⑧ Mathematical Foundations of Computer Science.

SECTION-A

① What is Binary Relation? what are the properties of Binary Relations? Give an Example for each.

Binary Relation:

Relation is defined in terms of ordered paires (x, y) of elements, where 'x' is the first element and 'y' is the second element. Let X, Y be two sets. A binary relation from X to Y is a subset of $X \times Y$

$$\Rightarrow X \times Y = \{(x,y) / x \in X, y \in Y\}$$

Properties of Binary Relation:

Some of the properties of binary relations are:

i) Reflexive, non-reflexive, irreflexive

ii) Symmetric, asymmetric, antisymmetric

iii) Transitive, nontransitive.

i) Reflexive Relations:

If R is any relations defined on any given set A then R is said to be reflexive if $\forall x \in A$ □a relation (x, x) such that $(x, x) \in R$ i.e., every element of A is in binary relation with itself and belong to R.

i.e., $\{x \in A \rightarrow xRx \ (x, x) \in R\}$

Example:

a) If the relation '$\leq$' is defined on the set of real numbers then the relation R satisfies reflexive property

(∵ for any real number $a \leq a \Rightarrow (a,a)$ □ R)

b) $A = \{a, b, c\}$ then $= \{(a,a), (b,b), (c,c), (a,c), (b,c)\}$

if R is defined such that for any x □ A, (x, x) does not belong to R then R is irreflexive relation.

ii) Symmetric relation:

If R is any relation defined on given set A, then □ (x, y) □ R if (y, x) □ R then the Relation R is said to be symmetric.

i.e., $R = \{(x,y)$ □ R □ x, y □ $A \Rightarrow (y,x)$ □ R) is a symmetric relation.

Example: If $A = (a,b,c)$, then

$R = \{(a,a), (a,b) (b,c) (a,c) (c,a) (b,c) (c,b)\}$ is a symmetric relation for any relation R defined on set A if □ (x,y) □ R and $(y,x) = R$ the R is non-symmetricor. or asymmetric relation.

Example

a) Relation "<" is defined on real numbers for any a, b □ real numbers if $a \leq b$, then the relation $b \leq a$ is not posible.

Hence it is non-symmetric relation.

ii) $A = \{a, b, c\}$, then,

$R = \{(a,b)\ (a,c)\ (c,c)\ (b,c)\}$ is non-symmetric,

(Therefore (a,b) □ R but $(b,a) \notin R$)

If for any relation defined on set A if

□ (a,b) □ R and (b,a) □ R and $a=b$, then the relation is said to be antisymmetric.

$R = \{(x,y)$ □ R, (y,x) □ R □ $x, y = A \Rightarrow (x=y)\}$.

Transitive Relation :

If R is a relation defined on any set A such that if x, y, z □ A. then (x,y) □ R, (y,z) □ R implies (x,z) □ R, then R is the transitive relation.

Example:

i) The relations " $\leq$ " and " $\geq$ " defined on real numbers are transitive relations

ii) $A = \{a, b, c\}$ then $R = \{(a,b)\ (b,c)\ (a,c)\ (c,a)\ (b,c)\}$ is a transitive relation.

If for any x, y, z □ A and R is a Relation such that (x,y) □ R, (y,z) □ R and if $(x,z) = R$ then R is non-transitive relation.

Ex: $A = \{a, b, c\}$

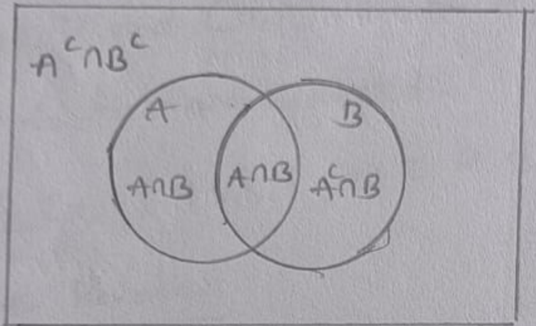
$R = \{(a,a)\ (a,b)\ (a,c)\}$ is non-transitive $(\because (b,c) \notin R)$

4. State and proof Principle of inclusion and Exclusion.

The sumrule which is applied to the non-disjoint sets is called priciple of inclusion-exclusion also called as 'Sieve method'.

<u>1. Statement</u> If A and B are two subsets of any set S (universal) then, $A \cup B = |A| + |B| - |A \cap B|$ ...... (1).

Proof :



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Given, For any Set S $A, B \subseteq S$

i.e., A and B are subsets of S

$|A \cup B|$ = All the elements of A + all the elements of B

$|A| = |A \cap B| + |A \cap B|$ (from Venn diagram) and

$|B| = |A' \cap B| + |A \cap B|$ (from Venn diagram)

$A \cap B$ Contains the elements which belong to both sets A and B

Therefore, $|A| + |B| = |A \cap B'| + |A' \cap B| + |A \cap B| + |A \cap B| = |A \cup B|$

Now from R.H.S of equation (1).

$$|A| + |B| - |A \cap B| = |A \cap B'| + 2|A \cap B| + |A' \cap B| - |A \cap B|$$
$$= |A \cap B'| + |A' \cap B| + |A \cap B|$$
$$= |A \cup B| \text{ (from Venn-diagram)}.$$

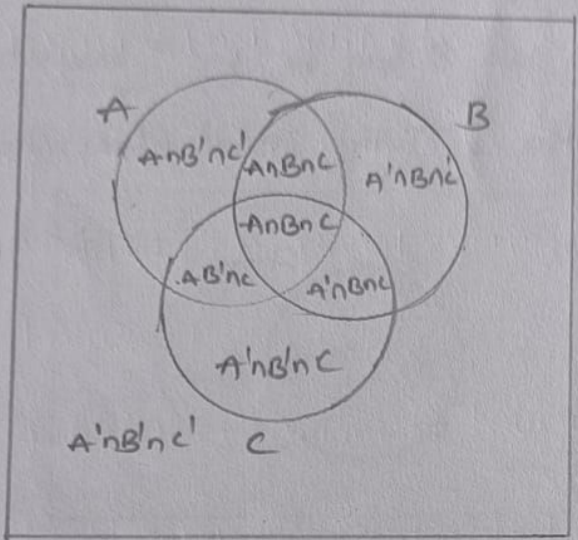
Therefore, $|A \cup B| = |A| + |B| - |A \cap B|$.

2. Statement:

If A, B, C are any three subsets of Set S, then $|A \cup B \cup C|$

$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.

Proof:



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$|A| = |A \cap B' \cap C'| + |A \cap B \cap C'| + |A \cap C \cap B'| + |A \cap B \cap C|$ (from Venn-diagram)

$|B| = |B \cap A' \cap C'| + |A \cap B \cap C'| + |A' \cap B \cap C| + |A \cap B \cap C|$ (from Venn-diagram)

$|C| = |C \cap A' \cap B'| + |A' \cap B \cap C| + |A \cap C \cap B'| + |A \cap B \cap C|$.

Therefore, $|A| + |B| + |C| = |A \cap B' \cap C'| + |B \cap A' \cap C'| + |C \cap A' \cap B| +$

$|A \cap B \cap C'| + |A \cap C \cap B'| + |A' \cap B \cap C| + |A \cap B \cap C'| +$

$|A \cap C \cap B'| + |A' \cap B \cap C| + 3|A \cap B \cap C| \ldots\ldots$ (2)

Now from venn-diagram.

$|A| + |B| + |C| = |A \cup B \cup C| + |A \cap B \cap C'| + |A \cap B \cap C| + |A' \cap B \cap C| +$

$|A \cap B' \cap C| + |A \cap B \cap C|$.

Replacing in equation (2) we get

$|A|+|B|+|C| = |A \cup B \cup C| + |A \cap B \cap C'| + |A \cap B \cap C| + |A' \cap B \cap C| +$

$|A \cap B' \cap C| + |A \cap B \cap C|$

$|A| + |B| + |C| = |A \cup B \cup C| + |A \cap B \cap C'| + |A \cap B \cap C| + |A' \cap B \cap C| +$

$|A \cap B' \cap C| + |A \cap B' \cap C|$ .... ③

Adding $A \cap B \cap C$ on both sides of Eqn ③, we get

$|A| + |B| + |C| + |A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap B' \cap C| +$

$|A \cap B \cap C|$

$|A| + |B| + |C| + |A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap C|$

$[\because |A \cap B \cap C| + |A' \cap B \cap C| = |B \cap C|]$

$\therefore$ Hence proved.

③ # what is Homogeneous Recurrence Relation? Give Example.

A recurrence relation of the form:

$a_{n+1} = da_n + f(n)$ $n >= 0$, where d is constant and $f(n)$ is a known as function is called linear recurrence relation of first order with constant coefficient.

If $f(n) = 0$, the relation is homogeneous otherwise non-homogeneous the solution to the above recurrence relation is given by $a_n = a_0 d^n$, $n \geq 0$ where $A = a_0$

or $[Ad^n = a_n]$

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Example: $a_{n+1} = 4a_n$ for $n \geq 0$, given that $a_0 = 3$.

The above Recurrence Relation is $a_{n+1} = 4a_n$ —— ①

$$a_{n+1} - 4a_n = 0 \quad [\because f(n) = 0]$$

So the given recurrence relation is first order homogenous recurrence relation.

Compare the given RR $a_{n+1} = 4a_n$ with

$$a_n = c.a_{n-1} + f(n)$$

$$a_{n+1} = 4.a_{(n+1)-1} + 0$$

$$a_{n+1} = 4a_n + 0$$

$\therefore$ The general solution of 1st order recurrence relation is

$$\boxed{a_n = c^n.a_0} \text{ —— ②}$$

If $a_n = c^n$ then $\boxed{a_{n+1} = c^{n+1}}$ —— ③

Substitute a_{n+1} value in ①

$$4a_n = c^{n+1} \quad [\because x^{n+y} = x^n.x^y]$$

$$= c^n.c$$

$$4(c^n) = c^n.c \quad \text{from — ③}$$

$$\boxed{c = 4}$$

Substitute c value in general solution

$$a_n = 4^n.a_0$$

③

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from given question $\boxed{a_0 = 3}$

$\therefore \boxed{a_n = 3 \cdot 4^n}$

③ ④ Explain the following terms (a) Semi-groups (b) monoids. (c) Groups.

<u>Semi-groups</u> : A semi group is a set A, together with a binary operation "." That satisfies the associative property.

For all a, b, c □ A, (a.b).c = a. (b.c) □ A.

<u>Example</u>: Consider the set o positive integers, apart from zero, with a binary operation of addition is said to be a semi group. Says S = {1, 2, 3, ... }. Find whether <S, +> is a semi group. for every pair (a,b) □ S, (a+b) is present the set, S. and hence this holds true the clouser property.

Let 1, 2, □ S, 1+2 = 3 □ S.

For every element a,b □ S, (a+b) + c = (a+b) + c.

Let 1, 2, 3, ... □ S,

(1+2) + 3 = 1+ (2+3) = 5.

<u>Monoids</u> : If a semi group {M, *} has an identity element with respect to the operation *, then {M, *} is called a monoids. For Example, if N is the set of natural numbers, then {N, +} & {N, x} are monoids with the identity elements 0 and 1. respectively.... The semigroups {E, +} and {E, x} are not monoids.

(c) Groups:

- A group $(G, *)$ is a set, together with an operation $*$ satisfying the following properties.

i) Closure:

For each $a, b \square G$, then $a * b \square G$.

ii) Associative property:

For each $a, b, c \square G$, then

$(a*b)*c = a*(b*c) \square G$.

iii) Identity: there exist an element $e \square G$, such that

$a*e = e*a = a$.

iv) Inverse property: for each $a \square G$, there is an element $a^{-} \square G$ such that $a * a^{-} = a^{-} * a = e$.

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(2) (5) Solve the Recurrence Relation.

$s(k) - 7s(k-1) + 12s(k-2) = 0$ where $s(0) = 4$, $s(1) = 4$.

Sol: The Given Recurrence Relation is

$$S_k - 7S_{k-1} + 12S_{k-2} = 0$$

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the above Equation compare with

$$\boxed{C_n \cdot a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0}$$

Here $C_n = 1$, $C_{n-1} = -7$, $C_{n-2} = 12$.

Now the charecteristic Equation is

$$k^2 - 7k + 12 = 0. \qquad [\because C_n k^2 + C_{n-1} k + C_{n-2} = 0]$$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

from the above Equation $a = 1$, $b = -7$, $c = 12$.

$$k = \frac{+7 \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{49 - 48}}{2}$$

$$k = \frac{7 \pm 1}{2} \Rightarrow k = \frac{7+1}{2} = 8/2 = 4$$

$$k = \frac{7-1}{2} = 6/2 = 3$$

$\boxed{k_1 = 4}$ and $\boxed{k_2 = 3}$.

Here the roots are different and real the second order homogeneous recurrence relation is

$$a_n = A(k_1)^n + B(k_2)^n \quad \text{—— (A)}$$

$$a_n = A(4)^n + B(3)^n \quad \text{—— (1)}$$

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from the given question

$$s(0) = 4 \Rightarrow \quad s_0 = A(4)^0 + B(3)^0$$

$$4 = A + B \quad \text{—— (2)}$$

$$s(1) = 4 \Rightarrow \quad s_1 = A(4)^1 + B(3)^1$$

$$4 = 4A + 3B \quad \text{—— (3)}$$

Solve $3 \times (2)$ and (3) Equations

$$\begin{aligned} 12 &= 3A + 3B \\ 4 &= 3A + 4B \\ \hline 8 &= -B \end{aligned}$$

$\therefore \boxed{B = -8}$

Substitute B value in Eqn (2) $\quad 4 = A - 8$

$$\boxed{A = 12}$$

A, B value Substitute in (A). Now the general solution of 2^{nd} order homogenous R.R is $\boxed{a_n = 12(4)^n - 8(3)^n}$.

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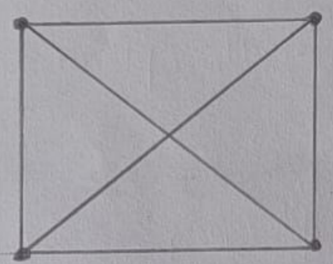
2> b) Explain the following terms i> Complete Graph
ii> Spanning Trees
iii> Binary Trees.

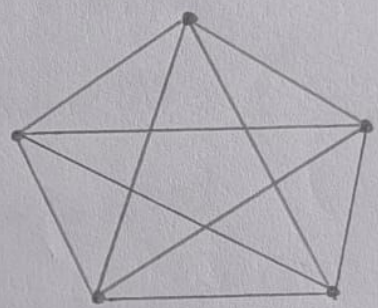
i> Complete graphs :

A complete graph is a graph that has an edge between every single vertex in the graph; we represent a complete graph with n vertices using the symbol K_n.

Examples :



K_4



K_5

ii> Binary Tree

Binary Search tree is a binary tree which satisfies the following property –

$\times x$ in left sub-tree of vertex v, value (x) $\leq$ value (v) v,

value (x) $\leq$ value (v)

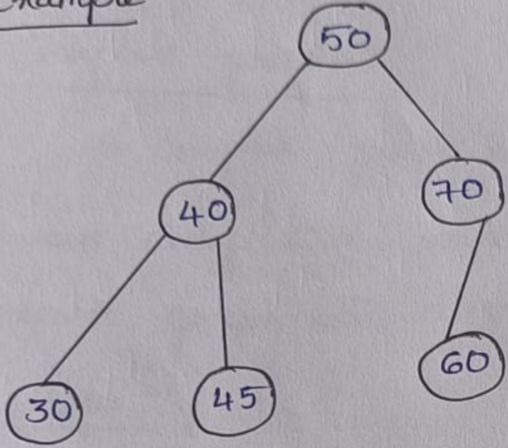
$\times y$ in right sub-tree of vertex v, value (Y) $\geq$ value (v) v,

value (Y) $\geq$ value (v)

So, the value of all the vertices of the left sub-tree of an internal node vv are less than or Equal to vv. and the value of all the vertices of the right sub-tree of the

of the internal node VV are greater than or Equal to VV.
The number of links from the root node to the deepest node is height of the Binary Search Tree.

Example



iii) Spanning Tree

A spanning tree is a sub-graph of an undirected graph, which includes all the vertices of the graph with a minimum possible number of edges. If a vertex is missed, then it is not a spanning tree.

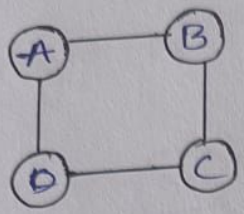
The edges may or may not have weights assigned to them.

The total number of spanning trees with n vertices that can be created from a complete graph is Equal to $n^{(n-2)}$.

If we have $n = 4$, the maximum number of possible spanning trees is Equal to $4^{4-2} = 16$. Thus, 16 spanning trees can be formed from a Complete graph with 4 vertices.

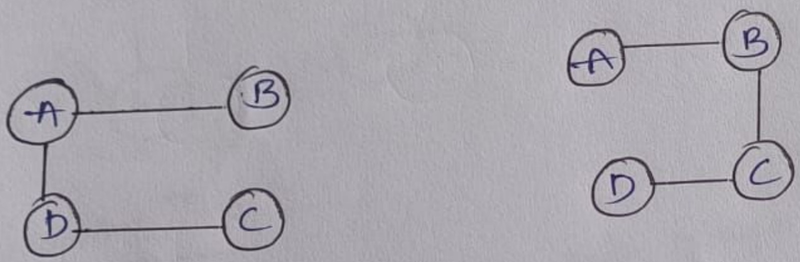
Let's understand the spanning tree with examples below:

Let the original graph be:



Normal graph.

Some of the possible spanning trees that can be created from the above graph are:



Spanning tree.

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