The G20 Currency Frontier

Group Assignment

IB9110: Asset Pricing and Risk

Group 1

Hema Srikar Ankem

Module Instructor: Prof. Roman Kozhan

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Abstract

This Project examines optimal portfolio construction for foreign currencies of G20 countries against the US dollar, as the G20 countries represent the world's largest economies and account for a substantial share of global currency trade. Markowitz Mean-Variance based Sample Estimates, Markowitz on Shrinkage estimator, Naive Equally weighted 1/N method, Constrained method and Resampling-based Estimation are the techniques used in optimizing the portfolios. The currency excess returns are computed using interest rate differentials and logarithm exchange rate changes. The common instability and estimation error associated with the covariance matrices in FX data are minimized using the shrinkage estimator and the results are compared with the sample estimates benchmark tangency portfolio. For our analysis, we have considered both the constrained and unconstrained scenarios for the weights of the assets. Our risk measure for optimizing portfolio is variance and we tend to maximize the Sharpe ratio in all the method except for Constrained where the we try to maximize the utility function, a function of weights of assets with a risk aversion coefficient.

The historical Forex currencies of G20 countries data is obtained from the Board of Governors of the Federal Reserve System, USA website [5] and from Federal Reserve Bank of St. Louis [4][3][2]

Keywords: Markowitz Mean-Variance optimization, Resampling-based Estimation, Shrinkage Estimator, Constrained portfolios, Sharpe Ratio, Rolling window method, Sharpe Ratio, Cumulative Wealth, Efficient frontiers, Annualized Expected returns, Boot strap window size, Rolling window methodology, Covariance matrix

1 | Introduction

Foreign Exchange (FX) markets play a pivotal role in the global financial system, providing liquidity and facilitating international trade and investment. The current domestic interest rates, international trade, and economic development all play a crucial role in determining the demand for a country's currency. In the early 2000s, we witnessed the influence of lower domestic interest rates on the global foreign exchange markets. Particularly from Japan where lower domestic interest rates pushed the investments in the foreign assets by borrowing cheaper Japanese yen seeking better returns.

In recent times, we have also witnessed the fluctuations in the foreign exchange markets due to the tariffs imposed by the United States of America. The country with heavy tariffs may see a downward pressure on their currency, as the demand for their currency to pay for exports declines. Foreign investors often tend to invest in assets of countries with growing economies, which involves selling their domestic currency for foreign currencies to make the purchase of assets. This increase in demand for foreign currencies tends to appreciate its value when compared to other currencies.

Managing a portfolio of the foreign currencies provides an opportunity as well as involves risk as exchange rates are subjected to volatility due to the macroeconomic development, domestic interest rates, geopolitical events and market sentiments. The aim to optimise the portfolio in FX markets is to mitigate risk while balancing the expected returns. Unlike the equities portfolios, optimising the FX portfolios needs attention to the characteristics like interest rate differentials, effects of currency carry which has the impact on excess returns. However, the high correlation and interdependence among the global currencies also demand a robust estimation of expected returns and covariance structures.

1.1 | The Markowitz Mean-Variance Method

The Mean-variance portfolio optimization, introduced by Markowitz (1952), provides a theoretical framework for constructing optimal portfolios that minimises portfolio risk level for a given vector of returns. It formalises the idea of a trade-off between risk and return as well as demonstrating the outcome by an efficient frontier.

The implementation of Markowitz presents difficulties. The main challenge is in its estimation error as the inputs required, expected returns and the variance-covariance matrix, are unobservable population parameters that must be estimated from finite historical samples, introducing substantial sampling variation. In the Michaud (1989)[8] paper, they remark that the mean-variance optimiser tends to maximise the impact of these estimation errors, overweighting assets with high estimated returns and low estimated correlations. The optimiser often overweighs the most extreme and unreliable parameter estimates, producing portfolios that perform poorly out of sample.

DeMiguel, Garlappi and Uppal (2009)[1] showed that with realistic estimation windows of 60 or 120 months, the out of sample Sharpe ratio of sample-based mean-variance portfolios drops dramatically. The estimation window required for mean-variance strategies to reliably outperform a naive 1/N allocation is approximately 3000 months for 25 assets and 6000 months for 50 assets, rendering practical implementation very difficult.

1.2 | Shrinkage estimation of the covariance matrix

Leidot,et al [7] propose shrinkage estimation as an approach to mitigating the estimation error in covariance matrices. When the number of parameters to estimate is large relative to the sample size, shrinking estimates towards a structured target can reduce mean squared error.

The Ledoit-Wolf estimator combines the sample covariance matrix with a structured shrinkage target which is the single-index model covariance matrix implied by Sharpe's (1963) market model. The optimal shrinkage intensity is derived by minimising expected quadratic loss under the Frobenius norm. The intensity can be consistently estimated from the data without requiring knowledge of the true covariance matrix. The resulting estimator pulls extreme covariance estimates towards more central values, systematically reducing the estimation error.

Leidot,et al [7] demonstrate that their shrinkage estimator produces portfolios with significantly lower out-of-sample variance compared to a range of alternative estimators, including multifactor models. The method's advantage is particularly visible when the number of assets is large relative to the estimation window.

Table 1.1: *Markowitz Mean-Variance Portfolios: Sample Estimates vs Shrinkage Estimator*

Feature	Sample Estimates	Shrinkage Estimator
Covariance Matrix	Raw sample covariance computed over the rolling window	Weighted combination of sample covariance and a structured target
Estimation Error	High arises due to the sensitivity to noise, especially when the number of assets is large relative to the sample size	Shrinkage stabilizes the covariance estimate and mitigates extreme noise effects results in reduced error
Portfolio Weights	Can be highly concentrated and unstable due to the extreme long or short positions are possible	More stable as the weights are less extreme and more diversified
Out-of-Sample Performance	Often poor, particularly in high-dimensional settings or small samples	Generally better because of improved Sharpe ratios and robustness out-of-sample
Sensitivity to Outliers	High; single extreme returns can heavily influence weights	Reduced; shrinkage partially pools information across assets, mitigating the effect of outliers
Numerical Stability	Poor performance if covariance matrix is ill-conditioned or nearly singular	Guaranteed invertibility and numerical stability
Adaptability	Fully responsive to recent market changes; can react strongly to volatility shifts	Moderately responsive; strong shrinkage may smooth over short-term fluctuations
Complexity	No extra parameters or targets are required, hence simple approach	Slightly more complex, requires selecting a shrinkage target and tuning shrinkage intensity α

1.3 | Naive equally-weighted portfolio

An alternative approach is the Naive Equally Weighted Portfolio. DeMiguel, et al [1] compare 14 different asset allocation models across seven empirical datasets. Their finding is that none of the more sophisticated optimization strategies consistently outperforms the 1/N method in terms of Sharpe ratio, certainty-equivalent return, or portfolio turnover.

The success of this parameter free strategy comes from its complete avoidance of estimation error. The lack of input data provides an advantage when estimation errors in the optimised portfolios are sufficiently large to offset the expected gains from optimal diversification. The 1/N approach also benefits from implicit rebalancing, automatically selling winners and buying losers.

1.4 | Constrained portfolios

Frost and Savarino [6] proposed imposing explicit constraints on portfolio weights to reduce estimation error impact. Estimation errors cause overinvestment in securities whose attractive characteristics may reflect measurement noise rather than genuine opportunity. Upper bound constraints limit this over investment and reduce bias in both expected return and variance estimates, with the bias approaching zero as constraints become more restrictive. The practical implication is that constraints which might appear restrictive actually benefit investors when information is measured with error.

1.5 | Portfolio Resampling

Resampling-based portfolio optimization, introduced by Michaud (1989) and examined by Scherer (2002)[9], offers an alternative simulation-driven approach to addressing estimation error. The method works by generating multiple samples from the estimated return distribution, computing optimal portfolio weights for each sample and then averaging across all resampled portfolios to obtain final allocations. This acts to reduce sensitivity to any single estimate.

However, Scherer notes theoretical limitations and that resampled portfolios inherit estimation errors from original parameter estimates. Despite these critiques, Scherer acknowledges that resampling tends to outperform unconstrained Markowitz portfolios in out-of-sample tests, primarily due to the implicit regularisation that diversification provides.

2 | Methodology

The Foreign currencies used for the portfolio construction are as follows,

Country	Australia	Euro Region	New Zealand	United Kingdom	Brazil
Currency	Australian Dollar	Euro	New Zealand Dollar	Pound Sterling	Brazilian Real
Symbol	AUD	EUR (€)	NZD	GBP (£)	BRL (R$)
Country	Canada	Denmark	Japan	Mexico	Norway
Currency	Canadian Dollar	Danish Krone	Japanese Yen	Mexican Peso	Norwegian Krone
Symbol	CAD ($)	DKK (kr)	JPY (¥)	MXN ($)	NOK (kr)
Country	South Africa	Singapore	Sweden	Switzerland	Taiwan
Currency	South African Rand	Singapore Dollar	Swedish Krona	Swiss Franc	New Taiwan Dollar
Symbol	ZAR (R)	SGD ($)	SEK (kr)	CHF	TWD (NT$)
Country	Turkey	Czechia	Hungary		
Currency	Turkish Lira	Czech Koruna	Hungarian Forint		
Symbol	TRY	CZk (Kč)	HUF (Ft)		

Table 2.1: *List of countries, currencies, and symbols*

The excess return is the return for holding the foreign currency from the interest rate in addition with the change in the foreign currency prices. The excess return of Foreign Exchange is calculated differently from the equities, which is given by,

FX Excess return (rx) = Interest Rate differential + Normal Forex returns

$$rx_t = (i_{\text{for,t}} - i_{\text{dom,t}}) - \Delta s_t \tag{2.1}$$

- $i_{\text{for,t}}$: Foreign interest rate
- $i_{\text{dom,t}}$: Domestic interest rate (USD)
- $s_t = \log(S_t)$: The log exchange rate expressed as domestic currency per foreign currency
- $\Delta s_t = \log(S_t) - \log(S_{t-1})$

We employ a rolling window methodology with a 120-month estimation period and monthly rebalancing. At each evaluation date, t, we estimate the sample mean vector $\hat{\mu}_t$ and sample covariance matrix $\hat{\Sigma}_t$ using the preceding 120 months of excess return data:

$$\hat{\mu}_t = \frac{1}{60}\sum_{\tau=t-60}^{t-1} r_\tau, \quad \hat{\Sigma}_t = \frac{1}{60}\sum_{\tau=t-60}^{t-1} (r_\tau - \hat{\mu}_t)(r_\tau - \hat{\mu}_t)^\top \tag{2.2}$$

Where r_τ is the $N \times 1$ vector of excess returns at time τ for the N currencies.

Table 2.2: *Portfolio Optimisation Regimes: Shorting Allowed vs No-Shorting*

Feature	Shorting Allowed	No-Shorting
Weight Constraints	$w_i \in \mathbb{R}$ (long or short positions permitted)	$w_i \geq 0$ (long-only portfolio)
Budget Constraint	$\mathbf{1}^\top \mathbf{w} = 1$	$\mathbf{1}^\top \mathbf{w} = 1$
QP Structure	Equality-constrained quadratic program	Quadratic program with both equality and inequality constraints
Flexibility	High (can exploit negative views and relative value trades)	Limited (cannot take negative exposure in currencies)
Risk Characteristics	May increase volatility due to leverage or net short positions	Typically more stable and less volatile
Return Potential	Higher theoretical return and Sharpe ratio potential	More conservative return profile
Estimation Sensitivity	More sensitive to covariance estimation errors	More robust to estimation noise
Practical Implication	Allows currency carry, trend-following, and hedging trades	Suitable for conservative or regulated investors

2.1 | Markowitz Mean-variance Method

The Markowitz framework selects portfolio weights to minimize variance for a given expected return. For a target expected return $r_{p,t}$, the optimization problem is:

$$\min_w \; w^\top \hat{\Sigma}_t w \quad \text{subject to} \quad w^\top \hat{\mu}_t = r_{p,t}, \; w^\top \mathbf{1} = 1 \tag{2.3}$$

Among all efficient portfolios, the tangency portfolio maximises the Sharpe ratio when a risk-free asset exists. At each evaluation date t, we implement the tangency portfolio with weights:

$$w_t = \frac{\hat{\Sigma}_t^{-1}\hat{\mu}_t}{1_N \hat{\Sigma}_t^{-1}\hat{\mu}_t} \tag{2.4}$$

Where $\mathbf{1}$ is an $N \times 1$ vector of ones. The Standard scalars in Mean-variance theory and standard deviation are given by,

$$\begin{aligned} A &= \mathbf{1}^\top \hat{\Sigma}^{-1}\mathbf{1}, \\ B &= \mathbf{1}^\top \hat{\Sigma}^{-1}\hat{\mu}, \\ C &= \hat{\mu}^\top \hat{\Sigma}^{-1}\hat{\mu}, \\ D &= AC - B^2. \end{aligned}$$

$$w(R) = \frac{C - BR}{D}\,\hat{\Sigma}^{-1}\mathbf{1} + \frac{AR - B}{D}\,\hat{\Sigma}^{-1}\hat{\mu}, \tag{2.5}$$

$$\sigma(R) = \sqrt{w(R)^\top \hat{\Sigma}\, w(R)}. \tag{2.6}$$

2.2 | Shrinkage estimation of the covariance matrix

The Ledoit-Wolf method replaces the sample covariance matrix with a shrinkage estimator

$$\hat{\Sigma}_{\text{shrink}} = \delta \mathbf{F} + (1-\delta)\hat{\Sigma}$$

Where $\mathbf{F}$ is the constant correlation target matrix with elements $f_{ii} = s_{ii}$ (sample variances on the diagonal) and $f_{ij} = \bar{r}\sqrt{s_{ii}s_{jj}}$ for $i \neq j$, using the average sample correlation $\bar{r}$.

$\delta \in [0,1]$ is the optimal shrinkage intensity, estimated as:

$$\delta = \tfrac{\hat{\pi}-\hat{\rho}}{\hat{\gamma}}$$

$\hat{\pi}$ measures the sum of asymptotic variances of the sample covariance matrix entries. $\hat{\rho}$ captures covariance between the target and sample covariance matrices and $\hat{\gamma}$ quantifies the misspecification of the target matrix. These parameters are estimated from the data. The tangency portfolio is then constructed using $\hat{\Sigma}_{\text{shrink}}$

2.3 | Naive equally-weighted portfolio

The naive strategy assigns equal weight to each currency:

$$w_i = \frac{1}{N}, \quad \forall i = 1, \ldots, N \tag{2.7}$$

This parameter-free approach is data independent and completely avoids the estimation error.

2.4 | Constrained portfolios

The constrained approach limits portfolio weights to reduce sensitivity to estimation error. We solve:

$$\max_{w} \; w^\top \hat{\Sigma}_t \, w - \lambda \times \text{risk}_i(w) \quad \text{subject to} \quad w^\top \hat{\mu}_t = r_{p,t}, \; w^\top \mathbf{1} = 1, \quad l \leq w_i \leq u \tag{2.8}$$

The bounds (l,u), depend on whether the short-selling is permitted. $l = 0$ when short selling is not considered. The λ is the risk aversion coefficient.

2.5 | Portfolio Resampling

The resampling method addresses estimation uncertainty through bootstrap simulation. At each rebalancing date t, we generate $B = 60$ bootstrap samples by randomly drawing from the 60 month estimation window.

For each sample $b = 1, \ldots, B$, we compute sample estimates $\hat{\mu}_t^{(b)}$ and $\hat{\Sigma}_t^{(b)}$, then calculte the tangency portfolio weights:

$$w_t^{(b)} = \frac{(\hat{\Sigma}_t^{(b)})^{-1}\hat{\mu}_t^{(b)}}{\mathbf{1}^\top(\hat{\Sigma}_t^{(b)})^{-1}\hat{\mu}_t^{(b)}} \tag{2.9}$$

The final resampled portfolio is the simple average across all bootstrap samples:

$$\bar{w}_t = \frac{1}{B}\sum_{b=1}^{B} w_t^{(b)} \tag{2.10}$$

2.6 | Performance Evaluation

We evaluate out-of-sample performance using:

- Annualised excess return: $\bar{r}_p \times 12$ where $\bar{r}_p$ is mean monthly excess return
- Annualised volatility: $\sigma_p \times 12$ where σ_p is mean monthly standard deviation
- Sharpe ratio: $\frac{\bar{r}_p \times 12}{\sigma_p \times \sqrt{12}}$

3 | Data and Empirical Results

The portfolio optimization methods were computed using python programming language, particularly using skfolio (0.15.2v)[10] package and also other common python scientific libraries like NumPy, Matplotlib, pandas. Using the Forex data obtained from Federal Reserve System[5], Federal Reserve Bank of St. Louis[2][3][4] and with the Risk-free monthly interest rates data we calculate the excess forex returns using the equation(2.1).

3.0.1 | Constraints and Programming Settings for the Optimization Methods:

Rolling window methodology: The window length is taken as 120 months (10 years). The deliberate choice of 120 months instead of 60 months is due to the very high noise in the data, to the extent which the optimization methods stopped working. Even with a 120 month window, the portfolio weights are of very small magnitude (in the range of $10^{-7} \sim 10^{-8}$). For meaningful analysis, based on different window size iterations, 120 windows size is decided for our optimization.

Another Important reason for choosing 120 month window is that, the noise caused by financial crisis during 2000s(Dot-Com Bubble) and the during 2008s (Housing Market Crises) will be included in estimating our weights which makes our portfolio construction robust any other financial crises in the foreseen future. A combination of Ledoit-Wolf shrinkage and resampling, mitigates the estimation risk and produces stable portfolios suitable for out-of-sample evaluation, so it is safe to consider a 120 month window in our Forex rates portfolio construction.

Constrained portfolio method: The lower bound constraint is set at -10% and the upper bound constraint is set at 40%. The Risk aversion coefficient (λ) is taken as 1 (Aggressive Risk taking) and 10 (Moderate Risk taking). The two results are compared in the above plots.

Resampling-based Estimation: The boot strap sample (B) is taken as 60, due to the constrain of the computational power. Although values between 100 and 500 is standard for resampling optimization, it has resulted in expected returns of all assets to be below 0% making the Sharpe ratio optimization not feasible. Using the 60 boot strap values may be a comprise, but it has produced comparable results, which are evident by the above plots.

> ***Note***
>
> *The Resampling optimization method is only for comparison with the other methods. Our optimization priority lies in other methods.*

Computational Constraints and Rounds offs: For some of the optimization methods, the weights for the assets are very small in magnitude, so a threshold of 10^{-10} is set, so that any weight smaller than the threshold is taken to be zero. Also, due to the long decimal values, for simplicity, the numbers are rounded off to 8 digits.

Plotting the Data: For detailed comparison and analysis, we have plotted several plots for the cases with short selling and without short selling. Since, Naive Equally weighted method does not depend and the constraint of our position in the market, we get the same results for both the cases, with and without short selling. So, only one plot 3.5 was given. For the constrained portfolio optimization 3.4b, we don't have the two cases as before, since we are fixing the bounds for the weights.

Before we construct the portfolios, we use all of the excess returns data to compute a benchmark portfolio using the Markowitz Mean-variance Method. We calculate the weights for all the countries in the data set and these are the weights that maximize the Sharpe ratio assuming zero risk-free rate. Since (i_dom, t) already included included it in the equation (2.1) of excess returns, we are not required to include it again in the Sharpe ratio numerator. So, zero risk-free rate is taken as zero. Using the standard scalars in the Markowitz mean-variance theory (2.5), we calculate the weight and using the weights we get the efficient frontier (2.6) (Risk - Return Curve) as follows,

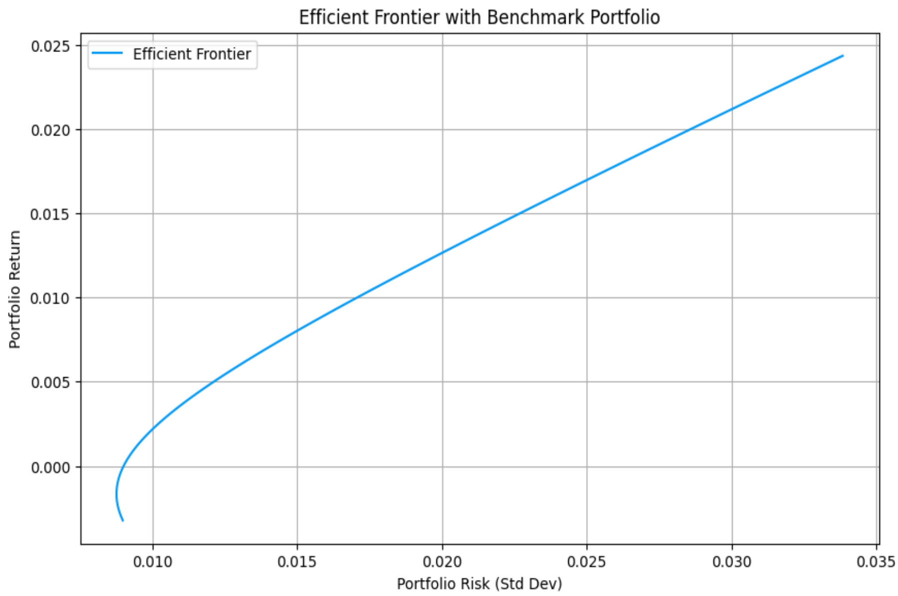


Figure 3.1: *Benchmark Tangency Portfolio*

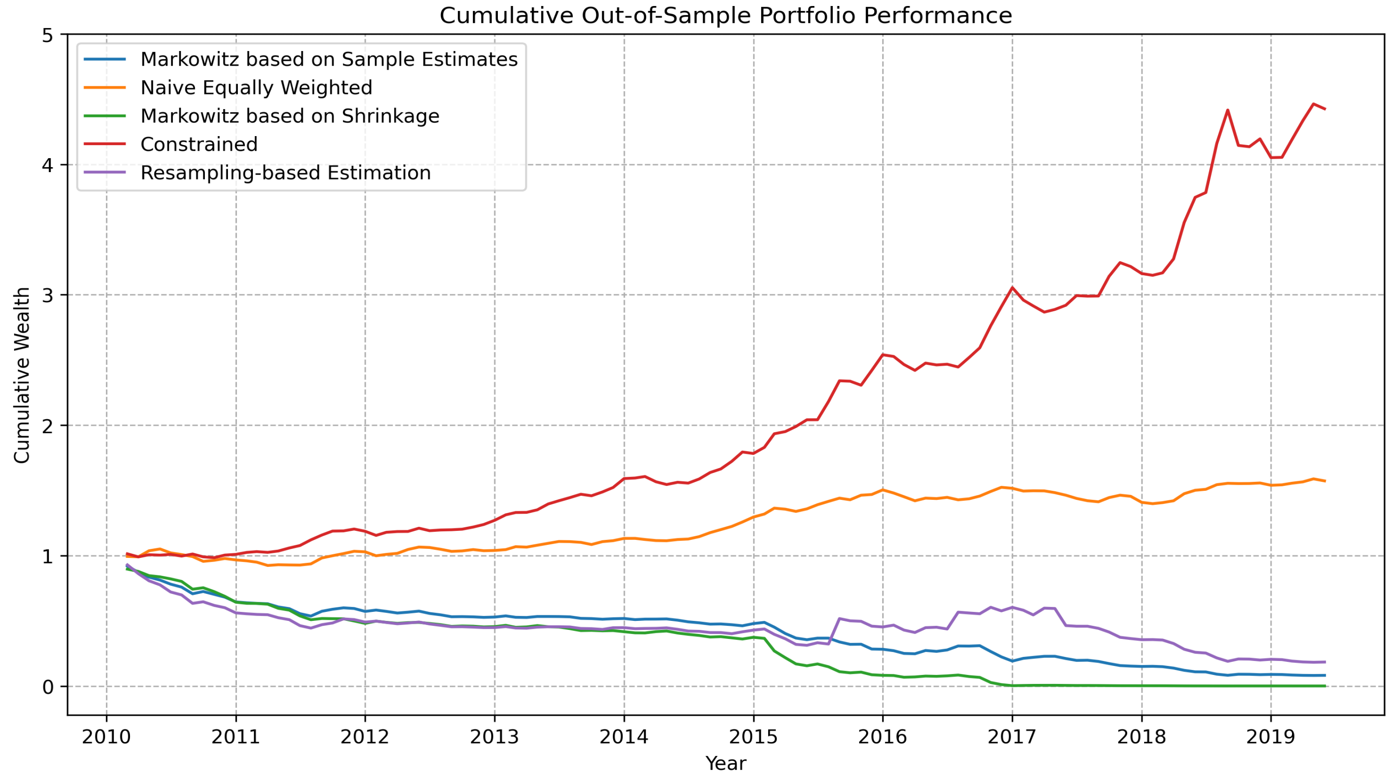
The above frontier is called the tangency or maximum Sharpe ratio portfolio, it serves as the benchmark for us to compare different methods of portfolio optimization. Using the all the portfolio optimization methods, we get the following annualized Sharpe Ratios for Out of sample data.

	Markowitz (Sample Estimates)	Markowitz (Shrinkage Estimator)	Naive Equally weighted	Constrained	Resampling
Without Short Selling	1.8566	1.8577	0.89230	1.9927	1.8430
With Short Selling	-1.4640	-1.1800	0.8930	1.9927	-0.5132

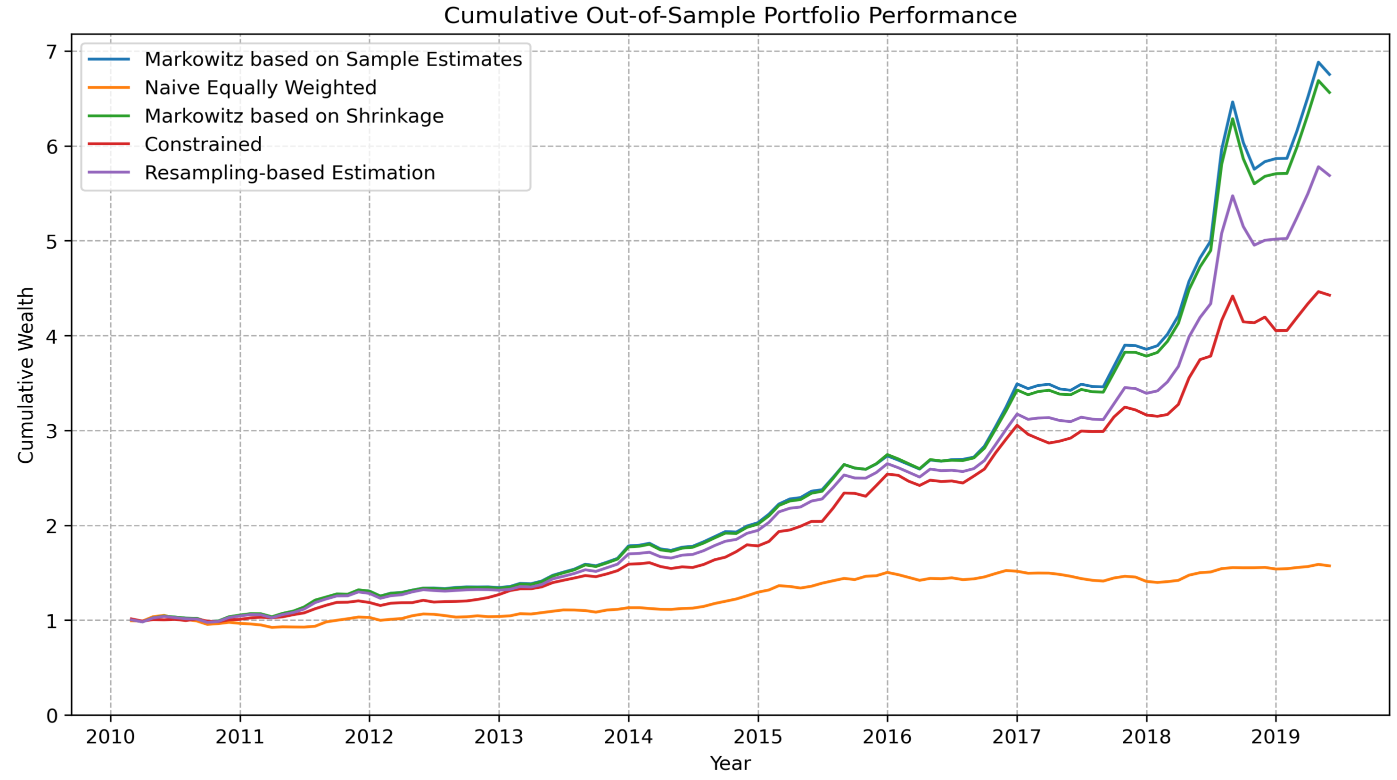
Table 3.1: *Annualized Out of Sample Sharpe Ratio Summary*

3.1 | Plots

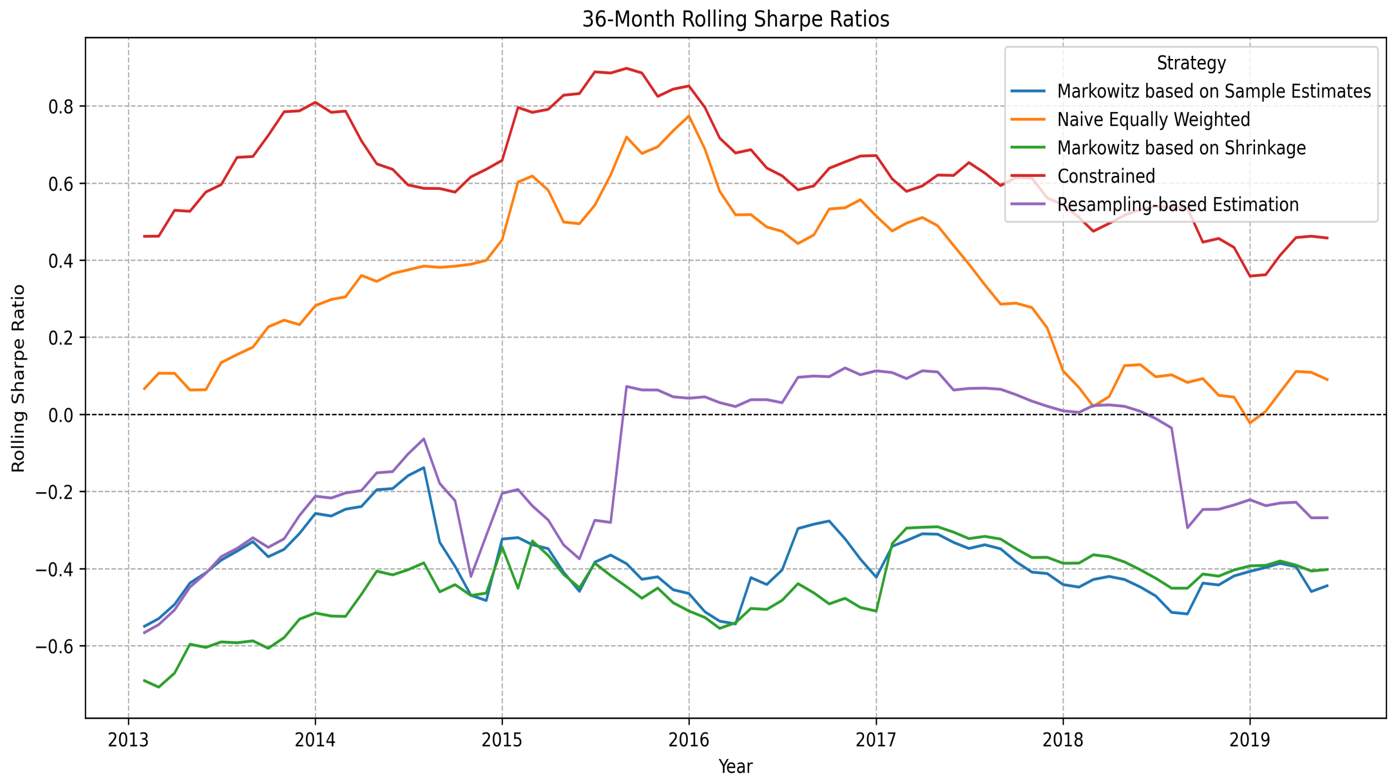
3.1.1 | Cumulative Wealth and Sharpe Ratio



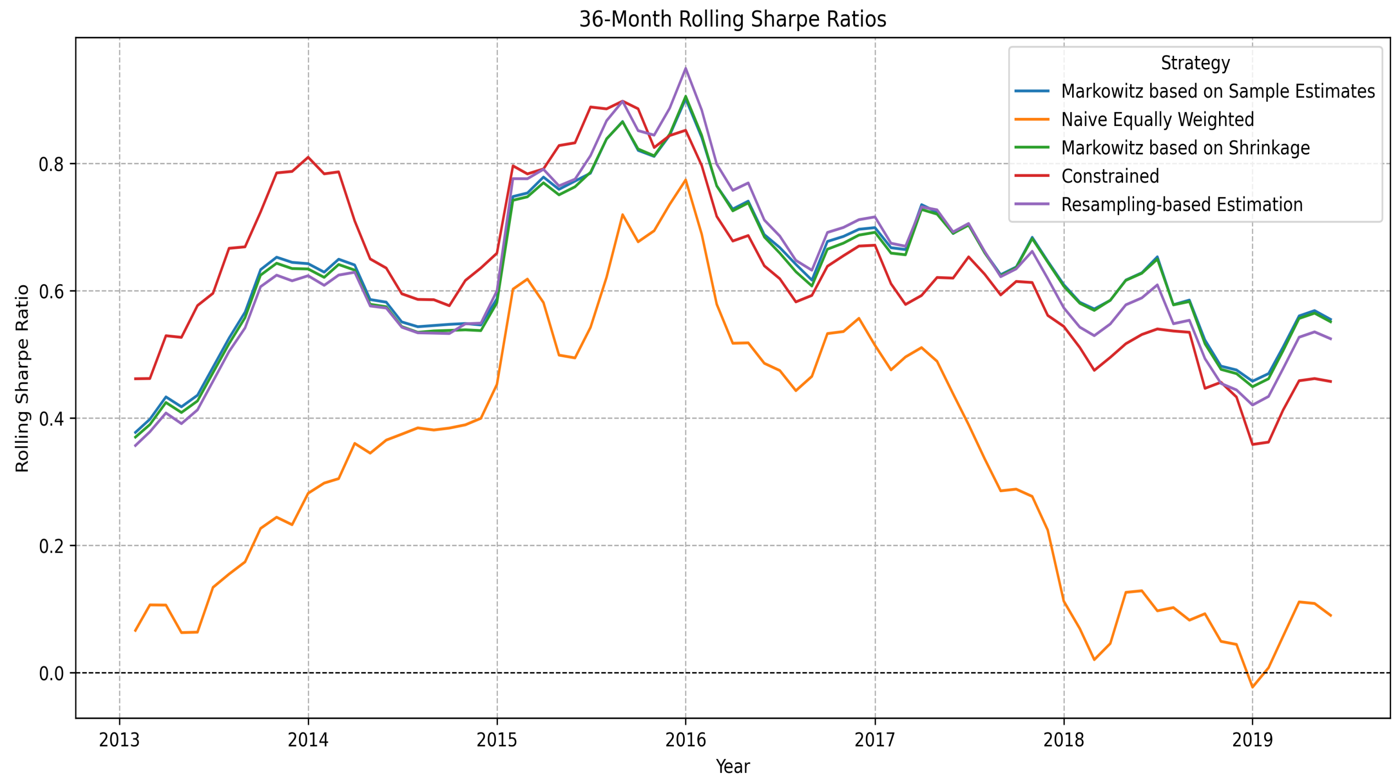
(a) *Cumulative Wealth - Short Selling*



(b) *Cumulative Wealth - Without Short Selling*

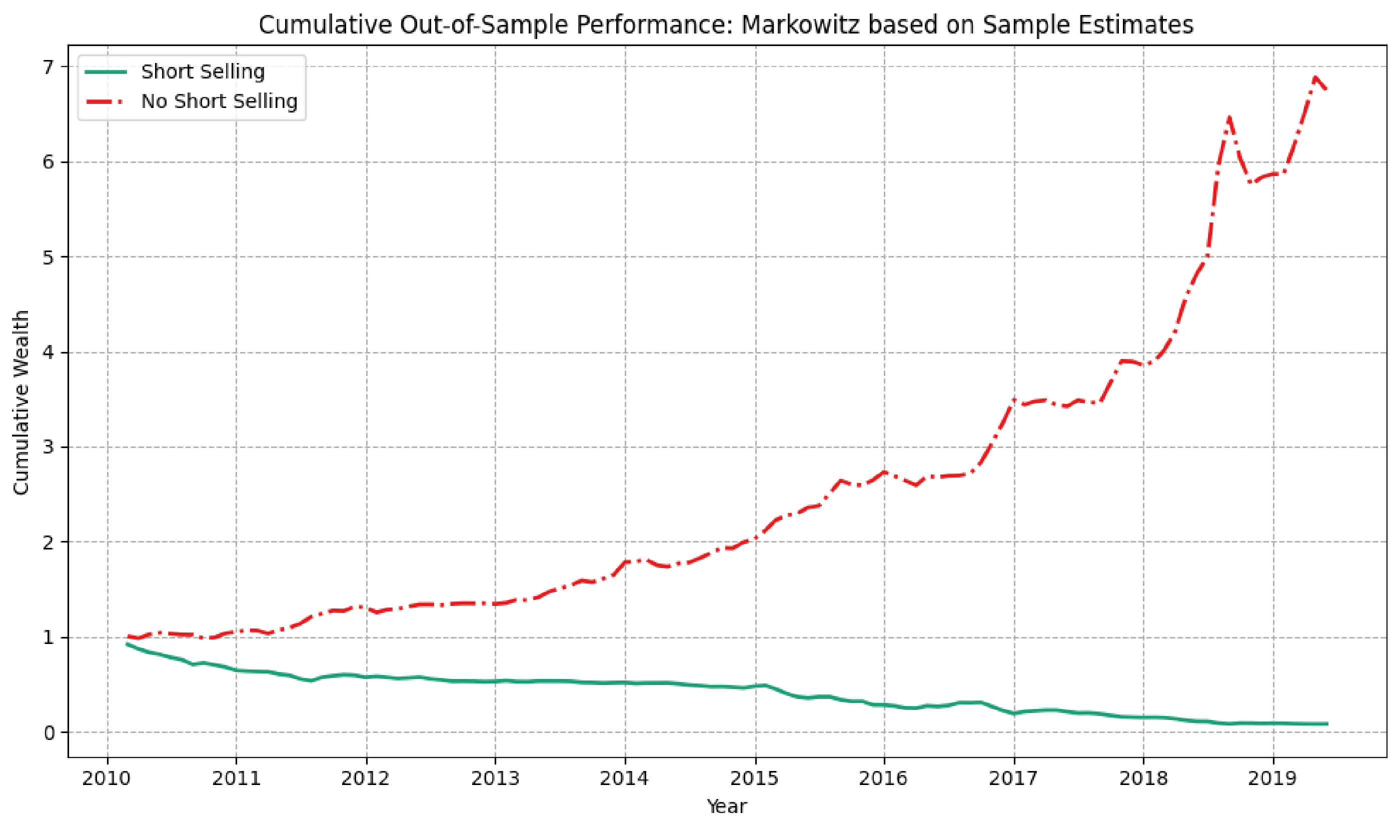


(c) *Sharpe Ratio - Short Selling*

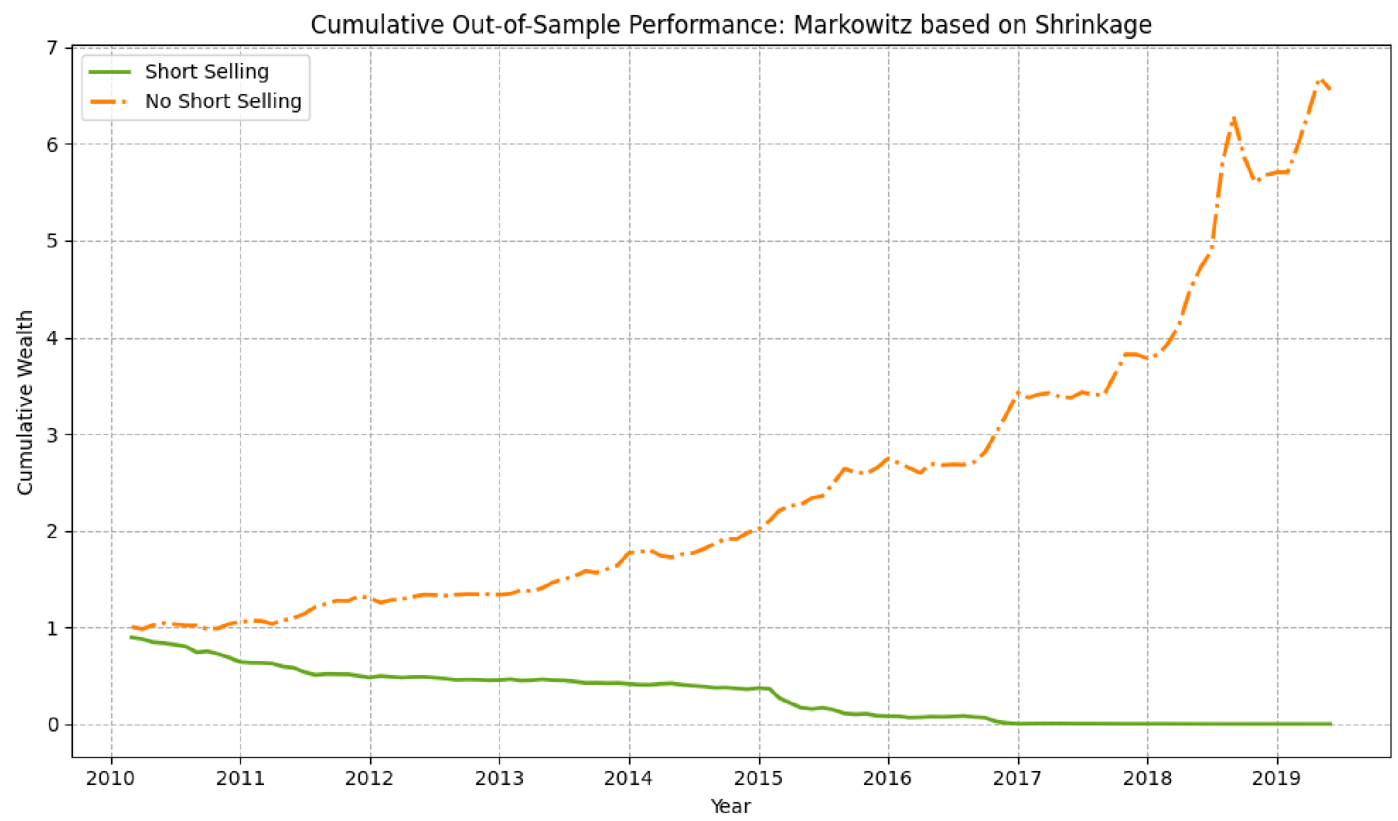


(d) *Sharpe Ratio - Without Short Selling*

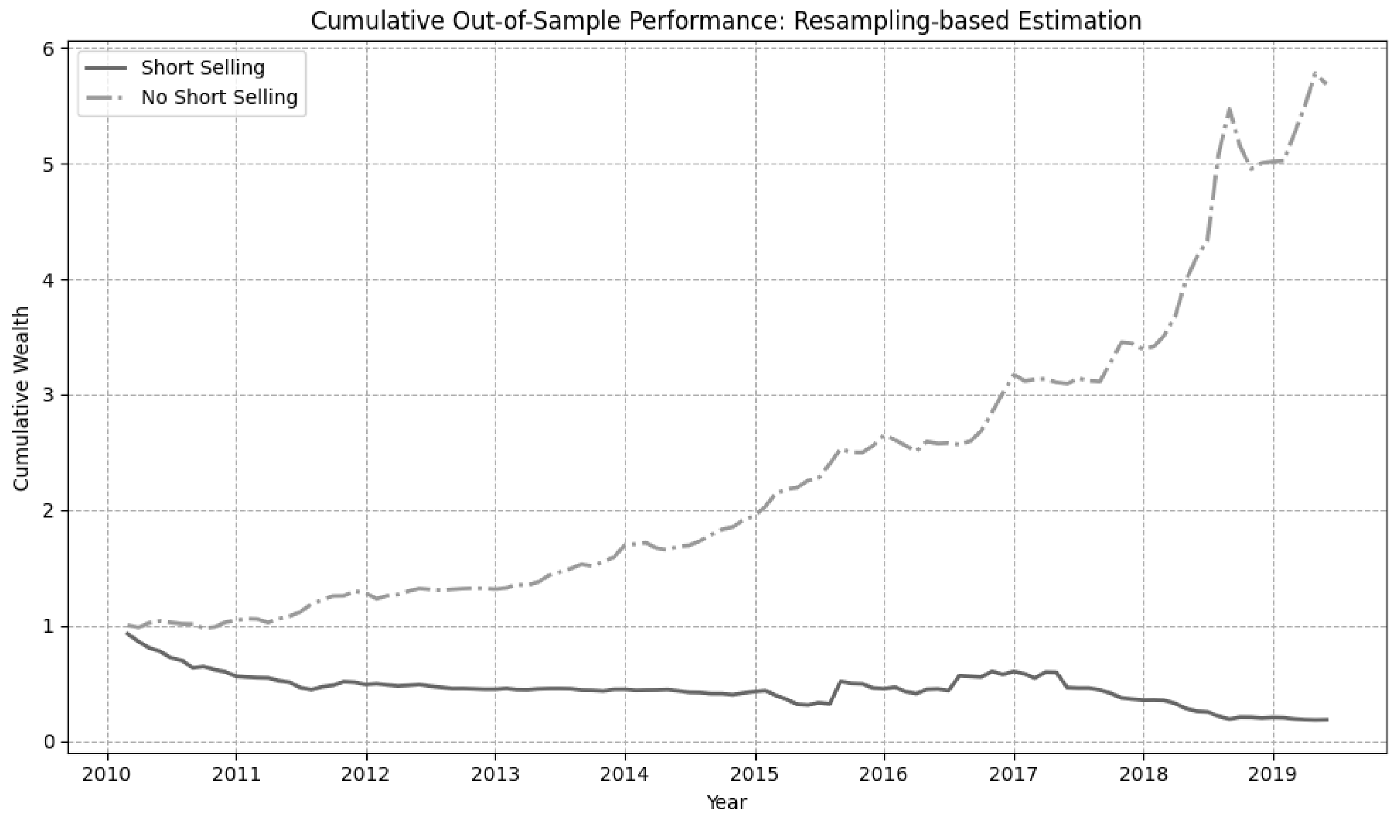
Figure 3.2: *Cumulative Wealth and Sharpe Ratio Results*



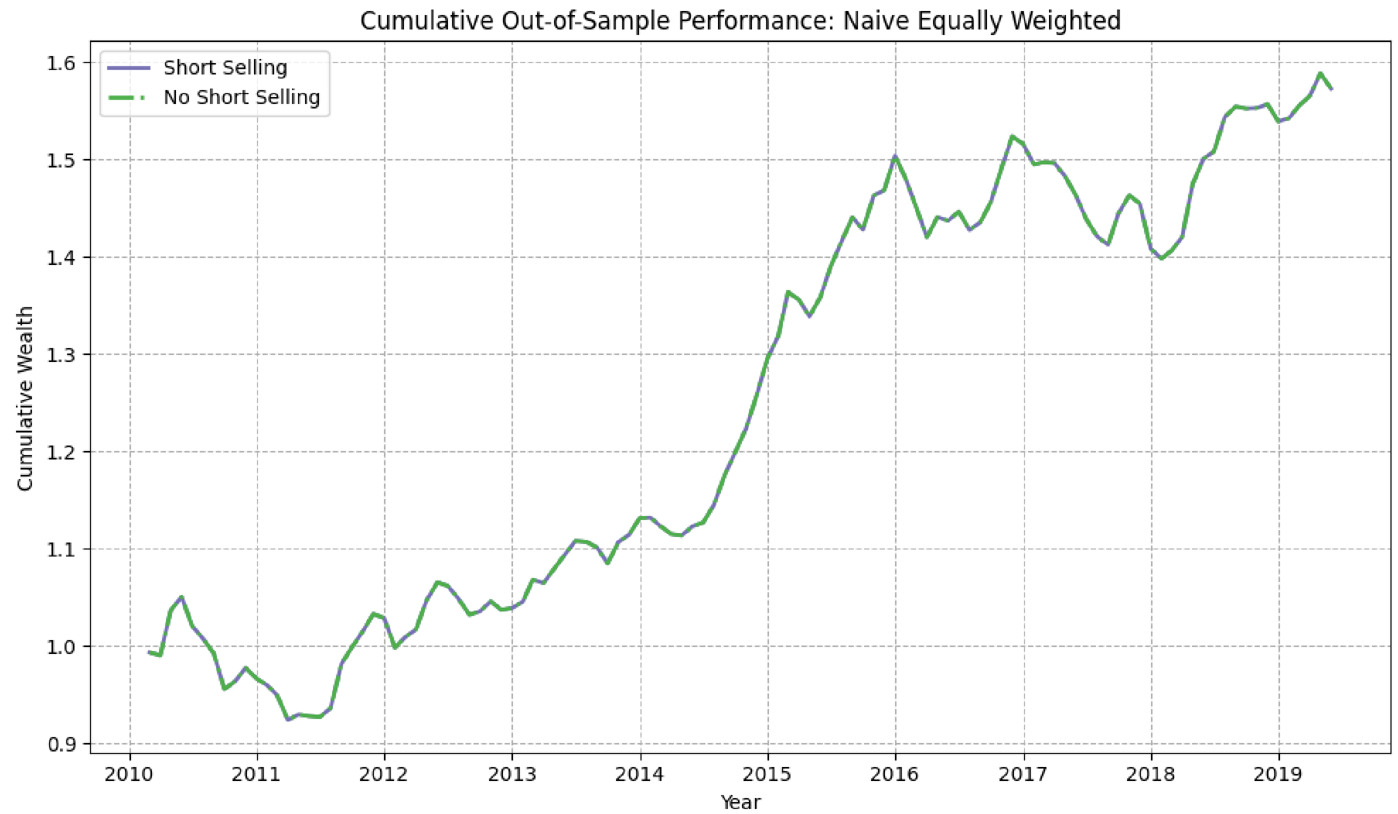
(a) *Markowitz based on sample estimates*



(b) *Markowitz based on the shrinkage estimator*



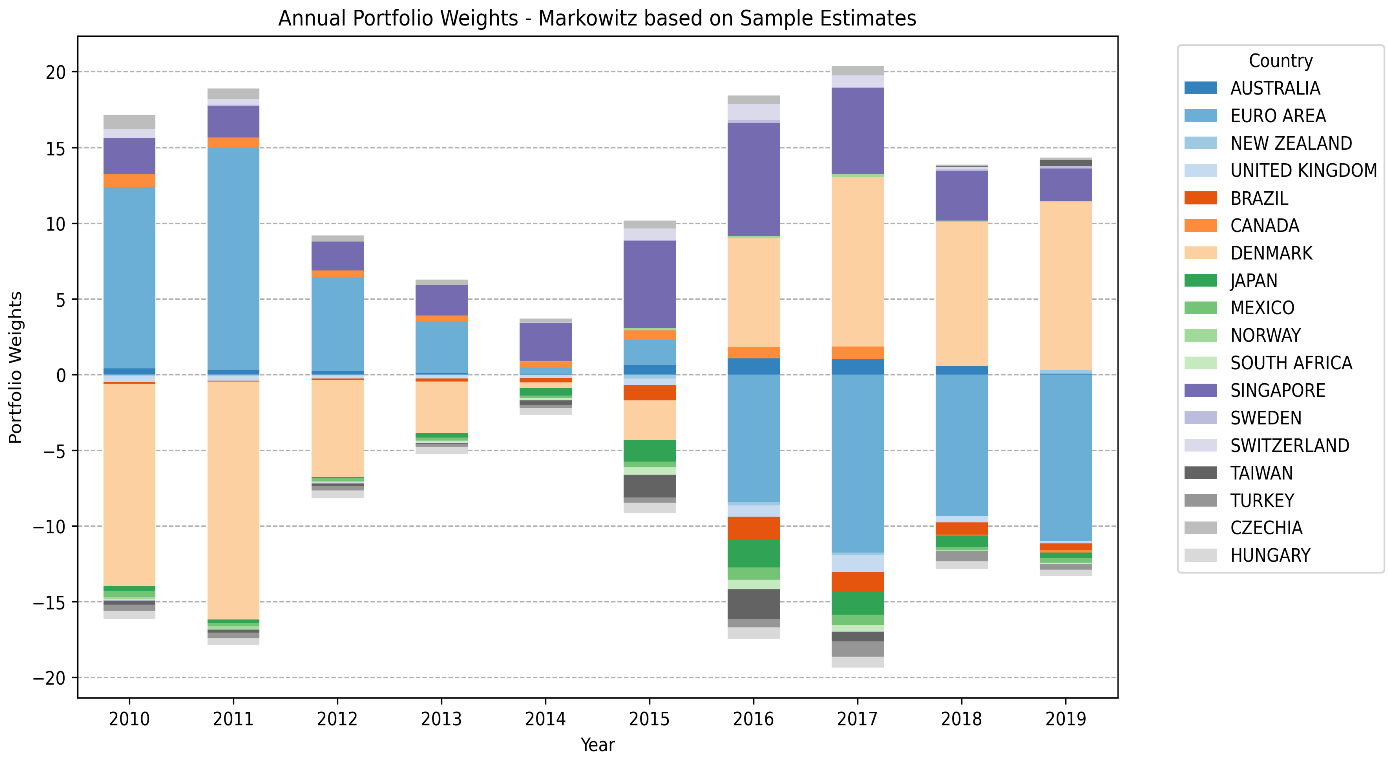
(c) *Resampling-based Estimation*



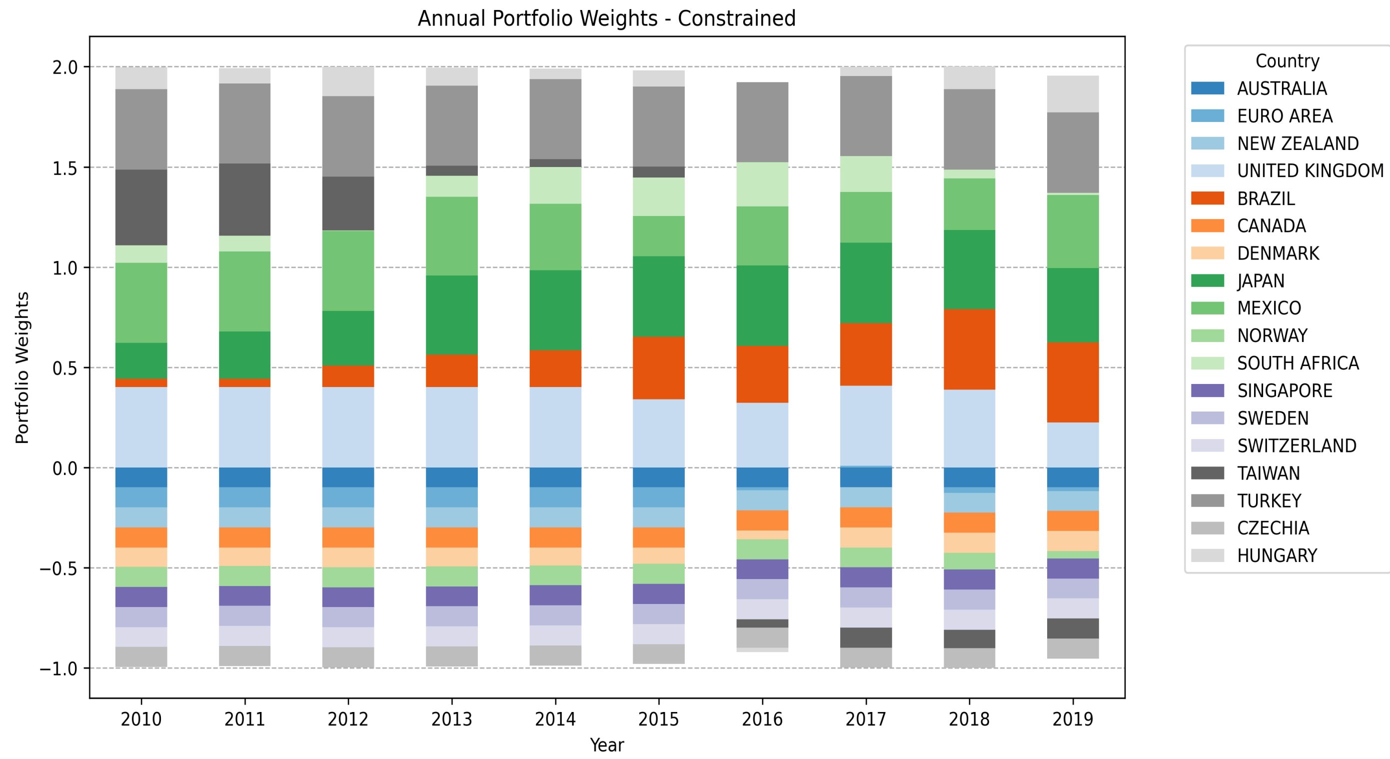
(d) *Constrained Portfolio*

Figure 3.3: *Cumulative Wealth Comparison between methods*

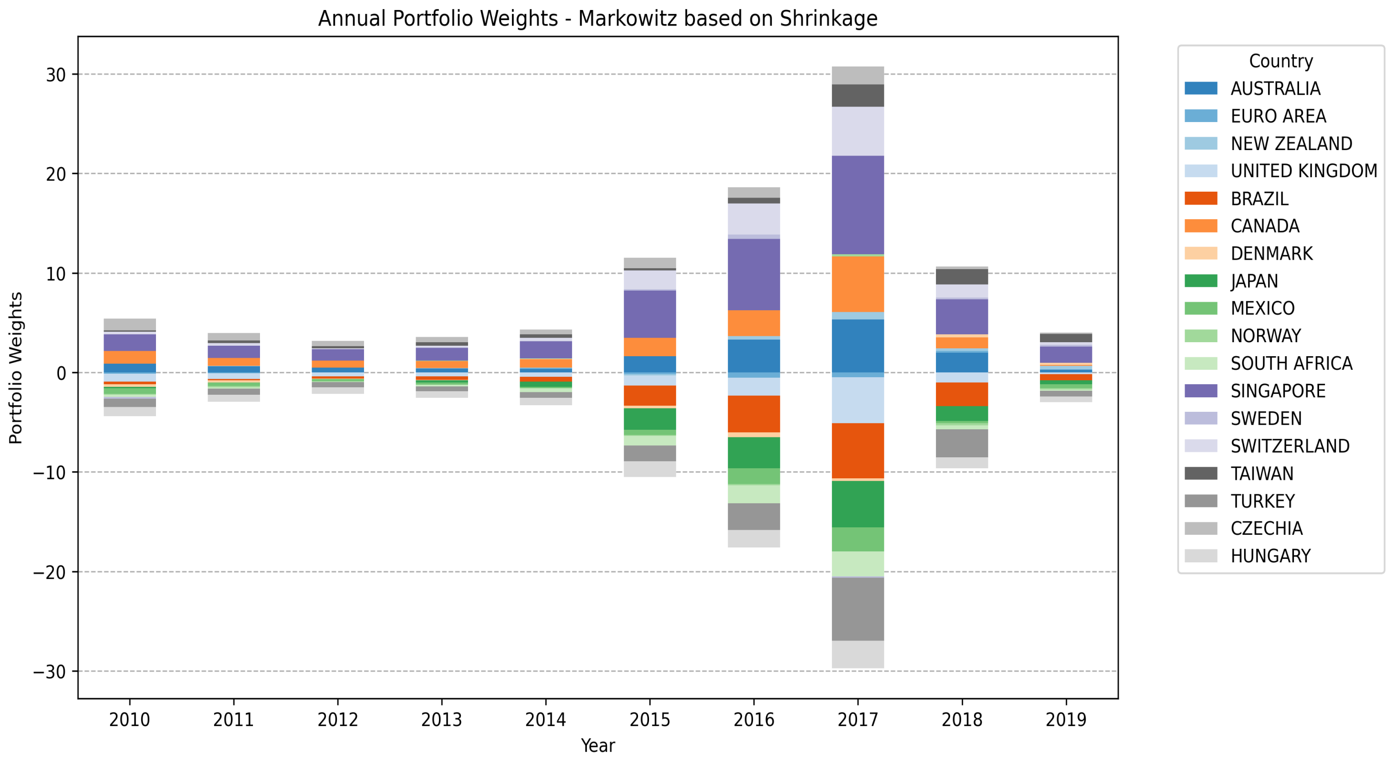
3.1.2 | Weight Distribution of Optimization methods



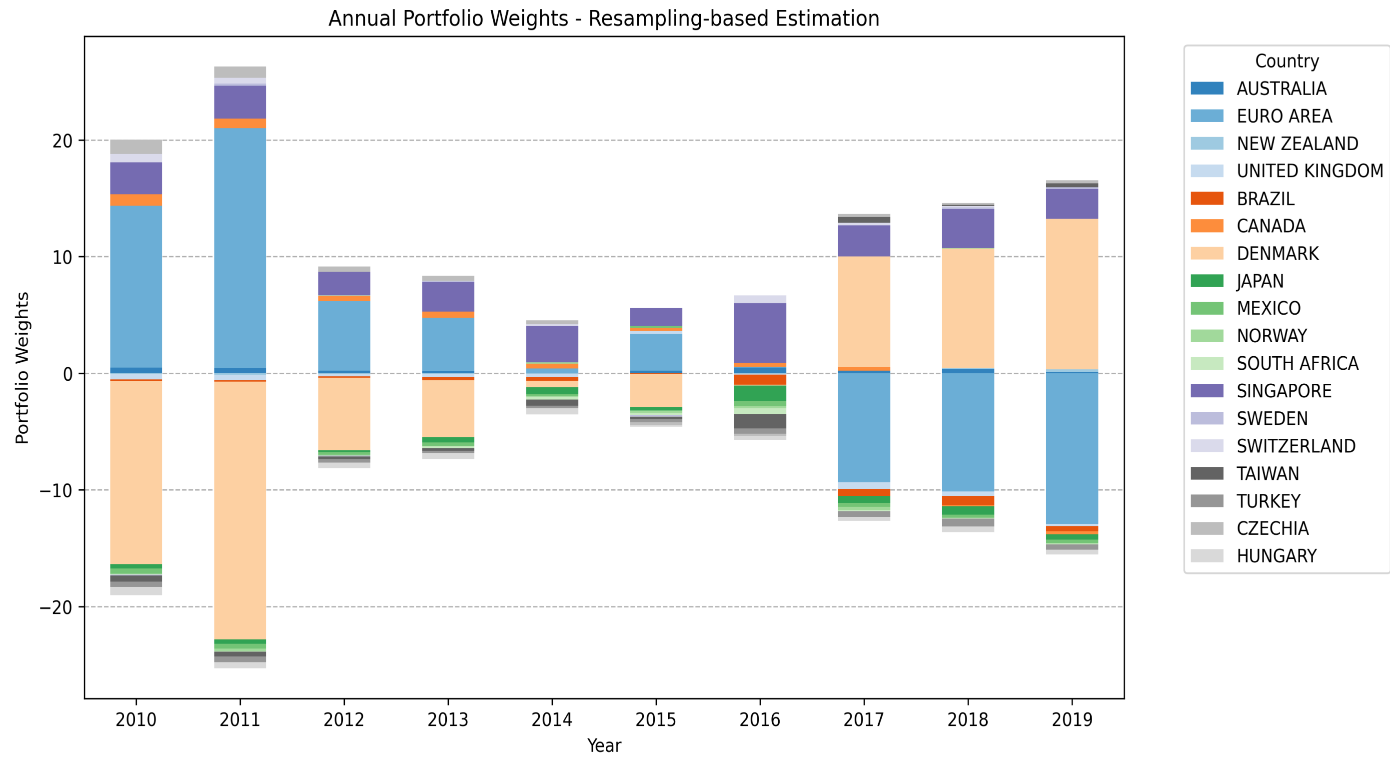
(a) *Markowitz based on sample estimates*



(b) *Constrained Portfolio*



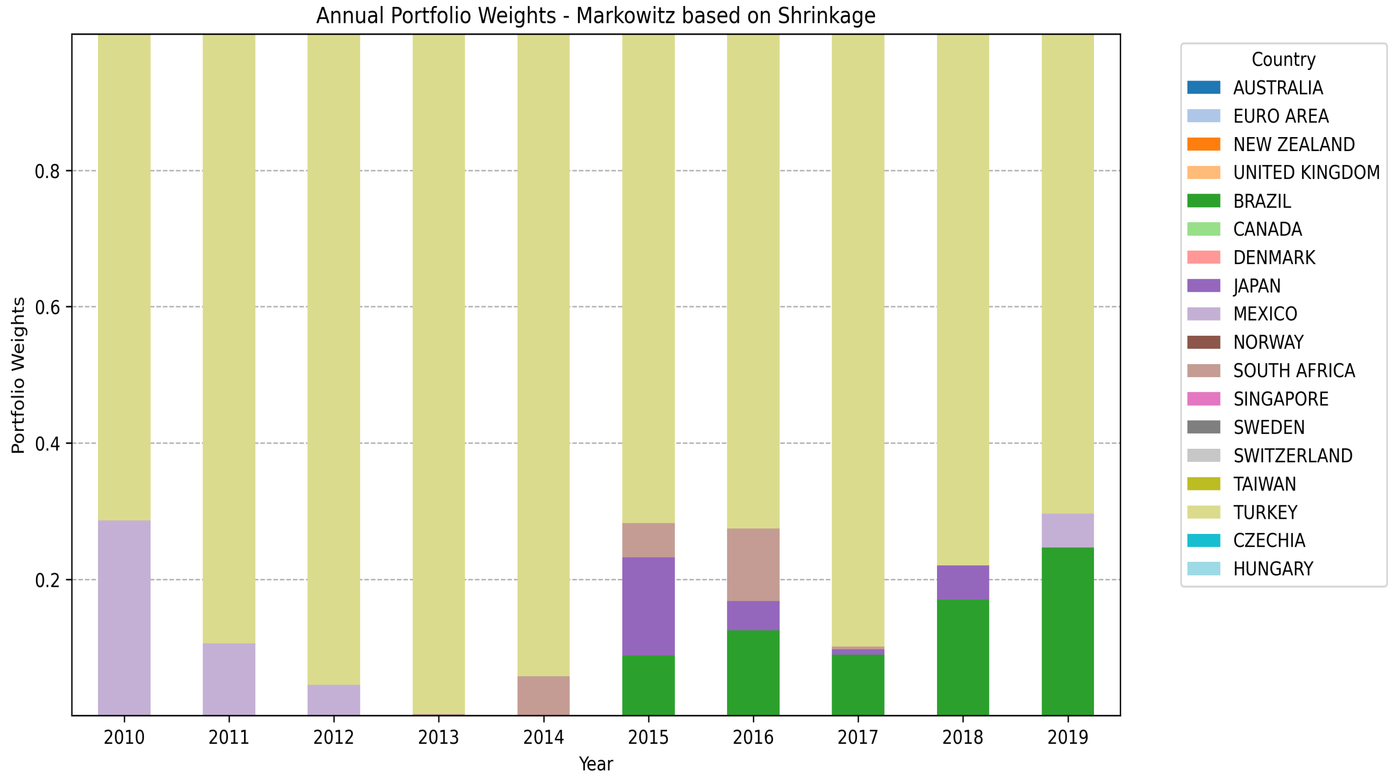
(c) *Markowitz based on the shrinkage estimator*



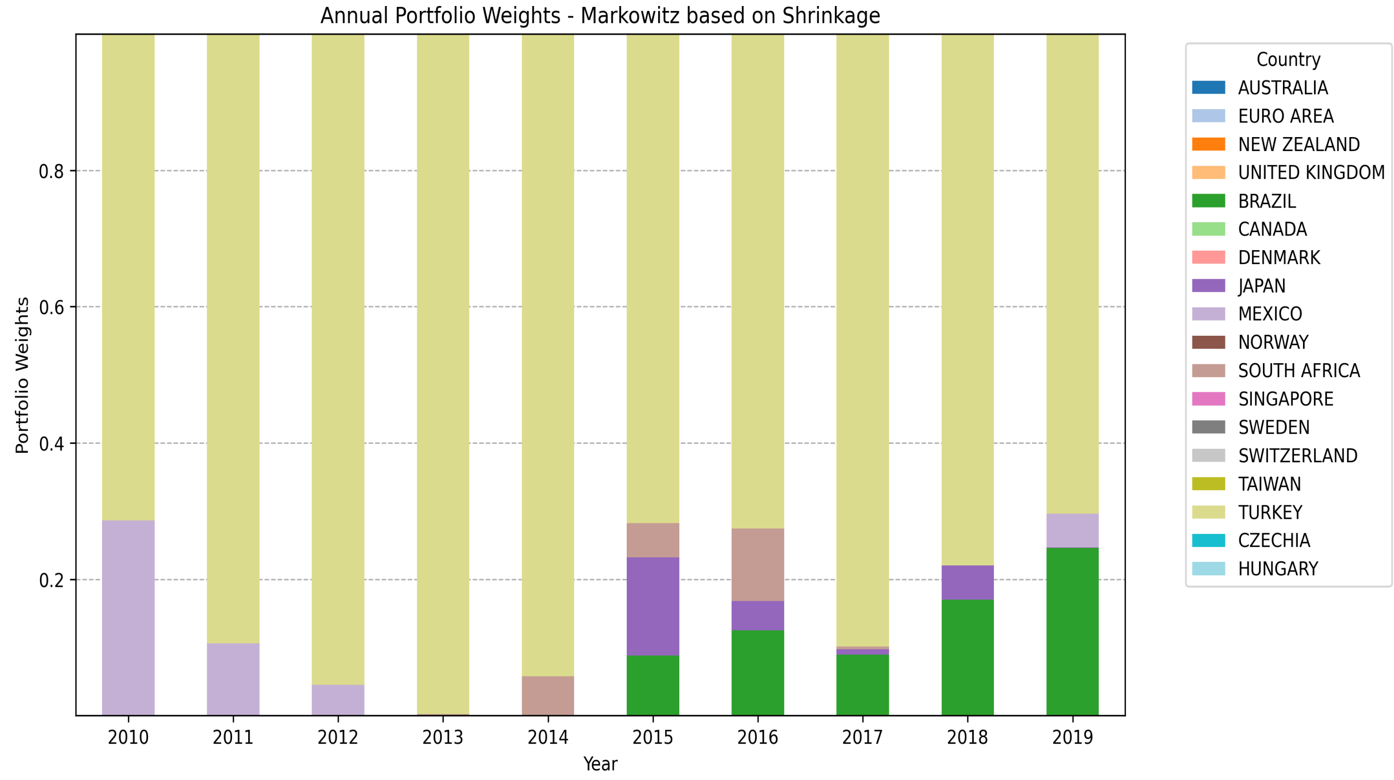
(d) *Resampling-based Estimation*

Figure 3.4: *Portfolio optimization Methods with Short Selling*

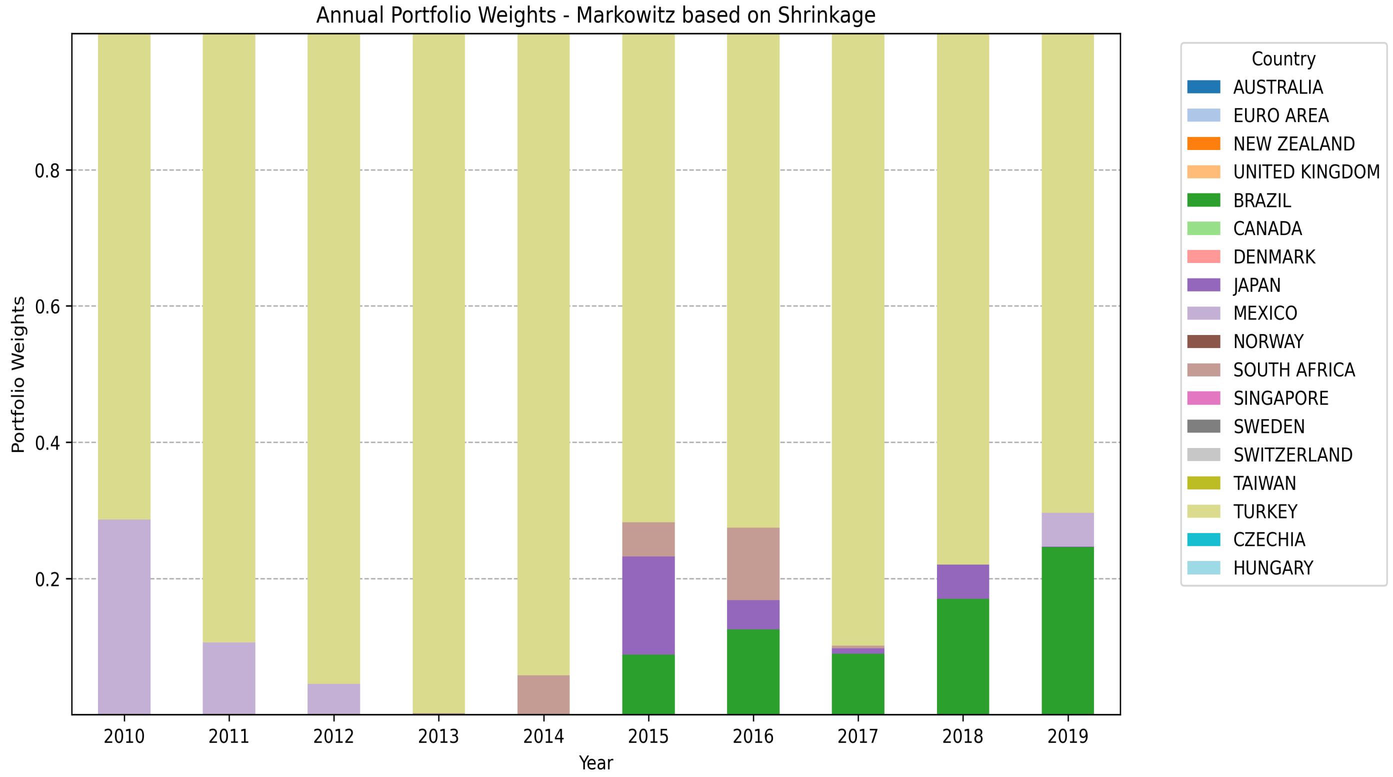
> ***Note***
>
> *The **Naive Equally weighted** portfolio optimization is same in both cases when short selling is considered and when it was not. The plot can be found in the next page (**??**)*



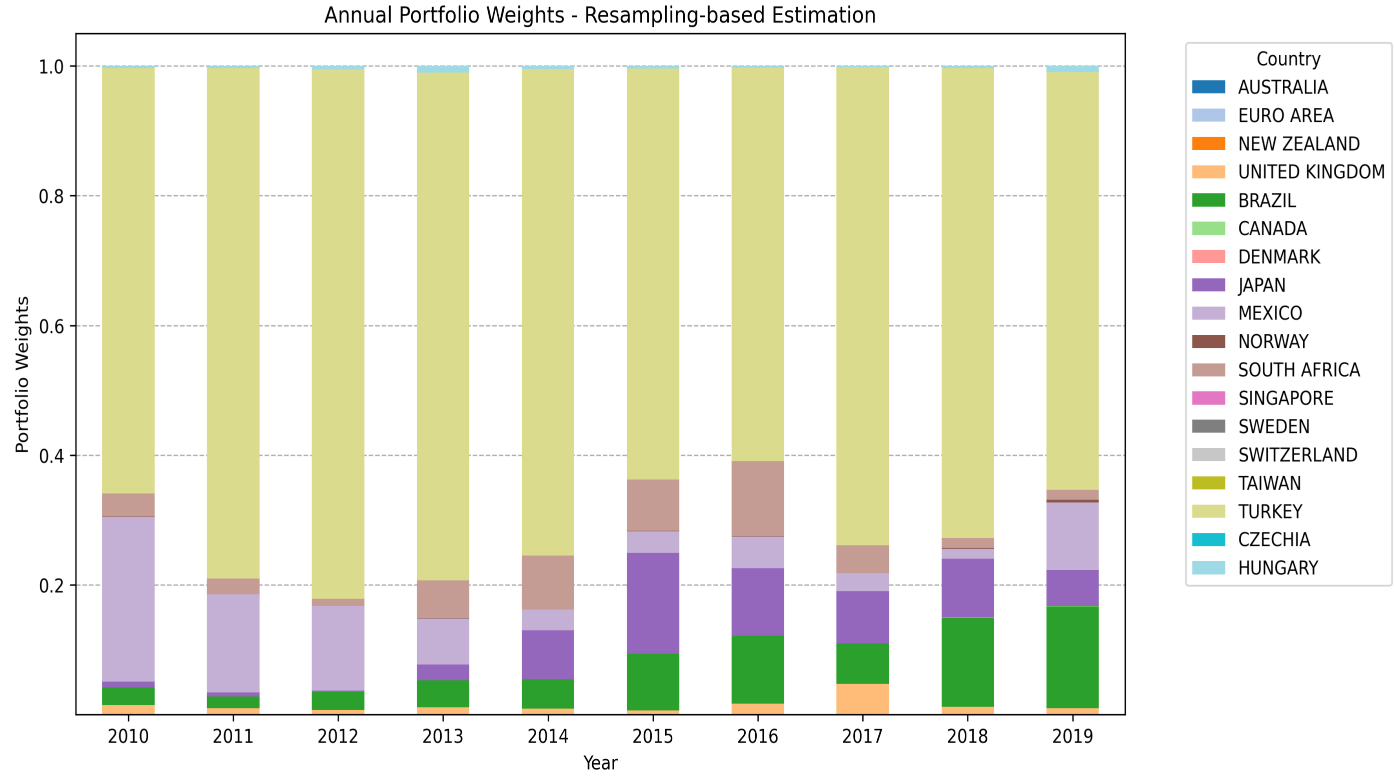
(a) *Markowitz based on sample estimates*



(b) *Markowitz based on sample estimates*



(c) *Markowitz based on the shrinkage estimator*



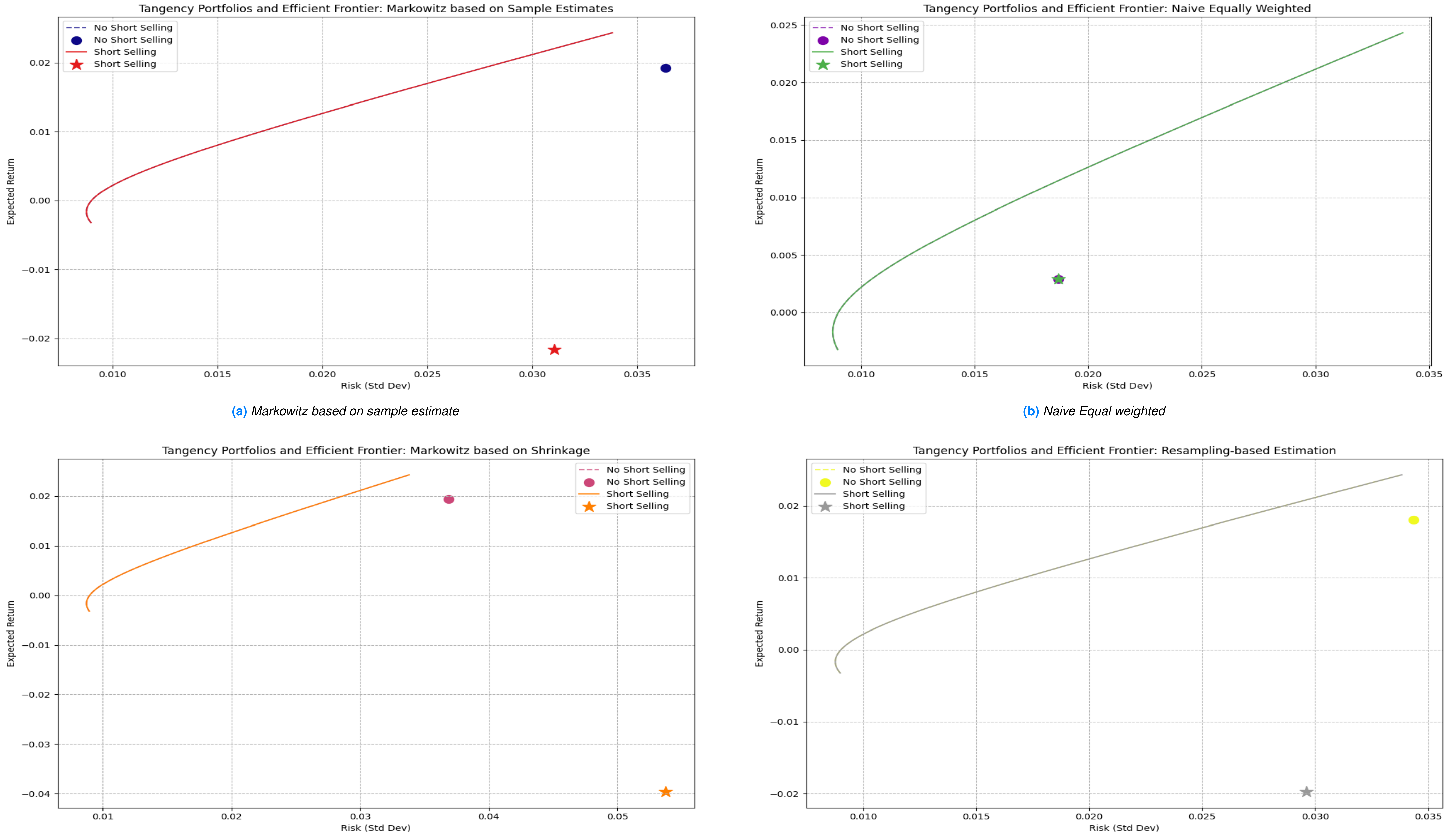
(d) *Resampling-based Estimation*

Figure 3.5: *Portfolio optimization Methods without Short Selling*

Note

*The **Constrained** portfolio optimization is same in both cases when short selling is considered and when it was not. The plot can be found in the previous page (3.4b)*

3.1.3 | Tangency Frontiers of Optimization methods



(a) *Markowitz based on sample estimate*

(b) *Naive Equal weighted*

(c) *Markowitz based on the shrinkage estimator*

(d) *Resampling-based Estimation*

Figure 3.6: *Portfolio Optimization - Frontiers*

4 | Inference

4.1 | Out-of-Sample Performance Analysis

From the above Annualized Out of Sample Sharpe Ratio, we can observe that,

Method	Observations
Markowitz (Sample Estimates)	Sharpe ratio drops from ∼1.86 to -1.46 because unconstrained short selling amplify estimation errors. The FX returns are being noisy makes the optimizer take extreme negative weights. The Sharpe ratio in similar to Shrinkage based estimation in case of without short selling, which indicates the sample estimates based estimation is stable in our case. The Unconstrained short selling increased volatility and reduces out-of-sample Sharpe. While the without short selling, sharpe ratio value indicates the model is to be strong in real markets as it has the value closest to ideal.
Markowitz (Shrinkage Estimator)	Shrinkage estimator reduces estimation error slightly but unconstrained shorts still create large negative weight which resulted in drops of Sharpe ration from ∼1.86 to -1.18 But the Shrinkage estimator helps which is evident from the slight increase in Sharpe ration in case of without short selling.
Resampling	Resampling reduces sensitivity to outliers but the Sharpe Ratio drops from 1.84 to -0.51 indicating shorting still allows extreme positions. It also mitigates some estimation error but negative Sharpe persists with shorting.
Naive Equal-Weighted	Sharpe ∼0.89 with or without short selling which is as expected as it does not depend on the information of data This method is robust to estimation error as weights are equal, so shorting does not create large negative exposure.
Constrained	Constraints limit leverage and extreme weights. So, a Sharpe ratio of ∼1.99 is achieved with or without shorting based on our chosen weight bounds

Table 4.1: *Impact of Short Selling on Out-of-Sample Sharpe Ratios*

Important Inference: As we can observe that the Sharpe ratio in case of Constrained portfolio and Resampling based method,is close to 2, which is an ideal situation in the financial markets. This was primarily due to the fact that we have used 120 months of rolling window size, which made the weights smooth. Rolling window choice is a trade-off, Shorter the window, more responsive to regime shifts, but noisy weights where as Longer window size, stable weights, but may lag in capturing structural changes. This trade off can be clearly observed in our case, but for the numerical stability we have considered 120 rolling window size.

> ***Note***
>
> *The Negative Sharpe ratios are due to the negative expected returns, which is primarily arised due to the fact that we are optimizing our portfolio based on the Sharpe Ratio. So, our portfolio construction optimized for the case where we lost the least amount of money when we are trying to optimize based on Sharpe ratio.*

4.2 | Cumulative Wealth Analysis:

The plots (3.2a,3.2b) demonstrates that without constraints (short selling allowed) the optimization methods are complete failure in generating wealth over time, which is the opposite of what an investor interested in.

Cumulative Wealth in Short Selling Scenario (3.2a)

- The cumulative wealth in case of Markowitz based on sample estimates, based on Shrinkage, and Resampling methods, all decay to zero. This is an example of **Error Maximization**. Without constraints, the optimization methods rely heavily on the noisy data , taking massive leverage that is eventually blown up.
- Markowitz based on Shrinkage estimator method should be performing better than the Markowitz based on Sample estimates, but in out due to the unconstrained scenario, the noisy data is more predominate in optimizing the covariance matrix. So, the cumulative wealth was not as we expected.
- The Markowitz methods are most unstable in practice when short shelling was allowed.
- The Constrained portfolio generated significant wealth of $\sim 4.5\times$ of the initial wealth, which the best optimization methods among others and it was followed by Naive Equally weighted method.

Cumulative Wealth in No Short Selling Scenario (3.2b)

- In case of no short selling, the Markowitz methods outperformed the other optimization methods in generating cumulative wealth, which is close to $\sim 7\times$ the initial wealth.
- Though the Markowitz performed well in this scenario, it is entirely due to the long positions where the estimation errors are diminished as the noisy data is filtered out based on the constraints.

4.3 | Sharpe Ratio Analysis

From the plots of Sharpe Ration in both the cases, with and without short selling, (3.2c)(3.2d) we can observe that the constrained portfolio maintained a positive risk-adjusted return due to the constraints. The Markowitz methods, and resampling based methods are below the zero in case of short selling because the fact that the optimization is done based on the maximizing the sharpe ratio, as there are no constraints the optimization methods gave us the scenario where we lost the least amount of money.

The movements in both the plots are attributed to the weight changes in the assets, which can be seen in the weights distribution plots of the assets (3.4)(3.5).

4.4 | Conclusion

Through this report, we have examined the stability of Mean-variance portfolio optimization under different techniques, and rolling window methodology. The Empirical results highlights the impact of the estimation errors and the noisy data on estimated returns and covariance estimator. Though the 60 window size is the standard choice for rolling window methodology, this has proved to be numerically unstable in our case and using a window size of 120 produced a stable results in non short selling scenario and gave us insights of the impact of over leveraging in unconstrained optimization.

Through our observations, we can understand the importation of the parameters, by careful attention to the parameters we can successfully optimize a portfolio even using a restrictive or conservative methods. Further study can be done based on a different risk measure as we have used variance in our study above.

5 | References

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