

Mathematical Foundation of Computer Science

MCA-I Semester

Assignment - II

① (a) Explain Division Algorithm.

If there exist an integer, such that $x = yn$ where $x, y \in \mathbb{Z}$ and $y \neq 0$, then it can be said that y divides x .
(i.e., $y|x$)

Division Algorithm: Let x, y (where $y > 0$) be two integers and let q (the quotient) and p (the remainder) $\in \mathbb{Z}$ such that $x = qy + p$, $0 \leq p < y$. Further more q and p are unique.

Proof: If y is a divisor of x (i.e., $y = xn$), then we say that y is divisible by x and so there main $\text{dop} = 0$.

Let us assume that y is not the divisor of x

$$\text{i.e., } y \neq xn \Rightarrow y \nmid x$$

Let $T = \{x - uy \mid u \in \mathbb{Z}, x - uy > 0\}$. If $x > 0$ and $u = 0$ then $x \in T$. T is non empty set and a subset of positive integers of $x \geq 0$.

$$\text{Let } u = x - 1$$

$$\begin{aligned}
 x - uy &= x - (x-1)y \\
 &= x - xy + y \\
 &= x(1-y) + y \text{ with } 1-y \leq 0 \\
 y &\geq 1
 \end{aligned}$$

Therefore, $x - uy > 0$ and $T \neq \emptyset$

From the definition of well ordering principle, T is a non-empty set consisting of smallest element p

$$\Rightarrow p = x - qy \text{ [for some } q \in \mathbb{Z} \text{ and } p > 0]$$

If $p = y$ then

$$y = x - qy$$

$$y + qy = x$$

$$y(q+1) = x$$

$$\Rightarrow y \mid x$$

Therefore, our assumption that $y \nmid x$ is false.

If $p > y$ then $p = y + t$ for some $t \in \mathbb{Z}^*$

$$x - qy = p = y + t$$

$$x - qy = y + t$$

$$x - qy - y = t$$



$$t = n - y(q+1) \in T$$

p is not the smallest element of T .

Therefore, $p < y$

Let $q_1, q_2, p_1, p_2 \in \mathbb{Z}$ such that,

$$n = q_1 y + p_1, 0 \leq p_1 < y$$

$$n = q_2 y + p_2, 0 \leq p_2 < y$$

Equating equations.

$$q_1 y + p_1 = q_2 y + p_2$$

$$q_1 y - q_2 y = p_2 - p_1$$

$$(q_1 - q_2) y = (p_2 - p_1)$$

$(q_1 - q_2)$ is an integral multiple of y .

$$p_2 - p_1 < y$$

If $q_1 \neq q_2$ then $y(q_1 - q_2) > y$

Therefore $q_1 = q_2, p_1 = p_2$

$\Rightarrow q$ and p are unique.

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(b) Explain pigeonhole Principle and Give an Example.

Pigeonhole principle :

Pigeonhole principle is one of the fundamental principles of counting. It states that, If 'q' pigeonholes are occupied by 'p' pigeons such that $p > q$ then at least one pigeonhole contains two or more pigeons.

If there are 11 pigeons occupying 8 pigeonhole then at least one pigeonhole contains more than one pigeon.

If p pigeons occupy q pigeonholes for $p > q$ then at least one pigeonhole contains $(m+1)$ or more pigeons where 'm' is the number of pigeons in each hole

$$\text{i.e., } m = p-1/q.$$

This principle is called generalized principle

Proof: Let us come contrary to the given statement that Pigeon hole contains $(m+1)$ or more pigeons

\Rightarrow every pigeonhole contains 'm' pigeons

\Rightarrow total number of pigeons $\leq qm$

$$qm = q \times (p-1)/q = p-1.$$



which is false since total number of pigeons = p
therefore, our assumption that every pigeonhole contains m pigeon is false.

Hence, when $p > q$, at least one pigeon hole contains $(m+1)$ pigeons.

Example: Prove that in a set of 16 children at least two have birthdays during the same month.

Sol: There are 12 months $\rightarrow q$

given total number of children = 16 $\rightarrow p$

therefore, by generalized principle, $m = \frac{p-1}{q} + 1$
 $= \frac{16-1}{12} + 1$

$$\Rightarrow 15 + 12/12 = 27/12 \approx 2.25 \approx 2.$$

\therefore At least two children have their birthday in the same month.

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② What is an Equivalence Relation?

Let $X = \{1, 2, 3, \dots, 73\}$ and $R = \{(x, y) \mid x - y \text{ is divisible by } 3\}$.

Show that R is an Equivalence Relation. Draw the graph of R .

Equivalence Relation :

A relation R on a set A is said to be an equivalence relation if and only if the relation R is reflexive, symmetric and transitive. The equivalence relation is a relationship on the set which is generally represented by the symbol " \sim ".

Reflexive: A relation is said to be reflexive, if $(a, a) \in R$, for every $a \in A$.

Symmetric: A relation is said to be symmetric, if $(a, b) \in R$, then $(b, a) \in R$.

Transitive: A relation is said to be transitive if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

In terms of equivalence relation notation, it is defined as follows.

A binary relation \sim on a set A is said to be an equivalence relation, if and only if it is reflexive, symmetric and transitive.

Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and

$R = \{(x, y) \mid x - y \text{ is divisible by } 3\}$

$4 - 1 = 3$ is divisible by 3

$5 - 2 = 3$ is divisible by 3

$6 - 3 = 3$ is divisible by 3

$7 - 4 = 3$ is divisible by 3.

And vice versa.

Also, $1 - 1 = 0$ is divisible by 3

$2 - 2 = 0$ is divisible by 3

and so on.

$R = \{(4, 1), (5, 2), (6, 3), (7, 4), (1, 4), (2, 5), (3, 6), (4, 7),$

$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)\}$.

Reflexive: clearly $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6),$
 $(7, 7)\}$.

so, $\{(a, a) \in R, \forall a \in A\}$.

Hence, it is reflexive.

Symmetric: $\{(1, 4), (4, 1), (2, 5), (5, 2), (3, 6), (6, 3),$

$(4, 7), (7, 4)\}$

so, $\{(a, b) \in R \Rightarrow (b, a) \in R, \forall a \in X\}$.

Hence it is symmetric.

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Transitive : clearly

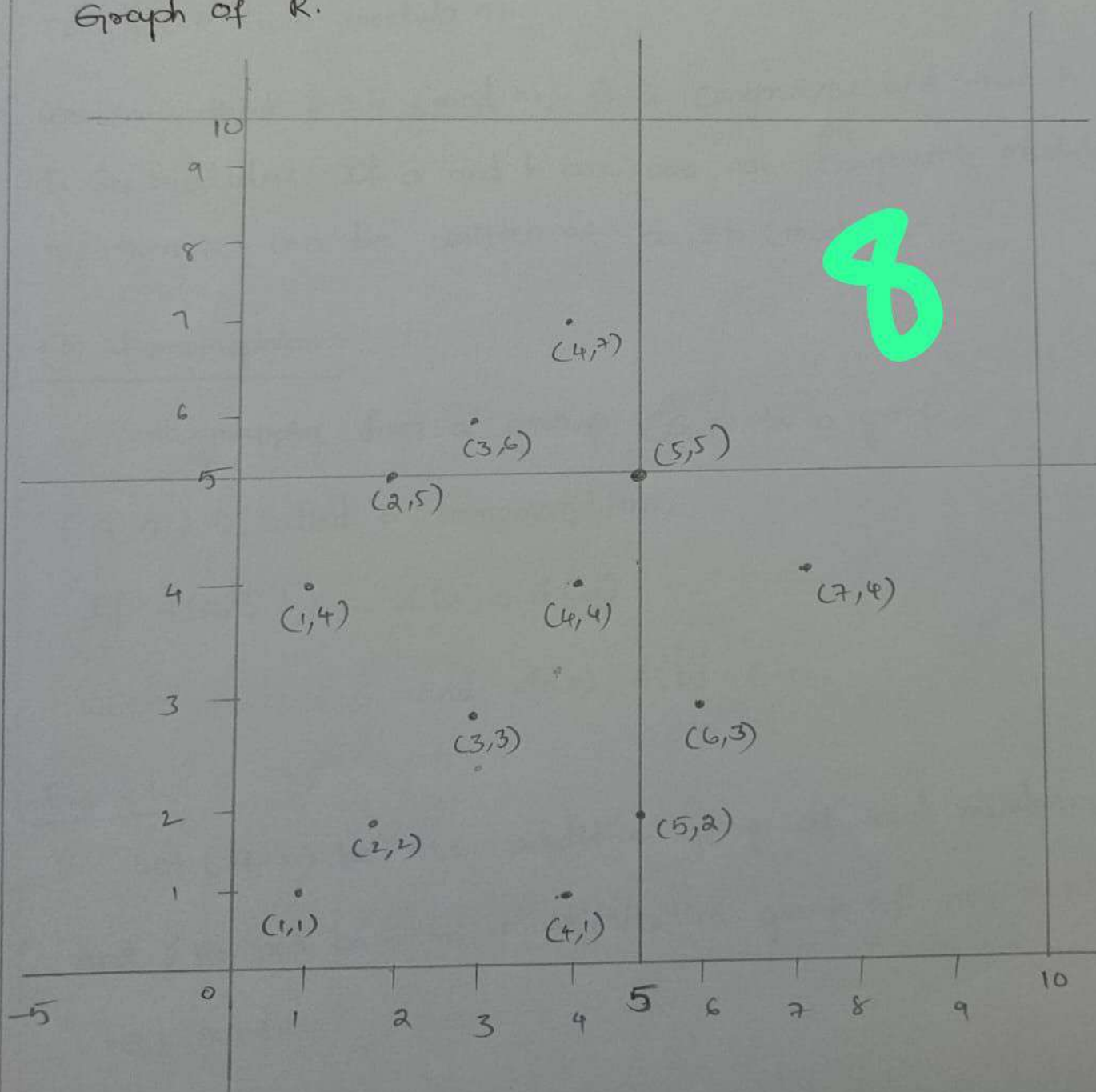
$\Sigma (1,4), (4,1), (1,1), (2,5), (5,2), (2,2), (3,6), (6,3), (3,3),$
 $(4,7), (7,4), (4,4) \}$

so, $\Sigma (a,b) \in R, \Sigma (b,c) \in R \Rightarrow (a,c) \in R, \forall a \in X \}$.

Hence it is transitive.

Thus the given relation is an equivalence relation.

Graph of R.



- ③ Explain the terms (a) Residue Arithmetic
(b) Homomorphism.

(a) Residue Arithmetic:

If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a - b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m .

We say that $a \equiv b \pmod{m}$ is a congruence and that m is its modulus. If a and b are not congruent modulo m , then it can be written as $a \not\equiv b \pmod{m}$.

(b) Homomorphism:

A mapping from a group (G, \circ) to a group (G', \circ') is called a homomorphism.

$$\phi(a \circ b) = \phi(a) \circ' \phi(b)$$

where $a, b \in G$ and $\phi(a), \phi(b) \in G'$.

Example:

Let $(\mathbb{R}, +)$ be the additive group of real numbers, and (\mathbb{R}_0, \times) be the multiplicative group of non-zero real numbers.

The mapping $(R, +) \rightarrow (R_0, \times)$;

$$f(x) = 2^x \text{ for every } x \in R,$$

is a homomorphism of R into R_0 because for any $x_1, x_2 \in R$.

$$f(x_1 + x_2) = 2^{x_1 + x_2} = 2^{x_1} \cdot 2^{x_2} = f(x_1) \cdot f(x_2)$$

Various Homomorphism

i) monomorphism, if f is injection (one to one)

ii) Epimorphism, if f is surjection (onto).

Here G' is called the homomorphism group of image G .

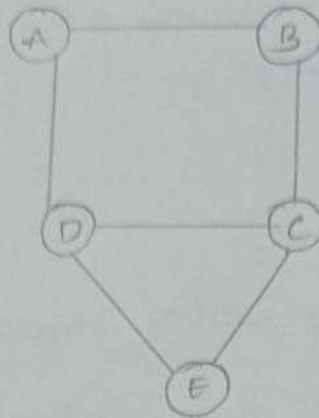
④ (a) What is Hamiltonian path and Cycle in graph?
Give an example.

Hamiltonian Path:

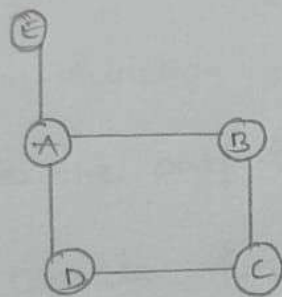
If there exists a walk in the connected graph that visits every vertex of the graph exactly once without repeating the edges. then such walk is called as Hamiltonian path.

If there exists a path in the connected graph that contains all the vertices of the graph, then such a path is called as a Hamiltonian path.

Examples:



Hamiltonian Path = ABCDE



Hamiltonian Path is EABCD.

Hamiltonian Cycle:

A Hamiltonian cycle, also called a Hamiltonian circuit, Hamilton cycle, or Hamilton circuit, is a graph cycle (i.e., closed loop) through a graph that visits each node exactly once. A graph possessing a Hamiltonian cycle is said to be a Hamiltonian graph.

(b) Bijective Function:

A function is said to be bijective or bijection, if a function $f: A \rightarrow B$ satisfies both the injective (one-to-one) function and surjective function (onto function) properties. It means that every element 'b' in the codomain B, there is exactly one element 'a' in the domain A such that $f(a) = b$. If the function satisfies this condition, then it is known as one-to-one correspondence.

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⑤ (a) Define Spanning Tree.

A spanning tree is a sub-graph of an undirected connected graph, which includes all the vertices of the graph with a minimum possible number of edges. If a vertex is missed, then it is not a spanning tree.

The edges may or may not have weights assigned to them. The total number of spanning trees with n vertices that can be created from a complete graph is equal to

$$n^{(n-2)}$$

If we have $n=4$, the maximum number of possible spanning trees is equal to $4^{4-2} = 16$. Thus, 16 spanning trees can be formed from a complete graph with 4 vertices.

(b) Define the following terms

(a) Inverse Function

(b) Bijective Function.

(a) Inverse Function:

A function accepts values, performs particular operations on these values and generates an output. The inverse function agrees with the resultant, operates and reaches back to the original function.

The inverse function returns the original value for which a function gave the output.

If you consider functions f and g are inverse, $f(g(x)) = g(f(x)) = x$. A function that consists of its inverse fetches the original value.

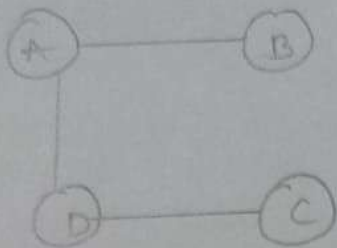
Example: $f(x) = 2x + 5 = y$

Then, $g(y) = (y - 5) / 2 = x$ is the inverse of $f(x)$.

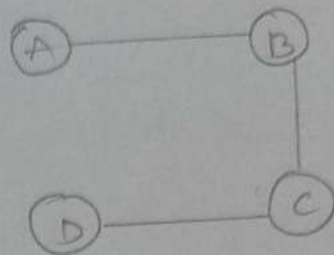
Note: The relation, developed when the independent variable is interchanged with the variable which is dependent on a specified equation and this inverse may or may not be a function.

If the inverse of a function itself, then it is known as inverse function, denoted by $f^{-1}(x)$.

Ex:



→ A spanning tree



→ A spanning tree.

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