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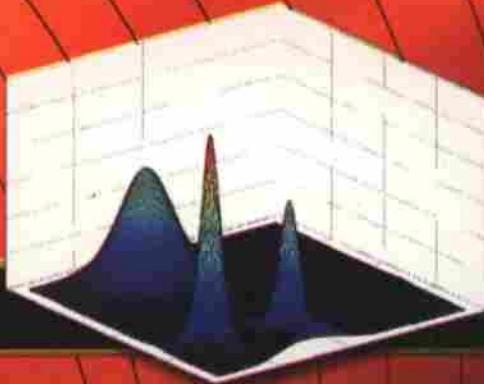
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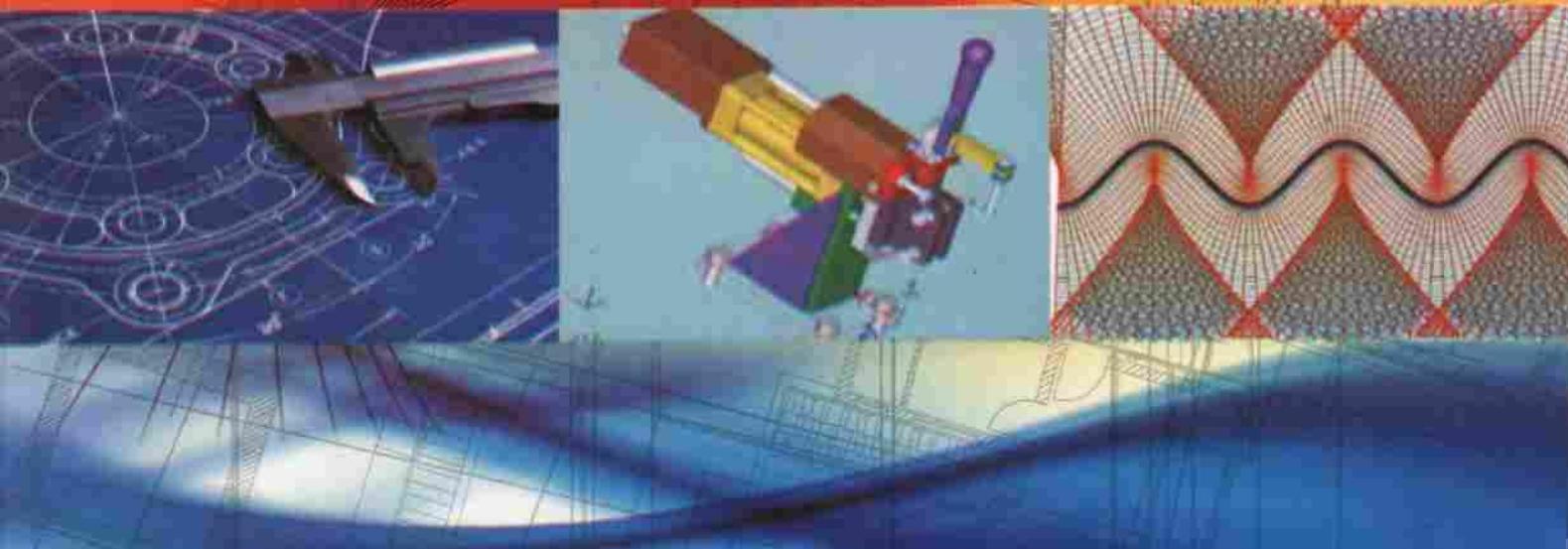
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42<sup>nd</sup>  
Edition



# Higher Engineering Mathematics

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## Fourier Series

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### 10.1 INTRODUCTION

In many engineering problems, especially in the study of periodic phenomena\* in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Most of the single-valued functions which occur in applied mathematics can be expressed in the form,

$$\frac{1}{2}a_0 \dagger + a_1 \cos x + a_2 \cos 2x + \dots \dagger \\ + b_1 \sin x + b_2 \sin 2x + \dots$$

within a desired range of values of the variable. Such a series is known as the **Fourier series**<sup>§</sup>.

### 10.2 EULER'S FORMULAE

The Fourier series for the function  $f(x)$  in the interval  $\alpha < x < \alpha + 2\pi$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\left. \begin{aligned} a_0 &= \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx \\ b_n &= \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx \end{aligned} \right\} \quad \dots(1)$$

These values of  $a_0, a_n, b_n$  are known as *Euler's formulae*<sup>\*\*</sup>.

\***Periodic functions.** If at equal intervals of abscissa  $x$ , the value of each ordinate  $f(x)$  repeats itself, i.e.,  $f(x) = f(x + a)$ , for all  $x$ , then  $y = f(x)$  is called a *periodic function* having **period**  $a$ , e.g.,  $\sin x, \cos x$  are periodic functions having a period  $2\pi$ .

† To write  $a_0/2$  instead of  $a_0$  is a conventional device to be able to get more symmetric formulae for the coefficients.

§ Named after the French mathematician and physicist *Jacques Fourier* (1768–1830) who was first to use Fourier series in his memorable work '*Theorie Analytique de la Chaleur*' in which he developed the theory of heat conduction. These series had a deep influence in the further development of mathematics and mathematical physics.

\*\*See footnote p. 205.

To establish these formulae, the following definite integrals will be required :

1.  $\int_{\alpha}^{\alpha+2\pi} \cos nx dx = \left| \frac{\sin nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0 \quad (n \neq 0)$
2.  $\int_{\alpha}^{\alpha+2\pi} \sin nx dx = - \left| \frac{\cos nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0 \quad (n \neq 0)$
3.  $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx dx$   
 $= \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x] dx$   
 $= \frac{1}{2} \left| \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right|_{\alpha}^{\alpha+2\pi} = 0 \quad (m \neq n)$
4.  $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx dx = \left| \frac{x}{2} + \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi \quad (n \neq 0)$
5.  $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx dx = - \frac{1}{2} \left[ \frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n} \right] = 0 \quad (m \neq n)$
6.  $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx dx = \left| \frac{\sin^2 nx}{2n} \right|_{\alpha}^{\alpha+2\pi} = 0 \quad (n \neq 0)$
7.  $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx dx = \frac{1}{2} \left| \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right|_{\alpha}^{\alpha+2\pi} = 0 \quad (m \neq n)$
8.  $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx dx = \left| \frac{x}{2} - \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi. \quad (n \neq 0)$

*Proof.* Let  $f(x)$  be represented in the interval  $(\alpha, \alpha + 2\pi)$  by the Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots(i)$$

To find the coefficients  $a_0, a_n, b_n$ , we assume that the series (i) can be integrated term by term from  $x = \alpha$  to  $x = \alpha + 2\pi$ .

To find  $a_0$ , integrate both sides of (i) from  $x = \alpha$  to  $x = \alpha + 2\pi$ . Then

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left( \sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_{\alpha}^{\alpha+2\pi} \left( \sum_{n=1}^{\infty} b_n \sin nx \right) dx \\ &= \frac{1}{2} a_0 (\alpha + 2\pi - \alpha) + 0 + 0 = a_0 \pi \end{aligned} \quad [\text{By integrals (1) and (2) above}]$$

Hence  $a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx.$

To find  $a_n$ , multiply each side of (i) by  $\cos nx$  and integrate from  $x = \alpha$  to  $x = \alpha + 2\pi$ . Then

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \left( \sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx dx \\ &\quad + \int_{\alpha}^{\alpha+2\pi} \left( \sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx dx \\ &= 0 + \pi a_n + 0 \end{aligned} \quad [\text{By integrals (1), (3), (4), (5) and (6)}]$$

Hence  $a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx.$

To find  $b_n$ , multiply each side of (i) by  $\sin nx$  and integrate from  $x = \alpha$  to  $x = \alpha + 2\pi$ . Then

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx + \int_{\alpha}^{\alpha+2\pi} \left( \sum_{n=1}^{\infty} a_n \cos nx \right) \sin nx \, dx \\ &\quad + \int_{\alpha}^{\alpha+2\pi} \left( \sum_{n=1}^{\infty} b_n \sin nx \right) \sin nx \, dx \\ &= 0 + 0 + \pi b_n \end{aligned} \quad [\text{By integrals (2), (5), (6), (7) and (8)}]$$

Hence

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx.$$

Cor. 1. Making  $\alpha = 0$ , the interval becomes  $0 < x < 2\pi$ , and the formulae (I) reduce to

$$\left. \begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \end{aligned} \right\} \quad \dots(\text{II})$$

Cor. 2. Putting  $\alpha = -\pi$ , the interval becomes  $-\pi < x < \pi$  and the formulae (I) take the form :

$$\left. \begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{aligned} \right\} \quad \dots(\text{III})$$

Example 10.1. Obtain the Fourier series for  $f(x) = e^{-x}$  in the interval  $0 < x < 2\pi$ .

(S.V.T.U., 2007)

Solution. Let

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots(i)$$

Then

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \, dx = \frac{1}{\pi} \left[ -e^{-x} \right]_0^{2\pi} = \frac{1 - e^{-2\pi}}{\pi}$$

and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx \, dx \\ &= \frac{1}{\pi(n^2 + 1)} \left[ e^{-x} (-\cos nx + n \sin nx) \right]_0^{2\pi} = \left( \frac{1 - e^{-2\pi}}{\pi} \right) \cdot \frac{1}{n^2 + 1} \end{aligned}$$

$$\therefore a_1 = \left( \frac{1 - e^{-2\pi}}{\pi} \right) \frac{1}{2}, a_2 = \left( \frac{1 - e^{-2\pi}}{\pi} \right) \cdot \frac{1}{5} \text{ etc.}$$

Finally,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin nx \, dx \\ &= \frac{1}{\pi(n^2 + 1)} \left[ e^{-x} (-\sin nx - n \cos nx) \right]_0^{2\pi} = \left( \frac{1 - e^{-2\pi}}{\pi} \right) \cdot \frac{n}{n^2 + 1} \end{aligned}$$

$$\therefore b_1 = \frac{1 - e^{-2\pi}}{\pi} \cdot \frac{1}{2}, b_2 = \left( \frac{1 - e^{-2\pi}}{\pi} \right) \cdot \frac{2}{5} \text{ etc.}$$

Substituting the values of  $a_0, a_n, b_n$  in (i), we get

$$e^{-x} = \frac{1 - e^{-2\pi}}{\pi} \left\{ \frac{1}{2} + \left( \frac{1}{2} \cos x + \frac{1}{5} \cos 2x + \frac{1}{10} \cos 3x + \dots \right) + \left( \frac{1}{2} \sin x + \frac{2}{5} \sin 2x + \frac{3}{10} \sin 3x + \dots \right) \right\}.$$

**Example 10.2.** Find a Fourier series to represent  $x - x^2$  from  $x = -\pi$  to  $x = \pi$ .

(V.T.U., 2011; Madras, 2006)

**Solution.** Let  $x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$  ... (i)

Then

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx = \frac{1}{\pi} \left| \frac{x^2}{2} - \frac{x^3}{3} \right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx dx^*$$

$$= \frac{1}{\pi} \left[ (x - x^2) \frac{\sin nx}{n} - (1 - 2x) \times \left( -\frac{\cos nx}{n^2} \right) + (-2) \left( \frac{-\sin nx}{n^3} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \quad [\because \cos n\pi = (-1)^n]$$

$$\therefore a_1 = 4/1^2, a_2 = -4/2^2, a_3 = 4/3^2, a_4 = -4/4^2 \text{ etc.}$$

Finally,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx dx$$

$$= \frac{1}{\pi} \left[ (x - x^2) \left( -\frac{\cos nx}{n} \right) - (1 - 2x) \times \left( -\frac{\sin nx}{n^2} \right) + (-2) \left( \frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi} = -2(-1)^n/n$$

$$\therefore b_1 = 2/1, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4 \text{ etc.}$$

Substituting the values of  $a$ 's and  $b$ 's in (i), we get

$$x - x^2 = -\frac{\pi^2}{3} + 4 \left[ \frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots \right] + 2 \left[ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right]$$

Obs. Putting  $x = 0$ , we find another interesting series  $0 = -\frac{\pi^2}{3} + 4 \left( \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right)$

i.e.,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}. \quad (\text{V.T.U., 2011})$$

**Note.** In the above example, we have used the results  $\sin n\pi = 0$  and  $\cos n\pi = (-1)^n$

Also  $\sin \left( n + \frac{1}{2} \right) \pi = (-1)^n$  and  $\cos \left( n + \frac{1}{2} \right) \pi = 0$ . The reader should remember these results.

**Example 10.3.** Expand  $f(x) = x \sin x$  as a Fourier series in the interval  $0 < x < 2\pi$ .

(S.V.T.U., 2009; Bhopal, 2009; Rohtak, 2006)

**Solution.** Let  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$  ... (i)

Then

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x \sin x dx = \frac{1}{\pi} \left| x(-\cos x) - 1.(-\sin x) \right|_0^{2\pi} = -2.$$

\* Apply the general rule of integration by parts which states that if  $u, v$  be two functions of  $x$  and dashes denote differentiations and suffixes integrations w.r.t.  $x$ , then

$$\int u v dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

In other words : Integral of the product of two functions

= 1st function  $\times$  integral of 2nd - go on differentiating 1st, integrating 2nd signs alternately +ve and -ve.

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx dx = \frac{1}{2\pi} \int_0^{2\pi} x (2 \cos nx \sin x) dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} x [\sin(n+1)x - \sin(n-1)x] dx \\
 &= \frac{1}{2\pi} \left[ x \left\{ \frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} - 1 \left\{ -\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left[ 2\pi \left\{ -\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1} \right\} \right] = \frac{2}{n^2-1} \cdot (n \neq 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{When } n = 1, a_1 &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos x dx = \frac{1}{2\pi} \int_0^{2\pi} x \sin 2x dx \\
 &= \frac{1}{2\pi} \left[ x \left( -\frac{\cos 2x}{2} \right) - 1 \cdot \left( -\frac{\sin 2x}{4} \right) \right]_0^{2\pi} = -\frac{1}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \text{Finally, } b_n &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} x [\cos(n-1)x - \cos(n+1)x] dx \\
 &= \frac{1}{2\pi} \left[ x \left\{ \frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right\} - 1 \cdot \left\{ -\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2} \right\} \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left[ \frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} \right] = 0 \quad (n \neq 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{When } n = 1, b_1 &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin x dx = \frac{1}{2\pi} \int_0^{2\pi} x (1 - \cos 2x) dx \\
 &= \frac{1}{2\pi} \left[ x \left( x - \frac{\sin 2x}{2} \right) - 1 \cdot \left( \frac{x^2}{2} + \frac{\cos 2x}{4} \right) \right]_0^{2\pi} = \pi
 \end{aligned}$$

Substituting the values of  $a$ 's and  $b$ 's, in (i), we get

$$x \sin x = -1 + \pi \sin x - \frac{1}{2} \cos x + \frac{2}{2^2-1} \cos 2x + \frac{2}{3^2-1} \cos 3x + \dots$$

**Example 10.4.** Expand  $f(x) = \sqrt{1-\cos x}$ ,  $0 < x < 2\pi$  in a Fourier series. Hence evaluate

$$\frac{I}{1.3} + \frac{I}{3.5} + \frac{I}{5.7} + \dots$$

(Mumbai, 2006 ; J.N.T.U., 2006)

**Solution.** We have  $f(x) = \sqrt{1-\cos x} = \sqrt{2 \sin x/2}$ .

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots(i)$$

$$\text{Then } a_0 = \frac{1}{\pi} \int_0^{2\pi} \sqrt{2 \sin x/2} dx = \frac{\sqrt{2}}{\pi} \left| -2 \cos \frac{\pi}{2} \right|_0^{2\pi} = \frac{4\sqrt{2}}{\pi}$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} \sqrt{2} \sin \frac{x}{2} \cos nx dx = \frac{\sqrt{2}}{2\pi} \int_0^{2\pi} 2 \cos nx \sin x/2 dx \\
 &= \frac{1}{\sqrt{2}\pi} \int_0^{2\pi} \left[ \sin \left( n + \frac{1}{2} \right)x - \sin \left( n - \frac{1}{2} \right)x \right] dx \\
 &= \frac{1}{\sqrt{2}\pi} \left| -\frac{2}{2n+1} \cos \left( \frac{2n+1}{2} \right)x + \frac{2}{2n-1} \cos \left( \frac{2n-1}{2} \right)x \right|_0^{2\pi}
 \end{aligned}$$

$$= \frac{2}{\sqrt{2}\pi} \left\{ -\frac{1}{2n+1} [\cos(2n+1)\pi - 1] + \frac{1}{2n-1} [\cos(2n-1)\pi - 1] \right\}$$

$$\begin{aligned}
 &= \frac{\sqrt{2}}{\pi} \left( \frac{2}{2n+1} - \frac{2}{2n-1} \right) = -\frac{4\sqrt{2}}{\pi(4n^2-1)} \quad [\because \cos(2n+1)\pi = \cos(2n-1)\pi = -1] \\
 b_n &= \frac{1}{\pi} \int_0^{2\pi} \sqrt{2} \sin \frac{x}{2} \sin nx dx = \frac{\sqrt{2}}{2\pi} \int_0^{2\pi} 2 \sin nx \sin x/2 dx \\
 &= \frac{1}{\sqrt{2}\pi} \int_0^{2\pi} \left[ \cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x \right] dx \\
 &= \frac{1}{\sqrt{2}\pi} \left| \frac{2}{2n-1} \sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1} \sin\left(\frac{2n+1}{2}\right)x \right|_0^{2\pi} \\
 &= \frac{\sqrt{2}}{\pi} \left[ \frac{1}{2n-1} \{\sin(2n-1)\pi - 0\} - \frac{1}{2n+1} \{\sin(2n+1)\pi - 0\} \right] = 0
 \end{aligned}$$

Substituting the values of  $a$ 's and  $b$ 's in (i), we get

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty} \frac{4\sqrt{2}}{(4n^2-1)\pi} \cos nx$$

When  $x = 0$ , we have

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} \quad i.e., \quad \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{1}{2}$$

### PROBLEMS 10.1

- Obtain a Fourier series to represent  $e^{-ax}$  from  $x = -\pi$  to  $x = \pi$ . Hence derive series for  $\pi/\sinh \pi$ .
- Prove that  $x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$ ,  $-\pi < x < \pi$ . (P.T.U., 2009 ; Bhopal, 2008 ; B.P.T.U., 2006)
- Hence show that (i)  $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ . (Anna, 2009 ; P.T.U., 2009 ; Osmania, 2003)  
 (ii)  $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$  (iii)  $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$  (S.V.T.U., 2008)  
 (iv)  $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$ . (Bhopal, 2008)
- If  $f(x) = \left(\frac{n-x}{2}\right)^2$  in the range 0 to  $2\pi$ , show that  $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ . (Delhi, 2002 ; Madras, 2000)
- Prove that in the range  $-\pi < x < \pi$ ,  $\cosh ax = \frac{2a^2}{\pi} \sinh a\pi \left[ \frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} \cos nx \right]$ .
- $f(x) = x + x^2$  for  $-\pi < x < \pi$  and  $f(x) = \pi^2$  for  $x = \pm \pi$ . Expand  $f(x)$  in Fourier series. (Kurukshetra, 2005 ; U.P.T.U., 2003)
- Hence show that  $x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{4}{n^2} \cos nx - \frac{2}{n} \sin nx \right\}$   
 and  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$  (V.T.U., 2008)

### 10.3 CONDITIONS FOR A FOURIER EXPANSION

The reader must not be misled by the belief that the Fourier expansion of  $f(x)$  in each case shall be valid. The above discussion has merely shown that if  $f(x)$  has an expansion, then the coefficients are given by Euler's formulae. The problems concerning the possibility of expressing a function by Fourier series and convergence

of this series are many and cumbersome. Such questions should be left to the curiosity of a pure-mathematician. However, almost all engineering applications are covered by the following well-known **Dirichlet's conditions\***:

Any function  $f(x)$  can be developed as a Fourier series  $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$  where  $a_0, a_n, b_n$  are constants, provided :

- (i)  $f(x)$  is periodic, single-valued and finite;
- (ii)  $f(x)$  has a finite number of discontinuities in any one period;
- (iii)  $f(x)$  has at the most a finite number of maxima and minima.

(Anna, 2009 ; P.T.U., 2009)

In fact the problem of expressing any function  $f(x)$  as a Fourier series depends upon the evaluation of the integrals.

$$\frac{1}{\pi} \int f(x) \cos nx dx; \frac{1}{\pi} \int f(x) \sin nx dx$$

within the limits  $(0, 2\pi)$ ,  $(-\pi, \pi)$  or  $(\alpha, \alpha + 2\pi)$  according as  $f(x)$  is defined for every value of  $x$  in  $(0, 2\pi)$ ,  $(-\pi, \pi)$  or  $(\alpha, \alpha + 2\pi)$ .

### PROBLEMS 10.2

State giving reasons whether the following functions can be expanded in Fourier series in the interval  $-\pi \leq x \leq \pi$ .

1.  $\text{cosec } x$
2.  $\sin 1/x$
3.  $f(x) = (m+1)/m, \pi/(m+1) < |x| \leq \pi/m, m = 1, 2, 3, \dots \infty,$

### 10.4 FUNCTIONS HAVING POINTS OF DISCONTINUITY

In deriving the Euler's formulae for  $a_0, a_n, b_n$ , it was assumed that  $f(x)$  was continuous. Instead a function may have a finite number of points of finite discontinuity i.e., its graph may consist of a finite number of different curves given by different equations. Even then such a function is expressible as a Fourier series.

For instance, if in the interval  $(\alpha, \alpha + 2\pi)$ ,  $f(x)$  is defined by

$$f(x) = \phi(x), \alpha < x < c \\ = \psi(x), c < x < \alpha + 2\pi, \text{ i.e., } c \text{ is the point of discontinuity, then}$$

$$a_0 = \frac{1}{\pi} \left[ \int_{\alpha}^c \phi(x) dx + \int_c^{\alpha+2\pi} \psi(x) dx \right]$$

$$a_n = \frac{1}{\pi} \left[ \int_{\alpha}^c \phi(x) \cos nx dx + \int_c^{\alpha+2\pi} \psi(x) \cos nx dx \right]$$

$$\text{and } b_n = \frac{1}{\pi} \left[ \int_{\alpha}^c \phi(x) \sin nx dx + \int_c^{\alpha+2\pi} \psi(x) \sin nx dx \right]$$

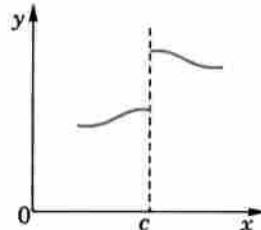


Fig. 10.1

At a point of finite discontinuity  $x = c$ , there is a finite jump in the graph of function (Fig. 10.1). Both the limit on the left [i.e.,  $f(c - 0)$ ] and the limit on the right [i.e.,  $f(c + 0)$ ] exist and are different. At such a point, Fourier series gives the value of  $f(x)$  as the arithmetic mean of these two limits,

$$\text{i.e., at } x = c, \quad f(x) = \frac{1}{2} [f(c - 0) + f(c + 0)].$$

**Example 10.5.** Find the Fourier series expansion for  $f(x)$ , if

$$f(x) = -\pi, -\pi < x < 0$$

$$x, 0 < x < \pi.$$

(Bhopal, 2008 S)

$$\text{Deduce that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

(Kottayam, 2005)

**Solution.** Let  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$  ... (i)

Then

$$a_0 = \frac{1}{\pi} \left[ \int_{-\pi}^0 (-\pi) dx + \int_0^\pi x dx \right] = \frac{1}{\pi} \left[ -\pi |x| \Big|_{-\pi}^0 + \left| x^2/2 \right| \Big|_0^\pi \right] = \frac{1}{\pi} \left( -\pi^2 + \frac{\pi^2}{2} \right) = -\frac{\pi}{2};$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[ \int_{-\pi}^0 (-\pi) \cos nx dx + \int_0^\pi x \cos nx dx \right] \\ &= \frac{1}{\pi} \left[ -\pi \left| \frac{\sin nx}{n} \right| \Big|_{-\pi}^0 + \left| \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right| \Big|_0^\pi \right] \\ &= \frac{1}{\pi} \left[ 0 + \frac{1}{n^2} \cos n\pi - \frac{1}{n^2} \right] = \frac{1}{\pi n^2} (\cos n\pi - 1) \end{aligned}$$

$$\therefore a_1 = \frac{-2}{\pi \cdot 1^2}, a_2 = 0, a_3 = -\frac{2}{\pi \cdot 3^2}, a_4 = 0, a_5 = -\frac{2}{\pi \cdot 5^2} \text{ etc.}$$

Finally,

$$\begin{aligned} b_n &= \frac{1}{\pi} \left[ \int_{-\pi}^0 (-\pi) \sin nx dx + \int_0^\pi x \sin nx dx \right] \\ &= \frac{1}{\pi} \left[ \left| \frac{\pi \cos nx}{n} \right| \Big|_{-\pi}^0 + \left| -x \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right| \Big|_0^\pi \right] \\ &= \frac{1}{\pi} \left[ \frac{\pi}{n} (1 - \cos n\pi) - \frac{\pi}{n} \cos n\pi \right] = \frac{1}{n} (1 - 2 \cos n\pi) \end{aligned}$$

$$\therefore b_1 = 3, b_2 = -\frac{1}{2}, b_3 = 1, b_4 = -\frac{1}{4}, \text{ etc.}$$

Hence substituting the values of  $a$ 's and  $b$ 's in (i), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi} \left( \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) + 3 \sin x - \frac{\sin 2x}{2} + \frac{3 \sin 3x}{3} - \frac{\sin 4x}{4} + \dots \quad \text{... (ii)}$$

which is the required result.

$$\text{Putting } x = 0 \text{ in (ii), we obtain } f(0) = -\frac{\pi}{4} - \frac{2}{\pi} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \infty \right) \quad \text{... (iii)}$$

Now  $f(x)$  is discontinuous at  $x = 0$ . As a matter of fact

$$f(0-0) = -\pi \text{ and } f(0+0) = 0 \quad \therefore f(0) = \frac{1}{2} [f(0-0) + f(0+0)] = -\pi/2.$$

Hence (iii) takes the form  $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi} \left[ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$  whence follows the result.

**Example 10.6.** If  $f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ \sin x, & 0 \leq x \leq \pi \end{cases}$ , prove that  $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$ .

Hence show that  $\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \dots - \infty = \frac{1}{4}(\pi - 2)$  (Bhopal, 2008; Mumbai, 2005 S; Rohtak, 2005)

**Solution.** Let  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 dx + \int_0^{\pi} \sin x dx \right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 \cdot dx + \int_0^{\pi} \sin x \cos nx dx \right]$$

$$= \frac{1}{2\pi} \int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] dx = \frac{1}{2\pi} \left[ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \left[ -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right] \\
 &= \frac{1}{2\pi} \left\{ \frac{1 - (-1)^{n+1}}{n+1} - \frac{(-1)^{n-1} - 1}{n-1} \right\} = 0, \text{ when } n \text{ is odd} \\
 &= -\frac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even.}
 \end{aligned} \tag{n \neq 1}$$

$$\text{When } n = 1, \quad a_1 = \frac{1}{\pi} \int_0^\pi \sin x \cos x \, dx = \frac{1}{2\pi} \int_0^\pi \sin 2x \, dx = \frac{1}{2\pi} \left[ -\frac{\cos 2x}{2} \right]_0^\pi = 0$$

$$\begin{aligned}
 \text{Finally, } b_n &= \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 \cdot dx + \int_0^\pi \sin x \sin nx \, dx \right] \\
 &= \frac{1}{2\pi} \int_0^\pi [\cos \overline{n-1}x - \cos \overline{n+1}x] \, dx = \frac{1}{2\pi} \left[ \frac{\sin \overline{n-1}x}{n-1} - \frac{\sin \overline{n+1}x}{n+1} \right]_0^\pi = 0 \quad (n \neq 1)
 \end{aligned}$$

$$\text{When } n = 1, \quad b_1 = \frac{1}{\pi} \int_0^\pi \sin x \sin x \, dx = \frac{1}{2\pi} \int_0^\pi (1 - \cos 2x) \, dx = \frac{1}{2\pi} \left[ x - \frac{\sin 2x}{2} \right]_0^\pi = \frac{1}{2}$$

$$\text{Hence } f(x) = \frac{1}{\pi} - \frac{2}{\pi} \left[ \frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots \right] + \frac{1}{2} \sin x \tag{i}$$

$$\text{Putting } x = \frac{\pi}{2} \text{ in (i), we get } 1 = \frac{1}{\pi} - \frac{2}{\pi} \left( -\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} - \frac{1}{5 \cdot 7} + \dots \infty \right) + \frac{1}{2}$$

$$\text{Whence } \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \dots \infty = \frac{1}{4}(\pi - 2).$$

**Example 10.7.** Find the Fourier series for the function

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$$

$$\text{Solution. Let } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \tag{i}$$

$$\text{Then } a_0 = \frac{1}{\pi} \left\{ \int_{-\pi}^{-\pi/2} (-1) dt + \int_{-\pi/2}^{\pi/2} (0) dt + \int_{\pi/2}^{\pi} (1) dt \right\}$$

$$= \frac{1}{\pi} \left\{ \left[ -x \right]_{-\pi}^{-\pi/2} + \left[ x \right]_{\pi/2}^{\pi} \right\} = \frac{1}{\pi} (\pi/2 - \pi + \pi - \pi/2) = 0$$

$$a_n = \frac{1}{\pi} \left\{ \int_{-\pi}^{-\pi/2} (-1) \cos nt dt + \int_{-\pi/2}^{\pi/2} (0) \cos nt dt + \int_{\pi/2}^{\pi} (1) \cos nt dt \right\}$$

$$= \frac{1}{\pi} \left\{ \left[ -\frac{\sin nt}{n} \right]_{-\pi}^{-\pi/2} + \left[ \frac{\sin nt}{n} \right]_{\pi/2}^{\pi} \right\} = \frac{1}{n\pi} \left( \frac{\sin n\pi}{2} - \frac{\sin n\pi}{2} \right) = 0$$

and

$$b_n = \frac{1}{\pi} \left\{ \int_{-\pi}^{-\pi/2} (-1) \sin nt dt + \int_{-\pi/2}^{\pi/2} (0) \sin nt dt + \int_{\pi/2}^{\pi} (1) \sin nt dt \right\}$$

$$= \frac{1}{\pi} \left\{ \left[ \frac{\cos nt}{n} \right]_{-\pi}^{-\pi/2} + \left[ -\frac{\cos nt}{n} \right]_{\pi/2}^{\pi} \right\} = \frac{2}{n\pi} \left( \cos \frac{n\pi}{2} - \cos n\pi \right)$$

$$\therefore b_1 = \frac{2}{\pi}, b_2 = -\frac{2}{\pi}, b_3 = \frac{2}{3\pi} \text{ etc.}$$

Hence substituting the values of  $a$ 's and  $b$ 's in (i), we get  $f(t) = \frac{2}{\pi} \left( \sin t - \sin 2t + \frac{1}{3} \sin 3t + \dots \right)$ .

### PROBLEMS 10.3

1. Find the Fourier series to represent the function  $f(x)$  given by

$$f(x) = x \text{ for } 0 \leq x \leq \pi, \text{ and } = 2\pi - x \text{ for } \pi \leq x \leq 2\pi.$$

(S.V.T.U., 2008; B.P.T.U., 2005 S)

$$\text{Deduce that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}.$$

(Madras 2000 S; V.T.U., 2000 S)

2. An alternating current after passing through a rectifier has the form

$$\begin{aligned} i &= I_0 \sin x && \text{for } 0 \leq x \leq \pi \\ &= 0 && \text{for } \pi \leq x \leq 2\pi \end{aligned}$$

where  $I_0$  is the maximum current and the period is  $2\pi$  (Fig. 10.2). Express  $i$  as a Fourier series and evaluate

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots \infty$$

(V.T.U., 2007; Calicut, 2005)

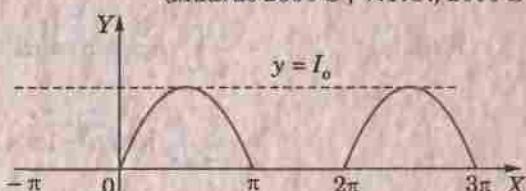


Fig. 10.2

3. Draw the graph of the function  $f(x) = 0, -\pi < x < 0$   
 $= x^2, 0 < x < \pi$ .

If  $f(2\pi + x) = f(x)$ , obtain Fourier series of  $f(x)$ .

4. Find the Fourier series of the following function:

$$\begin{aligned} f(x) &= x^2, && 0 \leq x \leq \pi, \\ &= -x^2, && -\pi \leq x \leq 0. \end{aligned}$$

(Mumbai, 2009)

(Hissar, 2007)

5. Find a Fourier series for the function defined by

$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < 0 \\ 0, & \text{for } x = 0 \\ 1, & \text{for } 0 < x < \pi \end{cases}$$

$$\text{Hence prove that } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \infty = \frac{\pi}{4}.$$

(U.P.T.U., 2005)

### 10.5 CHANGE OF INTERVAL

In many engineering problems, the period of the function required to be expanded is not  $2\pi$  but some other interval, say :  $2c$ . In order to apply the foregoing discussion to functions of period  $2c$ , this interval must be converted to the length  $2\pi$ . This involves only a proportional change in the scale.

Consider the periodic function  $f(x)$  defined in  $(\alpha, \alpha + 2c)$ . To change the problem to period  $2\pi$

$$\text{put } z = \pi x/c \quad \text{or} \quad x = cz/\pi \quad \dots(1)$$

$$\text{so that when } x = \alpha, \quad z = \alpha\pi/c = \beta \text{ (say)}$$

$$\text{when } x = \alpha + 2c, \quad z = (\alpha + 2c)\pi/c = \beta + 2\pi.$$

Thus the function  $f(x)$  of period  $2c$  in  $(\alpha, \alpha + 2c)$  is transformed to the function  $f(cz/\pi)$  [=  $F(z)$  say] of period  $2\pi$  in  $(\beta, \beta + 2\pi)$ . Hence  $f(cz/\pi)$  can be expressed as the Fourier series

$$f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \quad \dots(2)$$

$$\left. \begin{aligned} a_0 &= \frac{1}{\pi} \int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) dz \\ a_n &= \frac{1}{\pi} \int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \cos nz dz \\ b_n &= \frac{1}{\pi} \int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \sin nz dz \end{aligned} \right\} \quad \dots(3)$$

Making the inverse substitutions  $z = \pi x/c$ ,  $dz = (\pi/c) dx$  in (2) and (3) the Fourier expansion of  $f(x)$  in the interval  $(\alpha, \alpha + 2c)$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

where

$$\left. \begin{aligned} a_0 &= \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x) dx \\ a_n &= \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x) \cos \frac{n\pi x}{c} dx \\ b_n &= \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x) \sin \frac{n\pi x}{c} dx \end{aligned} \right\} \quad \dots(4)$$

**Cor.** Putting  $\alpha = 0$  in (4), we get the results for the interval  $(0, 2c)$  and putting  $\alpha = -c$  in (4), we get results for the interval  $(-c, c)$ .

**Example 10.8.** Expand  $f(x) = e^{-x}$  as a Fourier series in the interval  $(-l, l)$ .

(Kerala, 2005 ; V.T.U., 2004)

**Solution.** The required series is of the form

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \dots(i)$$

$$\text{Then } a_0 = \frac{1}{l} \int_{-l}^l e^{-x} dx = \frac{1}{l} \left[ -e^{-x} \right]_{-l}^l = \frac{1}{l} (e^l - e^{-l}) = \frac{2 \sinh l}{l}$$

$$\begin{aligned} \text{and } a_n &= \frac{1}{l} \int_{-l}^l e^{-x} \cos \frac{n\pi x}{l} dx & \left[ \because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right] \\ &= \frac{1}{l} \left| \frac{e^{-x}}{1 + (n\pi/l)^2} \left( -\cos \frac{n\pi x}{l} + \frac{n\pi}{l} \sin \frac{n\pi x}{l} \right) \right|_{-l}^l = \frac{2l(-1)^n \sinh l}{l^2 + (n\pi)^2} & [\because \cos n\pi = (-1)^n] \end{aligned}$$

$$\therefore a_1 = \frac{-2l \sinh l}{l^2 + \pi^2}, a_2 = \frac{2l \sinh l}{l^2 + 2^2 \pi^2}, a_3 = \frac{2l \sinh l}{l^2 + 3^2 \pi^2} \text{ etc.}$$

$$\begin{aligned} \text{Finally, } b_n &= \frac{1}{l} \int_{-l}^l e^{-x} \sin \frac{n\pi x}{l} dx & \left[ \because \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \right] \\ &= \frac{1}{l} \left| \frac{e^{-x}}{1 + (n\pi/l)^2} \left( -\sin \frac{n\pi x}{l} - \frac{n\pi}{l} \cos \frac{n\pi x}{l} \right) \right|_{-l}^l = \frac{2n\pi(-1)^n \sinh l}{l^2 + (n\pi)^2} \end{aligned}$$

$$\therefore b_1 = \frac{-2\pi \sinh l}{l^2 + \pi^2}, b_2 = \frac{4\pi \sinh l}{l^2 + 2^2 \pi^2}, b_3 = \frac{-6\pi \sinh l}{l^2 + 3^2 \pi^2} \text{ etc.}$$

Substituting the values of  $a$ 's and  $b$ 's in (i), we get

$$\begin{aligned} e^{-x} &= \sinh l \left\{ \frac{1}{l} - 2l \left( \frac{1}{l^2 + \pi^2} \cos \frac{\pi x}{l} - \frac{1}{l^2 + 2^2 \pi^2} \cos \frac{2\pi x}{l} + \frac{1}{l^2 + 3^2 \pi^2} \cos \frac{3\pi x}{l} - \dots \right) \right. \\ &\quad \left. - 2\pi \left( \frac{1}{l^2 + \pi^2} \sin \frac{n\pi}{l} - \frac{2}{l^2 + 2^2 \pi^2} \sin \frac{2\pi x}{l} + \frac{3}{l^2 + 3^2 \pi^2} \sin \frac{3\pi x}{l} - \dots \right) \right\} \end{aligned}$$

**Example 10.9.** Find the Fourier series expansion of  $f(x) = 2x - x^2$  in  $(0, 3)$  and hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \infty = \frac{\pi}{12}.$$

(Mumbai, 2005)

**Solution.** The required series is of the form

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{where } l = 3/2. \quad \dots(i)$$

Then  $a_0 = \frac{1}{l} \int_0^{2l} (2x - x^2) dx = \frac{2}{3} \left| x^2 - \frac{x^3}{3} \right|_0^3 = 0$

$$\begin{aligned} a_n &= \frac{1}{l} \int_0^{2l} (2x - x^2) \cos \frac{n\pi x}{l} dx = \frac{2}{3} \int_0^3 (2x - x^2) \cos \frac{2n\pi x}{3} dx \\ &= \frac{2}{3} \left[ (2x - x^2) \frac{\sin 2n\pi x/3}{2n\pi/3} - (2 - 2x) \frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} + (-2) \frac{-\sin 2n\pi x/3}{(2n\pi/3)^3} \right]_0^3 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{3} \cdot \frac{9}{4n^2\pi^2} [(2 - 6) \cos 2n\pi - 2] = -\frac{9}{n^2\pi^2} \\ b_n &= \frac{1}{l} \int_0^{2l} (2x - x^2) \sin \frac{n\pi x}{l} dx = \frac{2}{3} \int_0^3 (2x - x^2) \sin \frac{2n\pi x}{3} dx \\ &= \frac{2}{3} \left[ (2x - x^2) \frac{-\cos 2n\pi x/3}{2n\pi/3} - (2 - 2x) \frac{-\sin 2n\pi x/3}{(2n\pi/3)^2} + (-2) \frac{\cos 2n\pi x/3}{(2n\pi/3)^3} \right]_0^3 \\ &= \frac{2}{3} \left\{ -\frac{6}{n^2\pi^2} \cos 2n\pi - \frac{27}{4n^3\pi^3} (\cos 2n\pi - 1) \right\} = \frac{3}{n\pi} \end{aligned}$$

Substituting the values of  $a_0, a_n, b_n$  in (i), we get

$$2x - x^2 = - \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \cos \frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin \frac{2n\pi x}{3}$$

Putting  $x = 3/2$ , we get

$$3 - \frac{9}{4} = - \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \cos n\pi \quad \text{or} \quad - \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \frac{\pi^2}{9} \cdot \frac{3}{4}$$

or  $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots \infty = \frac{\pi^2}{12}$ .

**Example 10.10.** Obtain Fourier series for the function

$$f(x) = \begin{cases} \pi x, & 0 \leq x \leq 1 \\ \pi(2-x), & 1 \leq x \leq 2 \end{cases} \quad (\text{V.T.U., 2011; Bhopal, 2008; Mumbai, 2007})$$

Deduce that  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$ .

**Solution.** The required series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then  $a_0 = \int_0^1 \pi x dx + \int_1^2 \pi(2-x) dx = \pi \left[ \frac{x^2}{2} \right]_0^1 + \pi \left[ 2x - \frac{x^2}{2} \right]_1^2 = \pi \left( \frac{1}{2} \right) + \pi \left[ (4 - 2) - \left( 2 - \frac{1}{2} \right) \right] = \pi$

$$\begin{aligned} a_n &= \int_0^1 \pi x \cos nx dx + \int_1^2 \pi(2-x) \cos nx dx \\ &= \left| \pi x \cdot \frac{\sin nx}{n\pi} - \pi \left( -\frac{\cos nx}{n^2\pi^2} \right) \right|_0^1 + \left| \pi(2-x) \frac{\sin nx}{n\pi} - (-\pi) \left( -\frac{\cos nx}{n^2\pi^2} \right) \right|_1^2 \\ &= \left( \frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi^2} \right) - \left( \frac{\cos 2n\pi}{n^2\pi} - \frac{\cos n\pi}{n^2\pi} \right) = \frac{2}{n^2\pi} [(-1)^n - 1] \end{aligned}$$

$$= 0 \text{ when } n \text{ is even} ; -\frac{4}{n^2\pi} \text{ when } n \text{ is odd.}$$

$$\begin{aligned} b_n &= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx \\ &= \left| \pi x \left( -\frac{\cos n\pi x}{n\pi} \right) - \pi \left( -\frac{\sin n\pi x}{n^2\pi^2} \right) \right|_0^1 + \left| \pi(2-x) \left( -\frac{\cos n\pi x}{n\pi} \right) - (-\pi) \left( -\frac{\sin n\pi x}{n^2\pi^2} \right) \right|_1^2 \\ &= \left( -\frac{\cos n\pi}{n} \right) + \left( \frac{\cos n\pi}{n} \right) = 0 \end{aligned}$$

Hence  $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left( \frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \infty \right)$

Putting  $x = 2$ ,  $0 = \frac{\pi}{2} - \frac{4}{\pi} \left( \frac{\cos 2\pi}{1^2} + \frac{\cos 6\pi}{3^2} + \frac{\cos 10\pi}{5^2} + \dots \infty \right)$

Whence  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$ .

**Example 10.11.** Find the Fourier series for

$$\begin{aligned} f(t) &= 0, -2 < t < -1 \\ &= 1+t, -1 < t < 0 \\ &= 1-t, 0 < t < 1 \\ &= 0, \quad 1 < t < 2. \end{aligned}$$

Solution. Let  $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{2}$  ... (i)

[ $\because 2c = 2 - (-2)$  so that  $c = 2$ ]

Then  $a_0 = \frac{1}{2} \left\{ \int_{-2}^{-1} (0) dt + \int_{-1}^0 (1+t) dt + \int_0^1 (1-t) dt + \int_1^2 (0) dt \right\} = \frac{1}{2} \left\{ \left| t + \frac{t^2}{2} \right|_{-1}^0 + \left| t - \frac{t^2}{2} \right|_0^1 \right\}$

$$= \frac{1}{2} \left\{ -\left( -1 + \frac{1}{2} \right) + \left( 1 - \frac{1}{2} \right) \right\} = \frac{1}{2}$$

$$a_n = \frac{1}{2} \left\{ \int_{-1}^0 (1+t) \cos \frac{n\pi t}{2} dt + \int_0^1 (1-t) \cos \frac{n\pi t}{2} dt \right\}$$
 [Integrate by parts]
$$\begin{aligned} &= \frac{1}{2} \left\{ \left| (1+t) \left( \sin \frac{n\pi t}{2} \right) \frac{2}{n\pi} - (1) \left( -\cos \frac{n\pi t}{2} \right) \frac{4}{n^2\pi^2} \right|_{-1}^0 \right. \\ &\quad \left. + \left| (1-t) \left( \sin \frac{n\pi t}{2} \right) \frac{2}{n\pi} - (-1) \left( -\cos \frac{n\pi t}{2} \right) \frac{4}{n^2\pi^2} \right|_0^1 \right\} \\ &= \frac{4}{n^2\pi^2} (1 - \cos n\pi/2) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2} \left\{ \int_{-1}^0 (1+t) \sin \frac{n\pi t}{2} dt + \int_0^1 (1-t) \sin \frac{n\pi t}{2} dt \right\} \\ &= \frac{1}{2} \left\{ \left| (1+t) \left( -\cos \frac{n\pi t}{2} \right) \frac{2}{n\pi} - 1 \left( -\sin \frac{n\pi t}{2} \right) \frac{4}{n^2\pi^2} \right|_{-1}^0 \right. \\ &\quad \left. + \left| (1-t) \left( -\cos \frac{n\pi t}{2} \right) \frac{2}{n\pi} - (-1) \left( -\sin \frac{n\pi t}{2} \right) \frac{4}{n^2\pi^2} \right|_0^1 \right\} \end{aligned}$$

$$= \frac{1}{2} \left\{ \frac{4}{n^2 \pi^2} \sin \frac{n\pi}{2} \right\} - \left( \frac{4}{n^2 \pi^2} \sin \frac{n\pi}{2} \right) = 0$$

Substituting the values of  $a$ 's and  $b$ 's in (i), we get

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} \left( 1 - \cos \frac{n\pi}{2} \right) \cos \frac{n\pi t}{2}.$$

### PROBLEMS 10.4

1. Obtain the Fourier series for  $f(x) = \pi x$  in  $0 \leq x \leq 2$ .  
 2. (i) Find the Fourier series to represent  $x^2$  in the interval  $(0, a)$ .  
 (ii) Find a Fourier series for  $f(t) = 1 - t^2$  when  $-1 \leq t \leq 1$ .

(Mumbai, 2009)

(Mumbai, 2006)

3. If  $f(x) = 2x - x^2$  in  $0 \leq x \leq 2$ , show that  $f(x) = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} \cos n\pi x$ .

(V.T.U., 2006)

4. Find the Fourier series for  $f(x) = \begin{cases} x & \text{in } 0 \leq x \leq 3 \\ 6-x & \text{in } 3 \leq x \leq 6 \end{cases}$

(Anna, 2008)

5. A sinusoidal voltage  $E \sin \omega t$  is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function

$$\begin{aligned} U(t) &= 0 && \text{when } -T/2 < t < 0 \\ &= E \sin \omega t && \text{when } 0 < t < T/2, \end{aligned}$$

and  $T = 2\pi/\omega$ , in a Fourier series.

(Calicut, 1999)

6. Find the Fourier series of the function  $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x-2), & 1 < x < 2 \end{cases}$

$$\text{Hence show that } \frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

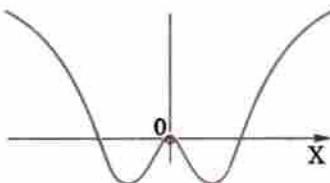
(Mumbai, 2008)

## 10.6 (1) EVEN AND ODD FUNCTIONS

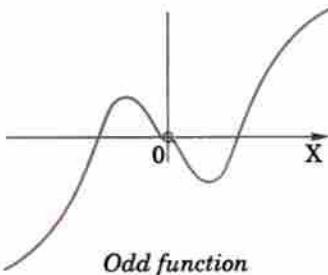
A function  $f(x)$  is said to be **even** iff  $f(-x) = f(x)$ ,

e.g.,  $\cos x$ ,  $\sec x$ ,  $x^2$  are all even functions. Graphically an even function is symmetrical about the  $y$ -axis.

A function  $f(x)$  is said to be **odd** iff  $f(-x) = -f(x)$ ,



Even function



Odd function

Fig. 10.3

e.g.  $\sin x$ ,  $\tan x$ ,  $x^3$  are odd functions. Graphically, an odd function is symmetrical about the origin.

We shall be using the following property of definite integrals in the next paragraph :

$$\int_c^c f(x) dx = 2 \int_0^c f(x) dx, \text{ when } f(x) \text{ is an even function.}$$

$$= 0, \text{ when } f(x) \text{ is an odd function.}$$

**(2) Expansions of even or odd periodic functions.** We know that a periodic function  $f(x)$  defined in  $(-c, c)$  can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c},$$

where

$$a_0 = \frac{1}{c} \int_{-c}^c f(x) dx, a_n = \frac{1}{c} \int_{-c}^c f(x) \cos \frac{n\pi x}{c} dx, b_n = \frac{1}{c} \int_{-c}^c f(x) \sin \frac{n\pi x}{c} dx.$$

**Case I.** When  $f(x)$  is an even function  $a_0 = \frac{1}{c} \int_{-c}^c f(x) dx = \frac{2}{c} \int_0^c f(x) dx$ .

Since  $f(x) \cos \frac{n\pi x}{c}$  is also an even function,

$$\therefore a_n = \frac{1}{c} \int_{-c}^c f(x) \cos \frac{n\pi x}{c} dx = \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c} dx$$

Again since  $f(x) \sin \frac{n\pi x}{c}$  is an odd function,  $\therefore b_n = \frac{1}{c} \int_{-c}^c f(x) \sin \frac{n\pi x}{c} dx = 0$ .

Hence, if a periodic function  $f(x)$  is even, its Fourier expansion contains only cosine terms, and

$$\left. \begin{aligned} a_0 &= \frac{2}{c} \int_0^c f(x) dx \\ a_n &= \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c} dx \end{aligned} \right\} \quad \dots(1)$$

**Case II.** When  $f(x)$  is an odd function,  $a_0 = \frac{1}{c} \int_{-c}^c f(x) dx = 0$ ,

Since  $\cos \frac{n\pi x}{c}$  is an even function, therefore,  $f(x) \cos \frac{n\pi x}{c}$  is an odd function.

$$\therefore a_n = \frac{1}{c} \int_{-c}^c f(x) \cos \frac{n\pi x}{c} dx = 0$$

Again since  $\sin \frac{n\pi x}{c}$  is an odd function, therefore,  $f(x) \sin \frac{n\pi x}{c}$  is an even function.

$$\therefore b_n = \frac{1}{c} \int_{-c}^c f(x) \sin \frac{n\pi x}{c} dx = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} dx$$

Thus, if a periodic function  $f(x)$  is odd, its Fourier expansion contains only sine terms and

$$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} dx \quad \dots(2)$$

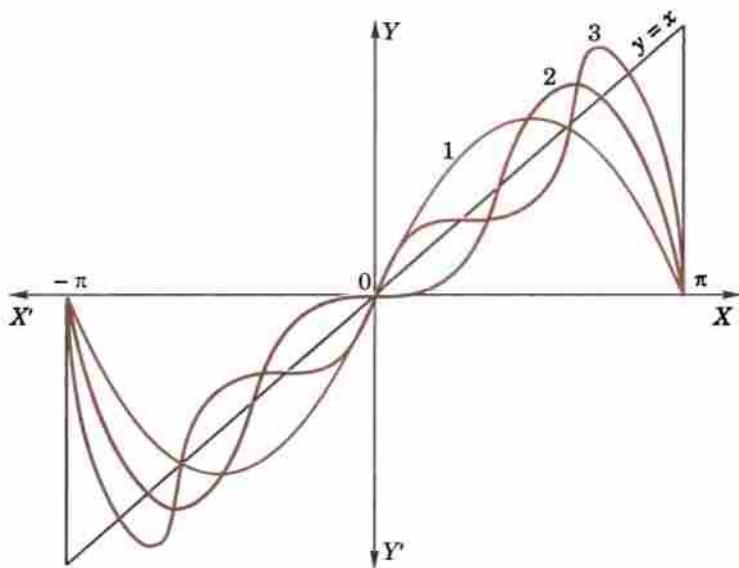


Fig. 10.4

**Example 10.12.** Express  $f(x) = x/2$  as a Fourier series in the interval  $-\pi < x < \pi$ .

(J.N.T.U., 2006)

**Solution.** Since

$$f(-x) = -x/2 = -f(x).$$

$\therefore f(x)$  is an odd function and hence  $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx = \frac{2}{\pi} \int_0^\pi \frac{x}{2} \sin nx dx$$

$$= \frac{1}{\pi} \left[ x \left( -\frac{\cos nx}{n} \right) - \left( \frac{-\sin nx}{n^2} \right) \right]_0^\pi = -\frac{\cos n\pi}{n}.$$

$\therefore b_1 = 1/1, b_2 = -1/2, b_3 = 1/3, b_4 = -1/4, \text{ etc.}$

Hence the series is  $x/2 = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots$  ... (i)

Obs. The graphs of  $y = 2 \sin x, y = 2(\sin x - \frac{1}{2} \sin 2x)$  and  $y = 2(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x)$  are shown in Fig. 10.4, by the curves 1, 2 and 3 respectively. These illustrate the manner in which the successive approximations to the series (i) approach more and more closely to  $y = x$  for all values of  $x$  in  $-\pi < x < \pi$ , but not for  $x = \pm \pi$ .

As the series has a period  $2\pi$ , it represents the discontinuous function, called *saw-toothed waveform*, shown in Fig. 10.5. It is important to note that the given function  $y = x$  is continuous and each term of the series (i) is continuous, but the function represented by the series (i) has finite discontinuities at  $x = \pm \pi, \pm 3\pi, \pm 5\pi$  etc.

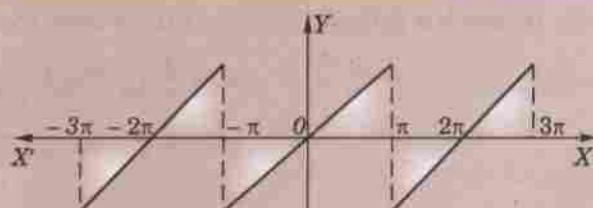


Fig. 10.5

**Example 10.13.** Find a Fourier series to represent  $x^2$  in the interval  $(-l, l)$ .

(S.V.T.U., 2008)

**Solution.** Since  $f(x) = x^2$  is an even function in  $(-l, l)$ ,

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \dots (i)$$

$$\text{Then } a_0 = \frac{2}{l} \int_0^l x^2 dx = \frac{2}{l} \left| \frac{x^3}{3} \right|_0^l = \frac{2l^2}{3}$$

$$a_n = \int_0^l x^2 \cos \frac{n\pi x}{l} dx \quad [\text{See footnote p. 398}]$$

$$= \frac{2}{l} \left[ x^2 \left( \frac{\sin n\pi x/l}{n\pi/l} \right) - 2x \left( -\frac{\cos n\pi x/l}{n^2\pi^2/l^2} \right) + 2 \left( -\frac{\sin n\pi x/l}{n^3\pi^3/l^3} \right) \right]_0^l \\ = 4l^2 (-1)^n / n^2 \pi^2$$

$\because \cos n\pi = (-1)^n$

$$\therefore a_1 = -4l^2/\pi^2, a_2 = 4l^2/2^2\pi^2, a_3 = -4l^2/3^2\pi^2, a_4 = 4l^2/4^2\pi^2 \text{ etc.}$$

Substituting these values in (i), we get

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2} \left( \frac{\cos \pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - \frac{\cos 4\pi x/l}{4^2} + \dots \right)$$

which is the required Fourier series.

**Example 10.14.** If  $f(x) = |\cos x|$ , expand  $f(x)$  as a Fourier series in the interval  $(-\pi, \pi)$ .

**Solution.** As  $f(-x) = |\cos(-x)| = |\cos x| = f(x)$ ,  $|\cos x|$  is an even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi} \int_0^\pi |\cos x| dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^\pi (-\cos x) dx$$

$\because \cos x$  is  $-ve$  when  $\pi/2 < x < \pi$

$$= \frac{2}{\pi} \left\{ |\sin x|_0^{\pi/2} - |\sin x|_{\pi/2}^\pi \right\} = \frac{2}{\pi} [(1-0) - (0-1)] = \frac{4}{\pi}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi |\cos x| \cos nx dx \\ &= \frac{2}{\pi} \left[ \int_0^{\pi/2} \cos x \cos nx dx + \int_{\pi/2}^\pi (-\cos x) \cos nx dx \right] \\ &= \frac{1}{\pi} \left\{ \int_0^{\pi/2} [\cos(n+1)x + \cos(n-1)x] dx - \int_{\pi/2}^\pi [\cos(n+1)x + \cos(n-1)x] dx \right\} \\ &= \frac{1}{\pi} \left\{ \left| \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right|_0^{\pi/2} - \left| \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right|_{\pi/2}^\pi \right\} \\ &= \frac{1}{\pi} \left[ \left\{ \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right\} + \left\{ \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right\} \right] \\ &= \frac{2}{\pi} \left( \frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1} \right) = \frac{-4 \cos n\pi/2}{\pi(n^2-1)} \quad (n \neq 1) \end{aligned}$$

$$\text{In particular } a_1 = \frac{2}{\pi} \left[ \int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^\pi \cos^2 x dx \right] = 0$$

$$\text{Hence } |\cos x| = \frac{2}{\pi} + \frac{4}{\pi} \left\{ \frac{1}{3} \cos 2x - \frac{1}{15} \cos 4x + \dots \right\}.$$

**Example 10.15.** Obtain Fourier series for the function  $f(x)$  given by

$$\begin{aligned} f(x) &= 1 + 2x/\pi, & -\pi \leq x \leq 0, \\ &= 1 - 2x/\pi, & 0 \leq x \leq \pi. \end{aligned}$$

$$\text{Deduce that } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

(V.T.U., 2010 ; Mumbai, 2007)

**Solution.** Since  $f(-x) = 1 - \frac{2x}{\pi}$  in  $(-\pi, 0) = f(x)$  in  $(0, \pi)$

and  $f(-x) = 1 + \frac{2x}{\pi}$  in  $(0, \pi) = f(x)$  in  $(-\pi, 0)$

$\therefore f(x)$  is an even function in  $(-\pi, \pi)$ . This is also clear from its graph A'BA (Fig. 10.6) which is symmetrical about the  $y$ -axis.

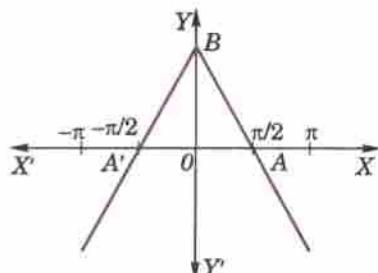


Fig. 10.6

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots(i)$$

$$\text{where } a_0 = \frac{2}{\pi} \int_0^\pi f(x) dx = \frac{2}{\pi} \int_0^\pi \left( 1 - \frac{2x}{\pi} \right) dx = \frac{2}{\pi} \left( x - \frac{x^2}{\pi} \right)_0^\pi = 0$$

$$\text{and } a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx = \frac{2}{\pi} \int_0^\pi \left( 1 - \frac{2x}{\pi} \right) \cos nx dx$$

$$= \frac{2}{\pi} \left[ \left( 1 - \frac{2x}{\pi} \right) \frac{\sin nx}{n} - \left( -\frac{2}{\pi} \right) \left( -\frac{\cos nx}{n^2} \right) \right]_0^\pi = \frac{2}{\pi} \left( -\frac{2 \cos n\pi}{\pi n^2} + \frac{2}{\pi n^2} \right) = \frac{4}{n^2 \pi^2} [1 - (-1)^n]$$

$$\therefore a_1 = 8/\pi^2, a_3 = 8/3^2 \pi^2, a_5 = 8/5^2 \pi^2, \dots$$

$$\text{and } a_2 = a_4 = a_6 = \dots = 0.$$

Thus substituting the values of  $a$ 's in (i), we get

$$f(x) = \frac{8}{\pi^2} \left[ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] \quad \dots(ii)$$

as the required Fourier expansion

Putting  $x = 0$  in (ii), we get  $1 = f(0) = \frac{8}{\pi^2} \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$

whence follows the desired result.

### PROBLEMS 10.5

1. Obtain the Fourier series expansion of  $f(x) = x^2$  in  $(0, a)$ . Hence show that

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

(Mumbai, 2009 ; S.V.T.U., 2008)

2. Show that for  $-\pi < x < \pi$ ,  $\sin ax = \frac{2 \sin a\pi}{\pi} \left( \frac{\sin x}{1^2 - a^2} - \frac{2 \sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} - \dots \right)$

3. Expand the function  $f(x) = x \sin x$  as a Fourier series in the interval  $-\pi \leq x \leq \pi$ .

(V.T.U., 2008 ; Anna, 2003)

Deduce that  $\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \dots = \frac{1}{4}(\pi - 2)$ .

(U.P.T.U., 2005)

4. Prove that in the interval  $-\pi < x < \pi$ ,  $x \cos x = -\frac{1}{2} \sin x + 2 \sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2 - 1} \sin nx$ .

(S.V.T.U., 2009)

5. For a function  $f(x)$  defined by  $f(x) = |x|$ ,  $-\pi < x < \pi$ , obtain a Fourier series.

(Bhopal, 2007 ; V.T.U., 2004)

Deduce that  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$ .

(S.V.T.U., 2009 ; Kerala, 2005 ; P.T.U., 2005)

6. Find the Fourier series to represent the function

(i)  $f(x) = |\sin x|$ ,  $-\pi < x < \pi$ .

(Mumbai, 2008)

(ii)  $f(x) = |\cos(\pi x/l)|$  in the interval  $(-1, 1)$ .

(P.T.U., 2009 S)

7. Given  $f(x) = \begin{cases} -x+1 & \text{for } -\pi \leq x \leq 0, \\ x+1 & \text{for } 0 \leq x \leq \pi. \end{cases}$

Is the function even or odd? Find the Fourier series for  $f(x)$  and deduce the value of

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

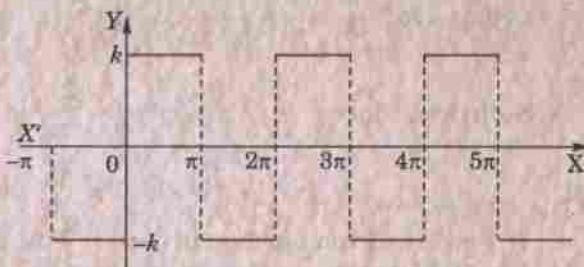


Fig. 10.7

8. Find the Fourier series of the periodic function  $f(x)$ :  $f(x) = -k$  when  $-\pi < x < 0$  and  $f(x) = k$  when  $0 < x < \pi$ , and  $f(x + 2\pi) = f(x)$ . Sketch the graph of  $f(x)$  and the two partial sums. (See Fig. 10.7)

Deduce that  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$ .

(Rohtak, 2005)

9. A function is defined as follows :

$$f(x) = -x \text{ when } -\pi < x \leq 0 = x \text{ when } 0 < x < \pi.$$

Show that  $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left[ \frac{1}{1^2} \cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right]$

Deduce that  $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ .

### 10.7 HALF RANGE SERIES

Many a time it is required to obtain a Fourier expansion of a function  $f(x)$  for the range  $(0, c)$  which is half the period of the Fourier series. As it is immaterial whatever the function may be outside the range  $0 < x < c$ , we extend the function to cover the range  $-c < x < c$  so that the new function may be odd or even. The Fourier expansion of such a function of half the period, therefore, consists of sine or cosine terms only. In such cases the

graphs for the values of  $x$  in  $(0, c)$  are the same but outside  $(0, c)$  are different for odd or even functions. That is why we get different forms of series for the same function as is clear from the examples 10.16 and 10.17.

**Sine series.** If it be required to expand  $f(x)$  as a sine series in  $0 < x < c$ ; then we extend the function reflecting it in the origin, so that  $f(x) = -f(-x)$ .

Then the extended function is odd in  $(-c, c)$  and the expansion will give the desired Fourier sine series :

$$\left. \begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \\ \text{where } b_n &= \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} dx \end{aligned} \right\} \quad \dots(1)$$

**Cosine series.** If it be required to express  $f(x)$  as a cosine series in  $0 < x < c$ , we extend the function reflecting it in the  $y$ -axis, so that  $f(-x) = f(x)$ .

Then the extended function is even in  $(-c, c)$  and its expansion will give the required Fourier cosine series :

$$\left. \begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} \\ \text{where } a_0 &= \frac{2}{c} \int_0^c f(x) dx \\ \text{and } a_n &= \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c} dx \end{aligned} \right\} \quad \dots(2)$$

**Example 10.16.** Express  $f(x) = x$  as a half-range sine series in  $0 < x < 2$ .

(U.P.T.U., 2004)

**Solution.** The graph of  $f(x) = x$  in  $0 < x < 2$  is the line OA. Let us extend the function  $f(x)$  in the interval  $-2 < x < 0$  (shown by the line BO) so that the new function is symmetrical about the origin and, therefore, represents an odd function in  $(-2, 2)$  (Fig. 10.8)

Hence the Fourier series for  $f(x)$  over the full period  $(-2, 2)$  will contain only sine terms given by

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \\ \text{where } b_n &= \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} dx = \int_0^2 x \sin \frac{n\pi x}{2} dx \\ &= \left| -\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right|_0^2 = -\frac{4(-1)^n}{n\pi} \end{aligned}$$

Thus  $b_1 = 4/\pi$ ,  $b_2 = -4/2\pi$ ,  $b_3 = 4/3\pi$ ,  $b_4 = -4/4\pi$  etc.

Hence the Fourier sine series for  $f(x)$  over the half-range  $(0, 2)$  is

$$f(x) = \frac{4}{\pi} \left( \sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \frac{1}{4} \sin \frac{4\pi x}{2} + \dots \right).$$

**Example 10.17.** Express  $f(x) = x$  as a half-range cosine series in  $0 < x < 2$ .

(S.V.T.U., 2009 ; Bhopal, 2007 ; Mumbai, 2006)

**Solution.** The graph of  $f(x) = x$  in  $(0, 2)$  is the line OA. Let us extend the function  $f(x)$  in the interval  $(-2, 0)$  shown by the line OB' so that the new function is symmetrical about the  $y$ -axis and, therefore, represents an even function in  $(-2, 2)$ . (Fig. 10.9)

Hence the Fourier series for  $f(x)$  over the full period  $(-2, 2)$  will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

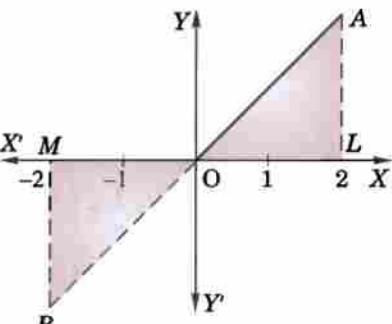


Fig. 10.8

where  $a_0 = \frac{2}{2} \int_0^2 f(x) dx = \int_0^2 x dx = 2$

and  $a_n = \frac{2}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx = \int_0^2 x \cos \frac{n\pi x}{2} dx$

$$= \left| \frac{2x}{n\pi} \sin \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi x}{2} \right|_0^2 = \frac{4}{n^2\pi^2} [(-1)^n - 1]$$

Thus  $a_1 = -8/\pi^2, a_2 = 0, a_3 = -8/3^2\pi^2, a_4 = 0, a_5 = -8/5^2\pi^2$  etc.

Hence the desired Fourier series for  $f(x)$  over the half-range  $(0, 2)$  is

$$f(x) = 1 - \frac{8}{\pi^2} \left[ \frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + \dots \right]$$

**Important Obs.** It must be clearly understood that we expand a function in  $0 < x < c$  as a series of sines or cosines, merely looking upon it as an odd or even function of period  $2c$ . It hardly matters whether the function is odd or even or neither.

**Example 10.18.** Obtain the Fourier expansion of  $x \sin x$  as a cosine series in  $(0, \pi)$ .

(V.T.U., 2003; U.P.T.U., 2002)

Hence show that  $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots \infty = \frac{\pi - 2}{4}$ .

(Anna, 2001)

**Solution.** Let  $x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then  $a_0 = \frac{2}{\pi} \int_0^\pi x \sin x dx = \frac{2}{\pi} [x(-\cos x) - 1(-\sin x)]_0^\pi = 2$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi x \sin x \cos nx dx = \frac{1}{\pi} \int_0^\pi x (\sin(n+1)x - \sin(n-1)x) dx \\ &= \frac{1}{\pi} \left[ x \left\{ \frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} - 1 \cdot \left\{ \frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^\pi \\ &= \frac{1}{\pi} \pi \left\{ \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \right\} (n \neq 1). \end{aligned}$$

When  $n = 1, a_1 = \frac{2}{\pi} \int_0^\pi x \sin x \cos x dx = \frac{1}{\pi} \int_0^\pi x \sin 2x dx$

$$= \frac{1}{\pi} \left[ x \left( \frac{-\cos 2x}{2} \right) - 1 \left( \frac{-\sin 2x}{2} \right) \right]_0^\pi = \frac{1}{\pi} \left( -\frac{\pi \cos 2\pi}{2} \right) = -\frac{1}{2}.$$

Hence  $x \sin x = 1 - \frac{1}{2} \cos x - 2 \left\{ \frac{\cos 2x}{1.3} - \frac{\cos 3x}{3.5} + \frac{\cos 4x}{5.7} - \dots \infty \right\}$

Putting  $x = \pi/2$ , we obtain  $\pi/2 = 1 + 2 \left\{ \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots \infty \right\}$

Hence  $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots \infty = \frac{\pi - 2}{4}$ .

**Example 10.19.** Obtain a half range cosine series for

$$f(x) = \begin{cases} kx, & 0 \leq x \leq l/2 \\ k(l-x), & l/2 \leq x \leq l. \end{cases} \quad (\text{Bhopal, 2008; V.T.U., 2008})$$

Deduce the sum of the series  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty$

(Rohtak, 2006; U.P.T.U., 2003)

**Solution.** Let the half-range cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

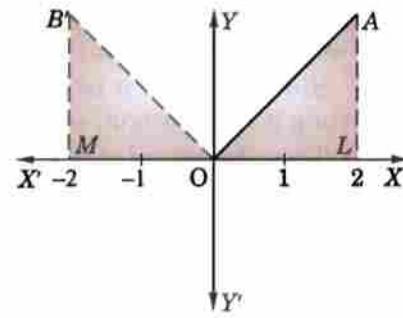


Fig. 10.9

Then 
$$\begin{aligned} a_0 &= \frac{2}{l} \left\{ \int_0^{l/2} kx dx + \int_{l/2}^l k(l-x) dx \right\} = \frac{2k}{l} \left\{ \left| \frac{x^2}{2} \right|_0^{l/2} - \left| \frac{(l-x)^2}{2} \right|_{l/2}^l \right\} \\ &= \frac{2k}{l} \cdot \frac{1}{2} \left\{ \frac{l^2}{4} - \left( 0 - \frac{l^2}{4} \right) \right\} = \frac{kl}{2} \\ a_n &= \frac{2}{l} \left\{ \int_0^{l/2} kx \cos \frac{n\pi x}{l} dx + \int_{l/2}^l k(l-x) \cos \frac{n\pi x}{l} dx \right\} \\ &= \frac{2k}{l} \left[ x \left( \frac{\sin n\pi x/l}{n\pi/l} \right) - 1 \left\{ -\cos \frac{n\pi x/l}{(n\pi/l)^2} \right\} \right]_0^{l/2} \\ &\quad + \frac{2k}{l} \left[ \left\{ \frac{(l-x) \sin n\pi x/l}{n\pi/l} \right\} - (-1) \left( \frac{-\cos n\pi x/l}{(n\pi/l)^2} \right) \right]_{l/2}^l \\ &= \frac{2k}{l} \left[ \left( \frac{l^2}{2n\pi} \cdot \sin \frac{n\pi}{2} \right) + \frac{l^2}{n^2\pi^2} \left( \cos \frac{n\pi}{2} - \cos 0 \right) \right] + \frac{2k}{l} \left[ \left( \frac{l}{n\pi} \left( -\frac{l}{2} \sin \frac{n\pi}{2} \right) \right. \right. \\ &\quad \left. \left. - \frac{l^2}{n^2\pi^2} \left( \cos n\pi - \cos \frac{n\pi}{2} \right) \right) \right] \\ &= \frac{2k}{l} \cdot \frac{l^2}{n^2\pi^2} \left[ 2 \cos \frac{n\pi}{2} - 1 - (-1)^n \right] = \frac{2kl}{n^2\pi^2} \left\{ 2 \cos \frac{n\pi}{2} - 1 - (-1)^n \right\} \end{aligned}$$

Hence the required Fourier series is

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2} \left[ \frac{1}{2^2} \cos \frac{2\pi x}{l} + \frac{1}{6^2} \cos \frac{6\pi x}{l} + \frac{1}{10^2} \cos \frac{10\pi x}{l} + \dots \right]$$

Putting  $x = l$ , we get

$$0 = \frac{kl}{4} - \frac{8kl}{\pi^2} \left( \frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \dots \infty \right)$$

$$\text{Thus } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}.$$

**Example 10.20.** Expand  $f(x) = \frac{1}{4} - x$ , if  $0 < x < \frac{1}{2}$ ,

$$= x - \frac{3}{4}, \text{ if } \frac{1}{2} < x < 1,$$

as the Fourier series of sine terms.

(V.T.U., 2011; Andhra, 2000)

**Solution.** Let  $f(x)$  represent an odd function in  $(-1, 1)$  so that  $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

where

$$\begin{aligned} b_n &= \frac{2}{1} \int_0^1 f(x) \sin n\pi x dx \\ &= 2 \left[ \int_0^{\frac{1}{2}} \left( \frac{1}{4} - x \right) \sin n\pi x dx + \int_{\frac{1}{2}}^1 \left( x - \frac{3}{4} \right) \sin n\pi x dx \right] \\ &= 2 \left| -\left( \frac{1}{4} - x \right) \frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2} \right|_0^{\frac{1}{2}} + 2 \left| -\left( x - \frac{3}{4} \right) \frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2} \right|_{\frac{1}{2}}^1 \\ &= 2 \left[ \frac{1}{4n\pi} \cos \frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin n\pi/2}{n^2\pi^2} \right] + 2 \left[ -\frac{1}{4n\pi} \cos n\pi - \frac{1}{4n\pi} \cos \frac{n\pi}{2} - \frac{\sin n\pi/2}{n^2\pi^2} \right] \\ &= \frac{1}{2n\pi} [1 - (-1)^n] - \frac{4 \sin n\pi/2}{n^2\pi^2} \end{aligned}$$

Thus

$$b_1 = \frac{1}{\pi} - \frac{4}{\pi^2}; b_2 = 0$$

$$b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2}; b_4 = 0$$

$$b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2}; b_6 = 0 \text{ etc.}$$

Hence

$$f(x) = \left( \frac{1}{\pi} - \frac{4}{\pi^2} \right) \sin \pi x + \left( \frac{1}{3\pi} + \frac{4}{3^2\pi^2} \right) \sin 3\pi x + \left( \frac{1}{5\pi} - \frac{4}{5^2\pi^2} \right) \sin 5\pi x + \dots$$

## PROBLEMS 10.6

1. Show that a constant  $c$  can be expanded in an infinite series  $\frac{4c}{\pi} \left\{ \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right\}$  in the range  $0 < x < \pi$ .

(Marathwada, 2008; Kerala, 2005)

2. Obtain cosine and sine series for  $f(x) = x$  in the interval  $0 \leq x \leq \pi$ . Hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

(Osmania, 2003 S)

3. Find the half-range cosine series for the function  $f(x) = x^2$  in the range  $0 \leq x \leq \pi$ . (B.P.T.U., 2005; Kottayam, 2005)

4. Find the Fourier cosine series of the function  $f(x) = \pi - x$  in  $0 < x < \pi$ . Hence show that

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$$

(West Bengal, 2004)

5. Find the half-range cosine series for the function  $f(x) = (x-1)^2$  in the interval  $0 < x < 1$ .

(V.T.U., 2010; J.N.T.U., 2006)

Hence show that  $\pi^2 = 8 \left( \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$

(Anna, 2003)

6. Find the half-range sine series for the function  $f(t) = t - t^2$ ,  $0 < t < 1$ .

7. Represent  $f(x) = \sin(\sin(\pi x/l))$ ,  $0 < x < l$  by a half-range cosine series.

(Mumbai, 2009)

8. Find the half range sine series for  $f(x) = x \cos x$  in  $(0, \pi)$ .

(Anna, 2008 S)

9. Obtain the half-range sine series for  $e^x$  in  $0 < x < 1$ .

10. Find the half range Fourier sine series of  $f(x) = x(\pi-x)$ ,  $0 \leq x \leq \pi$  and hence deduce that

$$(i) \sum \frac{1}{n^4} = \frac{\pi^4}{90} \quad (\text{Anna, 2009})$$

$$(ii) \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$$

(Mumbai, 2005)

11. If  $f(x) = x$ ,  $0 < x < \pi/2$

$$= \pi - x, \quad \pi/2 < x < \pi,$$

show that (i)  $f(x) = \frac{4}{\pi} \left[ \sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \dots \right]$

(Mumbai, 2008; S.V.T.U., 2008; V.T.U., 2004)

(ii)  $f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left[ \frac{1}{12} \cos 2x + \frac{1}{3^2} \cos 6x + \frac{1}{5^2} \cos 10x + \dots \right]$

(V.T.U., 2011)

12. Find the half-range cosine series expansion of the function  $f(x) = \begin{cases} 0, & 0 \leq x \leq l/2 \\ l-x, & l/2 \leq x \leq l \end{cases}$

(P.T.U., 2010)

13. If  $f(x) = \sin x$  for  $0 \leq x \leq \pi/4$

$$= \cos x \text{ for } \pi/4 \leq x \leq \pi/2, \text{ expand } f(x) \text{ in a series of sines.}$$

14. For the function defined by the graph OAB in Fig. 10.10, find the half-range Fourier sine series.

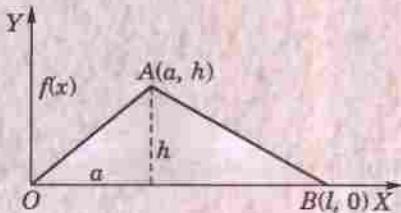


Fig. 10.10

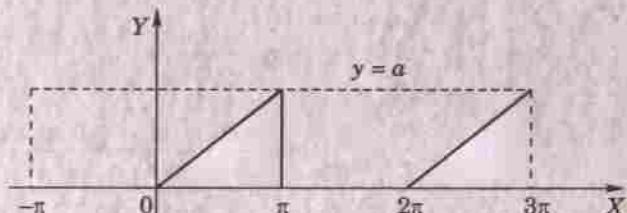


Fig. 10.11

## 10.8 TYPICAL WAVEFORMS

We give below six typical waveforms usually met with in communication engineering :

- (1) *Square waveform* (Fig. 10.7) is an extension of the function of Problem 8, page 412.
- (2) *Saw-toothed waveform* (Fig. 10.5) is an extension of the function in Ex. 10.12, page 409.
- (3) *Modified saw-toothed waveform* (Fig. 10.11) is extension of the function

$$\begin{aligned} f(x) &= 0, & -\pi < x \leq 0 \\ &= x, & 0 \leq x < \pi, \end{aligned}$$

Its Fourier expansion is

$$f(x) = \frac{a}{4} - \frac{2a}{\pi^2} \left( \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) + \frac{a}{\pi} \left( \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right)$$

- (4) *Triangular waveform* (Fig. 10.6) is an extension of the function of Ex. 10.15, page 411.
- (5) *Half-wave rectifier* (Fig. 10.2) is an extension of the function of Problem 2, page 412.
- (6) *Full-wave rectifier* (Fig. 10.12) is an extension of the function  $f(x) = a \sin x$ ,  $0 \leq x \leq \pi$ . Its Fourier expansion is

$$f(x) = \frac{4a}{\pi} \left\{ \frac{1}{2} - \frac{1}{1 \cdot 3} \cos 2x - \frac{1}{3 \cdot 5} \cos 4x - \frac{1}{5 \cdot 7} \cos 6x - \dots \right\}$$

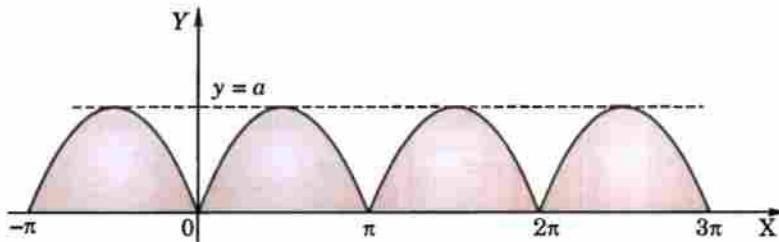


Fig. 10.12

## 10.9 (1) PARSEVAL'S FORMULA\*

$$\text{To prove that } \int_{-l}^l [f(x)]^2 dx = l \left\{ \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\},$$

provided the Fourier series for  $f(x)$  converges uniformly in  $(-l, l)$ .

$$\text{The Fourier series for } f(x) \text{ in } (-l, l) \text{ is } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \quad \dots(1)$$

Multiplying both sides of (1) by  $f(x)$  and integrating term by term from  $-l$  to  $l$  [which is justified as the series (1) is uniformly convergent – p. 389], we get

$$\int_{-l}^l [f(x)]^2 dx = \frac{a_0}{2} \int_{-l}^l f(x) dx + \sum_{n=1}^{\infty} \left\{ a_n \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx + b_n \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \right\} \quad \dots(2)$$

$$\text{Now } \int_{-l}^l f(x) dx = la_0,$$

$$\int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx = la_n \text{ and } \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx = lb_n, \text{ by (4) of p. 405}$$

$$\therefore (2) \text{ takes the form } \int_{-l}^l [f(x)]^2 dx = l \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\} \quad \dots(3)$$

which is the desired Parseval's formula.

(Mumbai, 2005 S)

\*Named after the French mathematician Marc Antoine Parseval (1755–1836).

**Cor. 1.** If  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$  in  $(0, 2l)$ , then

$$\int_0^{2l} |f(x)|^2 dx = l \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\} \quad \dots(4)$$

**Cor. 2.** If the half-range cosine series is  $(0, l)$  for  $f(x)$  is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{l} \right), \text{ then}$$

$$\int_0^l |f(x)|^2 dx = \frac{l}{2} \left( \frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \dots \infty \right) \quad \dots(5)$$

**Cor. 3.** If the half-range sine series in  $(0, l)$  for  $f(x)$  is  $f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{l} \right)$ , then

$$\int_0^l |f(x)|^2 dx = \frac{l}{2} (b_1^2 + b_2^2 + b_3^2 + \dots \infty) \quad \dots(6)$$

**(2) Root mean square (rms) value.** The root mean square value of the function  $f(x)$  over an interval  $(a, b)$  is defined as

$$[f(x)]_{\text{rms}} = \sqrt{\left\{ \frac{\int_a^b |f(x)|^2 dx}{b-a} \right\}} \quad \dots(7)$$

The use of root mean square value of a periodic function is frequently made in the theory of mechanical vibrations and in electric circuit theory. The r.m.s. value is also known as the effective value of the function.

**Example 10.21.** Obtain the Fourier series for  $y = x^2$  in  $-\pi < x < \pi$ . Using the two values of  $y$ , show that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{\pi^4}{90}$$

**Solution.** Let  $y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

We have  $a_0 = 2 \cdot \frac{n^2}{3}, a_n = \frac{4}{n^2} (-1)^n, b_n = 0$  for all  $n$  (See problem 2, p. 400)

If  $\bar{y}$  be the r.m.s. value of  $y$  in  $(-\pi, \pi)$ , then

$$\begin{aligned} (\bar{y})^2 &= \frac{\pi}{2\pi} \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right] && [\text{By (3) and (7) §10.9}] \\ &= \frac{1}{4} \left( \frac{2\pi^2}{3} \right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \frac{16}{n^4} (-1)^{2n} + 0 \right] = \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4} \end{aligned}$$

Also by definition,

$$(\bar{y})^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{\pi^4}{5}$$

Equating the two values of  $(\bar{y})^2$ , we get

$$\frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{5} \text{ i.e., } \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

## PROBLEMS 10.7

1. By using the sine series for  $f(x) = 1$  in  $0 < x < \pi$ , show that  $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

2. Prove that in  $0 < x < l$ ,  $x = \frac{l}{2} - \frac{4l}{\pi^2} \left( \cos \frac{\pi x}{l} + \frac{1}{3^2} \cos \frac{3\pi x}{l} + \frac{1}{5^2} \cos \frac{5\pi x}{l} + \dots \right)$

and deduce that  $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$ .

3. If  $\frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l}$  is the half-range cosine series of  $f(x)$  of period  $2l$  in  $(0, l)$ , then show that the mean square

value of  $f(x)$  in  $(0, l)$  is  $\frac{l}{2} \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right]$ .

Use this result to evaluate  $1^{-4} + 3^{-4} + 5^{-4} + \dots$  from the half-range cosine series of the function  $f(x)$  of period 4 defined in  $(0, 2)$  by

$$f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$$

## 10.10 COMPLEX FORM OF FOURIER SERIES

The Fourier series of a periodic function  $f(x)$  of period  $2l$ , is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \quad \dots(1)$$

Since  $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$  and  $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ ,

therefore, we can express (1) as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \left( \frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2} \right) + b_n \left( \frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i} \right) \right\} \\ &= c_0 + \sum_{n=1}^{\infty} \left\{ c_n e^{in\pi x/l} + c_{-n} e^{-in\pi x/l} \right\} \quad \dots(2) \end{aligned}$$

where

$$c_0 = \frac{1}{2} a_0, c_n = \frac{1}{2}(a_n - ib_n), c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$\begin{aligned} \text{Now } c_n &= \frac{1}{2l} \left\{ \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx - i \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \right\} \\ &= \frac{1}{2l} \int_{-l}^l f(x) \left( \cos \frac{n\pi x}{l} - i \sin \frac{n\pi x}{l} \right) dx = \frac{1}{2l} \int_{-l}^l f(x) e^{-in\pi x/l} dx \end{aligned}$$

and

$$c_{-n} = \frac{1}{2l} \int_{-l}^l f(x) \left( \cos \frac{n\pi x}{l} - i \sin \frac{n\pi x}{l} \right) dx = \frac{1}{2l} \int_{-l}^l f(x) e^{in\pi x/l} dx$$

Combining these, we have  $c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-in\pi x/l} dx$

where  $n = 0, \pm 1, \pm 2, \pm 3, \dots$  ...(3)

Then the series (2) can be compactly written as :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

which is the *complex form of Fourier series* and its coefficients are given by (3).

**Obs.** The complex form of a Fourier series is especially useful in problems on electrical circuits having impressed periodic voltage.

**Example 10.22.** Find the complex form of the Fourier series of  $f(x) = e^{-x}$  in  $-1 \leq x \leq 1$ .

(Mumbai, 2005 S ; Madras, 2000 S)

**Solution.** We have  $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$  ( $\because l = 1$ )

where

$$\begin{aligned} c_n &= \frac{1}{2} \int_{-1}^1 e^{-x} \cdot e^{-inx} dx = \frac{1}{2} \int_{-1}^1 e^{-(1+inx)x} dx = \frac{1}{2} \left| \frac{e^{-(1+inx)x}}{-(1+inx)} \right|_{-1}^1 = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)} \\ &= \frac{e(\cos n\pi + i \sin n\pi) - e^{-1}(\cos n\pi - i \sin n\pi)}{2(1+in\pi)} = \frac{e - e^{-1}}{2} (-1)^n \cdot \frac{1 - in\pi}{1 + n^2\pi^2} \\ &= \frac{(-1)^n (1 - in\pi) \sinh 1}{1 + n^2\pi^2} \end{aligned}$$

Hence

$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} \sinh 1 \cdot e^{inx}.$$

### PROBLEMS 10.8

Find the complex form of the Fourier series of the following periodic functions :

1.  $f(x) = e^{ax}, -l < x < l$ . (Madras, 2003)

2.  $f(t) = \sin t, 0 < t < \pi$

3.  $f(x) = \cos ax, -\pi < x < \pi$

(Anna, 2009 ; Mumbai, 2009)

4.  $f(x) = \cosh 3x + \sinh 3x$  in  $(-3, 3)$ . (Mumbai, 2008) 5.  $f(x) = \begin{cases} 0 & \text{when } 0 < x < l \\ a & \text{when } l < x < 2l \end{cases}$

## 10.11 PRACTICAL HARMONIC ANALYSIS

We have discussed at length, the problem of expanding  $y = f(x)$  as Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \dots(1)$$

where

$$\left. \begin{array}{l} a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx \\ a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \\ b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \end{array} \right\} \quad \dots(2)$$

So far, the function has always been defined by an explicit function of an independent variable. In practice, however, the function is often given not by a formula but by a graph or by a table of corresponding values. In such cases, the integrals in (2) cannot be evaluated and instead, the following alternative forms of (2) are employed.

Since the mean value of a function  $y = f(x)$  over the range  $(a, b)$  is  $\frac{1}{b-a} \int_a^b f(x) dx$ .

$\therefore$  the equations (2) give,

$$\left. \begin{array}{l} a_0 = 2 \times \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = 2[\text{mean value of } f(x) \text{ in } (0, 2\pi)] \\ a_n = 2 \times \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos nx dx = 2[\text{mean value of } f(x) \cos nx \text{ in } (0, 2\pi)] \\ b_n = 2 \times \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin nx dx = 2[\text{mean value of } f(x) \sin nx \text{ in } (0, 2\pi)] \end{array} \right\} \quad \dots(3)$$

There are numerous other methods of finding the value of  $a_0$ ,  $a_n$ ,  $b_n$  which constitute the field of harmonic analysis.

In (1), the term  $(a_1 \cos x + b_1 \sin x)$  is called the **fundamental or first harmonic**, the term  $(a_2 \cos 2x + b_2 \sin 2x)$  the **second harmonic** and so on.

**Example 10.23.** The displacement  $y$  of a part of a mechanism is tabulated with corresponding angular movement  $x^\circ$  of the crank. Express  $y$  as a Fourier series neglecting the harmonic above the third :

$x^\circ$	0	30	60	90	120	150	180	210	240	270	300	330
$y$	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

**Solution.** Let the Fourier series upto the third harmonic representing  $y$  in  $(0, 2\pi)$  be

$$y = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \quad \dots(i)$$

To evaluate the coefficients, we form the following table.

$x^\circ$	$\sin x$	$\cos x$	$\sin 2x$	$\cos 2x$	$\sin 3x$	$\cos 3x$	$y$	$y \sin x$	$y \cos x$	$y \sin 2x$	$y \cos 2x$	$y \sin 3x$	$y \cos 3x$
0	0	1	0	1	0	1	1.80	0.00	1.80	0.00	1.80	0.00	1.80
30	0.50	0.87	0.87	0.50	1	0	1.10	0.55	0.96	0.96	0.55	1.10	0.00
60	0.87	0.50	0.87	-0.50	0	-1	0.30	0.26	0.15	0.26	-0.15	0.00	-0.30
90	1.00	0	0	-1.00	-1	0	0.16	0.16	0.00	0.00	-0.16	-0.16	0.00
120	0.87	-0.50	-0.87	-0.50	0	1	0.50	0.43	-0.25	-0.43	-0.25	0.00	0.50
150	0.50	-0.87	-0.87	-0.50	1	0	1.30	0.65	-1.13	-1.13	0.65	1.30	0.00
180	0	-1.00	0	1.00	0	-1	2.16	0.00	-2.16	-0.00	2.16	0.00	-2.16
210	-0.50	-0.87	0.87	0.50	-1	0	1.25	-0.63	-1.09	1.09	0.63	-1.25	0.00
240	-0.87	-0.50	0.87	-0.50	0	1	1.30	-1.13	-0.65	1.13	-0.65	0.00	1.30
270	-1.00	0	0	-1.00	1	0	1.52	-1.52	0.00	0.00	-1.52	1.52	0.00
300	-0.87	0.50	-0.87	-0.50	0	-1	1.76	-1.53	0.88	-1.53	-0.88	0.00	-1.76
330	-0.50	0.87	-0.87	0.50	-1	0	2.00	-1.00	1.74	-1.74	1.00	-2.00	0.00
					$\Sigma =$		15.15	-3.76	0.25	-1.39	3.18	0.51	-0.62

$$\therefore a_0 = 2 \cdot \frac{\Sigma y}{12} = \frac{15.15}{6} = 2.53 ; a_1 = \frac{1}{6} \Sigma y \cos x = \frac{0.25}{6} = 0.04$$

$$a_2 = \frac{1}{6} \Sigma y \cos 2x = \frac{3.18}{6} = 0.53 ; a_3 = \frac{1}{6} \Sigma y \cos 3x = \frac{-0.62}{6} = -0.1$$

$$b_1 = \frac{1}{6} \Sigma y \sin x = \frac{-3.76}{6} = -0.63 ;$$

$$b_2 = \frac{1}{6} \Sigma y \sin 2x = \frac{-1.39}{6} = -0.23$$

$$b_3 = \frac{1}{6} \Sigma y \sin 3x = \frac{0.51}{6} = 0.085$$

Substituting the values of  $a$ 's and  $b$ 's in (i), we get

$$y = 1.26 + 0.04 \cos x + 0.53 \cos 2x - 0.1 \cos 3x - 0.63 \sin x - 0.23 \sin 2x + 0.085 \sin 3x.$$

**Example 10.24.** The following table gives the variations of periodic current over a period.

$t$ sec	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	$T$
A amp.	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a direct current part of 0.75 amp in the variable current and obtain the amplitude of the first harmonic. (V.T.U., 2010; S.V.T.U., 2009)

**Solution.** Here length of the interval is  $T$ , i.e.  $C = T/2$  ( $\S$  10.5)

$$\text{Then } A = \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} + a_2 \cos \frac{4\pi t}{T} + b_2 \sin \frac{4\pi t}{T} + \dots$$

The desired values are tabulated as follows :

$t$	$2\pi t/T$	$\cos 2\pi t/T$	$\sin 2\pi t/T$	$A$	$A \cos 2\pi t/T$	$A \sin 2\pi t/T$
0	0	1.0	0.000	1.98	1.980	0.000
$T/6$	$\pi/3$	0.5	0.866	1.30	0.650	1.126
$T/3$	$2\pi/3$	-0.5	0.866	1.05	-0.525	0.909
$T/2$	$\pi$	-1.0	0.000	1.30	-1.300	0.000
$2T/3$	$4\pi/3$	-0.5	-0.866	-0.88	0.440	0.762
$5T/6$	$5\pi/3$	0.5	-0.866	-0.25	-0.125	0.217
			$\Sigma =$	4.5	1.12	3.014

$$\therefore a_0 = 2 \cdot \frac{1}{6} \Sigma A = \frac{1}{3}(4.5) = 1.5$$

$$a_1 = 2 \cdot \frac{1}{6} \Sigma A \cos \frac{2\pi t}{T} = \frac{1}{3}(1.12) = 0.373$$

$$b_1 = 2 \cdot \frac{1}{6} \Sigma A \sin \frac{2\pi t}{T} = \frac{1}{3}(3.014) = 1.005$$

Thus the direct current part in the variable current  $= a_0/2 = 0.75$  and amplitude of the first harmonic

$$= \sqrt{(a_1^2 + b_1^2)} = \sqrt{(0.373)^2 + (1.005)^2} = 1.072$$

**Example 10.25.** Obtain the first three coefficients in the Fourier cosine series for  $y$ , where  $y$  is given in the following table :

$x :$	0	1	2	3	4	5	
$y :$	4	8	15	7	6	2	(V.T.U., 2009 ; V.T.U., 2006 ; J.N.T.U., 2004 S)

**Solution.** Taking the interval as  $60^\circ$ , we have

$\theta =$	$0^\circ$	$60^\circ$	$120^\circ$	$180^\circ$	$240^\circ$	$300^\circ$
$x =$	0	1	2	3	4	5
$y =$	4	8	15	7	6	2

$\therefore$  Fourier cosine series in the intervals  $(0, 2\pi)$  is

$$y = \frac{a_0}{2} + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \dots$$

$\theta^\circ$	$\cos \theta$	$\cos 2\theta$	$\cos 3\theta$	$y$	$y \cos \theta$	$y \cos 2\theta$	$y \cos 3\theta$
$0^\circ$	1	1	1	4	4	4	4
$60^\circ$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	8	4	-4	-8
$120^\circ$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	15	-7.5	-7.5	15
$180^\circ$	-1	1	-1	7	-7	7	-7
$240^\circ$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	6	-3	-3	6
$300^\circ$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	2	1	-1	-2
			$\Sigma =$	42	-8.5	-4.5	8

$$\text{Hence } a_0 = 2 \cdot \frac{42}{6} = 14, a_1 = 2 \left( \frac{-8.5}{6} \right) = -2.8, a_2 = 2 \left( \frac{-4.5}{6} \right) = -1.5,$$

$$a_3 = 2 \left( \frac{8}{6} \right) = 2.7.$$

**Example 10.26.** The turning moment  $T$  is given for a series of values of the crank angle  $\theta^\circ = 75^\circ$

$\theta^\circ :$	0	30	60	90	120	150	180
$T :$	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent  $T$  and calculate  $T$  for  $\theta = 75^\circ$ .

**Solution.** Let the Fourier sine series to represent  $T$  in  $(0, 180)$  be

$$T = b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + \dots$$

To evaluate the coefficients, we form the following table :

$\theta^\circ$	$T$	$\sin \theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$
0	0	0	0	0	0
30	5224	0.500	0.866	1	0.866
60	8097	0.866	0.866	0	-0.866
90	7850	1.000	0	-1	0
120	5499	0.866	-0.866	0	0.866
150	2626	0.500	-0.866	1	-0.866

$$\therefore b_1 = \frac{2}{6} \sum y \sin \theta = \frac{1}{3} [(5224 + 2626) 0.5 + (8097 + 5499) 0.866 + 7850] = 7850$$

$$b_2 = \frac{2}{6} \sum y \sin 2\theta = \frac{1}{3} [(5224 + 8097) 0.866 + (5499 + 2626)(-0.866)] = 1500$$

$$b_3 = \frac{2}{6} \sum y \sin 3\theta = \frac{1}{3} [5224 - 7850 + 2626] = 0.$$

$$b_4 = \frac{2}{6} \sum y \sin 4\theta = \frac{1}{3} [(5224 + 5499)(0.866) + (8097 + 2626)(-0.866)] = 0$$

Hence  $T = 7850 \sin \theta + 1500 \sin 2\theta$

For  $\theta = 75^\circ$ ,  $T = 7850 \sin 75^\circ + 1500 \sin 150^\circ$

$$= 7850(0.9659) + 1500(0.5) = 8332.$$

### PROBLEMS 10.9

1. The following values of  $y$  give the displacement in inches of a certain machine part for the rotation  $x$  of the flywheel. Expand  $y$  in terms of a Fourier series :

$x :$	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$
$y :$	0	9.2	14.4	17.8	17.3	11.7

2. Compute the first two harmonics of the Fourier series of  $f(x)$  given in the following table :

$x :$	0	$\pi/3$	$2\pi/3$	$\pi$	$4\pi/3$	$5\pi/3$	$2\pi$
$f(x) :$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

(Anna, 2009)

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of  $y$  as given in the following table :

$x :$	0	1	2	3	4	5
$y :$	9	18	24	28	26	20

(V.T.U., 2011; Anna, 2005 S)

4. In a machine the displacement  $y$  of a given point is given for a certain angle  $\theta$  as follows :

$\theta^\circ :$	0	30	60	90	120	150	180	210	240	270	300	330
$y :$	7.9	8.0	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the coefficient of  $\sin 2\theta$  in the Fourier series representing the above variation.

5. Determine the first two harmonics of the Fourier series for the following values :

$x^\circ :$	30	60	90	120	150	180	210	240	270	300	330	360
$y :$	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

(Madras, 2006; Cochin, 2005)

6. The turning moment  $T$  on the crankshaft of a steam engine for the crank angle  $\theta$  degrees is given as follows :

$\theta :$	0	15	30	45	60	75	90	105	120	135	150	165	180
$T :$	0	2.7	5.2	7.0	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

Expand  $T$  in a series of sines upto the fourth harmonics.

## 10.12 OBJECTIVE TYPE OF QUESTIONS

## PROBLEMS 10.10

Fill up the blanks or choose the correct answer in each of the following problems :

1. The period of  $\cos 3x$  is  $x = \dots$
2. If  $x = c$  is a point of discontinuity then the Fourier series of  $f(x)$  at  $x = c$  gives  $f(x) = \dots$
3. A function  $f(x)$  defined for  $0 < x < 1$  can be extended to an odd periodic function in  $\dots$
4. The mathematical function representing the following graph is  $\dots$
5. Fourier expansion of an odd function has only  $\dots$  terms.
6. Formulae for evaluation of Fourier coefficients for a given set of points  $(x_i, y_i) : i = 0, 1, 2, \dots, n$  are  $\dots$
7. If  $f(x) = x^4$  in  $(-1, 1)$ , then the Fourier coefficient  $b_n = \dots$
8. The period of a constant function is  $\dots$
9. If  $f(t) = \begin{cases} -1, & -1 < t < 0 \\ 1, & 0 < t < 1 \end{cases}$ , then  $f(t)$  is an  $\dots$
10. Fourier expansion of an even function  $f(x)$  in  $(-\pi, \pi)$  has only  $\dots$  terms.
11. If  $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ , then  $f(x)$  is an  $\dots$  function in  $(-\pi, \pi)$ .
12. The smallest period of the function  $\sin\left(\frac{2n\pi x}{k}\right)$  is  $\dots$
13. In the Fourier series expansion of  $f(x) = |\sin x|$  in  $(-\pi, \pi)$ , the value of  $b_n = \dots$
14. In the Fourier series for  $f(x) = x$  in  $(-\pi \leq x \leq \pi)$ , the  $\dots$  terms are absent.
15. If  $f(x)$  is an even function in  $(-l, l)$ , then the value of  $b_n = \dots$
16. If  $f(x) = x^2$  in  $-2 < x < 2$ ,  $f(x+4) = f(x)$ , then  $a_n$  is  $\dots$
17. If  $f(x)$  is a periodic function with period  $2T$ , then the value of the Fourier coefficient  $b_n = \dots$
18. Dirichlet conditions for the expansion of a function as a Fourier series in the interval  $c_1 \leq x \leq c_2$  are  $\dots$
19. If  $f(x) = x \sin x$  in  $(-\pi, \pi)$ , then the value of  $b_n = \dots$
20. The formulae for finding the half range cosine series for the function  $f(x)$  in  $(0, l)$  are  $\dots$
21. The half-range sine series for 1 in  $(0, \pi)$ , is  $\dots$
22. Period of  $|\sin t|$  is  $\dots$
23. The value of  $b_n$  in the Fourier series of  $f(x) = |x|$  in  $(-\pi, \pi) = \dots$
24. If  $f(x)$  is defined in  $(0, l)$  then the period of  $f(x)$  to expand it as a half range sine series is  $\dots$
25. The complex form of Fourier series for  $e^{-x}$  in  $(-1, 1)$  is  $\dots$
26.  $f(x)$  is an odd function in  $(-\pi, \pi)$ , then the graph of  $f(x)$  is symmetric about the  $x$ -axis. (True or False)
27.  $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi, \end{cases}$  then  $f(0) = \dots$
28. If  $f(x) = \begin{cases} \pi x & \text{in } 0 \leq x \leq 1 \\ \pi(2-x) & \text{in } 1 \leq x \leq 2, \end{cases}$  then it is  $\dots$  function. (odd or even)
29. If  $f(x)$  is an odd function in  $(-l, l)$ , then the values of  $a_0$  and  $a_n$  are  $\dots$
30. The root mean square value of  $f(t) = 3 \sin 2t + 4 \cos 2t$  over the range  $0 \leq t \leq \pi$  is  $\dots$  (Nagpur, 2009)
31. In the Fourier series expansion of the function  

$$f(x) = \begin{cases} -(x+\pi), & -\pi < x < 0 \\ -(x-\pi), & 0 < x < \pi, \end{cases}$$
 the value of  $b_n$  is  $\dots$  (P.T.U., 2010)
32. Let  $f(x)$  be defined in  $(0, 2\pi)$  by  

$$f(t) = \begin{cases} \frac{1 + \cos x}{\pi - x}, & 0 < x < \pi \\ \cos x, & \pi < x < 2\pi, \end{cases}$$
  $f(x) + 2\pi = f(x)$ . The value of  $f(\pi)$  is  $\dots$  (Anna, 2009)

33. The mean value of  $f(x) \cos nx$  in  $(0, 2\pi)$  = .....
34. Using sine series for  $f(x) = 1$  in  $0 < x < \pi$ , show that  $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \infty = \dots$
35. Fourier series representing  $f(x) = |x|$  in  $-\pi < x < \pi$ , is .....
36. Fourier series of  $f(x) = \cos^4 x$  in  $(0, 2\pi)$  is .....
37. If  $f(x) = x^2 + x$  in  $(0, l)$ , then the even extension of  $f(x)$  in  $(-l, 0)$  is .....
38. If  $f(x) = x(l-x)$  in  $(0, l)$ , then the extension of  $f(x)$  in  $(l, 2l)$  so as to get sine series is .....
39. A function  $f(x)$  defined in  $(-\pi, \pi)$  can be expanded into Fourier series containing both sine and cosine terms. (True or False)
40. The function  $f(x) = \begin{cases} 1-x & \text{in } -\pi < x < 0 \\ 1+x & \text{in } 0 < x < \pi, \end{cases}$  is an odd function. (True or False)
41. If  $f(x) = x^2$  in  $(-\pi, \pi)$ , then the Fourier series of  $f(x)$  contains only sine terms. (True or False)