PARUL UNIVERSITY - FACULTY OF ENGINEERING & TECHNOLOGY
Department of Applied Science & Humanities
3rd Semester B. Tech (CSE, IT)
Discrete Mathematics (203191202)
UNIT-1 Sets, Relation & Functions

Overview:

- Cartesian product of sets
- Relations and their properties
- Composition of relations
- POSets and equivalence sets
- Matrix representation of relations
- Digraph of relations
- Closures of relations
- n-tuples
- Database and Relations
- Cantor's diagonal argument
- The Power set theorem
- Schroder-Bernstein Theorem

Weightage: 11%

Teaching Hours: 5

Introduction:

Much of mathematics is about finding a pattern – a recognisable link between quantities that change. In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, teacher and student.

In mathematics also, we come across many relations such as

number m is **less than** number n,

line l is **parallel to** line m,

set $\boldsymbol{A}$ is **a subset of** set $\boldsymbol{B}$.

In all these, we notice that a relation involves pairs of objects in certain order.

In this Chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. Finally, we will learn about special relations which will qualify to be functions. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correspondence of one quantities with the other.

Prerequisites:

Theory of sets: **Definition**
Operations on sets
Cardinality of sets
Subset and Power set

Application:

The time required to manipulate information in a database depends on how this information is stored. The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database. Because of the importance of these operations, various methods for representing databases have been developed. One of these methods, called the **relational data model** is based on the concept of a relation.

The database query language SQL (short for Structured Query Language) can be used to carry out the operations we have described in this section. Example 12 illustrates how SQL commands are related to operations on n-ary relations.

COMPOSITE OF RELATIONS

Definition

Let R be a relation from a set A to a set B and S a relation from B to a set C.
The composite of R and Sgiven by $S \circ R$ is the relation from A to C consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
Thus, $S \circ R = \{(a, c) | (a, b) \in R, (b, c) \in S, for\ some\ b \in B\}$.
In other words $a(S \circ R)c$ if and only if aRb and bSc for some $b \in B$.

Note:

1. **The *powers of a relation R* can be recursively defined from the definition of a composite of two relations.**
   Let $\boldsymbol{R}$ be a relation on the set $\boldsymbol{A}$. The powers $\boldsymbol{R^n}, n = 1,2,3, \ldots$are defined recursively by $\boldsymbol{R^1} = \boldsymbol{R}$ and $\boldsymbol{R^{n+1}} = \boldsymbol{R^n} \circ \boldsymbol{R}$.
   Thus, $\boldsymbol{R^2 = R \circ R}$, $\boldsymbol{R^3 = R^2 \circ R = (R \circ R) \circ R}$, and so on.

Problem.1

Let $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$ and $S = \{ (1,0), (2,0), (3,1), (3,2), (4,1)\}$ be two relations on some sets. Check if $S \circ R$ is possible or not. If it is possible then write the elements of the relation $S \circ R$.

Solution

Here, $codom(R) = \{1,3,4\}$ is a subset of $dom(S) = \{1,2,3,4\}$
Hence, $S \circ R$ is Possible.
Further, $S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

Problem.2

Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers R^n, n= 2,3,4,....

Solution

$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$
Further, $R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$.
Similarly, $R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$.
It follows that $R^n = R^3$ for $n = 5,6,7, \ldots$

Exercise:

1. Let $R = \{(0,1), (1,2), (1,4), (2,3), (3,1), (4,3)\}$ and $S = \{ (1,0), (2,1), (3,1), (3,2)\}$ be two relations on some sets. Check if $R \circ S$ is possible or not. If it is possible then write the elements of the relation $R \circ S$.

PROPERTIES OF RELATIONS

Let A be a set. Let R be a relation on it.

- The relation $\boldsymbol{R}$ is said to be **reflexive** if $(a, a) \in R,$ for every $a \in A$ In other words, a relation on A is reflexive if every element of A is related to itself.
- The relation $\boldsymbol{R}$ is said to be **transitive,** if $(a, b) \in R \text{ and } (b, c) \in R \Rightarrow (a, c) \in R$, for all $a, b, c \in A$.
- The relation $\boldsymbol{R}$ is said to be **symmetric** $if\ (a, b) \in R \Rightarrow (b, a) \in R,$ for all $a, b \in A$
- The relation $\boldsymbol{R}$ is said to be **anti-symmetric** if $(a, b) \in R, (b, a) \in R \Rightarrow a = b$,for all $a, b \in A$

  In other words, a relation R on a set A is anti-symmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a.

  i.e. the only way to have $\boldsymbol{a}$ related to $\boldsymbol{b}$ and $\boldsymbol{b}$ related to $\boldsymbol{a}$ is for $\boldsymbol{a}$ and $\boldsymbol{b}$ to be the same element.

Definition

A relation R on a set A is said to be ***equivalence relation***, if R is reflexive, transitive and symmetric.
A relation R on a set A is said to be ***partially ordered relation***, if R is reflexive, transitive and anti-symmetric.
A set A with a partially ordered operation R,(i.e. (A, R)) is said to be ***Partially Ordered Set (POSet).***

Note:

(i) The terms symmetric and antisymmetric are not opposites.

(ii) If R is an equivalence relation, and $(a, b) \in R$, then a and b are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Problem 1: Consider $A = \{1,2,3\}$ and a relation R on A in each of the following cases.
Check whether they are reflexive, symmetric, anti-symmetric or transitive.
Also check which of them is equivalence relation or partially ordered relation.

(1) $aRb, if\ a = b$	[R, T, S, AS]
(2) $aRb\ if\ a \leq b$	[R, T, AS]
(3) $aRb\ if\ a \neq b$	[S]
(4) $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$	[R, T, S]
(5) $R = \{(1,1), (2,2), (1,2), (2,1)\}$	[T, S]
(6) $R = \{(1,1), (2,2)\}$	[T, S, AS]
(7) $R = \{(1,1), (2,2), (3,3), (2,1), (1,3)\}$	[R]

Problem.2

Show that the "greater than or equal" relation (≥) is a partial ordering on the set of integers.

Solution:

Because a ≥ a for every integer a, ≥ is reflexive.
If a ≥ b and b ≥ a, then a = b.
Hence, ≥is antisymmetric.
Finally, ≥is transitive because a ≥ b and b ≥ c imply that a ≥ c.
It follows that ≥is a partial ordering on the set of integers and **(Z,≥) is a poset**.

Problem.3

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution:

Because A $\subseteq$A whenever A is a subset of ***S***,$\subseteq$ is reflexive.

It is antisymmetric because A $\subseteq$ B and B $\subseteq$ A imply that A = B.

Finally, $\subseteq$is transitive, because A $\subseteq$ B and B $\subseteq$ C imply that A $\subseteq$ C.

Hence, $\subseteq$is a partial ordering on ***P(S)***, and ***(P(S),$\subseteq$)is a poset***.

Exercise

1. Check whether the following relations are equivalence relation or not on the set of all integers Where aRb if and only if 1) $a \neq b$ 2) $ab \geq 0$
2. Check whether from the following relation sets, which are satisfying the transitive, reflexive or symmetric property which relation is an Equivalence relation and partially ordered relation.
   1) R1 = {(1, 1), (2, 2), (3, 3)}
   2) R2 = {(1, 1), (2, 2), (1, 2), (2, 1), (3, 1)}
   3) R3= {(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)}
   4) R4= {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)}
3. Prove that $(\mathbb{Z}, \leq)$ is a partially ordered set where Z is the set of integers.
4. Check if $\mathbb{N}$ with the '$divides$' relation is a POSet.

REPRESENTION OF RELATIONS

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix.

Suppose that $\boldsymbol{R}$ is a relation from $\boldsymbol{A} = \{ a_1, a_2, \ldots, a_m\}$ to $\boldsymbol{B} = \{ b_1, b_2, \ldots, b_n\}$.

(Here the elements of the sets $\boldsymbol{A}$ and $\boldsymbol{B}$ have been listed in a particular, but arbitrary, order. Furthermore, when $\boldsymbol{A} = \boldsymbol{B}$ we use the same ordering for $\boldsymbol{A}$ and $\boldsymbol{B}$.)

The relation $\boldsymbol{R}$ can be represented by the matrix $\boldsymbol{M_R} = [\boldsymbol{m_{ij}}]$, where $\boldsymbol{m_{ij}} = \begin{cases} 1 \; if \;\; (a_i, b_j) \in \boldsymbol{R} \\ 0 \; if \;\; (a_i, b_j) \notin \boldsymbol{R} \end{cases}$

In other words, the zero–one matrix representing $\boldsymbol{R}$ has a **1** as its(i, j)th entry when $\boldsymbol{a_i}$**is related to** $\boldsymbol{b_j}$, and a **0** in this position if $\boldsymbol{a_i}$**is not related to** $\boldsymbol{b_j}$.

Note: Such a representation depends on the orderings used for $\boldsymbol{A}$ and $\boldsymbol{B}$.

Problem.1
Suppose that $A = \{1,2,3\}$and $B = \{ 1,2\}$.
Let R be the relation from A to B containing (a, b) if $a \in A, b \in B, and\ a > b$.
What is the matrix representing R?
Solution
Because $R = \{(2,1), (3,1), (3,2)\}$, the matrix for $\boldsymbol{R}$ is $\mathbf{M_R} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

Problem.2
Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1 , b_2, b_3, b_4, b_5\}$.
Which ordered pairs are in the relation R represented by the matrix $\mathrm{M_R} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$?
Solution:
Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Remark

- R is reflexive if and only if $m_{ii} = 1$ for $i = 1,2, \ldots, n$.
  In other words, R is reflexive if all the elements on the main diagonal of $\mathrm{M_R}$ are equal to 1.
  Note that the elements off the main diagonal can be either 0 or 1.

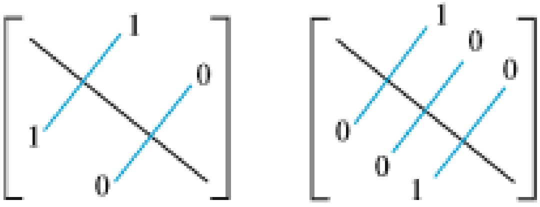
$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

(The matrix for a reflexive relation)

- The relation $\boldsymbol{R}$ is **symmetric** if and only if $\boldsymbol{m_{ji} = 1}$ whenever $\boldsymbol{m_{ij} = 1.}$
  This also means $\boldsymbol{m_{ji} = 0}$ whenever $\boldsymbol{m_{ij} = 0}$.

Consequently, $\boldsymbol{R}$ is symmetric if and only if $\boldsymbol{m_{ij}} = \boldsymbol{m_{ji}}$, for all i and j.
i.e. $\boldsymbol{R}$ is symmetric if and only if $M_R = M_R^T$,
i.e, $\boldsymbol{R}$ is symmetric if M_R is a symmetric matrix.

- The relation R is anti-symmetric if and only if $m_{ij} = 1$ with $i \neq j$,then $m_{ji} = 0$.
  In other words, if $i \neq j$ then either $m_{ij} = 0$ or $m_{ji} = 0$.
  The form of the matrix for an antisymmetric relation is illustrated in Figure.



(a) Symmetric (b) Antisymmetric

Matrix of Union and Intersection of two relations

Suppose that $\boldsymbol{R}$ and $\boldsymbol{S}$ are relations on a set **A** represented by the matrices $\boldsymbol{M_R}$ and $\boldsymbol{M_S}$, respectively.
The matrix representing the **union** of these relations has a 1 in the positions where **either $\boldsymbol{M_R}$ or $\boldsymbol{M_S}$ has a 1.**
The matrix representing the **intersection** of these relations has a 1 in the positions **where both $\boldsymbol{M_R}$ and $\boldsymbol{M_S}$ have a 1.**
Thus, the matrices representing the union and intersection of these relations are
$\boldsymbol{M_{R\cup S}} = \boldsymbol{M_R} \vee \boldsymbol{M_S}$ and $\boldsymbol{M_{R\cap S}} = \boldsymbol{M_R} \wedge \boldsymbol{M_S}$.

Problem.3

Suppose that the relations R and S on a set A are represented by the matrices
$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
What are the matrices representing $R \cup S$ and $R \cap S$?

Solution:

The matrices of these relations are

$$\boldsymbol{M_{R\cup S}} = \boldsymbol{M_R} \vee \boldsymbol{M_S} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \boldsymbol{M_{R\cap S}} = \boldsymbol{M_R} \wedge \boldsymbol{M_S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix of Composite of two relations:

Matrix of Composite of two relations can be found using the Boolean product of the matrices for these relations.
In particular, suppose that $\boldsymbol{R}$ is a relation from ***A to B*** and $\boldsymbol{S}$ is a relation from ***B to C.***
Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ have $\boldsymbol{m}, \boldsymbol{n}$, and $\boldsymbol{p}$ elements, respectively.
Let the zero– one matrices for $\boldsymbol{S \circ R}, \boldsymbol{R}$, and $\boldsymbol{S}$ be $\boldsymbol{M_{S\circ R}} = [\boldsymbol{t_{ij}}], \boldsymbol{M_R} = [\boldsymbol{r_{ij}}], \boldsymbol{and}\ \boldsymbol{M_S} = [\boldsymbol{s_{ij}}]$, ***respectively.*** (Note that these matrices have sizes $\boldsymbol{m} \times \boldsymbol{p}, \boldsymbol{m} \times \boldsymbol{n}$, and $\boldsymbol{n} \times \boldsymbol{p}$, respectively).
The ordered pair (a_i, c_j) belongs to $\boldsymbol{S \circ R}$ if and only if there is an element b_k such that $(a_i, b_k) \in \boldsymbol{R}$ and $(b_k, c_j) \in \boldsymbol{S}$.
It follows that $\boldsymbol{t_{ij} = 1}$ **if and only if** $\boldsymbol{r_{ik}} = \boldsymbol{s_{kj}} = \boldsymbol{1}$ for some k.

In other words, $t_{ij} = 1$ if and only if i^{th} row of M_R and j^{th} column of M_S has 1 at a same position. From the definition of the Boolean product, this means that $\boldsymbol{M_{S\circ R}} = \boldsymbol{M_R} \odot \boldsymbol{M_S}$.

Problem.4

Find the matrix representing the relation $S \circ R$, where the matrices representing R and S are

$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Solution:

The matrix for S∘R is $M_{S\circ R} = \boldsymbol{M_R} \odot \boldsymbol{M_S} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Exercise

1. Let R be the relation represented by the matrix $M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find the matrix representing

   (a) R^{-1} **(b)** R' ($\boldsymbol{c}$) R^2

2. Let R and S be relations on a set A represented by the matrices $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Find the matrices representing the following relations.

   (a) $R \cup S$ (b) $R \cap S$ (c) $S \circ R$ (d) $R \circ$ S (e) $R \oplus S$

Representing Relations Using Digraphs

There is another important way of representing a relation using a pictorial representation.

Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow.

We use such pictorial representations when we think of relations on a finite set as **directed graphs**, or **digraphs**.

Definition

A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex ‘a’ is called the initial vertex of the edge (a,b), and the vertex ‘ b’ is called the terminal vertex of this edge.

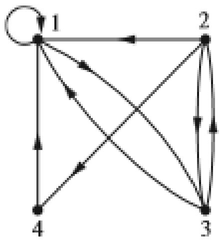
An edge of the form (a,a) is represented using an arc from the vertex ‘a’ back to itself. Such an edge is called a *loop*.

REMARKS

- A relation R is **reflexive** if and only if **there is a loop at every vertex** of the directed graph, so that every ordered pair of the form (x, x) occurs in the relation.
- A relation is **transitive** if and only if **whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z** (completing a triangle where each side is a directed edge with the correct direction).
- A relation is **symmetric** if and only if **for every edge between distinct vertices in its digraph there is an edge in the opposite direction**, so that (y, x) is in the relation whenever (x, y) is in the relation.
- A relation is **antisymmetric** if and only if **there are never two edges in opposite directions between distinct vertices**.

Problem.1 Draw the directed graph of the relation

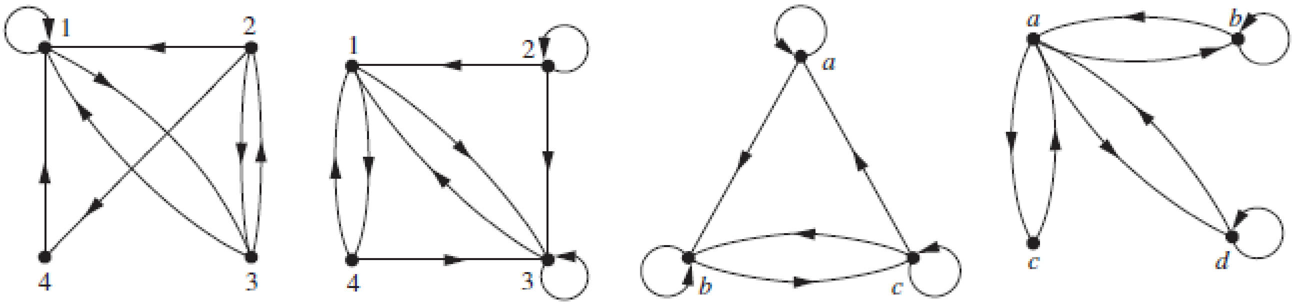
$$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$$



Problem.2 Draw the directed graph of the relation

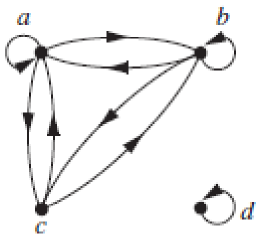
$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$

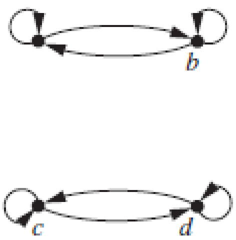
Problem.3 Determine whether the relations for the directed graphs shown in the following figure are reflexive, symmetric, antisymmetric, and/or transitive.



(a) **(b)** **(c)** **(d)**

Problem.4 Write the relation represented by the following digraph and also write the matrix representing this relation.

(a) 

(b) 

Solution.

(a) $R = \{(a,a),(a,b),(b,a),(b,b),(b,c),(c,b),(c,a),(a,c).(d,d)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $R = \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Exercise

1. Write the relation represented by the following matrices and also draw the corresponding digraph.

(a) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

CLOSURES OF RELATIONS

Introduction

Let R be a relation on a set A.

R may or may not have some property P, such as reflexivity, symmetry, or transitivity. If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R.

In other words, S is the smallest superset of R with the property P.

Reflexive closure of R

For given a relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$, the ***reflexive closure of R*** can be formed by adding to $\boldsymbol{R}$ all pairs of the form (a, a) with $a \in \boldsymbol{A}$, not already in $\boldsymbol{R}$.

The addition of these pairs produces a new relation that is reflexive, contains R, and is contained within any reflexive relation containing R. Consequently, it is the reflexive closure of R.

Thus, the reflexive closure of R can be given by R∪Δ, where $\Delta = \{(a, a) \mid a \in A\}$is the diagonal relation on A.

Problem.2

The relation $\boldsymbol{R} = \{(\boldsymbol{1}, \boldsymbol{1}), (\boldsymbol{1}, \boldsymbol{2}), (\boldsymbol{2}, \boldsymbol{1}), (\boldsymbol{3}, \boldsymbol{2})\}$on the set $\boldsymbol{A} = \{\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}\}$ is not reflexive. Obtain the reflexive closure of R.

Solution Here, diagonal relation on A is $\Delta = \{(1,1), (2,2), (3,3)\}$

Therefore, the reflexive closure of R is $S = R \cup \Delta = \{(1,1), (1,2), (2,1), (2,2), (3,2), (3,3)\}$

Problem.2

What is the reflexive closure of the relation $\boldsymbol{R} = \{(\boldsymbol{a}, \boldsymbol{b}) \mid \boldsymbol{a} < \boldsymbol{b}\}$on the set of integers?

Solution: The reflexive closure of R is

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in Z\} = \{(a, b) \mid a \leq b\}.$$

Symmetric closure of R

The symmetric closure of a relation $\boldsymbol{R}$ can be constructed by adding all ordered pairs of the form (b, a), for all (a, b) that are not already present in $\boldsymbol{R}$.

Adding these pairs produces a relation that is symmetric, that contains $\boldsymbol{R}$, and that is contained in any symmetric relation that contains $\boldsymbol{R}$. Consequently, it is the symmetric closure of R.

The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse
i.e., $R \cup R^{-1}$is the symmetric closure of R, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

Problem.1
Find the symmetric closure of the relation$\{(\boldsymbol{1}, \boldsymbol{1}), (\boldsymbol{1}, \boldsymbol{2}), (\boldsymbol{2}, \boldsymbol{2}), (\boldsymbol{2}, \boldsymbol{3}), (\boldsymbol{3}, \boldsymbol{1}), (\boldsymbol{3}, \boldsymbol{2})\}on\{\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}\}$.

Solution.

$$R^{-1} = \{(1,1), (2,1), (2,2), (3,2), (1,3), (2,3)\}$$

Therefore, symmetric closure of R is

$$S = R \cup R^{-1} = \{(\mathbf{1},\mathbf{1}),(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{2}),(\mathbf{2},\mathbf{3}),(\mathbf{3},\mathbf{2}),(\mathbf{3},\mathbf{1}),(\mathbf{1},\mathbf{3})\}$$

Problem.2

What is the symmetric closure of the relation $\boldsymbol{R} = \{(\boldsymbol{a},\boldsymbol{b}) \mid \boldsymbol{a} > \boldsymbol{b}\}$ on the set of positive integers?

Solution:

$$\boldsymbol{R}^{-1} = \{(\boldsymbol{b},\boldsymbol{a}) \mid \boldsymbol{a} > \boldsymbol{b}\} = \{(\boldsymbol{a},\boldsymbol{b}) | \boldsymbol{b} > \boldsymbol{a}\} = \{(\boldsymbol{a},\boldsymbol{b}) | \boldsymbol{a} < \boldsymbol{b}\}$$

The symmetric closure of R is the relation

$$\boldsymbol{R} \cup \boldsymbol{R}^{-1} = \{(\boldsymbol{a},\boldsymbol{b}) | \boldsymbol{a} > \boldsymbol{b}\} \cup \{(\boldsymbol{a},\boldsymbol{b}) | \boldsymbol{a} < \boldsymbol{b}\} = \{(\boldsymbol{a},\boldsymbol{b}) | \boldsymbol{a} \neq \boldsymbol{b}\}$$

<u>Transitive closure of R</u>

Suppose that a relation R is not transitive.

Let M_R be the zero–one matrix of the relation R on a set with n elements.

Let R^* be the transitive closure of R.

Then the zero–one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_{R^{[2]}} \vee M_{R^{[3]}} \vee \cdots \vee M_{R^{[n]}} .$$

Here R* is known as connectivity relation.

Problem.1

Find the zero–one matrix of the transitive closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

The zero–one matrix of $M_R$is $M_{R^*} = M_R \vee M_{R^{[2]}} \vee M_{R^{[3]}}$.

Now, $\boldsymbol{M}_{\boldsymbol{R}^{[2]}}$ **is the matrix of the composite relation** $\boldsymbol{R} \circ \boldsymbol{R}$.

$$\Rightarrow \boldsymbol{M}_{\boldsymbol{R}^{[2]}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and $\boldsymbol{M}_{\boldsymbol{R}^{[3]}}$ **is the matrix of the composite relation** $\boldsymbol{R} \circ \boldsymbol{R}^{\mathbf{2}}$

$$\Rightarrow \boldsymbol{M}_{\boldsymbol{R}^{[3]}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence,

$$M_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise:

1. Let R be the relation on the set {0,1,2,3} containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0). Find a) reflexive closure of R.

b) symmetric closure of R.

DATABASE & RELATIONS

Concepts of relations have a strong application in the theory of relational databases.

Definition:

Let $A1, A2, \dots, An$ be sets. An ***n-ary relation*** on these sets is a subset of $A1 \times A2 \times \dots \times An$. The sets $A1, A2, \dots, An$ are called the *domains* of the relation, and *n* is called its *degree*.

For example:

Let R be the relation on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ consisting of triples (a, b, c), where $a, b, and\ c$ are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

For example:

Let R be the relation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ consisting of all triples of integers (a, b, c) in which $a, b, and\ c$ form an arithmetic progression. That is, $(a, b, c) \in R$ if and only if there is an integer k such that $b = a + k$ and $c = a + 2k$, or equivalently, such that $b - a = k$ and $c - b = k$.

Note that $(1, 3, 5) \in R$ because $3 = 1 + 2$ and $5 = 1 + 2 \cdot 2$, but $(2, 5, 9) \notin R$ because $5 - 2 = 3$ while $9 - 5 = 4$. This relation has degree 3 and its domains are all equal to the set of integers.

For example:

Let R be the relation consisting of $5 - tuples$ (A, N, S, D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time.

For instance, if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) belongs to R.

The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

Database & relations:

The time required to manipulate information in a database depends on how this information is stored. The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database. Because of the importance of these operations, various methods for representing databases have been developed. We will discuss one of these methods, called the **relational data model**, based on the concept of a relation.

A database consists of **records**, which are *n*-tuples, made up of **fields**. The fields are the entries of the *n*-tuples. For instance, a database of student records may be made up of fields containing the name, student number, major, and grade point average of the student. The relational data model represents a database of records as an *n*-ary relation.

Definition: (Selection operator)

Let R be an $n - ary$ relation and C be a condition that elements in R may satisfy. Then the **selection operator** S_c maps the $n - ary$ relation R to the $n - ary$ relations of all n-tuples from R that satisfy the condition C.

Definition: (Projection operator*)*

The *projection* $P_{i_1 i_2, \dots, i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the *n*-tuple $(a_1, a_2, \dots, a_n)$ to the *m*-tuple $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$, where $m \le n$.

In other words, the projection $P_{i_1 i_2, \dots, i_m}$ deletes $n - m$ of the components of an *n*-tuple, leaving the $i_1^{th}, i_2^{th}, \dots$, and i_m^{th} components.

For example:

Consider the student records given by the following table.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

These student records can be given using $4-tuple$ of the form $(Student\quad name, ID\quad number, Major, GPA)$.
A sample database of six such records is

$$(Ackermann, 231455, Computer\ Science, 3.88)$$
$$(Adams, 888323, Physics, 3.45)$$
$$(Chou, 102147, Computer\ Science, 3.49)$$
$$(Goodfriend, 453876, Mathematics, 3.45)$$
$$(Rao, 678543, Mathematics, 3.90)$$
$$(Stevens, 786576, Psychology, 2.99).$$

To find the records of computer science majors in the $n-ary$ relation R shown in the above table, we use the operator S_{c_1}, where c_1 is the condition $Major = \text{“}Computer\ Science\text{”}$ The result is the two 4-tuples $(Ackermann, 231455, Computer\ Science, 3.88)$ and $(Chou, 102147, Computer\ Science, 3.49)$.
Similarly, to find the records of students who have a grade point average above 3.5 in this database, we use the operator S_{c_2}, where c_2 is the condition $GPA > 3.5$. The result is the two 4-tuples $(Ackermann, 231455, Computer\ Science, 3.88)$ and $(Rao, 678543, Mathematics, 3.90)$.
Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator S_{c_3}, where C_3 is the condition $(Major = \text{“}Computer\ Science\text{”} \wedge GPA > 3.5)$. The result consists of the single 4-tuple $(Ackermann, 231455, Computer\ Science, 3.88)$.
When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and grade point averages are obtained. The following table displays the results of this projection.

Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

Definition: (Join operator)
Let R be a relation of degree m and S be a relation of degree n. The ***join*** $J_P(R,S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m+n-p$ that consists of all $(m+n-p)-tuples$ $\left(a_1,a_2,\dots a_{m-p},c_1,c_2,\dots c_p,b_1,b_2,\dots,b_{n-p}\right)$, where the m-tuple $(a_1,a_2,\dots a_{m-p},c_1,c_2,\dots c_p)$ belongs to R and the n-tuple $\left(c_1,c_2,\dots c_p,b_1,b_2,\dots,b_{n-p}\right)$ belongs to S.
In other words, the join operator J_P produces a new relation from two relations by combining all m-tuples of the first relation with all n-tuples of the second relation, where the last p components of the m-tuples agree with the first p components of the n-tuples
For example: What relation results when the join operator $J2$ is used to combine the relation displayed in the following tables?

Teaching_assignments.

Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

Class_schedule.

Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

The join J_2 produces a member of relation as $(Cruz, Zoology, 335, A100, 9{:}00A.M.)$ by joining the members $(Cruz, Zoology, 335)$ and $(Zoology, 335, A100, 9{:}00A.M.)$
The relation thus produced is shown in the following table.

Teaching_schedule.

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

Exercise:

1. Consider the following Tables.
   a. What do you obtain when you apply the selection operator S_C, where C is the condition $(Project = 2) \wedge (Quantity \geq 50)$, to the database in the table of Parts_inventory.?
   b. Construct the table obtained by applying the join operator $J2$ to the relations in the following tables

Part_needs.		
Supplier	*Part_number*	*Project*
23	1092	1
23	1101	3
23	9048	4
31	4975	3
31	3477	2
32	6984	4
32	9191	2
33	1001	1

Parts_inventory.			
Part_number	*Project*	*Quantity*	*Color_code*
1001	1	14	8
1092	1	2	2
1101	3	1	1
3477	2	25	2
4975	3	6	2
6984	4	10	1
9048	4	12	2
9191	2	80	4

CANTOR'S DIAGONAL ARGUMENT

Finite and infinite sets:

Let *S* be a set. If there are exactly *n* distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is a *finite set* and that *n* is the *cardinality* of *S*. The cardinality of *S* is denoted by |*S*|.

A set is said to be *infinite* if it is not finite.

Countable and uncountable sets:

A set that is either finite or has the same cardinality as the set of positive integers is called *countable.*

A set that is not countable is called *uncountable.*

Note:

Cantor's diagonal argument

A set S is finite iff there is a bijection between S and $\{1, 2, \dots, n\}$ for some positive integer n, and infinite otherwise. (i.e., if it makes sense to count its elements.)

Two sets have the same cardinality iff there is a bijection between them.

A set S is called countably infinite if there is a bijection between S and $\mathbb{N}$. Such a set is countable because elements can be counted, but unlike a finite set, counting never ends.

On the other hand, not all infinite sets are countably infinite. In fact, there are infinitely many sizes of infinite sets.

Georg Cantor proved this astonishing fact in 1895 by showing that the set of real numbers is not countable. That is, it is impossible to construct a bijection between $\mathbb{N}$ and $\mathbb{R}$. In fact, it's impossible to construct a bijection between $\mathbb{N}$ and the interval $[0, 1]$.

Theorem:

The set of real numbers is not countable

Proof:

Suppose that $f : \mathbb{N} \to [0,1]$ is any function.

Make a table of values of f, where the 1st row contains the decimal expansion of $f(1)$, the 2nd row contains the decimal expansion of $f(2)$, . . . the nth row contains the decimal expansion of $f(n)$, …

Perhaps, $f(1) = \pi/10$, $f(2) = 37/99, f(3) = 1/7, f(4) = \sqrt{2}/2, f(5) = 3/8$, and so on, so that the table starts out like this.

n	$f(n)$											
1	0	.	3	1	4	1	5	9	2	6	5	3	…
2	0	.	3	7	3	7	3	7	3	7	3	7	…
3	0	.	1	4	2	8	5	7	1	4	2	8	…
4	0	.	7	0	7	1	0	6	7	8	1	1	…
5	0	.	3	7	5	0	0	0	0	0	0	0	…
⋮	⋮												

Highlighting the digits in the main diagonal of the table.

n	$f(n)$												
1	0	.	**3**	1	4	1	5	9	2	6	5	3	…
2	0	.	3	**7**	3	7	3	7	3	7	3	7	…
3	0	.	1	4	**2**	8	5	7	1	4	2	8	…
4	0	.	7	0	7	**1**	0	6	7	8	1	1	…
5	0	.	3	7	5	0	**0**	0	0	0	0	0	…
⋮	⋮												

The highlighted digits are 0.37210 . . . . Suppose that we add 1 to each of these digits, to get the number

0.48321 . . . . then this number can't be in the table. Because

- it differs from f(1) in its first digit;
- it differs from f(2) in its second digit;
- . . .
- it differs from f(n) in its nth digit;
- . . .

So it can't equal f(n) for any n — that is, it can't appear in the table
This looks like a trick, but in fact there are lots of numbers that are not in the table.
As long as we highlight at least one digit in each row and at most one digit in each column, we can change each the digits to get another number not in the table.
Therefore, there does not exist a bijection between ℕ and $[0, 1]$.
Hence, [0,1] is not a countable set.
Since, cardinality of ℝ and [0,1] is same, ℝ is also uncountable.

The Power set theorem
Statement: For every set S, |S| < |P(S)|
Proof:
Let $f: S \to P(S)$ be any function and define $X = \{ s \in S \mid s \notin f(s)\}$.
Suppose that $X = f(s)$ for some $s \in S$
If so, then either s belongs to X or it doesn't.
But by the very definition of X, if s belongs to X then it doesn't belong to $X = f(s)$.
And if it doesn't belong to X then it belong to $X = f(s)$.
This situation is impossible.
Hence,X cannot equal $f(s)$ for any s.
Using the Cantor's diagonal argument, this proves that f cannot be onto.|
Hence, $|S| \neq |P(S)|$. Which gives |S| < |P(S)|.

Schroder-Bernstein Theorem
If *A* and *B* are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions *f* from *A* to *B* and *g* from *B* to *A*, then there is a one-to-one correspondence between *A* and *B*.