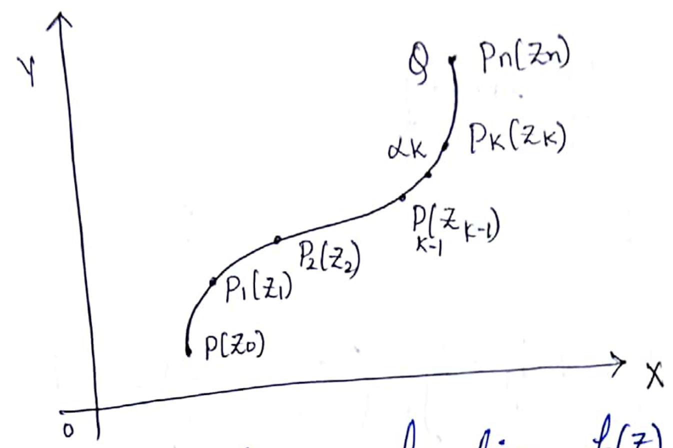
SEMESTER - 4

COMPLEX VARIABLES : MODULE - 3 (PART-2)

Complex line integral :-



Consider a continuous function, $f(z)$ of the complex variable $z = x+iy$ defined at all points of curve C extending from P to Q, dividing the curve C into n parts by arbitrarily taking points, $P = p(z_0), p(z_1) \ldots\ldots p_n(z_n) = Q$, then $\lim_{n\to\infty} \sum_{K=1}^{n} f(\alpha_K)\,\delta z_K$ where $\max|\delta z_K| \to 0$ as $n \to \infty$ is defined as complex line integral along path C denoted by : $\int_C f(z).dz$.

where ; $\delta z_K = z_K - z_{K-1}$. , $\alpha(K)$ is point on arc of Curve.

CAUCHY'S THEOREM :-

Statement :- If $f(z)$ is analytic at all points inside and on a simple closed curve C, then : $\boxed{\int_C f(z).dz = 0.}$

proof :- Let $f(z) = u+iv$. // $dz = dx+idy$.

Then, $\int_C f(z).dz = \int_C (u+iv).(dx+idy)$

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx+udy) \quad -(1)$$

We have ; Green's theorem in a plane stating that, if $M(x,y)$ & $N(x,y)$ are 2 real valued functions having continuous 1st order partial derivatives in a region R bounded by the curve C then ;

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \quad \sim (2)$$

Applying this theorem to the two line integrals in RHS of (1) ; we get

$$\int_C f(z).dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right).dx\,dy.$$

Since, $f(z)$ is analytic , we have Cauchy-Riemann Equations :

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

, hence we have ;

$$\int_C f(z)\,dz = \iint_R \underbrace{\left(\cancel{\frac{\partial u}{\partial y}} - \cancel{\frac{\partial u}{\partial y}}\right)}_{\to 0}.dx\,dy + i\iint_R \underbrace{\left(\cancel{\frac{\partial v}{\partial y}} - \cancel{\frac{\partial v}{\partial y}}\right)}_{\to 0}.dx\,dy.$$

Thus, we get ; $\boxed{\int_C f(z).dz = 0.}$

Hence, the proof of Cauchy's Theorem.

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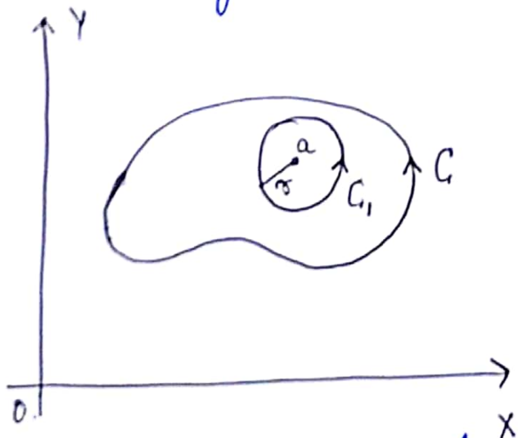
Derive the Cauchy's Integral Formula :-

Statement :- If $f(z)$ is analytic inside and on a simple closed curve C and if "a" is any point within C then;

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,.dz.}$$

proof :- Since "a" is a point within C, we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius such that C_1 lies entirely within C.

The function; $\frac{f(z)}{z-a}$ is analytic and on the boundary of the annular region b/w C & C_1.



Now, By the Consequence of the Cauchy's Theorem;

We have; $$\int_C \frac{f(z)}{z-a}\,.dz = \int_{C_1} \frac{f(z)}{z-a}\,.dz. \quad \text{--- (1)}$$

The Equation of C_1 (circle with centre "a" & radius "r"), can be written in the form :- $|z-a| = r$.

Which is equivalent to $\Rightarrow$ $z-a = re^{i\theta}$

$z = a+re^{i\theta}$, $0 \le \theta \le 2\pi$, $dz = ire^{i\theta}.d\theta$

$\Rightarrow$ using these results in Equation ①, RHS ; we get.

$$\int_C \frac{f(z)}{z-a} \cdot dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta} \cdot d\theta.$$

$$\int_C \frac{f(z)}{z-a} \cdot dz = i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\, d\theta.$$

This is true for any ; $r>0$, however small,

Hence ; $r \to 0$ we get...

$$\int_C \frac{f(z)}{z-a} \cdot dz = i\int_{\theta=0}^{2\pi} f(a) \cdot d\theta = i\, f(a)\, [\theta]_0^{2\pi}.$$

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

$$\Rightarrow \boxed{f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \cdot dz.}$$

Hence, the proof of Cauchy's Integral theorem

* Singularity / Singular point :-

A point $z=a$, where $f(z)$ fails to be analytic is called singularity or singular point of $f(z)$. $\rightarrow$ Ex:- $f(z) = \frac{z}{z-2}$.

* [Sing. pts are points which make Dr $\to$ terms $\to 0$]

$z=2$ is singular point.

Pole :- If the principal part of $f(z)$, consists of only a finite no of terms, say m ; then we say that ; $z=a$ is a "pole of order m".

Simple pole :- A pole of order 1 ($m=1$) is simple pole.

Residues :- The coefficient of $\frac{1}{z-a}$, that is a_{-1} in the expansion of $f(z)$ is called the residue of $f(z)$ at the pole $z=a$.

Ex :- $f(z) = \frac{\cos z}{z^5}$, then $f(z)$ can be expanded as;

$$f(z) = \frac{1}{z^5}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots\right]$$

$$f(z) = \left[\frac{1}{z^5} - \frac{1}{2!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z} \cdots\right]$$

$\therefore$ The residue of $f(z)$ at pole, $z=0$ is coefficent of $\frac{1}{z-0} = \frac{1}{z}$

$\Rightarrow \frac{1}{4!} = \frac{1}{24} \quad \Rightarrow \frac{1}{24}$ is Residue

$\longrightarrow$ Problems after Cauchy's Residue Theorem stmt :-

* [Problems & solns :-

1) For the function ; $f(z) = \frac{2z+1}{z^2-z-2}$, determine the poles & residue at the poles.

Soln :- In $f(z) = \frac{2z+1}{z^2-z-2}$

$\rightarrow z^2 - z - 2$
$z^2 - 2z + 1z - 2$
$z(z-2) + 1(z-2) = \boxed{(z+1)(z-2)}$

$-2 \cdot 1 = -2$, $-2+1$

WKT, [The poles are the points, by which by substituting them, we get $Dr \to 0$]

$$f(z) = \frac{2z+1}{(z-2)^1(z+1)^1} \quad m=1 \text{ (Simple pole)}$$

$\therefore$ The poles (simple) are :- $z=2$ & $z=-1$

Now, to find residue : at $z=a=2$]

* ## Cauchy's Residue Theorem :-

Statement :- If $f(z)$ is analytic inside and on the boundary of a simple closed curve C, except for a finite number of poles, $a, b, c \ldots$, then integral of $f(z)$ over C is equal to $2\pi i$ times the sum of residues at the poles inside C,

$$\Rightarrow \int_C f(z)dz = 2\pi i\left(a_{-1} + b_{-1} + c_{-1} + \ldots\right)$$

Important formulae :-

1) If we obtain simple pole.. then $\boxed{\underset{(\text{Residue})}{a_{-1}} = \lim_{z\to a} \{(z-a) f(z)\}}$

2) If $z=a$ is pole of order m, then : $\boxed{R(m,a) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]}$

3) $\int_C f(z)dz = 2\pi i(R_1+R_2)$

→ problem (1) continuation....

Now, to find Residue at $z=a=2$

$$\Rightarrow \lim_{z\to 2} (z-2).f(z) = \lim_{z\to 2} \cancel{(z-2)}.\frac{2z+1}{\cancel{(z-2)}(z+1)}$$

$$= \lim_{z\to 2} \frac{2z+1}{z+1}$$

$$= \frac{2(2)+1}{2+1} = \frac{5}{3} \text{ is Residue at } z=2$$

Now, to find residue at $z=a=-1$

$$\Rightarrow \lim_{z\to -1} (z+1).f(z) = \lim_{z\to -1} \cancel{(z+1)}\frac{(2z+1)}{(z-2)\cancel{(z+1)}}$$

$$= \lim_{z\to -1} \frac{(2z+1)}{(z-2)} = \frac{2(-1)+1}{(-1-2)} = \frac{-1}{-3}.$$

$\therefore \frac{1}{3}$ is Residue at $z=-1$

——*——

Determine ... $\sin z$ ... $(2z$ ... Soln

④

Determine residue at pole of function;

$f(z) = \frac{\sin z}{(2z-\pi)^2}$

Soln :- $f(z) = \frac{\sin z}{(2z-\pi)^2}$, $m=2$, It is a pole of order; $m=2$

Now, $2z-\pi = 0$, $2z = \pi$

$\boxed{z = \pi/2}$ is pole of order 2.

Now, to find the residue of $f(z)$ at $z = a = \pi/2$

$= \lim_{z\to\pi/2} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z-\pi/2)^2 \cdot f(z) \right\}$ // By formula..

$$\boxed{R(m,a) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}.}$$

$= \lim_{z\to\pi/2} \frac{1}{1!} \frac{d}{dz} \left\{ (z-\pi/2)^2 \cdot \frac{\sin z}{(2z-\pi)^2} \right\}$ $= \lim_{z\to\pi/2} \frac{d}{dz} \left\{ \frac{(2z-\pi)^2}{2^2} \cdot \frac{\sin z}{(2z-\pi)^2} \right\}$

$= \frac{1}{4} \lim_{z\to\pi/2} \frac{d}{dz} \left\{ \sin z \right\} = \frac{1}{4} \lim_{z\to\pi/2} (\cos z) = \frac{1}{4} \left[\cos \pi/2 \right]$

$= 1/4 (0)$

$= 0.$

$\therefore$ Thus, the residue of pole is 0.

3) Find the residues of the function :-

$f(z) = \frac{z}{(z+1)(z-2)^2}$ also find poles.

Soln:- $f(z) = \frac{z}{(z+1)(z-2)^2}$

here, $z+1 \Rightarrow z=-1$ is a pole of "order 1"

$(z-2)^2 \Rightarrow z=2$ is a pole of "order 2".

$\Rightarrow$ Case:1 To find Residue of $f(z)$ for a simple pole ; i.e, $z = -1$ // order "$m=1$"

$$\Rightarrow \lim_{z \to -1} (z+1) \cdot f(z) = \lim_{z \to -1} \cancel{(z+1)} \left[\frac{z}{\cancel{(z+1)}(z-2)^2} \right]$$

$$= \lim_{z \to -1} \left\{ \frac{z}{(z-2)^2} \right\} = \frac{-1}{(-1-2)^2} = \frac{-1}{9}$$ is Residue at pole $z=-1$

$\Rightarrow$ Case:2 To find Residue of $f(z)$ for pole ; $z=2$ of "order 2."

$$\Rightarrow \lim_{z \to 2} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z-\cancel{2})^2 \cdot \left[\frac{z}{(z+1)(z-\cancel{2})^2} \right] \right.$$

$$= \lim_{z \to 2} \frac{1}{1!} \frac{d}{dz} \left\{ \frac{z}{z+1} \right\}$$ // Quotient rule...

$$= \lim_{z \to 2} \left\{ \frac{1}{(z+1)^2} \left[(z+1)(1) - z(1+0) \right] \right\}$$

$$= \lim_{z \to 2} \left[\frac{z+1-z}{(z+1)^2} \right] = \frac{1}{(2+1)^2} = \frac{1}{9}$$ is Residue at pole $z=2$

Assmt

\#) Determine the poles & residue of functions given :-

① $f(z) = \dfrac{z}{(z-1)^2(z+2)}$.

② $f(z) = \dfrac{4z-1}{z^2-z-2}$.

③ $f(z) = \dfrac{z}{(z+2)^2}$.

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Problems :-

1) Evaluate : $\int_C \frac{e^{2z}}{(z+1)(z-2)} \cdot dz$, where C is the circle $|z|=3$.

Soln :- The poles of the function ; $f(z) = \frac{e^{2z}}{(z+1)(z-2)}$

are :- $\underline{\underline{z=-1}}$ & $\underline{\underline{z=2}}$ which are simple poles and both these poles lie within the circle ; $|z|=3$.

∴ Residue of $f(z)$ at $\underline{\underline{z=a=-1}}$ is given by ;

$$\lim_{z\to -1} (z+1).f(z) = \lim_{z\to -1} \cancel{(z+1)}\left[\frac{e^{2z}}{\cancel{(z+1)}(z-2)}\right] = \lim_{z\to -1}\left[\frac{e^{2z}}{z-2}\right]$$

$$= \frac{e^{-2}}{-3} = \boxed{\frac{-1}{3e^2} = R_1}, \text{(say)}$$

Also, Residue of $f(z)$ at $\underline{\underline{z=a=2}}$ is given by;

$$\lim_{z\to 2} \cancel{(z-2)}\left[\frac{e^{2z}}{(z+1)\cancel{(z-2)}}\right] = \lim_{z\to 2}\left[\frac{e^{2z}}{z+1}\right] = \boxed{\frac{e^4}{3} = R_2} \text{(say)}$$

Now, we have by Cauchy's Residue Theorem ;

$$\int_C f(z).dz = 2\pi i\,(R_1+R_2).$$

$$\int_C \frac{e^{2z}}{(z+1)(z-2)} dz = 2\pi i\left[\frac{-1}{3e^2}+\frac{e^4}{3}\right] = 2\pi i\left(e^4-\frac{1}{e^2}\right) //.$$

2) Evaluate : $\int_C \frac{z^2+5}{(z-2)(z-3)}$ using Residue theorem; $C : |z|=4$.

3) Evaluate :- $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where $C : |z| = 3$.

Soln :- $f(z) = \frac{e^{2z}}{(z+1)^4}$.

$z=-1$ is a pole of order 4, $m=4$ which lies inside C

$|z| = 3$.

$\therefore$ The residue of $f(z)$ at $z=a=-1$ is given by ;

$$= \lim_{z\to -1} \frac{1}{(4-1)!} \frac{d^3}{dz^3}\left\{ \cancel{(z+1)^4} \frac{e^{2z}}{\cancel{(z+1)^4}} \right\}$$

$$= \lim_{z\to -1} \frac{1}{3!} \frac{d^3}{dz^3}\left\{ e^{2z} \right\} = \lim_{z\to -1} (8 \cdot e^{2z})$$

$$= \frac{4}{3} e^{-2}.$$

By Applying Cauchy's Residue thm ;

$$\int_C f(z).dz = 2\pi i \left(\frac{4}{3} e^{-2} \right)$$

$$\therefore \boxed{\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i}{3e^2}}$$

Asmt Evaluate :- $\int_C \frac{z^2}{(z-1)^2(z+1)} \cdot dz$, $C : |z| = 2$. (Ans :- $4\pi i$)

Evaluate :- $\int_C \frac{z-3}{z^2+2z+5}$ $C : |z| = 1$ (Ans :- 0)

Evaluate :- $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)} . dz$, where

C is the circle $|z| = 3$.

Soln :- Let $f(z) = \int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}$; $C : |z| = 3$.

$\Rightarrow$ $z=1$ is a pole of order 2 & $z=2$ is a pole of order 1

Both of them lies within the circle; $|z|=3$.

$\therefore$ Residue at $z=1$ be denoted by R_1;

$$\Rightarrow R_1 = \lim_{z\to 1} \frac{1}{(2-1)!} \frac{d}{dz}\left\{ \cancel{(z-1)^2} . \frac{(\sin\pi z^2 + \cos\pi z^2)}{\cancel{(z-1)^2}(z-2)} \right\}$$

$$= \lim_{z\to 1} \frac{d}{dz}\left\{ \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} \right\}$$

$$= \lim_{z\to 1} (\sin\pi z^2 + \cos\pi z^2) . \frac{-1}{(z-2)^2} + \lim_{z\to 1} 2\pi z(\cos\pi z^2 - \sin\pi z^2) . \frac{1}{z-2}.$$

$\therefore$ $\boxed{R_1 = (1+2\pi)}$ // $\sin\pi = 0$, $\cos\pi = -1$

Now, Residue at $z=2$ be denoted by R_2,

$$R_2 = \lim_{z\to 2} \cancel{(z-2)} \left\{ \frac{\sin\pi z^2 + \cos\pi z^2}{\cancel{(z-2)}(z-1)^2} \right\} = \lim_{z\to 2} \left\{ \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2} \right\} = 1$$

$\therefore$ $\boxed{R_2 = 1}$

Hence, by Cauchy's Residue theorem, $\int_C f(z)dz = 2\pi i (R_1 + R_2)$

$$= 2\pi i (1+2\pi+1) = 4\pi i(1+\pi)$$

$$\therefore \boxed{\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)} . dz = 4\pi i(1+\pi).}$$ //..

Asmt

5) Evaluate; $f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$, C, $|z| = 3$

* **Conformal Transformations :-**

If a transformation preserves the angle b/w any 2 curves both in magnitude and sense then it is called a Conformal transformation.

Imp * Discuss the transformation ; $w = e^{z}$

Consider ; $w = e^{z}$

ie, $u+iv = e^{x+iy}$ // $w = u+iv$, $z = x+iy$.

$u+iv = e^{x} \cdot e^{iy}$ // $e^{iy} = \cos y + i\sin y$.

$u+iv = e^{x}[\cos y + i\sin y]$

$\therefore u = e^{x}\cos y$, $v = e^{x}\sin y$. —(1)

Now, we shall find the image in the w-plane corresponding to the straight lines parallel to the co-ordinate axes in the z plane

ie, $x =$ Constant , $y =$ Constant.

Let us eliminate x & y seperately from Eqn (1) ;

$\Rightarrow$ Squaring & Adding ; we get ; $u^2+v^2 = (e^{x}\cos y)^2 + (e^{x}\sin y)^2$

$\Rightarrow u^2+v^2 = e^{2x}\cos^2 y + e^{2x}\sin^2 y$.

$= e^{2x}[\sin^2 y + \cos^2 y]$ $\Rightarrow$ $u^2+v^2 = e^{2x}$ ~(2)

Also, by dividing ; we get...

$\Rightarrow \frac{u}{v} = \frac{e^{x}\cos y}{e^{x}\sin y} \Rightarrow \frac{u}{v} = \tan y$. ~(3)

Case:1 Let $x = c_1$ where c_1 is Constant.

Equation (2) becomes; $u^2 + v^2 = e^{2x} = e^{2c_1} = \text{Constant} = r^2$, (say)

This represents a circle with centre origin & radius r in the w-plane.

Case:2 Let $y = c_2$ where c_2 is Constant.

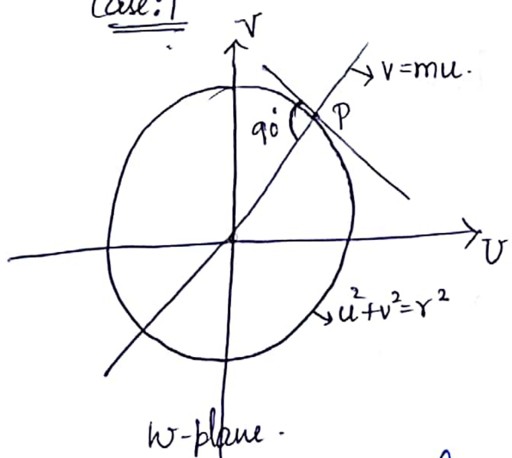
Equation (3) becomes; $\frac{u}{v} = \tan y = \tan c_2 = m$, (say)

$\therefore v = mu$

This represents a straight line passing through the origin in the w-plane.

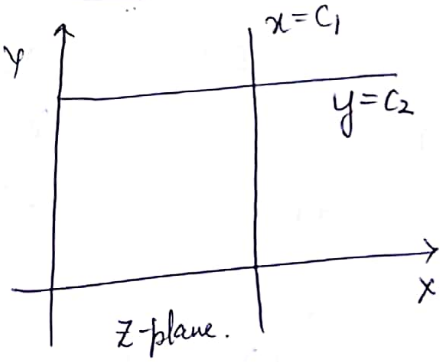
Conclusion:- The straight line parallel to the x-axis $(y = c_2)$ in

Case:1



w-plane.

Case:2



the plane maps onto straight line passing through origin in w-plane. The straight line parallel to y-axis $(x = c_1)$ in z-plane maps onto the cle with centre origin & radius r, where $r = e^{c_1}$ in w-plane.

Suppose, we draw tangent ~~to~~ at pt of intersection of these 2 curves in w-plane, the angle subtended is equal to 90°.

Hence 2 Curves can be regarded as orthogonal trajectories of each other.

* Discuss the transformation :- $w = z + a^2/z$, $z \neq 0$. since,

Consider, $w = z + (a^2/z)$.

putting; $z = re^{i\theta}$, we have, ... $w = u + iv$.

$\Rightarrow u + iv = re^{i\theta} + (a^2/r).e^{-i\theta}$ // $r = \cos\theta + i\sin\theta$.

ie, $u + iv = r[\cos\theta + i\sin\theta] + (a^2/r)(\cos\theta - i\sin\theta)$

$u + iv = [r\cos\theta + a^2/r \cos\theta] + i[r\sin\theta - \frac{a^2}{r}\sin\theta]$

$\Rightarrow u + iv = [r + a^2/r]\cos\theta + i[r - a^2/r]\sin\theta$

$\therefore u = [r + a^2/r]\cos\theta \qquad v = [r - a^2/r]\sin\theta$ ~ (1).

Now, we shall elliminate r & θ seperately from (1) ...

To eliminate θ, let us put (1) in the form..

$$\frac{u}{[r + a^2/r]} = \cos\theta \quad ; \quad \frac{v}{[r - (a^2/r)]} = \sin\theta.$$

$\Rightarrow$ squaring & adding, we obtain...

$$\frac{u^2}{[r + a^2/r]^2} + \frac{v^2}{[r - a^2/r]^2} = 1 \quad , r \neq a.$$

To eliminate r, let us put (1) in form...

$$\frac{u}{\cos\theta} = [r + a^2/r] \quad , \quad \frac{v}{\sin\theta} = [r - (a^2/r)]$$

$\Rightarrow$ squaring & subtracting, we obtain...

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = [r + (a^2/r)]^2 - [r - (a^2/r^2)]^2 = 4a^2$$

$$\frac{u^2}{(2a\cos\theta)^2} - \frac{v^2}{(2a\sin\theta)^2} = 1 \sim (3)$$

$z \neq 0$.

Since, $z = re^{i\theta}$, $|z| = r$ & $amp(z) = \theta$

$|z| = r \Rightarrow \sqrt{x^2+y^2} = r$ (or) $x^2+y^2 = r^2$

This represents a ~~straight~~ "circle" with centre origin & radius r, in the z-plane, when r is constant.

$\Rightarrow amp(z) = \theta$, $\tan^{-1}(y/x) = \theta$ or $y/x = \tan\theta$.

This represents "straight line" in z plane when θ is Constant.

We shall discuss the image in the w-plane, corresponding to $r = $ Const, (circle) & $\theta = $ Const. (straight line) in z-plane.

Case: 1 Let $r = $ Constant

Equation (2) is of the form;

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1, \text{ where } A = [r + (a^2/r)], \ B = [r - (a^2/r)]$$

This represents an ellipse in the w-plane with $f = (\pm\sqrt{A^2-B^2}, 0) = (\pm 2a, 0)$.

Since; $\sqrt{A^2-B^2} = \sqrt{(r+(a^2/r))^2 - [r-(a^2/r)]^2} = \sqrt{4a^2} = \pm 2a$

Hence, we conclude that $\bigcirc^{le}$, $|z| = r = $ Const in z-plane maps onto "ellipse" in w-plane

Case: 2 Let $\theta = $ Constant.

Equation (3) is of the form...

$$\Rightarrow \frac{u^2}{A^2} - \frac{v^2}{B^2} = 1, \text{ where } A = 2a\cos\theta, \ B = 2a\sin\theta.$$

This, represents hyperbola in w-plane with

$f = (\pm\sqrt{A^2+B^2}, 0) = (\pm 2a, 0)$.

Hence, we conclude, that "straight line" passing through in z-plane maps onto a "hyperbola in w-plane" ∴

3) Discuss the transformation; $w = z + 1/z$.

Conformal transformation :-

1) Discuss the transformation of : $w = z^2$

Soln :- $w = z^2$

$\Rightarrow u + iv = (x+iy)^2$ where ; $w = u+iv$
$z = x+iy.$

$\Rightarrow u+iv = x^2-y^2+i2xy$

$\Rightarrow$ Seperating Real & imaginary parts...

$\therefore u = x^2 - y^2 \sim (1)$

$\therefore v = 2xy \sim (2)$

Case : 1 $x = c_1$ (Constant) , Replace in Equation (1) & (2) ;

$\therefore u = c_1^2 - y^2$

$\therefore v = 2c_1 y.$

$\Rightarrow y = \frac{v}{2c_1}$

$\Rightarrow u = c_1^2 - \frac{v^2}{4c_1^2}$ $\Rightarrow 4uc_1^2 = 4c_1^2 \cdot c_1^2 - v^2$

$v^2 = -4c_1^2 \cdot c_1^2 - \cancel{v^2}\, 4uc_1^2$

$\Rightarrow v^2 = -4c_1^2 [u - c_1^2] \sim (3)$ // $y^2 = 4ax$ is parabola ..

$\Rightarrow$ Equation (3) represents the Equation of the parabola., which is symmentrical about real-axis & focus at the origin.

Case : 2 $y = c_1$,(Constant) Replace in Eqn (1) & (2),...

$\therefore u = x^2 - c_1^2$ & $\therefore v = 2c_1 x$

$\Rightarrow x = \frac{v}{2c_1}$

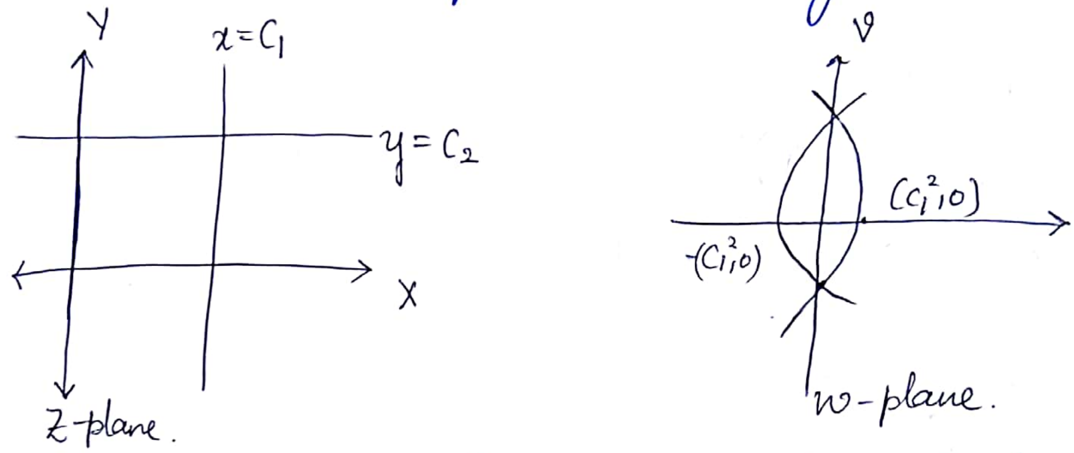
$\Rightarrow u = \frac{v^2}{4c_1^2} - c_1^2$

$u = \frac{v^2 - 4c_1c_1^2}{4c_1^2}$ $\Rightarrow 4c_1^2 u = v^2 - 4c_1c_1^2$

$v^2 = 4c_1^2 u + 4c_1c_1^2$

$$\therefore v^2 = 4c_1^2[u + c_1^2] \sim (4)$$

$\Rightarrow$ Eqn (4) represents equation of the parabola, which is symmetric about the real axis & focus at the origin.



Conclusion:- Hence, we conclude that the line which is parallel to co-ordinate axis in z-plane which maps onto equation of parabola in w-plane.

✓2) Discuss the transformation; $w = e^z$

Soln:- $u + iv = e^z$ // $w = u + iv$, $z = x + iy$.

$u + iv = e^{x+iy}$.

$u + iv = e^x \cdot e^{iy}$ // $e^{iy} = \cos y + i\sin y$.

$u + iv = e^x \cdot [\cos y + i\sin y]$

$\Rightarrow$ Separating real & imaginary parts:-

$\Rightarrow$ $u = e^x \cos y$ & $v = e^x \sin y$.

Consider; $u^2 + v^2 = e^{2x} \cdot \cos^2 y + e^{2x} \cdot \sin^2 y$.

$= e^{2x} (\cos^2 y + \sin^2 y)$

$\therefore \boxed{u^2 + v^2 = e^{2x}} \sim (1)$

Consider;

$\frac{u}{v} = \frac{e^x \cos y}{e^x \sin y}$.

$\therefore \boxed{\frac{u}{v} = \cancel{\tan y}}$ $\sim (2)$

$\cot y$

$\frac{v}{u} = \tan y \sim (2)$.

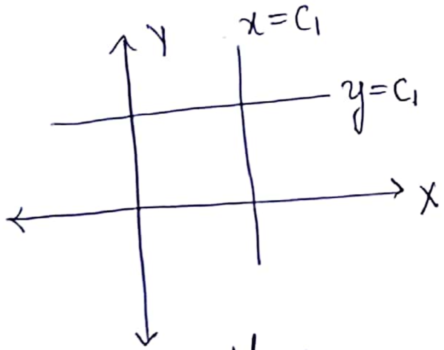
Case: 1 $x = c_1$, Constant, sub in Eqⁿ ①; we get...

$$\Rightarrow u^2+v^2 = e^{2c_1} = r^2 \text{ (say)}. \quad \text{--- (3)}$$

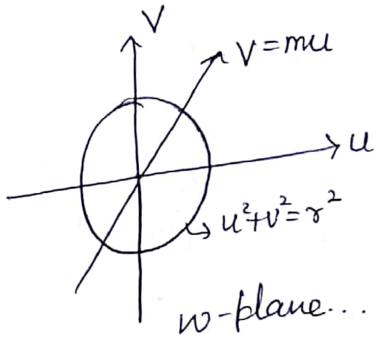
Case: 2 $y = c_1$, constant, sub in Eqⁿ ②, we get...

$$\Rightarrow \frac{v}{u} = \tan c_1$$

$$\Rightarrow v = mu, \text{ (say)} \quad \text{--- (4)}$$



z-plane.



w-plane...

Conclusion:- The line which is ∥ᵉˡ to y-axis $[x=c_1]$ in z-plane, maps onto Equation of ⊙ˡᵉ $[u^2+v^2=r^2]$ in w-plane. ∥ˡʸ, the line parallel to x-axis $[y=c_1]$ in z-plan maps on to Equation of straight line $[v=mu]$ in w-plane...

✓3) **Discuss the transformation of : $w = z + 1/z$.**

*[If we solve this in Cartesian form, it will be complicated, Hence we solve it by Polar form].

Soln:- $w = z + 1/z$. // $w = u+iv$
// $z = re^{i\theta}$

$$\Rightarrow u+iv = re^{i\theta} + 1/re^{i\theta}$$

$$\Rightarrow u+iv = re^{i\theta} + \frac{1}{r}\cdot e^{-i\theta} \qquad // \; e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow u+iv = r[\cos\theta + i\sin\theta] + \frac{1}{r}[\cos\theta - i\sin\theta]$$

$$u+iv = (r+1/r)\cos\theta + i[r - 1/r]\sin\theta.$$

⇒ Now, by seperating real & imaginary parts..

$\Rightarrow u = (r+1/r)\cos\theta \quad \& \quad v = (r-1/r)\sin\theta.$

$\Rightarrow \dfrac{u}{\cos\theta} = r + 1/r. \sim \circledast_1 \quad \& \quad \dfrac{v}{\sin\theta} = r - 1/r. \sim \circledast_2$

$\Rightarrow$ SBS $\qquad\qquad \Rightarrow$ SBS

$\Rightarrow \dfrac{u^2}{\cos^2\theta} = (r+1/r)^2 \quad \& \quad \dfrac{v^2}{\sin^2\theta} = (r-1/r)^2$

$\dfrac{u^2}{\cos^2\theta} = r^2 + 1/r^2 + 2.\cancel{r}.1/\cancel{r} \quad \& \quad \dfrac{v^2}{\sin^2\theta} = r^2 + 1/r^2 - 2.\cancel{r}.1/\cancel{r}$

Consider; $\dfrac{u^2}{\cos^2\theta} - \dfrac{v^2}{\sin^2\theta} = \cancel{r^2} + \cancel{1/r^2} + 2 - \cancel{r^2} - \cancel{1/r^2} + 2$

$\therefore \dfrac{u^2}{\cos^2\theta} - \dfrac{v^2}{\sin^2\theta} = 4$

$\Rightarrow \dfrac{u^2}{(2\cos\theta)^2} - \dfrac{v^2}{(2\sin\theta)^2} = 1. \sim (1).$

Since, by $\circledast_1$ & $\circledast_2$ $\Rightarrow \dfrac{u}{r+1/r} = \cos\theta \sim (2)$ & $\dfrac{v}{r-1/r} = \sin\theta \sim (3).$

$\Rightarrow$ using in (1) $\Rightarrow$ S.B.S of (3) & (2) $\quad \dfrac{u^2}{(r+1/r)^2} + \dfrac{v^2}{(r-1/r)^2} = \cos^2\theta + \sin^2\theta.$

$\Rightarrow \dfrac{u^2}{(r+1/r)^2} + \dfrac{v^2}{(r-1/r)^2} = 1 \sim (4).$

$\Rightarrow \underline{\underline{z = re^{i\theta}}}, \quad r = \sqrt{x^2+y^2}$

$\underline{\underline{r^2 = (x^2+y^2)}}$

$\underline{\underline{\theta = \tan^{-1}(y/x)}} \Rightarrow \dfrac{y}{x} = \tan\theta$

$\underline{\underline{y = x.\tan\theta.}}$

$\Rightarrow$ ∴ // Supp... Sele...

:1 Suppose : $r = c_1$ (or) $x^2+y^2 = c_1^2$ [Eqn of circle]

$\Rightarrow$ Substituting in Eqn ①...

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 \sim ⑤ \quad \text{where; } A^2 = \left[c_1 + \frac{1}{c_1}\right]^2$$

$$B^2 = \left[c_1 - \frac{1}{c_1}\right]^2$$

$\Rightarrow$ Eqn ⑤ represent Eqn of Ellipse with the focii $[\pm\sqrt{A^2-B^2}, 0] = [\pm 2a, 0]$

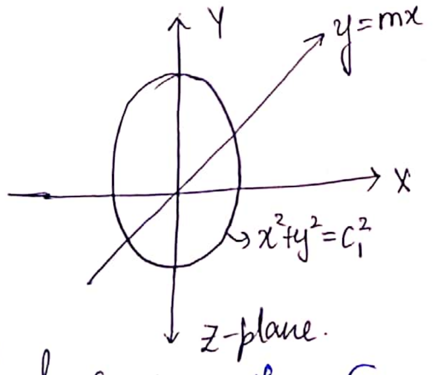
Case : 2 Suppose : $\theta = c_1$ (or) $y = x\tan c_1 = mx$, (say).

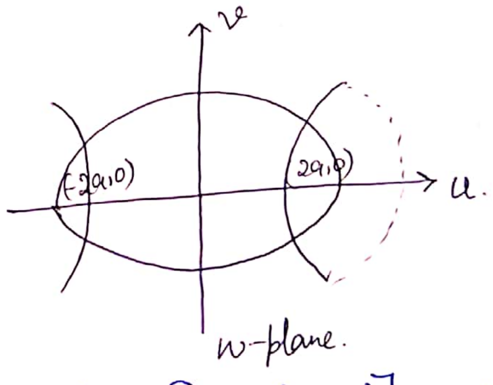
Eqn ① becomes ;

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1 \sim ⑥. \quad \text{where; } (2\cos c_1)^2 = A^2$$

$$(2\sin c_1) = B^2$$

Eqn ⑥ represents ~~Eqn~~ of the ellipse with focci $[\pm\sqrt{A^2+B^2}, 0] = [\pm 2a, 0]$.





Conclusion :- The Eqn of circle in z-plane $[x^2+y^2=c_1^2]$, maps onto the Equation of ellipse in $\left[\frac{u^2}{A^2}+\frac{v^2}{B^2}=1\right]$ in W-plane.

$|||^{ly}$, The Eqn of straight line in z-plane $[y=mx]$, maps onto the Equation of the Ellipse $\left[\left[\frac{u^2}{A^2}-\frac{v^2}{B^2}\right]=1\right]$ in w-plane.

—— * ——

BILINEAR TRANSFORMATION :- [B.T]

The transformation, $w = \dfrac{az+b}{cz+d}$, where a, b, c, d are real / Complex constant such that; $ad-bc \neq 0$. is called <u>bilinear transformation</u>.

<u>Invariant points</u> :-
If the point z, maps itself ie; $\underline{w=z}$ under the bilinear transformation, then the point is called as invariant point (or) fixed point.

To find the bilinear transformations;

We have;

$$w = \frac{az+b}{cz+d}.$$

$$\text{(or)} \quad \boxed{w = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}}$$

Problems : on Bilinear transformations :-

1) Find the bilinear transformation which maps the points, $z = 1, i, -1$ into $w = i, 0, -i$.

Soln:- Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

Now, we shall substitute the given values of z & w to obtain 3 equivalent equations as follows :-

$w_1 = i$, $w_2 = 0$, $w_3 = -i$

$z_1 = 1$, $z_2 = i$, $z_3 = -1$

$z=1, w=i$; $w_1 = \dfrac{az_1+b}{cz_1+d}$

$$\Rightarrow i = \frac{a(1)+b}{c(1)+d} \quad \Rightarrow a+b-ic-id = 0 \sim (1)$$

$z_2 = i$, $w_2 = 0$; $w_2 = \dfrac{az_2+b}{cz_2+d}$

$$\Rightarrow 0 = \frac{a(i)+b}{c(i)+d} \quad \Rightarrow ai+b = 0 \sim (2)$$

$z_3 = -1$, $w_3 = -i$; $w_3 = \dfrac{az_3+b}{cz_3+d}$

$$\Rightarrow -i = \frac{a(-1)+b}{c(-1)+d} \quad \Rightarrow -a+b-ic+id = 0 \sim (3)$$

Consider; Eqn (1) + (3) ... $\cancel{a}+b-ic-i\cancel{d}-\cancel{a}+b-ic+i\cancel{d} = 0$

$$\Rightarrow 2b-2ic = 0$$

$$b-ic = 0 \sim (4)$$

Now, we shall solve; (2) & (4);

$ia + b + 0c = 0$

$0a + 1b - ic = 0.$

Applying the rule of Cross multiplication, we have...

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}} = \frac{c}{\begin{vmatrix} i & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\Rightarrow \frac{a}{-i} = \frac{-b}{i^2} = \frac{c}{i} \quad \text{(or)} \quad \frac{a}{-1} = \frac{b}{-1} = \frac{c}{i} = K, \text{ say.}$$

$\therefore \underline{\underline{a = -iK}}$, $\underline{\underline{b = -K}}$, $\underline{\underline{c = iK}}$

$\Rightarrow$ sub in Eqn (1), we get..

(1) $\Rightarrow -iK - K + K - di = 0.$

$\Rightarrow -(di + iK) = 0.$

$\Rightarrow \underline{\underline{d = -K}}$ $\Rightarrow$ sub, values of a, b, c, d in assumed bilinear transformations, we get..

$$\therefore w = \frac{-iKz - K}{+iKz - K} = \frac{-K(1 + iz)}{-K(1 - iz)}.$$

Thus, $\boxed{w = \frac{1 + iz}{1 - iz}.}$ ie Required bilinear transformation

2) Find ... (cut off at the right edge)

2) Find the Bilinear transformation, which maps the points ; $z = 1, i, -1$, into $w = 2, i, -2$. , also find invariant points (or) fixed pt of transformation.

Soln :- Let $w = \dfrac{az+b}{cz+d}$.

$z = 1, i, -i$, $w = 2, i, -2$.

$\Rightarrow$ $z=1$, $w=2$ $\Rightarrow$ $w_1 = \dfrac{az_1+b}{cz_2+d}$.

$2 = \dfrac{a(1)+b}{c(1)+d}$ $\Rightarrow$ $2c+2d = a+b$.

$a+b-2c-2d=0 \sim (1)$.

$\Rightarrow$ $z=i$, $w=i$ $\Rightarrow$ $w_2 = \dfrac{az_2+b}{cz_2+d}$

$i = \dfrac{a(i)+b}{c(i)+d}$ $\Rightarrow$ $ci^2+di = ai+b$

$ai+b+c-di=0 \sim (2)$.

$\Rightarrow$ $z=-i$, $w=-2$ $\Rightarrow$ $-2 = \dfrac{-a+b}{-c+d}$.

$\Rightarrow a-b+2c-2d=0 \sim (3)$.

$\Rightarrow$ Sub Eqn in (1) & (3).

$a+\cancel{b}-\cancel{2c}-2d=0$.

$a-\cancel{b}+\cancel{2c}-2d=0$

$2a-4d=0 \sim (4)$

$\Rightarrow$ Sub Eqn in (2) & (4) ...

$ai+b+c-id=0$.

$2a+0b+0c-4d=0$.

2

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}.$$

$$\Rightarrow \frac{(w-2)(i+2)}{(2-i)(\underline{-2-w})} = \frac{(z-1)(i+1)}{(1-i)(\underline{-1-z})}. \rightarrow (-\text{Common})$$

$$\Rightarrow \frac{(w-2)(i+2)}{-(2-i)(2+w)} = \frac{(z-1)(i+1)}{-(1-i)(1+z)}$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{z-1}{z+1}\left\{\frac{(i+1)(2-i)}{(1-i)(i+2)}\right\}$$

$$\frac{w-2}{w+2} = \frac{z-1}{z+1}\left\{\frac{2i-i^2-2-i}{i+2-i^2-2i}\right\}$$

$$\frac{w-2}{w+2} = \frac{(z-1)}{(z+1)}\frac{(i+3)}{(-i+3)}.$$

$$= \frac{(w-2)(-i+3)}{(w+2)(i+3)} = \frac{z-1}{z+1}.$$

$$\Rightarrow (w-2)(-i+3)(z+1) = (z-1)(w+2)(i+3).$$

$$\Rightarrow (w-2)[-iz-i+3z+3] = (w+2)[zi+3z-i-3]$$

$$\Rightarrow -iwz - \cancel{iw} + 3\cancel{wz} + 3w + \cancel{2iz} + 6z - 2i - \cancel{6}.$$

$$-[iwz + 3\cancel{wz} - \cancel{iw} - 3w + \cancel{2iz} + 6z - 2i - \cancel{6}]$$

$$\Rightarrow -iwz + 3w + 2i - 6z = iwz - 3w - 2i + 6z. = 0.$$

$$\Rightarrow -2iwz + 6w + 4i - 12z = 0.$$

$$iwz - 3w - 2i + 6z = 0.$$

$$w(iz-3) = 2i - 6z.$$

$$\therefore \boxed{w = \frac{2i-6z}{iz-3}.}$$

For invariant pt,

Consider; $z = \dfrac{2i-6z}{iz-3}.$

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$$z = \frac{2i-6z}{iz-3}$$

$$z(iz-3) = 2i-6z$$

$$i z^2 - 3z = 2i - 6z$$

$$z^2 + 3z - 2i = 0$$

$b=3$, $a=1$, $c=-2$

$$z = \frac{-3 \pm \sqrt{9-4(i)(-2i)}}{2(i)} = \frac{-3 \pm \sqrt{9-8}}{2i}$$

$$z = \frac{-3 \pm 1}{2i} \quad \therefore z = \frac{-3+1}{2i} \quad (or) \quad z = \frac{-3-1}{2i}$$

$$\therefore \boxed{z = \frac{-1}{i}} \quad (or) \quad \boxed{z = \frac{-2}{i}}$$

Asmt

3) Find the Bilinear transformations; which maps; $z = 0, -i, -1$. & $w = i, 1, 0$. $\left[w = \frac{i(1+z)}{1-z} \right.$, 1

4) Find the Bilinear transformation; which maps: $z = 0, i, \infty$, $w = 1, -i, -1$, also find invariant point.

Soln :- Consider;

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$= \left\{ \frac{z_3\left[\frac{z_2}{z_3}-1\right]}{z_3\left[\frac{-z}{z_3}+1\right]} \right\} = \frac{\frac{1}{\infty}-1}{1-\frac{1}{\infty}} = \frac{0-1}{1-0} = -1$$

WKT; $z = 0, i, \infty$; $w = i, 1, 0$

$$\Rightarrow \frac{(w-1)(-i+1)}{(1+i)(-1-w)} = -1\left[\frac{z-0}{0-i}\right]$$

$$\Rightarrow \frac{(w-1)(1-i)}{-(i+1)(w+1)} = +\frac{z}{i} \qquad // \ \frac{1}{i} = -i$$

$$\Rightarrow \frac{(w-1)(1-i)}{-(i+1)(w+1)} = -zi$$

$$\Rightarrow (w-1)(1-i) = zi(1+i)(w+1)$$

$$\Rightarrow w - iw - 1 + i = (zi - z)(w+1)$$

$$\Rightarrow w - iw - 1 + i = zwi + zi - zw - z.$$

$\Rightarrow -w+1-iw+i = z+iz+wz+iwz.$

$w[-1-i-z-iz] = z+iz-1-i \Rightarrow -w(1+i+z+iz) = (z-1)(1+i)$

$\Rightarrow -w(1+i+z+iz) = (z+1)(i+1).$

$\Rightarrow -w[(1+i)(z+i)] = (z-1)(i+1)]$

$\therefore \boxed{w = \frac{(1-z)}{1+z}.}$

$\Rightarrow z = \frac{1-z}{1+z}. \quad \Rightarrow z+z^2-1+z=0.$

$z^2+2z-1=0.$

$z = \frac{-2\pm\sqrt{4+4}}{2} \Rightarrow z = -1\pm\sqrt{2}$

$\therefore \boxed{z=-1+\sqrt{2}}$ or $\boxed{z=-1-\sqrt{2}}$

—— * ——

Assignments

1) Find B.T. for :- $z = -1, i, 1$, $w = 1, i, -1$., also find inv pt.
2) Find B.T for :- $z = 0, 1, \infty$, $w = -5, -1, 3$. " —
3) Find B.T for :- $z = i, +1, -1$, $w = 1, 0, \infty$. " —
4) Find B.T for :- $z = -1, i, 1$, $w = 1, i, -i$.
5) Find B.T for :- $z = \infty, i, 0$, $w = -1, -i, 1$.

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