Factorial Models, Kronecker Products, and Interaction Terms

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Interaction Terms and Factorial Models

We may consider a model that includes only the mean of the response as a zero-order model, sometimes called an intercept-only model. A model with means of several categories, parameterized as a mean and offsets to that mean, is a model with multiple intercepts (only), and is also a zero-order model. Classical ANOVA models are zero-order models.

A model of a response with a continuous variable includes both an intercept and a slope. This is a first-order model, with a zero-order term and a first-order term. Adding categorical variables to the model adds more intercepts, or zero-order terms. Adding continuous variables adds more slopes, or first-order terms. Such additive models are all first-order models.

An interaction term is formed as the product of two variables. A product of categorical variables adds intercepts to the model. The interaction of a categorical variable and a continuous variable adds slopes to the model. In either case, the order of the model remains the same. But the interaction formed from the product of two continuous variables adds a second-order term to a model, a curvature.

A factorial model is formed by adding all the products of all the zero- and first-order terms, in all combinations. If we think of the terms in a model as its column space, then any linear model resides in a subspace of the factorial column space. The columns of any linear model are a subset of the columns in a full-factorial model.

An Example, Two Variables

Consider three models

- $y = b_0 + b_1 x$, also expressed in matrix form as $y = Xb$
- $y = c_0 + c_1 z$, or $y = Zc$
- $y = d_0 + d_1 x + d_2 z + d_3 xz$, or $y = Ad$

The column space a A may be formed as an outer product of the column spaces of X and Z. Denote $X = \begin{bmatrix} 1_x & x \end{bmatrix}$ and $Z = \begin{bmatrix} 1_z & z \end{bmatrix}$. Then

$$X \otimes Z = \begin{bmatrix} 1_x & x \end{bmatrix} \otimes \begin{bmatrix} 1_z & z \end{bmatrix}$$

This operation produces a matrix with the correct column space of A.

$$\begin{bmatrix} 1_x 1_z & 1_x z & x 1_z & xz \end{bmatrix} = \begin{bmatrix} 1 & z & x & xz \end{bmatrix}$$

(However, our row space is a subset of the outer product of the rows, which are also produced by any Kronecker operation. So the model matrix for a model with interaction terms is a submatrix of the Kronecker product(s) of the zero- and first-order terms. The full Kronecker product becomes useful in the design of experiments, see Vartak.)