Unit I

Logic And Proof:

Propositions On Statement:

A propositions on statment is a declarative sentence which either true (or) false. not both.

Ex:

$1 > 2$ — False

$5 > 2$ — True.

The following are not propositions (or) not statement:

(a) who is your friend?

(b) what a beauty!

Connectives:

Connective is an operation which is used to connect two (or) more than two stats. Simply It is called Sentencal Connective.

Basic Connectives:

i) Not – $\neg$ (or) $\sim$

ii) And – $\wedge$

iii) Or – $\vee$

iv) If ... then – $\rightarrow$

v) Iff – $\rightleftarrows$

Truth tables for Basic Connectives:

Not:

The negation of a stat is generally form by introducing the word "NOT" ($\sim$) at the proper place.

P	$\neg P$
T	F
F	T

Ex:

P : She is student.

And:

The Conjunction of two statment P and Q is the stat denoted by $P \wedge Q$ and read as "P and Q"

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

The disjunction of two stat P and Q is denoted by $P \vee Q$ and read

Conditional Stat:

$(\rightarrow)$

Let P and Q are any two statments then the stats $P \rightarrow Q$ is called a Conditional stat and read as "if P then Q".

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional Stat:

$(\leftrightarrow)$

Let P and Q are any two stats Then the stat $P \leftrightarrow Q$ is called biconditional stat and read as "P iff Q".

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
T		F

Excercise:

① Construct the truth table for (1) $\sim(P \wedge Q)$
(2) $(\sim P) \vee (\sim Q)$

Soln

(i)

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(ii)

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

② Construct the truth table for
$\sim(P \vee (Q \wedge R))$

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$\sim(P \vee (Q \wedge R))$
T	T	T	T	T	F
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	F
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

Tautology:

A stat formula which is true always irrespective of the truth value of the individual variable is called Tautology.

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Contradiction:

A statment formula which is false always irrespective of the truth value of the individual variable is called "Contradiction".

Ex:

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

Fallacy:

In the result Column there exist only one false stat is called fallacy.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Contigency:

A stat formula which is either tautology nor Contradiction is called Contigency.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Exercise

(1) S.T. $(P \vee Q) \leftrightarrow (Q \vee P)$.

Soln

P	Q	$P \vee Q$	$Q \vee P$	$(P \vee Q) \leftrightarrow (Q \vee P)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

$\therefore$ all the entries are "T".

$\Rightarrow$ Gn stat is tautology.

(2) Check the following is tautology

$$((P \to Q) \to R) \vee \sim P$$

Q	R	P → Q	(P → Q) → R	~P	Result
T	T	T	T	F	T
T	F	T	F	F	F
F	T	F	T	F	T
F	F	F	T	F	T
T	T	T	T	F	T
T	F	T	F	F	F
F	T	T	T	F	T
F	F	T	F	F	F

∴ Not all the entries are 'T'

⇒ Given stat is not tautology.

Logical Equivalences:

1. Idempotent Law
   - $P \wedge P \Leftrightarrow P$, $P \vee P \Leftrightarrow P$

2. Associative Law
   - $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$
   - $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

3. Commutative law
   - $P \wedge Q \Leftrightarrow Q \wedge P$, $P \vee Q \Leftrightarrow Q \vee P$

4. Demorgan's Law
   - $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$
   - $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$

5. Distributive Law
   - $P \wedge (q \vee r) \Leftrightarrow (P \wedge q) \vee (P \wedge r)$
   - $P \vee (q \wedge r) \Leftrightarrow (P \vee q) \wedge (P \vee r)$

6. Complement Law
   - $P \wedge \neg P \Leftrightarrow F$, $P \vee \neg P \Leftrightarrow T$

7. Dominance Law
   - $P \vee T \Leftrightarrow T$
   - $P \wedge F \Leftrightarrow F$

8. Identity Law
   - $P \wedge T \Leftrightarrow P$
   - $P \vee F \Leftrightarrow P$

.

(9) Absorption Law –

$$p \vee (p \wedge q) \Leftrightarrow p$$
$$p \wedge (p \vee q) \Leftrightarrow p$$

(10) Double Negation :–

$$\neg(\neg p) \Leftrightarrow p$$

(11) Contrapositive Law –

$$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$$

(12) C.D. –

$$p \rightarrow q \Leftrightarrow \neg p \vee q.$$

(13) Biconditional as Conditional –

$$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

(14) Exportation Law –

$$p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$$

① Determine whether $(\neg Q \wedge (P \to Q)) \to \neg P$ is a tautology.

Soln

$(\neg Q \wedge (P \to Q)) \to \neg P$	Reason
$\neg(\neg Q \wedge (\neg P \vee Q)) \vee \neg P$	C.D.
$\Rightarrow (Q \vee (P \wedge \neg Q)) \vee \neg P$	Demorgans
$\Rightarrow (\neg P \vee Q) \vee (P \wedge \neg Q)$	Commutative
$\Rightarrow \neg(P \wedge \neg Q) \vee (P \wedge \neg Q)$	C.D.
$\Rightarrow T$	$\neg P \vee P \Leftrightarrow T$

$\therefore$ Gn stat formula is tautology.

(2) Show that

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$.

Soln

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$	Reason
$\Rightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$	Distributive
$\Rightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$	Associative
$\Rightarrow ((\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R$	Distri.
$\Rightarrow (\neg (P \vee Q) \vee (P \vee Q)) \wedge R$	Demorgan
$\Rightarrow T \wedge R$	$\neg P \vee P \Leftrightarrow T$
$\Rightarrow R$	$T \wedge P \Leftrightarrow P$.

Equialance

Two stat formulas A and B are equialent iff $A \leftrightarrow B$ (or) $A \rightleftarrows B$ is a tautology.

It is denoted by the Symbol $A \Leftrightarrow B$ which is read as A is equalent to B

① Show that $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$

Proof

$P \to (Q \to P)$	Reasons
$\Rightarrow \neg P \vee (Q \to P)$	C.P.L.
$\Rightarrow \neg P \vee (\neg Q \vee P)$	C.P.L.
$\Rightarrow (\neg P \vee \neg Q) \vee P$	A.L.
$\Rightarrow$ ~~$\neg(P \wedge Q) \vee P$~~ $(\neg P \vee P) \vee \neg Q$	A.L.
$\Rightarrow$ ~~$(P \wedge Q) \to P$~~ .	$P \vee \neg P \Leftrightarrow T$
$T \vee \neg Q$.	$T \vee Q \Leftrightarrow T$.
$\Rightarrow T$.	
and	
$\neg P \to (P \to Q)$	C.P.L.
$\neg(\neg P) \vee (P \to Q)$	C.P.L.
$P \vee (P \to Q)$	
$P \vee (\neg P \vee Q)$	C.P.L.
$(P \vee \neg P) \vee Q$	A.L.
$T \vee Q$	$P \vee \neg P \Leftrightarrow T$
T	$T \vee Q \Leftrightarrow T$.

(2) $\neg(p \vee (\neg p \wedge Q))$ and $\neg p \wedge \neg Q$ are logically equailent

P	Q	$\neg p$	$\neg Q$	$\neg p \wedge Q$	$p \vee (\neg p \wedge Q)$	$\neg(p \vee (\neg p \wedge Q))$	$\neg p \wedge \neg Q$
T	T	F	F	F	T	F	F
T	F	F	T	F	T	F	F
F	T	T	F	T	T	F	F
F	F	T	T	F	F	T	T

Tautological Implication:

A stat formula A is said to be tatologically imply a stat formula B iff $A \to B$ is tautology.

Ex: $(p \to (q \to r)) \Rightarrow (p \to q) \to (q \to r)$

p	q	r	$p \to q$	$q \to r$	$p \to (q \to r)$	$(p \to q) \to (q \to r)$	$(p \to (q \to r)) \to ((p \to q) \to (q \to r))$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

② $((P \to q) \to q) \Longrightarrow (P \vee q)$.

Proof:

To Prove the above stat, it is enough to P.T. $((P \to q) \to q) \to (P \vee q)$ is a tautology.

$((P \to q) \to q) \to (P \vee q)$	Reasons
$\Rightarrow ((\neg p \vee q) \to q) \to (P \vee q)$	Complement Law
$\Rightarrow (\neg(\neg p \vee q) \vee q) \to (P \vee q)$	C.L.
$\Rightarrow ((\neg p \vee q) \wedge \neg q) \vee (P \vee q)$	Demorgan Law.
$\Rightarrow (\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee (P \vee q)$	Distributive Law.
$\Rightarrow$ ~~$\neg p \wedge q$~~	
$\neg(P \vee q) \vee F \vee (P \vee q)$	$P \wedge \neg P \Leftrightarrow F$
$\Rightarrow \neg(P \vee q) \vee (P \vee q)$	$F \vee P \Leftrightarrow P$
$\Rightarrow T$	$P \vee \neg P \Leftrightarrow T$.

$\therefore$ The above stats are tautologically implies

Normal forms:

1) A Product of the stat variables and their negations in a formula is called Elementary Product.

2) A sum of two stat variables and their negation is called Elementary Sum.

DNF

A stat formula which is equivalent to given formula and which consists of a sum of elementary products is called DNF of the given formula.

$$DNF = (EP) \vee (EP) \vee \cdots \vee (EP)$$

CNF

$$CNF = (ES) \wedge (ES) \wedge \cdots \wedge (ES)$$

(1) Find the DNF of the following

$$(P \wedge \neg(Q \vee R)) \vee (((P \wedge Q) \vee \neg R) \wedge P)$$

$(P \wedge \neg(Q \vee R)) \vee (((P \wedge Q) \vee \neg R) \wedge P)$	Reasons.
$\Rightarrow (P \wedge (\neg Q \wedge \neg R)) \vee ((P \wedge Q) \wedge P) \vee (\neg R \wedge P)$	Demorgan.
$\Rightarrow (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q) \vee (\neg P \wedge P)$	Distributive

PCNF

For a given stat formula an equivalent formula consisting of conjunction of max terms only is known as PCNF.

$$PCNF = (\text{Max Term}) \wedge (\text{Max Term}) \wedge \ldots \wedge (\text{Max Term})$$

PDNF

$$PDNF = (\text{Min Term}) \vee (\text{Min Term}) \vee \ldots \vee (\text{Min Term})$$

Working Rules:

PCNF	PDNF
1. By applying various equivalence rules, simplify gn stat formula, if possible	—— Do ——
2. Apply the fact "$\wedge T$"	2) Apply the fact "$\vee F$"
3. Instead of T apply $(P \vee \neg P)$	3) Instead of F apply "$P \wedge \neg P$".
4. Apply Distributive law	4) —— Do ——
5. Apply Commutative law.	5) —— Do ——

(1) obtain PDNF and PCNF of the following.

$S : (\neg P \rightarrow R) \wedge (Q \rightleftharpoons P)$.

$(\neg P \rightarrow R) \wedge (Q \rightleftharpoons P)$	Reason.
$\Rightarrow (P \vee R) \wedge ((Q \rightarrow P) \wedge (P \rightarrow Q))$	C.D.
$\Rightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$	Biconditional.
$\Rightarrow ((P \vee R) \vee F) \wedge ((\neg Q \vee P) \vee F) \wedge ((\neg P \vee Q) \vee F)$	C.D.
$\Rightarrow [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [(\neg Q \vee P) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$	Rule (2)
$\Rightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R) \wedge (\neg P \vee Q \vee R) \vee (\neg P \vee Q \vee \neg R)$	Rule (3)
$\Rightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \vee (\neg P \vee Q \vee \neg R)$	Distributive.
$\Rightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \vee (\neg P \vee Q \vee \neg R)$.	Associative

$\therefore$ The PCNF of given formula stat is

$S : (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$.

$\Rightarrow \quad \neg S : (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

$\Rightarrow \quad \neg(\neg S) = S : \neg\left[(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)\right]$

$S : (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

$\therefore$ PDNF is $(\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

Unit: II

Combinatorics

Stat of the Principle of Math. induction

Step: 1

We must prove that $P(1)$ is true.

Step: 2

By assuming $P(k)$ is true, we must prove that $P(k+1)$ is also true.

Ex:

Using mathematical induction. P.T.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Soln

Let $P(n) = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

(i) Assume

$$P(1) = 1^2 = \frac{1(1+1)(2\cdot1+1)}{6}$$

$$= \frac{1\times2\times3}{6} = 1$$

$P(1)$ is true

(ii) Assume

$= \frac{\quad}{6}$ is true.

where k is any integer.

(iii) Claim: $P(k+1)$ is true.

$$P(k+1) = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$= \frac{(k)(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k^2+2k+1)}{6}$$

$$= \frac{(k+1)(2k^2+k+6k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

$\therefore$ $P(k+1)$ is true.

Then $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all n.

(3)

Ex:

P.T. $8^n - 3^n$ is a multiple of 5.

Soln

$P(n) = 8^n - 3^n$ be a multiple of 5.

(i) $P(1) = 8 - 3 = 5$ is a multiple of 5.

$P(1)$ is true.

(ii) Assume $P(k) = 8^k - 3^k$ is a multiple of 5

ie) $8^k - 3^k = 5m$, where $m \in Z^+$

$\Rightarrow 8^k = 5m + 3^k$

(iii) Claim

$P(k+1)$ is true.

$$P(k+1) = 8^{k+1} - 3^{k+1}$$
$$= 8^k \cdot 8 - 3^k \cdot 3$$
$$= (5m + 3^k) \cdot 8 - 3^k \cdot 3$$
$$= 5 \cdot 8m + 8 \cdot 3^k - 3^k \cdot 3$$
$$= 5 \cdot 8^m + 3^k (8 - 3)$$
$$= 5 \cdot 8^m + 5 \cdot 3^k$$
$$= 5(8^m + 3^k)$$

which is a multiple of 5 for all m.

$\therefore P(k+1)$ is true

Ex:

S.T. $3^{2n}+4^{n+1}$ is divisible by 5, $n \geq 0$

Proof:

$$P(n) = 3^{2n}+4^{n+1}$$

(i) Assume

$$P(0) = 3^0+4^{0+1}$$
$$= 1+4$$
$$= 5$$

$P(0)$ is True.

(ii) Assume $P(k) = 3^{2k}+4^{k+1}$ is divisible by 5

(ie) $3^{2k}+4^{k+1} = 5l$

$$3^{2k} = (5l-4^{k+1})$$

(iii) $P(k+1) = 3^{2(k+1)}+4^{k+2}$

$$= 3^{2k}.3^2+4^{k+2}$$
$$= (5l-4^{k+1})\,3^2+4^{k+2}$$
$$= 5l.9 - 4^{k+1}.9 + 4^{k+1}.4$$
$$= 5l.9 - 4^{k+1}(9-4)$$
$$= 5(9l) - 5(4^{k+1})$$
$$= 5(9l-4^{k+1})$$
$$= \text{multiple of } 5$$

Strong Induction :

Pigeon Hole Principle :

If (n+1) pigeon occupies n holes then atleast one hole has more than 1 pigeon.

Proof Asume (n+1) pigeon occupies n holes

Claim Atleast one hole has more than one pigeon

Suppose not i.e) Atleast one hole has not more than one pigeon

∴ Each and every hole has exactly one pigeon.

Since there are n holes ⇒ we've totally n pigeon,

Which is a ⇒⇐ to our assumption that there are (n+1) pigeon.

∴ Atleast one hole has more than 1 pigeon.

Ex:

S.T. among 100 people, atleast 9 of them were born in the same month.

Soln:

Here no: of pigeon = m = 100

∴ Holes = n = 12

Then by generalised pigeon hole principle,

$$\left(\frac{100-1}{12}\right)+1 = 9$$

Ex:

S.T. if 7 colours are used in paint 50 bicycles, atleast 8 bicycles will be the same colour.

Soln:

No: of pigeon = 50

∴ Holes = 7

Then by generalised pigeon hole principle,

$$\frac{(50-1)}{7}+1 = 8.$$

Ex:

S.T. if any 11 no:s from 1-20 are chosen, then two of them will add upto 21.

Soln

Construct the following sets with two numbers that add upto 21.

$A_1 = \{1, 20\}$

$A_2 = \{2, 19\}$

$A_3 = \{3, 18\}$

$A_4 = \{4, 17\}$

$A_5 = \{5, 16\}$

$A_6 = \{6, 15\}$

$A_7 = \{7, 14\}$

$A_8 = \{8, 13\}$

$A_9 = \{9, 12\}$

$A_{10} = \{10, 11\}$

By P.H.P. if we select 11 no:s from the above 10 sets, we must have

Permutations & Combination

Permutation:

Each of the diff arrangements which can be made by taking some or all of a no. of things at a time is called permutation.

$$nP_r = n(n-1)(n-2)\ldots(n-r+1)$$

Ex:

If $12P_r = 1320$, find r.

Soln:

$$12P_r = 1320$$

$$(12 \times 11 \times 10 \times \ldots r \text{ factors}) = 1320,$$

i.e) $12 \times 11 \times 10 = 1320$

$\therefore r = 3$.

$$nP_r = \frac{n!}{(n-r)!}$$

Proof:

$$nP_r = n(n-1)(n-2)\ldots(n-r+1)$$

9

$$= n(n-1)\ldots(n-r+1) \times \frac{(n-r)!}{(n-r)!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r-1)\ldots\times 3\times 2\times 1}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

$$\therefore \boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

Ex:

$${}^nP_r = (n-r+1) \times {}^nP_{r-1}$$

Soln

WKT, $${}^nP_r = \frac{n!}{(n-r)!}$$

$${}^nP_{r-1} = \frac{n!}{(n-(r-1))!}$$

$$= \frac{n\times(n-1)!}{(n-r+1)(n-r)!}$$

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$= (n-r+1) \times \frac{n!}{(n-r+1)(n-r)!}$$

Ex:

In how many of the permutations of 10 things taken 4 at a time will (a) one thing always occur

(b) never occur.

Soln

(a) we can keep aside the particular thing which will always occur, the no: of permutations of 9 things taken 3 at a time is $^9P_3$.

Now this particular thing can take up any one of the 4 occur places and so can be arranged in 4 ways.

Hence the total no: of permutations

$$= {}^9P_3 \times 4$$

$$= 9 \times 8 \times 7 \times 4$$

$$= 2016.$$

(b) If we are keeping the particular thing aside which has never to occur, the no: of 9 things $(10-1=9)$

taken 4 at a time is

$$9P_4 = 9 \times 8 \times 7 \times 6$$

$$= 3024.$$

Combinations

$$nC_r = \frac{n!}{(n-r)!\,r!}$$

$$nC_r = nC_{n-r}.$$

Ex: How many ways are there to form a Committee, if the committee Consists of 3 educationalist and 4 socialist, if there are 9 educationalist and 11 socialist.

Soln The 3 educationalist can be Chosen from 9 education in $9C_3$.

The 4 socialist can be Chosen from 11 ~~educ~~ Socialist in $11C_4$

the no: of ways to select the Committee $\Big\} = {}^9C_3 \times {}^{11}C_4$

$$= \frac{9!}{3! \times 6!} \times \frac{11!}{4!\,7!}$$

$$= 84 \cdot 330$$

$$= 27720.$$

Ex:

If ${}^nC_5 = 20\,{}^nC_4$ find "n"

Soln

$${}^nC_5 = 20\,{}^nC_4$$

$$\Rightarrow \frac{\cancel{n}(\cancel{n-1})(\cancel{n-2})(\cancel{n-3})(n-4)}{5!} = 20 \times \frac{\cancel{n} \times (\cancel{n-1})(\cancel{n-2})(\cancel{n-3})}{4!}$$

$$n-4 = \frac{20 \times 5 \times \cancel{4 \times 3 \times 2}}{\cancel{4 \times 3 \times 2}}$$

$$= 100$$

$$\boxed{n = 104}$$

Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Ex:

If the Seq $a_n = 3 \cdot 2^n$, $n \geq 1$ then find the Corresponding reccurance relation.

Soln:

For $n \geq 1$

$$a_n = 3 \cdot 2^n.$$

$$a_{n-1} = 3 \cdot 2^{n-1}$$

$$= 3 \cdot \frac{2^n}{2}$$

$$a_{n-1} = \frac{a_n}{2}$$

$a_n = 2(a_{n-1})$, for $n \geq 1$ with $a_0 = 3$

Ex:

Solve the recurrence relation defined by $S_0 = 100$ and $S_k = (1.08) S_{k-1}$ for $k \geq 1$

Soln:

Gn $S_0 = 100$

$S_k = (1.08) S_{k-1}$, $k \geq 1$

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$$16(\alpha_1 + 2\alpha_2) = 16$$

$$\alpha_1 + 2\alpha_2 = 1 \longrightarrow (1)$$

$$S_3 = 80.$$

$$\Rightarrow (\alpha_1 + 3\alpha_2)\, 4^3 = 80$$

$$\Rightarrow 64(\alpha_1 + 3\alpha_2) = 80.$$

$$\Rightarrow \alpha_1 + 3\alpha_2 = \frac{80}{64} = \frac{5}{4}$$

$$\alpha_1 + 3\alpha_2 = \frac{5}{4} \longrightarrow (2)$$

$$(2) - (1) \Rightarrow \boxed{\alpha_2 = \frac{1}{4}}$$

$$\Rightarrow \alpha_1 + 2\left(\tfrac{1}{4}\right) = 1$$

$$\alpha_1 + \tfrac{1}{2} = 1$$

$$\boxed{\alpha_1 = \tfrac{1}{2}}$$

$$S(k) = (\alpha_1 + \alpha_2 k)\, 4^k.$$

$$= (2 + k)\, 4^{k-1}$$

which is the reqd soln

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Linear Non Homogeneous Recurrance Relation with Constant Coeff:

Find all the soln of the recurrence relation:

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$

Soln

Given non homogeneous eqn can be written as

$$a_n - 5a_{n-1} + 6a_{n-2} - 7^n = 0$$

Now its associated homogeo eqn is

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

Its Char. eqn is

$$r^2 - 5r + 6 = 0$$

$$r = 2, 3.$$

$$a_n = \alpha_1 (2)^n + \alpha_2 (3)^n.$$

P. Solution

Since $F(n) = 7^n$. Then the

(12)

$\therefore a_n^{(h)} = 5a_{n-1} + 6a_{n-2} + 7^n$ becomes

$$c\,7^n = 5c\,7^{n-1} - 6c\,7^{n-2} + 7^n.$$

Divide both sides of eqn by 7^{n-2}

$$c \cdot 7^2 = 5c \cdot 7 - 6c + 7^2$$

$$49c = 35c - 6c + 49$$

$$20c = 49$$

$$\boxed{c = \frac{49}{20}}$$

$$\therefore a_n^{(p)} = (49/20)\,7^n$$

General soln

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\therefore a_n = \alpha_1 (2)^n + \alpha_2 (3)^n + \left(\frac{49}{20}\right) 7^n.$$

Generating Function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Ex:

Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ given that $a_0 = 10$, $a_1 = 41$, using generating function.

Soln:

The given recurrence relation is

$$a_n + 7a_{n-1} + 10a_{n-2} = 0$$

Multiplying the above eqn by x^n and summing 2 to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7x\sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 10x^2\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$(G(x) - a_0 - a_1 x) - 7x(G(x) - a_0) + 10x^2 (G(x)) = 0$$

$$G(x)[10x^2 - 7x + 1] + 29x - 10 = 0 ,$$

(19)

$$G(x) = \frac{A}{1-2x} + \frac{B}{1-5x}$$

$$= A(1 + 2x + (2x)^2 + \cdots)$$

$$+ B(1 + 5x + (5x)^2 + \cdots)$$

$$G(x) = A \sum_{n=2}^{\infty} 2^n x^n + B \sum_{n=2}^{\infty} 5^n x^n .$$

a_n = Coeff of x^n in $G(x)$

$$a_n = A2^n + B5^n .$$

Given $a_0 = 10$, $a_1 = 41$

$$10 = A + B$$

$$\Rightarrow A + B = 10 \Rightarrow 2A + 2B = 20 \longrightarrow (1)$$

$$41 = A(2) + B(5)$$

$$\Rightarrow 2A + 5B = 41 \longrightarrow (2)$$

$$(2) - (1) \Rightarrow 3B = 21$$

$$\boxed{B = 7}$$

$$\Rightarrow A + 7 = 10$$

$$\Rightarrow \boxed{A = 3}$$

The Principle of inclusion & Exclusion:

Find the no: of integers between 1 to 250 that are not divisible by any of the integers 2, 3, 5 and 7.

Soln

Let A denote the integer from 1 to 250 that are divisible by 2.

B — integer from 1 to 250 divisible by 3

C — 5

D — 7

$$|A| = \left[\frac{250}{2}\right] = 125$$

$$|B| = \left[\frac{250}{3}\right] = 83$$

$$|C| = \left[\frac{250}{5}\right] = 50$$

$$|D| = \left[\frac{250}{7}\right] = 35.$$

The no: of integer between 1-250 that $\Big\} = |A \cap B| = \left[\frac{250}{2\times 3}\right] = 41$

(21)

The number of integers divisible by 2 & 5 $\Big\} = |A \cap C| = \left[\frac{250}{2\times5}\right]$

$= 25$.

III^{ly} $|A \cap D| = \left[\frac{250}{2\times7}\right] = 17$

$|B \cap C| = \left[\frac{250}{3\times5}\right] = 16$

$|B \cap D| = \left[\frac{250}{3\times7}\right] = 11$

$|C \cap D| = \left[\frac{250}{5\times7}\right] = 7$

The no: of integer divisible by 2,3,5 $\Big\} = |A \cap B \cap C|$

$= \left[\frac{250}{2\times3\times5}\right] = 8$

$|A \cap B \cap D| = \left[\frac{250}{2\times3\times7}\right] = 5$

$|A \cap C \cap D| = \left[\frac{250}{2\times5\times7}\right] = 3$.

$|B \cap C \cap D| = \left[\frac{250}{3\times5\times7}\right] = 2$.

The number of integers between 1-250 that are divisible 2, 3, 5 & 7 is

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - \{|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|\} + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap B \cap C| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= (125 + 83 + 50 + 35) - (41 + 25 + 17 + 16 + 11 + 7) + (8 + 5 + 3 + 2) - 1$$

$$\cancel{= 293}$$

$$= 293 - 117 + 8 - 1$$

$$= 193.$$

$\therefore$ The number of integer not divisible by any of 2, 3, 5 and 7

$$= \text{Total} - |A \cup B \cup C \cup D|$$

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Here total = 250

$|A \cup B \cup C \cup D| = 193$

$\therefore$ Ans = 250 − 193

= 57.

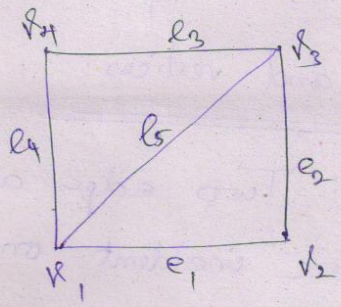
1

Unit III

Graphs

A graph $G=(V,E,\phi)$ consists a nonempty set $V=\{v_1,v_2,\ldots\}$ called the set of vertices of the graph, $E=\{e_1,e_2\ldots\}$ called the set of edges of the graph and ϕ is a mapping from the set of edges E to the set of ordered or unordered pairs of elts of vertices

Ex:



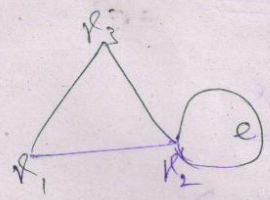
$V=\{v_1,v_2,v_3,v_4\}$ is set of vertices.

$E=\{e_1,e_2,e_3,e_4,e_5\}$ is set of edges.

Self Loop:

If there is an edge from v_i to v_i is called self loop (or) Simply loop.

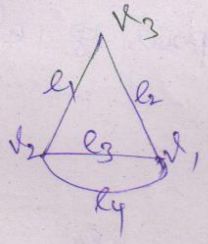
Ex:



Here 'e' is self loop.

Parallel edges:

If two edges have same end points, then the edges are called parallel edges

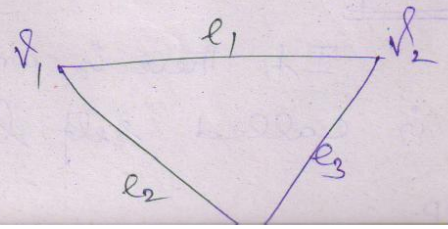


e_3 and e_4 are $\parallel^l$ edges

Adj edges and vertices:

Two edges are called as adj, if they are incident on a common vertex.

Two vertices are said to be adj if they are the end points of any one edge.

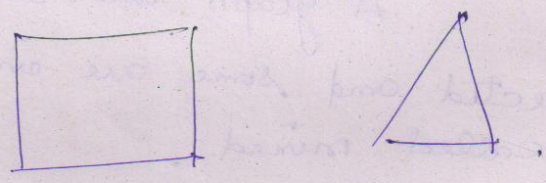


(3)

Here e_1 is adj to e_2 & e_3
v_1 is adj to v_2 & v_3

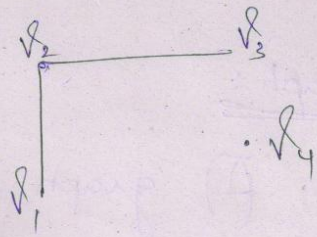
Simple Graph:

A graph which has neither self loops nor parallel edges is called simple.



Isolated Vertex:

A vertex having no edge incident on it is called isolated vertex.

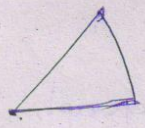


Directed Graph:

A graph in which every edge is directed is called digraph.

Undirected Graph:

A graph inwhich every edge is undirected is called undirected graph.



Mixed Graph:

A graph inwhich some edge are directed and some are undirected then it is called mixed.

Multigraph:

A graph which contains some parallel edges is called multi graph.

Pseudo Graph:

A graph inwhich loops and parallel edges are allowed is called pseudo graph.

Degree:

The number of edges incident at the vertex v_i is called degree of the

(5)

Indegree & Out degree

In a directed graph, the indegree of a vertex v_i denoted by $deg^-(v_i)$ and defined by the number of edges with v_i as their terminal vertex.

The outdegree of a vertex v_j is denoted by $deg^+(v_j)$ and defined by the number of edges with their initial.

Handshaking Thm:

Let $G = (V, E)$ be an undirected graph with 'e' edges. Then $\sum_{v \in V} deg(v) = 2e$.

Proof:

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of degree of the vertex.

$\therefore$ all the edges 'e' contribute '2e' to the sum of degree of the vertex.

$\therefore \sum deg(v) = 2e$.

Theorem:

The maximum no: of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

Proof

we prove this by induction method

For n=1

the graph with one vertex has no edge.

$$\frac{n(n-1)}{2} = \frac{1 \times 0}{2} = 0.$$

$\therefore$ The result is true for $n=1$

For n=2

The graph with 2 vertices has one edge atmost.

$$\frac{n(n-1)}{2} = \frac{2 \times 1}{2} = 1.$$

$\therefore$ The result is true for $n=2$.

Assume that the result is true for $n=k$.

A graph with k vertices has atmost $\frac{k(k-1)}{2}$ edges.

when $n=k+1$, let G be a graph having n vertices and G' be a graph obtained from G

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G' has k vertices, then by hypothesis G' has almost $\frac{k(k-1)}{2}$ edges, Now add the vertex v to G.

no. of edges $= \frac{k(k-1)}{2} + k$

$$= \frac{k^2 - k + 2k}{2}$$

$$= \frac{k^2 + k}{2}$$

$$= \frac{k(k+1)}{2}$$

$$= \frac{(k+1)k}{2}$$

$$= \frac{(k+1)(k+1-1)}{2}$$

$$= \frac{n(n-1)}{2}$$

Hence the max no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

① Can a simple graph exist with 15 vertices each of degree 5?

Soln

WKT "The no: of odd degree vertices is even".

Here no: of odd degree vertices is odd.

It is not possible.

∴ we can't say that a simple graph exist with 15 vertices each of degree 5.

② How many vertices does a regular graph of degree 4 with 10 edges have?

Soln

WKT, $\sum deg(v) = 2e$.

$n \times 4 = 2 \times 10$

$n = 20/4$

$n = 5$.

No: of vertices is 5.

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Regular Graph:

If every vertex of a simple graph has the same degree, then the graph is called a Regular Graph.

If every vertex in a regular graph has degree k, then the graph is called k-regular graph.

Complete Graph:

A simple graph with n vertices is said to be a Complete graph if the degree of every vertex is n-1.

(or)

In a graph G, every vertex v is adj to all other vertices, then G is called Complete.

The Complete graph with n

Bipartite Graph.

A graph G is said to be Bipartite if its vertex set $V(G)$ can be partitioned into two disjoint non-empty set V_1 & V_2, $V_1 \cup V_2 = V$ s.t. every edge in E has one end in V_1 and other end in V_2.

Complete Bipartite:

A Bipartite graph G with the bipartition V_1 and V_2 is called Complete Bipartite if every vertex in V_1 is adj to every vertex in V_2.

Subgraph:

A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

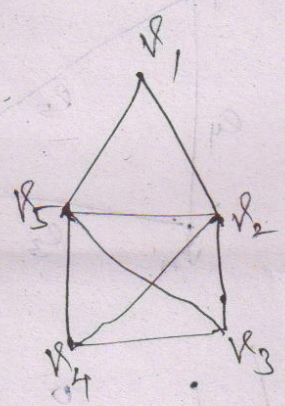
(11)

Adj matrix of a Simple Graph

Let $G = (V, E)$ be a simple graph with n vertices $\{v_1, v_2 \ldots v_n\}$ its adj matrix is denoted by $A = (a_{ij})$ and defined by.

$$A = \begin{cases} 1, & \text{if } v_i \text{ \& } v_j \text{ are adj.} \\ 0, & \text{otherwise.} \end{cases}$$

Ex:



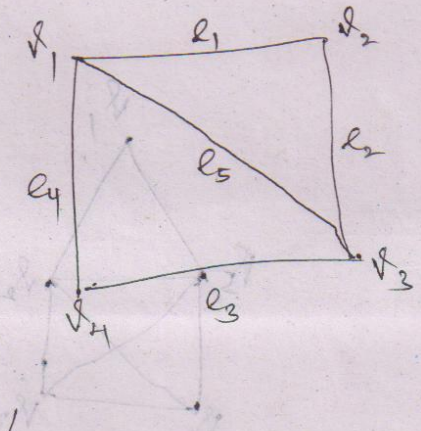
$$A = \begin{array}{c} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}$$

Incidence Matrix

Let $G = (V, E)$ be a graph with n vertices $\{v_1, v_2, \ldots v_n\}$ and m edges $\{e_1, e_2 \ldots e_m\}$ then the $n \times m$ matrix $B = (b_{ij})$ where

$$b_{ij} = \begin{cases} 1 & \text{if the edge } e_j \text{ incident on } v_i \\ 0 & \text{; otherwise} \end{cases}$$

Ex:



$$B = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \\ \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \end{array}$$

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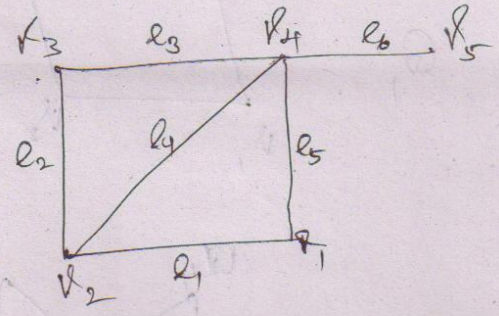
Path Matrix:

If $G = (V, E)$ be a simple digraph in which $|V| = n$ and the nodes of G are assumed to be ordered. An $n \times n$ matrix P whose elements are given by

$$P = (P_{ij}) = \begin{cases} 1 & \text{if there exist a path from } v_i \text{ to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

It is also called reachability.

Ex:



$$P = \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} \\ \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \end{array}$$

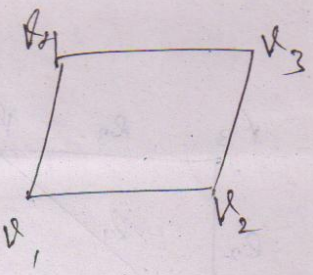
Graph Isomorphism:

Two graphs G_1 and G_2 are isomorphic then G_1 and G_2 have the following.

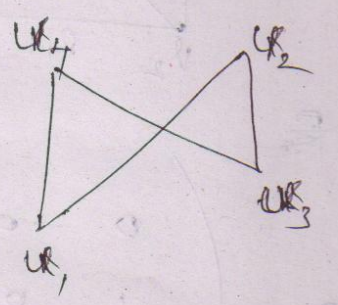
1) G_1 and G_2 have the same number of vertices
2) " " " have the same no: of edges.
3) An equal no: of vertices with a given degree

Ex:

G_1 :



G_2 :



Here $v_1 \to u_1$

$v_2 \to u_2$

$v_3 \to u_3$

The edges (v_1, v_2) (v_2, v_3) (v_3, v_4) (v_4, v_1) are mapped into (u_1, u_2) (u_2, u_3) (u_3, u_4) and (u_4, u_1)

$\therefore$ Adj ~~matri~~ of vertex sets are satisfied.

$\therefore$ The above graphs are isomorphic.

Paths, Reachability and Connectedness:

Path:

A path in a graph is a seq $v_1, v_2 \ldots v_k$ of vertices each adj to the next. (In other words, starting with the vertex v_1, one can walk along the edges and reach the vertex v_k).

Length of the Path:

The no: of edges appearing in the seq of a path is called the length of path.

Cycle:

A path which originates and ends in the same node is called a cycle.

Both starting point and end vertex are the same.

Connected Graph:

An undirected graph is said to be Connected if any pair of nodes are reachable from one another. That is, there is a path between any pair of nodes.

A graph which is not Connected is called disconnected graph.

Ex:

P.T. a simple graph with n-vertices must be Connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Soln:

Let G be a simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges.

Suppose if G is not Connected, then G must have atleast two Components.

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$|V_1| = m$ and $|V_2| = n-m$

Then

(i) $1 \le m \le n-1$

(ii) There is no edge joining a vertex of V_1 and a vertex of V_2

(iii) $|V_2| = n-m \geq 1$

Now $|E(G)| = |E(G_1 \cup G_2)|$

$$= |E(G_1)| + |E(G_2)|$$

$$\le \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$

$$= \frac{1}{2}\left[m^2 - m - nm - m(n-m-1) + m^2 - m\right]$$

$$= \frac{1}{2}\left[n(n-1) - nm - m(n-m-1) + m^2 - m\right]$$

$$= \frac{1}{2}\left[n(n-1) - 2(n-1) + 2n - 2 - nm - m(n-m-1) + m^2 - m\right]$$

$$= \frac{1}{2}\left[(n-1)(n-2) + 2n(1-m) + 2(m^2-1)\right]$$

$$= \frac{1}{2}\left[(n-1)(n-2) - 2(m-1)(n-m-1)\right]$$

$$|E(G)| \le \frac{(n-1)(n-2)}{2}$$

Since $(m-1), (n-m-1) \geq 0$

Components of a graph:

A Connected subgraphs of a graph G are called Components of the graph G.

Theorem:

A simple graph with n vertices and k Components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:

Let $n_1, n_2 \ldots n_k$ be the no: of vertices in each of k Components of the graph G.

Then $n_1 + n_2 + \ldots + n_k = n = |V(G)|$

i.e) $\sum_{i=1}^{k} n_i = n$

Now $\sum_{i=1}^{k} (n_i - 1) = (n_1 - 1) + \cdots + (n_k - 1)$

$= \sum_{i=1}^{k} (n_i - k) = n - k$

Squaring $\Rightarrow \left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$.

$(n_1 - 1)^2 + (n_2 - 1)^2 + \cdots (n_k - 1)^2 = n^2 + k^2 - 2nk$

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$$\sum n_i^2 + k - 2n \le n^2 + k^2 - 2nk$$

$$\sum n_i^2 \le n^2 + k^2 - k - 2nk - 2n$$

$$= n^2 + k(k-1) - 2n(k-1)$$

$$= n^2 + (k-2n)(k-1)$$

$$|E(G)| = \frac{1}{2}\sum n_i^2 - \frac{1}{2}\sum n_i$$

$$= \frac{1}{2}\left(n^2 + (k-1)(k-2n)\right) - \frac{n}{2}$$

$$= \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$

$$= \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

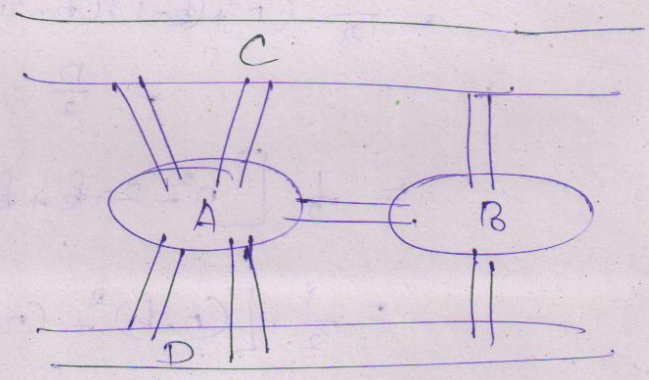
$$= \frac{1}{2}\left[(n-k)(n-k+1)\right]$$

$\therefore$ Max no: of edges

$$\le \frac{(n-k)(n-k+1)}{2}$$

Konisberg Bridge Problem:

There are two islands A & B formed by a river. They are Connected to each other and to the river banks C and D by means of 7-bridges

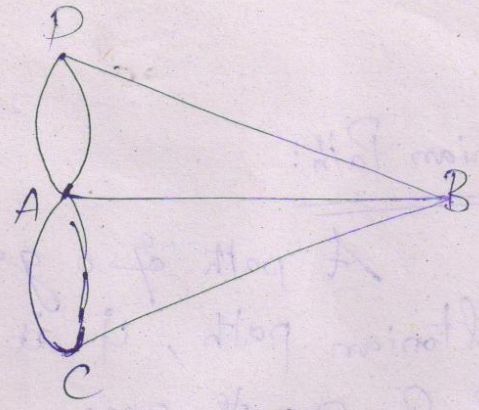


The pbm is to start from any one of the 4 land areas A, B, C, D walk accross each bridge exactly once and return to the starting pt.

This pbm has no soln.

When the situation is represented by a graph, with vertices representing the land

(21)



Here we can't find a Eulerian Circuit.

Hence KBP has no soln.

Euler ~~Graph~~ path

A path of a graph G is called an eulerian path if it contains each edge of the graph exactly once.

Euler Circuit

A circuit or cycle of a graph G is called an eulerian Circuit (or) Cycle if it includes each edge of G exactly once and also the starting pt and ending pt are same.

Hamiltonian Path:

A path of a graph G is called a hamiltonian path, if it includes each vertex of G exactly once.

Hamiltonian cycle:

A circuit of a graph G is called a Hamiltonian circuit if it includes each vertex of G exactly once, except the starting pt and end pt.

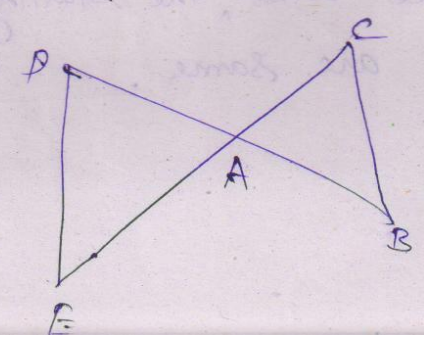
Ex:

Give an example of a graph which

(1) Eulerian but not Hamiltonian
(2) Hamiltonian but not Eulerian.
(3) Both Eulerian and Hamiltonian
(4) Non Eulerian non Hamiltonian.

Ex:

(1)



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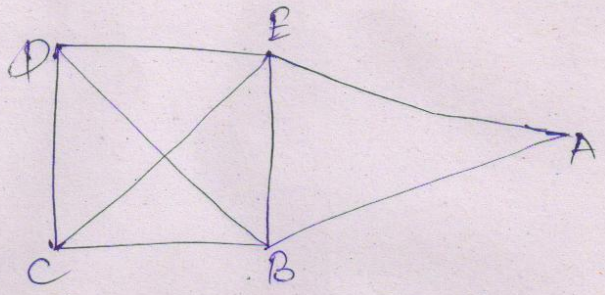
(2)



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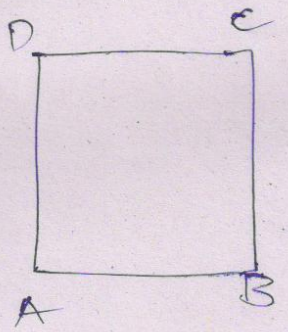

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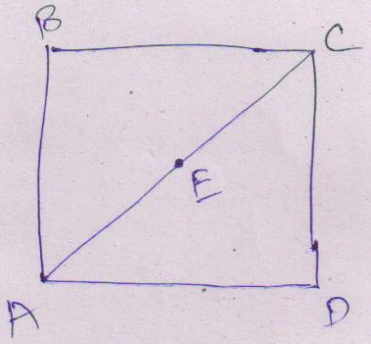

Unit IV

Algebraic Structures

Defn:

A non-empty set G together with one (or) more n-ary operations say $*$ (binary) is called an algebraic system (or) Algebra.

Properties:

Let the binary operation be $*: G \times G \to G$

(1) **Closure**

If $a, b \in G$, then $a * b \in G$

(2) **Commutativity:**

If $a, b \in G$, then $a * b = b * a$

(3) **Associativity:**

If $a, b, c \in G$ then $a * (b * c) = (a * b) * c$

(4) **Identity:**

For all $a \in G$, then $\exists$ $e \in G$ s.t.

(5) Inverse:

~~For all a, b, e~~

If $a \in G$, then $\exists$ $b \in G$ s.t.

$a * b = b * a = e$.

Here "b" is inverse of "a".

(6) Distributive Law:

$a * (b \cdot c) = (a * b) \cdot (a * c)$ (Left)

$(a \cdot b) * c = (a * c) \cdot (b * c)$ (Right)

for all $a, b, c \in G$.

(7) Cancellation:

$a * b = a * d \Rightarrow b = d$ ~~(Right)~~

(Left)

$a * b = c * b \Rightarrow a = c$

(Right)

for all $a, b, c \in G$.

		$(G,+)$	$(G,\times)$
1.	Associativity	$(a+b)+c = a+(b+c)$	$(a\times b)\times c = a\times(b\times c)$
2.	Commutativity	$a+b=b+a$	$a\times b = b\times a$
3	Identity	$a+0 = 0+a = a$	$a\times 1 = 1\times a = a$
4.	Inverse element	$a+(-a) = 0$ '$-a$' is add inverse	$a\times\frac{1}{a} = \frac{1}{a}\times a = 1$ '$\frac{1}{a}$' is inverse

Notation:

Z - the set of all integers

Q - the set of all rational.

R - . . . . . Real numbers.

R^+ - . . . . . +ve Real.

Q^+ - . . . . . +ve rational

C - . . . . . Complex numbers.

Ex:

1) Z is an algebraic system w.r. to the binary operations.

2) R with the usual + and x as binary operations is an algebraic.

Semigroup:

If a non-empty set S together with the binary operation '*' satisfying the following two properties.

(1) closure

(2) Associative. is called semigroup. (S, *)

Monoid:

If a semigroup M together with the binary operation '*' satisfy the ~~following property~~ Identity property, then (M, *) is called monoid

Ex:

1) (N, +) and (N, x) are both semigroup and monoids.

Cyclic Monoids:

A monoid $(M, *)$ is said to be cyclic, if every elt of m is of the form a^n, $a \in M$ and n is and integer.

i.e) $x = a^n$.

a is generator of cyclic Monoid.

Theorem:

Every cyclic monoid is Commutative.

Proof:

Let $(M, *)$ be a cyclic monoid whose generator is $a \in M$.

Then for $x, y \in M$.

$x = a^n$, $y = a^m$.

$$x * y = a^n * a^m$$
$$= a^{n+m}$$
$$= a^{m+n}$$
$$= a^m * a^n$$
$$= y * x$$

$\therefore$ $(M, *)$ is Commutative.

Morphisms of Semigroups:

Semigroup Homomorphism:

Let $(S,*)$ and (T,Δ) be any two semigroups with binary operation $*$ and Δ resp. A mapping $f: S \to T$ is said to be semigroup homomorphism if

$$f(a*b) = f(a)\Delta f(b), \text{ for all } a,b \in S.$$

Ex:

Let $(N,+)$ and $(Z_m,+_m)$ be any two semigp

Let us define a map

$$f: N \to Z_m \text{ by}$$

$f(a) = [a]_m$. for all $a \in N$

$$f(a+b) = [a+b]_m$$
$$= [a] +_m [b]$$
$$= f(a) +_m f(b)$$

f is semigroup homomorphism.

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Theorem:

Let $(S, *)$, (T, Δ) and $(V, \oplus)$ be semigroups and $h: S \to T$, $g: T \to V$ be semigp homomorphisms. S.T. $h \circ g : S \to V$ is a semigroup homomorphism from $(S, *)$ to $(V, \oplus)$

(or)

S.T. the composition of semigroup homomorphism is also a semigroup.

Proof:

Let $a, b \in S$, then

$$(h \circ g)(a * b) = h(g(a * b))$$
$$= h(g(a) \Delta g(b))$$
$$= h(g(a)) \oplus h(g(b))$$
$$= (h \circ g)(a) \oplus (h \circ g)(b)$$

Hence

$h \circ g$ is a semigroup homomorphism from $(S, *)$ to $(V, \oplus)$.

Example:

A semigroup homomorphism preserves the associative property.

Soln:

Let $(S, *)$ and $(T, \circ)$ be two semigroups.

Let $g: S \to T$ be semigroup homomorphism.

Let $a, b, c \in S$.

Now $g((a*b)*c) = g(a*b) \circ g(c)$

$= [g(a) \circ g(b)] \circ g(c)$

$g(a*(b*c)) = g(a) \circ g(b*c)$

$= g(a) \circ [g(b) \circ g(c)]$

NKT, $g((a*b)*c) = g(a*(b*c))$

$\Rightarrow (g(a) \circ g(b)) \circ g(c) = g(a) \circ (g(b) \circ g(c))$

$\therefore$ Semigroup homomorphism preserves associative property.

Morphism of Monoids:

Let $(M, *, e_m)$ and (T, Δ, e_T) be any two monoids with the identity elements e_m and e_T respectively.

A mapping $f: M \to T$ is said to be a monoid morphism if for any two elements $a, b \in M$,

$$f(a * b) = f(a) \Delta f(b) \text{ and}$$

$$f(e_m) = e_T.$$

Groups:

A nonempty set G together with the binary operation $*$, i.e) $(G, *)$ is called a group if $*$ satisfies the following property.

1) closure
2) Associative
3) Identity
4) Inverse

Abelian Group:

If a group $(G, *)$ is said to be abelian, then it is satisfies the

Order of Gp:-

The no: of elements in a group G is called the order of group. It is denoted by $O(G)$.

If $O(G)$ is finite, then G is finite.

If $O(G)$ is infinite, the G is infinite.

Ex:

S.T. the set $G = \{1, -1, i, -i\}$ Consisting of the 4th root of unity is a Commutative group under multiplication.

Soln:

Consider the multiplication (cayley table)

	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

All the elts in this table belongs to G. Hence G is closed.

Inverse of. 1 is 1

-1 is -1

i is $-i$

$-i$ is i

Ex: S.T. $(R-\{1\}, *)$ is an abel. group where $*$ is defined by $a*b = a+b+ab$ for all $a, b \in R$.

Proof:

Here $R-\{1\}$ means the set of real numbers except 1.

1) **Closure:**

clearly $a*b = a+b+ab \in R-\{1\}$

2) **Associative:**

$$(a*b)*c = (a+b+ab)*c$$
$$= (a+b+ac)+c+(a+b+ab)c$$
$$= a+b+c+ac+ac+bc+abc.$$

$$a*(b*c) = a*(b+c+bc)$$
$$= (a+b+c+bc)+(b+c+bc)a$$
$$= a+b+c+bc+ab+ac+abc.$$

$$\therefore (a*b)*c = a*(b*c).$$

Identity:

Let "e" be the identity

$\Rightarrow \quad a * e = a$

$\cancel{a*e =} \quad a + e + ae = a$

$\cancel{= a}$

$\cancel{\text{But by}} \quad e(1+a) = 0$

$e = 0$

$0 \in R - \{1\}$

Inverse:

Let the inverse of a be a^{-1}

$a * a^{-1} = 0$

$a + a^{-1} + aa^{-1} = 0$

$\cancel{a + a^{-1} + 1 = 0} \quad a + a^{-1}(1+a) = 0$

$a^{-1} = \frac{-a}{1+a}$

Clearly $\frac{-a}{1+a} \in R - \{1\}$

Commutative:

$a * b = a + b + ab$

$= b + a + ba$

$= b * a$

$\therefore \quad a * b = b * a$

$\therefore \quad R - \{1\}$ is abelian group.

P.T. the set $A = \{1, \omega, \omega^2\}$ is an abe. gp of order 3 under usual multi; where $1, \omega, \omega^2$ are cube roots of unity and $\omega^3 = 1$

Soln:

Composition table:

$\circ$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

① Closure:

All the elts in the above table are the elts of A.

$\therefore$ A is closed.

② Associative:

Multiplication of complex no:s are associative.

Identity:

Identity elt is 1.

Inverse:

Inverse of 1 is 1

ω is ω^2

ω^2 is ω

Commutative:

From the table, the Comm. property satisfies.

For Ex:

$\omega . \omega^2 = \omega^3 = \omega^2 . \omega$

(ie) $(A, \cdot)$ is abel. gp.

Here $o(A) = 3$.

$\therefore (A, \cdot)$ is finite abelian group.

Properties of Groups:

1) The identity elt of gp is unique.

2) In a gp $(G,*)$, the left and right cancellation laws are true.

That is $a*b = a*c \Rightarrow b=c$

$a*b = c*b \Rightarrow a=c$.

3) The inverse elt of a group is unique.

4) A gp cannot have any elt which is idempotent except the identity elt.

5) In a gp, $(a^{-1})^{-1} = a$, $a \in G$

6) If a has inverse b and b has inverse c, then $a=c$.

7) Let G be a group. If $a, b \in G$ then $(a*b)^{-1} = b^{-1}*a^{-1}$

8) If every elt of a gp G has its own inverse, then G is abelian. The converse need not be true.

9) ~~If~~ A group $(G,*)$ is

Ex: P.T. in an abel. gp $(ab)^2 = a^2b^2$

Soln:

$$(ab)^2 = (ab)(ab)$$
$$= a(ab)b$$
$$= a^2b^2$$

Ex: P.T. in a gp G, $x^2 = x$ iff $x = e$.

Soln:

clearly $e^2 = e$.

let $x^2 = x$.

$$x \cdot x = x$$
$$\Rightarrow x \cdot x = xe.$$
$$\Rightarrow x = e$$

By Left Cancellation.

Subgroups:

Let $(G,*)$ be a group. Let e be the identity in G. Let H be a subset of G. If $(H,*)$ itself is a group then H is called a subgroup of G.

$(H,*)$ is said to be subgroup of $(G,*)$ if

1) $e \in H$, where e is the identity in G

(19)

Check whether $H_1 = \{0, 5, 10\}$ and $H_2 = \{0, 4, 8, 12\}$ are subgroups of Z_{15} with respect to $+_{15}$.

Soln:

The addn tables (mod 15) for the sets H_1 and H_2 is gn below

For H_1

$+_{15}$	0	5	10
0	0	5	10
5	5	10	0
10	10	0	5

For H_2

$+_{15}$	0	4	8	12
0	0	4	8	12
4	4	8	12	1
8	8	12	1	5
12	12	1	5	9

Here all the entries in the addn table for H_1 are the elements of H_1

$\therefore H_1$ is a subgroup of Z_{15}

Also all the entries in the addition table for H_2 are not the elts of H_2.

$\therefore H_2$ is not closed under addition.

$\therefore H_2$ is not a subgroup of Z_{15}

Kernal of A Homomorphism:

Let $f: G \to G'$ be a gp of homomorphism. The set of elts of G which are mapped into e' (identity in G') is called kernal of f and it is denoted by $\ker(f)$.

i) $\ker(f) = \{x \in G / f(x) = e'\}$

Theorem:

The kernal of a homomorphism f form a gp $(G, *)$ to $(G', *)$ is a subgp of G.

Proof:

WKT, $\ker(f) = \{x \in G / f(x) e'\}$

Since $f(e) = e'$ is always true, atleast $e \in \ker(f)$.

Inotherwords, $\ker(f)$ is not empty in G.

Let the two elements $a, b \in \ker(f)$,

$\therefore f(a) = e'$ and $f(b) = e'$

$f(a * b^{-1}) = f(a) * f(b^{-1})$

$f(a*b^{-1}) = e'$

$\Rightarrow a*b^{-1} \in \ker(f)$

ii) $a, b \in \ker f \Rightarrow a*b^{-1} \in \ker(f)$

$\therefore \ker(f)$ is a subgp of G.

Isomorphism:

A mapping from a group $(G,*)$ to a gp (G', Δ) is said to be an isomorphism if

(1) f is a homomorphism

i) $f(a*b) = f(a) \Delta f(b)$.

for all $a, b \in G$.

2) f is 1-1 (Injective)

3) f is onto (Surjective)

Ex. P.T. if $f: G \to G'$ is a homomorphism then $\ker f = \{e\}$ iff f is 1-1.

Soln:

Assume f is 1-1

$$\ker f = \{e\}$$

Conversely, assume

$$\ker f = \{e\}$$

Now $f(x) = f(y) \Rightarrow f(x)[f(y)]^{-1} = e'$

$$\Rightarrow f(xy^{-1}) = e'$$

$$\Rightarrow xy^{-1} \in \ker(f)$$

$$\Rightarrow x^{-1}y = e.$$

$$\Rightarrow x = y.$$

$\therefore f(x) = f(y) \Rightarrow x = y.$

f is 1-1.

Cosets:

Let $(H, *)$ be a subgp of $(G, *)$

For any $a \in G$, the left Coset of H denoted by $a*H$ is the set $a*H = \{a*h : h \in H\}$

The right Coset of H denoted by $H*a$ is the set

$H*a = \{h*a : h \in H\}$

Lagrange's Thm:

The order of each subgp of a finite group is a divisor of the order of the group.

Proof:

Let $(G,*)$ be a gp whose order is n.

$\therefore O(G) = n$.

Let $(H,*)$ be a subgp of G whose order is m.

$\Rightarrow O(H) = m$

Let $h_1, h_2, \ldots h_m$ be the 'm' diff elts of H.

The right Cosets $H*a$ of H in G is defined by $H*a = \{h_1*a, h_2*a, \ldots h_m*a\}$ $a \in G$.

Since there is a 1-1 Correspondence between the elts of H and $H*a$,

The elts of $H*a$ are distinct.

Hence each right Coset of H in G has m distinct elts.

$$G = (H*a_1) \cup (H*a_2) \cup \ldots (H*a_k)$$

$$o(G) = o(H*a_1) + o(H*a_2) + \ldots$$

Lagrange's Thm:

The order of each subgp of a finite group is a divisor of the order of the group.

Proof:

Let $(G,*)$ be a gp whose order is n.

$\therefore O(G) = n$.

Let $(H,*)$ be a subgp of G whose order is m.

$\Rightarrow O(H) = m$

Let $h_1, h_2, \ldots h_m$ be the 'm' diff elts of H.

The right Cosets $H*a$ of H in G is defined by $H*a = \{h_1*a, h_2*a, \ldots h_m*a\}$ $a \in G$.

Since there is a 1-1 Correspondence between the elts of H and $H*a$,

The elts of $H*a$ are distinct.

Hence each right Coset of H in G has m distinct elts.

$$G = (H*a_1) \cup (H*a_2) \cup \ldots (H*a_k)$$

$$O(G) = O(H*a_1) + O(H*a_2) + \ldots$$

$$n = m + m + \cdots + m$$

$$= km$$

$$\Rightarrow k = \frac{n}{m}$$

$$\therefore \frac{O(G)}{O(H)} = k$$

Since k is an integer,

m is divisor of n.

$O(H)$ is divisor of $O(G)$

$\therefore O(H)$ divides $O(G)$.

Normal Subgps:

Let H be a subgp of G under $*$. The H is said to be a normal subgroup of G. For every $x \in G$ and for $h \in H$,

if $x * h * x^{-1} \in H$

$$x * H * x^{-1} \subseteq H$$

Alternatively, a subgp H of G is called a normal subgp of G, if

$x * h = h * x$ for all x

Natural Homomorphism:

Let H be a normal subgroup of a Group G.

The map $f: G \to G/H$ s.t. $f(x) = H * x$, $x \in G$ is called a Natural Homomorphism of group G.

Rings:

An algebraic System $(R, +, \cdot)$ is called a ring if the binary operations $+$ and $\cdot$ satisfies the following Conditions.

1) $(a+b)+c = a+(b+c)$, $a, b, c \in R$
2) $\exists$ an elt $0 \in R$ called zero elt s.t. $a+0 = 0+a = a$, for all $a \in R$
3) For all $a \in R$,

$$a+(-a) = (-a)+a = 0$$

$-a$ is negative of a

4) $a+b = b+a$, $a, b \in R$.
5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
6) $a \cdot (b+c) = a \cdot b + a \cdot c$

$(a+b) \cdot c = a \cdot c + b \cdot c$

In other words, if R is an abel. group

The ring $(R,+,\cdot)$ is called a Comm. ring, if $ab = ba$ for $a,b \in R$.

Field:

A Commu ring with identity $(R,+,\cdot)$ is called a field if every non-zero elt has a mult. inverse.

Thus $(R,+,\cdot)$ is a field

1) $(R,+)$ is abelian group and
2) $(R-\{0\},\cdot)$ is also abel. group.

Unit - 5

Lattices And Boolean Algebra

Relation:

Reflexive:

Let X be a set, R be a relation defined on X. Then R is said to be reflexive if it satisfies the following Condition

$$xRx, \ \forall x \in X.$$

R - is reflexive $= \{ x / (x,x) \in R \}, \ \forall x \in R.$

Symmetric:

Let X be any set, R be a relation defined on X. Then R is said to be symmetric, if it satisfies the following Condition

$$xRy \Rightarrow yRx. \quad \forall x, y \in X.$$

R - is Symmetric $= \{ (x,y) / (x,y) \in R \Rightarrow (y,x) \in R \}$

Transitive:

Let X be any set. R be a relation defined on X. Then R is said to be transitive, if R satisfies the following

$$xRy, yRz \Rightarrow xRz \quad , \forall x,y,z \in X.$$

R is transitive $\Rightarrow \{(x,z) / (x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R\}$

Antisymmetric

Let X be any set. R be a relation defined on X. Then R is said to be antisymmetric if it satifies the following Condition.

$$xRy, yRx \Rightarrow x = y \quad , \forall x,y \in X.$$

Equialence Relation:

Let X be any relation

R is said to be equialence if R satisfies Reflexive, Symmetric and

POSET

A set together with partiall order relation defined on it is called POSET.

Hasse Diagram:

~~Draw~~ pictorial representation of a poset is called Hasse diagram.

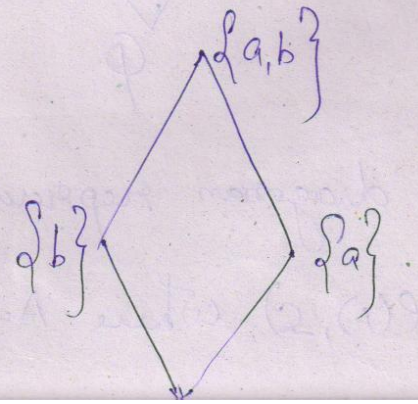
Ex Draw Hasse diagram for $(\rho(A), \subseteq)$ where

1) $A = \{a, b\}$ (2) $A = \{a, b, c\}$

Soln

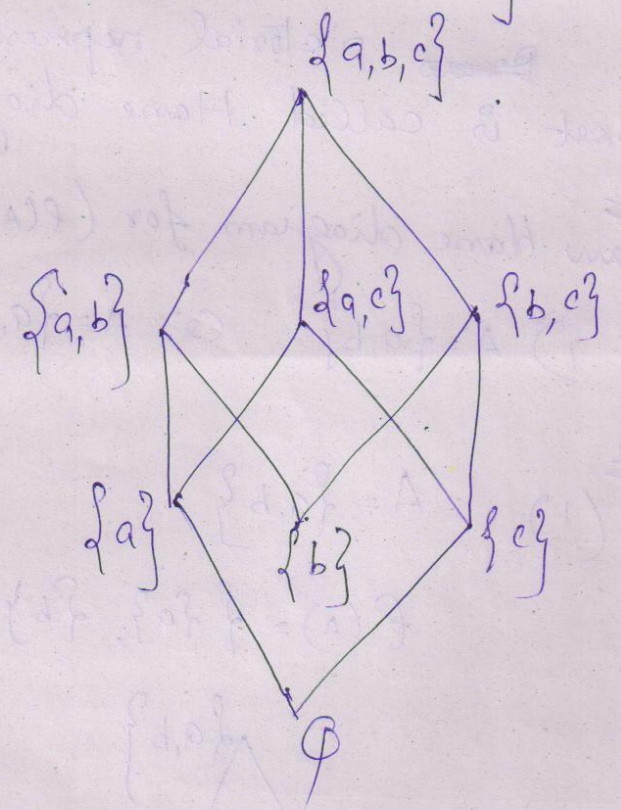
(1) $A = \{a, b\}$.

$\rho(A) = \{\{a\}, \{b\}, \{a, b\}, \{\phi\}\}$



(2) $A = \{a, b, c\}$

$P(A) = \{ \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \phi \}$



This diagram represented as.

$(P(A), \subseteq)$ where $A = \{a, b, c\}$

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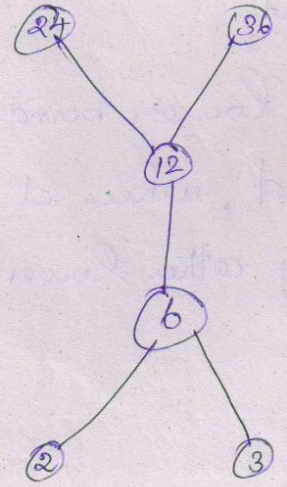
If $X = \{2, 3, 6, 12, 24, 36\}$ and the relation R defined on X by $R = \{\langle a,b \rangle \mid a/b\}$. Draw Hasse diagram for (X, R)

Soln

The relation

$$R = \{\langle 2,6\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 2,36\rangle, \langle 3,6\rangle, \langle 3,12\rangle, \langle 3,24\rangle, \langle 3,36\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 6,36\rangle, \langle 12,24\rangle, \langle 12,36\rangle\}$$

Hasse diagram for (X, R) is



LUB

Let $(P, \leq)$ be a poset and $A \subseteq P$. An elt $a \in P$ is said to be least upper bound (LUB) or supremum (sup) of A, if

1) a is a upper bound of A.
2) $a \leq c$, where c is any other upper bound of A.

GLB

Let $(P, \leq)$ be a poset $A \subseteq P$. An elt $b \in P$ is said to be greatest lower bound (GLB) (or) ~~min~~ infimum (inf) of A if

* b is lower bound of A.
* $b \geq d$, where d is any other lower bound of A.

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Consider $X = \{1, 2, 3, 4, 6, 12\}$

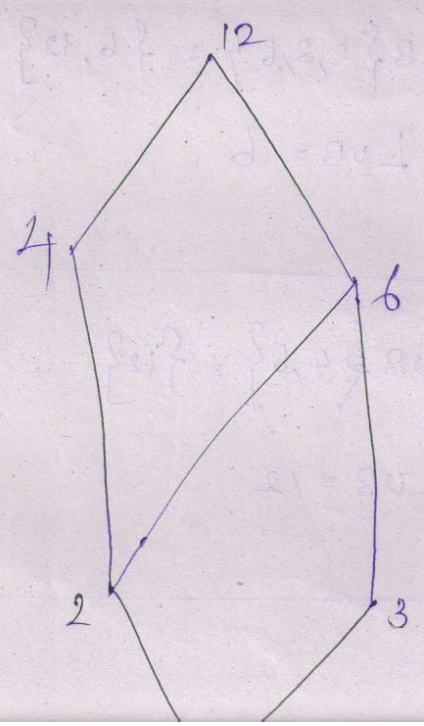
$R = \{ \langle a, b \rangle / a/b \}$.

Find LUB and GLB for the Poset (X,R)

Soln

The relation

$R = \{ \langle 1,2 \rangle, \langle 1,3 \rangle, \langle 1,4 \rangle, \langle 1,5 \rangle, \langle 1,6 \rangle, \langle 1,12 \rangle, \langle 2,4 \rangle, \langle 2,6 \rangle, \langle 2,12 \rangle, \langle 3,6 \rangle, \langle 3,12 \rangle, \langle 4,12 \rangle, \langle 6,12 \rangle \}$



1.	UB $\{1,3\} = \{3,6,12\}$ LUB $\{1,3\} = 3$.	LB $\{1,3\} = \{1\}$ G.L.B. $\{1,3\} = 1$.
2.	UB $\{1,2,3\} = \{6,12\}$ LUB $= 6$	LB $\{1,2,3\} = \{1\}$ GLB $= 1$.
3.	UB $\{2,3\} = \{6,12\}$ LUB $= 6$	LB $\{2,3\} = \{1\}$ GLB $= 1$.
4.	UB $\{2,3,6\} = \{6,12\}$ LUB $= 6$.	LB $\{2,3,6\} = \{1\}$ GLB $= 1$.
5.	UB $\{4,6\} = \{12\}$ LUB $= 12$	LB $\{4,6\} = \{1,2\}$ GLB $= 2$

Lattice

A lattice is Poset $(L, \leq)$ in which for every pair of elts $a, b \in L$, both greatest Lower bound (GLB) and LUB exists.

$$GLB\{a,b\} = a * b \text{ (or) } a \wedge b \text{ (or) } a \cdot b$$

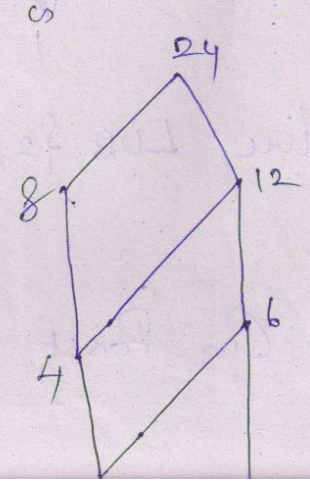
$$LUB\{a,b\} = a \oplus b \text{ (or) } a \vee b \text{ (or) } a + b.$$

$$S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$$

$$D = \{ \langle a,b \rangle \mid a/b \}$$

$$D = \{ \langle 1,2 \rangle, \langle 1,3 \rangle, \ldots \}$$

Hase diagram is



Determine whether the Posets

① $(\{1,2,3,4,5\}, /)$

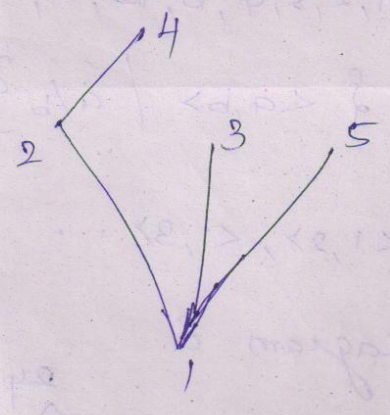
② $(\{1,2,4,8,16\}, /)$ are lattices.

Soln

① Consider the poset $(\{1,2,3,4,5\}, /)$

Then relation is $D = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle, \langle 2,4\rangle\}$

Hasse diagram is



Here LUB $\{2,3\} = 2 \vee 3$

= doesnot exist

Gn Poset is not a lattice

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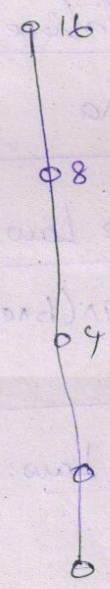
(2) Consider the poset $(\{1,2,4,8,16\}, /)$

The relation is

$$D = \{ \langle 1,2\rangle, \langle 1,4\rangle, \langle 1,8\rangle, \langle 1,16\rangle, \langle 2,4\rangle \langle 2,8\rangle \langle 2,16\rangle \langle 4,8\rangle, \langle 4,16\rangle, \langle 8,16\rangle \}$$



Here for every pair of elements, both GLB and LUB exists.

The poset $(\{1,2,4,8,16\}, /)$ is a lattice

Properties of Lattice:

1) Idempotent law	
$a \wedge a = a$	$a \vee a = a$
2) Commutative Law:	
$a \wedge b = b \wedge a$	$a \vee b = b \vee a$
3) Associative Law:	
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	$(a \vee b) \vee c = a \vee (b \vee c)$
4) Absorption Law:	
$a \wedge (a \vee b) = a$	$a \vee (a \wedge b) = a$

Theorem: 1

Let $(L, \wedge, \vee)$ be a lattice in which $\wedge$ and $\vee$ denote the operation of meet and join respectively.

For any $a, b \in L$

In other words,

(1) $a \vee b = b$ iff $a \leq b$

(2) $a \wedge b = a$ iff $a \leq b$

(3) $a \wedge b = a$ iff $a \vee b = b$.

Proof.

(i) $\Rightarrow$ (ii)

Let $a \leq b$

As $b \leq b$, from the defn of $a \vee b$, we have

$$a \vee b \leq b \longrightarrow (1)$$

Since $a \vee b$ is LUB of $\{a, b\}$, we've

$$b \leq a \vee b \longrightarrow (2)$$

From (1) & (2), we get

$$a \vee b = b.$$

(ii) $\Rightarrow$ (iii)

Let $a \vee b = b \longrightarrow (3)$

Now $a \wedge b = a \wedge (a \vee b)$

(ii) ⇒ (i)

Let $a \wedge b = a$.

The lower bound of $\{a, b\} = a$.

which implies $a \leq b$

Stat and prove distributive inequality of Lattice:

Distributive Inequality Lattice:

Let $(L, \wedge, \vee)$ be a gn lattice

For any $a, b, c \in L$, the following holds

1) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

2) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

Proof:

Claim (1)

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

From the defn of LUB, it is obvious that $a \leq a \vee b$.

(17)

$a \vee b$ is an upper bound of $\{a, b \wedge c\}$

Hence $a \vee b \geq a \vee (b \wedge c)$ —— (1)

From the defn of LUB, it is obvious

that $a \leq a \vee c$

$b \wedge c \leq c \leq a \vee c$.

$\Rightarrow b \wedge c \leq a \vee c$

$\Rightarrow$ $a \vee c$ is an upper bound of $\{a, b \wedge c\}$

$a \vee c \geq a \vee (b \wedge c)$ —— (2)

From (1) & (2) $a \vee (b \wedge c)$ is a LB of $\{a \vee b, a \vee c\}$

$\therefore a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Claim: 2

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

WKT, $a \geq a \wedge b$. —— (1)

And $b \vee c \geq b \geq a \wedge b$

From ① & ②,

$a \wedge b$ is an LB of $\{a, b \vee c\}$

$a \wedge b \leq a \wedge (b \vee c) \longrightarrow ③$

WKT, $a \geq a \wedge c \longrightarrow ④$

and $b \vee c \geq c \geq a \wedge c$

$\Rightarrow b \vee c \geq a \wedge c$ —— ⑤

From ④ & ⑤

$a \wedge c$ is LB of $\{a, b \vee c\}$

$\Rightarrow a \wedge c \leq a \wedge (b \vee c) \longrightarrow ⑥$

From ③ & ⑥, we get

$a \wedge (b \vee c)$ is an upper bound of $a \wedge c$ and $a \wedge b$.

$\therefore a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

Hence the proof

(19)

Theorem:

P.T. product of 2 lattices is also a Lattice.

(i) If $(L, \wedge, \vee)$ and $(M, *, \oplus)$ are 2 lattices, then P.T. $(L \times M, \cdot, +)$ is also a lattice.

Proof:

we show that the binary operations $\cdot$ and $+$ as defined above satisfies Commu. asso, and absorption laws in $L \times M$.

Let $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle, \langle x_3, y_3 \rangle \in L \times M$

(i) $\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 \vee x_2, y_1 \oplus y_2 \rangle$

$= \langle x_2 \vee x_1, y_2 \oplus y_1 \rangle$

$= \langle x_2, y_2 \rangle + \langle x_1, y_1 \rangle$

$\text{IIIly } \langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle = \langle x_2, y_2 \rangle \cdot \langle x_1, y_1 \rangle$

(ii) $\langle x_1, y_1 \rangle + [\langle x_2, y_2 \rangle + \langle x_3, y_3 \rangle]$

$= \langle x_1, y_1 \rangle + \langle x_2 \vee x_3, y_2 \oplus y_3 \rangle$

$= \langle (x_1 \vee x_2) \vee x_3 , (y_1 \oplus y_2) \oplus y_3 \rangle$

$= (\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle) + \langle x_3, y_3 \rangle$

(ii) ly $\langle x_1, y_1 \rangle \cdot (\langle x_2, y_2 \rangle \cdot \langle x_3, y_3 \rangle)$

$= (\langle x_1, y_1 \rangle \cdot (x_2, y_2 \rangle) \cdot \langle x_3 \cdot y_3 \rangle$

(iii) $\langle x_2 y_2 \rangle + [\langle x_2, y_2 \rangle \cdot \langle x_3, y_3 \rangle]$

$= \langle x_2 y_2 \rangle + (x_2 \wedge x_3 , y_2 \wedge y_3]$

$= \langle x_2 \vee (x_2 \wedge x_3) , y_2 \vee (y_2 \wedge y_3) \rangle$

$\cancel{= \langle (x_1 \vee x_2) \wedge (x_1 \vee x_3) , (y_1 \vee y_2) \wedge (y_1 \wedge y_3)]}$

$= \langle x_2 , y_2 \rangle$.

(iv) ly

$\langle x_1, y_1 \rangle \cdot (\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle)$

$= \langle x_1, y_1 \rangle$.

Boolean Algebra

(1) P.T. $(a+b)(a+c) = a+bc$.

$$L.H.S = (a+b)(a+c)$$
$$= aa + ac + ab + bc.$$
$$= a + ac + ab + bc.$$
$$= a(1+c) + ab + bc$$
$$= a + ab + bc.$$

(2) In any BA, S.T.

$$(a.\bar{b}) + (b.\bar{a}) = (a+b).(\bar{a}+\bar{b})$$

Soln:

Let $(B, \cdot, +, 0, 1)$ be any Boolean Algebra

Let $a, b \in B$

Now $(a.\bar{b}) + (b.\bar{a}) = (a + (b.\bar{a})) . (\bar{b} + (b.\bar{a}))$

$$= (a+b).(a+\bar{a}).(\bar{b}+b)(\bar{b}+\bar{a})$$

③ Reduce the expression $ab + abc + ab\bar{c} + \bar{a}bc$.

$ab + abc + ab\bar{c} + \bar{a}bc$.

$= ab(1+c) + ab\bar{c} + \bar{a}bc$

$= ab + ab\bar{c} + \bar{a}bc$

$= ab(1+\bar{c}) + \bar{a}bc$

$= ab + \bar{a}bc$.

$= b(a + \bar{a}c)$

$= b(a+c)$

④ P.T. $a\bar{b} + b\bar{c} + c\bar{a} = \bar{a}b + \bar{b}c + \bar{c}a$

Soln

$a\bar{b} + b\bar{c} + c\bar{a} = a\bar{b}.1 + b\bar{c}.1 + c\bar{a}.1$

$= a\bar{b}(c+\bar{c}) + b\bar{c}(a+\bar{a}) + c\bar{a}(b+\bar{b})$

$= a\bar{b}c + a\bar{b}\bar{c} + ab\bar{c} + \bar{a}b\bar{c}$

$+ \bar{a}bc + \bar{a}\bar{b}c$.

$= \bar{b}c(a+\bar{a}) + \bar{c}a(b+\bar{b})$

$+ \bar{a}b(c+\bar{c})$

$\bar{b}c + \bar{c}a + \bar{a}b$

Let $a, b, c \in B$. S.T.

(1) $a \cdot 0 = 0$

(2) $a + 1 = 1$

Proof:

(1) $a \cdot 0 = (a \cdot 0) + 0$

$= (a \cdot 0) + (a \cdot a')$

$= a \cdot (0 + a')$

$= a \cdot a'$

$= 0$

(2) $a + 1 = 1$

By taking dual, we get $a + 1 = 1$