MODULE-2 NUMERICAL METHODS-2 - SEM-4

SPECIAL FUNCTIONS :-

The solution of Laplace's Equation, $\nabla^2 u = 0$ in cylindrical system & spherical system leads to 2 important ordinary differential Equations namely; Bessel Differential Equation. &
Legendre Differential Equation.

The series solution of the Bessel's differential Equation is a "special function" known as "Bessel's function".

The special polynomial function that occurs in the process of solving series of Legendre's differential Equation is known as "Legendre's polynomial."

Applications :- The Bessel's function has various applications in solving boundary value problems with axial symmetry and the Legendre polynomial has various application in solving boundary value problems with spherical symmetry.

* SERIES SOLUTION OF DIFFERENTIAL EQUATION :-

* Series solution of Bessel's differential Equation leading to Bessel functions :-

The differential Equation of the form;

$$x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + (x^2 - n^2) y = 0 \quad \sim (1)$$

is known as "Bessel's differential Equation."

where, n is a non-negative real Constant. (parameter).

Now, let us assume that the series solution of (1) of form;

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \sim (2)$$

$\Rightarrow$ Diff Eqn (2) w.r.t x twice, we get.

$$\frac{dy}{dx} = \sum_{0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-1-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

$\Rightarrow$ using $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in Eqn (1)...

$$(1) \Rightarrow x^2 \cdot \sum_{0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} + x \cdot \sum_{0}^{\infty} a_r (k+r) x^{k+r-1}$$

$$+ x^2 \sum_{0}^{\infty} a_r x^{k+r} - n^2 \sum_{0}^{\infty} a_r x^{k+r} = 0.$$

Case: 1 Assume ; $K = n$ in Eqn (3).

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} \rightarrow // \cancel{n^2} + r^2 + 2nr - \cancel{n^2}$$

$$a_r = \frac{-a_{r-2}}{(2nr + r^2)} \sim (4) \quad , r \geq 2.$$

put ; $r = 2, 3, 4 \ldots$ in Eqn (4)

$r=2 \Rightarrow a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}$

$r=3 \Rightarrow a_3 = \frac{-a_1}{6n+9} = 0$, since, $a_1 = 0$

$r=4 \Rightarrow a_4 = \frac{-a_2}{8n+16} = \frac{-1}{8(n+2)}\left[\frac{-a_0}{4(n+1)}\right] = \frac{a_0}{32(n+1)(n+2)} = \frac{a_0}{2^5(n+1)(n+2)}$.

→ Substitute the values : $a_0, a_1, a_2, a_3, a_4 \ldots$ in Expanded form of Eqn (2) ;

$$y_1 = \left[x^K[a_0] + x^{K+1}(a_1) + x^{K+2}(a_2) + x^{K+3}(a_3) + x^{K+4}(a_4) + \cdots\right]$$

$$y_1 = x^K\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\}.$$

$$y_1 = x^K\left\{a_0 + 0 + \left[\frac{-a_0}{4(n+1)}\right]x^2 + \left[\frac{-a_1}{6n+9}\right]x^3 + \left[\frac{-1}{8n+16}\left[\frac{-a_0}{4(n+1)}\right]\right]\cdots\right\}.$$

put, $\boxed{n=K}$

$$y_1 = x^n a_0\left\{1 - \frac{1}{2^2(n+1)}x^2 + \frac{1}{2^5(n+1)(n+2)}x^4 + \cdots\right\}.$$

choose, $a_0 = \frac{1}{2^n \Gamma[n+1]}$

$$\Rightarrow \sum_0^\infty a_r (K+r)(K+r-1) x^{K+r-\cancel{2}+\cancel{2}} + \cancel{x}.\sum_0^\infty a_r (K+r) x^{K+r-1+1} + \sum_0^\infty a_r x^{K+r+2} - n^2 \sum_0^\infty a_r x^{K+r} = 0$$

Collecting the 1st, 2nd & 4th terms together, we have;

$$\Rightarrow \sum_0^\infty a_r x^{K+r}\left[\underline{(K+r)}(K+r-1) + \underline{(K+r)} - n^2\right] + \sum_0^\infty a_r x^{K+r+2} = 0$$

$$\Rightarrow \sum_0^\infty a_r x^{K+r}\left[(K+r)[K+r-\cancel{1}+\cancel{1}] - n^2\right] + \sum_0^\infty a_r x^{K+r+2} = 0.$$

$$\Rightarrow \underbrace{\sum_0^\infty a_r x^{K+r}\left[(K+r)^2 - n^2\right]} + \sum_0^\infty a_r x^{K+r+2} = 0.$$

Now, we shall equate the coefficient of lowest degree term in x, ie, $\underline{x^K \to 0}$, $\underline{\underline{r=0}}$

ie, $a_0 (K^2 - n^2) = 0$.

Setting $a_0 \neq 0$ ~~whe~~ we have; $K^2 - n^2 = 0$ & hence; $\underline{\underline{K = \pm n}}$

Also, we need to independently equate coefficient of $\underline{\underline{x^{K+1} \to 0}}$

ie, $\underline{\underline{x^{K+1} \to 0}}$, $\underline{\underline{r=1}}$

ie, $a_1 [(K+1)^2 - n^2] = 0$.

$\Rightarrow \underline{\underline{a_1 = 0}}$, ~~since~~ & $(K+1)^2 \neq n^2$ $\because$ we accepted already $\underline{\underline{K = \pm n}}$

$\Rightarrow$ Coefficient of x^{K+r}.

$$// \sum_0^\infty a_r \underline{\underline{x^{K+r}}} x^2 = a_{r-2}$$

$\Rightarrow x^{K+r}: a_r [(K+r)^2 - n^2] + \underline{\underline{a_{r-2}}} = 0.$

$$\Rightarrow \boxed{a_r = \frac{-a_{r-2}}{[(K+r)^2 - n^2]}} \sim (3), \quad \underline{\underline{r \geq 2}}$$

$\Rightarrow$ Equation (3) is known as Recurrence Relation.

$$\Rightarrow y_1 = \frac{x^n}{2^n \gamma(n+1)} \left\{ 1 - \left(\frac{x}{2}\right)^2 \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{2(n+1)(n+2)} \right\}$$

$$y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{n+1\ \gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\gamma(n+1)2} \right\}$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\gamma(n+1)\ 0!} \left(\frac{x}{2}\right)^0 + \left(\frac{x}{2}\right)^2 \frac{(-1)^1}{(n+1)\gamma(n+1)\ 1!} + \left(\frac{x}{2}\right)^4 \frac{(-1)^2}{(n+1)(n+2)\gamma(n+1)\ 2!} \right\}$$

$$y_1 = \cancel{\left(\frac{x}{2}\right)^n} \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\ r!} \left(\frac{x}{2}\right)^{n+2r} \quad \text{—(5)}$$

$\Rightarrow$ Equa^n (5) is known as "Bessel's function" of 1st kind of order n, denoted by; $J_n(x)$

$$\therefore \boxed{J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\ r!} \left(\frac{x}{2}\right)^{n+2r}}$$

$\therefore$ Complete soln is : $\boxed{y = A.J_n(x) + B\,J_{-n}(x)}$

Prove that :-

* $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$

Soln :- $J_{1/2}(x)$ // WKT; $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$

$\Rightarrow$ from, the definition of Bessel's fn

$$\therefore J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\ r!} \left(\frac{x}{2}\right)^{n+2r} \quad \text{—(1)}$$

put ; $\boxed{n = 1/2}$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(1/2+r+1)\, r!} \left(\frac{x}{2}\right)^{1/2+2r}$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \cdot \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(3/2+r)\, r!} \left(\frac{x}{2}\right)^{2r}.$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\gamma(3/2)} + (-1)\left(\frac{x}{2}\right)^2 \frac{1}{\gamma(3/2+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{\gamma(3/2+2)\cdot 2} + \cdots \right]$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\gamma(3/2)} - \frac{x^2}{4} \frac{1}{\gamma(5/2)} + \frac{x^4}{16} \frac{1}{\gamma(7/2)\, 2} \cdots \right]$$

$\gamma(1/2) = \sqrt{\pi}$

$\gamma(3/2) = \frac{1}{2}\sqrt{\pi}$

$\gamma(5/2) = \frac{3}{4}\sqrt{\pi}$

$\gamma(7/2) = \frac{15}{8}\sqrt{\pi}$

$$= \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4} \cdot \frac{4}{3\sqrt{\pi}} + \frac{x^4}{16} \frac{8}{15\sqrt{\pi}\, 2} \right]$$

$$= \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \left[1 - \frac{2x^2}{12} + \frac{4x^4}{16 \times 30} \right]$$

$$= \sqrt{\frac{x}{2}} \cdot \times \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{6} + \frac{4x^4}{120} - \cdots \right]$$

$$= \sqrt{\frac{2x}{\pi}} \cdot \frac{1}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$\therefore \boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sin x.}$$

2) Proove that ; $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x.$

Soln:- From, the defn of bessel's function ....

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r} \quad -① $$

put, $\boxed{n = -1/2}$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(1/2+r+1)\, r!} \left(\frac{x}{2}\right)^{-1/2+2r}.$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{\gamma(3/2+r)\, r!} \left(\frac{x}{2}\right)^{2r}.$$

$$= \left(\frac{x}{2}\right)^{-1/2} \left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{2}{\sqrt{\pi}} + \frac{x^4}{16}\left[\frac{4}{3\sqrt{\pi}\,2}\right]\right].$$

$$= \sqrt{\frac{2}{x\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right]$$

$$\therefore \boxed{J_{-1/2}(x) = \sqrt{\frac{2}{x\pi}}\cdot \cos x.}$$

* Orthogonal property of Bessel's function :-

If α & β are 2 distinct roots of $J_n(x) = 0$, then PT; $\int_0^1 x J_n(\alpha x).J_n(\beta x)dx = 0$. if $\alpha \neq \beta$.

Proof :- WKT; $y = J_n(\lambda x)$ is soln of Eqn :

$$\Rightarrow x^2 y'' + xy' + (\lambda^2 x^2 - n^2)y = 0.$$

if $u = J_n(\alpha x)$ & $v = J_n(\beta x)$, then Associated differential

Equations are :- $x^2 u'' + xu' + (\alpha^2 x^2 - n^2)u = 0 \sim (1)$

$x^2 v'' + xv' + (\beta^2 x^2 - n^2)v = 0 \sim (2)$.

$\Rightarrow$ x'ly (1) by $\frac{v}{x}$ & (2) by $\frac{u}{x}$, we get ..

(1) $\Rightarrow$ $xvu'' + vu' + \alpha^2 uvx - n^2\frac{uv}{x} = 0.$

(2) $\Rightarrow$ $xuv'' + uv' + \beta^2 uvx - n^2\frac{uv}{x} = 0.$

$\Rightarrow$ On subtracting the obtained Equations, we get -

$$\Rightarrow x(u''v - uv'') + (vu' - uv') + uvx(\alpha^2 - \beta^2) = 0.$$

$$\Rightarrow \frac{d}{dx}[x(vu' - uv')] = (\beta^2 - \alpha^2)\, uvx.$$

$\Rightarrow$ Integrating wrt x, b/w $0 \to 1$, we get ..

$$\Rightarrow \left[x(vu'-uv')\right]_{x=0}^{1} = (\beta^2-\alpha^2)\int_{x=0}^{1} uvx\,dx.$$

$\rightarrow$ we have;

$$\left[x(vu'-uv')\right]_{x=1} - 0 = (\beta^2-\alpha^2)\int_{x=0}^{1} uvx\,dx. \quad \text{(3)}$$

$\Rightarrow$ diff w.r.t. x $u = J_n(\alpha x)$, $v = J_n(\beta x)$ $\Rightarrow$ diff w.r.t. x

$\underline{u' = J_n'(\alpha x).\alpha.}$ $\quad$ $\underline{v' = J_n'(\beta x).\beta.}$

sub in (3)

$$\Rightarrow \left[x\{J_n(\beta x)\alpha.J_n'(\alpha x) - J_n(\alpha x).\beta.J_n'(\beta x)\}\right]_{x=1} = (\beta^2-\alpha^2)\int_{x=0}^{1} J_n(\alpha x).J_n(\beta x).x\,dx.$$

$$\Rightarrow \left[J_n(\beta)\alpha.J_n'(\alpha) - J_n(\alpha).\beta.J_n'(\beta)\right] = (\beta^2-\alpha^2)\int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x)\,.dx$$

$$\Rightarrow \int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x)\,dx = \frac{1}{(\beta^2-\alpha^2)}\left[\alpha.J_n(\beta).J_n'(\alpha) - \beta J_n(\alpha)J_n'(\beta)\right]$$

Since; α & β are distinct roots of $J_n(x)=0$.

$$\Rightarrow \underline{J_n(\alpha)=0} \;, \; \underline{J_n(\beta)=0}$$

$$\therefore \boxed{\int_{x=0}^{1} x.J_n(\alpha x).J_n(\beta x).dx = 0.} \quad \beta^2 \neq \alpha^2$$

$\Rightarrow$. Hence, Orthogonal property is satisfied.

Orthogonal property of bessel's function :-

If α & β are the two distinct root of the Equation ;

$J_n(x) = 0$, then ...

$$\int_0^1 x \cdot J_n(\alpha x) \cdot J_n(\beta x) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_n'(\alpha)]^2 = \frac{1}{2} J_{n+1}^2(\alpha) & \text{if } \alpha = \beta \end{cases}$$

* Series solution of Legendre' differential Equation :-

Consider a DE ;

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{—(1)}$$

Eqn (1) is legendre's diff. Equation.

Let us assume that the soln of Eqn (1) is of form ;

$$y = \sum_{r=0}^{\infty} a_r x^r \quad \text{—(2)}$$

$\Rightarrow$ diff Eqn (2) wrt x ...

$$y' = \sum_{r=0}^{\infty} a_r \, r \cdot x^{r-1}$$

$$y'' = \sum_{r=0}^{\infty} a_r \, r \cdot (r-1) \, x^{r-2}$$

$\therefore$ Eqn (1) becomes ...

$$= (1-x^2)\sum_{r=0}^{\infty} a_r \, r(r-1) x^{r-2} - 2x\sum_{r=0}^{\infty} a_r \, r \, x^{r-1} + n(n+1)\sum_{0}^{\infty} a_r x^r = 0$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2+2} - 2\sum_{r=0}^{\infty} a_r r x^{r-1+1}$$

$$+ n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1) x^r - 2\sum_{r=0}^{\infty} a_r r x^r$$
$$+ n(n+1) \sum_{r=0}^{\infty} a_r x^r = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r x^r \left[r(r-1) + 2r - n(n+1) \right] = 0.$$

$$= \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} - \sum_{r=0}^{\infty} a_r x^r \left[r^2 + r - n(n+1) \right] = 0.$$

$\Rightarrow$ Equate the coefficient of lowest power of x to 0.

$$\left[a_0 (0)(-1) x^{-2} + a_1 (1)(0) x^{-1} + \dots \right]$$

Coff of x^{-2} :- $a_0(0)(-1) = 0, \Rightarrow \underline{\underline{a_0 \neq 0}}$

x^{-1} :- $a_1(1)(0) = 0 \qquad \Rightarrow \underline{\underline{a_1 \neq 0}}$

$\Rightarrow$ Now, we equate coeff of $\underline{\underline{x^r \to 0}}$ & Replace $\underline{\underline{r \to r+2}}$ for $\underline{\underline{\text{1st term}}}$

we get,

$\underline{x^r}$: $+a_{r+2}(r+2)(r+1) - a_r\left[r(r+1) - n(n+1) \right] = 0.$

$$\Rightarrow a_{r+2} = \frac{r(r+1) - n(n+1)}{(r+1)(r+2)} a_r \sim (3).$$

$\Rightarrow$ Eqn (3) is called as Recurrance relation.

$\Rightarrow$ Substitute : $r = 0, 1, 2 \dots$ in Eqn (3), we get --

$\underline{\underline{r=0}}$, $\quad a_2 = \left[\dfrac{-n(n+1)}{(1)(2)} \right] a_0$, $\quad a_2 = \left[\dfrac{-n(n+1)}{2} \right] a_0$

$$-[n^2 + n - 2] \quad {}^{-2}_{+2\,-1}$$

$\underline{\underline{r=1}}$, $\quad a_3 = \left[\dfrac{2 - n(n+1)}{6} \right] a_1$, $\quad a_3 = \left[\dfrac{-(n-1)(n+2)}{3!} \right] a_1$

$\underline{\underline{r=2}}$, $\quad a_4 = \left[\dfrac{6 - n(n+1)}{12} \right] a_2 \;\; \cancel{a_4}\; \left\{ \left[\dfrac{-n(n+1)}{2} \right] a_0 \right\}$

$$a_4 = -\frac{(n^2+n-6)}{12}\left\{-\frac{n(n+1)a_0}{2}\right\}$$

n^2+n-6
$(n+3)(n-2)$

$$a_4 = \frac{n(n+1)(n+3)(n-2)\,a_0}{4!}$$

* (no need)

$r=3$,

$$a_5 = \left[\frac{12-n(n+1)}{20}\right] \times \left[\frac{-(n-1)(n+2)}{3!}\right] a_1$$

$$= \left[-\left[\frac{(n^2+n-12)-(n-1)(n+2)}{120}\right]\right] a_1$$

$$a_5 = \left[\frac{(n+4)(n-3)(n-1)(n+2)}{5!}\,a_1\right]$$

Substitute these values in the expanded form of eqn (2)

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$= a_0 + xa_1 - \frac{n(n+1)}{2!}a_0x^2 - \left[\frac{(n-1)(n+2)}{3!}\right]a_1x^3 + \left[\frac{n(n+1)(n+3)(n-2)a_0}{4!}\right]x^4 + \left[\frac{(n+4)(n-3)(n-1)(n+2)}{5!}\right]a_1x^5 \dots$$

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+3)(n-2)}{4!}x^4 + \dots\right] + a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n+4)(n-3)(n-1)(n+2)}{5!}x^5 - \dots\right] \quad (4)$$

Let $u(x)$ & $v(x)$ resp represent 2 infinite series in eqn (4)

$$\Rightarrow \boxed{y = a_0u(x) + a_1v(x)}$$

This is the series soln of "Legendre's differential equation"

→ [Leading to Legendre's polynomial & continued if asked]

Legendre Polynomial :-

The solution of Legendre's dE is Given by;

$$y = a_0 u(x) + a_1 v(x) \sim (*)$$

if 'n' is positive even integer ;
$a_0 u(x)$ reduces to polynomial of degree n.
if 'n' is ~~negative~~ posi odd integer. ;
$a_1 v(x)$ reduces to polynomial of degree n.

ie, $F(x) = \begin{cases} a_0, & \text{if } n \text{ even} \\ a_1 & \text{if } n \text{ odd} \end{cases}$

Consider Legendre's fn of Second Kind....

$$y = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \dots F(x) \sim (1)$$

where; $F(x) = \begin{cases} a_0 & \text{if } n \text{ is even} \\ a_1 & \text{if } n \text{ is odd.} \end{cases}$

WKT; the reccurrance of legendre's is Given by;

$$a_{r+2} = \frac{-[n(n+1) - r(r+1)]}{(r+1)(r+2)} . a_r \sim (2)$$

using Eqn (2) we have to find values of : $(a_{n-2}, a_{n-4} \dots)$

Replace, $r = n-2$ in Eqn (2) ...

$$a_{n-2+2} = -\left[\frac{n(n+1) - (n-2)(n-2+1)}{(n-2+1)(n-2+2)}\right] a_{n-2}$$

$$a_n = -\left[\frac{n(n+1) - (n-2)(n-1)}{n(n-1)}\right] a_{n-2}$$

$$a_n = -\left[\frac{n^2 + n - n^2 + 3n - 2}{n(n-1)}\right] a_{n-2}.$$

$$a_n = \frac{-2(2n-1)}{n(n-1)} a_{n-2}$$

$$\boxed{a_{n-2} = \frac{-n(n-1)}{2(2n-1)} . a_n .}$$

Replace, $\underline{r = n-4}$ in Eqn (2), we get ...

$$a_{n-\cancel{2}4} = -\left[\frac{n(n+1) - (n+4)(n-3)}{(n-3)(n-2)}\right] a_{n-4}$$

$$= -\left[\frac{n^2+n-n^2+3n+4n-12}{(n-3)(n-2)}\right] a_{n-4} = \frac{-4(2n-3)\ a_{n-4}}{(n-3)(n-2)}$$

$$\boxed{a_{n-4} = -\frac{(n-3)(n-2)}{4(2n-3)} . a_{n-2}}$$

$$a_{n-4} = \frac{-(n-3)(n-2)}{4(2n-3)} . \frac{n(n-1)}{2(2n-3)} . a_n .$$

$\Rightarrow$ Now, Eqn (1) becomes ...

$$y = a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-3)(2n-1)} a_n x^{n-4}$$

where; $G(x) = \begin{cases} a_0/a_n & \text{if } n \to \text{even} \\ a_1/a_n & \text{if } n \to \text{odd.} \end{cases}$

$$+ a_n \left\{ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-3)(2n-1)} x^{n-4} + \cdots \right\} \quad -(3)$$

$\Rightarrow$ Constant, an is so choosen such that; $y = f(x)$ becomes 1, when $\underline{x=1}$, the polynomials so obtained are called Legendre polynomials denoted by $\underline{P_n(x)}$

$\Rightarrow$ putting; $n = 0, 1, 2, 3, 4$ in Eqn (3)

we get : $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

(Problems :-

Express the following in terms of Legendre polynomial :-

(a) x),

* RODRIGUE'S FORMULA :-

The legendre polynomials $P_n(x)$ of the form :

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$ is known as "Rodrigue's formula".

Proof :- Let $u = (x^2-1)^n$.

We shall 1st establish that the n^{th} derivative of u, that is u_n is the solution of Legendre's differential Equation.

$$\Rightarrow (1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{—} (1)$$

$\Rightarrow$ diff u wrt x, we have...

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1} \cdot 2x$$

$$u_1 = 2nx \cdot \frac{(x^2-1)^n}{(x^2-1)}$$

$$(x^2-1)u_1 = 2nx(x^2-1)^n$$

$$\Rightarrow (x^2-1)u_1 = 2nxu$$

$\Rightarrow$ diff again wrt x, we get...

$$(x^2-1)u_2 + u_1(2x) = 2n[xu_1 + u(1)]$$

$\Rightarrow$ We shall now diff, the result n times by applying Leibnitz theorem for n^{th} derivative of product given by;

$$(uv)_n = uv_n + nu_1v_{n-1} + \frac{n(n-1)}{2!}u_2v_{n-2} + \dots + u_nv.$$

$$\therefore [(x^2-1)u_2]_n + 2[xu_1]_n = 2n[xu_1]_n + 2nu_n$$

$$\left\{[(x^2-1)u_2]_n + n[2x]u_{n+2-1} + \frac{n(n-1)}{2!}(2(1))u_{n+2-2}\right\} + 2[xu_1]_n + n(2(1))u_{1+n-1}]$$

1st term

$$= 2n[xu_{n+1} + nu_n] + 2nu_n.$$

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ie, $\left[(x^2-1)\,u_{n+2} + n.2x.u_{n+1} + \frac{n(n-1)}{2}.2u_n\right] + 2\left[x u_{n+1} + n.1.u_n\right]$
$= 2n\left[x u_{n+1} + n.1 u_n\right] + 2nu_n$

$\Rightarrow (x^2-1)u_{n+2} + 2nx\,u_{n+1} + (n^2-n)u_n + 2xu_{n+1} + 2nu_n = 2nx\,u_{n+1} + 2n^2u_n + 2nu_n$

ie, $(x^2-1)\,u_{n+2} + 2xu_{n+1} - n^2u_n - nu_n = 0$

$(x^2-1)\,u_{n+2} + 2xu_{n+1} - nu_n(n+1) = 0$

(or) $(1-x^2)\,u_{n+2} - 2x.u_{n+1} + n(n+1)u_n = 0$

This can be put in the form;

$\Rightarrow (1-x^2)\,u_n'' - 2xu_n' + n(n+1)u_n = 0 \sim (2)$

Comparing (2) with (1) we conclude that, u_n is a solution of Legendre's diff. Eqn. (It may be observed that u is a polynomial of degree $2n$ & hence u_n will be polynomial of deg n).

Also $P_n(x)$ which satisfies Legendre's dE is also a polynomial of deg. n. Hence u_n is same as $P_n(x)$

$\Rightarrow P_n(x) = Ku_n = K\left[(x^2-1)^n\right]_n$

$P_n(x) = K\left[(x-1)^n(x+1)^n\right]_n$

Applying Leibnitz thm for RHS;

$P_n(x) = K\left[(x-1)^n\{(x+1)^n\}_n + n.n(x-1)^{n-1}\{(x+1)^n\}_{n-1} + \frac{n(n-1)}{2!}n(n-1)(x-1)^{n-2}\{(x+1)^n\}_{n-2}\right.$
$\left. + \cdots\cdots \{(x-1)^n\}_n(x+1)^n\right] \sim (3)$

Since, $z = (x-1)^n$ then;

$Z_1 = n(x-1)^{n-1}$

$Z_2 = n(n-1)(x-1)^{n-2}$ .....

$Z_n = n(n-1)(n-2) \cdots 2.1 (x-1)^{n-n}$

$Z_n = n!(x-1)^0 = n!$

$\therefore \{(x-1)^n\}_n = n!$

putting, $x=1$ in eqn (3), all terms in RHS become 0, except last term, ie, $n!(1+1)^n = n!\,2^n$.

(3) ie $\Rightarrow P_n(1) = K n!\,2^n$, $P_n(1) = 1$ // By defn of $P_n(x)$.

$1 = K n!\,2^n$

$$\boxed{K = \frac{1}{n!\,2^n}}$$

Since, $P_n(x) = K u_n$, we have; $P_n(x) = \frac{1}{n!\,2^n}\{(x^2-1)^n\}_n$.

Therefore; $\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n}$ is "Rodrigue's Formula".

Note :-

1) $P_0(x) = 1$
2) $P_1(x) = x$
3) $P_2(x) = \frac{1}{2}(3x^2-1)$
4) $P_3(x) = \frac{1}{2}(5x^3-3x)$
5) $P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

* $1 = P_0(x)$
* $x = P_1(x)$
* $x^2 = 2/3\, P_2(x) + 1/3\, P_0(x)$
* $x^3 = 2/5\, P_3(x) + 3/5\, P_1(x)$.
* $x^4 = 8/35\, P_4(x) + 4/7\, P_2(x) + 1/5\, P_0(x)$

① Express :- $x^4 + 3x^3 - x^2 + 5x - 2$ in terms of legendre's polyno.

× Soln :- $x^4 + 3x^3 - x^2 + 5x - 2$.

$$= \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x) + \frac{6}{5} P_3(x) + \frac{9}{5} P_1(x) - \frac{2}{3} P_2(x)$$

$$- \frac{1}{3} P_0(x) + 5P_1(x) - 2P_0(x).$$

$$= \frac{8}{35} P_4(x) + \left(\frac{12-14}{21}\right) P_2(x) + \frac{6}{5} P_3(x) + \left(\frac{9+25}{5}\right) P_1(x)$$

$$+ \left(\frac{3-5-30}{15}\right) P_0(x).$$

$$= \frac{8}{35} . P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x) - \frac{32}{15} P_0(x)$$

* Using Rodrigue's formula, Compute : $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$:-

Soln :- WKT; Rodrigue's formula is ;

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

put ; $n = 0, 1, 2, 3, 4$.

put ; $n=0$,
$$P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2-1)^0.$$

$$\boxed{P_0(x) = 1}$$

$n=1$,
$$P_1(x) = \frac{1}{(2)^1 (1)!} \frac{d}{dx} (x^2-1)^1$$

$$= \frac{1}{2} \frac{d}{dx} (x^2-1)$$

$$= \frac{1}{2} (\cancel{2}x)$$

$$\boxed{P_1(x) = x.}$$

$n=2$ $\quad P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2} (x^2-1)^2$

$= \frac{1}{8} \cdot \frac{d}{dx} \, 2(x^2-1)\, 2x.$

$= \frac{1}{2} \cdot [3x^2-1]$

$\therefore \boxed{P_2(x) = \frac{1}{2}(3x^2-1)}$ (or) $\boxed{x^2 = \frac{2 \cdot P_2(x)+1}{3}}$

$n=3$ $\quad P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2-1)^3$

[Binomial Expression :- $(x-y)^n = x^n - nC_1 x^{n-1} \cdot y + nC_2 x^{n-2} \cdot y^2 + \dots + x^0 y^n$]

WKT; $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geq n.$

$3(x^2-1)^2(2x)$

$6(x^2-1)^2 x$

$= 6 \cdot [x \cdot 2(x^2-1)2x - (x^2-1)^2(1)]$

$= 6[4x^2(2x) + (x^2-1)(8x) - 2(x^2-1)(2x)]$

$= \frac{24 x^3 + 8x^3 - 8x - 4x^3 + 4x}{48}$

To find ; $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

$n=3$; $P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3}(x^2-1)^3$ // $(a-b)^3 =$ Binomial Expression.

$= \frac{1}{48} \frac{d^3}{dx^3} [(x^2)^3 - 3C_1 (x^2)^2 + 3C_2 (x^2)^1 1^2 - 3C_3 (x^2)^0]$

$= \frac{1}{48} \frac{d^3}{dx^3} [x^6 - 3x^4 + 3x^2 - 1]$

$= \frac{1}{48} [\frac{6!}{3!} x^3 - 3 \cdot \frac{4!}{1!} x^1 + 0]$

$= \frac{1}{48} [120x^3 - 72x]$

$\boxed{P_3(x) = 5/_2 x^3 - 3/_2 x.}$ (or) $\boxed{x^3 = \frac{2P_3(x)+3x}{5}}$

Problems

$\underline{n=4}$ $P_4(x) = \frac{1}{2^4 4!} \frac{d^4}{dx^4} (x^2-1)^4$

$= \frac{1}{384} \frac{d^4}{dx^4} [(x^2-1)^4]$

$= \frac{1}{384} \frac{d^4}{dx^4} [(x^3)^4 - 4C_1 (x^3)^3 + 4C_2 (x^3)^2 - 4C_3 (x^3)^1 + 4C_4 (x^3)^0]$

$= \frac{1}{384} \frac{d^4}{dx^4} [x^{12} - 4(x^9) + 6x^6 - 4x^3 + 1]$

$= \frac{1}{384} \frac{d^3}{dx^3} [12x^{11} - 9\times4\, x^8 + 6\times6x^5 - 4\times3.x^2 + 0]$

$= \frac{1}{384} \frac{d^2}{dx^2} [12\times11\times x^{10} - 9\times4\times8\times x^7 + 36\times5x^4 - 12\times2x]$

$= \frac{1}{384} \frac{d}{dx} [12\times11\times10\, x^9 - 36\times8\times7x^6 + 180\times4\, x^3 - 24]$

$= \frac{1}{384} [120\times11\times9x^8 - 36\times8\times7\times6\, x^5 + 180\times4\times3x^2 - 0]$

$nC_r = \frac{n!}{r!(n-r)!}$

$4C_2 = \frac{4!}{2!\,2!}$

$\boxed{4C_2 = 6}$

$\Rightarrow x^4 /= 8P.$

$$\boxed{P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]}$$

(or) $$\boxed{x^4 = \frac{8P_4(x) + 30x^2 - 3}{35}}$$

Formulas :-

1) $K = K.P_0(x)$

2) $x = P_1(x)$

3) $x^2 = \frac{2P_2(x)+1}{3}$

4) $x^3 = \frac{2P_3(x) + 3x}{5x}$

5) $x^4 = \frac{8P_4(x)+30x^2-3}{35}$

(4)

Problems & Solutions :-

1) Express : x^3+2x^2-x-3 interms of legendre's polynomials :-

Soln :- Let $f(x) = x^3+2x^2-x-3$.

We have ;

$1 = P_0(x)$ $\qquad x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$

$x = P_1(x)$ $\qquad x^4 = \frac{8}{35}P_4(x) + \frac{30}{35}x^2 - \frac{3}{35}$

$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] + 2\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] - \left[P_1(x)\right] - 3 \cdot P_0(x)$$

$$= \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x) - P_1(x) - 3P_0(x) \Rightarrow \text{Simplify}$$

$$\therefore f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - P_1(x) - \frac{2}{5}P_1(x) - \frac{7}{3}P_0(x)$$

2) Express : $f(x) = x^4+3x^3-x^2+5x-2$ in terms of legendre's polynomials.

Let ; $f(x) = x^4+3x^3-x^2+5x-2$

We have ; $P_0(x) = 1$, $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$ $\qquad x^4 = \frac{8}{35}P_4(x) + \frac{30}{35}x^2 - \frac{3}{35}$

$x = P_1(x)$ $\qquad x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$

$$\therefore f(x) = \left[\frac{8}{35}P_4(x) + \frac{30}{35}x^2 - \frac{3}{35}\right] + 3\left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - \left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + 5P_1(x) - 2 \cdot P_0(x)$$

$$= \frac{8}{35}P_4(x) + \frac{30}{35}\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - \frac{3}{35}P_0(x) + \frac{6}{5}P_3(x) + \frac{9}{5}P_1(x) - \frac{2}{3}P_2(x) - \frac{1}{3}P_0(x) + 5P_1(x) - 2P_0(x)$$

$$f(x) = \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{16}{21}P_2(x) + \frac{34}{105}P_1(x) - \frac{32}{105}P_0(x)$$

$\frac{20}{35} - \frac{2}{3}$

$\frac{60-35\times2}{105}$

$\frac{10}{35} - \frac{3}{35} - \frac{1}{3}$

3) If $x^3+2x^2-x+1 = a p_0(x) + b p_1(x) + c p_2(x) + b p_3(x)$

find values of a, b, c, d

$f(x) = x^3+2x^2-x+1$

we have; $1 = P_0(x)$ $\qquad x^2 = 2/3\, P_2(x) + 1/3$

$x = P_1(x)$ $\qquad x^3 = 2/5\, P_3(x) + 3/5\, x.$

$f(x) = 2/5\, P_3(x) + 3/5\, P_1(x) + 2\left[2/3\, P_2(x) + 1/3\, P_0(x)\right] - p_1(x) + p_0(x)$

$= 2/3\, P_3(x) + 3/5\, P_1(x) + 4/3\, P_2(x) + 2/3\, P_0(x) - p_1(x) + p_0(x)$

$f(x) = \underset{d}{\underset{\downarrow}{2/3\, P_3(x)}} + \underset{c}{\underset{\downarrow}{4/3\, P_2(x)}} - \underset{b}{\underset{\downarrow}{2/5\, p_1(x)}} + \underset{a}{\underset{\downarrow}{5/3\, p_0(x)}}.$

$\therefore \boxed{a = 5/3}$ $\boxed{b = -2/5}$ $\boxed{c = 4/3}$ $\boxed{d = 2/3}$

4) If $2x^3 - x^2 - 3x + 2 = a p_0(x) + b p_1(x) + c p_2(x) + d p_3(x)$ find: a, b, c, d ..

Soln:- $f(x) = 2x^3 - x^2 - 3x + 2$, LOKT; $1 = p_0(x)$, $x = p_1(x)$, $x^2 = 2/3\, p_2(x) + 1/3$

$x^3 = 2/5\, P_3(x) + 3/5\, x$

$\therefore f(x) = 2\left[2/5\, p_3(x) + 3/5\, x\right] - \left[2/3\, p_2(x) + 1/3\right] - 3\left[p_1(x)\right] + 2 p_0(x).$

$= 4/5\, p_3(x) + 6/5\, p_1(x) - 2/3\, p_2(x) - 1/3\, p_0(x) - 3 p_1(x) + 2 p_0(x)$

$\therefore f(x) = \underset{d}{\underset{\downarrow}{4/3\, P_3(x)}} - \underset{c}{\underset{\downarrow}{2/3\, P_2(x)}} - \underset{b}{\underset{\downarrow}{9/5\, p_1(x)}} + \underset{a}{\underset{\downarrow}{5/3\, p_0(x)}}$

$\therefore \boxed{a = 5/3}$ $\boxed{b = -9/5}$ $\boxed{c = -2/3}$ $\boxed{d = 4/3}$