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Subject: Foundations of Quantum information and computation

Q1 $A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$ $\langle A \rangle_\psi = ?$

$\psi = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Soln:

$\therefore |A - \lambda I| = 0$

$\therefore \begin{vmatrix} -\lambda & 0 & i \\ 0 & 1-\lambda & 0 \\ -i & 0 & -\lambda \end{vmatrix} = 0$

$\therefore (-\lambda)(-\lambda)(1-\lambda) + i(i)(1-\lambda) = 0$

$\therefore \lambda^2 - \lambda^3 - 1(1-\lambda) = 0$

$\therefore \lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\therefore (\lambda-1)^2(\lambda+1) = 0$

$\Rightarrow \underline{\lambda = 1, 1, -1}$ $\Rightarrow$ Eigen values of A.

→ For $\lambda = 1$:

$$\begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} iz \\ y \\ -ix \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{aligned} x &= zi \\ y &= y \\ -ix &= z \end{aligned} \quad \Rightarrow \underline{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}} \text{ and } \underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

Eigen vectors for $\lambda = 1$

→ For $\lambda = -1$:

$$\begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} iz \\ y \\ -ix \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

$$\Rightarrow \begin{aligned} x &= -iz \\ y &= -y \Rightarrow y = 0 \\ -ix &= -z \end{aligned}$$

$$\Rightarrow \underline{\underline{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ +i \end{pmatrix}}} \Rightarrow \text{Eigen vector for } \lambda = -1$$

→ Spectral decomposition of A is given as

$$\therefore A = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

where λ_i = eigen values
ψ_i = eigen vectors

$$\therefore A = (1)\left(\frac{1}{2}\right)\begin{pmatrix}1\\0\\-i\end{pmatrix}(1\ 0\ i)$$

$$+ 1\begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 1\ 0)$$

$$+ (-1)\frac{1}{2}\begin{pmatrix}1\\0\\i\end{pmatrix}(1\ 0\ -i)$$

→ Now to get $\langle A\rangle_\psi$:

$$\therefore \langle A\rangle_\psi = \langle\psi|A|\psi\rangle$$

$$|\psi\rangle = \frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} + \left(\frac{-i}{2}\right)\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1/2\\-i/2\\1/\sqrt{2}\end{pmatrix}$$

$$\Rightarrow \langle\psi| = (1/2\quad i/2\quad 1/\sqrt{2})$$

$\therefore \langle A \rangle_\psi = \langle \psi | A | \psi \rangle$

$$= \begin{pmatrix} \frac{1}{2} & \frac{i}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ -i/2 \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{i}{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} i/\sqrt{2} \\ -i/2 \\ -i/2 \end{pmatrix}$$

$$= \frac{i}{2\sqrt{2}} + \frac{1}{4} - \frac{i}{2\sqrt{2}}$$

$\Rightarrow \underline{\underline{\langle A \rangle_\psi = 1/4}}$

$\underline{\underline{Q2}}$ $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\rightarrow$ $|R - \lambda I| = 0$ $\quad$ (λ = eigen value)

$\therefore \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$

$\therefore (\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$\therefore \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta = 0$

$$\therefore \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4(\cos^2\theta + \sin^2\theta)}}{2}$$

$$= \frac{2\cos\theta \pm i\,2\sin\theta}{2}$$

$$\therefore \lambda = \cos\theta \pm i\sin\theta$$

→ For $\lambda = \cos\theta + i\sin\theta$;

$$\begin{pmatrix} \cos\theta \cancel{+i\sin} & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (\cos\theta + i\sin\theta)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \begin{pmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left.\begin{aligned} &\Rightarrow -i\sin\theta\, x + \sin\theta\, y = 0 \\ &\Rightarrow -\sin\theta\, x - i\sin\theta\, y = 0 \end{aligned}\right\} \quad \begin{aligned} &-x = iy \\ &\text{or } ix = y \end{aligned}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$ ⇒ Eigen vector for $\lambda = \cos\theta + i\sin\theta$

→ For $\lambda = \cos\theta - i\sin\theta$;

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (\cos\theta - i\sin\theta)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \begin{pmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left.\begin{aligned}(i\sin\theta)x + \sin\theta y = 0\\ (-\sin\theta)x + i\sin\theta y = 0\end{aligned}\right\} \quad \begin{aligned}-y = ix\\ \text{or } x = iy\end{aligned}$$

$$\therefore |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \Rightarrow \text{Eigen vector for } \lambda = \cos\theta - i\sin\theta$$

$$\Rightarrow U = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Verification :

$$U^{+}RU = \frac{1}{4}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \cos\theta + i\sin\theta & i\cos\theta + \sin\theta \\ -\sin\theta + i\cos\theta & -i\sin\theta + \cos\theta \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2(\cos\theta + i\sin\theta) & 0 \\ 0 & 2(\cos\theta - i\sin\theta) \end{pmatrix}$$

$\hookrightarrow$ diagonal matrix under unitary transformation

Q3

→ Spectral decomposition of σ_x operator:

$\therefore \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\therefore |\sigma_x - \lambda I| = 0$

$\therefore \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$

$\therefore \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$

$\therefore \lambda^2 - 1 = 0$

$\Rightarrow \underline{\underline{\lambda = \pm 1}} \Rightarrow$ Eigen values of σ_x

→ For $\lambda = 1$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow x = y$

$\Rightarrow \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}} \Rightarrow$ Eigen vector for $\lambda = 1$

For $\lambda = -1$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

$\Rightarrow x = -y$

$\Rightarrow \underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}} \Rightarrow$ Eigen vector for $\lambda = -1$

$\rightarrow \sigma_x = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$

$$= (1)\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}$$

$$+ (-1)\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}$$

$$\therefore \sigma_x = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} + -\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix}$$

Q3

→ spectral decomposition of σ_y operator :

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\therefore \; |\sigma_y - \lambda I| = 0$

$\therefore \; \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$

$\therefore \; \lambda^2 - 1 = 0$

$\therefore \; \underline{\underline{\lambda = \pm 1}} \Rightarrow$ Eigen value of σ_y

→ $\underline{\text{for } \lambda = 1}$;

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -iy \\ ix \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow y = ix$

$\Rightarrow \underline{\underline{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}}} \Rightarrow$ Eigen vector for $\underline{\lambda = 1}$

→ For $\lambda = -1$:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-1)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \quad -iy = -x$$

$$\Rightarrow x = iy$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \Rightarrow \text{Eigen vector for } \lambda = -1$$

$$\Rightarrow \sigma_y = \sum_i \lambda_i |i\rangle \langle i|$$

$$\sigma_y = \frac{(1)}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}\begin{pmatrix} 1 & -i \end{pmatrix} + \frac{(-1)}{2}\begin{pmatrix} i \\ 1 \end{pmatrix}\begin{pmatrix} -i & 1 \end{pmatrix}$$

Q4 $U_{CNOT}(X \otimes I_2)U_{CNOT} = ?$

$\rightarrow X \otimes I_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\rightarrow U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$\therefore U_{CNOT}(X \otimes I_2)U_{CNOT}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Leftarrow \underline{\text{ans}}.$$

Q5

→ Let's consider Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$

$$\therefore \rho = |\beta_{00}\rangle\langle\beta_{00}|$$

$$= \frac{1}{2}\left[|0\rangle|0\rangle + |1\rangle|1\rangle\right]\left[\langle 0|\langle 0| + \langle 1|\langle 1|\right]$$

$$= \frac{1}{2}\Big[(|0\rangle\langle 0|)_1 (|0\rangle\langle 0|)_2 + (|0\rangle\langle 1|)_1 (|0\rangle\langle 1|)_2 + (|1\rangle\langle 0|)_1 (|1\rangle\langle 0|)_2 + (|1\rangle\langle 1|)_1 (|1\rangle\langle 1|)_2\Big]$$

$$\therefore \rho_1 = Tr_2(\rho)$$

$$= \frac{Tr_2}{2}\Big[(|0\rangle\langle 0|)_1 (|0\rangle\langle 0|)_2 + (|0\rangle\langle 1|)_1 (|0\rangle\langle 1|)_2 + (|1\rangle\langle 0|)_1 (|1\rangle\langle 0|)_2 + (|1\rangle\langle 1|)_1 (|1\rangle\langle 1|)_2\Big]$$

$$= \frac{1}{2}\Big[(|0\rangle\langle 0|)_1\, Tr_2(|0\rangle\langle 0|)_2 + (|0\rangle\langle 1|)_1\, Tr_2(|0\rangle\langle 1|)_2 + (|1\rangle\langle 0|)_1\, Tr_2(|1\rangle\langle 0|)_2 + (|1\rangle\langle 1|)_1\, Tr_2(|1\rangle\langle 1|)_2\Big]$$

$$= \frac{1}{2}\Big[(|0\rangle\langle 0|)(\overset{=1}{\langle 0|0\rangle}) + (|0\rangle\langle 1|)(\overset{=0}{\langle 0|1\rangle}) + (|1\rangle\langle 0|)(\overset{=0}{\langle 1|0\rangle}) + (|1\rangle\langle 1|)(\overset{=1}{\langle 1|1\rangle})\Big]$$

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$$\therefore \rho_1 = \frac{1}{2}\left[|0\rangle\langle 0| + |1\rangle\langle 1|\right]$$

$$\therefore \underline{\rho_1} = \frac{I_2}{2}$$

Q6 $$U_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore U_Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$= \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\left(\frac{\langle +| + \langle -|}{\sqrt{2}}\right)$$

$$- \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)\left(\frac{\langle +| - \langle -|}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\Big[\cancel{|+\rangle\langle +|} + |+\rangle\langle -| + |-\rangle\langle +| + \cancel{|-\rangle\langle -|} - \cancel{|+\rangle\langle +|} + |+\rangle\langle -| + |-\rangle\langle +| - \cancel{|-\rangle\langle -|}\Big]$$

$$\therefore U_Z = |+\rangle\langle -| + |-\rangle\langle +|$$

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Spectral decomposition of σ_z operator:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\rightarrow |\sigma_z - \lambda I| = 0$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & +\lambda \end{pmatrix} = 0$

$\therefore \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = 0$

$\therefore -(1-\lambda^2) = 0$

$\Rightarrow \underline{\underline{\lambda = \pm 1}} \rightarrow$ Eigen values of σ_z

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For $\lambda = 1$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow y = 0$

$x = x$

$\therefore \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$ Eigen vector for $\lambda = 1$

For $\lambda = -1$

$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$\Rightarrow y = y$

$x = 0$

$\therefore \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$ Eigen vector for $\lambda = -1$

$$\Rightarrow \therefore \sigma_2 = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

$$\therefore \sigma_2 = (1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\sigma_2 = |0\rangle\langle 0| - |1\rangle\langle 1|$$

→ Spectral decomposition of the H-gate:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\therefore |H - \lambda I| = 0$

$$\therefore \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow -\left(\frac{1}{2} - \lambda^2\right) - \frac{1}{2} = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$\Rightarrow \underline{\lambda = \pm 1}$ ⇒ Eigen value of H-gate

→ For $\underline{\lambda = 1}$:

$$\begin{bmatrix} 1/\sqrt{2} - 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} - 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}} - 1\right)x + \frac{1}{\sqrt{2}}y = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}}x - \left(\frac{1}{\sqrt{2}} + 1\right)y = 0 \rightarrow \frac{1}{\sqrt{2}}x = \frac{1}{\sqrt{2}}y + y$$

$$\therefore x = (\sqrt{2} + 1)y$$

$\Rightarrow \begin{pmatrix} \sqrt{2}+1 \\ 1 \end{pmatrix} \Rightarrow$ Eigen vector for $\lambda = 1$

↳ on normalizing,

$$\frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$$

→ for $\lambda = -1$:

$$\begin{pmatrix} \frac{1}{\sqrt{2}}+1 & 1/\sqrt{2} \\ 1/\sqrt{2} & \frac{-1}{\sqrt{2}}+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \left(\frac{1}{\sqrt{2}}+1\right)x + \frac{1}{\sqrt{2}}y$ and $\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y + y = 0$

$\Rightarrow x = (1-\sqrt{2})y$

$\Rightarrow \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow$ Eigen vector for $\lambda = -1$

→ Spectral decomposition

$$H = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

$$\therefore H = \frac{1}{(4+2\sqrt{2})} \begin{pmatrix} \sqrt{2}+1 \\ 1 \end{pmatrix} \begin{pmatrix} \sqrt{2}+1 & 1 \end{pmatrix}$$

$$- \left(\frac{1}{4-2\sqrt{2}}\right) \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \begin{pmatrix} 1-\sqrt{2} & 1 \end{pmatrix}$$

Q7

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\delta}\sin\frac{\theta}{2}|1\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\delta}\sin\frac{\theta}{2} \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\therefore \sigma_y|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\theta/2 \\ e^{i\delta}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -ie^{i\delta}\sin\theta/2 \\ i\cos\theta/2 \end{pmatrix}$$

↳ Effect of ygate in terms of the Bloch sphere

Q8 To show: Controlled NOT gate is hermitian and unitary

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

→ For hermitian, $A^+ = A$;

$\therefore$ $$U^+_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow U^+_{CNOT} = U_{CNOT}$

$\Rightarrow$ U_{CNOT} is hermitian.

→ For unitary, $A^+ A = I$

$\therefore U^+_{CNOT} U_{CNOT} = (U_{CNOT})^2$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

$\Rightarrow$ U_{CNOT} is unitary

Q9 $X \otimes Z |\beta_{00}\rangle$ is entangled?

$$\therefore X \otimes Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\therefore |\beta_{00}\rangle = \frac{1}{\sqrt{2}} [|00\rangle + |11\rangle]$$

$$= \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$\therefore X \otimes Z |\beta_{00}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

→ The state $|\psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ is seperable iff $ad = bc$

Here, $\left.\begin{matrix} ad = 0 \\ bc = -1 \end{matrix}\right\}$ $ad \neq bc$

$\Rightarrow$ It is not seperable

$\Rightarrow$ $X \otimes Z |\beta_{00}\rangle$ is entangled.

Q10

$$\rho = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

→ Pauli representation is given as:

$$\rho = \frac{1}{2}\sum_{i=0}^{3} c_i \sigma_i$$

where, $c_i = Tr(\rho\sigma_i) = \langle\sigma_i\rangle$

$[\sigma_0 = I, \ \sigma_1 = \sigma_x, \ \sigma_2 = \sigma_y, \ \sigma_3 = \sigma_z]$

→

$\therefore c_0 = Tr(\rho\sigma_0)$

$$= Tr\left[\begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= Tr\left[\begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\right]$$

$$= \sin^2\theta + \cos^2\theta$$

$\therefore \underline{\underline{c_0 = 1}}$

$\rightarrow \therefore C_1 = Tr(\rho\sigma_1)$

$$= Tr\left[\begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= Tr\begin{bmatrix} e^{-i\phi}\sin\theta\cos\theta & \sin^2\theta \\ \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \end{bmatrix}$$

$$= e^{-i\phi}\sin\theta\cos\theta + e^{i\phi}\sin\theta\cos\theta$$

$$\therefore C_1 = 2\sin\theta\cos\theta\cos\phi$$

$\rightarrow C_2 = Tr(\rho\sigma_2)$

$$= Tr\left[\begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]$$

$$= Tr\begin{bmatrix} ie^{-i\phi}\sin\theta\cos\theta & -i\sin^2\theta \\ i\cos^2\theta & -ie^{i\phi}\sin\theta\cos\theta \end{bmatrix}$$

$$= i\sin\theta\cos\theta\,(e^{-i\phi} - e^{i\phi})$$

$$\therefore C_2 = 2\sin\theta\cos\theta\sin\phi$$

$\rightarrow c_3 = Tr(\rho\sigma_3)$

$$= Tr\left[\begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$= Tr\begin{bmatrix} \sin^2\theta & -e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & -\cos^2\theta \end{bmatrix}$$

$$\therefore c_3 = \sin^2\theta - \cos^2\theta$$

$\rightarrow$ Thus,

$$\therefore \rho = \frac{1}{2}\sum_i c_i \sigma_i$$

$$\therefore \rho = \frac{1}{2}\Big[(1)\sigma_0 + 2\sin\theta\cos\theta\cos\phi\,(\sigma_1)$$
$$+ 2\sin\theta\cos\theta\sin\phi\,(\sigma_2)$$
$$+ (\sin^2\theta - \cos^2\theta)(\sigma_3)\Big]$$

Q11

$$\rho = \begin{pmatrix} 1/2 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/8 & 0 & 0 & 1/2 \end{pmatrix}$$

$$\therefore \rho^2 = \begin{pmatrix} 1/2 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/8 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/8 & 0 & 0 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{64} & 0 & 0 & -1/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/8 & 0 & 0 & 17/64 \end{pmatrix}$$

$$Tr(\rho^2) = \frac{17}{64} + \frac{17}{64} = \frac{17}{32} < 1$$

$\Rightarrow$ ρ is an entangled state.