Linear Algebra

Binary Operations:

A ***binary operation*** on a set is a calculation involving two elements of the set to produce another element of the same set.

(or)

A binary operation on a set S is a mapping of the elements of the Cartesian product $S \times S$ to S.

$$* : S \times S \rightarrow S$$

(or)

An operation * on a non-empty set A is said to be a binary operation if

$a \in A$ & $b \in A$, then $a * b \in A$ **(Closure Property)**

Group:

A non empty set S with the binary operation * is said to be ***group*** if it satisfies the following conditions;

1. **Closure property:** $a \in S$ & $b \in S$, then $a * b \in S$;
2. **Associative property:** **u * (v * w) = (u * v) * w** for all u, v, w ∈ S; **(Semi group)**
3. **Identity property:** there exists an element e ∈ S such that **u * e = e * u = u** for all u ∈ S. **(Monoid)**
4. **Inverse property:** For every u ∈ S, there exists an element v ∈ S such that **u * v = v * u = e.** Then v is said to an inverse of u and denote by u^{-1}. **(Group)**
5. **Commutative property:** **u * v = v * u** for all u, v ∈ S. **(Abelian Group)**

Vector Space:

A ***vector space*** over a field F (in this entire course it is $\mathbb{R}$) is a non empty set V together with two operations vector addition '+' (just for name it is no need to usual addition) and scalar multiplication that satisfy the ten axioms listed below.

I. Abelian group under addition;

1. **Closure property**: u & $v \in V$, then $u + v \in V$;
2. **Associative property**: $\mathbf{u+(v+w) = (u+v)+w}$ for all $u,v,w \in V$;
3. **Identity property:** there exists an element $e \in V$ such that $\mathbf{u + e = e + u = u}$ for all $u \in V$;
4. **Inverse property:** For every $u \in V$, there exists an element $-u \in V$ such that $\mathbf{u + (-u) = (-u) + u = e}$. Then $-u$ is said to an additive inverse of u;
5. **Commutative property: $\mathbf{u + v = v + u}$** for all $u, v \in V$.

II. Scalar multiplication;

6. Closure property: $u \in V$ & $\alpha \in F$, then $\alpha u \in V$;

7. Distributive property of scalar multiplication over vector addition : $\alpha(u+v) = \alpha u + \alpha v \quad \forall\ u, v \in V$ & $\alpha \in F$

8. Distributive property of vector addition over scalar multiplication: $(\alpha+\beta)u = \alpha u + \beta u \quad \forall\ u \in V$ & $\alpha, \beta \in F$

9. Associative property:

$$(\alpha\beta)u = \alpha(\beta u) = \alpha\beta u \quad \forall\ u \in V \ \&\ \alpha, \beta \in F$$

10. **1.u = u** for all $u \in V$.

Examples:

The set $V = \mathbb{R}^2$ will form a vector space over $\mathbb{R}$ with usual vector addition and scalar multiplication given by

$$(i)\ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(ii)\ k(x_1, y_1) = (kx_1, ky_1)$$

Proof: **I. Abelian group under addition**

1. Closure property: Let $u, v \in V = \mathbb{R}^2$, where $u = (x_1, y_1), \&\ v = (x_2, y_2)$,

we need to prove that $u + v \in \mathbb{R}^2$

Now

$$u + v = (x_1, y_1) + (x_2, y_2)$$

$$u + v = (x_1 + x_2, y_1 + y_2) \quad \in \mathbb{R}^2$$

Therefore

$$\boxed{u, v \in \mathbb{R}^2 \text{ implies } u + v \in \mathbb{R}^2}$$

2. Associativity property:

Let $u, v, w \in V = \mathbb{R}^2$, where $u = (x_1, y_1)$, $v = (x_2, y_2)$ & $w = (x_3, y_3)$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$u + (v + w) = (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in \mathbb{R}^2$, such that

$$u + e = e + u = u \text{ for all } u \in \mathbb{R}^2$$

Let $e = (0,0)$ such that

$$u + e = (x_1, y_1) + (0,0)$$
$$= (x_1 + 0, y_1 + 0)$$
$$= (x_1, y_1)$$
$$\boxed{u + e = u}$$

$$e + u = (0,0) + (x_1, y_1)$$
$$= (0 + x_1, 0 + y_1)$$
$$= (x_1, y_1)$$
$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in \mathbb{R}^2}$$

4. Inverse Property:

We need to prove that, for every element $u \in \mathbb{R}^2$, there exists $-u \in \mathbb{R}^2$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = (-x_1, -y_1)$ such that

$$u + (-u) = (x_1, y_1) + (-x_1, -y_1)$$

$$= (0,0)$$

$$\boxed{u + (-u) = e}$$

$$(-u) + u = (-x_1, -y_1) + (x_1, y_1)$$

$$= (0,0)$$

$$\boxed{(-u) + u = e}$$

Therefore

$$\boxed{u + (-u) = (-u) + u = e}$$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in \mathbb{R}^2$$

Now,

$$u + v = (x_1, y_1) + (x_2, y_2)$$

$$= (x_1 + x_2, y_1 + y_2)$$

$$= (x_2 + x_1, y_2 + y_1)$$

$$= (x_2, y_2) + (x_1, y_1)$$

$$\boxed{u + v = v + u}$$

Therefore

$$\boxed{u + v = v + u \text{ for all } u, v \in \mathbb{R}^2}$$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$$u \in V = \mathbb{R}^2 \ \& \ \alpha \in K \text{ such that } \alpha u \in V = \mathbb{R}^2$$

Now,

$$\alpha u = \alpha(x_1, y_1)$$

$$\alpha u = (\alpha x_1, \alpha y_1) \ \in \mathbb{R}^2$$

Therefore

$$\text{for any } u \in \mathbb{R}^2 \ \& \ \alpha \in K \text{ we have } \alpha u \in \mathbb{R}^2$$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V = \mathbb{R}^2$ & $\alpha \in K$

$$\alpha(u + v) = \alpha u + \alpha v$$

Now,

$$\alpha(u + v) = \alpha\{(x_1, y_1) + (x_2, y_2)\}$$

$$= \alpha(x_1 + x_2, y_1 + y_2)$$

$$= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2)$$

$$= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2)$$

$$= \alpha(x_1, y_1) + \alpha(x_2, y_2)$$

$$\boxed{\alpha(u + v) = \alpha u + \alpha v}$$

Therefore

$$\boxed{\alpha(u + v) = \alpha u + \alpha v \text{ for all } u, v \in V = \mathbb{R}^2 \text{ \& } \alpha \in K}$$

8. Distributive property of vector addition over scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^2$ & $\alpha, \beta \in K$

$$(\alpha + \beta)u = \alpha u + \beta u$$

Now,

$$(\alpha + \beta)u = (\alpha + \beta)(x_1, y_1)$$

$$(\alpha + \beta)u = \big((\alpha + \beta)x_1, (\alpha + \beta)y_1\big)$$

$$= (\alpha x_1 + \beta x_1, \alpha y_1 + \beta y_2)$$

$$= (\alpha x_1, \alpha y_1) + (\beta x_1, \beta y_1)$$

$$= \alpha(x_1, y_1) + \beta(x_1, y_1)$$

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u}$$

Therefore

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u \text{ for all } u \in V = \mathbb{R}^2 \text{ \& } \alpha, \beta \in K}$$

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^2$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(\beta x_1, \beta y_1)$$

$$= (\alpha\beta x_1, \alpha\beta y_1)$$

$$= \alpha\beta(x_1, y_1)$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

Therefore $\boxed{\alpha(\beta u) = (\alpha\beta)u \text{ for all } u \in V = \mathbb{R}^2 \ \& \ \alpha, \beta \in K}$

10. Property 10:

$$1(u) = 1(x_1, y_1)$$

$$= ((1)x_1, (1)y_1)$$

$$= (x_1, y_1)$$

$$\boxed{1(u) = u}$$

The set of all order pairs $V = \mathbb{R}^2$ with usual vector addition and scalar multiplication is vector space

2. Examples:

Is the set $V = \mathbb{R}^2$ with vector addition and scalar multiplication given below form a vector space

$$(i)\ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(ii)\ k(x_1, y_1) = (kx_1, 0)$$

Proof: **I. Abelian group under addition**

1. Closure property: Let $u, v \in V = \mathbb{R}^2$, where $u = (x_1, y_1), \&\ v = (x_2, y_2)$,

we need to prove that $u + v \in \mathbb{R}^2$

Now

$$u + v = (x_1, y_1) + (x_2, y_2)$$

$$u + v = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$$

Therefore

$$\boxed{u, v \in \mathbb{R}^2 \text{ implies } u + v \in \mathbb{R}^2}$$

2. Associativity property:

Let $u, v, w \in V = \mathbb{R}^2$, where $u = (x_1, y_1)$, $v = (x_2, y_2)$ & $w = (x_3, y_3)$,

we need to prove that $u + (v + w) = (u + v) + w$

Now

$$u + (v + w) = (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$$

$$\boxed{u + (v + w) = (u + v) + w}$$

3. Identity Property:

We need to prove that, there exists an element $e \in \mathbb{R}^2$, such that

$$u + e = e + u = u \text{ for all } u \in \mathbb{R}^2$$

Let $e = (0,0)$ such that

$$u + e = (x_1, y_1) + (0,0)$$

$$= (x_1 + 0, y_1 + 0)$$

$$= (x_1, y_1)$$

$$\boxed{u + e = u}$$

$$e + u = (0,0) + (x_1, y_1)$$

$$= (0 + x_1, 0 + y_1)$$

$$= (x_1, y_1)$$

$$\boxed{e + u = u}$$

Therefore

$$\boxed{u + e = e + u = u \text{ for all } u \in \mathbb{R}^2}$$

4. Inverse Property:

We need to prove that, for every element $u \in \mathbb{R}^2$, there exists $-u \in \mathbb{R}^2$ such that

$$u + (-u) = (-u) + u = e$$

Let $-u = (-x_1, -y_1)$ such that

$$u + (-u) = (x_1, y_1) + (-x_1, -y_1)$$

$$= (0,0)$$

$$\boxed{u + (-u) = e}$$

$$(-u) + u = (-x_1, -y_1) + (x_1, y_1)$$

$$= (0,0)$$

$$\boxed{(-u) + u = e}$$

Therefore

$$\boxed{u + (-u) = (-u) + u = e}$$

5. Commutative Property:

We need to prove that,

$$u + v = v + u \text{ for all } u, v \in \mathbb{R}^2$$

Now,

$$u + v = (x_1, y_1) + (x_2, y_2)$$

$$= (x_1 + x_2, y_1 + y_2)$$

$$= (x_2 + x_1, y_2 + y_1)$$

$$= (x_2, y_2) + (x_1, y_1)$$

$$\boxed{u + v = v + u}$$

Therefore $$\boxed{u + v = v + u \text{ for all } u, v \in \mathbb{R}^2}$$

II Scalar Multiplication

6. Closure Property:

We need to prove that,

$$u \in V = \mathbb{R}^2 \ \& \ \alpha \in K \text{ such that } \alpha u \in V = \mathbb{R}^2$$

Now,

$$\alpha u = \alpha(x_1, y_1)$$

$$\alpha u = (\alpha x_1, 0) \quad \in \mathbb{R}^2$$

Therefore

$$\text{for any } u \in \mathbb{R}^2 \ \& \ \alpha \in K \text{ we have } \alpha u \in \mathbb{R}^2$$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V = \mathbb{R}^2$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now,

$$\alpha(u+v) = \alpha u + \alpha v$$

$$\alpha\{(x_1, y_1) + (x_2, y_2)\} = \alpha(x_1, y_1) + \alpha(x_2, y_2)$$

$$\alpha(x_1 + x_2, y_1 + y_2) = (\alpha x_1, 0) + (\alpha x_2, 0)$$

$$(\alpha x_1 + \alpha x_2, 0) = (\alpha x_1 + \alpha x_2, 0)$$

$$\boxed{LHS = RHS}$$

Therefore

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in V = \mathbb{R}^2 \ \& \ \alpha \in K}$$

7. Distributive property of scalar multiplication over vector addition:

We need to prove that, for all $u, v \in V = \mathbb{R}^2$ & $\alpha \in K$

$$\alpha(u+v) = \alpha u + \alpha v$$

Now,
$$\alpha(u+v) = \alpha\{(x_1, y_1) + (x_2, y_2)\}$$

$$= \alpha(x_1 + x_2, y_1 + y_2)$$

$(i)\ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$= (\alpha x_1 + \alpha x_2, 0 + 0)$$

$(ii)\ k(x_1, y_1) = (kx_1, 0)$

$$= (\alpha x_1, 0) + (\alpha x_2, 0)$$

$$= \alpha(x_1, y_1) + \alpha(x_2, y_2)$$

$$\boxed{\alpha(u+v) = \alpha u + \alpha v}$$

Therefore

$$\boxed{\alpha(u+v) = \alpha u + \alpha v \text{ for all } u, v \in V = \mathbb{R}^2 \ \&\ \alpha \in K}$$

8. Distributive property of vector addition over scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^2$ & $\alpha, \beta \in K$

$$(\alpha + \beta)u = \alpha u + \beta u$$

Now,

$$(\alpha + \beta)u = \big((\alpha + \beta)x_1, 0\big)$$

$$= (\alpha x_1 + \beta x_1, 0 + 0)$$

$$= (\alpha x_1, 0) + (\beta x_1, 0)$$

$$= \alpha(x_1, y_1) + \beta(x_1, y_1)$$

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u}$$

Therefore

$$\boxed{(\alpha + \beta)u = \alpha u + \beta u \text{ for all } u \in V = \mathbb{R}^2 \text{ \& } \alpha, \beta \in K}$$

9. Associative property of vector with scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^2$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

$$\alpha(\beta u) = \alpha(\beta x_1, 0)$$

$$= (\alpha\beta x_1, 0)$$

$$= \alpha\beta(x_1, y_1)$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

Therefore $\boxed{\alpha(\beta u) = (\alpha\beta)u \text{for all } u \in V = \mathbb{R}^2 \text{ \& } \alpha, \beta \in K}$

9. Associative property of vector addition and scalar multiplication :

We need to prove that, for all $u \in V = \mathbb{R}^2$ & $\alpha, \beta \in K$

$$\alpha(\beta u) = (\alpha\beta)u$$

Now,

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$$= (\alpha\beta x_1, 0)$$

$$(\alpha\beta)u = (\alpha\beta x_1, 0)$$

$$\boxed{\alpha(\beta u) = (\alpha\beta)u}$$

10. Property 10:

$$1(u) = ((1)x_1, 0)$$
$$= (x_1, 0)$$
$$\boxed{1(u) \neq u}$$

The set of all order pairs $V = \mathbb{R}^2$ with the vector addition and scalar multiplication

$$(i)\ u + v = (x_1, y_1) + (x_2, y_2)$$
$$(ii)\ ku = (kx_1, 0)$$

is not a vector space, since Axiom 10 fails.

THANK YOU