8. Solve $\frac{dy}{dx} + y = x^2$; $y(0) = 1$ by obtaining Taylor's polynomial of order 4. Evaluate y at $x = 0.1$, 0.2, 0.3 and by using these values obtain y at $x = 0.4$ applying Adams - Bashforth predictor and corrector formulae.

ANSWERS

1. 1.22	2. 1.0234	3. 1.7003
4. 6.873	5. 0.3046, 0.4555	6. 3.0793
7. 0.949	8. 0.6897	

ENGINEERING MATHEMATICS - IV

Module - 2

Numerical Methods - 2

2.1 Numerical solution of second order ordinary differential equations

2.11 Introduction and pre-amble

The given second order ODE with two initial conditions will reduce to two first order simultaneous ODEs which can be solved.

We present the method explicitly.

Let $y'' = g(x, y, y')$ with the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ be the given second order DE.

Now, let $y' = \frac{dy}{dx} = z$. This gives $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}$

The given second order DE assumes the form : $\frac{dz}{dx} = g(x, y, z)$ with the conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ where y_0' is be denoted by z_0.

Hence we now have two first order simultaneous ODEs,

$$\frac{dy}{dx} = z \text{ and } \frac{dz}{dx} = g(x, y, z) \text{ with } y(x_0) = y_0 \text{ and } z(x_0) = z_0$$

Taking $f(x, y, z) = z$, we now have the following system of equations for solving.

$$\frac{dy}{dx} = f(x, y, z), \quad \frac{dz}{dx} = g(x, y, z) \ ; \quad y(x_0) = y_0 \text{ and } z(x_0) = z_0$$

2.12 Runge - Kutta Method

We have to compute $y(x_0+h)$ and if required $y'(x_0+h) = z(x_0+h)$

We need to first compute the following :

$$k_1 = hf(x_0, y_0, z_0) \quad ; \quad l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right) \quad ; \quad l_2 = hg\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right) \quad ; \quad l_3 = hg\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$$

$$k_4 = hf(x_0+h, y_0+k_3, z_0+l_3) \quad ; \quad l_4 = hg(x_0+h, y_0+k_3, z_0+l_3)$$

The required $y(x_0+h) = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

and $$y'(x_0+h) = z(x_0+h) = z_0+\frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

WORKED PROBLEMS

1. *Given* $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, $y(0) = 1$, $y'(0) = 0$. *Evaluate* $y(0.1)$ *using Runge-Kutta method of order 4.*

>> By data, $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$; $y = 1$, $y' = 0$ at $x = 0$.

Putting $\frac{dy}{dx} = z$ and differentiating w.r.t. x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form : $\frac{dz}{dx} - x^2z - 2xy = 1$

Hence, we have a system of equations,

$$\frac{dy}{dx} = z \; ; \; \frac{dz}{dx} = 1 + 2xy + x^2z \text{ where } y = 1, z = 0, x = 0$$

Let $f(x, y, z) = z$, $g(x, y, z) = 1 + 2xy + x^2z$

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$ and let us take $h = 0.1$

We shall first compute the following:

$$k_1 = hf(x_0, y_0, z_0) = (0.1)f(0, 1, 0) = (0.1)(0) = 0$$

$$l_1 = (0.1)[1 + (2)(0)(1) + (0^2)(0)] = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)f(0.05, 1, 0.05) = (0.1)(0.05) = 0.005$$

$$l_2 = (0.1)[1 + (2)(0.05)(1) + (0.05)^2(0.05)] = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 1.0025, 0.055) = (0.1)(0.055) = 0.0055$$

$$l_3 = (0.1)[1 + (2)(0.05)(1.0025) + (0.05)^2(0.055)] = 0.11004$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.1)f(0.1, 1.0055, 0.11004) = (0.1)(0.11004) = 0.011$$

$$l_4 = (0.1)[1 + (2)(0.1)(1.0055) + (0.1)^2(0.11004)] = 0.12022$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.1) = 1 + \frac{1}{6}[0 + 2(0.005) + 2(0.0055) + 0.011]$$

Thus, $y(0.1) = \mathbf{1.0053}$

2. *By Runge-Kutta method, solve* $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$ *for* $x = 0.2$ *correct to four decimal places, using the initial conditions* $y = 1$ *and* $y' = 0$ *when* $x = 0$.

>> By data, $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$

Putting $\frac{dy}{dx} = z$ and differentiating w.r.t. x, we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

The given equation becomes,

$$\frac{dz}{dx} = xz^2 - y^2 \text{ with } y = 1, z = 0 \text{ when } x = 0.$$

Hence, we have a system of equations $\frac{dy}{dx} = z, \ \frac{dz}{dx} = xz^2 - y^2$

Let $f(x, y, z) = z, \ g(x, y, z) = xz^2 - y^2, \ x_0 = 0, \ y_0 = 1, \ z_0 = 0$ and $h = 0.2$

We shall first compute the following.

$$k_1 = hf(x_0, y_0, z_0) = (0.2) f(0, 1, 0) = (0.2)(0) = 0$$

$$l_1 = (0.2)[(0)(0)^2 - (1)^2] = -0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2) f(0.1, 1, -0.1) = (0.2)(-0.1) = -0.02$$

$$l_2 = (0.2)[(0.1)(-0.1)^2 - (1)^2] = -0.1998$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2) f(0.1, 0.99, -0.0999) = (0.2)(-0.0999) = -0.01998$$

$$l_3 = (0.2)[(0.1)(-0.0999)^2 - (0.99)^2] = -0.1958$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.2) f(0.2, 0.98002, -0.1958) = (0.2)(-0.1958) = -0.03916$$

$$l_4 = (0.2)[(0.2)(-0.1958)^2 - (0.98002)^2] = -0.19055$$

We have $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.2) = 1 + \frac{1}{6}\left[0 + 2(-0.02) + 2(-0.01998) - 0.03916\right]$

Thus $\quad \mathbf{y(0.2) = 0.9801}$

3. *Compute $y(0.1)$ given $\frac{d^2y}{dx^2} = y^3$ and $y = 10$, $\frac{dy}{dx} = 5$ at $x = 0$ by Runge-Kutta method of fourth order.*

>> Putting $\frac{dy}{dx} = z$ and differentiating w.r.t x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form $\frac{dz}{dx} = y^3$. Hence we have a system of equations :

$$\frac{dy}{dx} = z \ ; \ \frac{dz}{dx} = y^3 \ \text{ where } \ y = 10, \ z = 5, \ x = 0.$$

Let $\quad f(x, y, z) = z, \ g(x, y, z) = y^3, \ x_0 = 0, \ y_0 = 10, \ z_0 = 5$ and $h = 0.1$

We shall first compute the following.

$$k_1 = hf(x_0, y_0, z_0) = (0.1) f(0, 10, 5) = (0.1)5 = 0.5$$

$$l_1 = (0.1)[10^3] = 100$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_2 = (0.1) f(0.05, 10.25, 55) = (0.1)(55) = 5.5$$

$$l_2 = (0.1)[(10.25)^3] = 107.7$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1) f(0.05, 12.75, 58.85) = (0.1)(58.85) = 5.885$$

$$l_3 = (0.1)(12.75)^3 = 207.27$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.1) f(0.1, 15.885, 212.27) = (0.1)(212.27) = 21.227$$

$$l_4 = (0.1)(15.885)^3 = 400.83$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.1) = 10 + \frac{1}{6}\left[0.5 + 2(5.5) + 2(5.885) + 21.227\right]$

Thus $\quad \mathbf{y(0.1) = 17.4162}$

4. *Given $y'' - xy' - y = 0$ with the initial conditions $y(0) = 1$, $y'(0) = 0$, compute $y(0.2)$ and $y'(0.2)$ using fourth order Runge-Kutta method.*

>> Putting $y' = z$ we obtain $y'' = \frac{dz}{dx}$. The given equation becomes

$$\frac{dz}{dx} = xz + y \ ; \ y(0) = 1, \ z(0) = 0$$

Hence we have a system of equations,

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = xz + y \quad \text{where } y = 1,\ z = 0,\ x = 0$$

Let $f(x, y, z) = z$, $g(x, y, z) = xz + y$, $x_0 = 0$, $y_0 = 1$, $z_0 = 0$ and $h = 0.2$

We shall first compute the following.

$$k_1 = hf(x_0, y_0, z_0) = (0.2)\,f(0, 1, 0) = (0.2)0 = 0$$

$$l_1 = (0.2)[0 \times 0 + 1] = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2)f(0.1, 1, 0.1) = (0.2)(0.1) = 0.02$$

$$l_2 = (0.2)[0.1 \times 0.1 + 1] = 0.202$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2)f(0.1, 1.01, 0.101) = (0.2)(0.101) = 0.0202$$

$$l_3 = (0.2)[0.1 \times 0.101 + 1.01] = 0.204$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$k_4 = (0.2)f(0.2, 1.0202, 0.204) = (0.2)(0.204) = 0.0408$$

$$l_4 = (0.2)[0.2 \times 0.204 + 1.0202] = 0.2122$$

We have $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Substituting the appropriate values we obtain $y(0.2) = 1.0202$ and $z(0.2) = 0.204$

Thus $y(0.2) = 1.0202$ and $y'(0.2) = 0.204$

5. *Obtain the value of x and $\frac{dx}{dt}$ when $t = 0.1$ given that x satisfies the*

$\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$ *and* $x = 3$, $\frac{dx}{dt} = 0$ *when $t = 0$ initially. Use fourth*

Runge-Kutta method.

>> Putting $y = \frac{dx}{dt}$ we obtain $\frac{dy}{dt} = \frac{d^2x}{dt^2}$ The given equation becomes

$$\frac{dy}{dt} = ty - 4x, \quad x = 3,\ y = 0 \ \text{ when } t = 0$$

Hence we have a system of equations,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = ty - 4x;\ x = 3,\ y = 0 \ \text{ when } t = 0.$$

Let $f(t, x, y) = y$, $g(t, x, y) = ty - 4x$, $t_0 = 0$, $x_0 = 3$, $y_0 = 0$ and t

We shall first compute the following.

$$k_1 = hf(t_0, x_0, y_0) = (0.1)f(0, 3, 0) = (0.1)(0) = 0$$

$$l_1 = (0.1)[0 - 12] = -1.2$$

$$k_2 = hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_1}{2},\ y_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)f(0.05, 3, -0.6) = (0.1)(-0.6) = -0.06$$

$$l_2 = (0.1)[(0.05)(-0.6) - 12] = -1.203$$

$$k_3 = hf\left(t_0 + \frac{h}{2},\ x_0 + \frac{k_2}{2},\ y_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 2.97, -0.6015) = (0.1)(-0.6015) = -0.06015$$

$$l_3 = (0.1)[(0.05)(-0.6015) - 4 \times 2.97] = -1.191$$

$$k_4 = hf(t_0 + h,\ x_0 + k_3,\ y_0 + l_3)$$

$$k_4 = (0.1)f(0.1, 2.93985, -1.191) = (0.1)(-1.191) = -0.1191$$

$$l_4 = (0.1)[(0.1)(-1.191) - 4 \times 2.93985] = -1.18785$$

We have $x(t_0+h) = x_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$$y(t_0+h) = y_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4) \quad \text{where} \quad y = \frac{dx}{dt}$$

Substituting the appropriate values we obtain $x(0.1) = 2.9401$, $y(0.1) = -1.196$

Thus $x = 2.9401$ and $\frac{dx}{dt} = -1.196$ when $t = 0.1$

2.13 Milne's Method

Preamble: We recall [*Module-I, Article-1.31*] Milne's predictor and corrector formulae for solving first order ODE : $y' = f(x, y)$; $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$. Here x_0, x_1, x_2, x_3 are equidistant values of x distant h.

We have to compute $y(x_4)$ where $x_4 = x_0 + 4h$

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad \text{[Predictor formula]}$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \qquad \text{[Corrector formula]}$$

Method to solve $y'' = f(x, y, y')$ given $y(x_0) = y_0$ and $y'(x_0) = y_0'$

- We put $y' = z$ which gives $y'' = \frac{dz}{dx} = z'$.
  The given DE becomes $z' = f(x, y, z)$
- We equip with the following table of values.

x	x_0	x_1	x_2	x_3
y	y_0	y_1	y_2	y_3
$y' = z$	$y_0' = z_0$	$y_1' = z_1$	$y_2' = z_2$	$y_3' = z_3$
$y'' = z'$	$y_0'' = z_0'$	$y_1'' = z_2'$	$y_2'' = z_2'$	$y_3'' = z_3'$

- We first apply predictor formula to compute $y_4^{(P)}$ and $z_4^{(P)}$ where,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3), \quad \text{since } y' = z.$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

- We compute $z_4' = f(x_4, y_4, z_4)$ and then apply corrector formula where,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

- Corrector formula can be applied repeatedly for better accuracy.

WORKED PROBLEMS

6. *Apply Milne's method to solve* $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ *given the follwoing table of initial values.*

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.399
y'	1	1.2103	1.4427	1.699

compute $y(0.4)$ *numerically and also theoretically.*

>> Putting $y' = \frac{dy}{dx} = z$, we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}$

The given equation becomes $\frac{dz}{dx} = 1 + z$ or $z' = 1 + z$.

Further $z' = 1 + z$ will give us the following values

$z'(0) = 1 + z(0) = 1 + 1 = 2$

$\therefore$ $z'(0.1) = 1 + z(0.1) = 2.2103$

$z'(0.2) = 1 + z(0.2) = 2.4427$

$z'(0.3) = 1 + z(0.3) = 2.699$

Now we tabulate these values.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 1.1103$	$y_2 = 1.2427$	$y_3 = 1.399$
$y' = z$	$z_0 = 1$	$z_1 = 1.2103$	$z_2 = 1.4427$	$z_3 = 1.699$
$y'' = z'$	$z_0' = 2$	$z_1' = 2.2103$	$z_2' = 2.4427$	$z_3' = 2.699$

We first consider Milne's predictor formulae :

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

Hence, $y_4^{(P)} = 1 + \frac{4(0.1)}{3}[2(1.2103) - 1.4427 + 2(1.699)]$

$$z_4^{(P)} = 1 + \frac{4(0.1)}{3}[2(2.2103) - 2.4427 + 2(2.699)]$$

$\therefore \quad y_4^{(P)} = 1.5835$ and $z_4^{(P)} = 1.9835$

Next we consider Milne's corrector formulae :

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have, $z_4' = 1 + z_4^{(P)} = 1 + 1.9835 = 2.9835$

Hence, $y_4^{(C)} = 1.2427 + \frac{0.1}{3}[1.4427 + 4(1.699) + 1.9835]$

$$z_4^{(C)} = 1.4427 + \frac{0.1}{3}[2.4427 + 4(2.699) + 2.9835]$$

$\therefore \quad y_4^{(C)} = 1.58344$ and $z_4^{(C)} = 1.98344$

Applying the corrector formula again for y_4 we obtain $y_4^{(C)} = 1.583438$

Thus the required $y(0.4) = 1.5834$

Theoretical solution of the problem is as follows.

$$y'' = 1 + y' \quad \text{or} \quad y'' - y' = 1 \quad \text{or} \quad (D^2 - D)y = 1$$

A.E : $m^2 - m = 0$ or $m(m-1) = 0 \Rightarrow m = 0, 1$

$$y_c = c_1 + c_2 e^x$$

$$y_p = \frac{1}{D^2 - D} = \frac{e^{0x}}{D^2 - D} = x \cdot \frac{e^{0x}}{2D - 1} = -x$$

Complete solution : $y = y_c + y_p$ is given by $y = c_1 + c_2 e^x - x$

This gives $y' = c_2 e^x - 1$ and by using $y(0) = 1$, $y'(0) = 1$ we have,

$$1 = c_1 + c_2 \text{ and } 1 = c_2 - 1 \quad \therefore \quad c_2 = 2 \text{ and } c_1 = -1$$

Hence, $y = -1 + 2e^x - x$ is the theoretical solution.

Now $y(0.4) = -1 + 2e^{0.4} - 0.4 = 1.58365$, **theoretically.**

7. *Apply Milne's method to compute $y(0.8)$ given that* $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ *and th... following table of initial values.*

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

Apply the corrector formula twice in presenting the value of y at $x = 0.8$

>> Putting $y' = \frac{dy}{dx} = z$ we obtain $y'' = \frac{d^2y}{dx^2} = z'$.

The given equation becomes $z' = 1 - 2yy' = 1 - 2yz$

Now, $z_0' = 1 - 2(0)(0) = 1$

$z_1' = 1 - 2(0.02)(0.1996) = 0.992$

$z_2' = 1 - 2(0.0795)(0.3937) = 0.9374$

$z_3' = 1 - 2(0.1762)(0.5689) = 0.7995$

We have the following table.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.02$	$y_2 = 0.0795$	$y_3 = 0.1762$
$y' = z$	$z_0 = 0$	$z_1 = 0.1996$	$z_2 = 0.3937$	$z_3 = 0.5689$
$y'' = z'$	$z_0' = 1$	$z_1' = 0.992$	$z_2' = 0.9374$	$z_3' = 0.7995$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 0.3049 \quad \text{and} \quad z_4^{(P)} = 0.7055$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have $z_4' = 1 - 2y_4^{(P)} z_4^{(P)} = 1 - 2(0.3049)(0.7055) = 0.5698$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 0.3045 \quad \text{and} \quad z_4^{(C)} = 0.7074$$

Applying the corrector formula again for y_4 we have

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7074] = 0.3046$$

Thus the required $y(0.8) = 0.3046$

8. *Obtain the solution of the equation* $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ *by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data*

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

>> Dividing the given equation by 2 we have,

$$\frac{d^2y}{dx^2} = 2x + \frac{1}{2}\frac{dy}{dx} \quad \text{or} \quad y'' = 2x + \frac{y'}{2}$$

Putting $y' = z$ we obtain $y'' = z'$ and the given equation becomes $z' = 2x + \frac{z}{2}$

Now, $z_0' = 2(1) + \frac{2}{2} = 3$

$$z_1' = 2(1.1) + \frac{2.3178}{2} = 3.3589$$

$$z_2' = 2(1.2) + \frac{2.6725}{2} = 3.73625$$

$$z_3' = 2(1.3) + \frac{3.0657}{2} = 4.13285$$

We have the following table.

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = z$	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 3.0657$
$y'' = z'$	$z_0' = 3$	$z_1' = 3.3589$	$z_2' = 3.73625$	$z_3' = 4.13285$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 3.0793 \quad \text{and} \quad z_4^{(P)} = 3.4996$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have, $z_4' = 2x_4 + \frac{z_4^{(P)}}{2} = 2(1.4) + \frac{3.4996}{2} = 4.5498$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 3.0794 \quad \text{and} \quad z_4^{(C)} = 3.4997$$

Thus the required value of y is 3.0794 at $x = 1.4$.

9. *Given the ODE $y'' + xy' + y = 0$ and the following table of initial values, compute $y(0.4)$ by applying Milne's method.*

x	0	0.1	0.2	0.3
y	1	0.995	0.9801	0.956
y'	0	-0.0995	-0.196	-0.2867

>> Putting $y' = z$, we get $y'' = z'$.

Also we have $z' = -(xz + y)$ from the given equation.

Further, $z'(0) = -[0+1] = -1$

$z'(0.1) = -[(0.1)(-0.0995) + 0.995] = -0.985$

$z'(0.2) = -[(0.2)(-0.196) + 0.9801] = -0.941$

$z'(0.3) = -[(0.3)(-0.2867) + 0.956] = -0.87$

We also have by using data the following table.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 0.995$	$y_2 = 0.9801$	$y_3 = 0.956$
$y' = z$	$z_0 = 0$	$z_1 = -0.0995$	$z_2 = -0.196$	$z_3 = -0.2867$
$y'' = z'$	$z_0' = -1$	$z_1' = -0.985$	$z_2' = -0.941$	$z_3' = -0.87$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 0.9231 \quad \text{and} \quad z_4^{(P)} = -0.3692$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have, $z_4' = -(x_4 z_4^{(P)} + y_4^{(P)}) = -[(0.4)(-0.3692) + 0.9231] = -0.7754$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 0.9230 \quad \text{and} \quad z_4^{(C)} = -0.3692$$

Thus the required $y(0.4) = 0.923$

10. *Applying Milne's predictor and corrector formulae compute $y(0.8)$ given that y satisfies the equation $y'' = 2yy'$ and y & y' are governed by the following values*

$y(0) = 0$, $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6841$

$y'(0) = 1$, $y'(0.2) = 1.041$, $y'(0.4) = 1.179$, $y'(0.6) = 1.468$

Apply corrector formula twice.

>> Putting $y' = z$ we obtain $y'' = \frac{dz}{dx} = z'$ & the given equation becomes $z' = 2yz$

Now, $z'(0) = 0$, $z'(0.2) = 2(0.2027)(1.041) = 0.422$

$z'(0.4) = 2(0.4228)(1.179) = 0.997$

$z'(0.6) = 2(0.6841)(1.468) = 2.009$

Now we tabulate all the values.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.2027$	$y_2 = 0.4228$	$y_3 = 0.6841$
$y' = z$	$z_0 = 1$	$z_1 = 1.041$	$z_2 = 1.179$	$z_3 = 1.468$
$y'' = z'$	$z_0' = 0$	$z_1' = 0.422$	$z_2' = 0.997$	$z_3' = 2.009$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 1.0237 \quad \text{and} \quad z_4^{(P)} = 2.0307$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have $z_4' = 2y_4^{(P)} z_4^{(P)} = 4.1577$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 1.0282 \text{ and } z_4^{(C)} = 2.0584$$

Applying the corrector formula again we have

$$y_4^{(C)} = 0.4228 + \frac{0.2}{3}[1.179 + 4(1.468) + 2.0584] = 1.03009$$

Thus the required $y(0.8) = 1.0301$

EXERCISES

1. Use fourth order Runge-Kutta method to solve the equation $\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$ given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. Compute y and $\frac{dy}{dx}$ at $x = 0.2$

2. Solve $y'' + 4y = xy$ given that $y(0) = 3$ and $y'(0) = 0$. Compute $y(0.1)$ using Runge-Kutta method of order 4.

3. Apply Milne's method to compute $y(0.4)$ given the equation $y'' + y' = 2e^x$ and the following table of initial values. Compare the result with the theoretical value.

x	0	0.1	0.2	0.3
y	2	2.01	2.04	2.09
y'	0	0.2	0.4	0.6

4. Solve the equation $y'' + y' = 2x$ at $x = 0.4$ by applying Milne's method given that $y = 1$, $y' = -1$ at $x = 0$. Requisite initial values be generated from Taylor's series method.

ANSWERS

1. $y(0.2) = 1.0202$, $y'(0.2) = 0.204$
2. $y(0.1) = 2.94$
3. $y(0.4) = 2.16$
4. $y(0.4) = 0.6897$

2.2 Special Functions

2.21 Introduction

The Laplace equation $\nabla^2 u = 0$ in various orthogonal curvilinear coordinate systems is of great importance in many physical and engineering problems. The solution of Laplace's equation in cylindrical system and spherical system leads to two important ordinary differential equations namely the Bessel differential equation and Legendre differential equation respectively. The series solution of the Bessel's differential equation is a special function known as the Bessel function. The special polynomial function that occurs in the process of solving in series the Legendre's differential equation is known as Legendre polynomial.

The Bessel function has various applications in solving boundary value problems with axial symmetry and the Legendre polynomial has various applications in solving boundary value problems with spherical symmetry.

2.22 Solution of Laplace Equation in Cylindrical System Leading to Bessel Differential Equation

The coordinates (ρ, ϕ, z) are called the cylindrical coordinates and the relationship with the cartesian coordinates (x, y, z) is given by $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

The Laplace equation $\nabla^2 f = 0$ in the cylindrical system is given by

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 \quad \text{[Refer Vol-II, Page -410]} \quad \ldots(1)$$

We shall solve this by the method of separation of variables (*Product method*)

Let $f = f_1 f_2 f_3$ be the solution of (1), where $f_1 = f_1(\rho)$, $f_2 = f_2(\phi)$, $f_3 = f_3(z)$.

Substituting this in (1) we have,

$$\frac{\partial^2}{\partial \rho^2}(f_1f_2f_3) + \frac{1}{\rho}\frac{\partial}{\partial \rho}(f_1f_2f_3) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}(f_1f_2f_3) + \frac{\partial^2}{\partial z^2}(f_1f_2f_3) = 0$$

That is, $$f_2f_3\frac{d^2f_1}{d\rho^2} + \frac{f_2f_3}{\rho}\frac{df_1}{d\rho} + \frac{f_1f_3}{\rho^2}\frac{\partial^2 f_2}{d\phi^2} + f_1f_2\frac{d^2f_3}{dz^2} = 0$$

Dividing by $f_1\ f_2\ f_3$ we have,

$$\frac{1}{f_1}\frac{d^2f_1}{d\rho^2} + \frac{1}{\rho f_1}\frac{df_1}{d\rho} + \frac{1}{\rho^2 f_2}\frac{d^2f_2}{d\phi^2} + \frac{1}{f_3}\frac{d^2f_3}{dz^2} = 0$$

i.e.,
$$\frac{1}{f_1}\frac{d^2 f_1}{d\rho^2}+\frac{1}{\rho f_1}\frac{df_1}{d\rho}+\frac{1}{\rho^2 f_2}\frac{d^2 f_2}{d\phi^2}=-\frac{1}{f_3}\frac{d^2 f_3}{dz^2} \qquad \ldots(2)$$

The LHS is a function of ρ, ϕ and RHS is a function of z. Therefore they must be equal to a constant.

Let us set $\frac{1}{f_3}\frac{d^2 f_3}{dz^2}=1$, so that (2) becomes

$$\frac{1}{f_1}\frac{d^2 f_1}{d\rho^2}+\frac{1}{\rho f_1}\frac{df_1}{d\rho}+\frac{1}{\rho^2 f_2}\frac{d^2 f_2}{d\phi^2}=-1$$

Now multiplying by ρ^2 we get,

$$\frac{\rho^2}{f_1}\frac{d^2 f_1}{d\rho^2}+\frac{\rho}{f_1}\frac{df_1}{d\rho}+\frac{1}{f_2}\frac{d^2 f_2}{d\phi^2}=-\rho^2$$

or
$$\frac{\rho^2}{f_1}\frac{d^2 f_1}{d\rho^2}+\frac{\rho}{f_1}\frac{df_1}{d\rho}+\rho^2=\frac{-1}{f_2}\frac{d^2 f_2}{d\phi^2} \qquad \ldots(3)$$

Again, LHS is a function of ρ and RHS is a function of ϕ. Therefore they must be equal to a constant.

Now setting $\frac{-1}{f_2}\frac{d^2 f_2}{d\phi^2}=n^2$, (3) becomes

$$\frac{\rho^2}{f_1}\frac{d^2 f_1}{d\rho^2}+\frac{\rho}{f_1}\frac{df_1}{d\rho}+\rho^2=n^2$$

or
$$\frac{\rho^2}{f_1}\frac{d^2 f_1}{d\rho^2}+\frac{\rho}{f_1}\frac{df_1}{d\rho}+(\rho^2-n^2)=0$$

i.e.,
$$\rho^2\frac{d^2 f_1}{d\rho^2}+\rho\frac{df_1}{d\rho}+(\rho^2-n^2)f_1=0$$

This equation can be written in the form

$$x^2\frac{d^2 y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

This is the *Bessel's differential equation of order n* in the standard form originating from the Laplace equation in the cylindrical system.

2.23 Solution of Laplace Equation in Spherical System Leading to Legendre Differential Equation

The coordinates (r,θ,ϕ) are called the spherical polar coordinates and the relationship with the cartesian coordinates (x,y,z) is given by $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$.

The Laplace equation $\nabla^2 f=0$ in the spherical system is given by

$$\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}=0 \qquad \ldots(1)$$

[*Refer Vol-2, Page-410*]

We shall solve this by the method of separation of variables (*Product method*).

Let $f=f_1f_2f_3$ be the solution of (1) where $f_1=f_1(r), f_2=f_2(\theta), f_3=f_3(\phi)$.

Substituting this in (1) we have,

$$\frac{\partial^2}{\partial r^2}(f_1f_2f_3)+\frac{2}{r}\frac{\partial}{\partial r}(f_1f_2f_3)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}(f_1f_2f_3)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}(f_1f_2f_3)=0$$

i.e.,
$$f_2f_3\frac{d^2 f_1}{dr^2}+\frac{2f_2f_3}{r}\frac{df_1}{dr}+\frac{f_1f_3}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df_2}{d\theta}\right)+\frac{f_1f_2}{r^2\sin^2\theta}\frac{d^2 f_3}{d\phi^2}=0$$

Dividing by $f_1f_2f_3$ we have,

$$\frac{1}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2}{rf_1}\frac{df_1}{dr}+\frac{1}{r^2\sin\theta f_2}\left(\sin\theta\frac{d^2 f_2}{d\theta^2}+\frac{df_2}{d\theta}\cos\theta\right)+\frac{1}{r^2\sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2}$$

i.e.,
$$\left(\frac{1}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2}{rf_1}\frac{df_1}{dr}\right)+\left(\frac{1}{r^2 f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{r^2 f_2}\frac{df_2}{d\theta}\right)+\frac{1}{r^2\sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2}=0$$

Multiplying by r^2 we have,

$$\left(\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}\right)+\left(\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right)+\frac{1}{\sin^2\theta f_3}\frac{d^2 f_3}{d\phi^2}=0$$

i.e.,
$$\sin^2\theta\left(\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2}+\frac{2r}{f_1}\frac{df_1}{dr}\right)+\sin^2\theta\left(\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2}+\frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right)=-\frac{1}{f_3}\frac{d^2 f_3}{d\phi^2} \quad \ldots$$

Since LHS is a function of r, θ and RHS is a function of ϕ, they must be equal to a constant.

Setting $\frac{1}{f_3}\frac{d^2 f_3}{d\phi^2} = 0$, equation (2) on dividing by $\sin^2\theta$ will give us

$$\left(\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2} + \frac{2r}{f_1}\frac{df_1}{dr}\right) + \left(\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2} + \frac{\cot\theta}{f_2}\frac{df_2}{d\theta}\right) = 0$$

i.e.,
$$\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2} + \frac{\cot\theta}{f_2}\frac{df_2}{d\theta} = -\left(\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2} + \frac{2r}{f_1}\frac{df_1}{dr}\right) \qquad \ldots(3)$$

Again LHS is a function of θ and RHS is a function of r, with the result they must be equal to a constant.

Now setting $\frac{r^2}{f_1}\frac{d^2 f_1}{dr^2} + \frac{2r}{f_1}\frac{df_1}{dr} = n(n+1)$ we obtain

$$\frac{1}{f_2}\frac{d^2 f_2}{d\theta^2} + \frac{\cot\theta}{f_2}\frac{df_2}{d\theta} = -n(n+1)$$

or
$$\frac{d^2 f_2}{d\theta^2} + \cot\theta\,\frac{df_2}{d\theta} + n(n+1)f_2 = 0 \qquad \ldots(4)$$

Now by taking $x = \cos\theta$, we shall convert the differential equation given by (4) in terms of f_2 and x as follows.

$$\frac{df_2}{d\theta} = \frac{df_2}{dx}\cdot\frac{dx}{d\theta} = \frac{df_2}{dx}(-\sin\theta) = -\sin\theta\,\frac{df_2}{dx} \qquad \ldots(5)$$

$$\frac{d^2 f_2}{d\theta^2} = \frac{d}{d\theta}\left(\frac{df_2}{d\theta}\right) = \frac{d}{d\theta}\left(-\sin\theta\,\frac{df_2}{dx}\right)$$

$$= -\sin\theta\,\frac{d}{d\theta}\left(\frac{df_2}{dx}\right) + \frac{df_2}{dx}(-\cos\theta)$$

$$= -\sin\theta\cdot\frac{d}{dx}\left(\frac{df_2}{dx}\right)\frac{dx}{d\theta} - \frac{df_2}{dx}\cos\theta$$

$$= -\sin\theta\,\frac{d^2 f_2}{dx^2}(-\sin\theta) - \frac{df_2}{dx}\cos\theta$$

i.e.,
$$\frac{d^2 f_2}{d\theta^2} = \sin^2\theta\,\frac{d^2 f_2}{dx^2} - \cos\theta\,\frac{df_2}{dx} \qquad \ldots(6)$$

Hence (4) as a consequence of (5) and (6) becomes,

$$\sin^2\theta\,\frac{d^2 f_2}{dx^2} - \cos\theta\,\frac{df_2}{dx} + \cot\theta\left(-\sin\theta\,\frac{df_2}{dx}\right) + n(n+1)f_2 = 0$$

i.e.,
$$\sin^2\theta\,\frac{d^2 f_2}{dx^2} - \cos\theta\,\frac{df_2}{dx} - \cos\theta\,\frac{df_2}{dx} + n(n+1)f_2 = 0$$

i.e.,
$$(1-\cos^2\theta)\frac{d^2 f_2}{dx^2} - 2\cos\theta\,\frac{df_2}{dx} + n(n+1)f_2 = 0$$

Since $x = \cos\theta$, the equation becomes

$$(1-x^2)\frac{d^2 f_2}{dx^2} - 2x\,\frac{df_2}{dx} + n(n+1)f_2 = 0$$

This equation can be written in the following standard form.

$$(1-x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

This is the ***Legendre's differential equation*** in the standard form originating from the Laplace equation in the spherical system.

2.24 Series Solution of Differential Equation

We discuss the method of finding solution of a second order homogeneous differential equation in the form of convergent infinite power series.

Subsequently we extend / generalize this method which is referred to as generalized power series method or **Frobenius method.**

- **Power series solution of a second order ODE**

Consider the DE in the form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0 \qquad \ldots(1)$$

where $P_0(x), P_1(x)$ and $P_2(x)$ are polynomials in x with $P_0(x) \neq 0$ at $x = 0$.

The method is explained step wise.

- We assume the solution of (1) in the form of a power series,

$$y = \sum_{r=0}^{\infty} a_r x^r \qquad \ldots(2)$$

- Then, $\dfrac{dy}{dx} = y' = \sum_0^\infty a_r\, r x^{r-1}$ and $\dfrac{d^2y}{dx^2} = y'' = \sum_0^\infty a_r\, r(r-1)\, x^{r-2}$

- We substitute these along with $y = \sum_0^\infty a_r x^r$ in (1) which results in an infinite series with various powers of x equal to zero.

[*It is evident that this will be satisfied only when the coefficients of the various powers of x are equal to zero.*]

- We equate the coefficients of various powers of x (*starting from the lowest power of x*) to zero. In general when the coefficient of x^r is equated to zero, we obtain a recurrence relation which will helps us to determine the constants $a_2, a_3, a_4, a_5, \cdots$ in terms of a_0 and a_1.

- We substitute the values of $a_2, a_3, a_4, \ldots$ in the expanded form of (2):

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

- Thus we get the power series solution of the ODE in the form,

$$y = a_0\, F(x) + a_1\, G(x)$$

where $F(x)$ and $G(x)$ are convergent infinite series in x.

Remarks :

1. *The method is adoptable for first order DEs also.*
2. *Sometimes we can recognize the functions $F(x)$ and $G(x)$ represented by convergent infinite series. This will help us to compare the series solution with that of analytical solution. We recall the following* ***standard convergent infinite series*** *in ascending powers of x.*

1. $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$
2. $e^{-x} = 1 - \dfrac{x}{1!} + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots$
3. $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$
4. $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$
5. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$
6. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

Note : ***The first three of the worked problems that follows are presented with the intention of giving an insight to the series solution method by taking DEs having analytical solution. Various steps are also explained in detail.***

WORKED PROBLEMS

11. *Obtain the series solution of the equation* $\dfrac{dy}{dx} - 2xy = 0$

>> We have, $y' - 2xy = 0$ ...(1)

[*Note that the coefficient of* $y' = 1 = P_0(x) \neq 0$ *at* $x = 0$]

Let $y = \sum_{r=0}^\infty a_r x^r$ ...(2)

be the series solution of the given d.e.

$\therefore \quad y' = \sum_0^\infty a_r\, r x^{r-1}$

Now (1) becomes,

$$\sum_0^\infty a_r r x^{r-1} - 2x \sum_0^\infty a_r x^r = 0$$

ie., $$\sum_0^\infty a_r r x^{r-1} - 2 \sum_0^\infty a_r x^{r+1} = 0$$

We equate the coefficients of various powers of x to zero.

[*Note that on giving values for $r = 0, 1, 2, 3, \cdots$ the first summation has terms with powers of $x : x^{-1}, x^0, x^1, x^2, \cdots$ and the second summation has terms with powers $x^1, x^2, \cdots$ We independently equate the coefficient of x^{-1} and x^0 to zero first and subsequently equate the coefficient of x^r in general to zero*]

Coeff. of $x^{-1} : a_0(0) = 0 \quad \Rightarrow a_0 \neq 0$

Coeff. of $x^0 : a_1(1) = 0 \quad \Rightarrow a_1 = 0$

Now we shall equate the coefficient of x^r $(r \geq 1)$ to zero.

ie., $a_{r+1}(r+1) - 2a_{r-1} = 0$

or $a_{r+1} = \dfrac{2a_{r-1}}{r+1}\ ;\ r \geq 1$ ...(3)

(*This is the recurrence relation which helps us to find $a_2, a_3, a_4, \ldots$*)

By putting $r = 1, 2, 3, 4, 5, \ldots$ in (3) we obtain,

$$a_2 = \frac{2a_0}{2} = a_0 \; ; \; a_3 = \frac{2a_1}{3} = 0, \text{ since } a_1 = 0$$

$$a_4 = \frac{2a_2}{4} = \frac{1}{2}a_0 ; \; a_5 = \frac{2a_3}{5} = 0, \text{ since } a_3 = 0$$

$$a_6 = \frac{2a_4}{6} = \frac{1}{3} \cdot \frac{1}{2}a_0 = \frac{a_0}{6} ; \; a_7 = \frac{2a_5}{7} = 0, \text{ since } a_5 = 0 \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

ie., $y = a_0 + a_2 x^2 + a_4 x^4 + \cdots$, since $a_1 = 0 = a_3 = a_5 = \cdots$

ie., $$y = a_0 \left[1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots \right] \qquad \ldots (4)$$

This is the required series solution of the given DE.

Remark : *Comparison with the analytical solution*

We have, $\dfrac{dy}{dx} = 2xy$

or $\dfrac{dy}{y} = 2x\,dx$, *by separating the variables,*

$\Rightarrow$ $\displaystyle\int \frac{dy}{y} = \int 2x\,dx + c$

ie., $\log_e y = x^2 + c$ *or* $y = e^{x^2 + c} = e^c e^{x^2}$

Denoting $e^c = k$, $y = ke^{x^2}$ is the analytical solution of (1).

The series solution obtained by us as in (4) can be written in the form

$$y = a_0 \left[1 + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots \right]$$

ie., $y = a_0 e^{x^2}$ *[Refer standard series (1)] This is same as the series solution (4) obtained.*

12. *Obtain the series solution of the equation* $\dfrac{d^2 y}{dx^2} + y = 0$

>> We have, $y'' + y = 0 \qquad \ldots (1)$

(Note that the coefficient of $y'' = 1 = P_0(x) \neq 0$ *at* $x = 0$ *).*

Let $$y = \sum_{r=0}^{\infty} a_r x^r \qquad \ldots (2)$$

be the series solution of (1).

$\therefore$ $$y' = \sum_0^{\infty} a_r r x^{r-1}, \; y'' = \sum_0^{\infty} a_r r(r-1) x^{r-2}$$

Now (1) becomes,

$$\sum_0^{\infty} a_r r(r-1) x^{r-2} + \sum_0^{\infty} a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

[Note that the first summation has terms with powers of x : $x^{-2}, x^{-1}, x^0, x^1, \ldots$ and the second summation has terms with powers of x : $x^0, x^1, x^2, \ldots$]

Coeff. of x^{-2} : $a_0(0)(-1) = 0 \quad \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} : $a_1(1)(0) = 0 \quad \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r ($r \geq 0$) to zero.

ie., $a_{r+2}(r+2)(r+1) + a_r = 0$

or $$a_{r+2} = \frac{-a_r}{(r+2)(r+1)} \quad (r \geq 0) \qquad \ldots (3)$$

(This is the recurrence relation which helps us to find $a_2, a_3, a_4, \ldots$)

By putting $r = 0, 1, 2, 3, 4, \ldots$ in (3) we obtain,

$$a_2 = -\frac{a_0}{2} \quad ; \quad a_3 = -\frac{a_1}{6}$$

$$a_4 = \frac{-a_2}{12} = \frac{a_0}{24} \quad ; \quad a_5 = \frac{-a_3}{20} = \frac{a_1}{120}$$

$$a_6 = -\frac{a_4}{30} = \frac{-a_0}{720} \quad ; \quad a_7 = -\frac{a_5}{42} = \frac{-a_1}{5040} \quad \text{and so on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

i.e., $$y = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right] + a_1\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots\right]$$

Thus $$y = a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right] + a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right] \quad \dots(4)$$

is the required series solution of the given d.e.

Remark : *Comparison with the analytical solution.*

We have, $(D^2+1)\,y = 0$, where $D = \dfrac{d}{dx}$

Auxilary equation is $m^2+1 = 0$ and $m = \pm i$ are its roots.

The analytical solution is given by

$$y = c_1 \cos x + c_2 \sin x$$

The series solution obtained by us can be represented as

$$y = a_0 \cos x + a_1 \sin x \quad \text{[Refer standard series (5) and (6)]}$$

This is same as the series solution.

---

13. *Obtain the power series solution of the equation* $\dfrac{d^2y}{dx^2} - y = 0$

>> This example is very much similar to the previous example. Proceeding on the same lines we obtain $a_0 \neq 0, a_1 \neq 0$ and the recurrence relation is as follows.

$$a_{r+2} = \frac{a_r}{(r+2)(r+1)} \quad (r \geq 0)$$

By putting $r = 0, 1, 2, 3, 4, \dots$ we obtain,

$$a_2 = \frac{a_0}{2},\quad a_3 = \frac{a_1}{6},\quad a_4 = \frac{a_0}{24},\quad a_5 = \frac{a_1}{120},\quad a_6 = \frac{a_0}{720}, \dots$$

Substituting these values in the expanded form of $y = \sum_{r=0}^{\infty} a_r x^r$ we obtain,

$$y = a_0\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right] + a_1\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right] \quad \dots(1)$$

This is the required series soution of the given d.e.

Remark : *Comparison with the analytical solution.*

We have $(D^2-1)\,y = 0$, where $D = \dfrac{d}{dx}$

Auxilary equation is $m^2 - 1 = 0$ and hence $m = \pm 1$ are its roots.

The analytical soution is given by $y = c_1 e^x + c_2 e^{-x}$

The series solution obtained by us can be represented as

$$y = a_0 \cosh x + a_1 \sinh x \quad \text{[Refer standard series (3) and (4)]}$$

i.e., $$y = a_0\left[\frac{e^x + e^{-x}}{2}\right] + a_1\left[\frac{e^x - e^{-x}}{2}\right]$$

or $$y = \left[\frac{a_0 + a_1}{2}\right] e^x + \left[\frac{a_0 - a_1}{2}\right] e^{-x}$$

i.e., $$y = c_1 e^x + c_2 e^{-x}, \text{ where } c_1 = \frac{a_0 + a_1}{2} \text{ and } c_2 = \frac{a_0 - a_1}{2}$$

This is same as the analytical solution given by (2).

---

14. *Solve* $\dfrac{d^2y}{dx^2} + xy = 0$ *by obtaining the solution in the form of series.*

>> We have $y'' + xy = 0$

The coefficient of $y'' = 1 = P_0(x) \neq 0$ at $x = 0$.

Let $$y = \sum_{r=0}^{\infty} a_r x^r$$

be the series solution of (1).

$$\therefore \quad y' = \sum_{0}^{\infty} a_r r x^{r-1}, \quad y'' = \sum_{0}^{\infty} a_r r(r-1) x^{r-2}$$

Now (1) becomes,

$$\sum_{0}^{\infty} a_r r(r-1) x^{r-2} + \sum_{0}^{\infty} a_r x^{r+1} = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the cofficients of x^{-2}, x^{-1} and x^0 available only in the first summation to zero.

Coeff. of x^{-2} ; $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} ; $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Coeff. of x^0 : $a_2(2)(1) = 0$ or $2a_2 = 0 \Rightarrow a_2 = 0$

Now we shall equate the coefficient of x^r $(r \geq 1)$ to zero.

ie., $a_{r+2}(r+2)(r+1) + a_{r-1} = 0$

or $$a_{r+2} = \frac{-a_{r-1}}{(r+2)(r+1)} \quad (r \geq 1) \qquad \ldots(3)$$

By putting $r = 1, 2, 3, 4, \ldots$ in (3) we obtain,

$$a_3 = \frac{-a_0}{6} \; ; \; a_4 = \frac{-a_1}{12} \; ; \; a_5 = \frac{-a_2}{20} = 0$$

$$a_6 = -\frac{a_3}{30} = \frac{a_0}{180} \; ; \; a_7 = \frac{-a_4}{42} = \frac{a_1}{504} \; ; \; a_8 = 0 = a_{11} = a_{14} \ldots$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

Thus, $y = a_0 \left[1 - \frac{x^3}{6} + \frac{x^6}{180} - \cdots \right] + a_1 \left[x - \frac{x^4}{12} + \frac{x^7}{504} - \cdots \right]$

is the required solution in the form of series.

---

15. *Obtain the series soution of the equation* $\dfrac{d^2 y}{dx^2} + x^2 y = 0$

>> We have $y'' + x^2 y = 0 \qquad \ldots(1)$

The coefficient of $y'' = 1 = P_0(x) \neq 0$ at $x = 0$.

Let $$y = \sum_{r=0}^{\infty} a_r x^r \qquad \ldots(2)$$

be the series solution of (1).

$\therefore$ $$y' = \sum_0^\infty a_r r x^{r-1}, \quad y'' = \sum_0^\infty a_r r(r-1) x^{r-2}$$

Now (1) becomes,

$$\sum_0^\infty a_r r(r-1) x^{r-2} + \sum_0^\infty a_r x^{r+2} = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2}, x^{-1}, x^0, x^1 available only in the first summation to zero.

Coeff. of x^{-2} : $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} : $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Coeff. of x^0 : $a_2(2)(1) = 0$ or $2a_2 = 0 \Rightarrow a_2 = 0$

Coeff. of x^1 : $a_3(3)(2) = 0$ or $6a_3 = 0 \Rightarrow a_3 = 0$

Now we shall equate the coefficient of x^r $(r \geq 2)$ to zero.

ie., $a_{r+2}(r+2)(r+1) + a_{r-2} = 0$

or $$a_{r+2} = \frac{-a_{r-2}}{(r+2)(r+1)} \quad (r \geq 2) \qquad \ldots(3)$$

By putting $r = 2, 3, 4, 5 \cdots$ in (3) we obtain,

$$a_4 = -\frac{a_0}{4 \cdot 3} \; ; \; a_5 = \frac{-a_1}{5 \cdot 4} \; ; \; a_6 = \frac{-a_2}{6 \cdot 5} = 0 \; ; \; a_7 = \frac{-a_3}{7 \cdot 6} = 0$$

$$a_8 = -\frac{a_4}{8 \cdot 7} = \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3} \; ; \; a_9 = -\frac{a_5}{9 \cdot 8} = \frac{a_1}{9 \cdot 8 \cdot 5 \cdot 4} \; ;$$

$a_{10} = 0$; $a_{11} = 0$ and so-on.

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $y = a_0 \left[1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \cdots \right] + a_1 \left[x - \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \cdots \right]$

is the required series solution.

16. *Solve $y'' + xy' + y = 0$ in series.*

>> $y'' + xy' + y = 0$ ...(1)

The coefficient of $y'' = 1 = P_0(x) \neq 0$ at $x = 0$.

Let $y = \sum_{r=0}^{\infty} a_r x^r$ ...(2)

be the series solution of (1).

$\therefore \quad y' = \sum_0^{\infty} a_r r x^{r-1}, \quad y'' = \sum_0^{\infty} a_r r(r-1) x^{r-2}$

Now (1) becomes,

$$\sum_0^{\infty} a_r r(r-1) x^{r-2} + \sum_0^{\infty} a_r r x^r + \sum_0^{\infty} a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2}, x^{-1} available only in the first summation to zero.

Coeff. of x^{-2} : $\quad a_0(0)(-1) = 0 \quad \Rightarrow \quad a_0 \neq 0$

Coeff. of x^{-1} : $\quad a_1(1)(0) = 0 \quad \Rightarrow \quad a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero.

ie., $\quad a_{r+2}(r+2)(r+1) + a_r r + a_r = 0$

ie., $\quad a_{r+2}(r+2)(r+1) + (r+1)a_r = 0$

ie., $\quad a_{r+2}(r+2) + a_r = 0$

or $\quad a_{r+2} = \dfrac{-a_r}{r+2} \quad (r \geq 0)$ ...(3)

By putting $r = 0, 1, 2, 3, 4, \cdots$ in (3) we obtain,

$$a_2 = -\frac{a_0}{2};\ a_3 = -\frac{a_1}{3};\ a_4 = -\frac{a_2}{4} = \frac{a_0}{2 \cdot 4};\ a_5 = -\frac{a_3}{5} = \frac{a_1}{3 \cdot 5}$$

$$a_6 = -\frac{a_4}{6} = \frac{-a_0}{2 \cdot 4 \cdot 6};\ a_7 = -\frac{a_5}{7} = \frac{-a_1}{3 \cdot 5 \cdot 7} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $\quad y = a_0\left[1 - \dfrac{x^2}{2} + \dfrac{x^4}{2 \cdot 4} - \dfrac{x^6}{2 \cdot 4 \cdot 6} + \cdots\right] + a_1\left[x - \dfrac{x^3}{3} + \dfrac{x^5}{3 \cdot 5} - \dfrac{x^7}{3 \cdot 5 \cdot 7} + \cdots\right]$

is the required series solution.

17. *Solve $y'' - xy' + y = 0$ in series.*

>> This example is very much similar to the previous one. Proceeding on the same lines we do obtain $a_0 \neq 0$, $a_1 \neq 0$. The coefficient of x^r $(r \geq 0)$ when equated to zero will result in,

$$a_{r+2}(r+2)(r+1) - a_r r + a_r = 0$$

ie., $\quad a_{r+2}(r+2)(r+1) = a_r(r-1)$

or $\quad a_{r+2} = \dfrac{(r-1)a_r}{(r+2)(r+1)} \quad (r \geq 0)$

By putting $r = 0, 1, 2, 3, 4, \ldots$ we obtain,

$$a_2 = -\frac{a_0}{2};\ a_3 = 0;\ a_4 = \frac{a_2}{4 \cdot 3} = \frac{-a_0}{2 \cdot 3 \cdot 4};\ a_5 = 0;\ a_6 = \frac{3a_4}{6 \cdot 5} = \frac{-3a_0}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

$a_7 = 0$ and so-on.

We substitute these values in the series,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $y = a_0\left[1 - \dfrac{x^2}{2} - \dfrac{x^4}{24} - \dfrac{x^6}{240} - \cdots\right] + a_1 x$ is the required series solution.

18. *Develop the series solution of the equation* $y'' + xy' + (x^2+2)y = 0$

>> $\quad y'' + xy' + (x^2+2)y = 0$

The coefficient of $y'' = 1 = P_0(x) \neq 0$ at $x = 0$.

Let $\quad y = \sum_{r=0}^{\infty} a_r x^r$

be the series solution of (1).

$$\therefore \quad y' = \sum_0^\infty a_r r x^{r-1}, \quad y'' = \sum_0^\infty a_r r(r-1)x^{r-2}$$

Now (1) becomes,

$$\sum_0^\infty a_r r(r-1)x^{r-2} + \sum_0^\infty a_r r x^r + \sum_0^\infty a_r x^{r+2} + 2\sum_0^\infty a_r x^r = 0$$

We equate the coefficients of various powers of x to zero. We first equate the coefficients of x^{-2}, x^{-1}, x^0, x^1 *(available in various summations except the third one)* to zero.

Coeff. of x^{-2} : $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} : $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Coeff. of x^0 : $a_2(2)(1) + a_0(0) + 2a_0 = 0$

ie., $2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$

Coeff. of x^1 : $a_3(3)(2) + a_1(1) + 2a_1 = 0$

ie., $6a_3 + 3a_1 = 0 \Rightarrow a_3 = -a_1/2$

Now we shall equate the coefficient of x^r $(r \geq 2)$ to zero.

ie., $a_{r+2}(r+2)(r+1) + a_r r + a_{r-2} + 2a_r = 0$

ie., $$a_{r+2} = \frac{-[a_{r-2} + (r+2)a_r]}{(r+2)(r+1)}$$

or $$a_{r+2} = \frac{-a_{r-2}}{(r+2)(r+1)} - \frac{a_r}{(r+1)} \quad (r \geq 2) \qquad \ldots (3)$$

By putting $r = 2, 3, 4, 5, \ldots$ in (3) we obtain,

$$a_4 = \frac{-a_0}{12} - \frac{a_2}{3} = -\frac{a_0}{12} + \frac{a_0}{3} = \frac{a_0}{4}$$

$$a_5 = \frac{-a_1}{20} - \frac{a_3}{4} = -\frac{a_1}{20} + \frac{a_1}{8} = \frac{3a_1}{40}$$

$$a_6 = \frac{-a_2}{30} - \frac{a_4}{5} = \frac{a_0}{30} - \frac{a_0}{20} = -\frac{a_0}{60}$$ and so on.

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $$y = a_0\left[1 - x^2 + \frac{x^4}{4} - \frac{x^6}{60} + \cdots\right] + a_1\left[x - \frac{x^3}{2} + \frac{3x^5}{40} - \cdots\right]$$

is the required series solution.

---

19. *Develop the series solution of the equation* $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$

>> We have $(1+x^2)y'' + xy' - y = 0 \qquad \ldots (1)$

The coefficient of $y'' = 1 + x^2 = P_0(x)$ and we have at $x = 0$, $P_0(x) = 1 \neq 0$

Let $$y = \sum_{r=0}^\infty a_r x^r \qquad \ldots (2)$$

be the series soluiton of (1).

$$\therefore \quad y' = \sum_0^\infty a_r r x^{r-1}, \quad y'' = \sum_0^\infty a_r r(r-1)x^{r-2}$$

Now (1) becomes,

$$\sum_0^\infty a_r r(r-1)x^{r-2} + \sum_0^\infty a_r r(r-1)x^r + \sum_0^\infty a_r r x^r - \sum_0^\infty a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2}, x^{-1} available only in the first summation to zero.

Coeff. of x^{-2} : $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} : $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero.

ie., $a_{r+2}(r+2)(r+1) + a_r r(r-1) + a_r r - a_r = 0$

ie., $a_{r+2}(r+2)(r+1) + a_r(r^2 - r + r - 1) = 0$

ie., $a_{r+2}(r+2)(r+1) + a_r(r^2 - 1) = 0$

ie., $a_{r+2}(r+2)(r+1) + a_r(r+1)(r-1) = 0$

or $$a_{r+2} = -\frac{(r-1)a_r}{(r+2)} \quad (r \geq 0) \qquad \ldots(3)$$

By putting $r = 0, 1, 2, 3, \ldots$ in (3) we obtain.

$$a_2 = \frac{a_0}{2}\ ;\ a_3 = 0\ ;\ a_4 = -\frac{a_2}{4} = -\frac{a_0}{8}\ ;\ a_5 = \frac{-2a_3}{5} = 0\ ;\ a_6 = \frac{-3a_4}{6} = \frac{a_0}{16}\ ;$$

$a_7 = 0$; and so-on.

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Thus $y = a_0\left[1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} \cdots\right] + a_1 x$ is the required series solution.

---

20. *Solve in series the equation* $(x-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ *subject to the conditions* $y(0) = 2$ *and* $y'(0) = -1$

>> We have, $(x-1)y'' + xy' + y = 0 \qquad \ldots(1)$

The coefficient of $y'' = (x-1) = P_0(x)$ and at $x = 0$, $P_0(x) = -1 \neq 0$.

Let $$y = \sum_{r=0}^{\infty} a_r x^r \qquad \ldots(2)$$

be the series solution of (1).

$$\therefore \quad y' = \sum_0^{\infty} a_r r x^{r-1},\quad y'' = \sum_0^{\infty} a_r r(r-1)x^{r-2}$$

Now (1) becomes,

$$\sum_0^{\infty} a_r r(r-1)x^{r-1} - \sum_0^{\infty} a_r r(r-1)x^{r-2} + \sum_0^{\infty} a_r r x^r + \sum_0^{\infty} a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2}, x^{-1} to zero.

Coeff. of x^{-2} : $-a_0(0)(1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1} : $a_0(0)(-1) - a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero.

ie., $a_{r+1}(r+1)r - a_{r+2}(r+2)(r+1) + a_r r + a_r = 0$

ie., $a_{r+1}(r+1)r - a_{r+2}(r+2)(r+1) + a_r(r+1) = 0$

ie., $a_{r+1}r - a_{r+2}(r+2) + a_r = 0$

or $$a_{r+2} = \frac{r a_{r+1} + a_r}{r+2} \quad (r \geq 0)$$

By putting $r = 0, 1, 2, 3, \ldots$ in (3) we obtain,

$$a_2 = \frac{a_0}{2}\ ;\ a_3 = \frac{a_2 + a_1}{3} = \frac{a_0/2 + a_1}{3} = \frac{a_0}{6} + \frac{a_1}{3}$$

$$a_4 = \frac{2a_3 + a_2}{4} = \frac{a_0/3 + 2a_1/3 + a_0/2}{4} = \frac{5a_0}{24} + \frac{a_1}{6}$$

$$a_5 = \frac{3a_4 + a_3}{5} = \frac{5a_0/8 + a_1/2 + a_0/6 + a_1/3}{5} = \frac{19a_0}{120} + \frac{a_1}{6} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

ie., $$y = a_0\left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{5x^4}{24} + \frac{19x^5}{120} + \cdots\right] + a_1\left[x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{6} + \cdots\right] \ldots($$

This is the general solution of (1) in series.

To apply the given initial conditions, we differentiate (4) w.r.t. x.

$$\therefore \quad y' = a_0\left[x + \frac{x^2}{2} + \frac{5x^3}{6} + \cdots\right] + a_1\left[1 + x^2 + \frac{2x^3}{3} + \cdots\right] \ldots($$

Using the conditions, $y = 2$ and $y' = -1$ and $x = 0$, (4) and (5) respectively become

$2 = a_0$ and $-1 = a_1$

Hence (4) becomes,

$$y = 2\left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{5x^4}{24} + \frac{19x^5}{120} + \cdots\right] - \left[x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{6} \cdots\right]$$

Thus $y = 2 - x + x^2 + \frac{x^4}{4} + \frac{3x^5}{20} + \cdots$ **is the required particular solution in series.**

- **Generalized Power Series Method or Frobenius Method**

Consider a second order DE in the form

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0 \qquad \ldots(1)$$

where $P_0(x), P_1(x), P_2(x)$ are polynomials in x with $P_0(x)=0$ at $x=0$

The method is explained stepwise.

- We assume the series solution of (1) in the form

$$y=\sum_{r=0}^{\infty} a_r x^{k+r} \qquad \ldots(2)$$

where $k, a_0, a_1, a_2, \ldots$ are all constants and $a_0 \neq 0$

- Then $\dfrac{dy}{dx}=y'=\displaystyle\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$

$$\frac{d^2y}{dx^2}=y''=\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)\,x^{k+r-2}$$

- We substitute these along with $y=\sum_{r=0}^{\infty} a_r x^{k+r}$ in (1) which results in an infinite series with various powers of x equal to zero.
- We equate the coefficient of the lowest degree term in x to zero. This will give us a quadratic equation in k known as the ***indicial equation***. Let k_1 and k_2 be the roots of this equation.
- We need to equate the coefficients of various other powers of x also to zero. In general when we equate the coefficient of x^{k+r} to zero, we obtain a recurrence relation which helps to determine the constants $a_1, a_2, a_3, a_4, \ldots$ in terms of a_0 only.
- We substitute the values of $a_1, a_2, a_3, \ldots$ in the expanded form of (2) given by

$$y=x^k\,(a_0+a_1x+a_2x^2+a_3x^3+\cdots)$$

- Hence we obtain $y=a_0x^kF(x)$ where $F(x)$ is an infinite series.
- We suppose that k_1, k_2 are real, distinct and do not differ by an integer, that is $k_1-k_2 \neq 0, 1, 2, 3, \ldots$

- Then $y_1=a_0x^{k_1}F(x)$ & $y_2=a_0x^{k_2}F(x)$ are two independent solutions of (1).
- Thus $y=Ay_1+By_2$ constitutes the general / complete solution of (1) in series, where A and B are arbitrary constants.

Remark : *It should be noted that the roots k_1, k_2 of the indicial equation can also be (a) real, distinct, differing by a non zero integer, that is $k_1-k_2=1, 2, 3, \ldots$ (b) coincident, that is $k_1=k_2=c$ (say)*

The complete / general solution in the case (a) will be

$$y=A[y]_{k=k_1}+B\left[\frac{\partial y}{\partial k}\right]_{k=k_1} \qquad (A, B \text{ are arbitrary constants})$$

where $k_1<k_2$ and $k=k_2$ will not result in a new independent solution.

The complete / general solution in the case (b) will be

$$y=A[y]_{k=c}+B\left[\frac{\partial y}{\partial k}\right]_{k=c}$$

Note : *Problems on these two cases are not presented as the focus of attention is mainly on the power series method and Frobenius method.*

WORKED PROBLEMS

21. *Solve by Frobenius method the equation* $4x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=0$

>> We have $4xy''+2y'+y=0 \qquad \ldots(1)$

The coefficient of $y''=4x=P_0(x)$ and $P_0(x)=0$ at $x=0$.

Let $\quad y=\displaystyle\sum_{r=0}^{\infty} a_r x^{k+r} \qquad \ldots(2)$

be the series solution of (1).

$$\therefore \quad y'=\sum_0^{\infty} a_r(k+r)x^{k+r-1}, \quad y''=\sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Now (1) becomes,

$$4\sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r-1}+2\sum_0^{\infty} a_r(k+r)x^{k+r-1}+\sum_0^{\infty} a_r x^{k+r}=0$$

We first equate the coefficient of the lowest degree term in x to zero.

On equating the coefficient of x^{k-1} to zero we have,

$4\,a_0\,k\,(k-1) + 2a_0\,k = 0$, which is the indicial equation.

ie., $\quad 2a_0\,k\,(2k-2+1) = 0 \quad$ or $\quad a_0\,k\,(2k-1) = 0$

Since $a_0 \neq 0$ we have $k = 0$ and $k = 1/2$

[*Note that these roots donot differ by an integer.*]

Next, we shall equate the coefficient of x^{k+r} $(r \geq 0)$ to zero.

ie., $\quad 4\,a_{r+1}(k+r+1)(k+r) + 2a_{r+1}(k+r+1) + a_r = 0$

ie., $\quad 2a_{r+1}\,(k+r+1)\,[\,2k+2r+1\,] + a_r = 0$

or $\quad a_{r+1} = \dfrac{-a_r}{2\,(k+r+1)\,(2k+2r+1)} \quad (r \geq 0)$ ...(3)

Case-(1) : Let $k = 0$

$\therefore \quad a_{r+1} = \dfrac{-a_r}{2\,(r+1)\,(2r+1)} \quad (r \geq 0)$

Putting $r = 0, 1, 2, 3, \ldots$ in this relation we obtain

$$a_1 = -\frac{a_0}{2}\;;\quad a_2 = -\frac{a_1}{12} = \frac{a_0}{24}\;;\quad a_3 = -\frac{a_2}{30} = -\frac{a_0}{720} \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = x^k\,(a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots)$$

ie., $\quad y = a_0\,x^k\left[1 - \dfrac{x}{2} + \dfrac{x^2}{24} - \dfrac{x^3}{720} + \cdots\right]$

The series solution obtained by putting $k = 0$ be denoted by y_1.

$\therefore \quad y_1 = a_0\left[1 - \dfrac{x}{2} + \dfrac{x^2}{24} - \dfrac{x^3}{720} + \cdots\right]$ ...(4)

Case-(2) : Let $k = 1/2$ and hence (3) assumes the form

$$a_{r+1} = \frac{-a_r}{2\,(r+3/2)\,(2r+2)} = \frac{-a_r}{(2r+3)\cdot 2\,(r+1)}$$

ie., $\quad a_{r+1} = \dfrac{-a_r}{2\,(r+1)\,(2r+3)} \quad (r \geq 0)$

Putting $r = 0, 1, 2, 3$ in this relation we obtain.

$$a_1 = -\frac{a_0}{6}\;;\quad a_2 = -\frac{a_1}{20} = \frac{a_0}{120}\;;\quad a_3 = -\frac{a_2}{42} = -\frac{a_0}{5040} \text{ and so-on.}$$

We again substitute these values in the expanded form of (2) :

$$y = x^k\,(a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots)$$

ie., $\quad y = a_0\,x^k\left[1 - \dfrac{x}{6} + \dfrac{x^2}{120} - \dfrac{x^3}{5040} + \cdots\right]$

The series solution obtained by putting $k = 1/2$ be denoted by y_2.

$\therefore \quad y_2 = a_0\,x^{1/2}\left[1 - \dfrac{x}{6} + \dfrac{x^2}{120} - \dfrac{x^3}{5040} + \cdots\right]$ ...(

The complete solution of (1) is given by

$$y = Ay_1 + By_2$$

ie., $\quad y = Aa_0\left[1 - \dfrac{x}{2} + \dfrac{x^2}{24} - \dfrac{x^3}{720} + \cdots\right] + Ba_0\,\sqrt{x}\left[1 - \dfrac{x}{6} + \dfrac{x^2}{120} - \dfrac{x^3}{5040} + \cdots\right]$

Let us denote $c_1 = Aa_0$ and $c_2 = Ba_0$

Thus $y = c_1\left[1 - \dfrac{x}{2!} + \dfrac{x^2}{4!} - \dfrac{x^3}{6!} + \cdots\right] + c_2\,\sqrt{x}\left[1 - \dfrac{x}{3!} + \dfrac{x^2}{5!} - \dfrac{x^3}{7!} + \cdots\right]$

is the required series solution.

Note : *The series solution obtained can be put in the following form :*

$$y = c_1\left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots\right] + c_2\left[\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} - \frac{(\sqrt{x})^7}{7!} + \cdots\right]$$

ie., $\quad y = c_1\,\cos(\sqrt{x}) + c_2\,\sin(\sqrt{x})$ [*Refer standard series (5) and (6) given earlier*]

22. *Solve in series the equation* $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (1 - x^2)\,y = 0$

We have, $\quad 2x^2\,y'' - xy' + (1 - x^2)\,y = 0$ ...(

The coefficient of $y'' = 2x^2 = P_0(x)$ and $P_0(x) = 0$ at $x = 0$.

Let $\quad y = \displaystyle\sum_{r=0}^{\infty} a_r\,x^{k+r}$ ...(

be the seris soution of (1).

$$\therefore \quad y' = \sum_0^\infty a_r (k+r) x^{k+r-1}, \quad y'' = \sum_0^\infty a_r (k+r)(k+r-1) x^{k+r-2}$$

Now (1) becomes,

$$2\sum_0^\infty a_r (k+r)(k+r-1) x^{k+r} - \sum_0^\infty a_r (k+r) x^{k+r} + \sum_0^\infty a_r x^{k+r}$$

$$- \sum_0^\infty a_r x^{k+r+2} = 0$$

We shall first equate the coefficient of the lowest degree term in x, that is x^k to zero.

ie., $\quad 2a_0 k(k-1) - a_0 k + a_0 = 0 \quad$ or $\quad a_0(2k^2 - 2k - k + 1) = 0$

ie., $\quad a_0(2k^2 - 3k + 1) = 0 \qquad$ or $a_0(k-1)(2k-1) = 0$

Since $a_0 \neq 0$, we have $k = 1$ and $k = 1/2$.

(*The roots donot differ by an integer.*)

Next, we shall equate the coefficient of x^{k+1} to zero.

ie., $\quad 2a_1(k+1)k - a_1(k+1) + a_1 = 0$

ie., $\quad a_1(2k^2 + k) = 0$ or $a_1 k(2k+1) = 0 \Rightarrow a_1 = 0, k = 0, k = -1/2$

Rejecting $k = 0$ and $-1/2$ (*since we already have* $k = 1$ *and* $1/2$ *) we must have* $a_1 = 0$.

Now, we shall equate the coefficient of x^{k+r} $(r \geq 2)$ to zero.

ie., $\quad 2a_r(k+r)(k+r-1) - a_r(k+r) + a_r - a_{r-2} = 0$

ie., $\quad 2a_r(k+r)(k+r-1) - a_r(k+r-1) - a_{r-2} = 0$

ie., $\quad a_r(k+r-1)[2k+2r-1] = a_{r-2}$

or $$a_r = \frac{a_{r-2}}{(k+r-1)(2k+2r-1)} \quad (r \geq 2) \qquad \ldots (3)$$

Case - (1) : Let $k = 1$

$$\therefore \quad a_r = \frac{a_{r-2}}{r(2r+1)} \quad (r \geq 2)$$

Putting $r = 2, 3, 4, \ldots$ in this relation we obtain,

$$a_2 = \frac{a_0}{2\cdot 5}\ ;\ a_3 = \frac{a_1}{3\cdot 7} = 0\ ;\ a_4 = \frac{a_2}{4\cdot 9} = \frac{a_0}{2\cdot 4\cdot 5\cdot 9}\ ;\ a_5 = 0$$

$$a_6 = \frac{a_4}{6\cdot 13} = \frac{a_0}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13} \quad \text{and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = x^k(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

ie., $$y = a_0 x^k\left[1 + \frac{x^2}{2.5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} + \frac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13} + \cdots\right]$$

Putting $k = 1$, we have

$$y_1 = a_0 x\left[1 + \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} + \frac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13} + \cdots\right] \qquad \ldots (4)$$

Case-(2) : Let $k = 1/2$ and hence (3) assumes the form

$$a_r = \frac{a_{r-2}}{(r-1/2)2r} = \frac{a_{r-2}}{r(2r-1)} \quad (r \geq 2)$$

Putting $r = 2, 3, 4, \ldots$ we obtain

$$a_2 = \frac{a_0}{2\cdot 3},\quad a_3 = 0 = a_5 = \cdots$$

$$a_4 = \frac{a_2}{4\cdot 7} = \frac{a_0}{2\cdot 4\cdot 3\cdot 7};\quad a_6 = \frac{a_4}{6\cdot 11} = \frac{a_0}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11} \quad \text{and so-on}$$

We substitute these values in the expanded form of (2) to obtain,

$$y = a_0 x^k\left[1 + \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 4\cdot 3\cdot 7} + \frac{x^6}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11} + \cdots\right]$$

Putting $k = 1/2$ we have,

$$y_2 = a_0\sqrt{x}\left[1 + \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 4\cdot 3\cdot 7} + \frac{x^6}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11} + \cdots\right] \qquad \ldots (5)$$

The complete solution of (1) is given by

$$y = Ay_1 + By_2$$

Let us denote $Aa_0 = c_1$ and $Ba_0 = c_2$

Thus the required series solution is given by

$$y = c_1 x\left[1+\frac{x^2}{2\cdot 5}+\frac{x^4}{2\cdot 4\cdot 5\cdot 9}+\frac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}+\cdots\right]$$
$$+ c_2\sqrt{x}\left[1+\frac{x^2}{2\cdot 3}+\frac{x^4}{2\cdot 4\cdot 3\cdot 7}+\frac{x^6}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11}+\cdots\right]$$

23. *Obtain the series solution of the equation* $4xy''+2(1-x)y'-y=0$

>> $4xy''+(2-2x)y'-y=0$ ... (1)

The coefficient of $y'' = 4x = P_0(x)$ and $P_0(x)=0$ at $x=0$.

Let $y = \sum_{r=0}^{\infty} a_r x^{k+r}$ ... (2)

be the series solution of (1).

$$\therefore \quad y' = \sum_0^{\infty} a_r(k+r)x^{k+r-1},\quad y'' = \sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Now (1) becomes,

$$4\sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r-1}+2\sum_0^{\infty} a_r(k+r)x^{k+r-1}$$
$$-2\sum_0^{\infty} a_r(k+r)x^{k+r}-\sum_0^{\infty} a_r x^{k+r}=0$$

We shall equate the coefficient of the lowest degreee term in x, that is x^{k-1} to zero.

ie., $4a_0k(k-1)+2a_0k=0$

ie., $2a_0k(2k-2+1)=0$ or $a_0k(2k-1)=0$

Since $a_0 \neq 0$ we have $k=0$ and $k=1/2$

Next, we shall equate the coefficient of x^{k+r} $(r\geq 0)$ to zero.

ie., $4a_{r+1}(k+r+1)(k+r)+2a_{r+1}(k+r+1)-2a_r(k+r)-a_r=0$

ie., $2a_{r+1}(k+r+1)[2(k+r)+1]-a_r[2(k+r)+1]=0$

ie., $2a_{r+1}(k+r+1)-a_r=0$

or $a_{r+1} = \dfrac{a_r}{2(k+r+1)}$ $(r\geq 0)$ ...(3)

Case-(1): Let $k=0$

$$\therefore \quad a_{r+1} = \frac{a_r}{2(r+1)} \quad (r\geq 0)$$

Putting $r = 0, 1, 2, 3, 4, \ldots$ we obtain,

$$a_1=\frac{a_0}{2};\ a_2=\frac{a_1}{4}=\frac{a_0}{8};\ a_3=\frac{a_2}{2\cdot 3}=\frac{a_0}{48};\ a_4=\frac{a_3}{2\cdot 4}=\frac{a_0}{384}\ \text{and so-on}$$

We substitute these values in the expanded form of (2):

$$y = x^k(a_0+a_1x+a_2x^2+a_3x^3+\cdots)$$

ie., $$y = a_0x^k\left[1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\cdots\right]$$

Putting $k=0$, we have

$$y_1 = a_0\left[1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\cdots\right]$$

Case-(2): Let $k=1/2$ and hence (3) assumes the form

$$a_{r+1}=\frac{a_r}{2(r+3/2)}=\frac{a_r}{2r+3}\quad(r\geq 0)$$

Putting $r = 0, 1, 2, 3, \ldots$ we obtain,

$$a_1=\frac{a_0}{3};\ a_2=\frac{a_1}{5}=\frac{a_0}{15};\ a_3=\frac{a_2}{7}=\frac{a_0}{105};\ a_4=\frac{a_3}{9}=\frac{a_0}{945}\ \text{and so-on.}$$

We substitute these values in the expanded form of (2) to obtain,

$$y = a_0x^k\left[1+\frac{x}{3}+\frac{x^2}{15}+\frac{x^3}{105}+\frac{x^4}{945}+\cdots\right]$$

Putting $k=1/2$ we have,

$$y_2 = a_0\sqrt{x}\left[1+\frac{x}{3}+\frac{x^2}{15}+\frac{x^3}{105}+\frac{x^4}{945}+\cdots\right]$$

The complete solution of (1) is given by

$$y = Ay_1+By_2$$

Let us denote $Aa_0 = c_1$ and $Ba_0 = c_2$

Thus the required series solution is given by

$$y = c_1\left[1+\frac{x}{2\cdot 1!}+\frac{x^2}{2^2\cdot 2!}+\frac{x^3}{2^3\cdot 3!}+\frac{x^4}{2^4\cdot 4!}+\cdots\right]$$
$$+\, c_2\sqrt{x}\left[1+\frac{x}{3}+\frac{x^2}{3\cdot 5}+\frac{x^3}{3\cdot 5\cdot 7}+\frac{x^4}{3\cdot 5\cdot 7\cdot 9}+\cdots\right]$$

EXERCISES

1. Obtain the series solution of the equation $\frac{dy}{dx} = ky$ and verify with the analytical solution.
2. Solve in series the equation : $(1-x^2)y' - y = 0$
3. Obtain the power series solution of the equation : $(1-x^2)y'' - 2xy' + 2y = 0$
4. Use Frobenius method to solve the equation $3xy'' + 2y' + y = 0$
5. Obtain the power series solution of the equation $16x^2y'' + 16xy' + (16x^2 - 1)y = 0$ by Frobenious method.

ANSWERS

1. $y = a_0\left[1+\frac{kx}{1!}+\frac{k^2x^2}{2!}+\frac{k^3x^3}{3!}+\cdots\right]; \; y = a_0 e^{kx}$
2. $y = a_0\left[1+x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{3x^4}{8}+\frac{11x^5}{40}+\cdots\right]$
3. $y = a_0\left[1-x^2-\frac{x^4}{3}-\frac{x^6}{5}-\frac{x^8}{7}-\cdots\right]+a_1x$
4. $y = c_1\left[1+x+\frac{x^2}{4}+\frac{x^3}{4.7}+\cdots\right] = c_2x^{2/3}\left[1+\frac{x}{3}+\frac{x^2}{3.6}+\frac{x^3}{3.6.9}+\cdots\right]$
5. $y = c_1c^{1/4}\left[1-\frac{x^2}{5}+\frac{x^4}{90}-\cdots\right]+c_2x^{-1/4}\left[1-\frac{x^2}{3}+\frac{x^4}{42}-\cdots\right]$

2.25 Series Solution of Bessel's Differential Equation Leading to Bessel Functions

The Bessel differential equation of order n is in the form,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \qquad \ldots(1)$$

where n is a non negative real constant. (*parameter*)
We employ Frobenius method to solve this equation as we have,
coefficient of $y'' = x^2 = P_0(x)$ (*say*) and $P_0(x) = 0$ at $x = 0$.

We assume the series solution of (1) in the form

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \qquad \ldots(2)$$

$$\therefore \quad \frac{dy}{dx} = \sum_{0}^{\infty} a_r(k+r)x^{k+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Now (1) becomes,

$$\sum_0^\infty a_r(k+r)(k+r-1)x^{k+r} + \sum_0^\infty a_r(k+r)x^{k+r} + \sum_0^\infty a_r x^{k+r+2} - n^2\sum_0^\infty a_r x^{k+r} = 0$$

Collecting the first, second and fourth terms together we have,

$$\sum_0^\infty a_r x^{k+r}[(k+r)(k+r-1)+(k+r)-n^2] + \sum_0^\infty a_r x^{k+r+2} = 0$$

i.e., $$\sum_0^\infty a_r x^{k+r}[(k+r)\{\overline{k+r-1}+1\}-n^2] + \sum_0^\infty a_r x^{k+r+2} = 0$$

i.e., $$\sum_0^\infty a_r x^{k+r}[(k+r)^2-n^2] + \sum_0^\infty a_r x^{k+r+2} = 0$$

We shall equate the coefficient of the lowest degree term in x, that is x^k to zero.

i.e., $a_0(k^2-n^2) = 0$.

Setting $a_0 \neq 0$ we have $k^2 - n^2 = 0$ and hence $k = \pm n$

Also we need to independently equate the coefficient of x^{k+1} to zero.

i.e., $a_1[(k+1)^2 - n^2] = 0$.

This implies $a_1 = 0$ since $(k+1)^2 - n^2 = 0$ would mean $(k+1)^2 = n^2$ or $(k+1) = \pm n$ which cannot be accepted as we have already $k = \pm n$.

Next, we shall equate the coefficient of x^{k+r} $(r \geq 2)$ to zero.

i.e., $a_r[(k+r)^2 - n^2] + a_{r-2} = 0$

or $$a_r = \frac{-a_{r-2}}{[(k+r)^2 - n^2]} \quad (r \geq 2) \qquad \ldots (3)$$

When $k = +n$, (3) becomes,

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} = \frac{-a_{r-2}}{2nr + r^2}$$

Putting $r = 2, 3, 4, \ldots$ we obtain,

$$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)} \; ; \quad a_3 = \frac{-a_1}{6n+9} = 0 \text{ since } a_1 = 0.$$

Similarly $a_5, a_7, \ldots$ are all equal to zero.

i.e., $a_1 = 0 = a_3 = a_5 = a_7 = \ldots$

Next, $$a_4 = \frac{-a_2}{8n+16} = \frac{-a_2}{8(n+2)} = \frac{a_0}{32(n+1)(n+2)} \text{ and so on.}$$

We substitute these values in the expanded form of (2):

$$y = x^k(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots)$$

Also let the solution for $k = +n$ be denoted by y_1.

$$\therefore \quad y_1 = x^n \left[a_0 - \frac{a_0}{4(n+1)} x^2 + \frac{a_0}{32(n+1)(n+2)} x^4 - \ldots \right]$$

i.e., $$y_1 = a_0 x^n \left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \ldots \right] \qquad \ldots (4)$$

Since we also have $k = -n$, let the solution for $k = -n$ be denoted by y_2.

Replacing n by $-n$ in (4) we have,

$$y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \ldots \right] \qquad \ldots (5)$$

The complete (*general*) solution of (1) is given by

$y = A y_1 + B y_2$, where A, B are arbitrary constants.

We shall now standardize the solution as in (4) by choosing

$$a_0 = \frac{1}{2^n \Gamma(n+1)}$$ and the same be denoted by Y_1.

$$Y_1 = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2) \cdot 2} - \ldots \right]$$

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma(n+1) \cdot 2} - \ldots \right]$$

We have a property of gamma functions,

$\Gamma(n) = (n-1)\Gamma(n-1)$

$\therefore$ $\Gamma(n+2) = (n+1)\Gamma(n+1)$ and

$\Gamma(n+3) = (n+2)\Gamma(n+2) = (n+2)(n+1)\Gamma(n+1)$

As a consequence of these results we now have,

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(n+3) \cdot 2} - \ldots \right]$$

This can further be put in the form

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\Gamma(n+1) \cdot 0!} \left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2) \cdot 1!} \left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3) \cdot 2!} \left(\frac{x}{2}\right)^4 + \ldots \right]$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1) \cdot r!} \left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

This function is called the ***Bessel function of the first kind of order n*** de... by $J_n(x)$.

Thus $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!}$ ...(6)

Further, the solution for $k = -n$, (*in respect of* y_2) be denoted by $J_{-n}(x)$.

Hence the general solution of the Bessel's equation is given by

$$y = aJ_n(x) + bJ_{-n}(x)$$

where a and b are arbitrary constants and n is not an integer.

(*See the Remark in the next article regarding n not to be an integer.*)

- **Equation reducible to the form of Bessel's equation and solution**

Consider the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0 \quad \text{...(1)}$$

We shall show that this equation is reducible to the form of Bessel equation.

Putting $t = \lambda x$ we have,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \lambda\frac{dy}{dt}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\lambda\frac{dy}{dt}\right) = \frac{d}{dt}\left(\lambda\frac{dy}{dt}\right)\frac{dt}{dx} = \lambda^2\frac{d^2 y}{dt^2}$$

Substituting these results along with $x = t/\lambda$ in (1) we obtain,

$$\frac{t^2}{\lambda^2}\cdot\lambda^2\frac{d^2 y}{dt^2} + \frac{t}{\lambda}\lambda\frac{dy}{dt} + (t^2 - n^2)y = 0$$

i.e., $$t^2\frac{d^2 y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$$

This is in the form of Bessel differential equation whose solution is given by $y = aJ_n(t) + bJ_{-n}(t)$.

Thus $y = aJ_n(\lambda x) + bJ_{-n}(\lambda x)$ is the solution of equation (1).

- **Properties of Bessel functions**

Property 1. $J_{-n}(x) = (-1)^n J_n(x)$ *where n is a positive integer.*

Proof : By the definition of Bessel function we have,

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!} \quad \text{...(1)}$$

$$\therefore \quad J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\frac{1}{\Gamma(-n+r+1)\,r!} \quad \text{...(2)}$$

In (2) $\Gamma(-n+r+1) = \Gamma[r-(n-1)]$ is of the form $\Gamma(-k)$ for $r = 0, 1, 2, \cdots (n-2)$ and $\Gamma(0)$ for $r = (n-1)$.

Noting that $\Gamma(-k) \to \infty$ or $\frac{1}{\Gamma(-k)} \to 0$, k being a positive integer we can say that $\frac{1}{\Gamma[r-(n-1)]} \to 0$ for $r = 0, 1, 2 \cdots (n-1)$.

Hence, $$J_{-n}(x) = \sum_{r=n}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\frac{1}{\Gamma(-n+r+1)\,r!} \quad \text{...(3)}$$

Let $r - n = s$ or $r = s + n$ so that we have when $r = n$, $s = 0$.

Now (3) assumes the form

$$J_{-n}(x) = \sum_{s=0}^{\infty}(-1)^{s+n}\left(\frac{x}{2}\right)^{-n+2s+2n}\frac{1}{\Gamma(s+1)(s+n)!}$$

$$= \sum_{s=0}^{\infty}(-1)^{s+n}\left(\frac{x}{2}\right)^{n+2s}\frac{1}{\Gamma(s+1)(s+n)!}$$

Using the properties of gamma function we can write $\Gamma(s+1) = s!$ and $(s+n)! = \Gamma(s+n+1)$

$$\therefore \quad J_{-n}(x) = \sum_{s=0}^{\infty}(-1)^{s+n}\left(\frac{x}{2}\right)^{n+2s}\frac{1}{s!\,\Gamma(s+n+1)}$$

$$= (-1)^n\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{n+2s}\frac{1}{\Gamma(n+s+1)\,s!}$$

Comparing with (1) we observe that the summation in the RHS is $J_n(x)$.

Thus we have proved that $J_{-n}(x) = (-1)^n J_n(x)$, n being a positive integer.

Remark : *From this property we can easily conclude that $J_{-n}(x)$ and $J_n(x)$ are not linearly independent when n is an integer. Hence the general solution of the Bessel's differential equation is $y = a\,J_n(x) + b\,J_{-n}(x)$ when n is not an integer. Equivalently, we can say that $J_n(x)$ and $J_{-n}(x)$ are linearly independent solutions of the Bessel's equation when n is not an integer.*

Property 2. $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$ *where n is a positive integer.*

Proof : We have $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$

$$\therefore \quad J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \left(-\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$= \sum_{r=0}^{\infty} (-1)^r (-1)^{n+2r} \frac{x^{n+2r}}{2^{n+2r}} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} \{(-1)^3\}^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

Thus $\quad J_n(-x) = (-1)^n J_n(x)$

Since $\quad (-1)^n J_n(x) = J_{-n}(x)$ we have,

$$J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$$

- **Recurrence relations / Recurrence formulae**

We derive recurrence relations relating to Bessel function of different orders from the basic definition of $J_n(x)$.

1. $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

Proof : We have by the definition

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$\therefore \quad 2nJ_n(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2n}{\Gamma(n+r+1)\,r!}$$

We shall write $2n = 2(n+r) - 2r$ and split the summation into two terms

i.e., $\quad 2nJ_n(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2(n+r)}{\Gamma(n+r+1)\,r!} - \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2r}{\Gamma(n+r+1)\,r!}$

$$= \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2(n+r)}{(n+r)\,\Gamma(n+r)\,r!} - \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2r}{\Gamma(n+r+1)\,r\,(r-1)!}$$

$$= \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{2}{\Gamma(n+r)\,r!} - \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{2}{\Gamma(n+r+1)(r-1)!}$$

$$= x\sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\,r!} - x\sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)(r-1)!}$$

Putting $r-1 = s$ or $r = s+1$ in the second term of the RHS we obtain,

$$2nJ_n(x) = x\sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\,r!} - x\sum_{s=0}^{\infty} (-1)^{s+1} \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\,s!}$$

$$= x\sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\,r!} + x\sum_{0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\,s!}$$

Thus $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

2. $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$

Differentiating w.r.t. x, we have,

$$J_n'(x) = \sum_{0}^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2} \frac{1}{\Gamma(n+r+1)\,r!}$$

We shall write $n+2r = (n+r)+r$ and split the summation into two terms.

$$2J_n'(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{(n+r)}{(n+r)\,\Gamma(n+r)\,r!} + \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{r}{\Gamma(n+r+1)\,r\cdot(r-1)!}$$

$$2J_n'(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n+r)\,r!} + \sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)\,(r-1)!}$$

Putting $(r-1) = s$ in the second term of the RHS we have,

$$2J_n'(x) = \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\,r!} - \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\,s!}$$

Thus $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

3. $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$

$$\therefore \quad x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r x^{2n+2r} \frac{1}{2^{n+2r}\,\Gamma(n+r+1)\,r!}$$

Now, $$\frac{d}{dx}[x^n J_n(x)] = \sum_{0}^{\infty} (-1)^r (2n+2r)\, x^{2n+2r-1} \frac{1}{2^{n+2r}\,(n+r)\,\Gamma(n+r)\,r!}$$

$$= \sum_{0}^{\infty} (-1)^r 2(n+r) x^{2n+2r-1} \frac{1}{2^{n+2r}(n+r)\,\Gamma(n+r)\,r!}$$

$$= \sum_{0}^{\infty} (-1)^r \cdot x^n x^{n+2r-1} \cdot \frac{1}{2^{n+2r-1}\,\Gamma(n+r)\,r!}$$

$$= x^n \sum_{0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\,r!}$$

$$= x^n J_{n-1}(x)$$

Thus $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

4. $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$

$$\therefore \quad x^{-n} J_n(x) = \sum_{0}^{\infty} (-1)^r x^{2r} \frac{1}{2^{n+2r}\,\Gamma(n+r+1)\,r!}$$

Now, $$\frac{d}{dx}[x^{-n} J_n(x)] = \sum_{0}^{\infty} (-1)^r 2r x^{2r-1} \cdot \frac{1}{2^{n+2r}\,\Gamma(n+r+1)\,r\cdot(r-1)!}$$

$$= \sum_{r=1}^{\infty} (-1)^r x^{-n} x^{n+2r-1} \frac{1}{2^{n+2r-1}\,\Gamma(n+r+1)\,(r-1)!}$$

Putting $r-1 = s$ or $r = s+1$ we have,

$$\frac{d}{dx}[x^{-n} J_n(x)] = \sum_{s=0}^{\infty} (-1)^{s+1} x^{-n} x^{n+2s+1} \frac{1}{2^{n+2s+1}\,\Gamma(n+1+s+1)\,s!}$$

$$= -x^{-n} \sum_{0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\,s!}$$

Thus $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

5. $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

Proof : $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$\therefore \quad J_n'(x) = \sum_0^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2} \frac{1}{\Gamma(n+r+1)\, r!}$$

Hence $xJ_n'(x) = \sum_0^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

We shall write $n+2r = 2(n+r) - n$ and split the summation into two terms.

$$xJ_n'(x) = \sum_0^{\infty} (-1)^r 2(n+r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{(n+r)\Gamma(n+r)\, r!} + \sum_0^{\infty} (-1)^r (-n) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$= \sum_0^{\infty} (-1)^r \cdot 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r)\, r!} - n\sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$= x\sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \cdot \frac{1}{\Gamma(n-1+r+1)\, r!} - nJ_n(x)$$

Thus $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

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6. $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

Proof : $J_n(x) = \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$\therefore \quad J_n'(x) = \sum_0^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2} \frac{1}{\Gamma(n+r+1)\, r!}$$

Hence $xJ_n'(x) = \sum_0^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$xJ_n'(x) = n\sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} + \sum_0^{\infty} (-1)^r 2r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r \cdot (r-1)!}$$

$$= nJ_n(x) + \sum_{r=1}^{\infty} (-1)^r \cdot 2\left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)(r-1)!}$$

Putting $r - 1 = s$ or $r = s + 1$ in the second term we have,

$$xJ_n'(x) = nJ_n(x) + \sum_{s=0}^{\infty} (-1)^{s+1} \cdot 2\left(\frac{x}{2}\right)^{n+2s+2} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$= nJ_n(x) - \sum_{s=0}^{\infty} (-1)^s \cdot 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2s+1} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$= nJ_n(x) - x\sum_0^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

Thus $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

WORKED PROBLEMS

24. *Prove that*

(a) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ (b) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

>> By the definition,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \quad \ldots(1)$$

Putting $n = 1/2$ in (1) we have,

$$J_{1/2}(x) = \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{1/2+2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

$$= \sqrt{\frac{x}{2}} \sum_0^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{\Gamma(r+3/2)\, r!}$$

On expanding we have,

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(5/2)\,1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(7/2)\,2!} - \cdots\right] \quad \ldots (2)$$

We know that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(n) = (n-1)\Gamma(n-1)$

Putting $n = 3/2,\ 5/2,\ 7/2 \ldots$ we get the following values.

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}\ ;\ \Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma(7/2) = \frac{5}{2}\Gamma(5/2) = \frac{15\sqrt{\pi}}{8}$$

Substituting these values in the RHS of (2) we have,

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{8}{15\sqrt{\pi}\cdot 2} - \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\left[2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\cdot\frac{2}{x}\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right]$$

(We have taken $2/x$ as a common factor keeping in view of the desired result)

ie., $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

Thus $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

Also by putting $n = -1/2$ in (1) we have,

$$J_{-1/2}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-1/2+2r}\frac{1}{\Gamma(r+1/2)\,r!}$$

$$= \left(\frac{x}{2}\right)^{-1/2}\sum_{0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{1}{\Gamma(r+1/2)\,r!}$$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma(1/2)} - \left(\frac{x}{2}\right)^2\frac{1}{\Gamma(3/2)\,1!} + \left(\frac{x}{2}\right)^4\frac{1}{\Gamma(5/2)\,2!} - \cdots\right]$$

Using the computed values of $\Gamma(3/2)$, $\Gamma(5/2)$ along with the value of $\Gamma(1/2)$ we have,

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{2}{\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{4}{3\sqrt{\pi}\cdot 2} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

Thus $$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

25. *Obtain expressions for $J_{1/2}(x)$ and $J_{-1/2}(x)$. Then use a suitable recurrence relation to deduce expressions for $J_{3/2}(x)$ and $J_{-3/2}(x)$*

>> We have already obtained the following results.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x\ ;\ J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Let us consider the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \quad \ldots (1)$$

Putting $n = 1/2$ in this relation we have,

$$J_{-1/2}(x) + J_{3/2}(x) = \frac{1}{x}J_{1/2}(x)$$

$$\therefore\quad J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

Thus $$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x - x\cos x}{x}\right]$$

Also by putting $n = -1/2$ in (1) we have,

$$J_{-3/2}(x) + J_{1/2}(x) = \frac{-1}{x}J_{-1/2}(x)$$

$$\therefore\quad J_{-3/2}(x) = -\left[J_{1/2}(x) + \frac{1}{x}J_{-1/2}(x)\right]$$

$$J_{-3/2}(x) = -\left[\sqrt{\frac{2}{\pi x}}\sin x + \frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x\right]$$

Thus $$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{x\sin x + \cos x}{x}\right]$$

---

26. *Show that* $J_{3/2}(x)\sin x - J_{-3/2}(x)\cos x = \sqrt{2/\pi x^3}$

>> **Note** : *Assuming the expression for* $J_{1/2}(x)$ *and* $J_{-1/2}(x)$ *we have to first establish expressions for* $J_{3/2}(x)$ *and* $J_{-3/2}(x)$ *as in Problem-25.*

Now $J_{3/2}(x)\sin x - J_{-3/2}(x)\cos x$

$$= \sqrt{\frac{2}{\pi x}}\cdot\frac{1}{x}\left[\sin^2 x - x\sin x\cos x + x\sin x\cos x + \cos^2 x\right]$$

$$= \sqrt{\frac{2}{\pi x}}\cdot\frac{1}{x}\cdot 1 = \sqrt{\frac{2}{\pi x^3}}$$

---

27. *Prove the following results.*

(a) $$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}.\cos x\right]$$

(b) $$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right]$$

>> [*Note of Problem-26 continue to hold good for this problem also*]

We consider the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

Putting $n = 3/2$ in this relation we have,

$$J_{1/2}(x) + J_{5/2}(x) = \frac{3}{x}J_{3/2}(x)$$

$$\therefore \quad J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$$

i.e., $$J_{5/2}(x) = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x - x\cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x - 3x\cos x - x^2\sin x}{x^2}\right]$$

Thus $$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{(3-x^2)\sin x}{x^2} - \frac{3\cos x}{x}\right]$$

Also by putting $n = -3/2$ in the recurrence relation we have,

$$J_{-5/2}(x) + J_{-1/2}(x) = -\frac{3}{x}J_{-3/2}(x)$$

$$\therefore \quad J_{-5/2}(x) = \frac{-3}{x}J_{-3/2}(x) - J_{-1/2}(x)$$

$$J_{-5/2}(x) = \frac{-3}{x}\cdot -\sqrt{\frac{2}{\pi x}}\left[\frac{x\sin x + \cos x}{x}\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3x\sin x + 3\cos x - x^2\cos x}{x^2}\right]$$

Thus $$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x} + \frac{3-x^2}{x^2}\cos x\right]$$

---

28. *Starting from the expressions of* $J_{1/2}(x)$ *and* $J_{-1/2}(x)$ *in the standard form the following results.*

(a) $$J'_{1/2}(x)J_{-1/2}(x) - J'_{-1/2}(x)J_{1/2}(x) = \frac{2}{\pi x}$$

(b) $$\int_0^{\pi/2}\sqrt{x}\,J_{1/2}(2x)\,dx = \frac{1}{\sqrt{\pi}}$$

>> (a) We have results,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Differentiating these results w.r.t. x we obtain,

$$J_{1/2}'(x) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{\sqrt{x}}\cos x + \sin x \cdot -\frac{1}{2}x^{-3/2}\right]$$

i.e., $$J_{1/2}'(x) = \sqrt{\frac{2}{\pi}}\left[\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x\sqrt{x}}\right]$$

$$J_{1/2}'(x) = \sqrt{\frac{2}{\pi x}}\left(\cos x - \frac{\sin x}{2x}\right)$$

Also $$J_{-1/2}'(x) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{\sqrt{x}}(-\sin x) + \cos x \cdot \frac{-1}{2}x^{-3/2}\right]$$

i.e., $$J_{-1/2}'(x) = \sqrt{\frac{2}{\pi}}\left[-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{2x\sqrt{x}}\right]$$

$$J_{-1/2}'(x) = -\sqrt{\frac{2}{\pi x}}\left[\sin x + \frac{\cos x}{2x}\right]$$

Consider $J_{1/2}'(x)\, J_{-1/2}(x) - J_{-1/2}'(x)\, J_{1/2}(x)$

$$= \sqrt{\frac{2}{\pi x}}\left[\cos x - \frac{\sin x}{2x}\right] \cdot \sqrt{\frac{2}{\pi x}}\cos x + \sqrt{\frac{2}{\pi x}}\left[\sin x + \frac{\cos x}{2x}\right] \cdot \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \frac{2}{\pi x}(\cos^2 x + \sin^2 x) = \frac{2}{\pi x} \quad \textit{(other terms cancel out)}$$

This proves the required result.

(b) We also have, $J_{1/2}(2x) = \sqrt{\dfrac{2}{\pi(2x)}}\sin 2x = \dfrac{1}{\sqrt{\pi x}}\sin 2x$

$\therefore$ $$\sqrt{x}\, J_{1/2}(2x) = \sqrt{x} \cdot \frac{1}{\sqrt{\pi x}}\sin 2x = \frac{\sin 2x}{\sqrt{\pi}}$$

Now $$\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(2x)\, dx = \int_0^{\pi/2} \frac{\sin 2x}{\sqrt{\pi}}\, dx$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{-\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{-1}{2\sqrt{\pi}}(\cos \pi - \cos 0) = \frac{-1}{2\sqrt{\pi}}(-2) = \frac{1}{\sqrt{\pi}}$$

Thus $$\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(2x)\, dx = \frac{1}{\sqrt{\pi}}.$$

29. *Starting from the series expression for $J_n(x)$ prove that*

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x}J_n(x). \text{ Hence deduce that}$$

$$J_4(x) = \frac{8}{x}\left(\frac{6}{x^2} - 1\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$$

>> *We have to derive the recurrence relation (1) as in the article 5.8.*

That is $J_{n+1}(x) + J_{n-1}(x) = \dfrac{2n}{x}J_n(x)$

or $$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \quad \ldots (1)$$

Putting $n = 3$ we obtain,

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) \quad \ldots (2)$$

We have to put $n = 1, 2$ in (1) to obtain $J_2(x)$ and $J_3(x)$ respectively.

$\therefore$ $$J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \text{ and}$$

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x)$$

i.e., $$J_3(x) = \frac{4}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x)$$

or $$J_3(x) = \frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x)$$

Using these results in the RHS of (2) we obtain

$$J_4(x) = \frac{6}{x}\left[\frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x)\right] - \left[\frac{2}{x}J_1(x) - J_0(x)\right]$$

$$J_4(x) = J_1(x)\left[\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right] + J_0(x)\left[1 - \frac{24}{x^2}\right]$$

Thus $$J_4(x) = \frac{8}{x}\left(\frac{6}{x^2} - 1\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$$

30. *Starting from the definition of $J_n(x)$ prove that*

(a) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

(b) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Hence deduce that

(c) $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

(d) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

Further deduce that

(e) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

(f) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

>> Proving *(a)* and *(b)* is obtaining the relations (3) and (4) as in the article 5.43.

Applying product rule in the LHS of (a) and (b) we obtain

$$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

and $$x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

Dividing these two equations respectively by x^n and x^{-n} we have,

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \qquad \ldots (1)$$

and $$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x) \qquad \ldots (2)$$

Multiplying both these equations by x, we get

$$\boldsymbol{x J_n'(x) = x J_{n-1}(x) - n J_n(x)}$$

and $$\boldsymbol{x J_n'(x) = n J_n(x) - x J_{n+1}(x)}$$

This proves the second pair of the desired relations.

Further adding and subtracting (1) and (2) we obtain

$$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

and $$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

or $$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

and $$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

This proves the third pair of the desired relations.

31. *Show that*

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[n J_n^2 - (n+1) J_{n+1}^2\right]$$

>> Consider LHS

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2 J_n(x) J_n'(x) + 2 J_{n+1}(x) J_{n+1}'(x)$$

We have recurrence relations

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x) \quad \text{or}$$

$$x J_{n+1}'(x) = x J_n(x) - (n+1) J_{n+1}(x)$$

Let us consider the RHS of (1) and use (2) for $J_n'(x)$ and (3) for $J_{n+1}'(x)$

Hence $$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2 J_n\left[\frac{n}{x} J_n - J_{n+1}\right] + 2 J_{n+1}\left[J_n - \frac{n+1}{x} J_{n+1}\right]$$

$$= \frac{2}{x} n J_n^2 - 2 J_n J_{n+1} + 2 J_{n+1} J_n - 2\frac{n+1}{x} J_{n+1}^2$$

$$= \frac{2}{x} n J_n^2 - 2\frac{(n+1)}{x} J_{n+1}^2$$

Thus we have proved that,

$$\boldsymbol{\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[n J_n^2 - (n+1) J_{n+1}^2\right]}$$

32. *Show that* $\frac{d}{dx}\left[x J_n J_{n+1}\right] = x\left[J_n^2 - J_{n+1}^2\right]$

>> $$\frac{d}{dx}\left[x J_n J_{n+1}\right] = x\left[J_n J_{n+1}' + J_{n+1} J_n'\right] + J_n J_{n+1}$$

by applying the product rule

We have recurrence relations

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \qquad \ldots(2)$$

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x) \qquad \ldots(3)$$

Replacing n by $(n+1)$, (3) becomes,

$$xJ_{n+1}'(x) = xJ_n(x) - (n+1)J_{n+1}(x) \qquad \ldots(4)$$

Substituting (4) and (2) in the RHS of (1) we get,

$$J_n[xJ_n - (n+1)J_{n+1}] + J_{n+1}[nJ_n - xJ_{n+1}] + J_nJ_{n+1}$$

$$= xJ_n^2 - nJ_nJ_{n+1} - J_nJ_{n+1} + nJ_nJ_{n+1} - xJ_{n+1}^2 + J_nJ_{n+1}$$

$$= x[J_n^2 - J_{n+1}^2]$$

Thus we have proved that

$$\frac{d}{dx}\left[xJ_nJ_{n+1}\right] = x\left[J_n^2 - J_{n+1}^2\right]$$

33. *Prove that* $J_0''(x) = \frac{1}{2}\left[J_2(x) - J_0(x)\right]$

>> We have the recurrence relation

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \qquad \ldots(1)$$

Putting $n = 0$, $J_0'(x) = \frac{1}{2}\left[J_{-1}(x) - J_1(x)\right]$

$$= \frac{1}{2}\left[-J_1(x) - J_1(x)\right] = -J_1(x)$$

i.e., $J_0'(x) = -J_1(x)$ and differentiating this w.r.t. x we get,

$$J_0''(x) = -J_1'(x)$$

Also from (1) when $n = 1$, $J_1'(x) = \frac{1}{2}\left[J_0(x) - J_2(x)\right]$

Hence $J_0''(x) = -\frac{1}{2}\left[J_0(x) - J_2(x)\right] = \frac{1}{2}[J_2(x) - J_0(x)]$

Thus we have proved that $J_0''(x) = \frac{1}{2}\left[J_2(x) - J_0(x)\right]$

34. *Starting from a suitable recurrence relation show that*

(a) $4J_n'' = J_{n-2} - 2J_n + J_{n+2}$

(b) $8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}$

>> We have the recurrence relation

$$2J_n' = J_{n-1} - J_{n+1} \qquad \ldots(1)$$

Differentiating w.r.t x we have,

$$2J_n'' = J_{n-1}' - J_{n+1}'.$$

Multiplying by 2 we obtain,

$$4J_n'' = 2J_{n-1}' - 2J_{n+1}'$$

We use (1) in the RHS of this equation replacing n by $(n-1)$ and also by $(n+1)$.

i.e., $4J_n'' = (J_{n-2} - J_n) - (J_n - J_{n+2})$

Thus $4J_n'' = J_{n-2} - 2J_n + J_{n+2}$ as required.

Differentiating this relation w.r.t. x again we have,

$$4J_n''' = J_{n-2}' - 2J_n' + J_{n+2}'$$

Multiplying by 2 we obtain,

$$8J_n''' = 2J_{n-2}' - 2(2J_n') + 2(J_{n+2}')$$

Now using (1) in the RHS by replacing n by $(n-2)$ and n by $(n+2)$ for the first and third terms along with (1) for the second term we obtain,

$$\therefore\ 8J_n''' = (J_{n-3} - J_{n-1}) - 2(J_{n-1} - J_{n+1}) + (J_{n+1} - J_{n+3})$$

Thus $8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}$ as required.

35. *Prove that* $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

>> We have established in the previous problem the result,

$$8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}$$

Putting $n = 0$ in this equation we have,

$$8J_0''' = J_{-3} - 3J_{-1} + 3J_1 - J_3$$

Using $J_{-n} = (-1)^n J_n$ this equation becomes

$$8J_0''' = -J_3 + 3J_1 + 3J_1 - J_3$$

i.e., $8J_0''' = 6J_1 - 2J_3$ or $4J_0''' = 3J_1 - J_3$

But $-J_1 = \dfrac{d}{dx}(J_0)$ or $J_1 = -J_0'$

Hence we have $4J_0''' = -3J_0' - J_3$

Thus $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

Note : *The result can be obtained independently also.*

36. *Show that* $\int J_3(x)\,dx = c - J_2(x) - \dfrac{2}{x}J_1(x)$

>> Consider the recurrence relation

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow \quad \int x^{-n}J_{n+1}(x)\,dx = -x^{-n}J_n(x) \qquad \ldots(1)$$

Let us write $J_3(x) = x^2\left[x^{-2}J_3(x)\right]$

$$\therefore \quad \int J_3(x)\,dx = \int x^2\left[x^{-2}J_3(x)\right]dx$$

Integrating RHS by parts we get

$$\int J_3(x)\,dx = x^2\int x^{-2}J_3(x)\,dx - \int\left[\int x^{-2}J_3(x)\right]2x\,dx \qquad \ldots(2)$$

From (1) we have when $n = 2$, $\int x^{-2}J_3(x)\,dx = -x^{-2}J_2(x)$

Using this in the RHS of (2) we get

$$\int J_3(x)\,dx = x^2\left\{-x^{-2}J_2(x)\right\} - \int -\left\{x^{-2}J_2(x)\right\}\cdot 2x\,dx$$

$$i.e., \quad \int J_3(x)\,dx = -J_2(x) + 2\int x^{-1}J_2(x)\,dx \qquad \ldots(3)$$

From (1) we have when $n = 1$, $\int x^{-1}J_2(x)\,dx = -x^{-1}J_1(x)$

Hence (3) becomes,

$$\int J_3(x)\,dx = -J_2(x) + 2\left\{-x^{-1}J_1(x)\right\}$$

Thus $\int J_3(x)\,dx = -J_2(x) - \dfrac{2}{x}J_1(x) + c$, c being the constant of integration.

37. *Show that* $\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x)$

>> We have the recurrence relation

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$$\Rightarrow \quad \int_0^x x^n J_{n-1}(x)\,dx = \left[x^n J_n(x)\right]_0^x = x^n J_n(x) - 0$$

Thus $\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x)$

38. *Show that* $\int_0^x x^{-n}J_{n+1}(x)\,dx = \dfrac{1}{2^n\,\Gamma(n+1)} - \dfrac{J_n(x)}{x^n}$

>> We have the recurrence relation

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow \quad \int_0^x x^{-n}J_{n+1}(x)\,dx = -\left[x^{-n}J_n(x)\right]_0^x$$

$$i.e., \quad \int_0^x x^{-n}J_{n+1}(x)\,dx = -x^{-n}J_n(x) + \left[x^{-n}J_n(x)\right]_{x=0}$$

The second term in the RHS is an indeterminate form 0/0 and hence we use the [concept] of limit for evaluating the same.

$$\lim_{x\to 0}\frac{J_n(x)}{x^n} = \lim_{x\to 0}\frac{1}{x^n}\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!}$$

$$= \lim_{x\to 0}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{2^{n+2r}}\frac{1}{\Gamma(n+r+1)\,r!}$$

$$\lim_{x \to 0} x^{-n} J_n(x) = \lim_{x \to 0} \left\{ \frac{1}{2^n \Gamma(n+1)} - \frac{x^2}{2^{n+2} \Gamma(n+2)} + \frac{x^4}{2^{n+4} \Gamma(n+3) 2!} \cdots \right\}$$

$$= \frac{1}{2^n \Gamma(n+1)} - 0 + 0 - \cdots = \frac{1}{2^n \Gamma(n+1)}$$

$$\therefore \quad \lim_{x \to 0} x^{-n} J_n(x) = \frac{1}{2^n \Gamma(n+1)}$$

Thus by using this result in (1) we have,

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \Gamma(n+1)} - \frac{J_n(x)}{x^n}$$

---

39. *Show that* $\int_0^x x J_0(ax)\, dx = \frac{x}{a} J_1(ax)$

>> We have the recurrence relation : $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Putting $n = 1$ we have, $\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x)$

Put $x = at$ so that we have

$$\frac{d}{dt}\left[at\, J_1(at)\right] \frac{dt}{dx} = at\, J_0(at)$$

i.e., $$\frac{d}{dt}\left[at\, J_1(at)\right] \frac{1}{a} = at\, J_0(at)$$

or $$\frac{d}{dt}\left[t J_1(at)\right] = at\, J_0(at)$$

$$\Rightarrow \quad \int_0^t t J_0(at)\, dt = \frac{1}{a}\left[t J_1(at)\right]_0^t = \frac{t}{a} J_1(at)$$

Now replacing the variable t by arbitrary variable x we have

$$\int_0^x x J_0(ax)\, dx = \frac{x}{a} J_1(ax)$$

---

40. *Show that* $\frac{d}{dx}\left[J_n^2(x)\right] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$

>> $$\frac{d}{dx}\left[J_n^2(x)\right] = 2 J_n(x) J_n'(x) \quad \ldots (1)$$

We have recurrence relations

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

and $$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Using these relations in the RHS of (1) we have,

$$\frac{d}{dx}[J_n^2(x)] = 2 \cdot \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \cdot \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Thus $$\frac{d}{dx}[J_n^2(x)] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

Prove the following :

41. $x J_n(x) = 2\left[(n+1) J_{n+1}(x) - (n+3) J_{n+3}(x) + (n+5) J_{n+5}(x) - \cdots\right]$

42. $x J_{n-1}(x) = 2\left[n J_n(x) - (n+2) J_{n+2}(x) + (n+4) J_{n+4}(x) - \cdots\right]$

43. $x J_n'(x) = n J_n(x) - 2\left[(n+2) J_{n+2}(x) - (n+4) J_{n+4}(x) + \cdots\right]$

41. >> We have the recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

or $$J_n(x) + J_{n+2}(x) = \frac{2(n+1)}{x} J_{n+1}(x)$$

$$\therefore \quad x J_n(x) = 2(n+1) J_{n+1}(x) - x J_{n+2}(x) \quad \ldots (1)$$

Also $x J_{n+2}(x) = 2(n+3) J_{n+3}(x) - x J_{n+4}(x)$

$x J_{n+4}(x) = 2(n+5) J_{n+5}(x) - x J_{n+6}(x)$ and so on.

By back substitution (1) becomes

$$x J_n(x) = 2(n+1) J_{n+1}(x) - 2(n+3) J_{n+3}(x) + 2(n+5) J_{n+5}(x) - \cdots$$

Thus $$x J_n(x) = 2\left[(n+1) J_{n+1}(x) - (n+3) J_{n+3}(x) + (n+5) J_{n+5}(x) - \cdots\right]$$

42. >> Again we have from the recurrence relation (1)

$$x J_{n-1}(x) = 2n J_n(x) - x J_{n+1}(x) \quad \ldots (2)$$

Also $x J_{n+1}(x) = 2(n+2) J_{n+2}(x) - x J_{n+3}(x)$

$x J_{n+3}(x) = 2(n+4) J_{n+4}(x) - x J_{n+5}(x)$ and so on.

By back substitution (2) becomes

$$x J_{n-1}(x) = 2n J_n(x) - 2(n+2) J_{n+2}(x) + 2(n+4) J_{n+4}(x) - \cdots$$

Thus $x J_{n-1}(x) = 2[n J_n(x) - (n+2) J_{n+2}(x) + (n+4) J_{n+4}(x) - \cdots]$

43. >> We have recurrence relations

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \ldots (3)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

or $J_{n+1}(x) + J_{n+3}(x) = \dfrac{2(n+2)}{x} J_{n+2}(x)$

or $x J_{n+1}(x) = 2(n+2) J_{n+2}(x) - x J_{n+3}(x) \quad \ldots (4)$

Also $x J_{n+3}(x) = 2(n+4) J_{n+4}(x) - x J_{n+5}(x)$

$x J_{n+5}(x) = 2(n+6) J_{n+6}(x) - x J_{n+7}(x)$ and so on.

By back substitution (3) becomes,

$$x J_n'(x) = n J_n(x) - 2(n+2) J_{n+2}(x) + 2(n+4) J_{n+4}(x) - \cdots$$

Thus $x J_n'(x) = n J_n(x) - 2[(n+2) J_{n+2}(x) - (n+4) J_{n+4}(x) + \cdots]$

44. *Verify that* $y = x^n J_n(x)$ *is a solution of the differential equation*

$$x y'' + (1-2n) y' + x y = 0$$

>> By data $y = x^n J_n(x)$

$\therefore \quad y' = x^n J_n'(x) + n x^{n-1} J_n(x)$

$$y'' = x^n J_n''(x) + 2n x^{n-1} J_n'(x) + n(n-1) x^{n-2} J_n(x)$$

Substituting in $xy'' + (1-2n)y' + xy$ we obtain

$$x\left\{x^n J_n''(x) + 2n x^{n-1} J_n'(x) + n(n-1) x^{n-2} J_n(x)\right\} + (1-2n)\left\{x^n J_n'(x) + n x^{n-1} J_n(x)\right\} + x \cdot x^n J_n(x)$$

$$= x^{n+1} J_n''(x) + 2n x^n J_n'(x) + n^2 x^{n-1} J_n(x) - n x^{n-1} J_n(x) + x^n J_n'(x) + n x^{n-1} J_n(x) - 2n x^n J_n'(x) - 2n^2 x^{n-1} J_n(x) + x^{n+1} J_n(x)$$

$$= x^{n+1} J_n''(x) + x^n J_n'(x) + x^{n+1} J_n(x) - n^2 x^{n-1} J_n(x)$$

$$= x^{n-1}\left\{x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x)\right\}$$

$$= x^{n-1} \cdot 0 = 0$$

since $J_n(x)$ is a solution of $x^2 y'' + xy' + (x^2 - n^2) y = 0$.

Thus we have proved that $y = x^n J_n(x)$ is a solution of the equation

$$x y'' + (1-2n) y' + x y = 0$$

45. *Verify that* $y = \sqrt{x}\, J_{3/2}(x)$ *is a solution of the differential equation given by*

$$x^2 y'' + (x^2 - 2) y = 0$$

>> By data $y = \sqrt{x}\, J_{3/2}(x)$

$\therefore \quad y' = \sqrt{x}\, J_{3/2}'(x) + J_{3/2}(x) \cdot \dfrac{1}{2\sqrt{x}}$

$$y'' = \sqrt{x}\, J_{3/2}''(x) + J_{3/2}'(x) \cdot \frac{1}{2\sqrt{x}} + J_{3/2}(x) \cdot \frac{1}{2} \cdot \frac{-1}{2} x^{-3/2} + \frac{1}{2\sqrt{x}} J_{3/2}'$$

i.e., $\quad y'' = \sqrt{x}\, J_{3/2}''(x) + \dfrac{J_{3/2}'(x)}{\sqrt{x}} - \dfrac{1}{4x\sqrt{x}} J_{3/2}(x)$

$\therefore \quad x^2 y'' + (x^2 - 2) y = \left\{x^{5/2} J_{3/2}''(x) + x^{3/2} J_{3/2}'(x) - \dfrac{\sqrt{x}}{4} J_{3/2}(x)\right\} + \left\{x^{5/2} J_{3/2}(x) - 2\sqrt{x}\, J_{3/2}(x)\right\}$

$$= \sqrt{x}\left[x^2 J_{3/2}''(x) + x J_{3/2}'(x) - \frac{1}{4} J_{3/2}(x) + x^2 J_{3/2}(x) - 2 J_{3/2}(x)\right]$$

$$= \sqrt{x}\left\{x^2 J_{3/2}''(x) + x J_{3/2}'(x) + \left[x^2 - (3/2)^2\right] J_{3/2}(x)\right\}$$

$$= \sqrt{x}\left[x^2u'' + xu' + (x^2-n^2)u\right] \text{ where } u = J_{3/2}(x) \text{ and } n = 3/2.$$

Since $J_{3/2}(x)$ is a solution of the Bessel's equation in the standard form, we have

$$x^2u'' + xu' + (x^2-n^2)u = 0$$

Thus $x^2y'' + (x^2-2)y = 0$ as required.

---

46. *Show by the substitution $u = 2n\sqrt{x}$ that the differential equation $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + n^2y = 0$ is transformed into Bessel's equation of order zero. Hence show that $y = AJ_0(2n\sqrt{x})$ is a general solution of the equation.*

>> We have, $u^2 = 4n^2x$ since $u = 2n\sqrt{x}$ by data.

Differentiating w.r.t. x we have,

$$2u\frac{du}{dx} = 4n^2 \text{ or } \frac{du}{dx} = \frac{2n^2}{u}$$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{dy}{du}\cdot\dfrac{2n^2}{u}$

Differentiating this again w.r.t. x we have,

$$\frac{d^2y}{dx^2} = 2n^2\left[\frac{dy}{du}\frac{-1}{u^2}\frac{du}{dx} + \frac{1}{u}\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]$$

$$= 2n^2\left[\frac{-2n^2}{u^3}\frac{dy}{du} + \frac{2n^2}{u^2}\frac{d^2y}{du^2}\right]$$

i.e., $\dfrac{d^2y}{dx^2} = \dfrac{4n^4}{u^2}\left[\dfrac{-1}{u}\dfrac{dy}{du} + \dfrac{d^2y}{du^2}\right]$

Consider $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + n^2y = 0$, where $x = \dfrac{u^2}{4n^2}$

i.e., $\dfrac{u^2}{4n^2}\cdot\dfrac{4n^4}{u^2}\left[\dfrac{-1}{u}\dfrac{dy}{du} + \dfrac{d^2y}{du^2}\right] + \dfrac{2n^2}{u}\dfrac{dy}{du} + n^2y = 0$

i.e., $\dfrac{-n^2}{u}\dfrac{dy}{du} + n^2\dfrac{d^2y}{du^2} + \dfrac{2n^2}{u}\dfrac{dy}{du} + n^2y = 0$

i.e., $n^2\left[\dfrac{d^2y}{du^2} + \dfrac{1}{u}\dfrac{dy}{du} + y\right] = 0$ or $\dfrac{d^2y}{du^2} + \dfrac{1}{u}\dfrac{dy}{du} + y = 0$

Multiplying by u^2 we get $u^2\dfrac{d^2y}{du^2} + u\dfrac{dy}{du} + u^2y = 0$

i.e., $u^2\dfrac{d^2y}{du^2} + u\dfrac{dy}{du} + (u^2 - 0^2)y = 0$

This is Bessel's equation of order zero and its general solution is

$$y = aJ_n(u) + bJ_{-n}(u) \text{ where } n = 0$$

i.e., $y = aJ_0(2n\sqrt{x}) + bJ_0(2n\sqrt{x}) = (a+b)J_0(2n\sqrt{x})$

Thus by denoting $A = a+b$, we conclude that

$y = AJ_0(2n\sqrt{x})$ is a general solution of the given equation.

---

47. *Prove that* $\dfrac{d}{dx}\left[x\left\{J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x)\right\}\right] = 0$

>> We know that $J_n(x)$ and $J_{-n}(x)$ are solutions of the Bessel's equation

$$x^2y'' + xy' + (x^2-n^2)y = 0$$

If $u = J_n(x)$ and $v = J_{-n}(x)$ we have,

$$x^2u'' + xu' + (x^2-n^2)u = 0 \quad \ldots(1)$$

and $x^2v'' + xv' + (x^2-n^2)v = 0 \quad \ldots(2)$

Multiplying (1) by v and (2) by u we have,

$$x^2vu'' + xvu' + x^2uv - n^2uv = 0$$

and $x^2v''u + xv'u + x^2uv - n^2uv = 0$

On subtracting and dividing by x we obtain,

$$x(vu'' - v''u) + (vu' - v'u) = 0$$

or $\dfrac{d}{dx}\left[x(vu' - v'u)\right] = 0$

Thus $\dfrac{d}{dx}\left[x\left\{J_{-n}(x)J_n'(x) - J_{-n}'(x)J_n(x)\right\}\right] = 0$

48. *Prove that*

$$x\left[J_n'(x)J_{-n}(x)-J_{-n}'(x)J_n(x)\right]=\frac{2\sin n\pi}{\pi}$$

>> $$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_{-n}(x)=\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{-n+2s}\frac{1}{\Gamma(-n+s+1)\,s!}$$

$\therefore$ $$J_n'(x)=\sum_{r=0}^{\infty}(-1)^r(n+2r)\left(\frac{x}{2}\right)^{n+2r-1}\cdot\frac{1}{2}\cdot\frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_{-n}'(x)=\sum_{s=0}^{\infty}(-1)^s(-n+2s)\left(\frac{x}{2}\right)^{-n+2s-1}\cdot\frac{1}{2}\cdot\frac{1}{\Gamma(-n+s+1)\,s!}$$

Consider $x\,[J_n'(x)J_{-n}(x)-J_{-n}'(x)J_n(x)]$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\left(\frac{x}{2}\right)^{2(r+s)}\frac{(n+2r)}{\Gamma(-n+s+1)\Gamma(n+r+1)\,r!\,s!}$$

$$-\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}\left(\frac{x}{2}\right)^{2(r+s)}\frac{(-n+2s)}{\Gamma(-n+s+1)\Gamma(n+r+1)\,r!\,s!}$$

It can be easily seen that on expansion all the terms cancel out except the terms when $r=0$ and $s=0$. Hence we have,

$$\frac{n}{\Gamma(-n+1)\Gamma(n+1)}-\frac{-n}{\Gamma(-n+1)\Gamma(n+1)}$$

$$=\frac{n}{\Gamma(1-n)\,n\,\Gamma(n)}+\frac{n}{\Gamma(1-n)\,n\,\Gamma(n)}=\frac{2}{\Gamma(1-n)\Gamma(n)}$$

But we know that $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$

Thus $x\,[J_n'(x)J_{-n}(x)-J_{-n}'(x)J_n(x)]=\dfrac{2\sin n\pi}{\pi}$

Note : *Alternative version of the problem. Show that* $\dfrac{d}{dx}\left[\dfrac{J_{-n}(x)}{J_n(x)}\right]=\dfrac{-2\sin n\pi}{\pi x J_n^2(x)}$

49. *Obtain the solution of the following equation in terms of Bessel functions.*

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{1}{9x^2}\right)y=0$$

>> Multiplying the given equation by x^2 we have,

$$x^2y''+xy'+(x^2-1/9)\,y=0$$

We have Bessel's equation in the form

$$x^2y''+xy'+(x^2-n^2)\,y=0$$

The solution is given by $y=a\,J_n(x)+b\,J_{-n}(x)$

Comparing (1) and (2) we have $n^2=1/9$ or $n=\pm 1/3$

Thus the solution of the given equation in terms of Bessel functions is

$$y=a\,J_{1/3}(x)+b\,J_{-1/3}(x)$$

50. *Solve :* $16x^2y''+16xy'+(16x^2-1)\,y=0$ *in terms of Bessel functions.*

>> The given equation on dividing by 16 becomes,

$$x^2y''+xy'+(x^2-1/16)\,y=0$$

We have Bessel's equation in the form,

$$x^2y''+xy'+(x^2-n^2)\,y=0$$

The solution is given by $y=a\,J_n(x)+bJ_{-n}(x)$

Comparing (1) and (2) we have, $n^2=1/16$ or $n=\pm 1/4$.

Thus the solution of the given equation in terms of Bessel functions is,

$$y=a\,J_{1/4}(x)+b\,J_{-1/4}(x)$$

51. *Solve* : $y'' + \dfrac{y'}{x} + \left(1 - \dfrac{1}{6.25x^2}\right) y = 0$ *in terms of Bessel functions.*

>> Multiplying the given equation by x^2 and writing 6.25 as 25/4 = $(5/2)^2$ we have,

$$x^2 y'' + xy' + [\,x^2 - (2/5)^2\,]\, y = 0$$

This problem is similiar to the previous one where $n = 2/5$.

Thus the solution of the given equation in terms of Besel functions is,

$$y = a J_{2/5}(x) + b J_{-2/5}(x)$$

---

52. *Show that the solution of the equation*

$$x^2 y'' + x y' + (x^2 - 1/4)\, y = 0 \quad is \quad c_1 \frac{\sin x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}}$$

>> Comparing the given equation with Bessel's equation

$$x^2 y'' + x y' + (x^2 - n^2)\, y = 0, \quad \text{we have } n^2 = 1/4 \quad \text{or} \quad n = \pm 1/2$$

The solution of the given equation in terms of Bessel functions is

$$y = a J_{1/2}(x) + b J_{-1/2}(x)$$

But $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$ and $J_{-1/2}(x) = \sqrt{2/\pi x}\,\cos x$ [*Refer Problem - 24*]

Hence $y = a\sqrt{2/\pi x}\,\sin x + b\sqrt{2/\pi x}\,\cos x$

ie., $y = \left(a\sqrt{2/\pi}\right)\dfrac{\sin x}{\sqrt{x}} + \left(b\sqrt{2/\pi}\right)\dfrac{\cos x}{\sqrt{x}}$

Let $c_1 = a\sqrt{2/\pi}$ and $c_2 = b\sqrt{2/\pi}$; c_1 and c_2 are arbitrary constants.

Thus $y = c_1 \dfrac{\sin x}{\sqrt{x}} + c_2 \dfrac{\cos x}{\sqrt{x}}$

- **Orthogonal Property of Bessel Functions**

If α *and* β *are two distinct roots of* $J_n(x) = 0$ *then*

$$\int_0^1 x J_n(\alpha x)\, J_n(\beta x)\, dx = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \dfrac{1}{2}[J_n'(\alpha)]^2 = \dfrac{1}{2}[J_{n+1}(\alpha)]^2 & \text{if } \alpha = \beta \end{cases}$$

Proof : We know that $J_n(\lambda x)$ is a solution of the equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)\, y = 0 \qquad [\ \textit{Article 5.41}\ ]$$

If $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ the associated differential equations are

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2)\, u = 0 \qquad \ldots (1)$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2)\, v = 0 \qquad \ldots (2)$$

Multiplying (1) by $\dfrac{v}{x}$ and (2) by $\dfrac{u}{x}$ we obtain,

$$x v u'' + v u' + \alpha^2 u v x - \frac{n^2 u v}{x} = 0$$

and $$x u v'' + u v' + \beta^2 u v x - \frac{n^2 u v}{x} = 0$$

On subtracting we obtain

$$x(v u'' - u v'') + (v u' - u v') + (\alpha^2 - \beta^2)\, u v x = 0$$

i.e., $$\frac{d}{dx}\, x(v u' - u v') = (\beta^2 - \alpha^2)\, x u v$$

Integrating both sides w.r.t. x between 0 and 1 we have

$$\Big[x(v u' - u v')\Big]_{x=0}^{1} = (\beta^2 - \alpha^2)\int_{x=0}^{1} x u v\, dx$$

i.e., $$(v u' - u v')_{x=1} - 0 = (\beta^2 - \alpha^2)\int_0^1 x u v\, dx \qquad \ldots (3)$$

Since $u = J_n(\alpha x)$, $v = J_n(\beta x)$ we have $u' = \alpha J_n'(\alpha x)$ $v' = \beta J_n'(\beta x)$ and as a consequence of these (3) becomes

$$\Big[J_n(\beta x)\,\alpha J_n'(\alpha x) - J_n(\alpha x)\,\beta J_n'(\beta x)\Big]_{x=1} = (\beta^2-\alpha^2)\int_0^1 x J_n(\alpha x)\,J_n(\beta x)\,dx$$

Hence $$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \frac{1}{\beta^2-\alpha^2}\left\{\alpha J_n(\beta)J_n'(\alpha) - \beta J_n(\alpha)J_n'(\beta)\right\} \qquad \ldots (4)$$

Since α & β are distinct roots of $J_n(x)=0$ we have $J_n(\alpha)=0$ & $J_n(\beta)=0$, with the result the RHS of (4) becomes zero provided $\beta^2-\alpha^2 \neq 0$ or $\beta \neq \alpha$.

Thus we have proved that if $\alpha \neq \beta$,

$$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = 0 \qquad \ldots (5)$$

We shall now discuss the case when $\alpha = \beta$

The RHS of (3) becomes an indeterminate form of the type $\frac{0}{0}$ when $\alpha = \beta$.

We shall evaluate by taking limits on both sides as $\beta \to \alpha$, keeping α fixed, by applying L' Hospital's rule.

i.e., $$\underset{\beta\to\alpha}{lt}\int_0^1 xJ_n(\alpha x)\,J_n(\beta x)\,dx$$

$$= \underset{\beta\to\alpha}{lt}\ \frac{1}{\beta^2-\alpha^2}\left\{\alpha J_n(\beta)J_n'(\alpha) - \beta J_n(\alpha)J_n'(\beta)\right\}$$

Since α is fixed we must have $J_n(\alpha)=0$ as α is a root of $J_n(x)=0$

$$\therefore \quad \underset{\beta\to\alpha}{lt}\int_0^1 xJ_n(\alpha x)\,J_n(\beta x)\,dx = \underset{\beta\to\alpha}{lt}\ \frac{1}{\beta^2-\alpha^2}\left\{\alpha J_n(\beta)J_n'(\alpha)\right\}$$

$$= \underset{\beta\to\alpha}{lt}\ \frac{1}{2\beta}\left\{\alpha J_n'(\beta)\,J_n'(\alpha)\right\} \text{ by L' Hospital's rule.}$$

The numerator and denominator are differentiated separately w.r.t. β

We now have,

$$\int_0^1 x\,[J_n(\alpha x)]^2\,dx = \frac{1}{2\alpha}\,\alpha\,J_n'(\alpha)J_n'(\alpha) = \frac{1}{2}\,[J_n'(\alpha)]^2$$

$$\therefore \quad \int_0^1 x\,J_n^2(\alpha x)\,dx = \frac{1}{2}\,[J_n'(\alpha)]^2$$

Further we have the recurrence relation $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$

$$\therefore \quad J_n'(\alpha) = \frac{n}{\alpha}J_n(\alpha) - J_{n+1}(\alpha)$$

Since $J_n(\alpha)=0$, we obtain $J_n'(\alpha) = -J_{n+1}(\alpha)$ and (6) now becomes,

$$\int_0^1 x\,J_n^2(\alpha x)\,dx = \frac{1}{2}\,[J_{n+1}(\alpha)]^2$$

This result is known as the ***Lommel integral formula.***

Note : *The orthogonal property is also presented in the form :*

$$\int_0^a x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{a^2}{2}\,[J_{n+1}(a\alpha)]^2 & \text{if } \alpha = \beta \end{cases}$$

This result can be established working on similar lines as before.

2.26 Series Solution of Legendre's Differential Equation

We have Legendre differential equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

The coefficient of $y'' = (1-x^2) = P_0(x)$ and $P_0(x) \neq 0$ at $x=0$.

We employ power series method to solve this equation.

We assume the series solution of (1) in the form

$$y = \sum_{r=0}^{\infty} a_r\,x^r$$

$$\therefore \quad \frac{dy}{dx} = \sum_0^\infty a_r\, r\, x^{r-1}, \quad \frac{d^2y}{dx^2} = \sum_0^\infty a_r\, r(r-1)x^{r-2}$$

Now (1) becomes,

$$(1-x^2)\sum_0^\infty a_r\, r(r-1)x^{r-2} - 2x\sum_0^\infty a_r\, r\, x^{r-1} + n(n+1)\sum_0^\infty a_r x^r = 0$$

i.e., $$\sum_0^\infty a_r\, r(r-1)x^{r-2} - \sum_0^\infty a_r\, r(r-1)x^r - \sum_0^\infty 2a_r\, r\, x^r + n(n+1)\sum_0^\infty a_r x^r = 0$$

We equate the coefficients of various powers of x to zero.

We first equate the coefficients of x^{-2} and x^{-1} available only in the first summation to zero.

Coeff. of x^{-2}: $\quad a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

Coeff. of x^{-1}: $\quad a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Now we shall equate the coefficient of x^r $(r \geq 0)$ to zero.

i.e., $\quad a_{r+2}(r+2)(r+1) - a_r\, r(r-1) - 2a_r\, r + n(n+1)a_r = 0$

i.e., $\quad a_{r+2}(r+2)(r+1) = a_r[r(r-1) + 2r - n(n+1)]$

or $$a_{r+2} = \frac{-[n(n+1) - r^2 - r]}{(r+2)(r+1)}\, a_r \qquad \ldots (3)$$

Putting $r = 0, 1, 2, 3, \ldots$ in (3) we obtain,

$$a_2 = \frac{-n(n+1)}{2}a_0 \; ; \; a_3 = \frac{-(n^2+n-2)}{6}a_1 = \frac{-(n-1)(n+2)}{6}a_1$$

$$a_4 = \frac{-(n^2+n-6)}{12}\cdot a_2 = \frac{-(n-2)(n+3)}{12}\cdot\frac{-n(n+1)}{2}a_0$$

i.e., $$a_4 = \frac{n(n+1)(n-2)(n+3)}{24}a_0$$

$$a_5 = \frac{-(n^2+n-12)}{20}\cdot a_3 = \frac{-(n-3)(n+4)}{20}\cdot\frac{-(n-1)(n+2)}{6}a_1$$

i.e., $$a_5 = \frac{(n-1)(n+2)(n-3)(n+4)}{120}a_1 \text{ and so-on.}$$

We substitute these values in the expanded form of (2) :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

Since constants $a_2, a_4, \ldots$ are in terms of a_0 and $a_3, a_5, \ldots$ are in terms of a_1 we rearrange the RHS in the form

$$y = (a_0 + a_2 x + a_4 x^2 + \cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$$

On substituting for $a_2, a_3, a_4, a_5 \ldots$ we obtain

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 - \cdots\right]$$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 - \cdots\right] \qquad \ldots (4)$$

Let $u(x)$ & $v(x)$ respectively represent the two infinite series in (4) so that we have

$$y = a_0 u(x) + a_1 v(x) \qquad \ldots (5)$$

This is the series solution of Legendre's differential equation.

• Legendre Polynomials

If n is a positive even integer, $a_0 u(x)$ reduces to a polynomial of degree n and if n is a positive odd integer $a_1 v(x)$ reduces to a polynomial of degree n. Otherwise these will give infinite series called ***Legendre functions of second kind.***

It may be observed that the polynomials $u(x)$, $v(x)$ contain alternate powers of x and a general form of the polynomial that represents either of them in descending powers of x can be presented in the form

$$y = f(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \cdots + F(x) \qquad \ldots (1)$$

where $$F(x) = \begin{cases} a_0 \text{ if } n \text{ is even} \\ a_1 x \text{ if } n \text{ is odd} \end{cases}$$

We note that a_r is the coefficient of x^r in the series solution of the differential equation and we have obtained, [*Refer (3) in the previous article*]

$$a_{r+2} = \frac{-[n(n+1) - r(r+1)]}{(r+2)(r+1)}\, a_r \qquad \ldots (2)$$

We plan to express $a_{n-2}, a_{n-4}, \ldots$ present in (1) in terms of a_n. Replacing r by $(n-2)$ in (2) we obtain

[illegible]

$$a_n = \frac{-[n(n+1)-(n-2)(n-1)]}{n(n-1)} a_{n-2}$$

i.e., $$a_n = \frac{-(4n-2)}{n(n-1)} a_{n-2}$$

or $$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

Again from (2), on replacing r by $(n-4)$ we obtain

$$a_{n-2} = \frac{-[n(n+1)-(n-4)(n-3)]}{(n-2)(n-3)} a_{n-4}$$

i.e., $$a_{n-2} = \frac{-(8n-12)}{(n-2)(n-3)} a_{n-4}$$

or $$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$\therefore$ $$a_{n-4} = \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} a_n$$ by using the value of a_{n-2} and so-on.

Using these values in (1) we have,

$$y = f(x) = a_n\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \dots + G(x)\right]$$

where $$G(x) = \begin{cases} a_0/a_n & \text{if } n \text{ is even.} \\ a_1 x/a_n & \text{if } n \text{ is odd.} \end{cases}$$

If the constant a_n is so chosen such that $y = f(x)$ becomes 1 when $x = 1$, the polynomials so obtained are called Legendre polynomials denoted by $P_n(x)$.

Let us choose $a_n = \dfrac{1\cdot 3\cdot 5\cdots 2n-1}{n!}$ to meet the said requirement. That is

$$P_n(x) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \dots\right] \quad \dots(3)$$

We obtain first few Legendre polynomials by putting $n = 0, 1, 2, 3, 4$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{1!}[x] = x$$

$$P_2(x) = \frac{1\cdot 3}{2!}\left[x^2 - \frac{2(2-1)}{2\cdot 3}x^0\right] = \frac{3}{2}\left(x^2 - \frac{1}{3}\right) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1\cdot 3\cdot 5}{3!}\left[x^3 - \frac{3(2)}{2\cdot 5}x\right] = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1\cdot 3\cdot 5\cdot 7}{4!}\left[x^4 - \frac{4(3)}{2(7)}x^2 + \frac{4(3)(2)(1)}{2\cdot 4(7)(5)}\right]$$

i.e., $$P_4(x) = \frac{35}{8}\left[x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right] = \frac{1}{8}[35x^4 - 30x^2 + 3] \text{ etc.}$$

It can be easily seen that all these expressions give 1 at $x = 1$ in accordence definition of Legendre polynomials.

- **Rodrigue's formula**

We derive a formula for the Legendre polynomials $P_n(x)$ in the form

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$ *known as* ***Rodrigue's formula.***

Proof : Let $u = (x^2-1)^n$

We shall first establish that the n^{th} derivative of u, that is u_n is a solutic Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Differentiating u w.r.t. x, we have,

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1}\cdot 2x \quad \text{or} \quad (x^2-1)u_1 = 2nx(x^2-1)^n$$

i.e., $$(x^2-1)u_1 = 2nxu$$

Differentiating w.r.t. x again we have,

$$(x^2-1)u_2 + 2xu_1 = 2n(xu_1+u)$$

We shall now differentiate the result n times by applying Leibnitz theorem for the n^{th} derivative of a product given by

$$(UV)_n = UV_n + nU_1V_{n-1} + \frac{n(n-1)}{2!}U_2V_{n-2} + \cdots + U_nV$$

$\therefore$ $[(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$

i.e., $\left[(x^2-1)u_{n+2} + n\cdot 2x\cdot u_{n+1} + \frac{n(n-1)}{2}\cdot 2\cdot u_n\right] + 2\left[xu_{n+1} + n\cdot 1\cdot u_n\right]$

$$= 2n\left[xu_{n+1} + n\cdot 1\cdot u_n\right] + 2nu_n$$

i.e., $(x^2-1)u_{n+2} + 2nxu_{n+1} + (n^2-n)u_n + 2xu_{n+1} + 2nu_n$

$$= 2nxu_{n+1} + 2n^2u_n + 2nu_n$$

i.e., $(x^2-1)u_{n+2} + 2xu_{n+1} - n^2u_n - nu_n = 0$

i.e., $(x^2-1)u_{n+2} + 2xu_{n+1} - nu_n(n+1) = 0$

or $(1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0$

This can be put in the form,

$$(1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0 \qquad \ldots(2)$$

Comparing (2) with (1) we conclude that u_n is a solution of the Legendre's differential equation. It may be observed that u is a polynomial of degree $2n$ and hence u_n will be a polynomial of degree n.

Also $P_n(x)$ which satisfies the Legendre differential equation is also a polynomial of degree n. Hence u_n must be the same as $P_n(x)$ but for some constant factor k.

i.e., $P_n(x) = ku_n = k[(x^2-1)^n]_n$

i.e., $P_n(x) = k[(x-1)^n(x+1)^n]_n$

Applying Leibnitz theorem for the RHS we have,

$$P_n(x) = k\Big[\{(x-1)^n\}(x+1)^n_n + n\cdot n(x-1)^{n-1}(x+1)^n_{n-1} + \frac{n(n-1)}{2!}n(n-1)(x-1)^{n-2}(x+1)^n_{n-2} + \cdots + \{(x-1)^n\}_n(x+1)^n\Big] \qquad \ldots(3)$$

It should be observed that if $Z = (x-1)^n$ then

$Z_1 = n(x-1)^{n-1}$, $Z_2 = n(n-1)(x-1)^{n-2}$ etc.

$Z_n = n(n-1)(n-2)\cdots 2\cdot 1(x-1)^{n-n}$ or $Z_n = n!(x-1)^0 = n!$

$\therefore$ $\{(x-1)^n\}_n = n!$

We proceed to find k by choosing a suitable value for x.
Putting $x=1$ in (3) all the terms in RHS become zero except the last term which becomes $n!(1+1)^n = n!2^n$.

i.e., $P_n(1) = k\cdot n!\cdot 2^n$ and $P_n(1) = 1$ by the definition of $P_n(x)$.

$\therefore$ $1 = k\cdot n!\cdot 2^n$ or $k = \frac{1}{n!\,2^n}$

Since $P_n(x) = ku_n$, we have, $P_n(x) = \frac{1}{n!\,2^n}\{(x^2-1)^n\}_n$

Thus we have proved that $P_n(x) = \frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$ [*Rodrigue's Formula*]

WORKED PROBLEMS

53. *Using Rodrigue's formula obtain expressions for $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$. Hence express x^2, x^3, x^4, x^5 in terms of Legendre polynomials.*

>> We have Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

We shall put $n = 0, 1, 2, 3, 4, 5$ successively in this formula.

$$P_0(x) = \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0 = 1$$

$\therefore$ $P_0(x) = 1$

$$P_1(x) = \frac{1}{2^1\,1!}\frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = x$$

$$\therefore \quad P_1(x) = x$$

$$P_2(x) = \frac{1}{2^2\,2!}\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{8}(12x^2-4) = \frac{1}{2}(3x^2-1)$$

$$\therefore \quad P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3\,3!}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$$

$$P_3(x) = \frac{1}{48}(120x^3-72x) = \frac{24}{48}(5x^3-3x)$$

$$\therefore \quad P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{2^4\,4!}\frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{16\times 24}\frac{d^4}{dx^4}(x^8-4x^6+6x^4-4x^2+1)$$

where we have used the binomial expansion

$$(x-a)^n = x^n - {}^nC_1\,x^{n-1}a + {}^nC_2\,x^{n-2}a^2 - {}^nC_3\,x^{n-3}a^3 + \cdots + (-1)^n a^n$$

We shall also use, $\dfrac{d^n}{dx^n}(x^m) = \dfrac{m!}{(m-n)!}x^{m-n}$, where $m > n$

Now, $$P_4(x) = \frac{1}{16\times 24}\left[\frac{8!}{4!}x^4 - 4\times\frac{6!}{2!}x^2 + 6\times\frac{4!}{0!}x^0\right]$$

$$= \frac{1}{16\times 24}[1680x^4 - 1440x^2 + 144]$$

$$= \frac{48}{16\times 24}[35x^4 - 30x^2 + 3]$$

$$\therefore \quad P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$$

$$P_5(x) = \frac{1}{2^5\,5!}\frac{d^5}{dx^5}(x^2-1)^5$$

$$= \frac{1}{32\times 120}\frac{d^5}{dx^5}(x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1)$$

$$= \frac{1}{32\times 120}\left(\frac{10!}{5!}x^5 - 5\times\frac{8!}{3!}x^3 + 10\times\frac{6!}{1!}x\right)$$

$$= \frac{1}{32\times 120}(30240x^5 - 33600x^3 + 7200x)$$

$$\therefore \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \text{ on simplification.}$$

We now express x^2, x^3, x^4, x^5 in terms of Legendre polynomials.

Consider, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

or $\quad 2P_2(x) = 3x^2 - 1$ or $3x^2 = 1 + 2P_2(x)$. But $P_0(x) = 1$

$$\therefore \quad x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$$

Next consider, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

or $\quad 2P_3(x) = 5x^3 - 3x$ or $5x^3 = 2P_3(x) + 3x$. But $P_1(x) = x$

$$\therefore \quad x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

Now consider, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

i.e., $\quad 8P_4(x) = 35x^4 - 30x^2 + 3 \quad$ or $\quad 35x^4 = 8P_4(x) + 30x^2 - 3$

i.e., $\quad 35x^4 = 8P_4(x) + 10[2P_2(x) + 1] - 3$, by using (1).

i.e., $\quad 35x^4 = 8P_4(x) + 20P_2(x) + 7$

$$\therefore \quad x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$$

Next consider, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

i.e., $8P_5(x) = 63x^5 - 70x^3 + 15x$ or $63x^5 = 8P_5(x) + 70x^3 - 15x$

Using $x = P_1(x)$ and (2) we have,

$$63x^5 = 8P_5(x) + 14\,[2P_3(x) + 3P_1(x)] - 15P_1(x)$$

i.e., $63x^5 = 8P_5(x) + 28P_3(x) + 27P_1(x)$

$$\therefore \quad x^5 = \frac{8}{63}P_5(x) + \frac{4}{9}P_3(x) + \frac{3}{7}P_1(x) \qquad \ldots (4)$$

Note : *Expressing $x^2, x^3, x^4, \cdots$ in terms of Legendre polynomials helps us to express any given polynomial $f(x)$ in terms of Legendre polynomials.*

Working procedure for problems

- We write down from memory the expression for $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)\cdots$ in correlation with the degree of the given polynomial.
- We express, $x^2, x^3, x^4, \cdots$ in terms of Legendre polynomials.
- We substitute these expressions in the given polynomial function $f(x)$ and simplify to obtain $f(x)$ in the form : $a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+a_3P_3(x)+\cdots$ where $a_0, a_1, a_2, a_3\cdots$ are constants.

54. *Express $x^3 + 2x^2 - 4x + 5$ in terms of Legendre polynomials.*

>> Let $f(x) = x^3 + 2x^2 - 4x + 5$

We have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3-3x)$

Hence $x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$; $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

Substituting these in $f(x)$ along with $x = P_1(x)$ and $1 = P_0(x)$ we have,

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 2\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right] - 4P_1(x) + 5P_0(x)$$

Thus $f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{17}{5}P_1(x) + \frac{17}{3}P_0(x)$

55. *If $x^3 + 2x^2 - x + 1 = aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x)$ find the values of a, b, c, d*

>> Let $f(x) = x^3 + 2x^2 - x + 1$

As in the previous example substituting for $x^3, x^2, x, 1$ in terms of Legendre polynomials we have,

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 2\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right] - P_1(x) + P_0(x)$$

i.e., $f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) + \frac{5}{3}P_0(x)$

Hence we have,

$$aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x) = \frac{5}{3}P_0(x) - \frac{2}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{5}P_3(x)$$

Thus by comparing both sides we obtain

$$a = \frac{5}{3},\quad b = -\frac{2}{5},\quad c = \frac{4}{3},\quad d = \frac{2}{5}$$

56. *Show that $x^4 - 3x^2 + x = \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x)$*

>> Let $f(x) = x^4 - 3x^2 + x$ and we have obtained,

$$x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x), \quad x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x).$$

Substituting these in $f(x)$ with $x = P_1(x)$ we have,

$$f(x) = \left[\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)\right] - 3\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right] + P_1(x)$$

Thus $f(x) = \frac{8}{35}P_4(x) - \frac{10}{7}P_2(x) + P_1(x) - \frac{4}{5}P_0(x)$

57. Show that

(i) $P_2(\cos\theta) = \frac{1}{4}(1+3\cos 2\theta)$ (ii) $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

>> (i) We have $P_2(x) = \frac{1}{2}(3x^2-1)$

Now $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$.

But $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\therefore \quad P_2(\cos\theta) = \frac{1}{2}\left[\frac{3}{2}(1+\cos 2\theta) - 1\right] = \frac{1}{4}[3+3\cos 2\theta - 2]$

Thus $P_2(\cos\theta) = \frac{1}{4}(1+3\cos 2\theta)$

(ii) We also have $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Now $P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

But $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$

$$\therefore \quad P_3(\cos\theta) = \frac{1}{2}\left[5\cdot\frac{1}{4}(\cos 3\theta + 3\cos\theta) - 3\cos\theta\right]$$
$$= \frac{1}{8}[5\cos 3\theta + 15\cos\theta - 12\cos\theta]$$

Thus $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

58. *Obtain $P_3(x)$ from Rodrigue's formula and verify that the same satisfies the Legendre's equation in the standard form.*

>> $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ [*Refer problem-53*]

We have Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

We have to verify that,

$$(1-x^2)P_3''(x) - 2xP_3'(x) + 3(3+1)P_3(x) = 0, \text{ since } n = 3$$

From the expression of $P_3(x)$ we obtain,

$$P_3'(x) = \frac{1}{2}(15x^2-3) \text{ and } P_3''(x) = 15x$$

Consider $(1-x^2)P_3''(x) - 2xP_3'(x) + 12P_3(x)$

$$= (1-x^2)(15x) - 2x\cdot\frac{1}{2}(15x^2-3) + 12\cdot\frac{1}{2}(5x^3-3x)$$
$$= 15x - 15x^3 - 15x^3 + 3x + 30x^3 - 18x = 0$$

Thus we have verified that $P_3(x)$ satisfies Legendre's equation.

59. *State Rodrigue's formula for Legendre polynomials and obtain the expression for P_4 from it. Verify the property of Legendre polynomials in respect of $P_4(x)$ and also*

$$\int_{-1}^{+1} x^3 P_4(x)\,dx$$

>> Rodrigue's formula : $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$

By putting $n = 4$ we obtain $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ [*Refer problem-*

Now $P_4(1) = \frac{1}{8}(35-30+3) = 1$

The property $P_n(x) = 1$ in respect of Legendre polynomials is satisfied for P_4

$$\text{Also,} \quad \int_{-1}^{+1} x^3 P_4(x)\,dx = \int_{-1}^{+1} x^3\cdot\frac{1}{8}(35x^4 - 30x^2 + 3)\,dx$$
$$= \frac{1}{8}\int_{-1}^{+1}(35x^7 - 30x^5 + 3x^3)\,dx$$
$$= \frac{1}{8}\left\{35\left[\frac{x^8}{8}\right]_{-1}^{1} - 30\left[\frac{x^6}{6}\right]_{-1}^{1} + 3\left[\frac{x^4}{4}\right]_{-1}^{1}\right\}$$
$$= \frac{1}{8}\left\{\frac{35}{8}(1-1) - \frac{1}{5}(1-1) + \frac{3}{4}(1-1)\right\} = 0$$

Thus $\int_{-1}^{+1} x^3 P_4(x)\,dx = 0$

EXERCISES

1. Verify that $y = x J_n(x)$ is a solution of the DE $x^2 y'' - xy' + (1 + x^2 - n^2) y = 0$

2. Verify that $y = J_0(2\sqrt{ax})$ satisfies the DE $x y'' + y' + a y = 0$

3. Show that $J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$

4. Show that $J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) J_1(x) - \left(\frac{192}{x^3} - \frac{12}{x}\right) J_0(x)$

Express the following polynomials interms of Legendre polynomials

5. $x^3 + x^2 + x + 1$
6. $4x^3 - 2x^2 - 3x + 8$
7. $x^4 + 3x^3 - x^2 + 5x - 2$
8. $(x+1)(x+2)(x+3)$

9. Show that $P_4(\cos\theta) = \frac{1}{64}(35\cos 4\theta + 20\cos\theta + 9)$

10. Obtain the expression for $P_5(x)$ from Rodrigue's formula and verify that the same satisfies the associated Legendre's equation. Also show that

$$\int_{-1}^{+1} x\, P_5(x)\, dx = 0$$

ANSWERS

5. $\frac{4}{3}P_0(x) + \frac{8}{5}P_1(x) + \frac{2}{3}P_2(x) + \frac{2}{5}P_3(x)$

6. $\frac{22}{3}P_0(x) - \frac{3}{5}P_1(x) - \frac{4}{3}P_2(x) + \frac{8}{5}P_0(x)$

7. $\frac{-224}{105}P_0(x) + \frac{34}{5}P_1(x) - \frac{2}{21}P_2(x) + \frac{6}{5}P_2(x) + \frac{8}{35}P_4(x)$

8. $8P_0(x) + \frac{58}{5}P_1(x) + 4P_2(x) + \frac{2}{5}P_3(x)$

ENGINEERING MATHEMATICS - IV

Module - 3

Complex Variables

3.1 Introduction

We are well acquainted with several concepts associated with a real valued function $y = f(x)$. We introduce *complex valued function* $w = f(z)$ [*function of a complex variable* z] and discuss some topics associated with it.

3.2 Recapitulation of Basic Concepts

A number of the form $z = x + iy$ where x, y are real numbers and $i = \sqrt{-1}$ or $i^2 = -1$ is called a *complex number*. x is called the *real part* of z and y is called the *imaginary part* of z.

Also $\bar{z} = x - iy$ is called the *complex conjugate* of z.

We have $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$\therefore$ $e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots$

ie., $e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$

or $e^{ix} = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$

Thus $e^{ix} = \cos x + i\sin x$ ... (1)

by Maclaurin's series.

Hence $e^{i(-x)} = \cos(-x) + i\sin(-x)$

or $e^{-ix} = \cos x - i\sin x$ ... (2)

Adding and subtracting (1) with (2) we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$