frac{1}{2}\sin x$$

$$= \frac{1}{2}\left[x\sin x - \frac{2(D+1)}{(D+1)^2}\sin x\right]$$

$$= \frac{1}{2}\left[x\sin x - 2\frac{(D-1)}{D^2-1}\sin x\right]$$

$$= \frac{1}{2}\left[x\sin x - \frac{(D-1)\sin x}{-2}\right]$$

$$= \frac{1}{2}\left[x\sin x + \cos x - \sin x\right]$$

$\therefore \; y = y_c + y_p$

$$y = (c_1 + c_2 x)e^{-x} + \frac{1}{2}\left[x\sin x + \cos x - \sin x\right]$$

(2) Solve $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = xe^x \sin x$

Given eqn can be written as

$$(D^2 - 2D + 1)y = xe^x \sin x \quad \text{--- (1)}$$

A.E. of (1) is $f(m) = 0$

$$\Rightarrow m^2 - 2m + 1 = 0$$

$$\Rightarrow m = 1, 1$$

$$y_c = (c_1 + c_2 x)e^x$$

<u>To find y_p</u>

$$y_p = \frac{1}{D^2-2D+1} xe^x \sin x = \frac{1}{(D-1)^2} xe^x \sin x$$

$$= e^x \left[\frac{1}{(D+1-1)^2} x\sin x\right]$$

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$$= e^x \left[\frac{1}{D^2} x\sin x\right]$$

$$= e^x \left[x - \frac{2D}{D^2}\right]\frac{1}{D^2}\sin x$$

$$= e^x \left[x - \frac{2}{D}\right](-\sin x)$$

$$= e^x \left[-x\sin x + \frac{2}{D}\sin x\right]$$

$$= e^x\left[-x\sin x - 2\cos x\right]$$

$$y_p = -e^x\left[x\sin x + 2\cos x\right]$$

$\therefore$ The general soln is

$$y = y_c + y_p$$

$$\boxed{y = (c_1 + c_2 x)e^x - e^x\left[x\sin x + 2\cos x\right]}$$

Hw ③ Solve $(D^2 - 4D + 4)y = 8x^2 e^{2x}\sin 2x$

Hw ④ Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x\sin x$

Hw ⑤ Solve $\frac{d^2y}{dx^2} + 9y = x\sin 2x$

Hw ⑥ Solve $(D^4 + 2D^2 + 1)y = x^2\cos^2 x$

Hw ⑦ Solve $(D^2+1)y = t\cos 2t$ given $x=0$, $\frac{dx}{dt} = 0$ at $t=0$

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Method of variation of parameters

General soln of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ by the method of variation of parameters

Working rule

→ Given equ is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ — (1)

→ Find the soln of (1)

Let the soln is $y_c = c_1 U(x) + c_2 V(x)$

→ Take $y_p = AU(x) + BV(x)$

where A & B are functions of x.

→ Find $W(x) = UV' - VU'$

(or)

$$W(x) = \begin{vmatrix} U & V \\ U' & V' \end{vmatrix}$$

→ Find $A = -\int \frac{VR}{W(x)}dx \quad B = \int \frac{UR}{W(x)}dx$

→ The general soln is $y = y_c + y_p$

Problem

(1) Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$$

Sol:- Given eqn ~~can be written~~ is $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ — (1)

eq (1) comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = 0, \quad Q = 1, \quad R = \operatorname{cosec} x$

A.E. of (1) is $\Rightarrow f(m) = 0 \Rightarrow m^2 + 1 = 0$

$\Rightarrow m = \pm i$

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$y_c = c_1 \cos x + c_2 \sin x$ — (2)

eq (2) Comparing with $y_c = c_1 U(x) + c_2 V(x)$

$U(x) = \cos x$, $V(x) = \sin x$

<u>To find y_p</u>

$y_p = A\cos x + B\sin x$

where $A = -\int \frac{VR}{U\frac{dV}{dx} - V\frac{dU}{dx}}$

$= -\int \frac{\sin x \operatorname{cosec} x}{\cos x \cos x - \sin x(-\sin x)} dx$

$= -\int \frac{\cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}}}{\cos^2 x + \sin^2 x} dx$

$A = -x$

$B = \int \frac{UR}{U\frac{dV}{dx} - V\frac{dU}{dx}}$

$= \int \frac{\cos x \operatorname{cosec} x}{(1)} dx$

$= \int \cot x \, dx$

$= \log|\sin x|$

$y_c = -x\cos x + \log|\sin x| \sin x$

$\therefore$ The general soln is

$y = y_c + y_p$

$= (c_1 \cos x + c_2 \sin x) - x\cos x + \log|\sin x| \sin x$

(2) Solve by the method of variation of parameters

$(D^2 - 2D)y = e^x \sin x$ — (1)

Sol: A.E. of (1) is

$f(m) = 0 \Rightarrow m^2 - 2m = 0$

$\Rightarrow m(m-2) = 0$

$\Rightarrow m = 0, 2$

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$$y_c = c_1 e^{0x} + c_2 e^{2x}$$

$$y_c = c_1 + c_2 e^{2x} \quad \text{—— (2)}$$

eq (2) Comparing with $y_c = c_1 U(x) + c_2 V(x)$

$$U(x) = 1, \quad V(x) = e^{2x}$$

<u>To find y_p</u>

$$y_p = A + B e^{2x}$$

where $A = -\int \frac{VR}{U\frac{dV}{dx} - V\frac{dU}{dx}} dx$

$$A = -\int \frac{e^{2x} \cdot e^{x}\sin x}{1 \cdot 2e^{2x} - e^{2x}(0)} dx$$

$$= -\int \frac{\cancel{e^{2x}} \cdot e^{x}\sin x}{2\cancel{e^{2x}}} dx$$

$$= -\frac{1}{2}\int e^{x}\sin x \, dx$$

$$= -\frac{1}{2}\frac{e^{x}}{1^2+1^2}\left[\sin x - \cos x\right]$$

$$= -\frac{1}{4} e^{x}(\sin x - \cos x)$$

$$B = \int \frac{UR}{U\frac{dV}{dx} - V\frac{dU}{dx}} dx$$

$$= \int \frac{1 \cdot e^{x}\sin x}{2 \cdot e^{2x}} dx$$

$$= \frac{1}{2}\int e^{-x}\sin x \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{2} e^{-x}(\sin x + \cos x)\right]$$

$$= -\frac{1}{4} e^{-x}(\sin x + \cos x)$$

$$y_p = -\frac{1}{4} e^{x}(\sin x - \cos x) - \frac{e^{-x}}{4}(\sin x + \cos x)\, e^{2x}$$

$$= -\frac{1}{2} e^{x}\sin x$$

$$y = y_c + y_p$$

$$\boxed{y = c_1 + c_2 e^{2x} - \frac{1}{2} e^{x}\sin x}$$

(3) $\frac{d^2y}{dx^2} + y = \tan x$

(4) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^{x}\tan x$

(5) $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

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Equations reducible to linear ODE with Constant Coefficients

Cauchy-Euler Equation

An equation is of the form $x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_{n-1} x \frac{dy}{dx} + P_n y = \phi(x)$ ① is called Cauchy-Euler equation (or) Cauchy's homogeneous linear equation.

Where $P_1\ P_2\ P_3 \cdots P_n$ are real constants and $\phi(x)$ is a function of x

The operator form of eq ① is

$$\left(x^n D^n + P_1 x^{n-1} D^{n-1} + \cdots + P_{n-1} x D + P_n\right) y = \phi(x)$$

where $D = \frac{d}{dx}$

This equation can be transformed into a linear equation with constant coefficients by the change of independent variable with the substitution $x = e^z$ (or) $z = \log x$, $xD = \theta$, $x^2 D^2 = \theta(\theta - 1)$

problems ②

① Solve $(x^2 D^2 - 4xD + 6) y = x^2$

Sol:- This is a Cauchy-Euler equation.

Let $x = e^z \Rightarrow \log x = z$ $\quad \frac{d}{dx} = D,\ \frac{d}{dz} = \theta$

$xD = \theta,\ x^2 D^2 = \theta(\theta - 1)$

Given eq^n can be written as

$$(\theta(\theta-1) - 4\theta + 6) y = e^{2z}$$

$$(\theta^2 - 5\theta + 6) y = e^{2z} \quad \text{—— ①}$$

eq ① is a D.E. with constant coefficients.

A.E. of ① is $f(m) = 0 \Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3$

$$y_c = c_1 e^{2z} + c_2 e^{3z}$$

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To find y_p

$$y_p = \frac{e^{2z}}{(\theta-3)(\theta-2)} = \frac{e^{2z}}{(2-3)}\frac{z}{1!} = -z e^{2z}$$

The general soln is

$$y = y_c + y_p$$

$$y = c_1 e^{2z} + c_2 e^{3z} - z e^{2z}$$

$$\boxed{y = c_1 x^2 + c_2 x^3 - \log x \cdot x^2}$$

② Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Given eqn can be written as

$$(x^2 D^2 - xD + 1)y = \log x \quad \text{— ①}$$

Let $x = e^z \Rightarrow z = \log x$

$$xD = \theta, \quad x^2 D^2 = \theta(\theta-1)$$

eq ① becomes

$$(\theta(\theta-1) - \theta + 1)y = z$$

$$(\theta^2 - 2\theta + 1)y = z \quad \text{— ②}$$

This is a linear D.E. with constant coefficients

A.E. of ② is $f(m) = 0$

$$\Rightarrow m^2 - 2m + 1 = 0$$

$$\Rightarrow m = 1, 1$$

$$y_c = (c_1 + c_2 z)e^{z}$$

To find y_p

$$y_p = \frac{z}{(\theta-1)^2} = (1-\theta)^{-2}(z)$$

$$= (1 + 2\theta + 3\theta^2 + \cdots)z$$

$$= z + 2$$

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The general soln is

$$y = y_c + y_p$$

$$= (c_1 + c_2 z)\, e^{z} + z + 2$$

$$\boxed{y = (c_1 + c_2 \log x)\, x + \log x + 2}$$

(3) Solve $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

Sol:- Given eqn can be written as

$$(x^2D^2 - xD - 4)\, y = x^4$$

Let $x = e^z \Rightarrow \log x = z$

$xD = \theta$, $x^2D^2 = \theta(\theta - 1)$

$$(\theta(\theta-1) - 2\theta - 4)\, y = e^{4z}$$

$$(\theta^2 - 3\theta - 4)\, y = e^{4z} \quad \text{--- (1)}$$

A.E of (1) is $m^2 - 3m - 4 = 0$

$$\Rightarrow m = -1, 4$$

$$y_c = c_1 e^{-z} + c_2 e^{4z}$$

To find y_p

$$y_p = \frac{1}{\theta^2 - 3\theta - 4} e^{4z} = \frac{e^{4z}}{(\theta+1)(\theta-4)} = \frac{e^{4z} \cdot z}{5} = \frac{x^4 \log x}{5}$$

$\therefore$ The general soln is

$$y = y_c + y_p$$

$$= c_1 e^{-z} + c_2 e^{4z} + \frac{x^4 \log x}{5}$$

$$= \frac{c_1}{x} + c_2 x^4 + \frac{x^4 \log x}{5}$$

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(4) Solve $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right)$

Sol:- Given eqn can be written as

$$(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x+\frac{1}{x}\right) \quad \text{--- (1)}$$

Let $x = e^z \Rightarrow \log x = z$

$xD = \theta$, $x^2D^2 = \theta(\theta-1)$ $\quad x^3D^3 = \theta(\theta-1)(\theta-2)$

eqn (1) can be written as

$$(\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2)y = 10(e^z + e^{-z})$$

$$(\theta^3 - \theta^2 + 2)y = 10(e^z + e^{-z}) \quad \text{--- (2)}$$

This is a linear D.E with constant coefficients

A.E of (2) is $m^3 - m^2 + 2 = 0$

$\Rightarrow m = -1, 1+i, 1-i$

$$y_c = c_1e^{-z} + e^z(c_2\cos z + c_3\sin z)$$

$$y_p = \frac{1}{\theta^3-\theta^2+2}10(e^z + e^{-z})$$

$$= 10\frac{e^z}{\theta^3-\theta^2+2} + 10\frac{e^{-z}}{\theta^3-\theta^2+2}$$

$$= 10\frac{e^z}{1-1+2} + 10\frac{e^{-z}}{(\theta+1)(\theta^2-2\theta+2)}$$

$$= 5e^z + \frac{10ze^{-z}}{(1+2+2)}$$

$$y_p = 5e^z + 2ze^{-z}$$

The general soln is $y = y_c + y_p = c_1e^{-z} + e^z(c_2\cos z + c_3\sin z) + 5e^z + 2ze^{-z}$

where $z = \log x$

Hw (1) Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Hw (2) Solve $(x^2D^2 - 3xD + 1)y = \frac{\log x\sin\log x + 1}{x}$

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Legendre's Linear equation

An equation of the form $(a+bx)^n \frac{d^n y}{dx^n} + P_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = Q(x)$

is called Legendre's linear equation.

where $P_1\ P_2\ P_3 \cdots P_n$ are real constants and

$Q(x)$ is a function of x.

The soln can be obtained by substituting $a+bx = e^z$

$\Rightarrow z = \log(a+bx)$

Here $(a+bx)D = b\theta$, $(a+bx)^2 D^2 = b^2\theta(\theta-1)$,

$(a+bx)^3 D^3 = b^3\theta(\theta-1)(\theta-2) \cdots\cdots$

problems

① Solve the D.E. $(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = x$

Sol:- Given D.E. $(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = x$ — ①

Let $2x-1 = e^z \Rightarrow z = \log(2x-1)$

$(2x-1)D = 2\theta$, $(2x-1)^2 D^2 = 2^2\theta(\theta-1)$

$(2x-1)^3 D^3 = 2^3\theta(\theta-1)(\theta-2)$

eq ① can be written as

$$\left[2^3\theta(\theta-1)(\theta-2) + 2(\theta-1)\right]y = \frac{1}{2}(1+e^z)$$

$$(8\theta^3 - 24\theta^2 + 18\theta - 2)y = \frac{e^z+1}{2} \quad \text{— ②}$$

A.E of ② is $8m^3 - 24m^2 + 18m - 2 = 0$

$\Rightarrow m = 1,\ 1 \pm \frac{\sqrt{3}}{2}$

$$y_c = c_1 e^z + c_2 e^{\left(1+\frac{\sqrt{3}}{2}\right)z} + c_3 e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$$

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To find y_p

$$\theta = 1 \left| \begin{array}{ccc} 8 & -24 & 18 & -2 \\ 0 & 8 & -16 & 2 \\ \hline 8 & -16 & 2 & 0 \end{array} \right.$$

$$(\theta - 1)(8\theta^2 - 16\theta + 2)$$

$$y_p = \frac{1}{2}\left[\frac{e^z + 1}{8\theta^3 - 24\theta^2 + 18\theta - 2}\right]$$

$$= \frac{1}{2}\left[\frac{e^z}{8\theta^3 - 24\theta^2 + 18\theta - 2}\right] + \frac{1}{2}\left[\frac{1}{8\theta^3 - 24\theta^2 + 18\theta - 2}\right]$$

$$= \frac{1}{2}\,\frac{e^z \cdot z}{2(8 - 16 + 2)} + \frac{1}{2}\left[\frac{e^{0} \cdot z}{8(0) - 24(0) + 18(0) - 2}\right]$$

$$= \frac{-1}{12} \cdot z e^z + \frac{1}{2} \cdot \left(\frac{-1}{2}\right)$$

$$= \frac{-z e^z}{12} - \frac{1}{4}$$

$$= \frac{-\log(2x-1)(2x-1)}{12} - \frac{1}{4}$$

$\therefore$ The general soln is

$$y = y_c + y_p$$

$$= c_1 e^z + c_2 e^{\left(1 + \frac{\sqrt{3}}{2}\right)z} + c_3 e^{\left(1 - \frac{\sqrt{3}}{2}\right)z} - \frac{\log(2x-1)(2x-1)}{12} - \frac{1}{4}$$

where $z = \log(2x-1)$

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UNIT – III

MULTIPLE INTEGRALS

Multiple Integral: – A double (or) triple integral is known as multiple integral.

It is an extension of a definite integral of a function of single variable to a function of two or three variables.

Multiple integrals are useful in evaluating area, volume, mass, centroid and moments of inertia in plane and solid regions.

Double integral: – The definite integral can be extended to functions of more than one variable (two variables) is called double integral.

Let $z = f(x, y)$ be a function of two variables then double integral of $f(x, y)$ is denoted with $\iint_R f(x, y)\,dx\,dy$ over R.

Properties

i) $\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$

ii) $\iint_R kf\,dx\,dy = k\iint_R f\,dx\,dy$ where 'k' is constant.

iii) $\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$

Evaluation of Double integrals

Suppose R can be described by inequalities of the form $a \le x \le b$, $y_1(x) \le y \le y_2(x)$ represented the boundary of R then

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\,dy\,dx = \int_{x=a}^{b} \left[\int_{y=y_1(x)}^{y_2(x)} f(x, y)\,dy \right] dx$$

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Note:- ① If all the four limits of integration are constants, then the double integral can be evaluated from the following way.

i.e; We first integrate with respect to x and then w.r.to. y' (OR) we first integrate w.r.to y and then w.r.to x'.

Note:- ② Suppose if the limits for one variable is a function of the other and constants for other variable, then first we have to integrate the variable w.r.to variable for which limits are function of the other.

Note:- ③ While evaluating a multiple integral w.r.to one variable then other variables ~~two~~ are treated as constants.

Problems

① Evaluate $\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy$.

Sol:- $$\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy = \int_{x=0}^{3}\int_{y=1}^{2}(xy+x^2y+xy^2)\,dx\,dy$$

$$= \int_{x=0}^{3}\left(\frac{xy^2}{2}+\frac{x^2y^2}{2}+\frac{xy^3}{3}\right)_{y=1}^{2}dx$$

$$= \int_{x=0}^{3}\left[\left(2x+2x^2+\frac{8}{3}x\right)-\left(\frac{x}{2}+\frac{x^2}{2}+\frac{x}{3}\right)\right]dx$$

$$= \int_0^3\left[\left(\frac{14x}{3}+2x^2\right)-\left(\frac{5x}{6}+\frac{x^2}{2}\right)\right]dx$$

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$$= \int_0^3 \left(\frac{23}{6}x + \frac{3}{2}x^2\right) dx = \left(\frac{23}{6}\frac{x^2}{2} + \frac{3}{2}\frac{x^3}{3}\right)_0^3 = \frac{123}{4}$$

(2) Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy$

Sol:- $$\int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2 + y^2)\, dx\, dy = \int_{x=0}^{1} \left(x^2 y + \frac{y^3}{3}\right)_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \left(x^2\sqrt{x} + \frac{x\sqrt{x}}{3} - x^3 - \frac{x^3}{3}\right) dx$$

$$= \int_{x=0}^{1} \left(x^{5/2} + \frac{x^{3/2}}{3} - \frac{4}{3}x^3\right) dx$$

$$= \left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{5/2} - \frac{4}{3}\frac{x^4}{4}\right]_{x=0}^{1}$$

$$= \frac{9}{105} = \frac{3}{35}$$

(3) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

Sol:- $$\int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2}\, dx\, dy = \int_0^\infty e^{-x^2} \left[\int_0^\infty e^{-y^2} dy\right] dx$$

$$= \int_0^\infty e^{-x^2} \frac{\sqrt{\pi}}{2}\, dx \qquad \left(\because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2} \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

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(OR)

By changing into polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$

$dxdy = r\,dr\,d\theta$

$\theta \to 0$ to $\pi/2$

$r \to 0$ to ∞

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

put $r^2 = u$

$2r\,dr = du$

$r\,dr = \frac{1}{2}du$

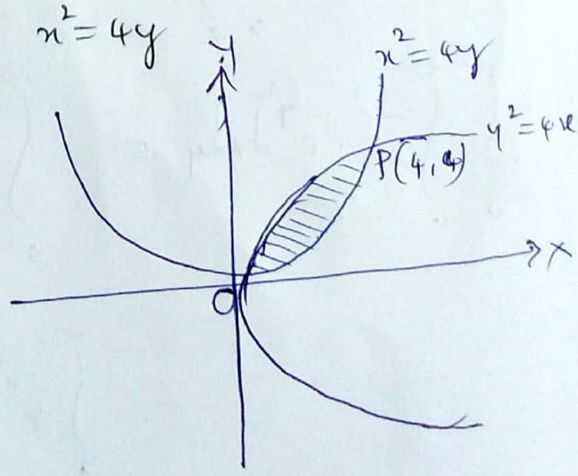
$$= \int_0^{\pi/2}\int_{r=0}^{\infty}\left(\frac{1}{2}\right)e^{-u}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{e^{-u}}{-1}\right)_{r=0}^{\infty}d\theta$$

$$= \frac{1}{-2}\int_0^{\pi/2}\left[0-1\right]d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2} = \frac{\pi}{4}$$

(4) Evaluate i) $\iint_R y\,dx\,dy$ ii) $\iint_R y^2\,dx\,dy$ where R is the region bounded by parabolas $y^2 = 4x$ and $x^2 = 4y$



Given parabolas

$y^2 = 4x$ — (1)

$x^2 = 4y$ — (2)

Solving (1) & (2)

$\left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x^4 = 64x$

$\Rightarrow x(x^3 - 64) = 0$

$\Rightarrow x = 0,\ x = 4$

$y^2 = 4x$

$y^2 = 16 \Rightarrow y = \pm 4$

Thus the two parabolas intersect at the points (0,0) (4,4)

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i) $\iint_R y\,dx\,dy = \int_{x=0}^{4}\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dx\,dy = \int_{x=0}^{4}\left(\frac{y^2}{2}\right)_{y=\frac{x^2}{4}}^{2\sqrt{x}} dx$

$= \int_{x=0}^{4}\left(\frac{\cancel{4}^{2}x}{\cancel{2}} - \frac{1}{2}\,\frac{1}{16}x^4\right)dx$

$= \left(\frac{\cancel{2}x^2}{\cancel{2}} - \frac{1}{32}\,\frac{x^5}{5}\right)_{x=0}^{4}$

$= 16 - \frac{1}{\cancel{32}_{\cancel{4}}}\,\frac{\cancel{64\times 8}^{\,2}}{5}\,4$

$= 16 - \frac{32}{5} = \frac{80-32}{5} = \frac{48}{5}$

ii) $\iint_R y^2\,dx\,dy =$

(5) Evaluate $\int_0^4\int_0^{x^2} e^{y/x}\,dy\,dx$

$\int_0^4\int_0^{x^2} e^{y/x}\,dx\,dy = \int_0^4\left[e^{y/x}\cdot x\right]_{y=0}^{x^2} dx$

$= \int_0^4\left(e^{\cancel{x^2}/\cancel{x}}\cdot x - x\right)dx$

$= \left(xe^x - e^x - \frac{x^2}{2}\right)_0^4$

$= (4e^4 - e^4 - 8) - (0 - 1 - 0)$

$= 3e^4 - 7$

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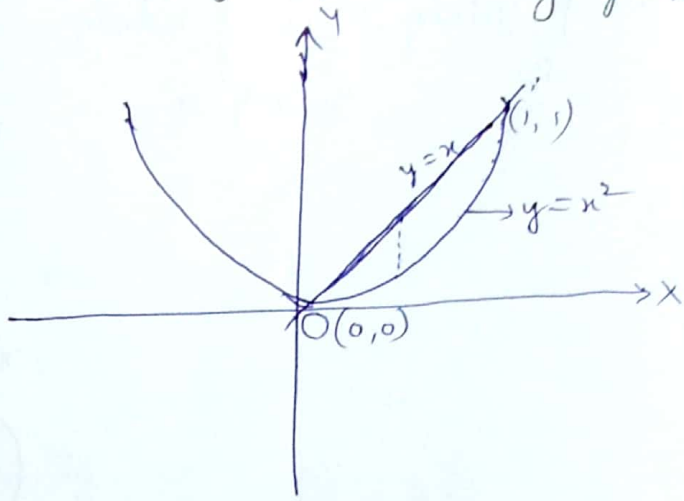
⑥ Evaluate $\iint xy(x+y)\,dx\,dy$ over the region R bounded by $y=x^2$ and $y=x$

Sol:-



$y=x^2,\ y=x$

$\Rightarrow x^2=x$

$\Rightarrow x^2-x=0$

$\Rightarrow x(x-1)=0$

$\Rightarrow x=0,1$ & $y=0,1$

The point of intersection of the curves are $(0,0)\ (1,1)$

We have to fix 'x' first, y varies from x^2 to x. & x from 0 to 1

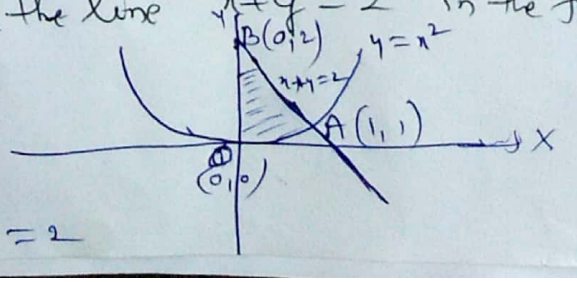
$$\iint xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x}(x^2y+xy^2)\,dx\,dy$$

$$= \int_{x=0}^{1}\left(\frac{x^2y^2}{2}+\frac{xy^3}{3}\right)_{y=x^2}^{x}dx$$

$$= \int_{x=0}^{1}\left[\left(\frac{x^2x^2}{2}+\frac{x^4}{3}\right)-\left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right]dx$$

$$= \frac{3}{56}$$

⑦ Evaluate $\iint_R y\,dx\,dy$ where R is the domain bounded by y-axis the curve $y=x^2$ and the line $x+y=2$ in the first quadrant.



Sol:- Y-axis $\Rightarrow x=0$

So given curves are

$x=0,\ y=x^2,\ x+y=2$

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The point of intersection of curves are

$$y = x^2 \quad , \quad x + y = 2$$

$$x + x^2 = 2$$

$$x^2 + x - 2 = 0$$

$$\Rightarrow (x+2)(x-1) = 0$$

$$\Rightarrow x = -2, \quad x = 1$$

When $x = 1$, $y = 1$

When $x = 0$ $y = 0$

When $x = 0$ $y = 2$

The given curves intersect at $O(0,0)$ $A(1,1)$ $B(0,2)$

Hence $$\iint_R y\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{2-x} y\,dx\,dy = \int_{x=0}^{1}\left(\frac{y^2}{2}\right)_{y=x^2}^{2-x} dx$$

$$= \int_{x=0}^{1}\left[\frac{(2-x)^2}{2} - \frac{x^4}{2}\right] dx$$

$$= \left(\frac{(2-x)^3}{(-6)} - \frac{x^5}{10}\right)_{x=0}^{1}$$

$$= \left(-\frac{1}{6} - \frac{1}{10}\right) - \left(\frac{8}{-6}\right)$$

$$= -\frac{1}{6} + \frac{8}{6} - \frac{1}{5}$$

$$= \frac{7}{6} - \frac{1}{10}$$

$$= \frac{70 - 6}{60} = \frac{64}{60} = \frac{16}{15}$$

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Double integrals in Polar Co-ordinates

To evaluate over the region bounded by the lines $\theta = \theta_1$, $\theta = \theta_2$ and the curves $r = r_1$ $r = r_2$, we first integrate w.r.to r between limits $r = r_1$ and $r = r_2$ keeping θ fixed. The resulting expression is integrated w.r.to θ from θ_1 to θ_2.

① Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$ (2017)

Sol:- $$\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$$

$$= \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[(\pi - 0) - (0 - 0)\right]$$

$$= \frac{a^2\pi}{4}$$

② Evaluate $\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r\,d\theta\,dr$

Sol:- $$\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr = \int_0^{\infty} e^{-r^2} r\left[\theta\right]_0^{\pi/2} dr = \frac{\pi}{2}\int_0^{\infty} r e^{-r^2} dr$$

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$$= \frac{-\pi}{4}\int_0^\infty (-2r)\, e^{-r^2}\, dr$$

$$= \frac{-\pi}{4}\left[e^{-r^2}\right]_0^\infty = \frac{-\pi}{4}\left[0-1\right] = \frac{\pi}{4}$$

(3) Evaluate $\displaystyle\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol:- $$\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta = \int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\right\}d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4}\left(2\sqrt{a^2-r^2}\right)_0^{a\sin\theta} d\theta \qquad \because \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

$$= (-1)\int_0^{\pi/4}\left(\sqrt{a^2-a^2\sin^2\theta} - \sqrt{a^2-0}\right)d\theta$$

$$= (-1)\int_0^{\pi/4}(a\cos\theta - a)\,d\theta$$

$$= (-a)\left[\sin\theta - \theta\right]_0^{\pi/4}$$

$$= (-a)\left[\left(\sin\frac{\pi}{4} - \frac{\pi}{4}\right) - (0-0)\right]$$

$$= (-a)\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

$$= a\left[\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right]$$

(4) Evaluate $\displaystyle\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} \frac{r}{(r^2+a^2)^2}\,dr\,d\theta$

Sol:- $$\int_0^{\pi/2}\int_0^{\infty}\frac{r}{(r^2+a^2)^2}\,dr\,d\theta = \int_0^{\pi/2}\left\{\int_0^{\infty}\frac{r}{(r^2+a^2)^2}\,dr\right\}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left\{\int_0^{\infty}\frac{2r}{(r^2+a^2)^2}\,dr\right\}d\theta$$

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$$= \frac{1}{2}\int_0^{\pi/2}\left\{\frac{-1}{(r^2+a^2)}\right\}_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[0+\frac{1}{a^2}\right]d\theta$$

$$= \frac{1}{2a^2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2a^2}\left(\theta\right)_0^{\pi/2}$$

$$= \frac{1}{2a^2}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{4a^2}$$

(5) Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta \, dr\, d\theta$

$$\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r^2\cos\theta\, dr\, d\theta = \int_{\theta=0}^{\pi}\cos\theta\, d\theta \int_{r=0}^{a(1+\cos\theta)} r^2\, dr$$

$$= \int_0^{\pi}\cos\theta\left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi}\cos\theta(1+\cos\theta)^3\, d\theta$$

Let $I = \frac{a^3}{3}\int_0^{\pi}\cos\theta(1+\cos\theta)^3 d\theta$

$$= \frac{a^3}{3}\int_0^{\pi}\cos(\pi-\theta)(1+\cos(\pi-\theta))^3 d\theta \qquad \because \int_0^a f(x)dx = \int_0^a f(a-x)$$

$$= -\frac{a^3}{3}\int_0^{\pi}\cos\theta(1-\cos\theta)^3 d\theta = I$$

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$$I + I = \frac{a^3}{3}\int_0^{\pi} \cos\theta\left[(1+\cos\theta)^3 - (1-\cos\theta)^3\right]d\theta$$

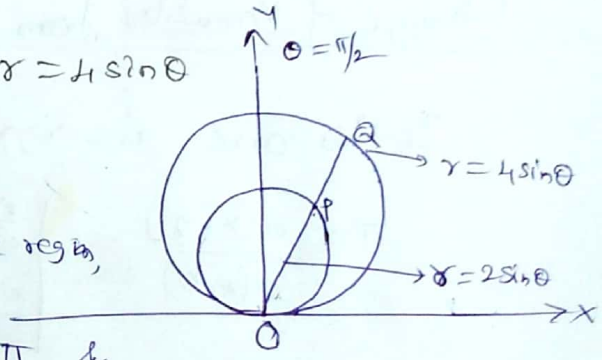
$$= \frac{2a^3}{3}\int_0^{\pi}\left(3\cos^2\theta + \cos^4\theta\right)d\theta$$

$$= \frac{2a^3}{3}\cdot 2\int_0^{\pi/2}\left(3\cos^2\theta + \cos^4\theta\right)d\theta$$

$$= \frac{4a^3}{3}\left[3\cdot\frac{\pi}{4} + \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$I = \frac{5\pi a^3}{8}$$

(6) Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$



Sol:

Here circle cover the whole region,

So θ varies from 0 to π &

r varies from $2\sin\theta$ to $4\sin\theta$.

$$\iint r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\left(256\sin^4\theta - 16\sin^4\theta\right)d\theta$$

$$= 60\int_0^{\pi}\sin^4\theta\,d\theta$$

$$= 60\times 2\int_0^{\pi/2}\sin^4\theta\,d\theta = 120\times\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{45\pi}{2}$$

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Change of variables in Double Integral

Transformation of Coordinates

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations between the old variables (x,y) with the new variables (u,v) of the new coordinate system.

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv \qquad \text{where } J = \frac{\partial(x,y)}{\partial(u,v)}$$

which is called the Jacobian of the coordinate transformation.

Change of variables from Cartesian to Polar Co-ordinates

Change of Variables from Cartesian to Polar Coordinates

In this case $u = r, v = \theta$ and $x = r\cos\theta$, $y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Hence

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

~~ie; $\iint_R F(x,y)\,dx\,dy =$~~

ie; $$\iint F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\,r\,dr\,d\theta$$

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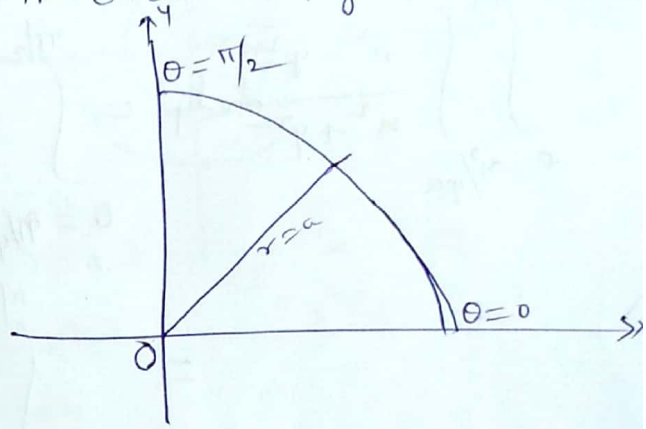
① Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dy\, dx$ by changing into polar coordinates.

Sol:- The given ~~can~~ region can be written as

$$0 \le x \le \sqrt{a^2-y^2} \ \& \ 0 \le y \le a$$

ie; R is the region bounded by the circle $x^2+y^2=a^2$ in the first quadrant.



By changing into polar form,

$$x = r\cos\theta, \quad y = r\sin\theta$$

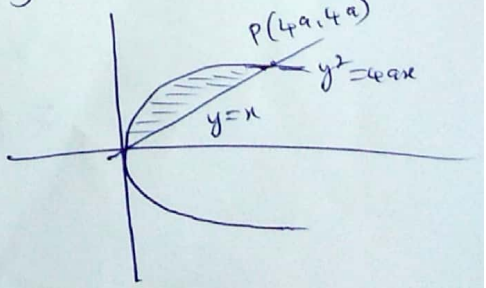
$$dx\,dy = r\,dr\,d\theta.$$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2\, r\, dr\, d\theta = \frac{\pi a^4}{8}$$

② Evaluate $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy$ by changing into polar coordinates.

Sol:- The region of integration is given by $x = \frac{y^2}{4a}$, $x = y$ and $y=0$, $y=4a$.

The region is bounded by the parabola $y^2=4ax$ and the straight line $y=x$.



Let $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$

The limits for r are $r=0$ at 'O' and for P on the parabola

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$\cancel{r^2\sin^2\theta} = y^2 = 4ax$

$r^2\sin^2\theta = 4a(\cancel{r}\cos\theta) \Rightarrow r = \dfrac{4a\cos\theta}{\sin^2\theta}$

For the line $y = x$

$\tan\theta = 1 \Rightarrow \theta = \pi/4$

$\therefore$ The limits for θ varies from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

Also $x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$ and $x^2 + y^2 = r^2$

$$\therefore \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{\cancel{r^2}(\cos^2\theta-\sin^2\theta)}{\cancel{r^2}}\, r\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{\overset{8}{\cancel{16}}a^2\cos^2\theta}{\cancel{2}\sin^4\theta}\right) d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta)\,d\theta$$

$$= 8a^2\left[\frac{\cot^3\theta}{3} - \cot\theta + \theta\right]_{\theta=\pi/4}^{\pi/2} \quad \left(\because \int \cot^4\theta\,d\theta = \frac{\cot^3\theta}{3} - \cot\theta + \theta + c\right)$$

$$- 8a^2\int_{\pi/4}^{\pi/2} \cot^2\theta\,d\theta$$

$$= 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

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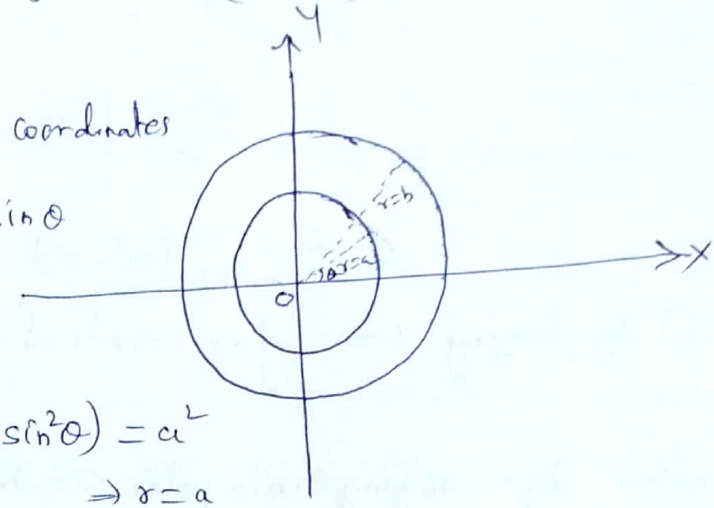
(3) By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region between the ~~circula~~ circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$

Sol:-



By changing into polar coordinates

by taking $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$

$$x^2+y^2=a^2 \Rightarrow r^2(\cos^2\theta+\sin^2\theta) = a^2$$
$$\Rightarrow r = a$$

$$x^2+y^2=b^2 \Rightarrow r^2(\cos^2\theta+\sin^2\theta) = b^2$$
$$\Rightarrow r = b$$

θ is from 0 to 2π

$$\therefore \iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{\cancel{r^2}\cos^2\theta\, r^2\sin^2\theta}{\cancel{r^2}(\cos^2\theta+\sin^2\theta)}\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\sin^2\theta \left(\frac{r^4}{4}\right)_{r=a}^{b} d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta\sin^2\theta \left(\frac{b^4-a^4}{4}\right)$$

$$= \frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi} (2\sin\theta\cos\theta)^2\,d\theta$$

$$= \frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi} \sin^2 2\theta\,d\theta$$

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$$= \frac{b^4 - a^4}{32} \int_{\theta=0}^{2\pi} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{b^4 - a^4}{32} \left[\theta - \frac{\sin 4\theta}{4} \right]_{\theta=0}^{2\pi}$$

$$= \frac{b^4 - a^4}{32} \left[(2\pi - 0) - (0 - 0) \right]$$

$$= \frac{\pi}{16} (b^4 - a^4)$$

④ By changing into polar coordinates, evaluate $\int_0^\infty \int_0^\infty \frac{x^2}{(x^2+y^2)^{3/2}}\, dx\, dy$.

Sol:- By changing into polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta, \quad x^2 + y^2 = r^2$$

The region of integration is in first quadrant of xy plane,

so $r \to 0$ to ∞ and $\theta \to 0$ to $\pi/2$

Hence $$\int_0^\infty \int_0^\infty \frac{x^2}{(x^2+y^2)^{3/2}}\, dx\, dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} \frac{r^2 \cos^2\theta}{(r^2)^{3/2}}\, r\, dr\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} (1 + \cos 2\theta)\, dr\, d\theta$$

$$= \frac{1}{2} \int_{r=0}^{\infty} \left[\theta + \frac{\sin 2\theta}{2} \right]_{\theta=0}^{\pi/2} dr$$

$$= \frac{1}{2} \int_{r=0}^{\infty} \left[\left(\frac{\pi}{2} + 0\right) - (0 + 0) \right] dr$$

$$= \frac{1}{2} \frac{\pi}{2} \int_{r=0}^{\infty} dr = \frac{\pi}{4} \left[r \right]_0^\infty = \infty$$

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Change of order of integration

Procedure

① ⇒ First identify the variables for the limits

② ⇒ Draw a rough sketch of the given region of integration.

③ ⇒ If we are evaluating the integral w.r.to y first, then take a vertical strip ie; a strip parallel to y-axis. Otherwise, take a horizontal strip ie; a strip parallel to x-axis.

④ ⇒ Now rotate the strip by an angle of 90° in the anti-clockwise direction and identify the starting and ending points of the strip, which will be the lower and upper limits of that variable.

⑤ ⇒ Identify the limits for other variables for the region of Consideration

⑥ ⇒ Evaluate the double integral with new order of integration.

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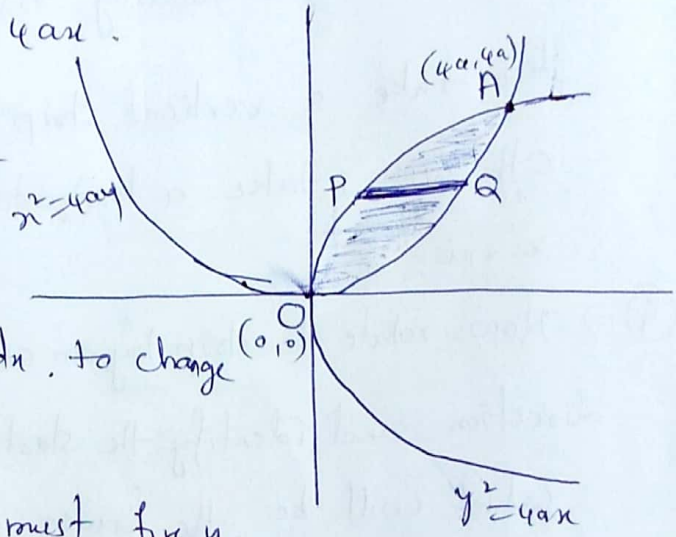
① Change the order of integration and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol:- In the given integral, for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$.

and then x varies from 0 to $4a$.

Let us draw the curves $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$

ie; $x^2 = 4ay$ and $y^2 = 4ax$.



The region of integration is the shaded region in figure.

The given integral $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$. to change the order of integration, we must fix y first, x varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$ then y varies from 0 to $4a$.

Hence the integral

$$\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right] dy$$

$$= \int_{y=0}^{4a} \left[x\right]_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_{y=0}^{4a} \left[\left(2\sqrt{ay} - \left(\frac{y^2}{4a}\right)\right)\right] dy$$

$$= \left[\frac{2\sqrt{a}\,y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_{y=0}^{4a} = \frac{2\sqrt{a}\,4a\sqrt{4a}}{3/2} - \frac{64a^3}{12a}$$

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$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

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$$= \frac{16a^2}{3}$$

(2) Change the order of integration and hence evaluate the double integral $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

Sol:- The region of integration is given by $y : x^2 \longrightarrow 2-x$ & $x : 0$ to 1

ie; $y = x^2$, $y = 2-x$ and $x = 0$, $x = 1$

When $x = 0$

$y = 0$ $(0, 0)$

When $x = 1$

$y = 1$ $(1, 1)$

When $x = 0$

$y = 2$ $(0, 2)$

When $x = 1$

$y = 1$ $(1, 1)$

The point of intersection of $y = x^2$, $y = 2-x$ is

$$x^2 = 2 - x$$

$$x^2 + x - 2 = 0$$

$$\Rightarrow (x+2)(x-1) = 0$$

$$\Rightarrow x = -2, 1$$

When $x = -2$ When $x = 1$

$y = 4$ $y = 1$

Hence the point of intersection of the curves are $(-2, 4)$ and $(1, 1)$.

The shaded region is shown in figure.

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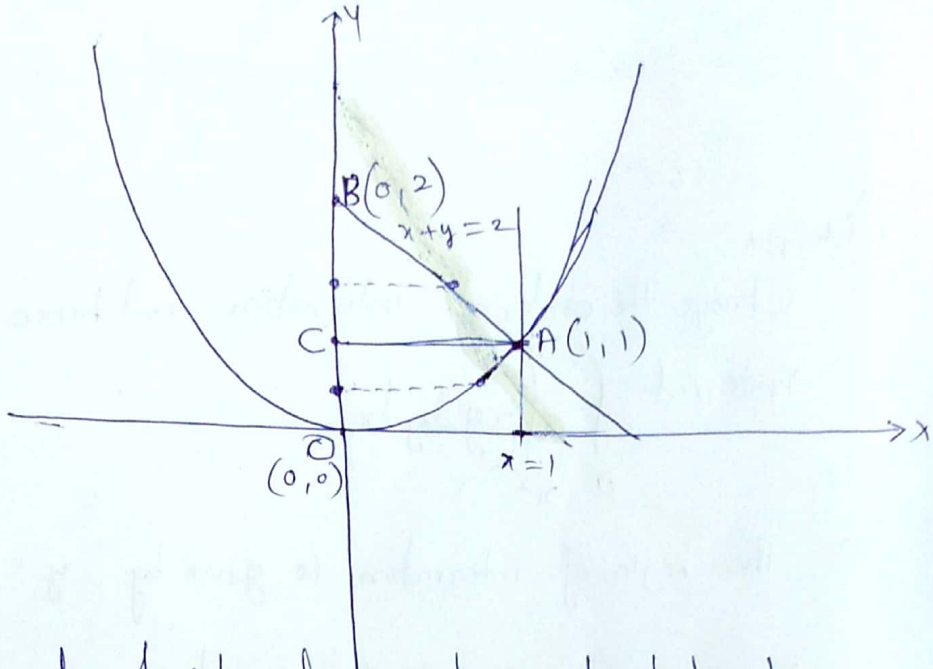

Suppose we change the order of integration, we have to take two horizontal strips, since during sliding one edge of the strip remains on $x=0$, but the other edge of the strip does not remain on a single curve.

$\therefore$ The regions can be

Area OAB = Area OAC + Area CAB

We shall fix y first, then for the region OACO, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1.

For the region CABC, for fixed y, x varies from 0 to $2-y$, then y varies from 1 to 2.

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \iint_{OACO} xy\,dx\,dy + \iint_{CABC} xy\,dx\,dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \frac{3}{8}$$

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Vector Differentiation

* Vector point functions and Scalar point functions
* Gradient, Divergence and curl of a vector
* Directional derivative.
* Tangent plane and normal line
* Vector Identities
* Scalar potential functions, Solenoidal and Irrotational vectors

---

Vector function :— A vector function is a function that assigns a vector to a set of real variables.

The general form is $F = f_1(x_1\ x_2 \cdots x_n)i + f_2(x_1\ x_2 \cdots x_n)j + f_3(x_1\ x_2 \cdots x_n)k$

where $x_1\ x_2\ x_3 \cdots x_n$ are real numbers.

Scalar function :— A scalar function is a function that assigns a real number (ie; scalar) to a set of real variables.

The general form is $u = u(x_1\ x_2 \cdots x_n)$

where $x_1\ x_2 \cdots x_n$ are real numbers

Vector point function :— A vector point function is a function that assigns a vector to each point of some region of space. If to each point (x, y, z) of a region R in space there is assigned a vector $F = F(x, y, z)$ then F is called a vector point function.

ie; $F = f_1(x, y, z)i + f_2(x, y, z)j + f_3(x, y, z)k$

Scalar point function :— A scalar point function is a function that assigns a real number (ie; a scalar) to each point of some region of space.

If to each point (x, y, z) of a region R in space there is assigned a real number $u = \phi(x, y, z)$ then ϕ is called a scalar point function.

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Vector differential Operator:- The vector differential operator is denoted with ∇ and it is defined as $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$.

This operator used in defining "gradient", ~~divge~~ "divergence", and "curl" of a vector function.

Gradient of a scalar point function:- Let $\phi(x, y, z)$ be a scalar point function of position defined in some region of space.

Then the vector function $\frac{i\partial\phi}{\partial x} + \frac{j\partial\phi}{\partial y} + \frac{k\partial\phi}{\partial z}$ is known as the gradient of ϕ and is denoted by gradϕ or $\nabla\phi$

e; $\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Properties

1) $\text{grad}(f \pm g) = \text{grad} f \pm \text{grad} g$

2) $\nabla f = \bar{0}$

3) $\text{grad}(fg) = f\,\text{grad}\, g + g\,\text{grad} f$

4) $\text{grad}(cf) = c\,\text{grad} f$ if c is a constant

5) $\text{grad}\left(\frac{f}{g}\right) = \frac{g\,\text{grad} f - f\,\text{grad} g}{g^2}$ $(g \neq 0)$

If $\bar{r} = xi + yj + zk$ and $d\bar{r} = (dx)i + (dy)j + (dz)k$ then

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Directional derivative :- The directional derivative of a scalar point function ϕ at a point $P(x, y, z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e} \cdot \text{grad}\phi = \bar{e} \cdot \nabla\phi$

Note :- If $xi + yj + zk$ be the any vector and it's ~~directional derivative~~ unit normal vector $(\bar{e})$ is given by $\bar{e} = \dfrac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$

Gradient of a function of a function

Let $v = f(u)$ where $u = u(x, y, z)$ then

$$\nabla V = \nabla[f(u)] = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(u)$$

$$= i\frac{\partial}{\partial x} f(u) + j\frac{\partial}{\partial y} f(u) + k\frac{\partial}{\partial z} f(u)$$

$$= f'(u)\left[i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right]$$

$$= f'(u)\,\nabla u$$

$$\therefore \boxed{\nabla f(u) = f'(u)\,\nabla u}$$

① If $a = x + y + z$, $b = x^2 + y^2 + z^2$, $c = xy + yz + zx$ then prove that $[\text{grad}\,a, \text{grad}\,b, \text{grad}\,c] = 0$

Sol:- $a = x + y + z$

$$\text{grad}\,a = \nabla a = \left(i\frac{\partial a}{\partial x} + j\frac{\partial a}{\partial y} + k\frac{\partial a}{\partial z}\right) = i + j + k$$

$$\text{grad}\,b = \nabla b = \left(i\frac{\partial b}{\partial x} + j\frac{\partial b}{\partial y} + k\frac{\partial b}{\partial z}\right) = 2xi + 2yj + 2zk$$

$$\text{grad}\,c = \nabla c = \left(i\frac{\partial c}{\partial x} + j\frac{\partial c}{\partial y} + k\frac{\partial c}{\partial z}\right) = (y+z)i + (z+x)j + (x+y)k$$

$$[\text{grad}\,a, \text{grad}\,b, \text{grad}\,c] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0$$

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(2) Prove that $\nabla(r^n) = n r^{n-2}\,\bar{r}$.

Sol:- Let $\bar{r} = xi + yj + zk$ and let $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

then $r^2 = x^2+y^2+z^2$

Diff w.r.to x

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

By $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\Rightarrow \nabla(r^n) = \sum i \frac{\partial}{\partial x}(r^n)$$

$$= \sum i . n r^{n-1} \frac{\partial r}{\partial x}$$

$$= \sum i . n r^{n-1} \frac{x}{r}$$

$$= \sum i . n r^{n-2} x$$

$$= n r^{n-2} . \sum x i$$

$$\boxed{\nabla(r^n) = n r^{n-2}\,\bar{r}}$$

(3) Find a unit normal vector to the given surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$.

Sol:- Let $f = x^2y + 2xz - 4$.

$$\text{grad} f = \nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = (2xy + 2z)\,i + x^2 j + 2x k.$$

$$\text{grad} f \text{ at } (2,-2,3) = (-8+6)\,i + 4j + 4k$$

$$= -2i + 4j + 4k$$

The unit normal vector along grad f is $= \frac{\nabla f}{|\nabla f|} = \frac{-2i+4j+4k}{\sqrt{4+16+16}}$

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(4) Evaluate the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Sol:-

$$\text{grad} f = yi + xj - 2zk$$

Let $n_1 = (\text{grad} f)_{(4,1,2)} = i + 4j - 4k$

$n_2 = (\text{grad} f)_{(3,3,-3)} = 3i + 3j + 6k$

Let θ be the angle between the two normals

$$\therefore \cos\theta = \frac{n_1 n_2}{|\bar{n}_1||\bar{n}_2|} = \frac{(i+4j-4k).(3i+3j+6k)}{\sqrt{1+16+16}\sqrt{9+9+36}}$$

$$\cos\theta = \frac{-9}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{3}\sqrt{11}\sqrt{9}\sqrt{2}\sqrt{3}}$$

$$= \frac{-9}{3\sqrt{22}} = \frac{-1}{\sqrt{22}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right)$$

(5) Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $i + 2j + 2k$ at the point $(1,2,0)$.

Sol:- Directional derivative of f along the given direction is $= \bar{e}.\nabla f$

$$\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xy + yz + zx) = (y+z)i + (z+x)j + (x+y)k$$

If $\bar{e}$ is the unit vector along $i+2j+2k$ then

$$\bar{e} = \frac{i+2j+2k}{\sqrt{1+4+4}} = \frac{1}{3}(i+2j+2k)$$

$\therefore$ Directional derivative $= \bar{e}.\nabla f$

$$= \frac{1}{3}(i+2j+2k)\left((y+z)i + (z+x)j + (x+y)k\right)$$

$$= \frac{1}{3}\left[(y+z) + 2(z+x) + 2(x+y)\right]$$

at $(1,2,0)$

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Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P = (1, 2, 3)$ in the direction of the line PQ where $Q = (5, 0, 4)$.

Sol:- The position vectors of P and Q with respect to the origin are

$\overline{OP} = i + 2j + 3k$ and

$\overline{OQ} = 5i + 4k$

$\therefore \overline{PQ} = \overline{OQ} - \overline{OP} = 4i - 2j + k$

Let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$ then $\bar{e} = \dfrac{4i - 2j + k}{\sqrt{21}}$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 2xi - 2yj + 4zk$$

$\therefore$ Directional derivative $= \bar{e} \cdot \nabla f$

$$= \left(\frac{4i - 2j + k}{\sqrt{21}}\right)(2xi - 2yj + 4zk)$$

$$= \frac{1}{\sqrt{21}}(8x + 4y + 4z)$$

at $(1, 2, 3)$

$$= \frac{1}{\sqrt{21}}(8(1) + 4(2) + 4(3))$$

Directional derivative $= \dfrac{28}{\sqrt{21}}$ //

(7) Find the directional derivative of $1/r$ in the direction of $\bar{r} = xi + yj + zk$ at $(1, 1, 2)$.

Sol:- Given $\bar{r} = xi + yj + zk$ and $r = \sqrt{x^2 + y^2 + z^2}$

$$\therefore \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \frac{-xi}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-yj}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-zk}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{-\bar{r}}{r^{3/2}}$$

$\therefore$ Directional derivative $= \bar{e} \cdot \nabla\left(\frac{1}{r}\right)$

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$$= \frac{-\bar{r}}{r^{3/2}} \cdot \frac{\bar{r}}{|\bar{r}|}$$

$$= \frac{-(\bar{r})^2}{r^{3/2}\, r}$$

$$= -\frac{(x^2+y^2+z^2)}{r^{3/2}} = \frac{-r^2}{r^{5/2}} = \frac{1}{r^{5/2-2}} = -\frac{1}{r^{1/2}} = -\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= -\frac{1}{\sqrt{r}} \quad \text{at } (1,1,2) = \frac{1}{\sqrt{6}}$$

(8) Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the normal to the surface $f(x,y,z) = x\log z - y^2$ at $(-1,2,1)$.

Sol:- Given $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,-1)$

& $f(x,y,z) = x\log z - y^2$ at $(-1,2,1)$

Now $\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$$= (2xyz + 4z^2)\bar{i} + (x^2z)\bar{j} + (x^2y + 8xz)\bar{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = 8\bar{i} - \bar{j} - 10\bar{k}$$

Unit normal to the surface $f(x,y,z)$ is $\frac{\nabla f}{|\nabla f|} = \frac{\left(\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right)}{|\nabla f|}$

at $(-1,2,1) \rightarrow$ $= \frac{\log z\,\bar{i} + (-2y)\bar{j} + \frac{x}{z}\bar{k}}{\sqrt{16+1}}$

$$= \frac{-4j - k}{\sqrt{17}}$$

$\therefore$ Directional derivative $= \nabla\phi \cdot \nabla f \rightarrow = \frac{(8\bar{i} - j - 10k)(-4j - k)}{\sqrt{17}}$

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H.W

1) Find the directional derivative of $\nabla \cdot \nabla \phi$ at the point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $\phi = 2x^3y^2z^4$.

Ans : $1724/\sqrt{21}$

2) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(-1, 1, 2)$

Ans : $a = 5/2$ $b = 1$

3) Find the unit normal vector to the surface $z = x^2 + y^2$ at $(-1, -2, 5)$

Ans : $= -\frac{1}{\sqrt{21}}(2i + 4j + k)$

4) Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$

Ans : $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

5) If $\bar{a}$ is constant vector then prove that $\text{grad}(\bar{a} \cdot \bar{r}) = \bar{a}$.

proof:- Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ where a_1, a_2, a_3 are constants.

$$\bar{a} \cdot \bar{r} = (a_1 i + a_2 j + a_3 k)(xi + yj + zk) = a_1 x + a_2 y + a_3 z$$

$$\text{grad}(\bar{a} \cdot \bar{r}) = \nabla(\bar{a} \cdot \bar{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(a_1 x + a_2 y + a_3 z)$$

$$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

$$\therefore \boxed{\text{grad}(\bar{a} \cdot \bar{r}) = \bar{a}}$$

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Divergence of a vector

Let $\bar{f}$ be any continuously differentiable vector point function, then $\bar{i}\cdot\frac{\partial \bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as $\operatorname{div}\bar{f}$.

i.e; $$\operatorname{div}\bar{f} = \bar{i}\cdot\frac{\partial \bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{f}$$

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f} \quad \text{but}$$

properties

1) $\operatorname{div}\bar{f} = \Sigma\bar{i}\left(\frac{\partial \bar{f}}{\partial x}\right)$

2) $\operatorname{div}(\bar{f}\pm\bar{g}) = \operatorname{div}\bar{f} \pm \operatorname{div}\bar{g}$

3) $(\bar{a}\cdot\nabla)\phi = \Sigma(\bar{a}\cdot i)\frac{\partial\phi}{\partial x}$ if ϕ is a scalar & $\bar{f}$ is a vector function

4) $(\bar{a}\cdot\nabla)\bar{f} = \Sigma(\bar{a}\cdot\bar{i})\frac{\partial \bar{f}}{\partial x}$ " "

Solenoidal vector

A vector point function $\bar{f}$ is said to be solenoidal if $\operatorname{div}\bar{f} = 0$

This equation is also called the equation of continuity (or) conservation of mass.

Note:- If the vector function $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

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Problems

① If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

Sol:- Given $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$\operatorname{div}\bar{f} = y^2 + 2x^2z - 6yz$$

$$(\operatorname{div}\bar{f})_{(1,-1,1)} = 1 + 2 + 6 = 9$$

② If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal then find p.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$

$= f_1 i + f_2 j + f_3 k$ then

$$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1 + 1 + p$$

$$= 2 + p$$

Since $\bar{f}$ is solenoidal, we have $\operatorname{div}\bar{f} = 0$

$\Rightarrow 2 + p = 0$

$\Rightarrow p = -2$

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③ Find div $\bar{f}$ where $\bar{f} = r^n \bar{r}$. Find n if it is solenoidal?

(or)

Prove that $r^n \bar{r}$ is solenoidal if $n = -3$.

(or)

Prove that $\text{div}(r^n \bar{r}) = (n+3) r^n$. Hence show that $\frac{\bar{r}}{r^3}$ is solenoidal.

Sol:- Given $\bar{f} = r^n \bar{r}$.

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x$

$\Rightarrow x\frac{\partial r}{\partial x} = \frac{x^2}{r}$.

$$\bar{f} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\text{div}\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n + x n r^{n-1}\frac{\partial r}{\partial x} + r^n + y n r^{n-1}\frac{\partial r}{\partial y} + r^n + z n r^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + \frac{n r^{n-1}}{r}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3r^n + n r^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right]$$

$$= 3r^n + \frac{n r^{n-1}}{r}(r^2)$$

$$= 3r^n + n r^n$$

$$\boxed{\text{div}\bar{f} = (3+n) r^n}$$

If $n = -3$

$$\boxed{\text{div}\bar{f} = 0} \Rightarrow \text{solenoidal.}$$

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(4) Show that $\frac{\bar{r}}{r^3}$ is solenoidal

(or)

Evaluate $\nabla \cdot \left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\bar{r}}{r^3} = \bar{r} \cdot \bar{r}^{-3} = \bar{r}^{-3} x i + \bar{r}^{-3} y j + \bar{r}^{-3} z k = f_1 i + f_2 j + f_3 k$$

Hence $\nabla \cdot \left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ —— (1)

$$f_1 = \bar{r}^{-3} x$$

$$\frac{\partial f_1}{\partial x} = \bar{r}^{-3} \cdot 1 + x \cdot (-3) \bar{r}^{-4} \frac{\partial r}{\partial x} = \bar{r}^{-3} - 3x^2 \bar{r}^{-5}$$

~~Similarly $\frac{\partial f_2}{\partial y} =$~~

From (1)

$$\nabla \left(\frac{\bar{r}}{r^3}\right) = \sum \frac{\partial f}{\partial x} = 3\bar{r}^{-3} - 3\bar{r}^{-5} \sum x^2$$

$$= 3\bar{r}^{-3} - 3\bar{r}^{-5}(x^2+y^2+z^2)$$

$$= 3\bar{r}^{-3} - 3\bar{r}^{-5} r^2 \quad (\because r^2 = x^2+y^2+z^2)$$

$$= 3\bar{r}^{-3} - 3\bar{r}^{-3}$$

$$\therefore \nabla \left(\frac{\bar{r}}{r^3}\right) = 0$$

$\therefore$ Hence $\frac{\bar{r}}{r^3}$ is solenoidal.

---

* Find $\operatorname{div}\left(\frac{\bar{r}}{r}\right)$

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(5) Find div $\bar{f}$ when $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$

$$\text{grad}\,\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\bar{f} = \text{grad}\,\phi = \bar{i}(3x^2-3yz) + j(3y^2-3xz) + k(3z^2-3xy)$$

Now $\text{div}\,\bar{f} = \nabla\cdot\bar{f}$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 6x + 6y + 6z$$

$$= 6(x+y+z) \quad //$$

Curl of a vector

Def:- Let $\bar{f}$ be any differentiable vector point function, then the vector function defined by $\bar{i}\times\frac{\partial\bar{f}}{\partial x} + \bar{j}\times\frac{\partial\bar{f}}{\partial y} + \bar{k}\times\frac{\partial\bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ (or) $(\nabla\times\bar{f})$.

$$\therefore \text{curl}\,\bar{f} = \bar{i}\times\frac{\partial\bar{f}}{\partial x} + \bar{j}\times\frac{\partial\bar{f}}{\partial y} + \bar{k}\times\frac{\partial\bar{f}}{\partial z}$$

$$\boxed{\text{curl}\,\bar{f} = \sum \bar{i}\times\frac{\partial\bar{f}}{\partial x}}$$

properties

1) If $\bar{f}$ is a differiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

then ~~$\text{curl}\,\bar{f} = i\times\frac{\partial\bar{f}}{\partial x} + j\times\frac{\partial\bar{f}}{\partial y} + k\times\frac{\partial\bar{f}}{\partial z}$~~

$$\text{curl}\,\bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

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2) $\text{curl}(\bar{a} \pm \bar{b}) = \text{curl}\,\bar{a} \pm \text{curl}\,\bar{b}$

Note:- If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ & $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ then

$$\text{curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational vector:- A vector $\bar{f}$ is said to be irrotational vector then $\text{curl}\,\bar{f} = 0$ //

problems

① If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ then find $\text{curl}\,\bar{f}$ at the point $(1,-1,1)$

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ then

$$\text{curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right) - \bar{j}\left(\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right)$$
$$+ \bar{k}\left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$\text{curl}\,\bar{f} = \bar{i}(-3z^2 - 2x^2y) + \bar{k}(4xyz - 2xy)$$

at $(1,-1,1)$

$$\text{curl}\,\bar{f} = \bar{i}(-3+2) + \bar{k}(-4+2)$$
$$= -\bar{i} - 2\bar{k}$$ //

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(2) Find curl $\bar{f}$ where $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol:-

Let $\phi = x^3+y^3+z^3-3xyz$ then

$$\bar{f} = \text{grad}\,\phi = \sum i \frac{\partial \phi}{\partial x}$$

$$= i\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + j\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$

$$+ k\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = i(3x^2-3yz) + j(3y^2-3xz) + k(3z^2-3xy)$$

Now

$$\text{Curl}\,\bar{f} = \nabla \times \bar{f}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i(-3x-(-3x)) - j(-3y+3y) + k(-3z+3z)$$

$$= 0$$

$$\text{Curl}\,\bar{f} = 0$$

ie; $\bar{f}$ is a irrotational vector

(3) If $\bar{F} = (x+y+1)i + j - (x+y)k$ then show that $\bar{F}\cdot\text{curl}\,\bar{F} = 0$

Sol:- $\text{Curl}\,\bar{F} = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$

$$= -i + j - k$$

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$\bar{F} \cdot \text{curl} F = \left[(x+y+1)i + j - (x+y)k\right] \cdot \left[-i + j - k\right]$

$= -x - y - 1 + 1 + x + y = 0$ //

(4) Prove that if $\bar{r}$ is the position vector of any point in space, then $r^n \bar{r}$ is irrotational.

(or)

Show that $\text{curl}(r^n \bar{r}) = 0$

proof:-

$\bar{r} = xi + yj + zk \quad r = |\bar{r}|$

$\Rightarrow r^2 = x^2 + y^2 + z^2 \quad \Rightarrow 2r\frac{\partial r}{\partial y} = 2y$

$\frac{\partial r}{\partial y} = \frac{y}{r}$

Now $r^n \bar{r} = r^n(xi + yj + zk)$

$\text{Curl}(r^n \cdot \bar{r}) = \nabla \times (r^n \bar{r})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y} zr^n - \frac{\partial}{\partial z} yr^n\right) - j\left(\frac{\partial}{\partial x} zr^n - \frac{\partial}{\partial z} xr^n\right) + k\left(\frac{\partial}{\partial x} yr^n - \frac{\partial}{\partial y} xr^n\right)$$

$$= i\left(znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right) - j\left(zn.r^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial z}\right) + k\left(ynr^{n-1}\frac{\partial r}{\partial y} - xn.r^{n-1}\frac{\partial r}{\partial y}\right)$$

$\cancel{= i\left(zn r^{n-1} \frac{y}{r}\right.}$

$= nr^{n-2}\left[(zy - yz)i + (xz - zx)j + (xy - yz)k\right]$

$= 0$ //

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(5) If $\bar{A}$ is irrotational vector, evaluate $\operatorname{div}(A \times \bar{r})$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Given $\bar{A}$ is an irrotational vector

i.e. $\nabla \times \bar{A} = 0$

Now ~~div (∇×~~

$a.(b \times c) = c.(a \times b)$
$= b.(a \times c)$

$$\operatorname{div}(A \times \bar{r}) = \nabla.(\bar{A} \times \bar{r})$$

$$= \bar{r}.(\nabla \times A) - \bar{A}.(\nabla \times \bar{r}) \quad (\because \nabla.(a \times b) = b.(\nabla \times a) - a.(\nabla \times b))$$

$$= \bar{r}.(0) - \bar{A}.(\nabla \times \bar{r})$$

$$= -\bar{A}.(\nabla \times \bar{r})$$

Here $\nabla \times \bar{r} = i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - j\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$

$\therefore A.(\nabla \times \bar{r}) = 0$

$\therefore \boxed{\operatorname{div}(A \times \bar{r}) = 0}$

(6) Find constants a, b, c so that the vector $\bar{A} = (x + 2y + az)\bar{i} + (bx - 3y - z)\bar{j} + (4x + cy + 2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$

Sol:- Given $\bar{A} = (x + 2y + az)\bar{i} + (bx - 3y - z)\bar{j} + (4x + cy + 2z)\bar{k}$

$\bar{A}$ is irrotational $\Rightarrow \operatorname{Curl}\bar{A} = 0$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

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$= \bar{i}(c+1) + \bar{j}(a-4) + \bar{k}(b-2) = 0 = 0\bar{i} + 0\bar{j} + 0\bar{k}$

Comparing $\longrightarrow$ $c+1 = 0 \Rightarrow c = -1$

$a = 4, \quad b = 2$

$\therefore \quad \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$

Now $\quad \bar{A} = \nabla\phi$

$\Rightarrow (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = x+2y+4z \qquad \frac{\partial\phi}{\partial y} = 2x-3y-z$

$\Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz \qquad \phi = 2xy - \frac{3y^2}{2} - zy$

$\frac{\partial\phi}{\partial z} = 4x - y + 2z \Rightarrow \phi = 4xz - yz + z^2$ //

H.W

① If $\bar{a}$ is a Constant vector, prove that $\text{Curl}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}\cdot\bar{r})$

② Find whether the function $\bar{F} = (x^3-y^3)i + (y^2-3x)j + (z^2-xy)k$ is irrotational and hence find scalar potential function.

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Operators

① Vector differential operator :- (∇)

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

ie; $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

ie; $\nabla\phi = \text{grad}\,\phi = \sum i\frac{\partial\phi}{\partial x}$

② Scalar differential operator ($\bar{a}.\nabla$)

$\bar{a}.\nabla = (\bar{a}.i)\frac{\partial}{\partial x} + (\bar{a}.j)\frac{\partial}{\partial y} + (\bar{a}.\bar{k})\frac{\partial}{\partial z}$ is defined s.t

ie; i) $(\bar{a}.\nabla)\phi = (\bar{a}.i)\frac{\partial\phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$

ii) $(\bar{a}.\nabla)\bar{f} = (\bar{a}.i)\frac{\partial\bar{f}}{\partial x} + (\bar{a}.j)\frac{\partial\bar{f}}{\partial y} + (\bar{a}.\bar{k})\frac{\partial\bar{f}}{\partial z}$ //

③ Vector differential operator ($\bar{a}\times\nabla$)

$\bar{a}\times\nabla = (\bar{a}\times i)\frac{\partial}{\partial x} + (\bar{a}\times j)\frac{\partial}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial}{\partial z}$ is defined s.t

i) $(\bar{a}\times\nabla)\phi = (\bar{a}\times\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\phi}{\partial z}$

ii) $(\bar{a}\times\nabla).\bar{f} = (\bar{a}\times\bar{i})\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\bar{f}}{\partial z}$

iii) $(\bar{a}\times\nabla)\times\bar{f} = (a\times i)\times\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times\bar{j})\times\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times\bar{k})\times\frac{\partial\bar{f}}{\partial z}$

④ Scalar differential operator ∇.

$\nabla = \bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}.\frac{\partial}{\partial z}$ is defined s.t

$\nabla.\bar{f} = \bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z} = \text{div}\,\bar{f}$.

⑤ Vector differential operator $\nabla\times$

$\nabla\times = i\times\frac{\partial}{\partial x} + j\times\frac{\partial}{\partial y} + k\times\frac{\partial}{\partial z}$ is defined s.t

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$$\nabla \times f = i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + k \times \frac{\partial \bar{f}}{\partial z} = \text{curl} \bar{f}$$

(6) Laplacian Operator ∇^2

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ is called Laplacian operator.

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \sum \frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi .$$

Note:- (1) $\nabla^2 \phi = \nabla \cdot \nabla \phi = \text{div}(\text{grad} \phi)$

(2) If $\nabla^2 \phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a ~~harmi~~ harmonic function.

Problems

(1) Prove that $\text{div}(\text{grad}\, r^m) = m(m+1) r^{m-2}$

(or)

$$\nabla^2 (r^m) = m(m+1) r^{m-2} \quad (\text{or}) \quad \nabla^2 (r^n) = n(n+1) r^{n-2}$$

proof:- Let $\bar{r} = xi + yj + zk$ and $r = |\bar{r}| \Rightarrow r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\text{grad}\, r^m = \sum i \frac{\partial}{\partial x} (r^m)$

$$= \sum i . m . r^{m-1} \frac{\partial r}{\partial x}$$

$$= \sum i . m r^{m-1} \left(\frac{x}{r} \right)$$

$$= \sum i m r^{m-2} x$$

Now $\text{div}(\text{grad}\, r^m) = \nabla . \text{grad}\, r^m$

$$= \sum \frac{\partial}{\partial x} m r^{m-2} x$$

$$= m \left[(m-2) r^{m-3} \frac{\partial r}{\partial x} . x + r^{m-2} \right]$$

$$= m \left[\sum (m-2) r^{m-4} . x^2 + r^{m-2} \right]$$

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$$= m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$$

$$= m\left[(m-2)r^{m-4}(r^2) + 3r^{m-2}\right]$$

$$= m\left[(m-2)r^{m-2} + 3r^{m-2}\right]$$

$$= m\left[(m-2+3)r^{m-2}\right]$$

$$\boxed{\text{div}(\text{grad}\, r^m) = m(m+1)r^{m-2}}$$

(or)

$$\boxed{\nabla^2(r^m) = m(m+1)r^{m-2}}$$ //

② Prove that i) $(\bar{f}\times\nabla).r = 0$ ii) $(\bar{f}\times\nabla)\times\bar{r} = -2\bar{f}$

Sol:- i) $(\bar{f}\times\nabla).r = \sum(f\times i).\frac{\partial\bar{r}}{\partial x}$

$$= \sum(f\times i).i = 0$$

ii) $\bar{f}\times\nabla = (f\times i)\frac{\partial}{\partial x} + (f\times j)\frac{\partial}{\partial y} + (f\times k)\frac{\partial}{\partial z}$

$$(\bar{f}\times\nabla)\times\bar{r} = (f\times i)\times\frac{\partial r}{\partial x} + (f\times j)\times\frac{\partial r}{\partial y} + (f\times k)\times\frac{\partial r}{\partial z}$$

$$= \sum(f\times i)\times i \quad (\because (a\times b)\times c = (a.c)b - (b.c)a$$

$$= \sum[(f.i)i - \bar{f}] \;?$$

$$= (\bar{f}.i)i + (\bar{f}.j)j + (\bar{f}.k)k - 3\bar{f}$$

$$= \cancel{3f}\;\bar{f} - 3\bar{f} = -2\bar{f}$$

③ If $f = (x^2+y^2+z^2)^{-n}$ then find div grad f and determine n if div grad f = 0

Sol:- Let $f = (x^2+y^2+z^2)^{-n}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

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$r = |\bar{r}| \Rightarrow r^2 = x^2 + y^2 + z^2$

$f(r) = (r^2)^{-n} = r^{-2n}$

$f'(r) = -2n\, r^{-2n-1}$

$f''(r) = (-2n)(-2n-1)\, r^{-2n-2} = 2n(2n+1)\, r^{-2n-2}$

$\text{div grad} f = \nabla^2 f(r)$

$= f''(r) + \frac{2}{r} f'(r)$? By Theorem $\left(\because \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)\right)$

$= (2n)(2n+1)\, r^{-2n-2} - 4n\, r^{-2n-2}$

$= r^{-2n-2}\left[2n(2n+1) - 4n\right]$

$= (2n)(2n-1)\, r^{-2n-2}$

Since $\text{div grad} \bar{f} = 0$

$\Rightarrow 2n(2n-1)\, r^{-2n-2} = 0$

$\Rightarrow 2n(2n-1) = 0$

$\Rightarrow 2n = 0$ or $2n - 1 = 0$

$n = 0$ or $n = 1/2$

H.W

① Prove that $\nabla \times \left(\frac{\bar{A} \times \bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}\cdot\bar{A})\bar{r}}{r^{n+2}}$

② ~~Find~~ Show that $\nabla^2[f(r)] = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f''(r) + \frac{2}{r} f'(r)$

where $r = |\bar{r}|$.

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Vector Identities

① Prove that $\text{curl}(\phi\bar{a}) = (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a}$

proof:- $\text{curl}(\phi\bar{a}) = \nabla\times(\phi\bar{a})$

$$= \sum i\times\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \sum i\times\left[\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum\left(i\frac{\partial\phi}{\partial x}\right)\times\bar{a} + \sum\left(i\times\frac{\partial\bar{a}}{\partial x}\right)\phi$$

$$= \cancel{\nabla\times\bar{a}}\ \nabla\phi\times\bar{a} + (\nabla\times\bar{a})\phi$$

$$= (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a}$$

② Prove that $\text{grad}(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times\text{curl}\,\bar{a} + \bar{a}\times\text{curl}\,\bar{b}$

(or)

P.T. $\nabla(\bar{A}\cdot\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} + (\bar{A}\cdot\nabla)\bar{B} + \bar{B}\times(\nabla\times\bar{A}) + \bar{A}\times(\nabla\times\bar{B})$

(or)

P.T. $\text{grad}(\bar{F}\cdot\bar{G}) = \bar{F}\times(\nabla\times\bar{G}) + \bar{G}(\nabla\times\bar{F}) + (\bar{F}\cdot\nabla)\bar{G} + (\bar{G}\cdot\nabla)\bar{F}$

proof:- Consider

$$\bar{a}\times\text{curl}\,\bar{b} = \bar{a}\times(\nabla\times\bar{b}) = \bar{a}\times\sum i\times\frac{\partial\bar{b}}{\partial x}$$

$(\because a\times(b\times c))$

$$= \sum\bar{a}\times\left(i\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum\left[\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right)i - (\bar{a}\cdot i)\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \sum i\left\{a\cdot\frac{\partial\bar{b}}{\partial x}\right\} - \left\{\bar{a}\sum i\frac{\partial}{\partial x}\right\}\bar{b}$$

$$\bar{a}\times\text{curl}\,\bar{b} = \sum i\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} \quad \text{—— ①}$$

Similarly

$$\bar{b}\times\text{curl}\,\bar{a} = \sum i\left(\bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a} \quad \text{—— ②}$$

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① + ② $\Rightarrow (\bar{a} \times \text{curl}\,\bar{b}) + (\bar{b} \times \text{curl}\,\bar{a}) = \sum i\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a} + \sum i\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b}$

$$\Rightarrow \text{R.H.S} = \cancel{(\bar{b}\cdot\nabla)\bar{a}} + \cancel{(\bar{a}\cdot\nabla)\bar{b}} + \sum i\left(\bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right) - \cancel{(\bar{b}\cdot\nabla)\bar{a}} + \sum i\left(\bar{a}\cdot\frac{\partial \bar{b}}{\partial x}\right) - \cancel{(\bar{a}\cdot\nabla)\bar{b}}$$

$$= \sum i\left(\bar{a}\frac{\partial \bar{b}}{\partial x} + \bar{b}\cdot\frac{\partial \bar{a}}{\partial x}\right)$$

$$= \sum i\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b})$$

$$= \nabla(\bar{a}\cdot\bar{b})$$

$$= \text{grad}(\bar{a}\cdot b) = \text{L.H.S}$$

③ P.T. $\text{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\text{curl}\,\bar{a} - \bar{a}\cdot\text{curl}\,\bar{b}$.

(or)

$$\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot(\nabla\times\bar{a}) - \bar{a}\cdot(\nabla\times\bar{b})$$

Proof

$$\text{div}(\bar{a}\times\bar{b}) = \sum i\cdot\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$$

$$= \sum i\cdot\left(\frac{\partial \bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial \bar{b}}{\partial x}\right)$$

$$= \sum i\cdot\left(\frac{\partial \bar{a}}{\partial x}\times\bar{b}\right) + \sum i\left(\bar{a}\times\frac{\partial \bar{b}}{\partial x}\right)$$

$$= \cancel{\sum\left(i\times\frac{\partial \bar{a}}{\partial x}\right)\cdot\bar{b} + \sum i\times\left(\bar{a}\times\frac{\partial}{}\right.}$$

$$= \sum\left(i\times\frac{\partial \bar{a}}{\partial x}\right)\cdot\bar{b} - \sum\left(i\times\frac{\partial \bar{b}}{\partial x}\right)\cdot\bar{a}$$

$$= (\nabla\times\bar{a})\cdot\bar{b} - (\nabla\times\bar{b})\cdot\bar{a}$$

$$= \bar{b}\cdot\text{curl}\,\bar{a} - \bar{a}\cdot\text{curl}\,\bar{b}$$

$= (a.c)b - (a.b)c$

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④ Prove that curl grad $\phi = 0$

Proof:- Let ϕ be any scalar point function, then $\text{grad}\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$\text{Curl grad}\,\phi = \nabla \times \text{grad}\,\phi$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)$$

$$+ \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \bar{0}$$

$\therefore$ curl grad $\phi = 0$

Thus grad ϕ is always irrotational.

Note: Similary we can prove div curl $\bar{f} = 0$

⑤ P.T. $\nabla \times (\nabla \times a) = \nabla(\nabla \cdot \bar{a}) - \nabla^2 \bar{a}$

proof:- $i \times \frac{\partial}{\partial x}(\nabla \times a)$ ~~$\nabla \times (\nabla \times a)$~~ $= i \times \frac{\partial}{\partial x}(\nabla \times \bar{a})$

$$= i \times \frac{\partial}{\partial x}\left(i \times \frac{\partial \bar{a}}{\partial x} + j \times \frac{\partial \bar{a}}{\partial y} + \bar{k} \times \frac{\partial \bar{a}}{\partial z}\right)$$

$$= i \times \left(\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2} + j \times \frac{\partial^2 \bar{a}}{\partial x\partial y} + k \times \frac{\partial^2 \bar{a}}{\partial x\partial z}\right)$$

$$= i \times \left[i \times \frac{\partial^2 \bar{a}}{\partial x^2}\right] + i \times \left[j \times \frac{\partial^2 \bar{a}}{\partial x\partial y}\right] + i \times \left[k \times \frac{\partial^2 \bar{a}}{\partial x\partial z}\right]$$

$$= \left(i \cdot \frac{\partial^2 \bar{a}}{\partial x^2}\right) i - \frac{\partial^2 \bar{a}}{\partial x^2} + \left(i \cdot \frac{\partial^2 \bar{a}}{\partial x\partial y}\right) j + \left(i \cdot \frac{\partial^2 \bar{a}}{\partial x\partial z}\right) \bar{k} \;?$$

$$= i\frac{\partial}{\partial x}\left(i \cdot \frac{\partial \bar{a}}{\partial x}\right) + j\frac{\partial}{\partial y}\left(i \cdot \frac{\partial \bar{a}}{\partial x}\right) + k\frac{\partial}{\partial z}\left(i \cdot \frac{\partial \bar{a}}{\partial x}\right) - \frac{\partial^2 \bar{a}}{\partial x^2}$$

$$= \nabla\left(i \cdot \frac{\partial \bar{a}}{\partial x}\right) - \frac{\partial^2 \bar{a}}{\partial x^2}$$

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$$= \sum i \times \frac{\partial}{\partial x}(\nabla \times \bar{a}) = \nabla \sum i \cdot \frac{\partial \bar{a}}{\partial x} - \sum \frac{\partial^2 \bar{a}}{\partial x^2}$$

$$= \nabla(\nabla \cdot \bar{a}) - \left[\frac{\partial^2 \bar{a}}{\partial x^2} + \frac{\partial^2 \bar{a}}{\partial y^2} + \frac{\partial^2 \bar{a}}{\partial z^2}\right]$$

$$\nabla \times (\nabla \times \bar{a}) = \nabla(\nabla \cdot a) - \nabla^2 \bar{a}$$

$$\boxed{\text{curl curl } \bar{a} = \text{grad div } \bar{a} - \nabla^2 \bar{a}}$$

(6) Prove that $(\nabla f \times \nabla g)$ is solenoidal.

proof:- w.k.t. $\text{div}(\bar{a} \times \bar{b}) = \bar{b}\,\text{curl}\,\bar{a} - \bar{a}\,\text{curl}\,\bar{b}$

~~Take~~ by taking $\bar{a} = \nabla f$ & $\bar{b} = \nabla g$

$$\text{div}(\nabla f \times \nabla g) = \nabla g\, \text{curl}(\nabla f) - \nabla f \cdot \text{curl}(\nabla g)$$

$$= 0 - 0 \qquad (\because \text{curl}(\nabla f) = \text{curl}(\nabla g) = 0)$$

$$\text{div}(\nabla f \times \nabla g) = 0$$

$\therefore$ $\nabla f \times \nabla g$ is solenoidal.

(7) Prove that $\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = -\frac{2}{r^3}\bar{r}$.

proof:- $\nabla\left(\frac{\bar{r}}{r}\right) = \sum i \frac{\partial}{\partial x}\left(\frac{\bar{r}}{r}\right)$

$$= \sum i \cdot \left[\frac{1}{r}\frac{\partial \bar{r}}{\partial x} + \bar{r}\left(-\frac{1}{r^2}\right)\left(\frac{x}{r}\right)\right]$$

$$= \sum i \left[\frac{1}{r} i - \frac{\bar{r}}{r^3} x\right]$$

$$= \frac{1}{r}\sum i \cdot i - \frac{1}{r^3} r^2$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = \nabla\left(\frac{2}{r}\right) = \sum i \frac{\partial}{\partial x}\left(\frac{2}{r}\right) = \sum i\left(-\frac{2}{r^2}\right)\left(\frac{x}{r}\right)$$

$$= \frac{-2}{r^3}\sum x i$$

$$= \frac{-2\bar{r}}{r^3} \quad //$$

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(8) Find $(A.\nabla)\phi$ at $(1,-1,1)$ if $A = 3xyz^2 i + 2xy^3 j - x^2yz k$ and $\phi = 3x^2 - yz$

Sol:- Given $A = 3xyz^2 i + 2xy^3 j - x^2yz k$ and $\phi = 3x^2 - yz$

W.K.T. $(A.\nabla)\phi = (A.i)\frac{\partial\phi}{\partial x} + (A.j)\frac{\partial\phi}{\partial y} + (A.k)\frac{\partial\phi}{\partial z}$

$$= (3xyz^2)(6x) + (2xy^3)(-z) + (-x^2yz)(-y)$$

$$(A.\nabla)\phi = 18x^2y^2z^2 - 2xy^3z + x^2y^2z$$

$(A.\nabla)\phi$ at $(1,-1,1) = 18(1)(-1)(1) - 2(1)(-1)(1) + 1(-1)(1)$

$$= -18 + 2 + 1$$

$$= -15.$$

(9) Evaluate $\nabla.\left[r.\nabla\left(\frac{1}{r^3}\right)\right]$ where $r = \sqrt{x^2+y^2+z^2}$

Sol:- Given $r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \nabla\left(\frac{1}{r^3}\right) = \text{grad}(r^{-3})$$

$$= \sum i \frac{\partial}{\partial x}(r^{-3})$$

$$= \sum i(-3r^{-4})\frac{\partial r}{\partial x}$$

$$= \sum i(-3r^{-4})\frac{x}{r}$$

$$\nabla\left(\frac{1}{r^3}\right) = -3r^{-5}(xi + yj + zk)$$

$$r.\nabla\left(\frac{1}{r^3}\right) = -3r^{-4}(xi + yj + zk) = f_1 i + f_2 j + f_3 k$$

$$\nabla\left[r.\nabla\left(\frac{1}{r^3}\right)\right] = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \sum \frac{\partial f_1}{\partial x}$$

$$f_1 = -3r^{-4}x$$

$$\frac{\partial f_1}{\partial x} = -3r^{-4}.1 + 12r^{-5}x.\frac{\partial r}{\partial x} = -3r^{-4} + 12r^{-5}.x.\frac{x}{r}$$

$$= -3r^{-4} + 12r^{-6}.x^2$$

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$$\nabla\cdot\left(r\cdot\nabla\left(\frac{1}{r^3}\right)\right) = \sum\left(-3r^{-4}+12r^{-6}x^2\right)$$

$$= -3(3r^{-4}) + 12r^{-6}\sum x^2$$

$$= -9r^{-4}+12r^{-6}r^2$$

$$= -9r^{-4}+12r^{-4}$$

$$= 3r^{-4}$$

$$\therefore \nabla\left(r\cdot\nabla\left(\frac{1}{r^3}\right)\right) = \frac{3}{r^4} \quad //$$

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– Vector Integration –

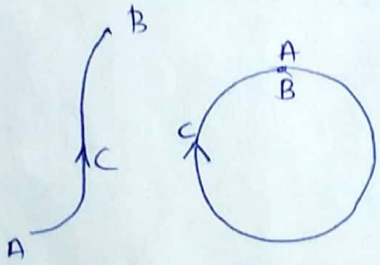
Definite integral :– If $\int \bar{f}(t)dt = \bar{F}(t) + c$ then $\int_a^b \bar{f}(t)dt = \bar{F}(b) - \bar{F}(a)$

This is called the definite integral of $\bar{f}(t)$ between the limits $t = a$ and $t = b$

If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ then

$$\int_a^b \bar{f}(t)dt = \bar{i}\int_a^b f_1(t)dt + \bar{j}\int_a^b f_2(t)dt + \bar{k}\int_a^b f_3(t)dt$$

Closed curve :– Let C be ~~the~~ a curve in space. Let A be the initial point and B be the terminal point of the curve C. When the direction along C oriented from A to B is positive then the direction from B to A is called negative direction. If the ~~two~~ two points A and B coinside the curve C is called the closed curve.



Smooth curve :– A curve $\bar{r} = \bar{f}(t)$ is called a smooth curve if $\bar{f}(t)$ is continuously differentiable.

Line integral :– Any integral which is to be evaluated along a curve is called a line integral.

Circulation :– If $\bar{v}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \bar{v}.d\bar{r}$ is called the circulation of $\bar{v}$ round the curve C.

If $\oint_C \bar{v}.d\bar{r} = 0$ then the field $\bar{v}$ is called Conservative.

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Work done by a force:- If $\bar{F}$ represents the force vector acting on a particle moving along an arc AB, then the total work done by $\bar{F}$ during displacement from A to B is given by the line integral

$$\int_A^B \bar{F}.d\bar{r}.$$

Note:- If $\bar{F}$ is conservative, $\bar{F} = \nabla\phi$ then the work done is independent of the path and vice-versa

ie; $\bar{F} = \nabla\phi$

$\Rightarrow \text{Curl}\,\bar{F} = 0$

problem

① If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$ evaluate $\int_C \bar{F}.d\bar{r}$ where C is the curve $y = 2x^2$ in the xy plane from $(0,0)$ to $(1,2)$

Sol:- The equ of the curve C is $y = 2x^2$

$\Rightarrow dy = 4x\,dx$

Given $\bar{F} = 3xy\bar{i} - y^2\bar{j}$

Since the integration is performed in the xy-plane, therefore $\bar{r} = x\bar{i} + y\bar{j}$ and

$\bar{F}.d\bar{r} = F_1dx + F_2dy$ and x varies from 0 to 1

$$\therefore \int_C \bar{F}.d\bar{r} = \int_C F_1dx + F_2dy$$

$$= \int_C 3xy\,dx - y^2dy$$

$$= \int_0^1 3x(2x^2)dx - 4x^4(4x)dx$$

$$= \int_0^1 (6x^3 - 16x^5)dx$$

$$= \left(\frac{6x^4}{4} - \frac{16x^6}{6}\right)_0^1 = -7/6$$

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(2) Find the work done by $\bar{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$ along a curve C in the xy-plane is given by

i) $x^2+y^2=9$, $z=0$

ii) $x^2+y^2=4$, $z=0$

Sol:- We have $\bar{F} = (2x-y-z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

In the xy-plane, $z=0 \Rightarrow dz = 0$

$$\therefore \bar{F}\cdot d\bar{r} = (2x-y)\,dx + (x+y)\,dy$$

i) Now work done $= \int_C \bar{F}\cdot d\bar{r}$ where C is the circle $x^2+y^2=9$

$$\text{Work done} = \int_C (2x-y)\,dx + (x+y)\,dy$$

Take $x = 3\cos\theta$, $y = 3\sin\theta$ so that

$dx = -3\sin\theta\,d\theta$, $dy = 3\cos\theta\,d\theta$

then θ varies from 0 to 2π

$$\text{Work done} = \int_0^{2\pi} (6\cos\theta - 3\sin\theta)(-3\sin\theta\,d\theta) + (3\cos\theta + 3\sin\theta)(3\cos\theta\,d\theta)$$

$$= \int_0^{2\pi} (-18\cos\theta\sin\theta + 9\sin^2\theta + 9\cos^2\theta + 9\cos\theta\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} (9(\cos^2\theta + \sin^2\theta) - 9\cos\theta\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} (9 - 9\cos\theta\sin\theta)\,d\theta$$

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$$= \frac{1}{2}\int_0^{2\pi}(18 - 18\sin\theta\cos\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(18 - 9\sin 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[18\theta + \frac{9\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left\{\left[36\pi + \frac{9\cos 4\pi}{2}\right] - \left[18(0) + \frac{9\cos 0}{2}\right]\right\}$$

$$W.D. = \frac{1}{2}\left[36\pi + \frac{9(0)}{2}\right] = 18\pi$$

Similarly

ii) ~~W.D.~~ H.W

③ Find the workdone by the force $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz - x)\bar{k}$ when it moves a particle from the point $(0,0,0)$ to $(2,1,1)$ along the curve $x = 2t^2$, $y = t$ and $z = t^3$

Sol:- Here $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz - x)\bar{k}$

$$x = 2t^2 \Rightarrow dx = 4t\,dt$$

$$y = t \Rightarrow dy = dt$$

$$z = t^3 \Rightarrow dz = 3t^2\,dt$$

t is from 0 to 1

$$W.D = \int_C \bar{F}\cdot d\bar{r} = \int (2y+3)\,dx + (xz)\,dy + (yz - x)\,dz$$

$$= \int_0^1 (2t+3)\,4t\,dt + (2t^2)(t^3)\,dt + (t^4 - 2t^2)\,3t^2\,dt$$

$$= \int_0^1 (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)\,dt$$

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$$= \left[\frac{3t^7}{7} + \frac{2t^6}{6} - \frac{6t^5}{5} + \frac{8t^3}{3} + \frac{2t^2}{2}\right]_0^1$$

$$= \frac{288}{35}.$$

(4) Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sol:- $W.D = \int_C \bar{F}\cdot d\bar{r}$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$$

$$\bar{F}\cdot d\bar{r} = 3x^2dx + dy + z\,dz$$

$$\int_{(0,0,0)}^{(2,1,3)} \bar{F}\cdot d\bar{r} = \int_{(0,0,0)}^{(2,1,3)} (3x^2dx + dy + dz)$$

$$= \left(x^3 + y + \frac{z^2}{2}\right)_{(0,0,0)}^{(2,1,3)}$$

$$= 8 + 1 + \frac{9}{2} = \frac{27}{2}$$

(5) If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in xy-plane $y = x^3$ from $(1,1)$ to $(2,8)$.

Sol:- Given $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$

Along the curve $y = x^3 \Rightarrow dy = 3x^2dx$

$$\therefore \bar{F} = (5x^4 - 6x^2)\bar{i} + (2x^3 - 4x)\bar{j}$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$d\bar{r} = dx\bar{i} + 3x^2dx\bar{j}$$

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$$\bar{F}\cdot d\bar{r} = \left[(5x^4-6x^2)\bar{i} + (2x^3-4x)\bar{j}\right]\cdot\left[dx\,\bar{i} + 3x^2dx\,\bar{j}\right]$$

$$= (5x^4-6x^2)dx + (2x^3-4x)\,3x^2dx$$

$$= (6x^5+5x^4-12x^3-6x^2)dx$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_1^2 (6x^5+5x^4-12x^3-6x^2)dx$$

$$= \left[\frac{6x^6}{6} + \frac{5x^5}{5} - \frac{12x^4}{4} - \frac{6x^3}{3}\right]_1^2$$

$$= 16\left[(4+2-3-1)\right] - \left[1+1-3-2\right] = 32+3 = 35$$

P.T, $\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$ is Conservative.

H.W

① If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t=0$ to $t=1$. Evaluate $\int_C \bar{F}\cdot d\bar{r}$.

Ans : $\frac{51}{70}$

② Find the workdone by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$. when it moves a particle along the arc of the curve $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\bar{k}$ from $t=0$ to $t=2\pi$

③ Evaluate $\int_C (yz\,dx + xz\,dy + xy\,dz)$ over arc of a helix $x = a\cos t$, $y = a\sin t$, $z = kt$ as ~~to~~ t varies from 0 to 2π

④ Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

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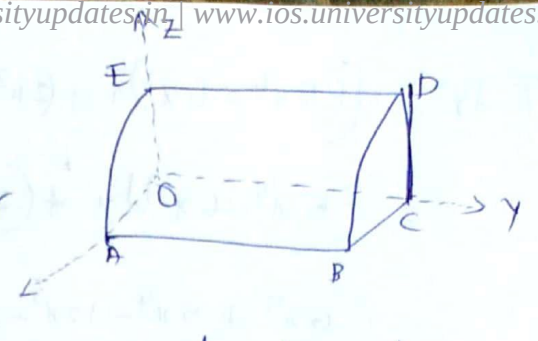
Surface integrals



The surface integral of a vector point function $\bar{F}$ expresses the normal flux through a surface. If $\bar{F}$ represents the velocity vector of a fluid then the surface integral $\int \bar{F}.\bar{n}\,ds$ over a closed surface S represents the rate of flow of fluid through the surface.

Note:- Let R be the projection of S on YZ plane then $\int \bar{F}.\bar{n}\,ds = \iint_R \bar{F}.\bar{n}\frac{dy\,dz}{|\bar{n}.i|}$

Problems

① Evaluate $\int \bar{F}.\bar{n}\,ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Sol:- Let $\phi = x^2+y^2-16$ be the surface.

then the normal to the surface S is grad ϕ.

$$\therefore \text{Normal } (\nabla\phi) = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = 2xi + 2yj + k(0)$$

$$= 2xi + 2yj$$

and unit normal $(\bar{n}) = \dfrac{2(xi+yj)}{2\sqrt{x^2+y^2}} = \dfrac{xi+yj}{\sqrt{16}} = \dfrac{xi+yj}{4}$

Let R be the projection of S on YZ plane then

$$\int_S \bar{F}.\bar{n}\,ds = \iint_R \bar{F}.\bar{n}.\frac{dy\,dz}{|n.i|}$$

$$\bar{F}.\bar{n} = (z\bar{i} + x\bar{j} - 3y^2z\bar{k})\left(\frac{x\bar{i}+\bar{y}\bar{j}}{4}\right)$$

$$= \frac{1}{4}(xz + xy)$$

$$\bar{n}.i = \frac{1}{4}(xi+yj).i = \frac{x}{4}$$

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Given

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\, \frac{dy\,dz}{(n\cdot i)}$$

$$= \iint_R \left(\frac{xz+xy}{4}\right)\left(\frac{4}{x}\right) dy\,dz$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} (y+z)\, dy\,dz$$

$$= \int_0^4 y\,dy \int_0^5 dz + \int_0^4 dy \int_0^5 z\,dz$$

$$= \left(\frac{y^2}{2}\right)_0^4 (z)_0^5 + (y)_0^4 \left(\frac{z^2}{2}\right)_0^5 = 40 + 50 = 90$$

$$= 90$$

(2) Evaluate $\iint_S \bar{F}\cdot\bar{n}\, d\bar{s}$ if ~~$\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$~~

$\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2+y^2=9$ contained in the first octant between the planes $z=0$ and $z=2$.

Sol:- Given $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$

Let $\phi = x^2+y^2-9 = 0$. then ~~$\frac{\partial\phi}{\partial x} = 2x, \frac{\partial\phi}{\partial}$~~

$$\text{grad}\,\phi = \sum \frac{\partial\phi}{\partial x}\, i = 2(xi+yj)$$

$$\bar{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2(xi+yj)}{\sqrt{x^2+y^2}} = \frac{\cancel{2}(xi+yj)}{\cancel{2}\sqrt{9}} = \frac{xi+yj}{3}$$

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$$\bar{F}\cdot\bar{n} = (yz\,i + 2y^2 j + xz^2 k)\left(\frac{x}{3} i + \frac{y}{3} j\right)$$

$$= \frac{xyz}{3} + \frac{2y^3}{3} = \frac{1}{3}(xyz + 2y^3)$$

Let R be the projection of S on xy plane

then $\int \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$

$$= \iint_R \frac{(xyz + 2y^3)/3}{x/3}\,dy\,dz$$

For the surface ~~in the~~ in the yz plane $x = 0 \Rightarrow y = 3$,

$z \to 0$ to 2

$$\int \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{3}\int_{z=0}^{2}\left(yz + \frac{2y^3}{x}\right)dy\,dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{2}\left(yz + \frac{2y^3}{\sqrt{9-y^2}}\right)dy\,dz$$

$$= \int_0^3 \left[2y + \frac{4y^3}{\sqrt{9-y^2}}\right]dy$$

$$= \cancel{\left(\frac{xy}{2}\right)_0} \cancel{\left(\frac{xy}{2}\right)_0^3} + 4\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\,dy$$

put $y = 3\sin\theta$

then $\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\,dy = 18$

$$= 9 + 4(18)$$

$$= 81$$

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(3) Evaluate $\int_S \bar{F}.\bar{n}\,ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant

Sol:- Let $\phi = 2x+3y+6z-12$

Normal to the plane is $\nabla\phi$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(2) + j(3) + k(6)$$

$$\nabla\phi = 2\bar{i} + 3\bar{j} + 6\bar{k}$$

Unit normal vector $(\bar{n}) = \dfrac{2i+3j+6k}{\sqrt{4+9+36}} = \dfrac{2i+3j+6k}{7}$

Let R be the projection of S on xy-plane, then

$$\int_S \bar{F}.\bar{n}\,ds = \iint_R \frac{\bar{F}.\bar{n}\,dx\,dy}{|\bar{n}.\bar{k}|} \quad \text{where } \bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$$

Now $\bar{F}.\bar{n} = (18z\bar{i} - 12\bar{j} + 3y\bar{k}).\dfrac{(2i+3j+6k)}{7}$

$$= \frac{36}{7}z - \frac{36}{7} + \frac{18y}{7}$$

$$\bar{n}.\bar{k} = \left(\frac{2i+3j+6k}{7}\right).\bar{k} = \frac{6}{7}$$

Given plane $2x+3y+6z=12$

on xy-plane

$$2x+3y=12 \Rightarrow y = \frac{12-2x}{3}$$

When $y=0 \Rightarrow \dfrac{12-2x}{3} = 0 \Rightarrow 12-2x = 0 \Rightarrow 2x = 12$

$\Rightarrow x = 6$

Now x varies from 0 to 6

y varies from 0 to $\dfrac{12-2x}{3}$

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$\therefore$ The surface integral

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\,dx\,dy = \iint_R \frac{\left(\frac{36}{7}z - \frac{36}{7} + \frac{18y}{7}\right)}{\frac{6}{7}}\,dx\,dy$$

$$= \iint_R (6z - 6 + 3y)\,dx\,dy$$

$$= \iint_R (12 - 2x - 3y - 6 + 3y)\,dx\,dy$$

$$(\because 6z = 12 - 2x - 3y)$$

$$= \iint_R (6 - 2x)\,dx\,dy$$

$$= 2\int_0^6 \int_0^{\frac{12-2x}{3}} (3 - x)\,dx\,dy$$

$$= 2\int_0^6 \left[(3-x)y\right]_0^{\frac{12-2x}{3}} dx$$

$$= 2\int_0^6 (3-x)\left\{\left(\frac{12-2x}{3}\right) - 0\right\} dx$$

$$= \frac{2}{3}\int_0^6 (36 - 18x + 2x^2)\,dx$$

$$= \frac{4}{3}\int_0^6 (18 - 9x + x^2)\,dx$$

$$= \frac{4}{3}\left[18x - \frac{9x^2}{2} + \frac{x^3}{3}\right]_0^6$$

$$= \frac{4}{3}\left[108 - 162 + 72\right]$$

$$= \frac{\cancel{153}^{\,51}}{\cancel{3}} \times 4 \qquad \frac{4}{\cancel{3}} \times \cancel{18}^{\,6} = 24$$

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(3) Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

Ans: $\frac{149}{24}$

~~Flux of a~~

Volume integrals *

Let $F(r) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x, y, z. We know that $dv = dx\,dy\,dz$.

The volume integral is given by

$$\int_V F\,dv = \iiint (F_1 i + F_2 j + F_3 k)\,dx\,dy\,dz$$

$$= i\iiint F_1\,dx\,dy\,dz + j\iiint F_2\,dx\,dy\,dz + k\iiint F_3\,dx\,dy\,dz$$

problem

(1) If $F = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ then evaluate i) $\int_V \nabla\cdot F\,dv$ and ii) $\int_V (\nabla\times F)\,dv$ where V is the closed region bounded by $x=0$, $y=0$, $z=0$, $2x+2y+z=4$

Sol:- i) $\nabla\cdot F = i\cdot\frac{\partial F}{\partial x} + j\cdot\frac{\partial F}{\partial y} + k\cdot\frac{\partial F}{\partial z} = 4x - 2x = 2x$

The limits are: $z=0$ to $z = 4-2x-2y$

$y=0$ to $y = \frac{4-2x}{2} = 2-x$

$x=0$ to $x=2$

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$$\therefore \int_V \nabla \cdot \bar{F}\, dv = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x \, dx\, dy\, dz$$

$$= \cancel{-8} \; 8/3$$

ii) $$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = \bar{j} - 2y\bar{k}$$

$$\int_V \nabla \times \bar{F}\, dv = \iiint_V (\bar{j} - 2y\bar{k})\, dx\, dy\, dz$$

$$= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (j - 2yk)\, dx\, dy\, dz$$

$$= \frac{8}{3}(\bar{j} - \bar{k})$$

11.ω

① If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ evaluate $\int_V \bar{F} \cdot d\bar{v}$ where V is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z = x^2$, $z=4$.

Ans: $\int_V \bar{F} \cdot d\bar{v} = 128\bar{i} - 24\bar{j} + 384\bar{k}$

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Vector Integral theorems

There are three important vector integral theorems

i) Gauss Divergence theorem

ii) Green's theorem in plane

iii) Stoke's theorem

These theorems deal with conversion of

i) $\int_S \bar{F}.\bar{n}ds$ into a volume integral where S is a closed surface

ii) $\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane where C is a closed curve in the plane and

iii) $\int_C (\text{Curl } \bar{F}).\bar{n}ds$ into a line integral around the boundary of an open two sided surface.

Gauss's Divergence theorem

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V \text{div}\bar{F}dv = \int_S \bar{F}.\bar{n}ds$$ where $\bar{n}$ is the normal vector of surface S.

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① Use Divergence theorem to evaluate $\iint_S \bar{F}\cdot d\bar{s}$ where

$\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and 'S' is the surface of the sphere

$x^2+y^2+z^2 = r^2$

Sol:- We have

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2+y^2+z^2)$$

$\therefore$ By Gauss Divergence theorem

$$\nabla\cdot\bar{F}\,dv = \iiint_V \nabla\cdot\bar{F}\,dv = \iiint_V 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

[Changing into spherical polar Coordinates $x = r\sin\theta\cos\phi$

$y = r\sin\theta\cdot\sin\phi$, $z = r\cos\theta$]

$$\therefore \iint_S \bar{F}\cdot dS = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\left[\int_{\phi=0}^{2\pi} d\phi\right] dr\,d\theta$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\,(2\pi-0)\,dr.d\theta =$$

$$= 6\pi\int_{r=0}^{a} r^4\left[\int_0^{\pi}\sin\theta\,d\theta\right]dr$$

$$= 6\pi\int_{r=0}^{a} r^4(-\cos\theta)_0^{\pi}\,dr = -6\pi\int_0^{a} r^4(\cos\pi - \cos 0)\,dr$$

$$= 12\pi\int_0^{a} r^4\,dr = 12\pi\left[\frac{r^5}{5}\right]_0^{5} = \frac{12\pi a^5}{5}$$

Note:- The polar coordinates for the sphere $x^2+y^2+z^2 = r^2$ are

$x = r\sin\theta\cos\phi$ $\quad y = r\sin\theta\sin\phi$ $\quad z = r\cos\theta$

The limits are $r = 0$ to a, $\theta = 0$ to π, $\phi = 0$ to 2π.

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

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(2) Use Divergence theorem to evaluate $\iint_S \bar{F}.d\bar{s}$ where $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ and 's' is the surface bounded by the region $x^2+y^2=4$, $z=0$ and $z=3$.

Sol:- We have $\operatorname{div} F = \nabla \cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(4xi - 2y^2j + z^2k\right)$

$$= 4 - 4y + 2z$$

$\therefore$ By Gauss Divergence theorem

$$\iint_S \bar{F}.ds = \iiint_V \nabla.\bar{F}\,dV$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4 - 4y + 2z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[(4-4y)z + \frac{\cancel{2}z^2}{\cancel{2}}\right]_0^3 dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[12(1-y) + 9\right]dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21 - 12y)\,dx\,dy$$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\,dy - 12\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\,dy\right]dx$$

$$= \int_{-2}^{2}\left[21 \times 2\int_0^{\sqrt{4-x^2}} dy - 12(0)\right]dx$$

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$$= 42 \int_{-2}^{2} (y)_0^{\sqrt{4-x^2}} dx$$

$$= 42 \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 42 \times 2 \int_{0}^{2} \sqrt{4-x^2}\, dx$$

$$= 84 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) \right]_0^2$$

$$= 84 \left[0 + 2 \cdot \frac{\pi}{2} - 0 \right] = 84\pi$$

③ Verify Gauss Divergence theorem for $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ taken over the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$.

Sol:- $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

$$\iiint_V \nabla \cdot F\, dv = \iiint_V (3x^2 + 3y^2 + 3z^2)\, dx\, dy\, dz$$

$$= 3 \int_0^a \int_0^a \left[\frac{x^3}{3} + xy^2 + z^2 x \right]_0^a dy\, dz$$

$$= 3 \int_0^a \int_0^a \left(\frac{a^3}{3} + ay^2 + az^2 \right) dy\, dz$$

$$= 3 \int_0^a \left(\frac{a^3}{3} y + a\frac{y^3}{3} + az^2 y \right)_0^a dz$$

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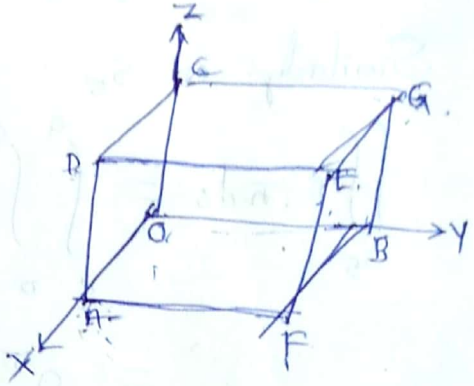
$$= 3\int_0^a \left(\frac{a^4}{3} + \frac{a^4}{3} + a^2 z^2\right) dz$$

$$= 3\int_0^a \left(\frac{2a^4}{3} + a^2 z^2\right) dz$$

$$= 3\left[\frac{2}{3}a^4 z + a^2 \frac{z^3}{3}\right]_0^a = 3\left[\frac{2}{3}a^5 + \frac{1}{3}a^5\right] = 3a^5$$

To evaluate the surface integral divide the closed surface S of the cube into 6 parts



Surfaces

S_1 : DEFA $\quad S_4$: OADC

S_2 : BGCO $\quad S_5$: GCDE

S_3 : BGEF $\quad S_6$: AFBO

On surface S_1 : DEFA, we have $\bar{n} = \bar{i}$, $x = a$

$$\iint_{S_1} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{a}\int_{y=0}^{a} (a^3\bar{i} + y^3\bar{j} + z^3 k)\cdot i\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} a^3\,dy\,dz = a^3\int_0^a (y)_0^a\,dz$$

$$= a^4 (z)_0^a = a^5$$

On S_2 : AGCO we have $\bar{n} = -\bar{i}$, $x = 0$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (y^3 j + z^3 k)(-i)\,dy\,dz = 0$$

On S_3 : AGEF, we have $\bar{n} = j$, $y = a$

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$$\iint_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (x^3\bar{i} + a^3\bar{j} + z^3\bar{k})\cdot\bar{j}\,dx\,dz$$

$$= a^5$$

On S_4, we have $\bar{n} = -\bar{j}$, $y = 0$

$$\iint_{S_4} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (x^3 i + z^3 k)(-j)\,dx\,dz = 0$$

Similarly on S_5 : $\bar{n} = \bar{k}$, $z = a$

$$\iint_{S_5} \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a (x^3 i + y^3 j + a^3 k)\cdot\bar{k}\,dx\,dy$$

$$= a^5$$

On S_6 ; $\bar{n} = -k$, $z = 0$

$$\iint_{S_6} \bar{F}\cdot\bar{n}\,ds = 0$$

Thus $$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,ds + \iint_{S_2} \bar{F}\cdot\bar{n}\,ds + \int_{S_3} \bar{F}\cdot\bar{n}\,ds$$

$$+ \iint_{S_4} \bar{F}\cdot\bar{n}\,ds + \iint_{S_5} \bar{F}\cdot\bar{n}\,ds + \iint_{S_6} \bar{F}\cdot\bar{n}\,ds$$

$$= a^5 + 0 + a^5 + 0 + a^5 + 0$$

$$= 3a^5$$

Hence $$\iiint_V \nabla\cdot F\,dv = \iint_S \bar{F}\cdot\bar{n}\,ds$$

Hence Gauss Divergence theorem proved.

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(*) Verify Gauss divergence theorem for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$, over the cube formed by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$

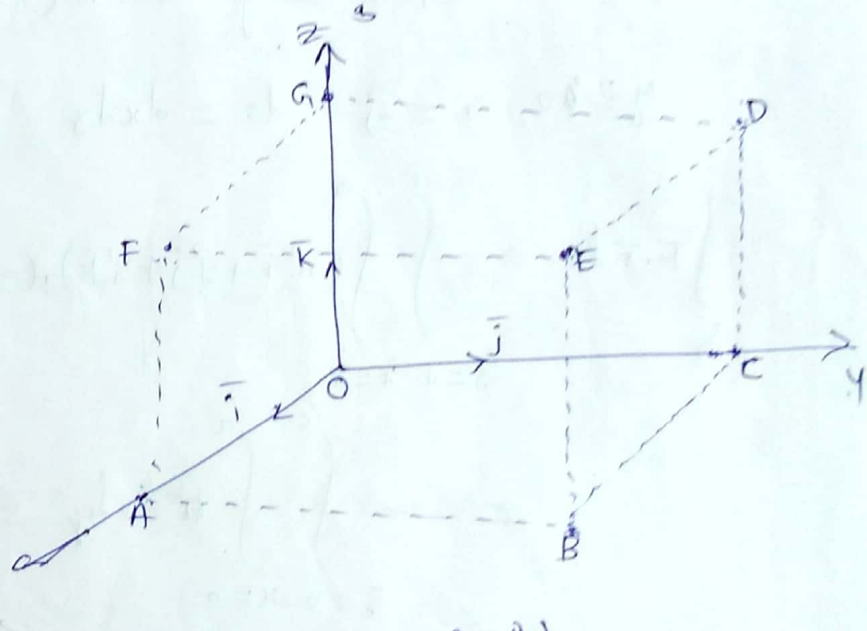
Sol:- By Gauss Divergence theorem

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

$$\nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\bar{i} + y^2\bar{j} + z^2\bar{k}\right) = 2(x+y+z)$$

$$R.H.S = \int_V \nabla\cdot\bar{F}\,dv = 2\int_0^c\int_0^b\int_0^a (x+y+z)\,dx\,dy = abc(a+b+c)$$

We will calculate the value of $\int_S \bar{F}\cdot\bar{n}\,ds$ over surfaces of the cube.



Let S_1 be the surface along OABC (XY plane) (opposite)

$z=0$, $\bar{n}=-k$, $ds = dx\,dy$

Now $\int_{S_1} \bar{F}\cdot\bar{n}\,ds = \int_0^b\int_0^a (x^2\bar{i} + y^2\bar{j} + z^2\bar{k})\cdot(-\bar{k})\,dx\,dy = -\int_0^b\int_0^a z^2\,dx\,dy = 0$

Let S_2 be the surface along DGFE (~~opposite~~ XY plane)

$z=c$, $\bar{n}=k$ $ds = dx\,dy$

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$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_0^b\int_0^a c^2\,dx\,dy = c^2ab$$

Let S_3 be the surface along BCDE (xz-opposite plane)

$y = b$, $\bar{n} = \bar{j}$ $\quad ds = dx\,dz$

Now $$\int_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} (x^2\bar{i}+y^2\bar{j}+z^2\bar{k})\cdot(\bar{j})\,dx\,dz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a} y^2\,dx\,dz = \int_{z=0}^{c}\int_{x=0}^{a} b^2\,dx\,dz = b^2ca$$

Let S_4 be the surface along OGFA (xz-plane)

$y = 0$, $\bar{n} = -\bar{j}$ $\quad ds = dx\,dz$

$$\int_{S_4} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} (x^2\bar{i}+y^2\bar{j}+z^2\bar{k})\cdot(-\bar{j})\,dx\,dy$$

$$= -\int_{z=0}^{c}\int_{x=0}^{a} 0\,dx\,dy = 0$$

$= \cancel{b^2ca}$

Let S_5 be the surface along ABEF (yz-plane opposite)

$x = a$, $\bar{n} = \bar{i}$ $\quad ds = dy\,dz$

$$\int_{S_5} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} (x^2\bar{i}+y^2\bar{j}+z^2\bar{k})\cdot(\bar{i})\,dy\,dz$$

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$$= \int_{z=0}^{c}\int_{y=0}^{b} x^2\,dy\,dz = \int_{z=0}^{c}\int_{y=0}^{b} a^2\,dy\,dz = a^2bc$$

Let S_6 be the surface along OCDG (YZ plane)

$x=0$, $\bar{n} = -\bar{i}$ $\quad ds = dy\,dz$

$$\int_{S_6} \bar{F}\cdot\bar{n}\,ds = -\int_{y=0}^{b}\int_{z=0}^{c} 0\,dy\,dz = 0$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,ds + \iint_{S_2} \bar{F}\cdot\bar{n}\,ds + \iint_{S_3} \bar{F}\cdot\bar{n}\,ds + \iint_{S_4} \bar{F}\cdot\bar{n}\,ds$$

$$+ \iint_{S_5} \bar{F}\cdot\bar{n}\,ds + \iint_{S_6} \bar{F}\cdot\bar{n}\,ds$$

$$= c^2ab + 0 + b^2ca + 0 + a^2bc + 0$$

$$L.H.S = \iint_S \bar{F}\cdot\bar{n}\,ds = abc(c+b+a)$$

$\therefore$ L.H.S = R.H.S

Hence Gauss Divergence theorem is verified.

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Green's theorem in a plane

(Transformation between Line integral and Double integral)

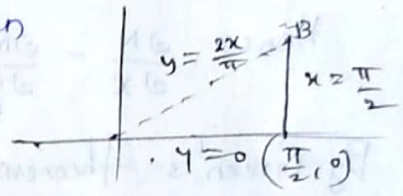
If R is a closed region in xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the positive direction.

① Evaluate by Green's theorem $\oint_C (y - \sin x)\,dx + \cos x\,dy$ where C is the triangle ~~enclo~~ enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$, $\pi y = 2x$

Sol: Let $M = y - \sin x$ and $N = \cos x$ then



$$\frac{\partial M}{\partial y} = 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = -\sin x$$

$\therefore$ By Green's theorem $\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\therefore \int_C (y - \sin x)\,dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dx\,dy$$

$$= -\int_0^{\pi/2}\int_0^{\frac{2x}{\pi}} (1 + \sin x)\,dx\,dy$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(\sin x + 1)\,dx$$

$\int uv = u\int v - \int[u'\int v\,dx]$

$$= -\frac{2}{\pi}\left[x(-\cos x + x)\Big|_0^{\pi/2} - \int_0^{\pi/2} 1(-\cos x + x)\,dx\right]$$

$$= -\frac{2}{\pi}\left[x(-\cos x + x) + \sin x - \frac{x^2}{2}\right]_0^{\pi/2}$$

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$$= -\frac{2}{\pi}\left[-x\cos x + \frac{x^2}{2} + \sin x\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8}+1\right] = -\left(\frac{\pi}{4}+\frac{2}{\pi}\right) = -\left[\frac{\pi^2+8}{4\pi}\right]$$

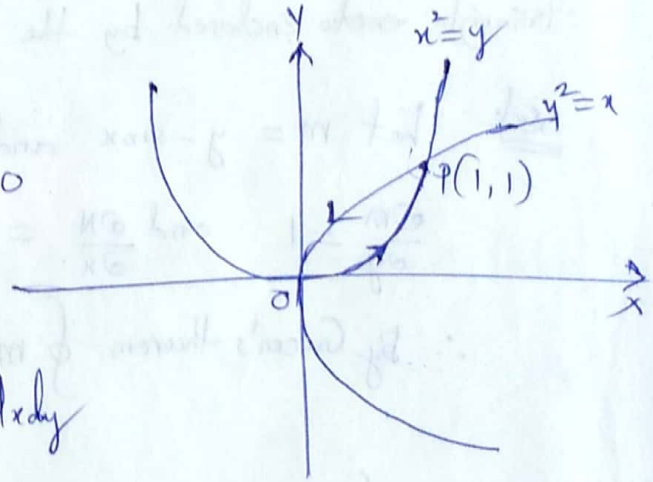
(2) Evaluate by Green's theorem $\oint (2xy - x^2)\,dx + (x^2+y^2)\,dy$, where 'c' is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$

Sol:- The two parabolas $y^2 = x$ and $y = x^2$ are intersecting at $O(0,0)$ and $P(1,1)$

Here $M = 2xy - x^2$ and $N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x$$

Hence $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2x = 0$



By Green's theorem

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (0)\,dx\,dy = 0$$

(3) Apply Green's theorem to evaluate $\oint_c (2x^2 - y^2)\,dx + (x^2+y^2)\,dy$, where c is the boundary of the area enclosed by the x-axis and w

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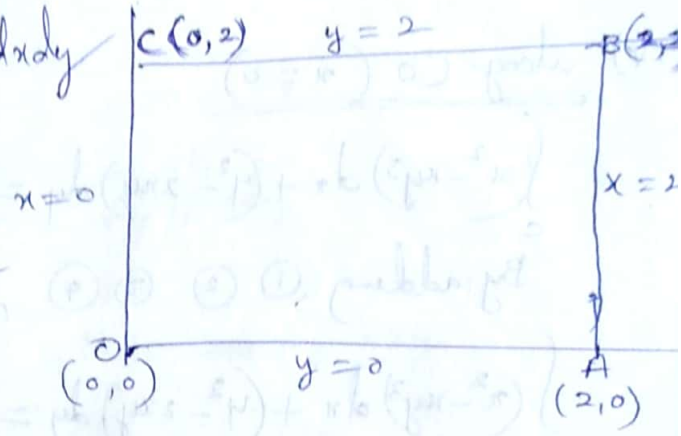
③ Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$ where C is a square with vertices (0, 0) (2, 0) (2, 2) (0, 2).

Sol:- The cartesian form of Green's theorem in the plane is

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$



Here $M = x^2 - xy^3$ and $N = y^2 - 2xy$

$$\frac{\partial M}{\partial y} = -3xy^2 \quad \& \quad \frac{\partial N}{\partial x} = -2y$$

To find $\int M dx + N dy$

To evaluate $\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$, we shall take C in four different segments

i) along OA (y = 0)

ii) along AB (x = 2)

iii) along BC (y = 2)

iv) along CO (x = 0)

i) along OA (y = 0)

$$\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3} \quad \text{--- (1)}$$

ii) along AB (x = 2)

$$\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy = \int_0^2 (y^2 - 4y)dy \quad (\because x = 2,\ dx = 0)$$

$$= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \frac{8}{3} - 8 = -\frac{16}{3} \quad \text{--- (2)}$$

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iii) along BC $(y=2)$

$$\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_2^0 (x^2-8x)\,dx \quad (\because y=2,\ dy=0)$$

$$= \left(\frac{x^3}{3} - 4x^2\right)_2^0 = -\left[\frac{8}{3}-16\right] = \frac{40}{3} \quad \text{--- (3)}$$

iv) along CO $(x=0)$

$$\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_2^0 y^2\,dy = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3} \quad \text{--- (4)}$$

By adding (1) (2) (3) (4), we get

$$\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8 \quad \text{--- (5)}$$

To find $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here $x \to 0$ to 2 & $y \to 0$ to 2

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^2\int_0^2 (-2y+3xy^2)\,dx\,dy$$

$$= \int_0^2 \left(-2xy + \frac{3x^2}{2}y^2\right)_0^2 dy = \int_0^2 (-4y+6y^2)\,dy$$

$$= \left(-2y^2+2y^3\right)_0^2 = 8 \quad \text{--- (6)}$$

$\therefore$ From (5) & (6)

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$ Hence the Green's theorem verified

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ANSWER BOOK

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Subject ..... Name .....

INSTRUCTIONS :

1) Fill in the Particulars mentioned above before your answers.
2) Write your answers on both sides of the paper.

Q. No	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	Total
Award													

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~~Evaluate by Green's theorem $\int (y - \sin x)dx + \cos x\,dy$ where C is the triangle enclosed by the lines $y = 0$, $x = \pi$~~

4. Verify Green's theorem for $\int (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by $x = 0$, $y = 0$ and $x + y =$

Sol:- By Green's theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = 3x^2 - 8y^2$ $\qquad$ $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\qquad$ $\frac{\partial N}{\partial x} = -6y$

R.H.S $$\iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \int_{y=0}^{1}\int_{x=0}^{1-x} (-6y + 16y)\,dx\,dy$$

$$= 10\int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{y=0}^{1-x}$$

$$= 5\int_{x=0}^{1} \left[(1-x)^2\right] dx$$

$$= 5\left[\frac{(1-x)^3}{3(-1)}\right]_{x=0}^{1}$$

$$= 5\left[0 - \left(\frac{-1}{3}\right)\right]$$

$$= \frac{5}{3}$$

Now L.H.S can be obtained by taking a triangle with the directions OA, AB, BO

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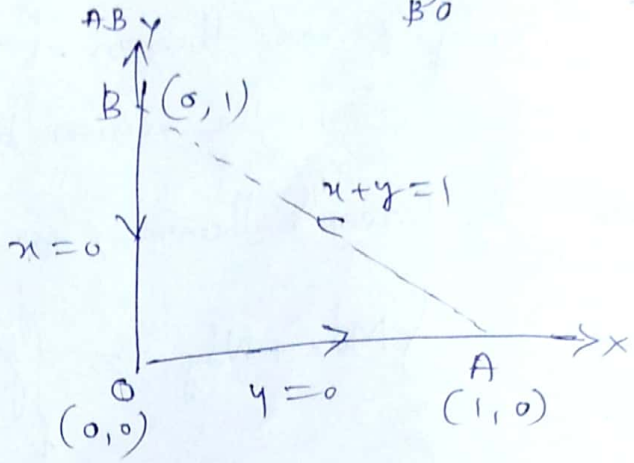
ie; $\int_{O} M dx + N dy = \int_{OA} M dx + N dy \Rightarrow \int_{AB} M dx + N dy + \int_{BO} M dx + N dy$ — (1)



Along OA

$y=0, \; dy=0$

$$\int_{OA} M dx + N dy = \int_0^1 3x^2 dx$$

$$= 3\left(\frac{x^3}{3}\right)_0^1$$

$$= 1$$

Along AB

$x+y=1 \quad \Rightarrow dy = -dx$

$\Rightarrow x = 1-y \qquad y \to 0 \text{ to } 1$

$$\int_{AB} M dx + N dy = \int_0^1 \left[3(y-1)^2 - 8y^2\right](-dy) + \left[4y - 6y(y-1)\right] dy$$

$$= \int_0^1 (11y^2 + 4y - 3)\, dy$$

$$= \left(\frac{11y^3}{3} + \frac{4y^2}{2} - 3y\right)_0^1$$

$$= \frac{11}{3} + 2 - 3$$

$$= \frac{8}{3}$$

Along BO

$x=0 \Rightarrow dx = 0 \qquad y \to 1 \text{ to } 0$

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$$\int_{BO} M\,dx + N\,dy = \int_{1}^{0} 4y\,dy = \left(\frac{4y^2}{2}\right)_1^0 = \left(2y^2\right)_1^0 = 0 - 2 = -2$$

$\therefore$ From (1)

$$\int_C M\,dx + N\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

$\therefore$ L.H.S = R.H.S.

Hence Green's theorem is verified

(15) Verify Green's theorem for $\bar{F} = (e^x \sin y)\,\bar{i} + (e^x \cos y)\,\bar{j}$ and C is the rectangle whose vertices are $(0,0)$ $(1,0)$ $(1,\pi/2)$ $(0,\pi/2)$

~~Sol: Here $M = e^x \sin y$ $N = e^x \cos y$~~

(OR)

Find the circulation of $\bar{F}$ round the curve where $\bar{F} = (e^x \sin y)\,\bar{i} + (e^x \cos y)\,\bar{j}$ and C is the rectangle whose vertices are $(0,0)$ $(1,0)$ $(1,\pi/2)$ $(0,\pi/2)$.

Sol: Here $M = e^x \sin y$ $N = e^x \cos y$

$$\frac{\partial M}{\partial y} = e^x \cos y \qquad \frac{\partial N}{\partial x} = e^x \cos y$$

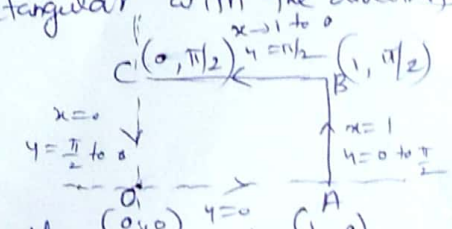
By Green's theorem

$$\oint M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

R.H.S $$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{y=0}^{\pi/2}\int_{x=0}^{1} \left(e^x \cos y - e^x \cos y\right) dx\,dy = 0$$

Now L.H.S can be obtained by taking a rectangular with the direction

OA, AB, BC, CO



$$\oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy + \int_{BC} M\,dx + N\,dy$$

$$+ \int_{CO} M\,dx + N\,dy \quad \text{——— (1)}$$

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Along OA : $y=0, dy=0 \quad x \to 0$ to 1

$$\int_{OA} M\,dx + N\,dy = \int_0^1 e^x \sin(0)\,dx + \cos(0)\int_0^1 e^x\,dx = 1.(e-1) = e-1$$

Along AB : $x=1,\ dx=0,\ y \to 0$ to $\pi/2$

$$\int_{AB} M\,dx + N\,dy = \int_{y=0}^{\pi/2} \left(e^x \sin y\,(0) + e^x \cos y\,dy\right) = e\int_{y=0}^{\pi/2} \cos y\,dy = e(\sin y)_0^{\pi/2} = e(1) = e$$

Along BC : $y = \frac{\pi}{2}\ dy=0 \quad x \to 1$ to 0

$$\int_{BC} M\,dx + N\,dy = \int_{x=1}^{0} \left(e^x(1) - 0\right) dx = \left(e^x\right)_1^0 = e^0 - e^1 = 1 - e$$

Along CO : $x=0 \quad dx=0 \quad y \to \frac{\pi}{2}$ to 0

$$\int_{CO} M\,dx + N\,dy = \int_{y=\pi/2}^{0} \left(1.\sin y(0) + (1)\cos y\right) dy = \left(\sin y\right)_{y=\frac{\pi}{2}}^{0} = 0 - \sin\frac{\pi}{2} = -1$$

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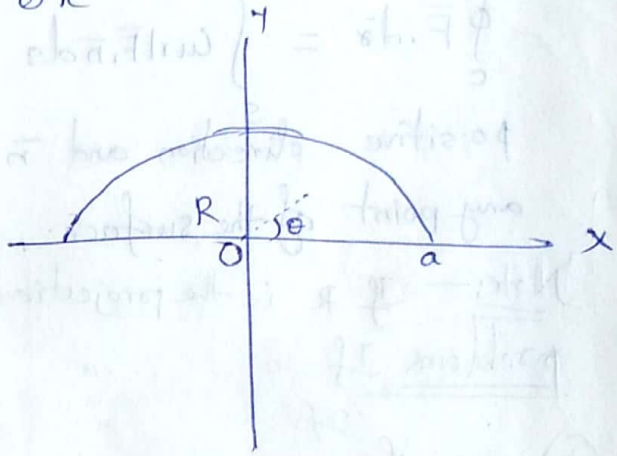
(4) Apply Green's theorem to evaluate $\oint_C (2x^2-y^2)dx + \int (x^2+y^2)dy$ where C is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2+y^2=a^2$.

Sol:- Let $M = 2x^2-y^2$ and $N = x^2+y^2$

$$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x$$



By Green's theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (2x+2y)\,dx\,dy$$

$$= 2\int_0^a\int_0^\pi (r\cos\theta + r\sin\theta)\,r\,d\theta\,dr$$

r varies from 0 to a
θ varies from 0 to π

$$= 2\int_0^a r^2\,dr \int_0^\pi (\cos\theta+\sin\theta)\,d\theta$$

$$= 2\,\frac{a^3}{3}(1+1) = \frac{4a^3}{3}$$

curl F

$$\int_S (\text{curl}\,F)\cdot n\,ds$$

Folium of Descartes $x^3+y^3=3axy$ using

lane for $\oint (3x^2-8y^2)dx + (4y-6xy)dy$

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Stoke's theorem

(Transformation between Line integral and surface integral)

Statement

Let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentiable vector point function then

$\oint_C \bar{F}.d\bar{r} = \int_S \text{curl}\bar{F}.\bar{n}ds$. where C is transversed in the positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface.

Note:— If R is the projection of S on xy plane then $\bar{k}.\bar{n}ds = dxdy$

problems If " " " " yz plane then $\bar{i}.\bar{n}ds = dydz$

If " " " " zx plane then $\bar{j}.\bar{n}ds = dzdx$

① Verify Stoke's theorem for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$, where S is the circular disc $x^2+y^2 \le 1$, $z=0$

Sol:- Given that $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

The boundary C of S is a circle in xy plane.

$x^2+y^2=1$, $z=0$

We use the parametric Co-ordinates $x=\cos\theta$, $y=\sin$

$z=0$, $0 \le \theta \le 2\pi$

$(3x^2+3y^2)$

$dx = -\sin\theta d\theta$ $dx = \cos\theta d\theta$

By stoke's theorem

$\oint_C \bar{F}.d\bar{r} = \int_S \text{curl}\bar{F}.\bar{n}.ds$

$+y^2)\bar{k}.\bar{n}ds$

we have $(\bar{k}.\bar{n})ds = dxdy$

and R is the region on xy

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$$\oint_C \bar{F}.dr = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_C -y^3 dx + x^3 dy$$

$$= \int_0^{2\pi} \left[-\sin^3\theta(-\sin\theta) + \cos^3\theta\cos\theta\right] d\theta$$

$$= \int_0^{2\pi} (\cos^4\theta + \sin^4\theta) d\theta$$

$$= \int_0^{2\pi} (1 - 2\sin^2\theta\cos^2\theta) d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} (\sin 2\theta)^2 d\theta$$

$$= 2\pi - \frac{1}{4}\int_0^{2\pi} (1 - \cos 4\theta) d\theta$$

$$= 2\pi - \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2} \quad \text{——— (1)}$$

$$\text{curl}F = \nabla \times F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$$

$$\int_S (\text{curl}F).\bar{n}.ds = 3\int (x^2+y^2)\,\bar{k}.\bar{n}\,ds$$

we have $(\bar{k}.\bar{n})ds = dx\,dy$
and R is the region on xy-plane

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$$\therefore \iint_S (\nabla \times F).\bar{n}\, ds = 3\iint_R (x^2+y^2)\, dx\, dy$$

Put $x = r\cos\phi$ & $y = r\sin\phi$ $\quad dx\, dy = r\, dr\, d\phi$

$r \to 0$ to 1 and $0 \le \phi \le 2\pi$

$$\int (\nabla \times \bar{F}).\bar{n}\, ds = 3\int_{\phi=0}^{2\pi}\int_0^1 r^2 . r\, dr\, d\phi = \frac{3\pi}{2} \longleftarrow (2)$$

From (1) & (2)

$$\oint_C \bar{F}.d\bar{r} = \int (\nabla \times \bar{F}).\bar{n}\, ds$$

Hence the theorem is verified.

(2) Verify Stokes theorem for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

Sol:- The boundary C of S is a circle in xy plane

ie; $x^2+y^2=1$, $z=0$

The parametric equations are $x = r\cos\theta$, $y = r\sin\theta$

$\theta = 0 \to 2\pi$

$dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$\int_C \bar{F}.d\bar{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$\cancel{= \int y\, dx + (x - 2xz)\, dy - xy\, dz}$$

~~Above xy plane the sphere is $x^2+y^2=a^2$, $z=0$~~

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$$\cancel{\int_C \bar{F}\cdot d\bar{r} = \int_C y\,dx + x\,dy}$$

put

$$= \int_C (2x-y)\,dx - yz^2\,dy - y^2 z$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta\,d\theta$$

$$= \int_0^{2\pi} \sin^2\theta - \int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta - \int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= \pi$$

Again $\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = i(-2yz + 2yz) - j(0-0) + \bar{k}(0+1)$

$$= \bar{k}$$

$$\int_S (\nabla\times F)\cdot\bar{n}\,ds = \int_S \bar{k}\cdot\bar{n}\,ds = \iint_R dx\,dy$$

where R is the projection of S on xy plane and

$$\bar{k}\cdot\bar{n}\,ds = dx\,dy$$

$$\iint_R dx\,dy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dy\,dx = 4\int_{x=0}^{1}\sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= \pi\left[\frac{1}{2}\sin^{-1}(1)\right] = \pi$$

∴ Hence ~~proved~~ theorem is verified

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③ Evaluate by Stoke's theorem $\int_C (e^x dx + 2y\,dy - dz)$ where c is the curve $x^2+y^2=9$ and $z=2$.

Sol:- Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and

$$\bar{F}.d\bar{r} = \bar{F}.(dx\bar{i}+dy\bar{j}+dz\bar{k}) = e^x dx + 2y\,dy - dz$$

then $\bar{F} = e^x\bar{i} + 2y\bar{j} - \bar{k}$

By stokes theorem $\int_C \bar{F}.d\bar{r} = \iint_S \text{curl}\bar{F}.\bar{n}\,ds$ —— ①

Now $\text{curl}\bar{F} = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$

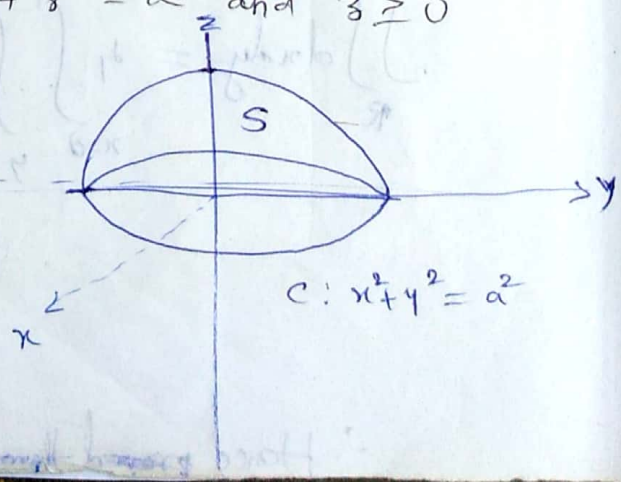
$$= i(0-0) - j(0-0) + k(0-0) = \bar{0}$$

$\therefore \int_C \bar{F}.d\bar{r} = \iint_S \text{curl}\bar{F}.\bar{n}\,ds = 0$ by ①

Hence $\int e^x dx + 2y\,dy - dz = \int_C \bar{F}.d\bar{r} = 0$

④ Verify Stoke's theorem for $F = y^2 i + yj - 3xk$ and S is the upper half of the sphere $x^2+y^2+z^2=a^2$ and $z \geq 0$

Sol:-



The curve c which is the boundary of the given hemisphere as the basic circle $x^2+y^2=a^2$

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By stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\bar{F}\cdot\bar{n}\,ds$

$$L.H.S = \oint_C \bar{F}\cdot d\bar{r}$$

$$= \oint_C \left(y^2\bar{i} + y\bar{j} - 3x\bar{k}\right)\cdot\left(dx\bar{i} + dy\bar{j} + dz\bar{k}\right)$$

$$= \oint_C y^2dx + ydy - x3dz$$

$x = a\cos\theta \Rightarrow dx = -a\sin\theta$

$y = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$

$\theta \to 0$ to 2π

Since $z = 0$

$$= \oint_C y^2dx + ydy$$

$$= \int_0^{2\pi} a^2\sin^2\theta\left(-a\sin\theta d\theta\right) + \left(a\sin\theta\right)\left(a\cos\theta\right)d\theta$$

$$= \int_0^{2\pi}\left[-a^3\sin^3\theta + a^2\sin\theta\cos\theta\right]d\theta$$

$$= -\frac{a^3}{4}\int_0^{2\pi}\left(3\sin\theta - \sin 3\theta\right)d\theta + \frac{a^2}{2}\int_0^{2\pi}\sin 2\theta\, d\theta$$

$$= -\frac{a^3}{4}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{2\pi} + \frac{a^2}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= -\frac{a^3}{12}\left(\cos 3\theta - 9\cos\theta\right)_0^{2\pi} - \frac{a^2}{4}\left[\cos 4\pi - \cos 0\right]$$

$$= -\frac{a^3}{12}(0) - \frac{a^2}{4}\left[1 - 1\right] = 0$$

Now

$$\text{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & y & -3x \end{vmatrix} = 3j - 2y\bar{k}$$

Let $\phi = x^2 + y^2 + z^2 - a^2$

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$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi+yj+zk)}{2\sqrt{x^2+y^2+z^2}} = \frac{xi+yj+zk}{a}$$

$$\text{curl}\bar{F}\cdot\bar{n} = (zj - 2yk)\cdot\left(\frac{xi+yj+zk}{a}\right) = \frac{yz-2yz}{a} = \frac{-yz}{a}$$

R.H.S. $\int_S \text{curl}\bar{F}\cdot\bar{n}\,ds = -\iint \frac{yz}{a}\,dx\,dy$ ~~For the~~

~~For the sphere, $x = a\sin\theta\cos\phi$ $y = a\sin\theta\sin\phi$~~

~~$z = a\cos\theta$~~

~~$dx\,dy\,dz =$~~

On XY plane $z = 0$

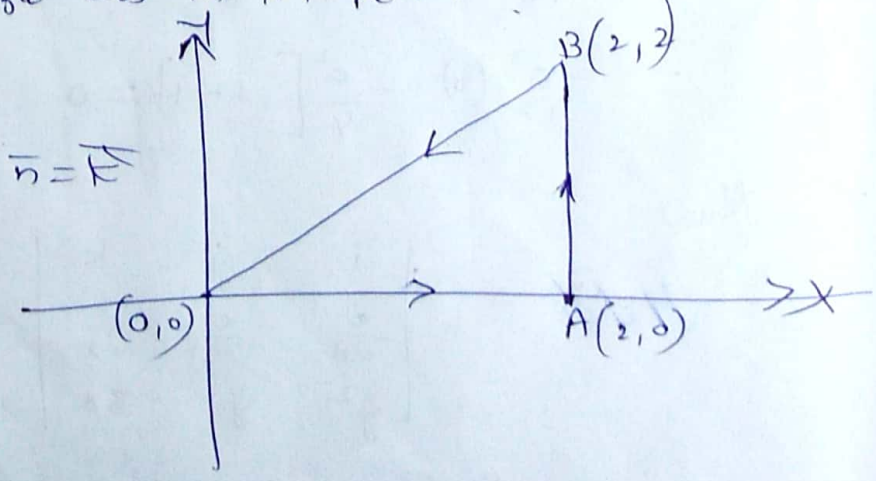
$$\int_S \text{curl}\bar{F}\cdot\bar{n}\,ds = 0$$

$\therefore$ L.H.S = R.H.S

Hence Stokes theorem is verified

(5) Using stoke's theorem evaluate the integral $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = 2y^2\bar{i} + 3x^2\bar{j} - (2x+z)\bar{k}$ and C is the boundary of the triangle whose vertices are $(0,0,0)$ $(2,0,0)$ $(2,2,0)$

Sol:- Here z-cordinates of each vertex of the triangle is zero, so the triangle lies in xy-plane.



Along xy-plane, $\bar{n} = \bar{k}$

~~curl F =~~

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$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & -2x-z \end{vmatrix} = 2\bar{j} + (6x-4y)\bar{k}$$

$$\text{curl}\,\bar{F}\cdot\bar{n} = \left(2j + (6x-4y)\bar{k}\right)\cdot(\bar{k}) = 6x-4y$$

<u>Along OA</u> :— $x \to 0$ to 2 $\quad y = 0 \Rightarrow dy = 0$

$$\iint_{OA} \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \iint (6x-4y)\,dx\,dy = 0$$

<u>Along AB</u> :— $x = 2$, $y = 0$ to 2
$\quad \downarrow$
$\quad dx = 0$

$$\iint_{AB} \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \iint (6x-4y)\,dx\,dy = 0$$

<u>Along BO</u> :— $y - 0 = 1(x-0) \Rightarrow y = x$
$\Rightarrow dy = dx$

$$\iint_{BO} \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_{y=2}^{0}\int_{x=2}^{0} (6x-4y)\,dx\,dy$$

$$= \int_{x=2}^{0} \left(6xy - \frac{4y^2}{2}\right)_{y=0}^{x} dx = \int_{0}^{2} (6x^2 - 2x^2)\,dx$$

$$= \int_{x=2}^{0} \left((0-0) - (12x - 8)\right) dx = \int_{0}^{2} 4x^2\,dx$$

$$= -\left[\frac{6x^2}{2} - 8x\right]_{x=2}^{0} \qquad = \left[\frac{4x^3}{3}\right]_{0}^{2}$$

$$= -\left[(0-0) - (24-16)\right] = 8 \qquad = \frac{32}{3}$$

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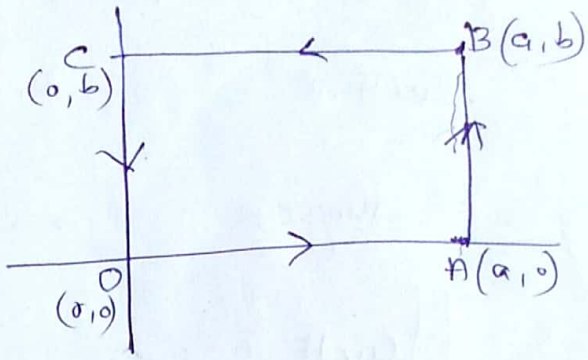
(b) Verify stoke's theorem for $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes $x=0, x=a, y=0, y=b$

Sol:- $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$



$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\bar{k}$$

$$R.H.S = \int \text{Curl}\,\bar{F}\cdot\bar{n}\,ds = \int 4y\bar{k}\cdot\bar{n}\,ds$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 4y\,dx\,dy = 2ab^2$$

L.H.S

$$\bar{F}\cdot d\bar{r} = (x^2 - y^2)\,dx + 2xy\,dy$$

Along OA

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{x=0}^{a} x^2\,dx = \frac{a^3}{3}$$

Along AB

$x=0,\ y=0$ to b

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{y=0}^{b} (2xy)\,dy = \left[2a\frac{y^2}{2}\right]_0^b = ab^2$$

Along BC

$y=b,\ dy=0$

$$\int_{BC} \bar{F}\cdot dr = \int_{x=a}^{0} (x^2 - y^2)\,dx = ab^2 - \frac{a^3}{3}$$

Along CO : $x=0, dx=0$ $\quad y \to b$ to 0

$$\int_{CO} \bar{F}\cdot d\bar{r} = \int_{y=b}^{0} 2xy\,dy = 0$$

$$\therefore \oint \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot dr = \cancel{\frac{a^3}{3}} + ab^2 + ab^2 - \cancel{\frac{a^3}{3}} = 2ab^2$$

$\therefore$ L.H.S = R.H.S

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