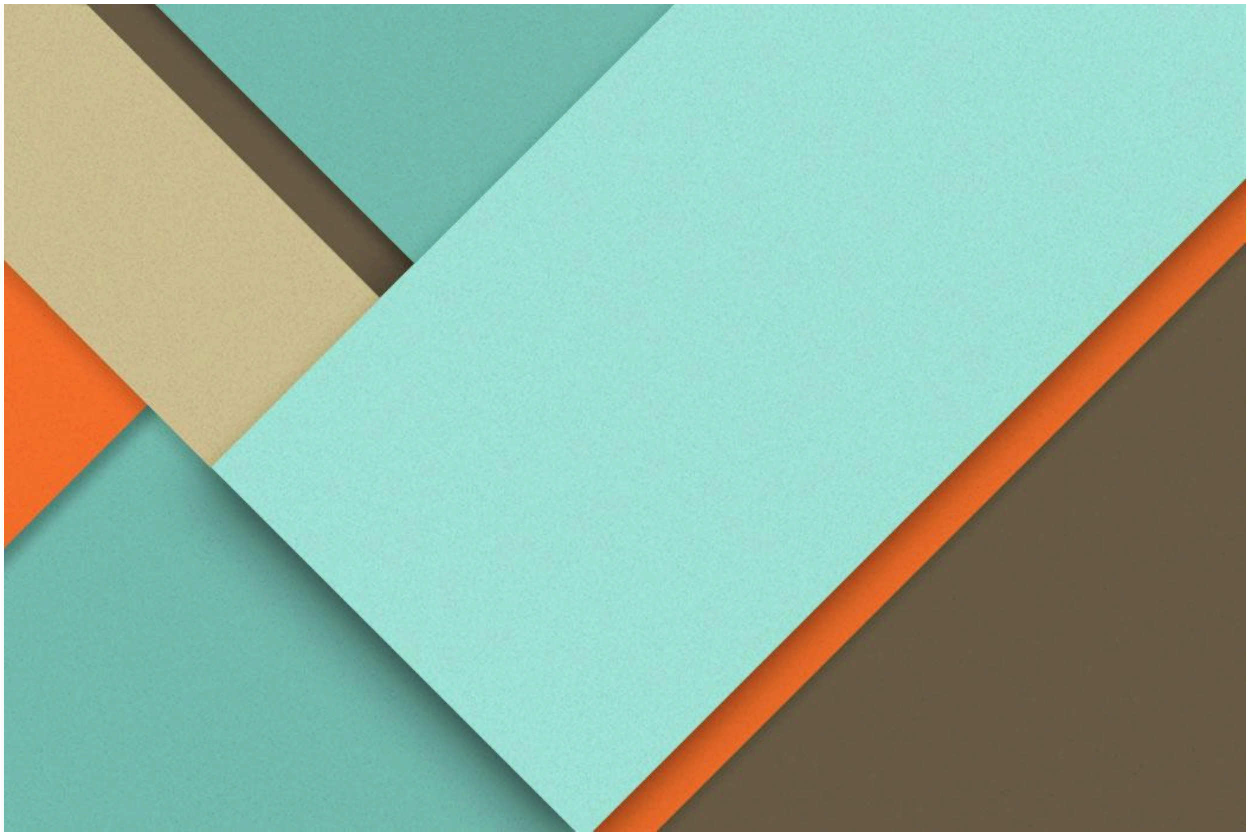

CFD LAB-2

Numerical solution of the 2D boundary layer equation

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Overview

In this report we use the Forward Difference Scheme, Central Difference Scheme and we refine the grid(mesh), to numerically solve the partial differential equation (PDE) that governs 2D boundary layer equation over a flat plate. Upon solving the equation we can view velocity profiles of boundary layers across the flat plate.

Goals

1. Objective of this task is to numerically solve the 2D boundary layer equation over a flat plate.
2. Computing the 2D boundary layer equation using given constraints and plot the results for 'u' and 'v'.

Context:

Given 2D boundary layer equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

Boundary Conditions

$$u(x, y = 0) = 0$$ (no slip conditions at walls)

$$u(x, y \to \infty) = 1$$ (free stream / free flow)

$$v(x, y = 0) = 0$$ (No penetration through walls)

Here, in this task, we numerically solve the 2D boundary layer equation over a flat plate , for this we discretize the given partial differential equations (PDE's) using finite difference method and apply on the nodal / grid points and obtain velocity profile across the boundary layer.

Task 2.0

A practical method of classifying PDEs is developed for a general second order PDE in two coordinates x and y.

$$a\frac{\partial^2\phi}{\partial x^2}+b\frac{\partial^2\phi}{\partial x\partial y}+c\frac{\partial^2\phi}{\partial y^2}+d\frac{\partial\phi}{\partial x}+e\frac{\partial\phi}{\partial y}+f\phi+g=0$$

For our steady x-momentum equation a=0, b=0, c=1/Re. So, $b^2-4ac=0$

Therefore the type of equation is parabolic (ref: Versteeig pg,32 eqn 2.53)

Boundary condition for u at x=0 is U_inf i.e., free stream velocity (except at y=0 i.e., no slip condition).

Boundary condition for v at x=0 is 0 (zero) , no slip condition and no penetration.

Boundary condition for u at x=1 and y = delta_99 is U_inf.

Boundary condition for v at x=1 and y = delta_99 is zero.

The boundary condition for v at y=inf is zero.

Task 2.1

Numerical Discretization:

Given 2D Boundary Layer equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

Numerical discretization of of Continuity equation and Momentum Equation

Continuity Equation :

We are gonna use the Forward Differencing Scheme to discretize the continuity equation.

$$\frac{u_{i+1,j}-u_{i,j}}{\Delta x}+\frac{v_{i,j+1}-v_{i,j}}{\Delta y}=0$$

Upon rearranging terms

$$v_{i,j+1}=v_{i,j}-\frac{\Delta y}{\Delta x}\left(u_{i+1,j}-u_{i,j}\right)$$

Momentum Equation :

For the momentum equation we are gonna use both forward and Central Differencing schemes.

$$u_{i,j}\left(\frac{u_{i+1,j}-u_{i,j}}{\Delta x}\right)+v_{i,j}\left(\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}\right)=\frac{1}{Re}\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}\right)$$

Upon rearranging terms

$$u_{i+1,j}=u_{i,j}+\left(\frac{\Delta x}{u_{i,j}}\right)\left[\frac{1}{Re}\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}\right)-v_{i,j}\left(\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}\right)\right]$$

As the nature of our equation is parabolic, for convective terms ($\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$) we used,forward difference scheme to ensure stability by considering the information from upstream nodes and they are first order accurate leading truncation errors are $\frac{\Delta x}{2}\cdot\frac{\partial^2 u}{\partial x^2}$ and $\frac{\Delta y}{2}\cdot\frac{\partial^2 v}{\partial y^2}$

(ref lecture-5,pg33).

For second order diffusive term like $\frac{\partial^2 u}{\partial y^2}$, generally central differencing scheme is used and it is second order accurate, leading truncation error is $\frac{\Delta y^2}{12}\cdot\frac{\partial^4 u}{\partial y^4}$
For the term $\frac{\partial u}{\partial y}$, CDS is used for better accuracy for the cross stream convection and it has second order accuracy . The truncation error is $\frac{\Delta^2 y}{6}\cdot\frac{\partial^3 u}{\partial y^3}$

(ref: CFD by J.D.Anderson pg no. 137,138,142)

Task 2.2

To solve the given partial differential equations we discretize it , apply boundary conditions and solve it in an iterative method where we will get our updated solution matrix which can be plotted in Matlab.

We will run our discretized PDEs in loops to update values and it will run until our errors are very small, less than or equal to epsilon i.e., the difference between u updated and u previous is less than or equal to epsilon. Similarly, for v also we use the same method.

Stability analysis:

$$u_{i+1,j}=u_{i,j}+\left(\frac{\Delta x}{u_{i,j}}\right)\left[\frac{1}{Re}\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}\right)-v_{i,j}\left(\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}\right)\right]$$

Discrete notation for the involved points

$$u(i,j) = V^n e^{I(k_x i\Delta x + k_y j\Delta y)}$$

$$u(i,j+1) = V^n e^{I(k_x i\Delta x + k_y (j+1)\Delta y)}$$

(ref: lec.7 , pg no.29)

$$u(i,j-1) = V^n e^{I(k_x i\Delta x + k_y (j-1)\Delta y)}$$

The amplification factor for this problem is defined as $G = \dfrac{u(i+1,j)}{u(i,j)}$

Leading to stability condition

$|G| = \dfrac{u(i+1,j)}{u(i,j)} \leq 1$ for all values of k or θ

(ref : lec.7, eqn.30)

Divide the discretized equation with u(i,j) so that we get

$$G = 1 + \frac{\Delta x}{u(i,j)\cdot Re \cdot \Delta y^2}\left[e^{Ik_y\Delta y} - 2 + e^{-Ik_y\Delta y}\right] - \frac{\Delta x \cdot v(i,j)}{2u(i,j)\Delta y}\left[e^{Ik_y\Delta y} - e^{-Ik_y\Delta y}\right]$$

From euler relations

$$e^{Ik_y\Delta y} + e^{-Ik_y\Delta y} = 2\cos k_y\Delta y$$

(ref: lec.7, eqn.34, 35)

$$e^{Ik_y\Delta y} - e^{-Ik_y\Delta y} = 2I\sin k_y\Delta y$$

Let

$$d = \frac{\Delta x}{u(i,j)\cdot Re \cdot \Delta y^2} \qquad d' = \frac{\Delta x \cdot v(i,j)}{u(i,j)\cdot \Delta y}$$

We get

$$G=[1-2d(1-\cos(k_y\Delta y)] - Id'\sin(k_y\Delta y)$$

The real part condition is true for d ≤ 1/2 (ref Lec.7, pg no,31)

The imaginary part condition is true for d’ ≤ 1

The stability constraints are $\Delta x \leq \frac{1}{2}\cdot Re \cdot \Delta y^2 \cdot u(i,j)$ and $\Delta x \leq \dfrac{u(i,j)\cdot\Delta y}{v(i,j)}$

The stability region is

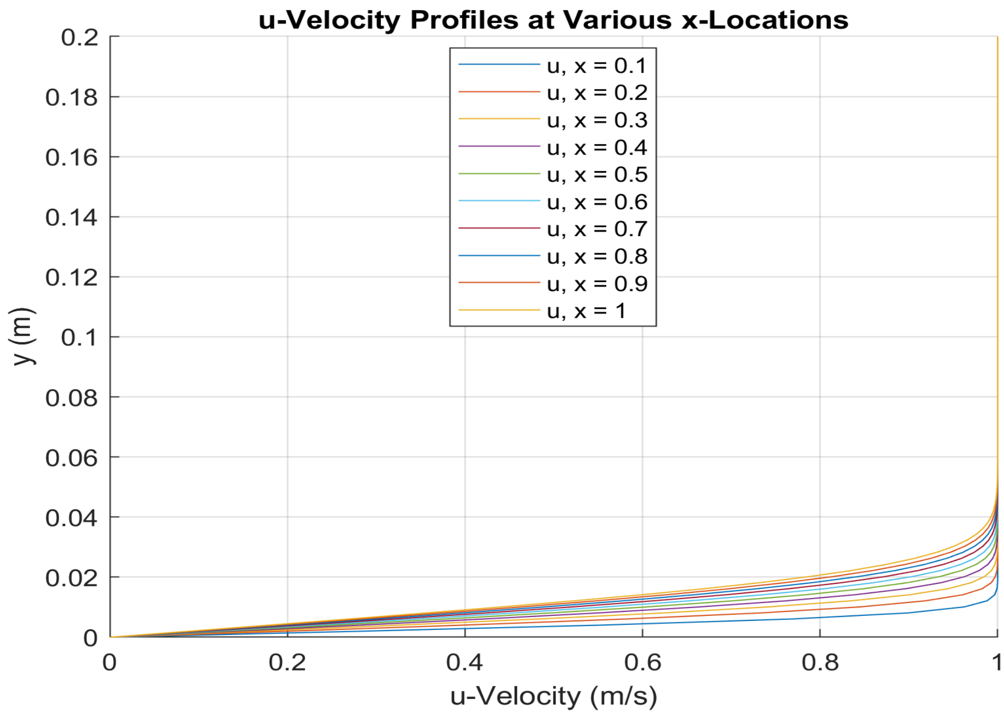
Let G* be conjugate of G

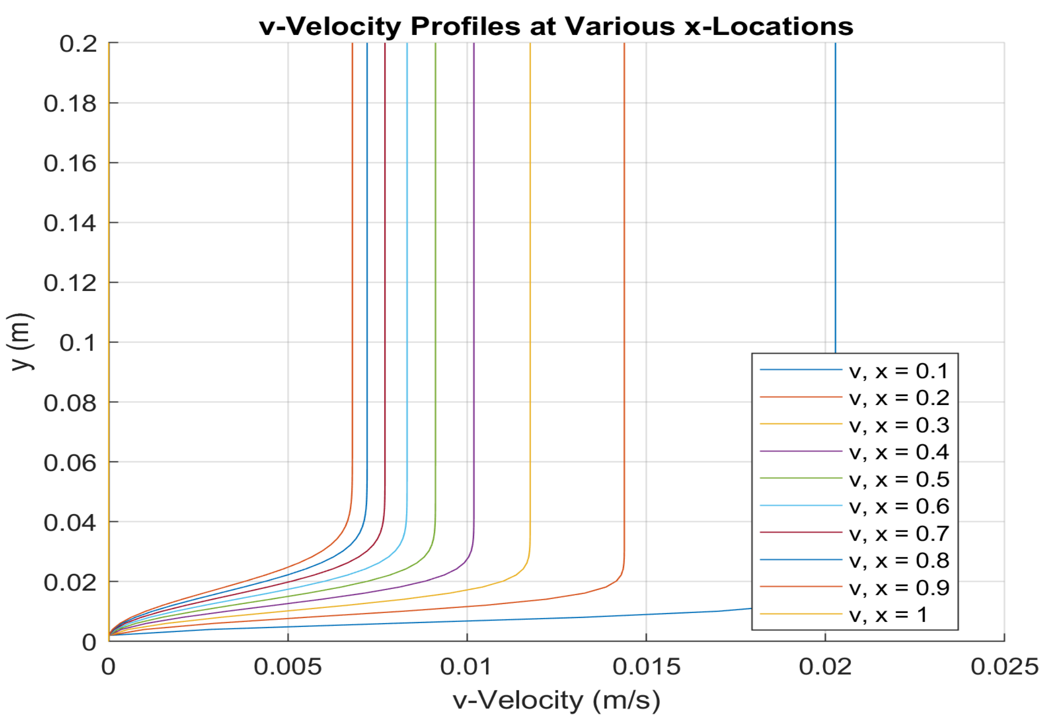
G.G* = $|G|^2 = [1-2d\,(1-\cos(k_y\Delta y))]^2 + [d'\sin(k_y\Delta y)]^2 \leq 1$

(ref:Lec.7 eqn 31 and CFD T.J.chung pg 77)

Task 2.3

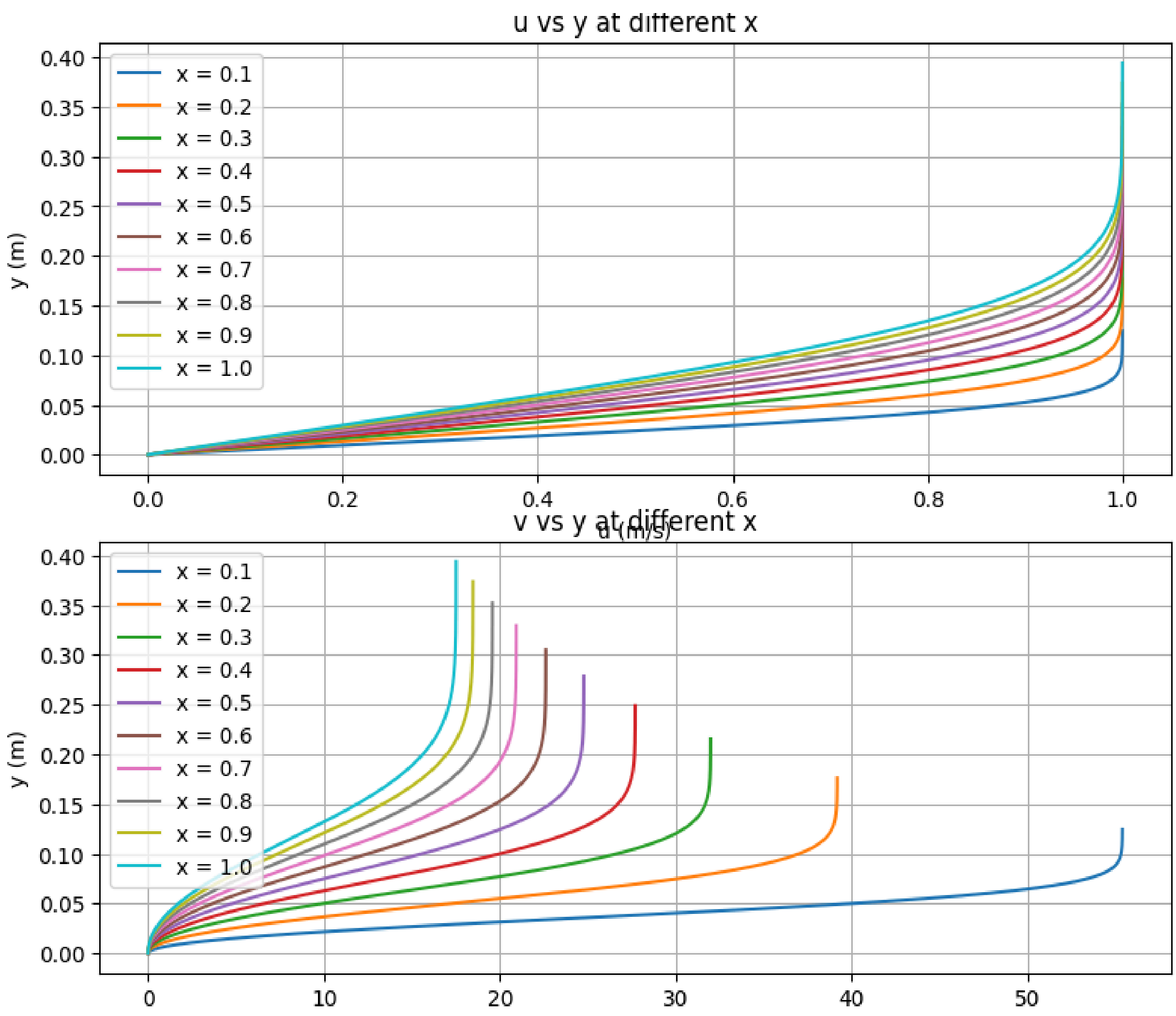
Plots for u and v as two separate plots are generated and images are as follows:





The above plots are generated using MATLAB at x =0,0.1,0.2…..1 for the given PDEs

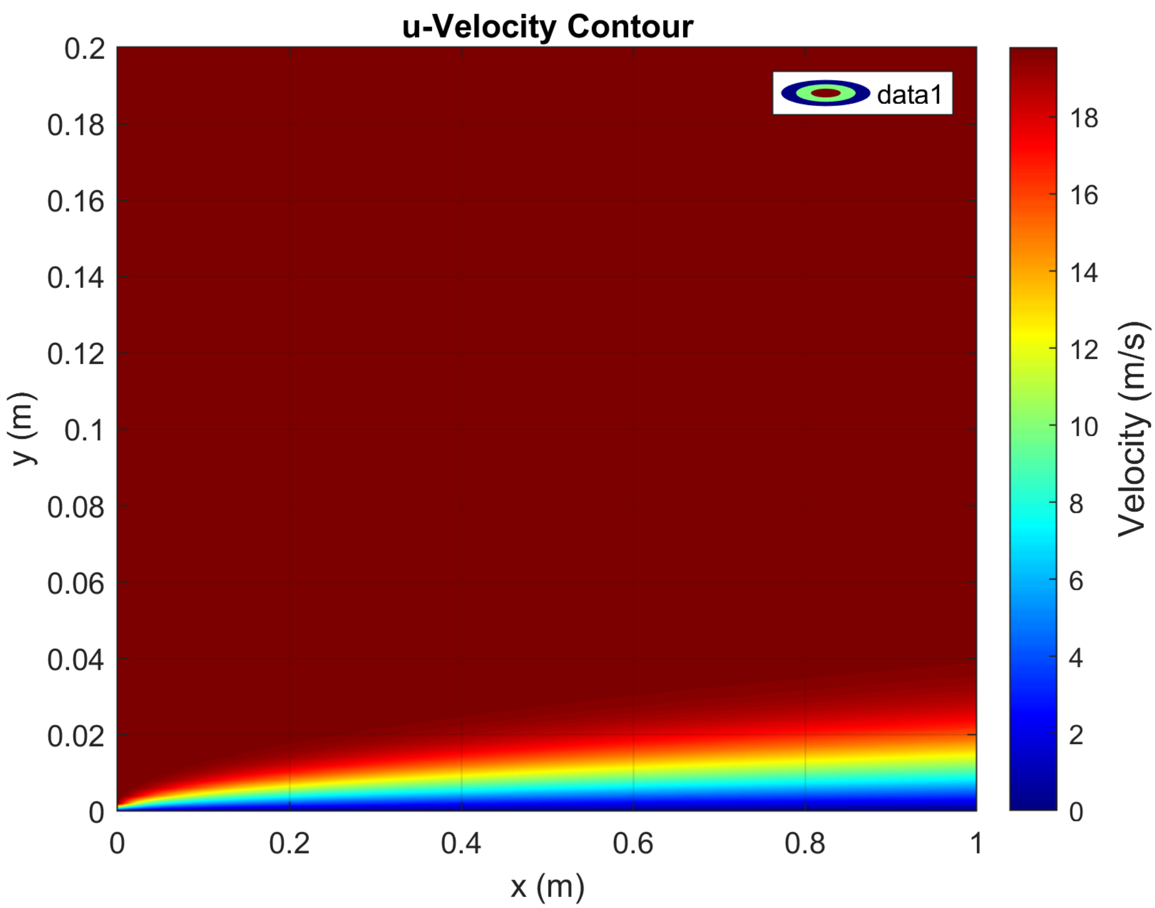
Task 2.4



The above shown image is a plot from Blasius equation in exercise 2, by cross plotting and comparing this image with our current lab task 2.3 plots we get approximately similar u and v profiles at different x locations i.e., at x=0, 0.1, 0.2, ….1. The boundary layer thickness increases with x in both plots and also the velocity gradient near the wall decreases as x increases indicating the presence of shear stress.

Task 2.5

Assuming our free stream velocity U_inf = 20 m/s working with air at 20℃. By using our discretized PDE we got a numerical solution of the velocity profile for the boundary layer formed on a flat plate.

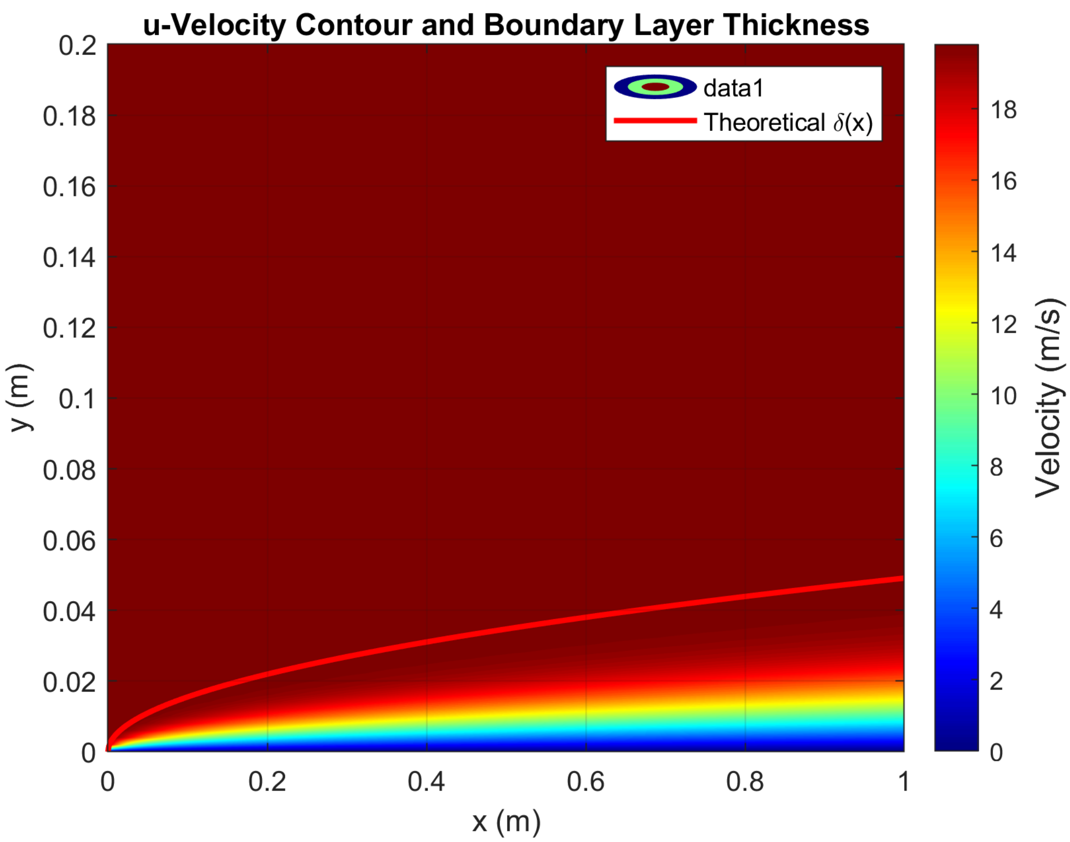


The streamwise length of the region that was considered in my computational domain is 1m, from observation of the plot we can see that boundary layer thickness increases along the length of the plate.

Maximum thickness of the boundary layer for our given domain is 0.049 m achieved at length Lx =1.

Task 2.6

The image attached below is the boundary layer formation along with boundary layer thickness which is the maximum thickness of boundary layer at particular x position in y direction. At the end of boundary layer thickness the velocity u will be 99% of the free stream velocity i.e., u at boundary layer edge = 0.99 * U_inf.



The boundary layer edge is a conceptual boundary where the velocity reaches a certain percentage (typically 99%) of the freestream velocity.

The streamlines do not cross this edge because in the boundary layer the fluid experiences velocity gradients but outside the boundary layer the flow is mostly considered to be inviscid.

Streamlines may appear to approach or align with this edge but do not actually cross it.