Mathematical Foundations of Computer Science.

SECTION-A

① What is Binary Relation? what are the properties of Binary Relations? Give an Example for each.

Binary Relation:

Relation is defined in terms of ordered paires (x, y) of elements, where 'x' is the first element and 'y' is the second element. Let X, Y be two sets. A binary relation from X to Y is a subset of $X \times Y$

$$\Rightarrow X \times Y = \{(x, y) / x \in X, y \in Y\}$$

Properties of Binary Relation:

Some of the properties of binary relations are:

i) Reflexive, non-reflexive, irreflexive

ii) Symmetric, asymmetric, antisymmetric

iii) Transitive, nontransitive.

i) Reflexive Relations:

If R is any relations defined on any given set A then R is said to be reflexive if $\forall x \in A$ □a relation (x, x) such that $(x, x) \in R$ i.e., every element of A is in binary relation with itself and belong to R.

i.e., $\{x \in A \rightarrow xRx \ (x, x) \in R\}$

(a)

Example:

a) If the relation '$\leq$' is defined on the set of real numbers then the relation R satisfies reflexive property

(∵ for any real number $a \leq a \Rightarrow (a,a) \square R$)

b) $A = \{a, b, c\}$ then $= \{(a,a), (b,b), (c,c), (a,c), (b,c)\}$

if R is defined such that for any $x \square A$, (x, x) does not belong to R then R is irreflexive relation.

ii) Symmetric relation:

If R is any relation defined on given set A, then $\square (x,y) \square R$ if $(y,x) \square R$ then the Relation R is said to be symmetric.

i.e., $R = \{(x,y) \square R \square x,y \square A \Rightarrow (y,x) \square R\}$ is a symmetric relation.

Example: If $A = (a,b,c)$, then

$R = \{(a,a), (a,b) (b,c) (a,c) (c,a) (b,c) (c,b)\}$ is a symmetric relation for any relation R defined on set A if

$\square (x,y) \square R$ and $(y,x) = R$ the R is non-symmetric or asymmetric relation.

Example

a) Relation "$<$" is defined on real numbers for any $a,b \square$ real numbers if $a < b$, then the relation $b < a$ is not possible.

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Hence it is non-symmetric relation.

ii) $A = \{a, b, c\}$, then,

$R = \{(a,b)\ (a,c)\ (c,c)\ (b,c)\}$ is non-symmetric,

(Therefore (a,b) □ R but $(b,a) \notin R$)

If for any relation defined on set A if

□ (a,b) □ R and (b,a) □ R and $a=b$, then the relation is said to be antisymmetric.

$R = \{(x,y)$ □ R, (y,x) □ R □ $x, y = A \Rightarrow (x=y)\}$.

Transitive Relation:

If R is a relation defined on any set A such that if x, y, z □ A. then (x,y) □ R, (y,z) □ R implies (x,z) □ R, then R is the transitive relation.

Example:

i) The relations " $\leq$ " and " $\geq$ " defined on real numbers are transitive relations

ii) $A = \{a, b, c\}$ then $R = \{(a,b)\ (b,c)\ (a,c)\ (c,a)\ (b,c)\}$ is a transitive relation.

If for any x, y, z □ A and R is a Relation such that (x,y) □ R, (y,z) □ R and if $(x,z) = R$ then R is non-transitive relation.

Ex: $A = \{a, b, c\}$

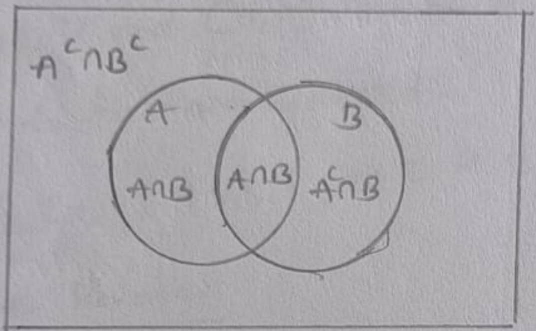
$R = \{(a,a)\ (a,b)\ (a,c)\}$ is non-transitive $(\because (b,c) \notin R)$

④ ② state and proof Principle of inclusion and Exclusion.

The sumrule which is applied to the non-disjoint sets is called priciple of inclusion-exclusion also called as 'Sieve method'.

1. Statement If A and B are two subsets of any set S(universal) then, $A \cup B = |A| + |B| - |A \cap B|$ ...... ①.

Proof :



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Given, For any set S $A, B \leq S$

i.e., A and B are subsets of S

$|A \cup B|$ = All the elements of A + all the elements of B

$|A| = |A \cap B| + |A \cap B|$ (from Venn diagram) and

$|B| = |A' \cap B| + |A \cap B|$ (from Venn diagram)

$A \cap B$ Contains the elements which belong to both sets A and B

Therefore, $|A| + |B| = |A \cap B'| + |A' \cap B] + |A \cap B| + |A \cap B| = |A \cup B|$

Now from R.H.S of equation (1).

$|A| + |B| - |A \cap B| = |A \cap B'| + 2|A \cap B| + |A' \cap B| - |A \cap B|$

$= |A \cap B'| + |A' \cap B| + |A \cap B|$

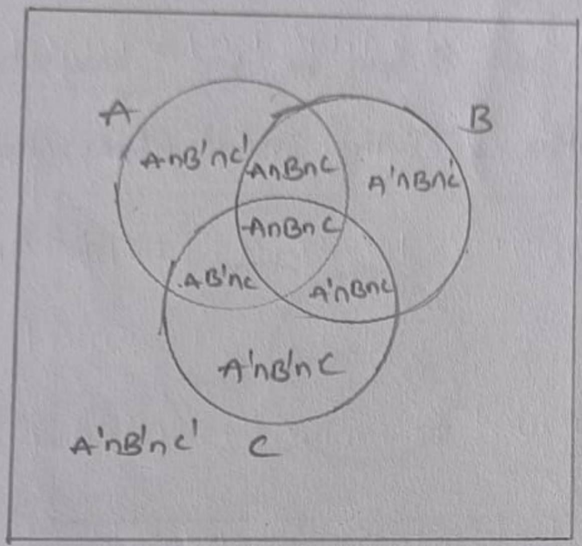
$= |A \cup B|$ (from Venn-diagram).

Therefore, $|A \cup B| = |A| + |B| - |A \cap B|$.

2. Statement:

If A, B, C are any three subsets of Set S, then $|A \cup B \cup C| = |A|+|B|+|C|-|A\cap B|-|B\cap C|-|C\cap A|+|A\cap B\cap C|$.

Proof:



$|A| = |A\cap B'\cap C'| + |A\cap B\cap C'| + |A\cap C\cap B'| + |A\cap B\cap C|$ (from Venn-diagram)

$|B| = |B\cap A'\cap C'| + |A\cap B\cap C'| + |A'\cap B\cap C| + |A\cap B\cap C|$ (from Venn-diagram)

$|C| = |C\cap A'\cap B'| + |A'\cap B\cap C| + |A\cap C\cap B'| + |A\cap B\cap C|$.

Therefore, $|A|+|B|+|C| = |A\cap B'\cap C'| + |B\cap A'\cap C'| + |C\cap A'\cap B| + |A\cap B\cap C'| + |A\cap C\cap B'| + |A'\cap B\cap C| + |A\cap B\cap C'| + |A\cap C\cap B'| + |A'\cap B\cap C| + 3|A\cap B\cap C|$ ...... (2)

Now from venn-diagram.

$|A|+|B|+|C| = |A\cup B\cup C| + |A\cap B\cap C'| + |A\cap B\cap C| + |A'\cap B\cap C| + |A\cap B'\cap C| + |A\cap B\cap C|$.

Replacing in equation (2) we get

$|A|+|B|+|C| = |A \cup B \cup C| + |A \cap B \cap C'| + |A \cap B \cap C| + |A' \cap B \cap C| +$

$|A \cap B' \cap C| + |A \cap B \cap C|$

$|A| + |B| + |C| = |A \cup B \cup C| + |A \cap B \cap C'| + |A \cap B \cap C| + |A' \cap B \cap C| +$

$|A \cap B' \cap C| + |A \cap B' \cap C|$ ...... (3)

Adding $A \cap B \cap C$ on both sides of Eqn (3), we get

$|A| + |B| + |C| + |A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap B' \cap C| +$

$|A \cap B \cap C|$

$|A| + |B| + |C| + |A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap C|$

$[\because |A \cap B \cap C| + |A' \cap B \cap C| = |B \cap C|]$

$\therefore$ Hence proved.

(3) # what is Homogeneous Recurrence Relation? Give Example.

A recurrence relation of the form:

$a_{n+1} = da_n + f(n)$ $n >= 0$, where d is constant and $f(n)$ is a known as function is called linear recurrence relation of first order with constant coefficient.

If $f(n) = 0$, the relation is homogeneous otherwise non-homogeneous the solution to the above recurrence relation is given by $a_n = a_0 d^n$, $n \geq 0$ where $A = a_0$

or $[Ad^n = a_n]$

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Example: $a_{n+1} = 4a_n$ for $n \geq 0$, given that $a_0 = 3$.

The above Recurrence Relation is $a_{n+1} = 4a_n$ —— ①

$$a_{n+1} - 4a_n = 0 \quad [\because f(n) = 0]$$

So the given recurrence relation is first order homogenous recurrence relation.

Compare the given RR $a_{n+1} = 4a_n$ with

$$a_n = c \cdot a_{n-1} + f(n)$$

$$a_{n+1} = 4 \cdot a_{(n+1)-1} + 0$$

$$a_{n+1} = 4a_n + 0$$

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$\therefore$ The general solution of 1^{st} order recurrence relation is

$$\boxed{a_n = c^n \cdot a_0} \quad \text{—— ②}$$

If $a_n = c^n$ then $\boxed{a_{n+1} = c^{n+1}}$ —— ③

Substitute a_{n+1} value in ①

$$4a_n = c^{n+1} \quad [\because x^{n+y} = x^n \cdot x^y]$$

$$= c^n \cdot c$$

$$4(c^n) = c^n \cdot c \quad \text{from — ③}$$

$$\boxed{c = 4}$$

Substitute c value in general solution

$$a_n = 4^n \cdot a_0$$

③ 8

from given question $\boxed{a_0 = 3}$

$\therefore \boxed{a_n = 3.4^n}$

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③ ④ Explain the following terms (a) Semi-groups (b) monoids. (c) Groups.

Semi-groups : A semi group is a set A, together with a binary operation "." That satisfies the assosiative property.

For all a, b, c □ A, (a.b).c = a. (b.c) □ A.

Example: Consider the set o positive integers, apart from zero, with a binary operation of addition is said to be a semi group. Find whether <S, + > is a semi group.

Says S = {1, 2, 3, ... }. For every pair (a,b) □ S, (a+b) is present the set, S and hence this holds true the clouske property.

Let 1, 2, □ S, 1+2 = 3 □ S.

For every element a,b □ S, (a+b)+c = (a+b)+c.

Let 1, 2, 3, ... □ S,

(1+2)+3 = 1+(2+3) = 5.

Monoids : If a semi group {M, *} has an identity element with respect to the operation *, then {M, *} is called a monoids. For Example, if N is the set of natural numbers, then {N, +} & {N, X} are monoids with the identity elements 0 and 1. respectively.... The semigroups {E, +} and {E, X} are not monoids.

(c) Groups:

A group $(G, *)$ is a set, together with an operation $*$ satisfying the following properties.

i) Clousure :

For each $a, b \square G$, then $a * b \square G$.

ii) Associative property:

For each $a, b, c \square G$, then

$(a*b)*c = a*(b*c) \square G$.

iii) Identity: there exist an element $e \square G$, such that

$a*e = e*a = a$.

iv) Inverse property : for each $a \square G$, there is an element $a^{-} \square G$ such that $a * a^{-} = a^{-} * a = e$.

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(2) Solve the Recurrence Relation.

(5) $s(k) - 7s(k-1) + 12s(k-2) = 0$ where $s(0) = 4$, $s(1) = 4$.

Sol: The Given Recurrence Relation is

$$S_k - 7S_{k-1} + 12S_{k-2} = 0$$

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the above Equation compare with

$$\boxed{C_n \cdot a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0}$$

Here $C_n = 1$, $C_{n-1} = -7$, $C_{n-2} = 12$.

Now the charecteristic Equation is

$$k^2 - 7k + 12 = 0. \qquad [\because C_n k^2 + C_{n-1} k + C_{n-2} = 0]$$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

from the above Equation $a = 1$, $b = -7$, $c = 12$.

$$k = \frac{+7 \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{49-48}}{2}$$

$$k = \frac{7 \pm 1}{2} \Rightarrow k = \frac{7+1}{2} = 8/2 = 4$$

$$k = \frac{7-1}{2} = 6/2 = 3$$

$\boxed{k_1 = 4}$ and $\boxed{k_2 = 3}$.

Here the roots are different and real the second order homogeneous recurrence relation is

$$a_n = A(k_1)^n + B(k_2)^n \text{ --- } (A)$$

$$a_n = A(4)^n + B(3)^n \text{ --- } (1)$$

from the given question

$s(0) = 4 \Rightarrow \quad s_0 = A(4)^0 + B(3)^0$

$4 = A + B$ --- (2)

$s(1) = 4 \Rightarrow \quad s_1 = A(4)^1 + B(3)^1$

$4 = 4A + 3B$ --- (3)

Solve $3 \times$ (2) and (3) Equations

$$12 = 3A + 3B$$
$$4 = 3A + 4B$$
$$8 = -B$$

$\therefore \boxed{B = -8}$

Substitute B value in Eqn (2) $\quad 4 = A - 8$

$\boxed{A = 12}$

A, B value Substitute in (A). Now the general solution of 2^{nd} order homogenous R.R is $\boxed{a_n = 12(4)^n - 8(3)^n}$.

Mathematical Foundation OF Computer Science

MCA – I Semister

Assignment – II

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① (a) Explain Division Algorithm.

If there exist an integer, such that $x = yn$ where $x, y \in Z$ and $y \neq 0$, then it can be said that y divides x. (i.e, $y|x$)

<u>Division Algorithm</u>: Let x, y (where $y > 0$) be two integers and let q (the quotient) and P (the remainder) $\in Z$ such that $x = qy + P$, $0 \leq P \leq y$ Further more q and P are unique.

<u>Proof</u>: If y is a divisor of x (i.e., $y = xn$), then we say that y is divisible by x and so there main drp $= 0$.

Let us assume that y is not the divisor of x

i.e., $y \neq xn \Rightarrow y|x$

Let $T = (x - uy / u \in Z, x - uy > 0)$, If $x > 0$ and $u = 0$ then $x \in T$ T is non empty set and a subset of positive integers of $x \leq 0$

Let $u = x - 1$

$$x - uy = x - (x-1)y$$

$$= x - xy + y$$

$$= x(1-y) + y \text{ with } 1-y \leq 0$$

$$y \geq 1$$

Therefore, $x - uy > 0$ and $T \neq \emptyset$

From the defination of well ordering principle, T is a non-empty set consisting of smallest element p

$\Rightarrow p = x - qy$ [for some $q \in \mathbb{Z}$ and $p > 0$]

If $p = y$ then

$$y = x - qy$$

$$y + qy = x$$

$$y(q+1) = x$$

$$\Rightarrow y \mid x$$

Therefore, our assumption that y/x is false.

If $p > y$ then $p = y + t$ for some $t \in \mathbb{Z}^*$

$$x - qy = p = y + t$$

$$x - qy = y + t$$

$$x - qy - y = t$$

$t = x - y(q+1) \in T$

p is not the smallest element of T.

Therefore, $p < y$

Let $q_1, q_2, p_1, p_2 \in \mathbb{Z}$ such that,

$x = q_1 y + p_1, 0 \leq p_1 < y$

$x = q_2 y + p_2, 0 \leq p_2 \leq y$

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Equating equations.

$q_1 y + p_1 = q_2 y + p_2$

$q_1 y - q_2 y = p_2 - p_1$

$(q_1 - q_2) y = (p_2 - p_1)$

$(q_1 - q_2)$ is an integral multiple of y.

$p_2 - p_1 < y$

If $q_1 = q_2$ then $y(q_1 - q_2) > y$

Therefore $q_1 = q_2$, $p_1 = p_2$

$\Rightarrow$ q and p are unique.

(b) Explain pigeonhole Principle and Give an Example.

<u>Pigeonhole principle :</u>

Pigeonhole principle is one of the fundamental principles of counting. It states that, If 'q' pigeonholes are occupied by 'p' pigeons such that $p > q$ then at least one pigeonhole contains two or more pigeons.

If there are 11 pigeons occupying 8 pigeonhole then at least one pigeonhole contains more than one pigeon.

If p pigeons occupy q pigeonholes for $p > q$ then at least one pigeonhole contains $(m+1)$ or more pigeons where 'm' is the number of pigeons in each hole

i.e., $m = p-1/q$.

This principle is called generalized principle

<u>Proof:</u> Let us come contrary to the given statement that pigeon hole contains $(m+1)$ or more pigeons

$\Rightarrow$ every pigeonhole contains 'm' pigeons

$\Rightarrow$ total number of pigeons $\leq qm$

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$qm = q \times (p-1)/q = p-1$.

which is false since total number of pigeons = p

therefore, our assumption that every pigeonhole contains m pigeon is false.

Hence, when $p > q$, at least one pigeon hole contains $(m+1)$ pigeons.

<u>Example:</u> Prove that in a set of 16 children at least two have birthdays during the same month.

<u>Sol:</u> There are 12 months $\rightarrow q$

given total number of children = 16 $\rightarrow p$

therfore, by generalized principle, $m = \frac{p-1}{q} + 1$

$$= \frac{16-1}{12} + 1$$

$$\Rightarrow 15 + 12/12 = 27/12 \cong 2.25 \cong 2.$$

$\therefore$ At least two children have there birthday in the same month.

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(2) What is an Equivalence Relation?

Let $X = \{1, 2, 3, \ldots 7\}$ and $R = \{(x, y) \mid x - y$ is divisible by $3\}$

Show that R is an equivalance Relation. Draw the graph of R.

<u>Equivalanence Relation :</u>

A relation R on a set A is said to be an equivalence relation if and only if the relation R is reflexive, symmetric and transitive. The equivalence relation is a relationship on the set which is generally represented by the symbol "~".

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Reflexive: A relation is said to be an reflexive, if

$(a, a) \in R$, for every $a \in A$.

Symmetric: A relation is said to be symmetric, if

$(a, b) \in R$, then $(b, a) \in R$.

Transitive: A relation is said to be transitive if

$(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

In terms of equivalence relation notation, it is defined as follows.

A binary relation ~ on a set A is said to be an equivalance relation, if and only if it is reflexive, symmetric and transitive.

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Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and

$R = \{(x,y) \mid x-y$ is divisible by $3\}$

$4-1 = 3$ is divisible by 3

$5-2 = 3$ is divisible by 3

$6-3 = 3$ is divisible by 3

$7-4 = 3$ is divisible by 3.

And vice versa.

Also, $1-1=0$ is divisible by 3

$2-2 = 0$ is divisble by 3

and so on.

$R = \{(4,1), (5-2), (6-3), (7-4), (1-4), (2,5), (3,6), (4,7),$
$(1,1), (2,2), (3,3), (4-4), (5-5), (6,6), (7,7)\}.$

Reflexive: clearly $\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6),$
$(7,7)\}.$

So, $\{(a,a) \in R, \forall\, a \in A\}.$

Hence, it is reflexive.

Symmetric: $\{(1,4), (4,1), (2,5), (5,2), (3,6), (6,3),$
$(4,7), (7,4)\}$

So, $\{(a,b) \in R \Rightarrow (b,a) \in R, \forall\, a \in X\}.$

Hence it is symmetric.

Hence

Transitive : clearly

$\{(1,4), (4,1), (1,1), (2,5), (5,2), (2,2), (3,6), (6,3), (3,3),$

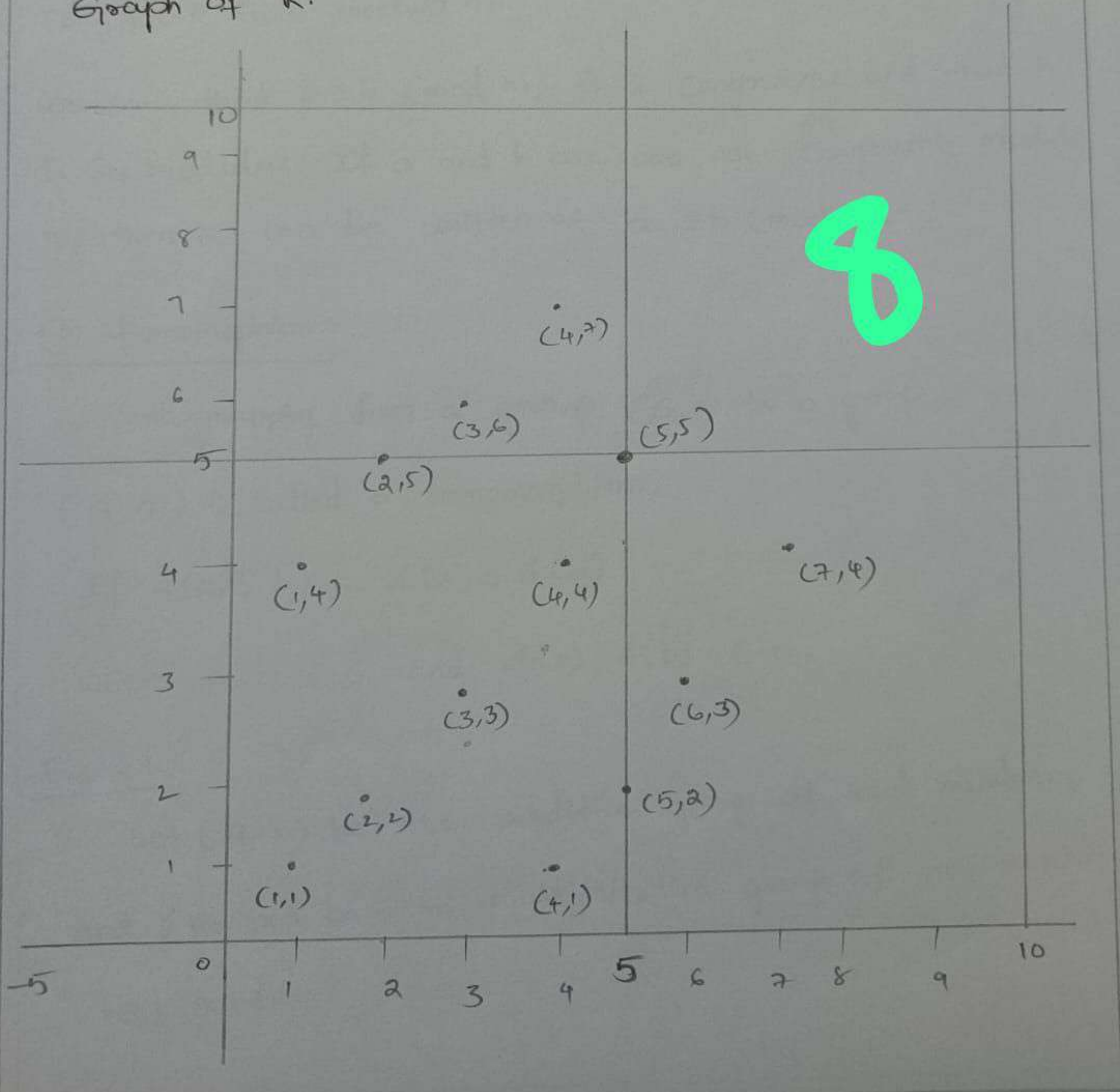
$(4,7), (7,4), (4,4)\}$

so, $\{(a,b) \in R, \& (b,c) \in R \Rightarrow (a,c) \in R, \forall a \in x\}$.

Hence it is transitive.

Thus the given relation is an equivalance relation.

Graph of R.



③ Explain the terms (a) Residue Arithmetic
(b) Homomorphism.

(a) Residue Arithmetic:

If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a \equiv b$. we use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m.

we say that $a \equiv b \pmod{m}$ is a congruence and that m is its modulus. If a and b are are not congruent modulo m, then it can be written as $a \not\equiv b \pmod{m}$.

(b) Homomophism:

A mapping from a group (G, o) to a group (G', o') is called a homomorphism.

If $f(a, o\, b) = f(a) \circ f(b)$

where $a, b, \in G$ and $f(a), f(b) \in G$.

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Example:

Let $(R, +)$ be the additive group of real numbers, and $(R_0, \times)$ be the multiplicative group of non-zero real numbers.

The mapping $(R, +) \to (R_0, \times)$;

$f(x) = 2^x$ for every $x \in R$,

is a homomorphism of R into R_0 because for any

$x_1, x_2 \in R$.

$f(x_1 + x_2) = 2^{x_1} \cdot 2^{x_2} = f(x_1) \cdot f(x_2)$

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Various Homomorphism

i) monomorphism, if f is injection (one to one)

ii) Epimorphism, if f is surjection (onto).

Here G' is called the homomorphism group of image G.

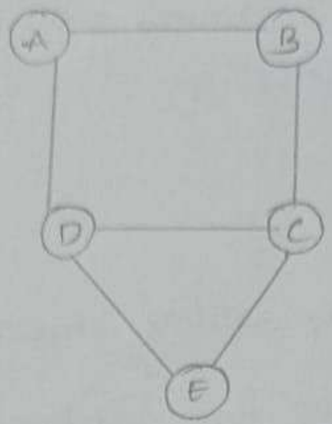
(4) (a) What is Hamiltonian path and Cycle in graph?
Give an example.

Hamitonion Path:

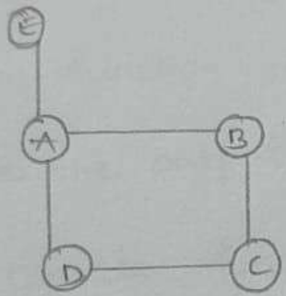
If there exists a walk in the connected graph that visits every vertex of the graph exactly once without repeating the edges. then such walk is called as Hamiltonian path.

If there exists a Path in the connected graph that contains all the vertices of the graph, then such a path is called as a Hamiltonian path.

Examples:



Hamiltonian Path = ABCDE



Hamiltonian Path is EABCD.

Hamiltonian Cycle:

A Hamiltonian cycle, also called a Hamiltonian circuit, Hamilton cycle, or Hamilton circuit, is a graph cycle (i.e, closed loop) through a graph that visits each node exactly once. A graph processing a Hamiltonian cycle is said to be a Hamiltonian graph.

(b) Bijective function:

A function is said to be bijective or bijection, if a function $f: A \to B$ satisfies both the injective (one-to-one) function and surjective function (onto function) properties. It means that every element 'b' in the condomain B, there is exactly one element 'a' in the domain A. Such that $f(a)=b$. If the function satisfies this condition, then it is known as one-to-one correspondence.

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(5) (a) Define Spanning Tree.

A spanning tree is a sub-graph of an undirected connected graph, which includes all the vertices of the graph with a minimum possible number of edges. If a vertex is missed, then it is a not a spanning tree. The edges may or may not have weights assigned to them. The total number of spanning trees with n vertices that can be created from a complete graph is equal to

$n^{(n-2)}$

If we have $n=4$, the maximum number of possible spanning trees is equal to $4^{4-2}=16$. Thus, 16 spanning trees can be formed from a complete graph with 4 vertices.

(b) Define the following terms

(a) Inverse Function (b) Bijective Function.

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(a) Inverse Function :

A function accepts values, performs particular operations on these values and generates an output. The inverse function agrees with the resultant, operates and reaches back to the original function.

The inverse function returns the original value for which a function gave the output.

If you consider functions, f and g are inverse, $f(g(x)) = g(f(x)) = x$. A function that consists of its inverse fetches the original value.

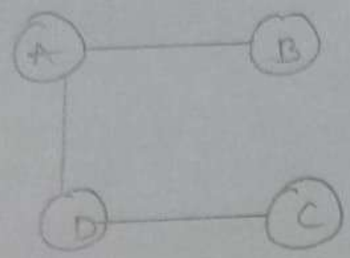
Example : $f(x) = 2x + 5 = y$

Then, $g(y) = (y-5)/2 = x$ is the inverse of $f(x)$.

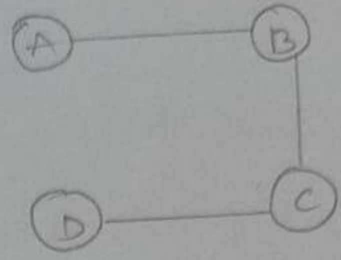
Note: The relation, developed when the independent variable is interchanged with the variable which is dependent on a specified equation and this inverse may or may not be a function.

If the inverse of a function itself, then it is known as inverse function, denoted by $f^{-1}(x)$.

Ex:



A spanning tree



A spanning tree.

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