MODULE -3 COMPLEX VARIABLES :- (PART-1)

Complex number :- A number of the form : $z = x+iy$, where x, y are real numbers & $i = \sqrt{-1}$ or $i^2 = -1$ is called a Complex number.

$\bar{z} = x - iy$ is called the Complex conjugate of z.

Note :- 1) If $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

then $\boxed{e^{ix} = \cos x + i\sin x}$, $\boxed{e^{-ix} = \cos x - i\sin x}$

2) $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2}$

3) $\cos(ix) = \cosh x$, $\sin(ix) = i\sinh x$

where, $\cosh x = \frac{e^{x} + e^{-x}}{2}$, $\sinh x = \frac{e^{x} - e^{-x}}{2}$

Complex no in polar form :- $e^{i\theta} = \cos\theta + i\sin\theta$

$z = r \cdot e^{i\theta}$

where; $\boxed{r = \sqrt{x^2+y^2}}$ is called modulus of z.

$\boxed{\theta = \tan^{-1}(y/x)}$

Argument of z :- $\text{Arg}(z) = \theta = \tan^{-1}(y/x)$

* **Function of a complex variable :-**

w is said to be a function of complex variable, if w is a function of z, defined for a domain D., where : $w = f(z)$

$$w = f(z) = u(x,y) + iv(x,y) \quad // \text{ Cartesian form}$$
$$w = f(z) = u(r,\theta) + iv(r,\theta) \quad // \text{ Polar form}$$

* **Limit of Complex variable :-** A complex valued function, $f(z)$ defined in the neighbourhood of a point z_0, is said to have a limit l as z tends to z_0, if for every $\varepsilon > 0$ however small, there exists a +ve real no δ $\ni$: $|f(z) - l| < \varepsilon$ when ; $|z - z_0| < \delta$.

ie ; $\boxed{\lim_{z \to z_0} f(z) = l}$

Continuity :- A complex valued function $f(z)$ is said to be continuous at $z = z_0$ if $\boxed{f(z_0) \text{ exists}}$ & $\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$

Differentiability :- A complex valued fn, $f(z)$ is said to be differentiable at $z = z_0$ if : $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is unique, this lim when exists is called derivative of $f(z)$ at $z = z_0$,

$$\boxed{f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}}$$

* Cauchy-Riemann Equations in Cartesian form :- (C-R Eqn's)

The necessary conditions that the function $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x+iy$ is that; there exists 4 continuous 1st order partial derivatives.

i.e : $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the Equations :-

$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ & $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$ These 2 Conditions are known as Cauchy-Riemann (C-R) Equations in Cartesian form.

* Cauchy-Riemann Equations in Polar form :-

If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z, then there exists four continuous 1st order partial derivatives : $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfies the equations;

$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}}$ & $\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}}$ These equations are known as Cauchy-Riemann Eq^ns in Polar form,

Harmonic function :- A function ϕ is said to be harmonic if it satisfies Laplace's Equation, $\nabla^2\phi = 0$

In Cartesian form : $\phi(x,y)$ is harmonic if : $\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.}$

In polar form : $\phi(r,\theta)$ is harmonic if : $\boxed{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.}$

Note :- i) The polar family of curves $u(r,\theta)=c$, $v(r,\theta)=c_2$ intersect orthogonally if : $\tan\phi_1 \tan\phi_2 = -1$.

Analytic function :-
A complex valued function ; $w = f(z)$ is said to be analytic at a point $z = z_0$, if $\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is unique at z_0 and in the neighbourhood of z_0.

(or)

A Complex valued function ; $w = f(z)$ is analytic at a point z_0, if it is differentiable at z_0 and in the neighbourhood of z_0.

* Derive Cauchy-Riemann equations in Cartesian form :-

Stmt : The necessary conditions that the function ; $w = f(z) = u(x,y) + i v(x,y)$ may be analytic at any point, $z = x+iy$ is that, there exist four continuous 1st order partial derivatives ;

: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the equations ;

: $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$, these equations are known as Cauchy-Riemann (C-R) Equations.

Proof :- Let $f(z)$ be analytic at a point ; $z = x+iy$ and hence by the definition ; $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is Unique.

In the Cartesian form, $f(z) = u(x,y) + iv(x,y)$.

Let "δz" be the increment in z corresponding to the increments : $\delta x, \delta y$ in x & y.

where ; $\boxed{f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)}$

Considering,

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta z}$$

$f'(z)$

where, $\delta x, \delta y \to$ small negligible

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \sim (1)$$

Now; $\delta z = (z+\delta z) - z$, where, $z = x+iy$.

$\delta z = [(x+\delta x) + i(y+\delta y)] - [x+iy]$

$\delta z = \delta x + i\delta y$. $\sim (*)_1$

Since, δz tends to zero, we have the following 2 possibilities;

Case: 1 : let $\delta y = 0$ so that; $(*)_1 \Rightarrow \delta z = \delta x$. & $\delta z \to 0$

$\Rightarrow \delta x \to 0$

Now, Eqn (1) becomes;

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}.$$

These limits from the basic definition are the partial derivatives of u and v wrt x...

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \sim (2)$$

Case: 2 let $\delta x = 0$ so that; $\delta z = i\delta y$ & $\delta z \to 0$

$\Rightarrow i\delta y \to 0$ (or) $\delta y \to 0$

Now, Eqn (1) becomes;

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y} \sim (*)_2$$

But; $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ $\therefore \boxed{\frac{1}{i} = -i}$

$\Rightarrow$ substitute i in $(*)_2$ ....

$$f'(z) = \lim_{\delta y \to 0} -i \cdot \frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y}$$

$$f'(z) = -i \cdot \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \sim (3)$$

$\therefore$ Equating Eqn (2) & (3) ; we get:

$$\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$\Rightarrow$ Now, Equating the real & imaginary parts ; we get;

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Thus, these Equations are : Cauchy-Riemann Equations (C-R) Eqns in Cartesian form.

Derive Cauchy-Riemann Equations in the polar form :-

Statement :- If $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives : $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the Equations :-

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}.$$

These are known as Cauchy-Riemann Equations in Polar form.

Proof:- Let $f(z)$ be analytic at a point $z = re^{i\theta}$

Hence, by defn ; $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists & is unique

In polar form, $f(z) = u(r,\theta) + iv(r,\theta)$

Let δz be the increment in z corresponding to increments $\delta r, \delta\theta$ in r, θ.

$$\Rightarrow f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

where ; $f(z+\delta z) = u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i \lim_{\delta z \to 0} \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \quad \sim (1).$$

Consider ; $z = re^{i\theta}$, since z is a function of 2 variables r, θ.

we have ; $\delta z = \frac{\partial z}{\partial r} . \delta r + \frac{\partial z}{\partial \theta} . \delta\theta.$

$$= \frac{\partial}{\partial r}(re^{i\theta}) . \delta r + \frac{\partial}{\partial \theta}(re^{i\theta}) . \delta\theta.$$

$$\therefore \delta z = e^{i\theta} . \delta r + ire^{i\theta} . \delta\theta.$$

Since, δz tends to zero, we have the following possibilities....

Case: 1 Let $\delta\theta = 0$ so that ; $\delta z = e^{i\theta} . \delta r$ & $\delta z \to 0$
$\Rightarrow \delta r \to 0.$

Now, Eqn (1) becomes...

$$f'(z) = \lim_{\delta r \to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta}.\delta r} + i \lim_{\delta r \to 0} \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta}.\delta r}$$

ie; $f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \sim (2)$.

Case: 2 Let $\delta r = 0$, so that; $\delta z = i r e^{i\theta}.\delta\theta$

$\delta z \to 0 \Rightarrow \delta\theta \to 0$.

Now, Eqn (1) becomes;

$$f'(z) = \lim_{\delta\theta \to 0} \frac{u(r, \theta+\delta\theta) - u(r,\theta)}{ir.e^{i\theta}\delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r, \theta+\delta\theta) - v(r,\theta)}{ir.e^{i\theta}.\delta\theta}$$

$$= \frac{1}{ir.e^{i\theta}}\left[\lim_{\delta\theta \to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{\delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{\delta\theta}\right]$$

$$\Rightarrow f'(z) = \frac{1}{ir.e^{i\theta}}\left[\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right] + \frac{1}{r.e^{i\theta}}\left[\frac{1}{i}.\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right]$$

But ; $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} \Rightarrow \boxed{\frac{1}{i} = -i}$

$$\Rightarrow f'(z) = \frac{1}{r.e^{i\theta}}\left[-i\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right] = e^{-i\theta}\left[\frac{-i}{r}.\frac{\partial u}{\partial\theta} + \frac{1}{r}.\frac{\partial v}{\partial\theta}\right]$$

$$\Rightarrow f'(z) = e^{-i\theta}\left[\frac{1}{r}.\frac{\partial v}{\partial\theta} - \frac{i}{r}.\frac{\partial u}{\partial\theta}\right] \sim (3).$$

Equating RHS of Eqn (2) & (3); we get;

$$\Rightarrow e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = e^{-i\theta}\left[\frac{1}{r}.\frac{\partial v}{\partial\theta} - \frac{i}{r}.\frac{\partial u}{\partial\theta}\right]$$

Equating real & img parts --

$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}.\frac{\partial v}{\partial\theta}}$ & $\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}.\frac{\partial u}{\partial\theta}}$ are C-R Equations in Polar form.

* Properties of Analytic function :-

Harmonic function :-

A function ϕ is said to be harmonic, if it satisfies Laplace's Equation :- $\nabla^2\phi = 0$.

In the cartesian form :- $\phi(x,y)$ is harmonic if : $\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0}$

In the polar form :- $\phi(r,\theta)$ is harmonic if :

$$\boxed{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.}$$

* Deduce / Proove that :- $\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} = 0.}$

(or) P.T : Real & imaginary parts of analytic function are harmonic.

Proof / soln :-

Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic.

We shall show that u & v satisfy Laplace's Equation in the polar form;

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

We have ~~from~~ : Cauchy-Riemann Equations in polar form;

$$r\cdot\frac{\partial u}{\partial r} = \frac{\partial v}{\partial\theta} \quad \sim (1)$$

$$r\cdot\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial\theta} \quad \sim (2)$$

Diff ① wrt r & ② wrt θ partially, we get-

$$① \Rightarrow r \cdot \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \cdot 1 = \frac{\partial^2 v}{\partial r \cdot \partial \theta}$$

$$② \Rightarrow r \cdot \frac{\partial^2 v}{\partial \theta \partial r} = \frac{-\partial^2 u}{\partial \theta^2} \Rightarrow \frac{-1}{r} \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 v}{\partial \theta \partial r}$$

But ; $\frac{\partial^2 v}{\partial r \cdot \partial \theta} = \frac{\partial^2 v}{\partial \theta \cdot \partial r}$ is always true and hence, we have ;

$$r \cdot \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = \frac{-1}{r} \frac{\partial^2 u}{\partial \theta^2}$$

$\Rightarrow$ Dividing by "r", and transposing the term in the RHS, to LHS, we obtain...

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

$\therefore$ u satisfies Laplace's Equation in polar form

$\therefore$ u is harmonic.

Hence, the result (proof) $\Rightarrow$ $\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0}$

* If $f(z) = u + iv$ is an analytic function, then proove that u & v both satisfy 2 dimensional laplace Equation.

ie, $\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \right]$

Proof :- Let $f(z) = u + iv$ is analytic.

$\Rightarrow u$ & v satisfies Cauchy-Reemann Equations...

ie, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ~ (1)

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ~ (2)

$\Rightarrow$ dwt (1) partially wrt x & (2) partially wrt y...

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ ~ (3) & $\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$ ~ (4)

- Now, Equating Eqn (3) & (4); we get..

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{-\partial^2 u}{\partial y^2}$

$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$ $\therefore$ u satisfies Laplace Equation.

Now, dwt (1) partially wrt y & (2) partially wrt x, ..

$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$ ~ (5) & $\frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 u}{\partial x \partial y}$ ~ (6).

$\Rightarrow$ Now, Equating Eqn (5) & (6) ..

$\Rightarrow \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$

$\Rightarrow \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$

$\therefore$ v satisfies Laplace Equations.

$\therefore$ u & v satisfies Laplace Equation.., Hence the proof.

— x —

* If $f(z) = u + iv$ is an analytic function, then prove that the Equations : $u(x,y) = c_1$ & $v(x,y) = c_2$ represent orthogonal family of curves.

Proof :- Let $f(z) = u + iv$ be analytic

$\Rightarrow$ u & v satisfies Cauchy - Riemann Equations.

ie, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Consider; $u(x,y) = c_1$ ~ (1)

$\Rightarrow$ Diff (1) w.r.t x, Eqn (1) treating y as a function of x [y is dep^t on x & u].

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0.$$

$$\frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = -\frac{\partial u}{\partial x}.$$

$$\Rightarrow \frac{dy}{dx} = \left\{ \frac{-\partial u/\partial x}{\partial u/\partial y} \right\} = m_1 \text{ (say)}.$$

III^{ly}, Consider; $v(x,y) = c_2$ ~ (2)

$\Rightarrow$ Diff (2) w.r.t x, keeping y as function of x

$$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = \left\{ \frac{-\frac{\partial v}{\partial x}}{\partial v/\partial y} \right\} = m_2 \text{ (say)}.$$

Now, Consider ;

$$m_1 \cdot m_2 = \left\{ \frac{-\partial u/\partial x}{\partial u/\partial y} \right\} \cdot \left\{ \frac{-\partial v/\partial x}{\partial v/\partial y} \right\}$$

$$\Rightarrow \left\{\frac{-\partial v/\partial y}{-\partial v/\partial x}\right\} \cdot \left\{\frac{-\partial v/\partial x}{\partial v/\partial y}\right\}$$ // [By C-R Equations ... * $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.]

$$\Rightarrow \boxed{m_1 \cdot m_2 = -1}$$

$\therefore u(x,y) = c_1$ & $v(x,y) = c_2$ represents the orthogonal family of curves. Hence, the proof.

Prob

Problems & Solutions :- [Construction of analytic function] - TYPE -1

1) Show that : $w = z + e^z$ is analytic and hence find : $\frac{dw}{dz}$ or $f'(z)$

Soln:- By data ; $w = z + e^z$ // $f(z) = w = u + iv$.

ie, $u + iv = (x+iy) + e^{x+iy}$.

$= (x+iy) + e^x . e^{iy}$ // $e^{ix} = \cos x + i \sin x$.

$= (x+iy) + e^x [\cos y + i \sin y]$

$\therefore u + iv = (x + e^x . \cos y) + i(y + e^x \sin y)$ ~ (1)

By seperating real & imaginary parts, we get--

$\Rightarrow u = x + e^x . \cos y$ & $v = y + e^x \sin y$.

$\Rightarrow$ diff u partially wrt x & y $\Rightarrow$ diff v partially wrt y & x.

$u_x = 1 + e^x \cos y$ $v_x = e^x \sin y$

$u_y = -e^x \sin y$. $v_y = 1 + e^x \cos y$.

We observe that cauchy-Riemann Equations in the cartesian form;

$\underline{u_x = v_y}$ & $\underline{v_x = -u_y}$ are satisfied--

Thus, $u_x = v_y = 1 + e^x \cos y$//

$v_x = -u_y = e^x \sin y = -- e^x \sin y$//

Hence; $w = z + e^z$ is Analytic

Now, to find : $\frac{dw}{dz} = f'(z) = u_x + iv_x$.

ie, $\frac{dw}{dz} = (1 + e^x \cos y) + i(e^x \sin y)$.

$$= 1+e^x\,[\underbrace{\cos y + i\sin y}]$$

$$= 1+e^x \cdot e^{iy}$$

$$\therefore \frac{dw}{dz} = 1+e^{x+iy}.$$

Since; $z = x+iy$

$$\therefore \boxed{\frac{dw}{dz} = 1+e^z}$$

Q) S.T ; $f(z) = \sin z$ is analytic.

Soln :- $f(z) = \sin z$.

$u+iv = \sin(x+iy)$ // $\sin(A+B) = \sin A.\cos B + \cos A \sin B$.

$u+iv = \sin x.\cos iy + \cos x\, i \sin iy$. // $\sin i\theta = i\sinh\theta$, $\cos i\theta = \cosh\theta$.

$u+iv = \sin x.\cosh y + i\cos x.\sinh y.$ ~ ①

Now, seperating real & imaginary parts..

$u = \sin x.\cosh y$ & $v = \cos x.\sinh y$.

⇒ diff u partially wrt x & y ⇒ diff v partially wrt x & y

$u_x = \cos x \cosh y$ $v_x = -\sin x.\sinh y$.

$u_y = \sin x\,\sinh y$ $v_y = \cos x.\cosh y$

Since, Cauchy-Riemann Equations $u_x = v_y$ & $v_x = -u_y$

ie, $u_x = v_y = \cos x \cosh y$ & $v_x = -u_y = \sin x.\sinh y = -\sin x \sinh y$

are satisfied--

$\therefore$ $f(z) = \sin z$ is analytic.

) Show that $f(z) = \cosh z$ is analytic, hence find $f'(z)$.

Soln:- $f(z) = \cosh z$, $z = x+iy$.

$u+iv = \cosh(x+iy)$ // $\cosh\theta = \cos i\theta$.

$u+iv = \cos i(x+iy)$

$= \cos(ix + i^2 y)$ // $i^2 = -1$

$= \cos(ix - y)$. // $\cos(A-B) = \cos A \cdot \cos B + \sin A \sin B$.

$= \underline{\cos ix} \cdot \cos y + \underline{\sin ix} \cdot \sin y$. // $\cos ix = \cosh x$. // $\sin ix = i\sinh x$.

$u+iv = \cosh x \cdot \cos y + i\sinh x \cdot \sin y$.

$\Rightarrow u = \cosh x \cdot \cos y$. $\quad v = \sinh x \cdot \sin y$

$\Rightarrow$ dwrt u & v partially wrt x & y, we get...

$u_x = \sinh x \cdot \cos y$. $\quad v_x = \cosh x \cdot \sin y$.

$u_y = -\cosh x \cdot \sin y$. $\quad v_y = \sinh x \cdot \cos y$.

Cauchy-Riemann Equations, $u_x = v_y$ & $v_x = -u_y$ are satisfied.

$\therefore$ $f(z) = \cosh z$ is analytic

$f'(z) = u_x + iv_x$.

$f'(z) = \sinh x \cos y + i \cosh x \sin y$.

$\Rightarrow$ x^{ly} & $\div$ by i in RHS...

$f'(z) = \frac{1}{i}[i\sinh x \cdot \cos y - \cosh x \sin y]$

$= \frac{1}{i}[\sin ix \cdot \cos y - \cos ix \cdot \sin y]$

$= \frac{1}{i} \sin(ix - y) = \frac{1}{i} \sin i(x+iy)$.

$\therefore f'(z) = \frac{1}{i}[i\sinh(x+iy)] = \boxed{\sinh(x+iy) = f'(z)}$

4) S.T ; $w = \log z$, $z \neq 0$ is analytic, hence find $\frac{dw}{dz}$ *

Soln :- $w = \log z$ [It is convenient to do problem in polar form as u & v can be found easily].

$w = \log z$, taking $z = re^{i\theta}$

$u + iv = \log(re^{i\theta}) = \log r + \log_e(e^{i\theta})$

$= \log r + i\theta \cdot \log_e e$ // $\log_e e = 1$

$u + iv = \log r + i\theta$.

$\Rightarrow u = \log r$ & $v = i\theta$.

$\Rightarrow$ diff u & v w.r.t r & θ, we get ...

$u_r = \frac{1}{r}$ & $v_r = 0$

$u_\theta = 0$ $v_\theta = i$

C-R Eqns in polar form ; $r u_r = v_\theta$ & $r v_r = -u_\theta$ are satisfied.

∴ $w = \log z$ ie Analytic

$f'(z) = e^{-i\theta}(u_r + i v_r) = e^{-i\theta}(1/r + i \cdot 0) = \frac{1}{re^{i\theta}} = f'(z) = \frac{1}{z}$

5) S.T ; $w = f(z) = z^n$, n is +ve integer , $f'(z)$. [polar form]

6) S.T ; $w = z + e^{-z}$ is Analytic.

7) S.T ; $w = z + \sinh z$ is Analytic, find :- $f'(z)$

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* Construction of Analytic function ;

f(z) given its real or imaginary part :- TYPE - 2

1) Construct analytic function whose real part is :- $u = \log\sqrt{x^2+y^2}$.

Soln :- Given ; Real part of Analytic function is ;

$$u = \log\sqrt{x^2+y^2} = \log(x^2+y^2)^{1/2}$$

$$u = \frac{1}{2}\cdot\log(x^2+y^2)$$

$\Rightarrow$ diff u partially wrt x & y, we get ...

$$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$$

$$u_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

Consider ; $f'(z) = u_x + i v_x$ ... , But : $v_x = -u_y$ By :- [C-R Equation.]

$$f'(z) = u_x - i u_y$$

$$\therefore f'(z) = \left[\frac{x}{x^2+y^2}\right] - i\left[\frac{y}{x^2+y^2}\right] \sim (1)$$

putting ; $x = z$ & $y = 0$, we get ... // To get $f'(z)$ in terms of z.

$$\therefore (1) \Rightarrow f'(z) = \left[\frac{z}{z^2+0^2}\right] - i\left[\cancel{\frac{0}{z^2+0^2}}\right]^{0}$$

$$\therefore f'(z) = \frac{1}{z} \rightarrow \text{Integrate} \dots$$

$$\int f'(z) = \int \frac{1}{z}\, dz$$

$\therefore$ $\boxed{f(z) = \log z + c.}$ is an Analytic fn

2) Determine the analytic function, $f(z) = u+iv$, given that the real part : $u = e^{2x}.[x.\cos 2y - y\sin 2y]$

Soln :- $u = e^{2x}[x\cos 2y - y\sin 2y]$ ~ (1)

$\Rightarrow$ diff (1) partially wrt x & y...

$\Rightarrow u_x = e^{2x}[1.\cos 2y - 0] + [x\cos 2y - y\sin 2y](2)e^{2x}$.

$\therefore u_x = e^{2x}[\cos 2y + 2x\cos 2y - 2y\sin 2y]$

$\Rightarrow u_y = e^{2x}[-2x\sin 2y - 2y\cos 2y - \sin 2y]$

$\therefore u_y = -e^{2x}[2x\sin 2y + 2y\cos 2y + \sin 2y]$

Consider, $f'(z) = u_x + iv_x = u_x - iu_y$ // $V_x = -u_y$ [By CR Equations]

$f'(z) \Rightarrow$ putting :- $x = z$, $y = 0$; we have...

$\Rightarrow f'(z) = e^{2z}(1+2z)$,

$\Rightarrow f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$ $\Rightarrow$ Integrate b.s.-

$\Rightarrow \int f'(z) = \int e^{2z}(1+2z).dz$

$\therefore f(z) = (1+2z)\frac{e^{2z}}{2} - 2.\frac{e^{2z}}{4} = \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2}$.

Thus, $f(z) = z.e^{2z} + c$

Also; $f(z) = u+iv = (x+iy)e^{2(x+iy)}$

$= e^{2x}(x+iy)(\cos 2y + i\sin 2y)$

$\therefore f(z) = e^{2x}(x\cos 2y - y\sin 2y) + ie^{2x}(x\sin 2y + y\cos 2y)$

Asmt

3) Determine analytic function, $f(z)$, whose ~~im~~ real part is :

$* u = e^{-2xy}.\sin(x^2-y^2)$

4) $u = \log(x^2+y^2)$

5) $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$.

6) Determine the analytic function, $f(z)$, whose imaginary part is; $\left[r - \frac{K^2}{r}\right] \sin\theta$, $r \neq 0$, Hence find the real part of $f(z)$ & PT it is harmonic.

Soln :- Let $v = \left[r - \frac{K^2}{r}\right] \sin\theta$ — (1)

$\Rightarrow$ diff (1) wrt r & θ partially, we get...

$$v_r = \left[1 + \frac{K^2}{r^2}\right]\sin\theta \quad , \quad v_\theta = \left[r - \frac{K^2}{r}\right]\cos\theta.$$

Consider; $f'(z) = e^{-i\theta}(u_r + i v_r)$, But ; $\boxed{\frac{1}{r} \cdot v_\theta = u_r}$ // C-R Equ in polar form

$$\therefore f'(z) = e^{-i\theta}\left[\frac{1}{r} \cdot v_\theta + i v_r\right]$$

$$f'(z) = e^{-i\theta}\left[\left(1 - \frac{K^2}{r^2}\right)\cos\theta + i\left(1 + \frac{K^2}{r^2}\right)\sin\theta\right]$$

$$= e^{-i\theta}\left[(\cos\theta + i\sin\theta) - \frac{K^2}{r^2}(\cos\theta - i\sin\theta)\right]$$

$$= e^{-i\theta}\left[e^{i\theta} - \frac{K^2}{r^2}e^{-i\theta}\right] = 1 - \frac{K^2}{(re^{i\theta})^2} = 1 - \frac{K^2}{z^2}$$

$f'(z) = 1 - \frac{K^2}{z^2}$ $\Rightarrow$ Integrate...

$$\int f'(z) = \int \frac{1-K^2}{z^2} dz \Rightarrow \boxed{f(z) = \left(z + \frac{K^2}{z}\right) + c}$$

Now, to find $u(r,\theta)$, put $z = re^{i\theta}$ in $f(z)$ ...

$$u + iv = (re^{i\theta}) + \frac{K^2}{re^{i\theta}} = r(\cos\theta + i\sin\theta) + \frac{K^2}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{K^2}{r^2}\right)\cos\theta + i\left(r - \frac{K^2}{r^2}\right)\sin\theta.$$

$\therefore \boxed{u = r + \frac{K^2}{r^2}\cos\theta.}$ is Real part.

— * —

Type:3 Finding the Conjugate harmonic function and analytic function :-

1) Show that; $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and find its harmonic conjugate, Also find corresponding analytic function, $f(z)$.

Soln :- $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$. ~(1)

⇒ diff (1) partially wrt x & y.. twice..

$u_x = 3x^2 - 3y^2 + 6x$. $\qquad u_y = -6xy - 6y$

$u_{xx} = 6x + 6$. $\qquad u_{yy} = -6x - 6$.

Consider; $u_{xx} + u_{yy} = \cancel{6x} + 6 - \cancel{6x} - 6 = 0$,

$u_{xx} + u_{yy} = 0$, Thus; u is harmonic

Consider; C-R Eqn :- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Substituting for :- $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ we have;

$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x$. $\qquad \frac{\partial v}{\partial x} = -(-6xy - 6y)$

⇒ Integrating wrt y.. $\qquad$ ⇒ Integrating wrt x..

$\int \frac{\partial v}{\partial y} = \int (3x^2 - 3y^2 + 6x)\,dy + f(x)$ $\qquad \int \frac{\partial v}{\partial x} = \int (6xy + 6y)\,dx + g(y)$

$v = \int (3x^2 - 3y^2 + 6x)\,dy + f(x)$ $\qquad v = \int (6xy + 6y)\,dx + g(y)$

$v = 3x^2y - y^3 + 6xy + f(x)$ $\qquad v = 3x^2y + 6xy + g(y)$.

we choose, $f(x) = 0$, $g(y) = -y^3$ (so that 1st & 2nd Equa^ns of v are same)

∴ $\boxed{v = 3x^2y - y^3 + 6xy.}$

put; $\boxed{x = z}$, $\boxed{y = 0}$

The analytic fn : $f(z) = (x^3 - 3xy^2 + 3x^2 + 3y^2 + 1) + i(3x^2y - y^3 + 6xy)$ ⇒ $\boxed{f(z) = z^3 + 3z^2 + 1}$