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Numerical Integration:

The process of evaluating a definite integral from a set of tabulated values of $f(x)$ is called numerical integration. This process when applied to a function of a single variable is known as quadrature. In numerical integration, $f(x)$ is represented by an interpolation formula and then it is integrated between the given limits. In this way, the quadrature formula is derived for approximate integration of a function defined by a set of numerical values only.

Newton-Cotes Methods:

Let the function $y = f(x)$ takes values $y_0, y_1, y_2, \ldots\ldots\ldots\ldots y_n$ for $x\ , x_1, x_2, \ldots\ldots\ldots\ldots x_n$ respectively.

Let $I = \int_a^b f(x)dx$. Dividing the interval (a,b) into n sub-intervals of width h such that

$x_0 = a, x_1 = x_0 + h\ , x_2 = x_0 + 2h\ , \ldots\ldots, x_n = x_0 + nh = b$

$$\therefore \int_a^b f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx$$

Putting $x = x_0 + rh$, $dx = hdr$

When $x = x_0, r = 0$

When $x = x_0 + nh, r = n$

$$\therefore \int_a^b f(x)dx = h\int_0^n f(x_0 + rh)dr$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + ... \right] dr$$

$$= h\left| ry_0 + \frac{r^2}{2}\Delta y_0 + \left(\frac{\frac{r^3}{3} - \frac{r^2}{2}}{2} \right)\Delta^2 y_0 + \left(\frac{\frac{r^4}{4} - r^3 + r^2}{6} \right)\Delta^3 y_0 + ... \right|_0^n$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ... \right]$$

This equation is known as the Newton-Cotes quadrature formula.

1) Trapezoidal Rule:

By the Newton-Cotes quadrature formula,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ... \right]$$

Putting $n = 1$ in above formula and ignoring the differences of order higher than one.

$$\therefore \int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly,

$$\therefore \int_{x_0+h}^{x_0+2h} f(x)dx == \frac{h}{2}(y_1 + y_2)$$

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$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx == \frac{h}{2}(y_{n-1} + y_n)$$

Adding all these integrals,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots.. + y_{n-1})\right]$$

This is known as the trapezoidal rule.

Example: Evaluate $\int_0^1 e^x\, dx$, with $n = 10$ using the trapezoidal rule.

Solution:

$$a = 0, b = 1, n = 10$$
$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$
$$y = f(x) = e^x$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$f(x)$	1	1.1052	1.2214	1.3499	1.4918	1.6487	1.8221	2.0138	2.2255	2.4596	2.7183
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\right]$$

$$\therefore \int_0^1 e^x dx = \frac{0.1}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + ... + y_9)\right]$$

$$= \frac{0.1}{2}\begin{bmatrix} (1+2.7183)+2(1.1052+1.2214+1.3499+1.4918+1.6487 \\ +1.8211+2.0138+2.2255+2.4596) \end{bmatrix}$$

$$= 1.7196$$

Example: Given the data below, find the isothermal work done on the gas if it is compressed from $v_1 = 22L \ to \ v_2 = 2L$ .Use $W = -\int_{v_1}^{v_2} p\,dv$

v_1 , L	2	7	12	17	22
P, atm	12.20	3.49	2.049	1.44	1.11

Solution:

$$v_1 = 22 \ , \ v_2 = 2 \ , h = 5$$

By the trapezoidal r

ule,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})\right]$$

$$\textit{and we have } W = -\int_{v_1}^{v_2} p\,dv$$

$$W = -\int_{22}^{2} p\,dv$$

$$W = \int_{2}^{22} p\,dv$$

$$= \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{5}{2}\left[(12.20 + 1.11) + 2(3.49 + 2.049 + 1.44)\right]$$

$$= 68.17$$

2) Simpson's $\frac{1}{3}$ Rule:

By the Newton-Cotes quadrature formula,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots\right]$$

Putting $n = 2$ in above formula and ignoring the differences of order higher than two.

$$\therefore \int_{x_0}^{x_0+2h} f(x)dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \left(\frac{y_2 - 2y_1 + y_0}{6}\right)\right]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\therefore \int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

.

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.

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})\right]$$

This is known as Simpson's $\frac{1}{3}$ Rule.

Example: Consider the following values:

x	10	11	12	13	14	15	16
y	1.02	0.94	0.89	0.79	0.71	0.62	0.55

Find $\int_{10}^{16} f(x)dx$ by Simpson's $\frac{1}{3}$ Rule.

Solution:

$$a = 10, b = 16, h = 1$$

By Simpson's $\frac{1}{3}$ Rule

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})\right]$$

$$\therefore \int_{10}^{16} f(x)dx = \frac{1}{3}\left[(1.02 + 0.55) + 4(0.94 + 0.79 + 0.62) + 2(0.89 + 0.71)\right]$$

$$= 4.7233$$

Example: A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given as follows.

$t(s)$	0	10	20	30	40	50	60	70	80
$a(m/s^2)$	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

By Simpson's $\frac{1}{3}$ rule, find the velocity $t = 80s$.

Solution:

$$a = 10, b = 16, h = 1$$

By Simpson's $\frac{1}{3}$ rule,

$$\therefore \; Velocity = \int_{0}^{80} a\,dt = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$\therefore \qquad = \frac{10}{3}\left[\begin{array}{r}(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) \\ + 2(33.34 + 37.75 + 43.25)\end{array}\right]$$

$$= 3086.1 \; m/s$$

3) Simpson's $\frac{3}{8}$ Rule:

By the Newton-Cotes quadrature formula,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ...\right]$$

Putting $n=3$ in above formula and ignoring the differences of order higher than three.

$$\therefore \quad \int_{x_0}^{x_0+3h} f(x)dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0 \right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\therefore \quad \int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

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$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

Adding all th

ese integrals,

$$\therefore \quad \int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 \ldots\ldots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 \ldots\ldots + y_{n-1})\right]$$

This is known as Simpson's $\frac{3}{8}$ Rule.

Example: Evaluate $\int_0^3 \frac{1}{1+x}dx$, with $n=6$ using Simpson's 3/8 rule.

Solution:

$$a = 0, b = 3, n = 6$$
$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$
$$y = f(x) = \frac{1}{1+x}$$

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	1	0.6667	0.5	0.4	0.3333	0.2857	0.25

By Simpson's 3/8 rule,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 \ldots\ldots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 \ldots\ldots + y_{n-1})\right]$$
$$\therefore \int_0^3 \frac{1}{1+x}dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$
$$\therefore \int_0^3 \frac{1}{1+x}dx = \frac{3(0.5)}{8}\left[(1+0.25) + 2(0.4) + 3(0.6667 + 0.5 + 0.3333 + 0.2875)\right]$$
$$= 1.3888$$

Example: Find the volume of a solid of revolution formed by rotating about the $x-$axis the area bounded by the lines $x = 0, x = 1.5, y = 0$ and the curve passing through the following points:

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50
y^2	1.00	0.9826	0.9589	0.9089	0.8415	0.7624	0.7589

Solution:

Volume is given by $V = \int \pi y^2 dx$

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50
y^2	1.00	0.9826	0.9589	0.9089	0.8415	0.7624	0.7589

$h = 0.25$

By Simpson's 3/8 rule,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6.....+y_{n-3})+3(y_1+y_2+y_4+y_5.....+y_{n-1})\right]$$

$$\therefore \int y^2 dx = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3(0.25)}{8}\begin{bmatrix}(1.00+0.5759)+2(0.8261)+3(0.9655\\ +0.9195+0.7081+0.5812)\end{bmatrix}$$

$$= 1.1954$$

$$Volume = \pi \int y^2 dx = \pi(1.1954) = 3.7555$$

Error formulae for Trapezoidal and Simpson's formulas:

1. **Error in Trapezoidal rule:**

$$E = -\frac{(b-a)h^2}{12}\, y''(\bar{x})$$

2. **Error in Simpson's $\frac{1}{3}$ rule:**

$$E = -\frac{(b-a)h^4}{180}\, y^{iv}(\bar{x})$$

3. **Error in Simpson's $\frac{3}{8}$ rule:**

$$E = -\frac{3h^5}{80}\, y^{iv}(\bar{x})$$

Gaussian Quadrature Formulae:

In Newton-Cotes quadrature formula, we require values at equally spaced points of the interval.

The Gaussian quadrature formula uses the values of x which are not all equidistant points.

Consider $I = \int_a^b f(t)dt$

The substitution $t = \frac{1}{2}[(b-a)x + (b+a)]$ transforms the above integral into $\int_{-1}^{1} f(x)dx$.

An $n-$ point Gaussian quadrature formula is a quadrature formula constructed to give an exact result for polynomials of degree $2n-1$ or less by a suitable choice of the point x_i and weights w_i for $i = 1, 2, 3, ...n$. Gauss quadrature formula can be expressed as

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

1) One-point Gaussian Quadrature Formula:

Consider a function $f(x)$ over the interval [-1, 1] with sampling point x_1 and weight w_1.

The one-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1)$$

This formula will be exact for polynomials of degree up to $2n-1 = 2(1)-1 = 1$, i.e. it is exact for

$f(x) = 1$ and x

Substituting $f(x)$ in the formula $\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$, we have

$$\int_{-1}^{1} 1\, dx = w_1$$

$$|x|_{-1}^{1} = w_1$$

$$2 = w_1 \quad \text{...(1)}$$

$$\int_{-1}^{1} x\, dx = w_1 x_1$$

$$\left|\frac{x^2}{2}\right|_{-1}^{1} = w_1 x_1$$

$$0 = w_1 x_1 \quad \text{...(2)}$$

Solving (1) and (2), we get

$$w_1 = 2$$
$$x_1 = 0$$

Hence , $$\int_{-1}^{1} f(x)dx = 2f(0)$$

This is known as one-point Gaussian quadrature formula. This formula is exact for polynomials up to degree one.

2) Two-point Gaussian Quadrature Formula:

Consider a function $f(x)$ over the interval [-1, 1] with sampling point x_1, x_2 and weight w_1, w_2 respectively.

The two-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

This formula will be exact for polynomials of degree up to $2n-1=2(2)-1=3$, i.e. it is exact for

$f(x)=1, x, x^2$ and x^3.

Similarly by above procedure we have

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

This is known as two-point Gaussian quadrature formula. This formula is exact for polynomials up to degree three.

3) Three-point Gaussian Quadrature Formula:

Consider a function $f(x)$ over the interval [-1, 1] with sampling point x_1, x_2, x_3 and weight w_1, w_2, w_3 respectively.

The three-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

This formula will be exact for polynomials of degree up to $2n-1=2(3)-1=5$, i.e. it is exact for

$f(x)=1, x, x^2, x^3, x^4$ and x^5.

Similarly by above procedure we have

$$\int_{-1}^{1} f(x)dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

This is known as two-point Gaussian quadrature formula. This formula is exact for polynomials up to degree three.

Example:

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ by one-point, two-point and three point Gaussian formulae.

Solution:

$$f(x) = \frac{1}{1+x^2}$$

By the one-point Gaussian formula,

$$\begin{aligned}\int_{-1}^{1} \frac{1}{1+x^2} dx &= 2f(0) \\ &= 2\left(\frac{1}{1+0}\right) \\ &= 2\end{aligned}$$

By the two-point Gaussian formula,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}}$$
$$= 1.5$$

By the three-point Gaussian formula,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$
$$= \frac{5}{9}\left(\frac{1}{1+\frac{3}{5}}\right) + \frac{8}{9}(1) + \frac{5}{9}\left(\frac{1}{1+\frac{3}{5}}\right)$$
$$= 1.5833$$

Example:

Evaluate $\int_{0}^{1} \frac{dt}{1+t}$ by three point Gaussian formula

Solution:

Here, the original limits 0 to 1 of the integration have to be changed to -1 to 1 for this we use the substitution,

$$t = \frac{1}{2}[(b-a)x + (b+a)] \quad \text{where } a = 0 \quad and \quad b = 1$$

$$\therefore t = \frac{1}{2}[(1-0)x + (1+0)]$$
$$= \frac{1}{2}(x+1)$$

$$\therefore dt = \frac{1}{2}dx \quad and \quad x = 2t - 1$$
$$t = 0 \quad \Rightarrow x = -1$$
$$t = 1 \quad \Rightarrow x = 1$$

$$Now, \quad I = \int_0^1 \frac{dt}{1+t}$$
$$= \int_{-1}^1 \frac{dx}{3+x}$$
$$Here, \quad f(x) = \frac{1}{3+x}$$

By the one-point Gaussian formula,

$$\int_{-1}^1 \frac{dx}{3+x} = 2f(0) = 2\left(\frac{1}{3}\right) = 0.6667$$

By the two-point Gaussian formula,

$$\int_{-1}^1 \frac{dx}{3+x} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.6923$$

By the three-point Gaussian formula,

$$\int_{-1}^1 \frac{dx}{3+x} = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) = 0.6931$$

Numerical Solution of Ordinary Differential Equations:-

The fundamental laws of physics, mechanics, electricity and thermodynamics are usually based on empirical observations that explain variations in physical properties and states of the systems. Rather than describing the state of physical system directly, the laws are usually couched in terms of spatial and temporal changes. The following table gives a few examples of such fundamental laws that are written in terms of the rate of change of variables (*t* = time and *x* = position)

Physical Law	Mathematical Expression	Variables and Parameters
Newton's second law of motion	$\frac{dv}{dt} = \frac{F}{m}$	Velocity(v), force (F) and mass (M)
Fourier's Law of Heat Conduction	$q = -k\frac{dT}{dx}$	Heat flux (q), thermal conductivity(k) and temperature (T)
Faraday's Law (Voltage drop across an inductor)	$\Delta V_L = L\frac{di}{dt}$	Voltage drop (ΔV_L), inductance4 (L) and current (i)

The above laws define mechanism of change. When combined with continuity laws for energy, mass or momentum, differential equation arises. The mathematical expression in the above table is an example of the Conversion of a Fundamental law to an Ordinary Differential Equation. Subsequent integration of these differential equations results in mathematical functions that describe the spatial and temporal state of a system in terms of energy, mass or velocity variations. In fact, such mathematical relationships are the basis of the solution for a great number of engineering problems. But, many ordinary differential equations arising in real-world applications and having lot of practical significance cannot be solved exactly using the classical analytical methods. These ode can be analyzed qualitatively. However,

qualitative analysis may not be able to give accurate answers. A numerical methodcan be used to get an accurate approximate solution to a differential equation. There are many programs and packages available for solving these differential equations. With today's computer, an accurate solution can be obtained rapidly. In this chapter we focus on basic numerical methods for solving initial value problems.

Analytical methods, when available, generally enable to find the value of *y* for all values of *x*. Numerical methods, on the other hand, lead to the values of *y* corresponding only to some finite set of values of *x*. That is the solution is obtained as a table of values,rather than as continuous function. Moreover, analytical solution, if it can be found, is exact, whereas a numerical solution inevitably involves an error which should be small but may, if it is not controlled, swamp the true solution. Therefore we must be concerned with two aspects of numerical solutions of ODEs: both the method itself and its accuracy.

In this chapter some methods for the numerical solution of ODEs are described.

The general form of first order differential equation, in implicit form, is $F(x, y, y') = 0$ and in the explicit form is $\frac{dy}{dx} = f(x, y)$. An Initial Value Problem (IVP) consists of a differential equation and a condition which the solution much satisfies (or several conditions referring to the same value of x if the differential equation is of higher order). In this chapter we shall consider IVPs of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 . \qquad (1)$$

Assuming f to be such that the problem has a unique solution in some interval containing x_0, we shall discuss the methods for computing numerical values of the solution. These methods are step-by-step methods. That is, we start from $y_0 = y(x_0)$ and proceed stepwise. In the first step, we compute an approximate value y_1 of the solution y of (1) at $x = x_1 = x_0 + h$. In the second step we compute an approximate value y_2 of the solution y at $x = x_2 = x_0 + 2h$, etc. Here h is fixed number for example 0.1 or 0.001 or 0.5 depends on the requirement of the problem. In each step the computations are done by the same formula.

The following methods are used to solve the IVP (1).

1. Taylor's Series Method
2. Euler and Modified Euler Method
3. Runge – Kutta Method
4. Milne's Method

1. Taylor's Series Method:

Consider an IVP $\dfrac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$. Let us approximate the exact solution y(x) to a power series in $(x - x_0)$ using Taylor's series. The Taylor's series expansion of y(x) about the point $x = x_0$ is

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y^{i}(x_0) + \frac{(x-x_0)^2}{2!} y^{ii}(x_0) + \frac{(x-x_0)^3}{3!} y^{iii}(x_0) + \frac{(x-x_0)^4}{4!} y^{iv}(x_0) + \cdots \cdots$$

.... (1)

From the differential equation, we have $y^{i}(x) = \dfrac{dy}{dx} = f(x,y)$. Differentiating this successively, we can get $y^{ii}(x), y^{iii}(x), y^{iv}(x), \text{etc.}$ Putting $x = x_0$ and $y = y_0$, the values of $y^{ii}(x_0), y^{iii}(x_0), y^{iv}(x_0), \text{etc.}$ can be obtained. Hence the Taylor's series (1) gives the values of y for every value of x for which (1) converges.

On finding the value of y_1 for $x = x_1$ from (1), $y^{ii}(x), y^{iii}(x), y^{iv}(x), \text{etc.}$ can be evaluated at $x = x_1$ from the differential equation

Example: Find by Taylor's series method the value of y at x = 0.1 and 0.2 five places of decimals for the IVP $\dfrac{dy}{dx} = x^2 y - 1, \quad y(0) = 1$

<u>Solution:</u>

Given $x_0 = 0, \; y_0 = 1$ and $f(x,y) = x^2 y - 1$

Taylor's series expansion about the point $x = 0 (= x_0)$ is

$$y(x) = y(0) + \frac{(x-0)}{1!} y^{i}(0) + \frac{(x-0)^2}{2!} y^{ii}(0) + \frac{(x-0)^3}{3!} y^{iii}(0) + \frac{(x-0)^4}{4!} y^{iv}(0) + \cdots \cdots$$

i.e. $$y(x) = y(0) + x y^{i}(0) + \frac{x^2}{2} y^{ii}(0) + \frac{x^3}{6} y^{iii}(0) + \frac{x^4}{24} y^{iv}(0) + \cdots \cdots \qquad (1)$$

It is given that $\qquad y(0) = 1$

$\dfrac{dy}{dx} = y^{i}(x) = x^2 y - 1 \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y^{i}(0) = -1$

Differentiating $y^{i}(x) = x^2 y - 1$ successively three times and putting x = 0 & y = 1, we get

$y^{i}(x) = 2xy + x^2 y^{i} \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y^{ii}(0) = 0$

$$\Rightarrow\Rightarrow\Rightarrow \qquad y^{iii}(0) = 2$$

$$y^{iv}(x) = 6y^{i} + 6xy^{ii} + x^2 y^{iii} \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y^{iv}(0) = -6$$

Putting the values of $y(0), y^{i}(0), y^{ii}(0), y^{iii}(0), y^{iv}(0)$ in (1), we get

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6)$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$$

Hence, y(0.1) = 0.90033 and y(0.2) = 0.80227.

Example: Using Taylor's series method solve $\frac{dy}{dx} = x^2 - y, \quad y(0) = 1$at 0.1 ≤ x ≤ 0.4. Compare the values with the exact solution.

Solution:

Given $x_0 = 0,\ y_0 = 1$ and $f(x, y) = x^2 - y$

Taylor's series expansion about the point $x = 0$ is

$$y(x) = y(0) + xy^{i}(0) + \frac{x^2}{2} y^{ii}(0) + \frac{x^3}{6} y^{iii}(0) + \frac{x^4}{24} y^{iv}(0) + \cdots\cdots \qquad (4)$$

It is given that $y(0) = 0$

$$\frac{dy}{dx} = y^{i}(x) = x^2 - y \qquad \Rightarrow \qquad y^{i}(0) = (0)^2 - 1 = -1$$

Differentiating $y^{i}(x) = x^2 - y$ successively and putting x =0, y = 1, we get

$$y^{ii}(x) = 2x - y^{i} \qquad \Rightarrow \qquad y^{ii}(0) = 0y^{i}(0) = 0 - (-1) = 1$$

$$y^{iii}(x) = 2 - y^{ii} \qquad \Rightarrow \qquad y^{iii}(0) = 2 - y^{ii}(0) = 1$$

$$y^{iv}(x) = -y^{iii} \qquad \Rightarrow \qquad y^{iv}(0) = -y^{iii}(0) = -1$$

Putting the values of $y(0), y^{i}(0), y^{ii}(0), y^{iii}(0), y^{iv}(0)$ in (4), we get

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$$

Hence, $y(0.1) = 1-(0.1)+\frac{(0.1)^2}{2}+\frac{(0.1)^3}{6}-\frac{(0.1)^4}{24}$ = 0.9051625

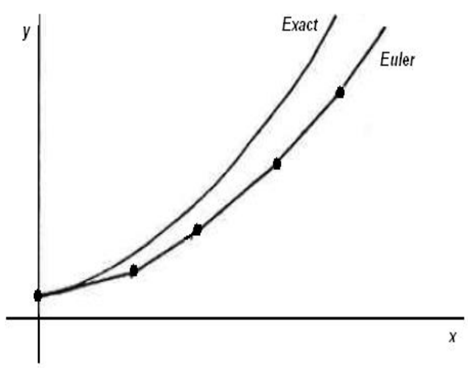
$$y(0.2) = 1-(0.2)+\frac{(0.2)^2}{2}+\frac{(0.2)^3}{6}-\frac{(0.2)^4}{24} = 0.8212667$$

$$y(0.3) = 1-(0.3)+\frac{(0.3)^2}{2}+\frac{(0.3)^3}{6}-\frac{(0.3)^4}{24} = 0.7491625$$

$$y(0.4) = 1-(0.4)+\frac{(0.4)^2}{2}+\frac{(0.4)^3}{6}-\frac{(0.4)^4}{24} = 0.6896.$$

2. Euler's Method:

Consider the IVP $\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$. The following two methods can be used to determine the solution at a point $x = x_n = x_0 + nh$.



The solution of the differential equation is represented by the curve as shown in the fig. The point $(x_{0,}y_0)$ lies on the curve. The equation of tangent to the curve at the point is given by

$$y - y_0 = \left(\left| \frac{dy}{dx} \right|_{x=x_0} \right)(x-x_0) = f(x_0, y_0)(x-x_0)$$
$$y = y_0 + f(x_0, y_0)(x-x_0)$$

But we know that $h = x_n - x_{n-1}$,

Hence, $y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0,1,2,3,\ldots\ldots$

<u>Euler's Method</u>:

$$y_{n+1}^{E} = y_n + hf(x_n, y_n), \qquad n = 0,1,2,3,\text{L L} \qquad (1)$$

Modified Euler's Method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{E})\right], \qquad n = 0,1,2,3,\Lambda\ \Lambda \qquad (2)$$

Remark:

1. The formulae (1) and (2) are also known as Euler's Predictor – Corrector formula.

When Modified Euler's method is applied to find the solution at a given point, we first find the solution at that point by using Euler's method and the same will be used in the calculation of Modified Euler's method. Also Modified Euler's method has to be applied repeatedly until the solution is stationary.

Example: Using Euler's method, find $y(0.2)$ given $\frac{dy}{dx} = y - \frac{2x}{y}, \quad y(0) = 1$ with h=0.1.

Solution :

$$\frac{dy}{dx} = y - \frac{2x}{y}$$

Given:

$$\begin{aligned}
&x_0 = 0,\ y_0 = 1,\ h = 0.1,\ x = 0.2 \\
&n = \frac{x - x_0}{h} = \frac{0.2 - 0}{0.1} = 2 \\
&x_1 = 0.1 \\
&y_1 = y_0 + hf(x_0, y_0) \\
&\quad = 1 + 0.1 f(0,1) \\
&\quad = 1.1 \\
&y_2 = y_1 + hf(x_1, y_1) \\
&\quad = 1 + 0.1 f(0.1, 1.1) \\
&\quad = 1.1918
\end{aligned}$$

Hence, $y_2 = y(0.2) = 1.1918$

Example: Determine the value of y when $x = 0.1$ correct up to four decimal places by taking $h = 0.05$. Given that $y(0) = 1$ and $\frac{dy}{dx} = x^2 + y$ using modified Euler's method.

Solution :

$$\frac{dy}{dx} = f(x, y) = x^2 + y$$

(i) Given: $x_0 = 0, \ \ y_0 = 1, \ \ h = 0.05, \ \ \ x_1 = 0.05$

$$f(x_0, y_0) = 0 + 1 = 1$$
$$y_1^0 = y_0 + hf(x_0, y_0) = 1 + 0.05(1) = 1.05$$

First approximation to y_1

$$y_1^1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^0) \right]$$
$$= 1 + \frac{0.05}{2}\left[1 + f(0.05, 1.05)\right]$$
$$= 1.0513$$

Second approximation to y_1

$$y_1^2 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^1) \right]$$
$$= 1 + \frac{0.05}{2}\left[1 + f(0.05, 1.0513)\right]$$
$$= 1.0513$$

Since the values of y_1^1 and y_1^2 are equal.

$$y_1 = y(0.05) = 1.0513$$

(ii) Now, $x_1 = 0.05, \ \ y_1 = 1.0513, \ \ h = 0.05, \ \ \ x_2 = 0.1$

$$f(x_1, y_1) = f(0.05, 1.0513) = 1.0538$$
$$y_2^0 = y_1 + hf(x_1, y_1) = 1.0513 + 0.05(1.0538) = 1.1040$$

First approximation to y_2

$$y_2^1 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^0) \right]$$
$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + f(0.1, 1.1040)\right]$$
$$= 1.1055$$

Second approximation to y_2

$$y_2^2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^1) \right]$$

$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + f(0.1, 1.1055)\right]$$

$$= 1.1055$$

Since the values of y_2^1 and y_2^2 are equal.

$$y_2 = y(0.1) = 1.1055$$

Example: Solve $\frac{dy}{dx} = -\frac{y^2}{1+x}, \quad y(0) = 1$ by Euler's method by choosing h = 0.1 and h = 0.05. Also solve the same problem by modified Euler's method by choosing h = 0.05. Compare the numerical solution with analytical solution.

Solution :

Analytical solution is : -

$$\frac{dy}{y^2} = -\frac{dx}{1+x} \quad \Rightarrow \quad \frac{1}{y} = \log(1+x) + c$$

Using the condition y(0) = 1, we get c = 1.

Hence the analytical solution is $y = \frac{1}{1+\log(1+x)} \Rightarrow$ y(0.2) = 0.84579

Now by Euler's method, we have $y_{n+1} = y_n - 0.1\left(\frac{y_n^2}{1+x_n}\right)$

$$y_1 = y(0.1) = 1 - 0.1\left(\frac{(1)^2}{1+0}\right) = 0.9$$

$$y_2 = y(0.2) = 0.9 - 0.1\left(\frac{(0.9)^2}{1+0.1}\right) = 0.82636$$

Error = 0.84579 – 0.82636 = 0.01943

Now taking h = 0.05, Euler's method is $y_{n+1} = y_n - 0.05\left(\frac{y_n^2}{1+x_n}\right)$

$$y_1 = y(0.05) = 1.0 - 0.05\left(\frac{(1)^2}{1+0}\right) = 0.95$$

$$y_2 = y(0.1) = 0.95 - 0.05\left(\frac{(0.95)^2}{1+0.05}\right) = 0.90702$$

$$y_3 = y(0.15) = 0.90702 - 0.05\left(\frac{(0.90702)^2}{1+0.1}\right) = 0.86963$$

$$y_4 = y(0.2) = 0.86963 - 0.05\left(\frac{(0.86963)^2}{1+0.15}\right) = 0.83675$$

Error = 0.84579 – 0.83675 = 0.00904

<u>Now we use modified Euler's method to find y(0.2) with h = 0.05</u>

Euler's Formula is $y_{n+1} = y_n - 0.05\left(\frac{y_n^2}{1+x_n}\right)$, n = 0,1,2 and 3

Modified Euler Formula is $y_{n+1} = y_n - 0.025\left(\frac{y_n^2}{1+x_n} + \frac{y_{n+1}^E}{1+x_{n+1}}\right)$, n = 0,1,2 and 3

<u>Stage – I: Finding y_1 = y(0.05)</u>

From Euler's formula (for n = 0),

$$y_1^E = y(0.05) = 1.0 - 0.05\left(\frac{(1)^2}{1+0}\right) = 0.95$$

From Modified Euler's formula, we have

$$y_1^{(1)} = y(0.05) = 1.0 - 0.025\left(\frac{(1)^2}{1+0} + \frac{(0.95)^2}{1+0.05}\right) = 0.95351$$

$$y_1^{(2)} = y(0.05) = 1.0 - 0.025\left(\frac{(1)^2}{1+0} + \frac{(0.95351)^2}{1+0.05}\right) = 0.95335$$

$$y_1^{(3)} = y(0.05) = 1.0 - 0.025\left(\frac{(1)^2}{1+0} + \frac{(0.95335)^2}{1+0.05}\right) = 0.95336$$

$$y_1^{(4)} = y(0.05) = 1.0 - 0.025\left(\frac{(1)^2}{1+0} + \frac{(0.95336)^2}{1+0.05}\right) = 0.95336$$

Hence $y_1 = y(0.05) = 0.95336$

Stage – II: Finding y_2 = y(0.1)

From Euler's formula (for n = 1), we get

$$y_2^E = y(0.1) = 0.95336 - 0.05\left(\frac{(0.95336)^2}{1+0.05}\right) = 0.91008$$

From Modified Euler's formula, we have

$$y_2^{(1)} = y(0.1) = 0.95336 - 0.025\left(\frac{(0.95336)^2}{1+0.05} + \frac{(0.91008)^2}{1+0.1}\right) = 0.91286$$

$$y_2^{(2)} = y(0.1) = 0.95336 - 0.025\left(\frac{(0.95336)^2}{1+0.05} + \frac{(0.91286)^2}{1+0.1}\right) = 0.91278$$

$$y_2^{(2)} = y(0.1) = 0.95336 - 0.025\left(\frac{(0.95336)^2}{1+0.05} + \frac{(0.91278)^2}{1+0.1}\right) = 0.91278$$

Hence $y_2 = y(0.1) = 0.91278$

Stage – III: Finding y_3 = y(0.15)

From Euler's formula (for n = 2), we get

$$y_3^E = y(0.15) = 0.91278 - 0.05\left(\frac{(0.91278)^2}{1+0.1}\right) = 0.87491$$

From Modified Euler's formula (for n = 2), we have

$$y_3^{(1)} = y(0.15) = 0.91278 - 0.025\left(\frac{(0.91278)^2}{1+0.1} + \frac{(0.87491)^2}{1+0.15}\right) = 0.87720$$

$$y_3^{(2)} = y(0.15) = 0.91278 - 0.025\left(\frac{(0.91278)^2}{1+0.1} + \frac{(0.87720)^2}{1+0.15}\right) = 0.87712$$

$$y_3^{(3)} = y(0.15) = 0.91278 - 0.025\left(\frac{(0.91278)^2}{1+0.1} + \frac{(0.87712)^2}{1+0.15}\right) = 0.87712$$

Hence $y_3 = y(0.15) = 0.87712$

Stage – IV: Finding y_4 = y(0.2)

From Euler's formula (for n = 3), we get

$$y_4^E = y(0.2) = 0.87712 - 0.05\left(\frac{(0.87712)^2}{1+0.15}\right) = 0.84367$$

From Modified Euler's formula (for n = 3), we have

$$y_4^{(1)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84367)^2}{1+0.2}\right) = 0.84557$$

$$y_4^{(2)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84557)^2}{1+0.2}\right) = 0.84550$$

$$y_4^{(2)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84550)^2}{1+0.2}\right) = 0.84550$$

Hence $\boxed{y_4 = y(0.2) = 0.84550}$

Hence $y_3 = y(0.15) = 0.87712$

Stage – IV: Finding y_4 = y(0.2)

From Euler's formula (for n = 3), we get

$$y_4^E = y(0.2) = 0.87712 - 0.05\left(\frac{(0.87712)^2}{1+0.15}\right) = 0.84367$$

From Modified Euler's formula(for n = 3), we have

$$y_4^{(1)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84367)^2}{1+0.2}\right) = 0.84557$$

$$y_4^{(2)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84557)^2}{1+0.2}\right) = 0.84550$$

$$y_4^{(2)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84550)^2}{1+0.2}\right) = 0.84550$$

Hence $y_4 = y(0.2) = 0.84550$

Error = 0.84579 – 0.84550 = 0.00029

Recall that the error from Euler's Method is 0.00904

3. Runge – Kutta Method:

The Taylor's series method to solve IVPs is restricted by the difficulty in finding the higher order derivatives. However, Runge – Kutta method do not require the calculations of higher order derivatives. Euler's method and modified Euler's method are Runge – Kutta methods of first and second order respectively.

Consider the IVP $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$. Let us find the approximate value of *y* at $x = x_{n+1}$, *n = 0,1,2,3,…..* of this numerically, using Runge – Kutta method, as follows:

Second Order (Heun) method:

First let us calculate the quantities k_1 and k_2 using the following formulae.

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h, y_n + k_1)$$
$$k = \frac{1}{2}(k_1 + k_2)$$

Finally, the required solution ***y*** is given by

$$y_{n+1} = y_n + k$$

Fourth Order method:

First let us calculate the quantities k_1, k_2, k_3 and k_4 using the following formulae.

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

+

$$k_4 = hf\left(x_n + h, y_n + k_3\right)$$

Finally, the required solution ***y*** is given by

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Example: Given that $y = 1.3$ when $x = 1$ and $\frac{dy}{dx} = 3x + y$. Use the second order R-K method to approximate y when $x = 1.2$. Use a step size of 0.1.

Solution:

$$\frac{dy}{dx} = f(x, y) = 3x + y.$$

(i) Given: $x_0 = 1, y_0 = 1.3, h = 0.1 \text{ and } n = 0$

$$\begin{aligned}
k_1 &= hf(x_0, y_0) \\
&= 0.1f(1, 1.3) \\
&= 0.43 \\
k_2 &= hf(x_0 + h, y_0 + k_1) \\
&= 0.1f(1 + 0.1, 1.3 + 0.43) \\
&= 0.503 \\
k &= \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.43 + 0.503) = 0.4665 \\
y_1 &= y_0 + k = 1.3 + 0.4665 = 1.7665
\end{aligned}$$

(ii) Now, $x_1 = 1.1, y_1 = 1.7665, h = 0.1 \text{ and } n = 1$

$$
\begin{aligned}
k_1 &= hf(x_1, y_1) \\
&= 0.1 f(1.1, 1.7665) \\
&= 0.5067 \\
k_2 &= hf(x_1 + h, y_1 + k_1) \\
&= 0.1 f(1.1 + 0.1, 1.7665 + 0.5067) \\
&= 0.5873 \\
k &= \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.5067 + 0.5873) = 0.5470 \\
y_2 &= y_1 + k = 1.7665 + 0.5470 = 2.3135
\end{aligned}
$$

Hence, $y_2 = y(1.2) = 2.3135$

Example: Apply Runge – Kutta fourth order method, to find an approximate value of y when x = 0.2 given that $\frac{dy}{dx} = x + y, \quad y(0) = 1$.

Solution:

Given: $x_0 = 0, y_0 = 1, h = 0.2 \text{ and } f(x, y) = x + y$

By R – K method (for n = 0) is: $y_1 = y(0.2) = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$ -------- (1)

Now

$$k_1 = hf(x_0, y_0) = 0.2 \times [0 + 1] \qquad = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.2}{2}\right)\right] \qquad = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.24}{2}\right)\right] \qquad = 0.2440$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = 0.2 \times \left[\left(0 + 0.2\right) + \left(1 + 0.2440\right)\right] \qquad = 0.2888$$

Using the values of k_1, k_2, k_3 and k_4 in (1), we get

$y_1 = y(0.2) = 1 + \frac{1}{6}(0.2 + 0.24 + 0.244 + 0.2888)$ = 1.2468

Hence the required approximate value of y is 1.2468.

Example: Using Runge – Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}, \quad y(0) = 1$

at x = 0.2 & 0.4.

Solution:

Given: $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

For $y_1 = y(0.2)$

By R – K method (for n = 0) is: $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ -------- (2)

$k_1 = hf(x_0, y_0) = 0.2 \times f(0,1)$ = 0.2

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times f(0.1, 1.1)$ = 0.19672

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times f(0.1, 1.0936)$ = 0.1967

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.1967)$ = 0.1891

Using the values of k_1, k_2, k_3 and k_4 in (2), we get

$y_1 = y(0.2) = 1 + \frac{1}{6}(0.2 + 2(0.19672) + 2(0.1967) + 0.1891)$

= 1+0.19599

= 1.19599

Hence the required approximate value of y is <u>1.19599</u>.

For $y_2 = y(0.4)$

We have $x_1 = 0.1, y_1 = 1.19599 \text{ and } h = 0.2$

By R – K method (for n = 1) is: $y_2 = y(0.4) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ -------- (3)

$$k_1 = hf(x_1, y_1) = 0.2 \times f(0.2, 1.19599) \qquad = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2 \times f(0.3, 1.2906) \qquad = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2 \times f(0.3, 1.2858) \qquad = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2 \times f(0.4, 1.3753) \qquad = 0.1688$$

Using the values of $k_1, k_2, k_3 \text{ and } k_4$ in (3), we get

$$y_2 = y(0.4) = 1.19599 + \frac{1}{6}(0.1891 + 2(0.1795) + 2(0.1793) + 0.1688)$$

= 1.19599 + 0.1792

= 1.37519

Hence the required approximate value of y is <u>1.37519</u>.