Mathematical Foundations of Computer Science.

SECTION-A

① What is Binary Relation ? what are the properties of Binary Relations ? Give an Example for each.

Binary Relation :

Relation is defined in terms of ordered paires (x, y) of elements, where 'x' is the first element and 'y' is the second element. Let X, Y be two sets. A binary relation from X to Y is a subset of $X \times Y$

$$\Rightarrow X \times Y = \{(x,y) / x \in X, y \in Y\}$$

Properties of Binary Relation:

Some of the properties of binary relations are:

i) Reflexive, non-reflexive, irreflexive

ii) Symmetric, asymmetric, antisymmetric

iii) Transitive, nontransitive.

i) Reflexive Relations :

If R is any relations defined on any given set A then R is said to be reflexive if $\forall x \in A$ □a relation (x, x) such that $(x,x) \in R$ i.e., every element of A is in binary relation with itself and belong to R.

$$\text{i.e., } \{x \in A \rightarrow xRx \ (x,x) \in R\}$$

Example:

a) If the relation '$\leq$' is defined on the set of real numbers then the relation R satisfies reflexive property

(∵ for any real number $a \leq a \Rightarrow (a,a)$ □ R)

b) $A = \{a, b, c\}$ then $= \{(a,a), (b,b), (c,c), (a,c), (b,c)\}$

if R is defined such that for any x □ A, (x, x) does not belong to R then R is irreflexive relation.

ii) Symmetric relation:

If R is any relation defined on given set A, then □ (x,y) □ R if (y,x) □ R then the Relation R is said to be symmetric.

i.e., $R = \{(x,y)$ □ R □ x,y □ A $\Rightarrow (y,x)$ □ R) is a symmetric relation.

Example: If $A = (a,b,c)$, then

$R = \{(a,a), (a,b) (b,c) (a,c) (c,a) (b,c) (c,b)\}$ is a symmetric relation For any relation R defined on set A if

□ (x,y) □ R and $(y,x) = R$ the R is non-Symmetricor. or asymmetric relation.

2

Example

a) Relation "$<$" is defined on real numbers for any a,b □ real numbers if $a \leq b$, then the relation $b \leq a$ is not pocible.

Hence it is non-symmetric relation.

ii) $A = \{a, b, c\}$, then,

$R = \{(a,b)\ (a,c)\ (c,c)\ (b,c)\}$ is non-symmetric,

(Therefore (a,b) □ R but $(b,a) \notin R$)

If for any relation defined on set A if

□ (a,b) □ R and (b,a) □ R and $a=b$, then the relation is said to be antisymmetric.

$R = \{(x,y)$ □ R, (y,x) □ R □ $x, y = A \Rightarrow (x=y)\}$.

Transitive Relation :

If R is a relation defined on any set A such that if x, y, z □ A. then (x,y) □ R, (y,z) □ R implies (x,z) □ R, then R is the transitive relation.

Example:

i) The relations "$\leq$" and "$\geq$" defined on real numbers are transitive relations

ii) $A = \{a,b,c\}$ then $R = \{(a,b)\ (b,c)\ (a,c)\ (c,a)\ (b,c)\}$ is a transitive relation.

If for any x, y, z □ A and R is a Relation such that (x,y) □ R, (y,z) □ R and if $(x,z) = R$ then R is non-transitive relation.

Ex: $A = \{a, b, c\}$

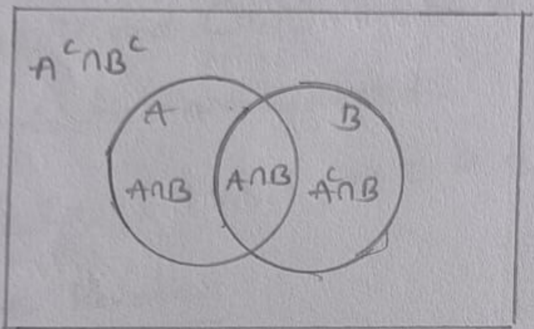
$R = \{(a,a)\ (a,b)\ (a,c)\}$ is non-transitive $(\because (b,c) \notin R)$

(4) (2) State and proof Principle of inclusion and Exclusion.

The sumrule which is applied to the non-disjoint sets is called priciple of inclusion-exclusion also called as 'Sieve method'.

1. Statement If A and B are two subsets of any set S (universal) then, $A \cup B = |A| + |B| - |A \cap B|$ ...... (1).

Proof :



4

Given, For any set S $A, B \leq S$

i.e., A and B are subsets of S

$|A \cup B|$ = All the elements of A + all the elements of B

$|A| = |A \cap B| + |A \cap B|$ (from Venn diagram) and

$|B| = |A' \cap B| + |A \cap B|$ (from Venn diagram)

$A \cap B$ Contains the elements which belong to both sets A and B

Therefore, $|A| + |B| = |A \cap B'| + |A' \cap B] + |A \cap B| + |A \cap B| = |A \cup B|$

Now from R.H.S of equation (1).

$|A| + |B| - |A \cap B| = |A \cap B'| + 2|A \cap B| + |A' \cap B| - |A \cap B|$

$= |A \cap B'| + |A' \cap B| + |A \cap B|$

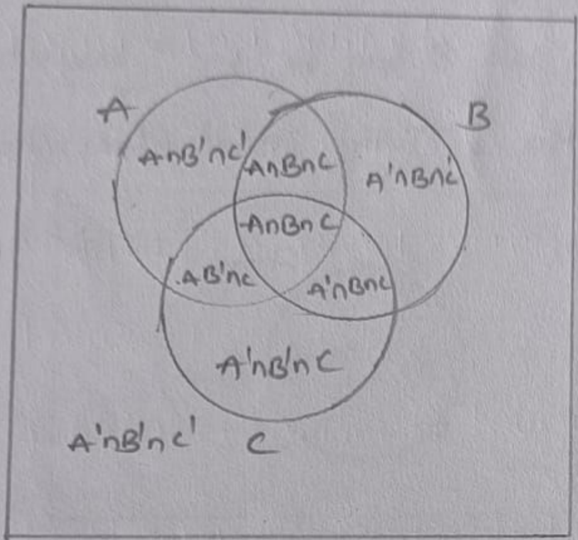
$= |A \cup B|$ (from Venn-diagram).

Therefore, $|A \cup B| = |A| + |B| - |A \cap B|$.

2. Statement:

If A, B, C are any three subsets of Set S, then $|A \cup B \cup C|$
$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.

Proof:



5

$|A| = |A \cap B' \cap C'| + |A \cap B \cap C'| + |A \cap C \cap B'| + |A \cap B \cap C|$ (from Venn-diagram)

$|B| = |B \cap A' \cap C'| + |A \cap B \cap C'| + |A' \cap B \cap C| + |A \cap B \cap C|$ (from Venn-diagram)

$|C| = |C \cap A' \cap B'| + |A' \cap B \cap C| + |A \cap C \cap B'| + |A \cap B \cap C|$.

Therefore, $|A| + |B| + |C| = |A \cap B' \cap C'| + |B \cap A' \cap C'| + |C \cap A' \cap B| +$
$|A \cap B \cap C'| + |A \cap C \cap B'| + |A' \cap B \cap C| + |A \cap B \cap C'| +$
$|A \cap C \cap B'| + |A' \cap B \cap C| + 3|A \cap B \cap C| \ldots\ldots$ (2)

Now from venn-diagram.

$|A| + |B| + |C| = |A \cup B \cup C| + |A \cap B \cap C'| + |A \cap B \cap C| + |A' \cap B \cap C| +$
$|A \cap B' \cap C| + |A \cap B \cap C|$.

Replacing in equation (2) we get

$|A|+|B|+|C| = |A \cup B \cup C| + |A \cap B \cap C'| + |A \cap B \cap C| + |A' \cap B \cap C| +$

$|A \cap B' \cap C| + |A \cap B \cap C|$

$|A| + |B| + |C| = |A \cup B \cup C| + |A \cap B \cap C'| + |A \cap B \cap C| + |A' \cap B \cap C| +$

$|A \cap B' \cap C| + |A \cap B' \cap C|$ ------ (3)

Adding $A \cap B \cap C$ on both sides of Eqn (3), we get

$|A| + |B| + |C| + |A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap B' \cap C| +$

$|A \cap B \cap C|$

$|A| + |B| + |C| + |A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap C|$

$[\because |A \cap B \cap C| + |A' \cap B \cap C| = |B \cap C|]$

$\therefore$ Hence proved.

(3) # what is Homogeneous Recurrence Relation? Give Example.

A recurrence relation of the form:

A recurrence relation of the form $a_{n+1} = da_n + f(n)$ $n >= 0$, where d is constant and $f(n)$ is a known as function is called linear recurrence relation of first order with constant coefficient.

If $f(n) = 0$, the relation is homogeneous otherwise non-homogeneous the solution to the above recurrence relation is given by $a_n = a_0 d^n$, $n \geq 0$ where $A = a_0$

or $[Ad^n = a_n]$

6

Example: $a_{n+1} = 4a_n$ for $n \geq 0$, given that $a_0 = 3$.

The above Recurrence Relation is $a_{n+1} = 4a_n$ —— ①

$$a_{n+1} - 4a_n = 0 \quad [\because f(n) = 0]$$

So the given recurrence relation is first order homogenous recurrence relation.

Compare the given RR $a_{n+1} = 4a_n$ with

$$a_n = c.a_{n-1} + f(n)$$

$$a_{n+1} = 4.a_{(n+1)-1} + 0$$

$$a_{n+1} = 4a_n + 0$$

$\therefore$ The general solution of 1^{st} order recurrence relation is

$$\boxed{a_n = c^n.a_0} \text{ —— ②}$$

If $a_n = c^n$ then $\boxed{a_{n+1} = c^{n+1}}$ —— ③

Substitute a_{n+1} value in ①

$$4a_n = c^{n+1} \quad [\because x^{n+y} = x^n.x^y]$$

$$= c^n.c$$

$$4(c^n) = c^n.c \quad \text{from — ③}$$

$$\boxed{c = 4}$$

Substitute c value in general solution

$$a_n = 4^n.a_0$$

③

from given question $\boxed{a_0 = 3}$

$\therefore \boxed{a_n = 3.4^n}$

8

③ ④ Explain the following terms (a) Semi-groups (b) monoids. (c) Groups.

Semi-groups : A semi group is a set A, together with a binary operation "." That satisfies the assosiative property.

For all a, b, c □ A, (a.b).c = a. (b.c) □ A.

Example: Consider the set o positive integers, apart from zero, with a binary operation of addition is said to be a semi group. Says S = {1, 2, 3, ... }. Find whether <S, +> is a Semi group. For every pair (a,b) □ S, (a+b) is present the set, S. and hence this holds true the clouser property.

Let 1, 2, □ S, 1+2 = 3 □ S.

For every element a,b □ S, (a+b)+c = (a+b) + c.

Let 1, 2, 3, ... □ S,

(1+2) + 3 = 1+ (2+3) = 5.

Monoids : If a semi group {M, *} has an identity element with respect to the operation *, then {M, *} is called a monoids. For Example, if N is the set of natural numbers, then {N, +} & {N, X} are monoids with the identity elements 0 and 1. respectively.... The semigroups {E, +} and {E, X} are not monoids.

(C) Groups:

A group $(G, *)$ is a set, together with an operation $*$ satisfying the following properties.

i> Closure :

For each $a, b \square G$, then $a * b \square G$.

ii> Associative property:

For each $a, b, c \square G$, then

$(a * b) * c = a * (b * c) \square G$.

iii> Identity: there exist an element $e \square G$, such that

$a * e = e * a = a$.

iv) Inverse property: for each $a \square G$, there is an element $a^{-} \square G$ such that $a * a^{-} = a^{-} * a = e$.

9

11

② Solve the Recurrence Relation.
⑤ $s(k) - 7s(k-1) + 12s(k-2) = 0$ where $s(0) = 4$, $s(1) = 4$.

Sol: The Given Recurrence Relation is

$$S_k - 7S_{k-1} + 12S_{k-2} = 0$$

10

the above Equation compare with

$$\boxed{C_n \cdot a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0}$$

Here $C_n = 1$, $C_{n-1} = -7$, $C_{n-2} = 12$.

Now the charecteristic Equation is

$$k^2 - 7k + 12 = 0. \qquad [\because C_n k^2 + C_{n-1} k + C_{n-2} = 0]$$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

from the above Equation $a = 1$, $b = -7$, $c = 12$.

$$k = \frac{+7 \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{49-48}}{2}$$

$$k = \frac{7 \pm 1}{2} \Rightarrow k = \frac{7+1}{2} = 8/2 = 4$$

$$k = \frac{7-1}{2} = 6/2 = 3$$

$\boxed{k_1 = 4}$ and $\boxed{k_2 = 3}$.

Here the roots are different and real the second order homogeneous recurrence relation is

$$a_n = A(k_1)^n + B(k_2)^n \text{ ——— (A)}$$

$$a_n = A(4)^n + B(3)^n \text{ ——— (1)}$$

11

from the given question

$s(0) = 4 \Rightarrow \quad s_0 = A(4)^0 + B(3)^0$

$$4 = A + B \text{ ——— (2)}$$

$s(1) = 4 \Rightarrow \quad s_1 = A(4)^1 + B(3)^1$

$$4 = 4A + 3B \text{ ——— (3)}$$

Solve $3 \times (2)$ and (3) Equations

$$12 = \cancel{3A} + 3B$$
$$4 = \cancel{3A} + 4B$$
$$8 = -B$$

$\therefore \boxed{B = -8}$

Substitute B value in Eqn (2) $\quad 4 = A - 8$

$$\boxed{A = 12}$$

A, B value Substitute in (A). Now the general solution of 2^{nd} order homogenous R.R is $\boxed{a_n = 12(4)^n - 8(3)^n}$.