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Assignment

(1) Here, we have

$$A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

Now, as we know, to find spectral decomposition, we need to find eigen values and eigen vectors.

Now,
For eigen values
$|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} -\lambda & 0 & i \\ 0 & 1-\lambda & 0 \\ -i & 0 & -\lambda \end{vmatrix} = 0.$$

$\therefore \lambda = -1, 1, 1.$

Now, we get eigen vectors as

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \text{ for}$$

$\lambda = -1, 1, 1$ respectively.

Thus, we get spectral decomposition as

$A = \sum a_i |u_i\rangle\langle u_i|$ where $a_i$ is eigen value and $|u_i\rangle$ is corresponding eigen vector.

Thus we get spectral decomposition as

$$A = (-1)\cdot\frac{1}{2}\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} i & 0 & 1 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} +$$

$$\frac{(1)}{2}\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}(-i\ 0\ 1)$$

$$\therefore \boxed{A = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0\ 1\ 0) - \frac{1}{2}\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}(i\ 0\ 1) + \frac{1}{2}\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}(-i\ 0\ 1)}$$

Now, we have

$$\psi = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$$

$$\therefore |\psi\rangle = \begin{pmatrix} 1/2 \\ -i/2 \\ 1/\sqrt{2} \end{pmatrix} \qquad \therefore \langle\psi| = (1/2\ \ i/2\ \ 1/\sqrt{2})$$

Now,

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$$

$$= (1/2\ \ i/2\ \ 1/\sqrt{2})\begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ -i/2 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\therefore \langle A\rangle_\psi = (-i/\sqrt{2}\ \ i/2\ \ i/2)\begin{pmatrix} 1/2 \\ -i/2 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\therefore \langle A\rangle_\psi = \left(\frac{-i}{\sqrt{2}} \times \frac{1}{2}\right) + \left(\frac{i}{2} \times \frac{-i}{2}\right) + \left(\frac{i}{2} \times \frac{1}{\sqrt{2}}\right)$$

$$= \frac{-\sqrt{2}i}{4} + \frac{1}{4} + \frac{\sqrt{2}i}{4}$$

$$\therefore \boxed{\langle A\rangle_\psi = \frac{1}{4}}$$

(2) We have

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Now, to convert to diagonal matrix, we use unitary matrix of eigen vectors.

$\therefore$ We find eigen values by $|R - \lambda I| = 0$.

$\therefore$ We get $\lambda_1 = \cos\theta + i\sin\theta$

$\lambda_2 = \cos\theta - i\sin\theta$.

Now, we get eigen vectors as $\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$ corresponding to eigen values $\lambda_1$ and $\lambda_2$ respectively.

Thus, we get

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \text{ and } U^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

Thus we get $R'$

$$R' = U^{-1} R U$$

$$= \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -i\cos\theta - \sin\theta & -i\sin\theta + \cos\theta \\ i\cos\theta - \sin\theta & i\sin\theta + \cos\theta \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \cos\theta - i\sin\theta - i\sin\theta + \cos\theta & 0 \\ 0 & \cos\theta + i\sin\theta + i\sin\theta + \cos\theta \end{pmatrix}$$

$$\therefore \boxed{R' = \begin{pmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos\theta + i\sin\theta \end{pmatrix}}$$

Thus, $\boxed{U = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}}$ is one of the unitary transformations that convert $R$ to diagonal matrix.

(3) a) We have $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

As we know, for $\sigma_x$, eigen vector are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for eigen values 1 and -1 respectively.

$\therefore \sigma_x = \sum_i a_i |u_i\rangle\langle u_i|$

$$= \frac{1(1)}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2}(-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

$\therefore \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\therefore \boxed{\sigma_x = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix}}$$

b) Similarly for $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

we have eigen vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for eigen values 1 and -1 respectively

$\therefore \sigma_z = \sum_i a_i |u_i\rangle\langle u_i|$

$$= (1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\therefore \boxed{\sigma_z = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}}$$

Thus, we get respective spectral decomposition for $\sigma_x$ and $\sigma_z$.

(4) We have

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus,

$$U_{CNOT}(X \otimes I_2)U_{CNOT} = U_{CNOT} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} U_{CNOT}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} U_{CNOT}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \boxed{U_{CNOT}(X \otimes I_2)U_{CNOT} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$$

(5) Let us consider the Bell State.

$|\psi^+\rangle = \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)$ $\langle\psi^+| = \left(\frac{\langle 01| + \langle 10|}{\sqrt{2}}\right)$

$\therefore \rho^{12} = |\psi^+\rangle\langle\psi^+|$

$= \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| + \langle 10|}{\sqrt{2}}\right)$

$= \frac{1}{2}\left(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\right)$

Now, taking partial trace with respect to first qubit

$\rho' = \frac{1}{2}\left(|0\rangle\langle 0|\left(\langle 1|1\rangle\right) + |0\rangle\langle 1|\left(\langle 1|0\rangle\right) + |1\rangle\langle 0|\left(\langle 0|1\rangle\right) + |1\rangle\langle 1|\left(\langle 0|0\rangle\right)\right)$

As we know,

$\langle 1|1\rangle = \langle 0|0\rangle = 1$ , $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

$\therefore \rho' = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$\therefore \rho' = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \boxed{\rho' = \frac{I_2}{2}}$

Repeating this with all Bell States, we get the same result.

Thus, partial trace of any Bell State with respect to one qubit is $\frac{I_2}{2}$.

(6) We have $U_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The eigen vectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigen values 1 and -1 respectively.

Now, the operator decomposition of $U_z$ is

$$U_z = \sum a_i |u_i\rangle\langle u_i|$$
$$= (1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \quad 0) + (-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \quad 1)$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore \boxed{U_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \quad 0) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \quad 1)}$$

Also, we have $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

The eigen vectors are $\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$ for eigen vectors 1 and -1.

Now, the operator decomposition for $H$ is

$$H = \sum a_i |v_i\rangle\langle v_i|$$
$$= (1)\frac{1}{4+2\sqrt{2}}\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}(1+\sqrt{2} \quad 1) + (-1)\frac{1}{4-2\sqrt{2}}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}(1-\sqrt{2} \quad 1)$$
$$= \frac{1}{8}\left(4-2\sqrt{2}\begin{pmatrix} 3+2\sqrt{2} & 1+\sqrt{2} \\ 1+\sqrt{2} & 1 \end{pmatrix} - 4+2\sqrt{2}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}\right)$$
$$= \frac{1}{8}\begin{bmatrix} 12-8+2\sqrt{2}-12+8+2\sqrt{2} & 4-4+2\sqrt{2}-4+4+2\sqrt{2} \\ 4-4+2\sqrt{2}-4+4+2\sqrt{2} & 4-2\sqrt{2}-4-2\sqrt{2} \end{bmatrix}$$
$$= \frac{1}{8}\begin{pmatrix} 4\sqrt{2} & 4\sqrt{2} \\ 4\sqrt{2} & -4\sqrt{2} \end{pmatrix}$$

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$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\therefore \boxed{H = \frac{1}{4+2\sqrt{2}}\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1+\sqrt{2} & 1 \end{pmatrix} - \frac{1}{4-2\sqrt{2}}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1-\sqrt{2} & 1 \end{pmatrix}}$$

Now, to express $U_2$ in $|+\rangle, |-\rangle$ form,

$$U_2|+\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$U_2|-\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$\therefore U_2 = \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\langle +| + \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)\langle -|$$

$$\therefore \boxed{U_2 = \frac{|+\rangle\langle +| + |-\rangle\langle +| + |+\rangle\langle -| - |-\rangle\langle -|}{\sqrt{2}}}$$

Thus, we represent $U_2$ in $|\pm\rangle$ basis.

(7) We have

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Let $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\delta}\sin(\theta/2)|1\rangle$

Now,

$$|\psi'\rangle = Y|\psi\rangle$$
$$= \cos(\theta/2)\, i|1\rangle + e^{i\delta}\sin(\theta/2)(-i)|0\rangle$$
$$= e^{i\pi/2}\left(\cos(\theta/2)|1\rangle - e^{i\delta}\sin(\theta/2)|0\rangle\right)$$
$$= -e^{i\pi/2}.e^{i\delta}\left(\sin(\theta/2)|0\rangle - e^{-i\delta}\cos(\theta/2)|1\rangle\right)$$

discarding global terms.

$$|\psi'\rangle = \sin(\theta/2)|0\rangle + e^{-i\delta}\cos(\theta/2)|1\rangle$$
$$= -\cos\left(\frac{\theta+\pi}{2}\right)|0\rangle + e^{-i\delta}\sin\left(\frac{\theta+\pi}{2}\right)|1\rangle$$
$$\therefore |\psi'\rangle = \cos\left(\frac{\theta+\pi}{2}\right)|0\rangle + e^{-i\delta}\sin\left(\frac{\theta+\pi}{2}\right)|1\rangle$$

Hence, the Y-gate changes $\theta$ (the angle with Z-axis) by $\pi$ and $\delta$ (the angle of projection in X-Y plane with X axis) to $-\delta$ or $2\pi-\delta$ in bloch sphere.

(8) We have

$$C_{NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\therefore C_{NOT}^{+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \boxed{C_{NOT} = C_{NOT}^{+}}$$

Thus, controlled NOT gate is hermitian.

Now,

$$C_{NOT}^{+} C_{NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \boxed{C_{NOT}^{+} C_{NOT} = I_4}$$

Thus, controlled NOT gate is unitary.

(9) As we know,

$$|\beta_{xy}\rangle = \left( \frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}} \right)$$

$$\therefore |\beta_{00}\rangle = \left( \frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

Also, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$X \otimes Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\therefore X \otimes Z |\beta_{00}\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\therefore \boxed{X \otimes Z |\beta_{00}\rangle = \left( \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right) = |\beta_{11}\rangle}$$

And, as we clearly know $|\beta_{11}\rangle$ is an entangled state.

Thus, $X \otimes Z |\beta_{00}\rangle$ is an entangled state.

(10) We have

$$\rho = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\cos\theta\sin\theta \\ e^{i\phi}\cos\theta\cdot\sin\theta & \cos^2\theta \end{pmatrix}$$

Now,

$$\rho\sigma_0 = \rho I = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho\sigma_0 = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

$$\rho\sigma_1 = \rho\sigma_x = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho\sigma_1 = \begin{pmatrix} e^{-i\phi}\sin\theta\cos\theta & \sin^2\theta \\ \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \end{pmatrix}$$

$$\rho\sigma_2 = \rho\sigma_y = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\rho\sigma_2 = \begin{pmatrix} ie^{-i\phi}\sin\theta\cos\theta & -i\sin^2\theta \\ i\cos^2\theta & -ie^{i\phi}\sin\theta\cos\theta \end{pmatrix}$$

$$\rho\sigma_3 = \rho\sigma_z = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho\sigma_3 = \begin{pmatrix} \sin^2\theta & -e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & -\cos^2\theta \end{pmatrix}$$

Now,

$C_0 = tr(\rho\sigma_0) = \sin^2\theta + \cos^2\theta = 1$

$C_1 = tr(\rho\sigma_1) = \sin\theta\cos\theta(2\cos\phi)$

$C_2 = tr(\rho\sigma_2) = i\sin\theta\cos\theta(-2i\sin\phi)$

$C_3 = tr(\rho\sigma_3) = \cancel{\cos^2\theta}\ \sin^2\theta - \cos^2\theta$

Thus, we get

$$\boxed{\rho = \frac{1}{2}\Big((1)\sigma_0 + (2\sin\theta\cos\theta\cos\phi)\sigma_1 + (2\sin\theta\cos\theta\sin\phi)\sigma_2 + (\sin^2\theta - \cos^2\theta)\sigma_3\Big)}$$

(ii) Here, we have

$$\rho = \begin{pmatrix} 1/2 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/8 & 0 & 0 & 1/2 \end{pmatrix}$$

Now,

$$C_{11} = tr\left(\rho\,(x \otimes x)\right)$$

$$= tr\left(\begin{pmatrix} 1/2 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/8 & 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right)$$

$$\therefore C_{11} = \frac{-1}{4}$$

$$C_{22} = tr\left(\rho\,(y \otimes y)\right)$$

$$= tr\left(\begin{pmatrix} 1/2 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/8 & 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\right)$$

$$\therefore C_{22} = \frac{1}{4}$$

$$C_{33} = tr\left(\rho\,(z \otimes z)\right)$$

$$= tr\left(\begin{pmatrix} 1/2 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/8 & 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$$

$$\therefore C_{33} = 1$$

Now,

$$|C_{11}| + |C_{22}| + |C_{33}| = 3/2 > 1$$

Hence, the state is entangled.