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A. Srinivas Yadav

Engineering Mathematics

(Theory with worked out Examples, Self and Classroom Practice Questions)

GATE – Civil, Mech & PI Engg.

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A word to the students

We start with a quotation
"For he who knows not mathematics can not know any other science, what is more, he can not discover his own ignorance, or find its proper remedy. Mathematics is door and key to science".

- Roger Bacon (1220-1212)

This book is prepared to benefit the students appearing for GATE examination. It contains necessary theory, worked out examples, Level-I and Level-II questions. Level –I questions are given for students practice. All Level-II questions are answered by the concerned faculty in the classroom.

To learn any subject in engineering, we have to be reasonably good in mathematics. We therefore advise all the students who are preparing for GATE to take it as an opportunity to strengthen their mathematical skills. To succeed in GATE, Mathematics plays very important role, because a student can easily score 15 to 20 marks, if he/she focus on Mathematics.

To succeed in GATE we advise the students to remember the following

- Complete clarity on the concepts and therefore subject is required, if you want to excel in GATE. Both theory and problems are important. Most of the questions in the GATE are very fundamental in nature and test your conceptual clarity on the basics.
- Solve as many new problems as possible on your own. Then only you will be in a position to realize the complexity in the subject and can solve new problems in GATE.
- Solve all the previous GATE papers to realize the pattern, complexity and standards of GATE.
- Realize the applications of mathematical concepts in your own branch of engineering and workout as many problems as possible. Knowing the concepts is not enough; you must be in a position to apply the concepts in new situations.
- Finally, Mathematics is the language of science and engineering. We therefore advise you to master the language so that you can enjoy the learning process at higher levels.

- Faculty of mathematics,
ACE Academy

With best wishes to all the Students

Y.V. Gopala Krishna Murthy,
M Tech. MIE,
Managing Director,
ACE Engineering Academy

Engineering Mathematics Syllabus

Linear Algebra: Matrix algebra, Systems of linear equations, Eigen values and eigenvectors.

Calculus: Functions of single variable, Limit, continuity and differentiability, Mean value theorems, Evaluation of definite and improper integrals, Partial derivatives, Total derivative, Maxima and minima, Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Differential equations: First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Cauchy's and Euler's equations, Initial and boundary value problems, Laplace transforms, Solutions of one dimensional heat and wave equations and Laplace equation.

Complex variables: Analytic functions, Cauchy's integral theorem, Taylor and Laurent series.

Probability and Statistics: Definitions of probability and sampling theorems, Conditional probability, Mean, median, mode and standard deviation, Random variables, Poisson, Normal and Binomial distributions.

Numerical Methods: Numerical solutions of linear and non-linear algebraic equations Integration by trapezoidal and Simpson's rule, single and multi-step methods for differential equations.

Books Recommended:

1. Higher Engineering Mathematics – B.S. Grewal
2. Advanced Engineering Mathematics – Erwin Kreyszig,
3. Advanced Engineering Mathematics – R.K. Jain , S.R.K Iyengar
4. Matrices – A.R. Vasishta
5. Vector Calculus – A.R. Vasishta
6. Fundamentals of Mathematical Statistics – Gupta & Kapoor
7. Complex Variables –J.N. Sharma, Murray, R. Speigial
9. Numerical Methods – S.S. Shastry

CONTENTS

1 Linear Algebra


Arthur Cayley (1821 – 1895)

"As for every thing else, so for a mathematical theory; beauty can be perceived but not explained".
– Arthur Cayley

1.0. Introduction

Applications of matrices are found in every branch of engineering, including classical mechanics, optics, electromagnetism, quantum mechanics, quantum electrodynamis and computer graphics.

Prerequisites

There are no formal prerequisites for this chapter the student is advised to read carefully and work through all the examples.

Matrix:

A set of mn numbers (real or complex) or functions arranged in the form of a rectangular array of m horizontal lines and n vertical lines is known as a matrix of order or type $m\times n$ (to be read as 'm' by 'n' matrix).

These numbers which form a matrix are called elements or entries of the matrix and the arrangement of numbers is commonly enclosed within square brackets [ ] or in parenthesis ( ) or some times in double bars || ||. The elements are denoted by small letters such as a_{ij}, b_{ij} or c_{ij} where the suffix i j indicates the position being in the i^{th} row and j^{th} column. The matrices are generally denoted by capital letters such as A,B,C etc.

The horizontal lines in a matrix are called rows and vertical lines are called columns of the matrix.

An $m\times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

The above matrix in a compact form is represented by $A = (a_{ij})_{m\times n}$ or $A = [a_{ij}]_{m\times n}$ where $i = 1,2,\dots\dots m$ and $j = 1,2,\dots\dots n$.

Equal matrices or Equality of matrices

Two matrices $A = (a_{ij})_{m\times n}$ and $B = (B_{ij})_{p\times q}$ are said to be equal if

(i) they are of same order, i.e $m = p$, $n = q$ and

(ii) their corresponding elements are equal, i.e $a_{ij} = b_{ij}$ for all i,j.

Note:

If the two matrices A and B are equal then we write $A = B$ otherwise $A \neq B$.

Ex: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ then $A = B \Leftrightarrow a = 1, b = 2, c = -1, d = 3$.

1.1. Algebra of matrices

Multiplication of a matrix by a scalar or scalar multiplication

If $A = (a_{ij})$ is an $m\times n$ matrix and k is any scalar (real number or complex number) then the matrix kA is obtained by multiplying every element of matrix A by scalar k.

Thus if $A = [a_{ij}]_{m\times n}$

then $kA = k[a_{ij}]_{m\times n} = [ka_{ij}]_{m\times n}$

Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & -2 & 4 \end{bmatrix}$

then $kA = \begin{bmatrix} k & 2k & 3k \\ 7k & -2k & 4k \end{bmatrix}$.

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices and k, l are two scalars then

(i) $k(A+B) = kA+kB$

(ii) $(k+l)A = kA+lA$

(iii) $(kl)A = k(lA) = l(kA)$

(iv) $(-k)A = -(kA) = k(-A)$

Arthur Cayley was probably the first mathematician to realize the importance of the notion of a matrix and in 1858 published book, showing the basic operations on matrices. He also discovered a number of important results in matrix theory.

Addition of matrices:

If A and B are two matrices of same order then the matrix obtained by adding the corresponding elements of A and B respectively is called the sum of A and B and it is denoted by A+B.

i.e., If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then $A+B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij}+b_{ij}]_{m\times n}$.

Ex: If $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & -1 & 0 \end{bmatrix}_{2\times 3}$

& $B = \begin{bmatrix} -2 & 1 & 7 \\ 3 & 0 & 4 \end{bmatrix}_{2\times 3}$ then

$A+B = \begin{bmatrix} 1-2 & 3+1 & 4+7 \\ -2+3 & -1+0 & 0+4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 4 & 11 \\ 1 & -1 & 4 \end{bmatrix}_{2\times 3}$

Note:

Addition of matrices is commutative: If A & B are two matrices of same order then A+B = B+A.

Addition of matrices is associative: If A,B,C are three matrices of same order then (A+B)+C = A+(B+C).

Existence of additive identity: If A is any matrix of order m×n and O is the null matrix of order m×n then A+O = O+A = A

Here, $O_{m\times n}$ is called an additive identity in the set of all m×n matrices with respect to addition of matrices.

Existence of the additive inverse: If for any given matrix $A_{m\times n}$ there exists a matrix '–A' such that A+(–A) = (–A)+A = O then matrix '–A' is called the additive inverse of matrix A.

Cancellation laws hold good in case of addition of matrices:
If A,B,C are three matrices of same order then
A+B=A+C⇒B=C (left cancellation law) &
B+A=C+A⇒ B=C (right cancellation law).

The equation A+X = O has a unique solution in the set of all m×n matrices.

Subtraction of matrices:

If A & B are two matrices of same order then the matrix obtained by subtracting each element of B form the corresponding elements of A is called the difference of A & B and it is denoted by A–B.

Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 4 \end{bmatrix}_{2\times 3}$

& $B = \begin{bmatrix} 4 & -2 & 4 \\ 2 & 3 & 1 \end{bmatrix}_{2\times 3}$ then

$A-B = \begin{bmatrix} 1-4 & 2+2 & 3-4 \\ 4-2 & 7-3 & 4-1 \end{bmatrix}$

$= \begin{bmatrix} -3 & 4 & -1 \\ 2 & 4 & 3 \end{bmatrix}_{2\times 3}$

Multiplication of matrices:
If $A=(a_{ik})_{m\times p}$ & $B = (b_{kj})_{p\times n}$ are any two matrices such that the number of columns in A is equal to the number of rows in B then the product of A and B is denoted by AB and defined as the matrix

$AB = C = [c_{ij}]_{m\times n}$ where

$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$

= the sum of the products of the elements of i^{th} row of A with the corresponding elements of j^{th} column of B.

Note:

- In the product AB, A is called pre-factor & B is called post-factor.
- If the product AB exists for two matrices A&B then the product BA may or may not exist.
- For example if A is 4×2 matrix & B is 2×3 matrix then the product AB is defined while the product BA is not defined.
- If A is any m×n matrix such that AB & BA are both defined then B is n×m matrix.

Matrix multiplication is not Commutative:
If AB and BA exist for any given two matrices A& B then AB & BA may not have same order and even if they have same order then they may not be equal.

Ex: If $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 3 & 6 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 1 \\ 2 & 7 \\ 4 & 5 \end{bmatrix}$ then

$AB = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 3 & 6 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 7 \\ 4 & 5 \end{bmatrix}$

$= \begin{bmatrix} (1\times3)+(2\times2)+(4\times4) & (1\times1)+(2\times7)+(4\times5) \\ (5\times3)+(3\times2)+(6\times4) & (5\times1)+(3\times7)+(6\times5) \end{bmatrix}$

$= \begin{bmatrix} 23 & 35 \\ 45 & 56 \end{bmatrix}_{2\times 2}$ and

$BA = \begin{bmatrix} 3 & 1 \\ 2 & 7 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 5 & 3 & 6 \end{bmatrix}$

$= \begin{bmatrix} (3\times1)+(1\times5) & (3\times2)+(1\times3) & (3\times4)+(1\times6) \\ (2\times1)+(7\times5) & (2\times2)+(7\times3) & (2\times4)+(7\times6) \\ (4\times1)+(5\times5) & (4\times2)+(5\times3) & (4\times4)+(5\times6) \end{bmatrix}$

$= \begin{bmatrix} 8 & 9 & 18 \\ 37 & 25 & 50 \\ 29 & 23 & 46 \end{bmatrix}$

Here both AB & BA exist but AB & BA are not same.

Ex: If $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ then

$AB = \begin{bmatrix} 7 & 2 \\ 3 & 6 \end{bmatrix}$ & $BA = \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$

Here both AB & BA exists but AB & BA are not same.

Multiplication of matrices is associative:
If A, B, C are matrices of orders m×n, n×p and p×q respectively then (AB)C = A(BC) is a matrix of order m×q

Matrix multiplication is distributive over matrix addition:
If A is a matrix of order m×n & B,C are matrices of order n×p then A(B+C)=AB+AC (left distributive law).
If A, B are matrices of order m×n & C is a matrix of order n×p then (A+B)C =AC+BC (right distributive law).

If A is a matrix of order m×n & B is a matrix of order n×p then α (AB) = A(αB) = (αA)B for any scalar α .

The product of two matrices can be the null matrix while neither of them is the null matrix.

Ex: If $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then

$BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

while neither A nor B is the null matrix.

If AB = O then BA may or may not be zero matrix.

Ex: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$ and $BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq O$

Thus AB = O while BA ≠ O

Types of Matrices

Real matrix: If all the elements of a matrix A are real numbers then the matrix A is called a real matrix.

Ex: [2] , $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ are real matrices of order 1×1, 2×3 and 2×2 respectively.

Complex matrix: If at least one of the elements of a matrix A is purely imaginary (or) complex then the matrix A is called complex matrix.

Ex: [2i], $\begin{bmatrix} 0 & 2 \\ 3i & 4+3i \end{bmatrix}$, $\begin{bmatrix} 2 & 3i & 0 \\ 0 & 9 & 3 \end{bmatrix}$,

$\begin{bmatrix} 3+2i & 4 \\ 2 & 0 \end{bmatrix}$ are complex matrices.

Row matrix: If a matrix A has only one row and any number of columns then the matrix A is called a row matrix.
Ex: $A = [1 \;\; -4 \;\; -7]$ is a row matrix of order 1×3.

Column matrix: If a matrix A has only one column and any number of rows then the matrix A is called a column matrix.

Ex: $A = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is a column matrix of order 3×1.

Vector: A vector of the form $\overline{A} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ in space can also be uniquely expressed as an ordered triad of three real numbers (a_1, a_2, a_3) where a_1, a_2, a_3 are called components of vector A. Here a_1, a_2, a_3 are called Ordered set of 3 real numbers a_1, a_2, a_3 or an ordered 3-tuple or 3-dimensional vector or vector of order 3 or a 3-vector.

If the set of n real numbers $x_1, x_2, \ldots x_n$ written in a particular order $(x_1, x_2, \ldots, x_n)$ then $(x_1, x_2, \ldots, x_n)$ is called an n-dimensional vector or a vector of order n (or) an ordered n-tuple. The numbers $x_1, x_2, \ldots x_n$ are called the components of the vector and a vector is denoted by single letter X, Y etc.

Note:
A vector may be written as either a row matrix $X = [x_1 \;\; x_2 \ldots x_n]$ which is called row vector (or) a column matrix $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ which is called column vector.

If A is a matrix of order $m\times n$ then each row of A is an n-vector and each column of A is an m-vector (m-tuple vector).

In particular, if m=1 then A is a row vector & if n=1 then A is a column vector.

An ordered n-tuple of numbers is called an n-vector. If $x_1, x_2, \ldots x_n$ are any n numbers then an ordered n-tuple $X = (x_1, x_2, \ldots x_n)$ is called an n-vector.

Null matrix or zero matrix:
If every element of a matrix is zero then the matrix is called null matrix and the null matrix of order m×n is denoted by $O_{m\times n}$.

Ex: $O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$

Rectangular matrix: If the number or rows is not equal to the number of columns in a matrix then the matrix is called a rectangular matrix.

Ex: $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & .5 & 6 & 7 \\ 4 & 7 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 0 & 7 \end{bmatrix}$

are rectangular matrices of order 2×3, 2×4, 3×2 respectively.

Square matrix:
If the number of rows is equal to the number of columns in a matrix then the matrix is called square matrix. A square matrix of order n×n is sometimes called as n-rowed matrix A or simply a Square matrix of order 'n'.

Ex: $[2]$, $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ are square matrices of order 1,2,3 respectively.

Diagonal or principal diagonal elements:
If $A = (a_{ij})_{n\times n}$ then the elements a_{ij} of a square matrix for which $i = j$ i.e., the elements $a_{11}, a_{22}, \ldots a_{nn}$ are called diagonal elements (or) leading diagonal elements.

Ex: If $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 3 \\ 7 & 3 & -4 \end{bmatrix}$ then $a_{11} = 1$, $a_{22} = 0$, $a_{33} = -4$ are diagonal elements of matrix A order 3×3.

Note:
The elements a_{ij} of a square matrix $A = (a_{ij})_{n\times n}$ for which $i \neq j$ are called off-diagonal elements.

Principal diagonal: -
The line along which the diagonal elements lie is called principal diagonal of the matrix.

Trace of a matrix:
The sum of the diagonal elements of a square matrix is called trace of A and it is denoted by trace (A) or tr(A).

Thus, if $A = (a_{ij})_{n\times n}$
then $tr(A_{n\times n}) = a_{11} + a_{22} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$.

Ex: If $A = \begin{bmatrix} 1 & 2 & 4 \\ 7 & -5 & 2 \\ 3 & 4 & 0 \end{bmatrix}$

then $tr(A_{3\times3}) = (1) + (-5) + (0) = -4$

Properties:
If A & B are square matrices of order n then
(i) $tr(A+B) = tr(A) + tr(B)$
(ii) $tr(A-B) = tr(A) - tr(B)$
(iii) $tr(AB) \neq tr(A)\,tr(B)$
(iv) $tr(BA) \neq tr(B)\,tr(A)$
(v) $tr(AB) = tr(BA)$
(vi) $tr(kA) = k\,tr(A)$ where k is a scalar.

Diagonal matrix: If all the non-diagonal or off-diagonal elements in a square matrix are zero then the matrix is called diagonal matrix.

Ex: $A = [4]$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are diagonal matrices of order 1,2,3 respectively.

Note:
- A matrix $A = [a_{ij}]_{n\times n}$ is diagonal matrix if $a_{ij} = 0 \;\forall i \neq j$
- A diagonal matrix of order n×n having $d_1, d_2, \ldots d_n$ as diagonal elements is denoted by $diag[d_1, d_2, \ldots d_n]$ or $diag\,[d_1\; d_2 \;\ldots d_n]$.

- If A & B are diagonal matrices then
  (i) A+B is a diagonal matrix.
  (ii) A–B is a diagonal matrix.
  (iii) A^2, B^2 are diagonal matrices.
  (iv) $A^n \pm B^n$ $(n \in N)$ are diagonal matrices.
  (v) A^T is a diagonal matrix.
  (vi) $A^T \pm B^T$ are diagonal matrices.
  (vii) adj(A), adj(B) are diagonal matrices.
  (viii) A^{-1}, B^{-1} are diagonal matrices.

Scalar matrix:
If all the diagonal elements of a diagonal matrix are same or equal then the matrix is called a scalar matrix.

Thus a matrix $A = [a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = \begin{cases} 0, \forall i \neq j \\ k, \forall i = j \end{cases}$

Ex: $A = [3]$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and

$C = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$ are scalar matrices of order 1,2, 3 respectively.

Unit matrix or identity matrix:
If all the diagonal elements of a diagonal matrix are one then the matrix is called an identity matrix.

Thus a matrix $A = [a_{ij}]_{n\times n}$ is an identity matrix if $a_{ij} = \begin{cases} 0, & \forall i \neq j \\ 1, & \forall i = j \end{cases}$.

A unit matrix of order n is denoted by I_n.

Ex: $I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1,2,3 respectively.

Upper triangular matrix:
If all the elements below the principal diagonal are zero in a square matrix then the matrix is called an upper triangular matrix.

Thus, $A = [a_{ij}]_{n\times n}$ is an upper triangular matrix if $a_{ij} = 0\ \forall\ i > j$.

Ex: $A = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 4 & 8 \\ 0 & 0 & 9 \end{bmatrix}$ is an upper triangular matrix of order 3.

Lower triangular matrix:
If all the elements above the principal diagonal are zero in a square matrix then the matrix is called lower triangular matrix.

Thus, $A = [a_{ij}]_{n\times n}$ is a lower triangular matrix if $a_{ij} = 0\ \forall\ i < j$.

Ex: $A = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 4 & 0 \\ 1 & 3 & 9 \end{bmatrix}$ is a lower triangular matrix.

Note:
A diagonal matrix is both upper and lower triangular matrix.

Triangular matrix:
A matrix which is either upper triangular or lower triangular is called a triangular matrix.

Strictly triangular matrix:
A triangular matrix $A = (a_{ij})_{n\times n}$ is called a strictly triangular matrix

if $a_{ii}=0$ for all $i = 1,2,\ldots n$

Ex: $A = \begin{bmatrix} 0 & 1.8 & 2 \\ 4 & 0 & 9 \\ 10 & 2 & 0 \end{bmatrix}$

Transpose of a matrix:
If a matrix $B_{n\times m}$ is obtained from a matrix $A_{m\times n}$ by changing its rows into columns and its columns into rows then the matrix $B_{n\times m}$ is called transpose of A and is denoted by A^T or A^1.

Thus, if $A = (a_{ij})_{m\times n}$ then $A^T = B = [b_{ij}]_{n\times m}$
where $b_{ij} = a_{ji}$

Ex: If $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 7 & -2 \end{bmatrix}_{2\times 3}$ then

$$A^T = \begin{bmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -2 \end{bmatrix}_{3\times 2}$$

Properties of Transpose:

If A & B are two matrices and A^T & B^T are transpose of A & B respectively then

(i) $(A^T)^T = A$
(ii) $(A \pm B)^T = A^T \pm B^T$
(iii) $(kA)^T = kA^T$ where k is a scalar (real or complex)
(iv) $(AB)^T = B^T A^T$
(v) $(A^2)^T = (A^T)^2$

Symmetric matrix:

If $a_{ij} = a_{ji}$ for all i,j in $A = (a_{ij})_{n\times n}$ then $A_{n\times n}$ is called a Symmetric matrix.

Ex: $A = \begin{bmatrix} 1 & 2i & -3 \\ 2i & 4 & 6 \\ -3 & 6 & 0 \end{bmatrix}$ is a symmetric matrix of order 3.

Note:
The necessary and sufficient condition for a square matrix A to be symmetric is that $A^T = A$.

Skew-symmetric matrix:
A square matrix $A = (a_{ij})_{n\times n}$ is said to be Skew-Symmetric matrix if $a_{ij} = -a_{ji}\ \ \forall$ i,j.

Ex: $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$ is a Skew symmetric matrix of order 3.

Note:
(i) The diagonal elements of a skew-symmetric matrix are all zero.
(ii) The necessary and sufficient condition for a square matrix A to be skew symmetric matrix is that $A^T = -A$.

Properties of symmetric & Skew-symmetric matrices:

If A and B are symmetric matrices then
(i) $A\pm B$ are symmetric
(ii) AB, BA are not symmetric
(iii) AB+BA is symmetric
(iv) AB−BA is skew-symmetric
(v) A^2, B^2, $A^2 \pm B^2$ are symmetric
(vi) A^3, A^4, B^3, B^4 are symmetric (In general A^k, B^k are symmetric when $k\in N$)
(vii) kA is symmetric when k is a scalar.

If A & A^T are any two square matrices then
(i) $A+A^T$ is symmetric
(ii) $A-A^T$, A^T-A are skew-symmetric.
(iii) AA^T, A^TA are symmetric.

If A & B are skew-symmetric then
(i) $A\pm B$ are skew-symmetric.
(ii) AB, BA are not skew-symmetric.
(iii) A^2, B^2 are symmetric.
(iv) $A^2 \pm B^2$ are symmetric.
(v) A^2, A^4, A^6 are symmetric
(vi) A^3, A^5, A^7 are skew-symmetric
(vii) kA is skew-symmetric.

Every square matrix A can be uniquely expressed as a sum of symmetric & Skew-symmetric matrices.

$$\text{i.e. } A = \frac{A+A^T}{2} + \frac{A-A^T}{2}.$$

If A and B are square symmetric matrices then AB is symmetric $\Leftrightarrow$ AB = BA.

If A and B are skew-symmetric matrices then AB is symmetric $\Leftrightarrow$ A and B are commute.

The matrix B^TAB is symmetric (or) skew-symmetric according as A is symmetric (or) skew-symmetric.

$O_{n\times n}$ is symmetric as well as skew-symmetric.
I_n is symmetric.

Positive integral powers of a square matrix:
If A is a square matrix then the product A.A is defined as A^2.

Similarly $A.A^2 = A^3$
$A^2A^2 = A^4$ and so on.

Idempotent matrix:
If $A^2 = A$ for a square matrix A of order n then $A_{n\times n}$ is called an idempotent matrix.

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$

$\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

are idempotent matrices.

Note:
(i) If AB = A and BA = B then A and B are idempotent.
(ii) If A is an idempotent matrix then I−A is also an idempotent matrix.
(iii) If AB = BA = O, then the sum of two idempotent matrices A & B is an idempotent.

Involutary matrix:
If $A^2 = I_n$ for a square matrix A of order n then matrix A is called an involutary matrix.

Ex: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}, \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix},$

$\begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$

are involutary matrices.

Note:
A is involuntary matrix $\Leftrightarrow$ (I+A)(I−A) = O

Nilpotent matrix:
If there exists a positive integer m for a square matrix A of order n such that $A^m = O$ then the matrix A is called a nilpotent matrix.

If m is a least positive integer for which $A^m = O$ then 'm' is called the index of the nilpotent matrix.

Ex:

1. $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ is a nilpotent matrix of index 2

2. $A=\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}$ is a nilpotent matrix of index 2

3. $A=\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$ is a nilpotent matrix of index 3

4. $A=\begin{bmatrix}1 & -3 & -4\\ -1 & 3 & 4\\ 1 & -3 & -4\end{bmatrix}$ is a nilpotent matrix of index 2

Orthogonal matrix:

A square matrix $A_{n\times n}$ is said to be an orthogonal matrix if $AA^T = A^TA = I$ (or) $A^{-1} = A^T$.

Ex:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix},\ \frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix},$$

$$\frac{1}{2}\begin{bmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\end{bmatrix},\ \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix},\ \frac{1}{3}\begin{bmatrix}1 & -2 & 2\\ -2 & 1 & 2\\ -2 & -2 & -1\end{bmatrix}$$

are orthogonal matrices

Note:

If A and B are orthogonal matrices then AB and BA are also orthogonal matrices.

Conjugate matrices:

If $A = (a_{ij})$ is a given matrix then the matrix obtained from a given matrix A by replacing all the elements by their corresponding conjugate complex number is called conjugate of matrix A and it is denoted by $\overline{A} = [\overline{a}_{ij}]$.

Ex: If $A=\begin{bmatrix}1+i & 2i & 4\\ 7 & 3-i & 2-3i\end{bmatrix}$

then $\overline{A}=\begin{bmatrix}1-i & -2i & 4\\ 7 & 3+i & 2+3i\end{bmatrix}$.

Properties of conjugate matrix

1. $\overline{(\overline{A})} = A$
2. $\overline{(A+B)} = \overline{A}+\overline{B}$
3. $\overline{(AB)} = \overline{A}\,\overline{B}$
4. $\overline{(kA)} = \overline{k}\,\overline{A}$, k is any complex number.

Transposed conjugate of a matrix:

If $\overline{A}$ is a conjugate matrix of a complex matrix A then $(\overline{A})^T$ is called transposed conjugate of A and it is denoted by A^θ.

$\therefore A^\theta = (\overline{A})^T = \overline{(A^T)}$.

Ex: If $A=\begin{bmatrix}2 & 7i & 4+3i\\ 2-4i & 3 & -4i\end{bmatrix}$ &

$\overline{A}=\begin{bmatrix}2 & -7i & 4-3i\\ 2+4i & 3 & 4i\end{bmatrix}$ then

$A^\theta = (\overline{A})^T = \begin{bmatrix}2 & 2+4i\\ -7i & 3\\ 4-3i & 4i\end{bmatrix}$.

Properties:

(i) $(A^\theta)^\theta = A$

(ii) $(A+B)^\theta = A^\theta + B^\theta$

(iii) $(kA) = \overline{k}A^\theta$

(iv) $(AB)^\theta = B^\theta A^\theta$

Hermitian matrix:

A square matrix $A = (a_{ij})_{n\times n}$ is said to be Hermitian if $a_{ij} = \overline{a}_{ij}$ for all i,j

Note:

(i) The necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^\theta$.

(ii) The diagonal element of a Hermitian matrix are purely real.

Ex: $A=\begin{bmatrix}2 & 2+3i\\ 2-3i & 3\end{bmatrix}$ is a hermitian matrix.

Skew-Hermitian matrix:

A square matrix $A = (a_{ij})$ is said to be skew-hermitian if $a_{ij} = -\overline{a}_{ij}$ for all i,j .

Note:

1. The necessary & sufficient condition for a matrix A to be Skew- Hermitian is that $A^\theta = -A$.

2. The diagonal elements of a skew-Hermitian matrix are either zero (or) purely imaginary.

   Ex: $A=\begin{bmatrix}2i & 4+3i & 2-5i\\ -(4-3i) & 0 & 3+4i\\ -(2+5i) & -(3-4i) & 0\end{bmatrix}$

   is a skew-Hermitian matrix.

3. Every complex square matrix A can be uniquely written as a sum of Hermitian and Skew-Hermitian matrices.

Unitary matrix:

A complex matrix $A = (a_{ij})$ is said to be unitary if $AA^\theta = A^\theta A = I$ (or) $A^\theta = A^{-1}$.

Ex:

$A=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$ and

$A=\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}\frac{1}{\sqrt{3}}$ are unitary matrices.

Properties of unitary matrix:

1. If A is unitary then A^T is also unitary
2. If A and B are unitary matrices then AB is also unitary.

Periodic matrix:

A square matrix A is called periodic if $A^{k+1} = A$ where k is a positive integer.

If k is a least positive integer for which $A^{k+1} = A$ then k is said to be period of A.

Note:

For k = 1 , we get $A^2 = A$ and A is called an **idempotent.**

1.2. Determinants

To every square matrix $A = (a_{ij})$ of order n, we can associate a number (real or complex) called determinant of the square matrix A where $a_{ij} = (i,j)^{th}$ element of A.

Determinant of a square matrix of order one

If $A = [a_{ij}]$ is a square matrix of order 1 then the expression $|a_{11}|$ (element a_{11} within vertical bars) is called determinant of first order and it is denoted by $|A|$ (or) det(A).

Here the unique number 'a_{11}' is called the expansion or value of the determinant.

$\therefore \Delta = \det(A) = |A| = |a_{11}| = a_{11}$

Ex: If $A = [-4]$ then

$\Delta = \det(A) = |A| = |-4| = -4$.

Determinant of a square matrix of order two:

If $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ is a square matrix of order 2 then

the expression $\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$ is called determinant of 2^{nd} order and it is denoted by $|A|$ (or) det(A) or Δ.

Here the unique number "$a_{11}a_{22} - a_{21}a_{12}$ is called the expansion or the value of the determinant of 2^{nd} order matrix.

$$\therefore \det(A) = |A| = \Delta = \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Ex: If $A=\begin{bmatrix}2 & 5\\ -3 & 4\end{bmatrix}$ then

$$|A| = \det(A) = \Delta = \begin{vmatrix}2 & 5\\ -3 & 4\end{vmatrix} = (2)(4) - (-3)5) = 23.$$

Determinant of a square matrix of order three:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3×3 then the expression $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is called determinant of 3[rd] order it is denoted by det(A) or $|A|$ or Δ.

Here unique number $(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ is called expansion or value of the determinant of 3[rd] order matrix.

$$\therefore \Delta = \det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

This method of evaluating a determinant is known as expansion along the first row.

Ex: If $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 5 \end{bmatrix}$ then

$$|A| = 1\begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 4 & 5 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= (1)(15-0) - 2(-5-0) + 4(-1-12)$$

$$\therefore |A| = -27$$

Elementary operations or transformations:

The following three types of operations applied on the rows (or columns) of a matrix are called elementary row (or column) transformations.

(i) Interchange of any two rows (or columns)

If the i[th] row (or column) of a matrix is interchange with the j[th] row (or column) then it is denoted $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$).

(ii) Multiplying all the elements of a row (or column) by a non-zero number or scalar 'k'.

If all the elements of i[th] row (or column) are multiplied by a non-zero scalar 'k' then it will be denoted by $R_i \to kR_i$ (or $C_i \to kC_j$).

(iii) Adding to the elements of a row (or column), the corresponding elements of any other row (or column) multiplied by any scalar 'k'.

If all the elements of i[th] row (or column) are multiplied by a number 'k' and added to the corresponding elements of j[th] row (or column) then it will be denoted by $R_j \to R_j + kR_i$ (or $C_j \to C_j + kC_i$).

Properties of determinants:

1. If A and B are two square matrices of some order then $|AB| = |A||B|$.
2. $|A^m| = |A|^m$ $(m = 2,3,4,\ldots\ldots)$
3. If $|A_{n\times n}| \neq 0$ then $|A^{-1}| = \frac{1}{|A|}$.
4. If every element of a row (column) of a determinant of A is zero then $|A| = 0$.

   Ex: $\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 2 & -4 & 5 \end{vmatrix} = 0$ ($\because R_2$ is a zero row)
5. If A is a square matrix of order n then $|A| = |A^T|$.
6. If any two rows (or columns) of a determinant are identical then the value of determinant is zero.

   Ex: $\begin{vmatrix} 1 & 3 & 7 \\ 2 & 4 & 5 \\ 2 & 4 & 5 \end{vmatrix} = 0$ $(\because R_2 = R_3)$

7. If the corresponding elements of any two rows (or columns) of a determinant are proportional (or in the same ratio) then the value of the determinant is zero.

   Ex: $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 1 \\ 4 & 2 & 6 \end{vmatrix} = 0$ $(\because R_3 = 3R_1)$
8. If any two rows (or columns) of a determinant are interchanged then the sign of determinant changes.

   Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = 27$

   $\begin{vmatrix} 7 & 8 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = (-1)\ (27) = -27$ $(\because R_1 \leftrightarrow R_3)$
9. If each element of a row (or a column) of a determinant is multiplied by a constant 'k' then the value of the determinant will be multiplied by 'k'. (or the value of the new determinant is 'k' times the value of the original determinant).

   Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 4k & 5k & 6k \\ 7 & 8 & 0 \end{vmatrix} = k\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = 27\,k$

   Note: $|kA_{n\times n}| = k^n|A_{n\times n}|$
10. If each element of a row (or column) of a determinant is multiplied by the same constant k and added to the corresponding elements of some other row (column) then the value of the determinant remains same.

    Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 1+4k & 2+5k & 3+6k \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}$
11. The each element of a row (or column) of a determinant is expressed as a sum of two or more terms then the determinant can be expressed as the sum of two or more determinants.

    Ex: $\begin{vmatrix} a+\lambda_1 & b+\lambda_2 & c+\lambda_3 \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ d & e & f \\ g & h & i \end{vmatrix}$
12. The determinant of upper triangular, lower triangular, diagonal, scalar (or) unit matrix is equal to product of its diagonal elements.
13. If A is an orthogonal matrix of order 'n' then $|A| = \pm 1$.
14. If A is an unitary matrix of order 'n' then $|A| = \pm 1$
15. $|\bar{A}| = \overline{|A|}$
16. $|A^\theta| = \overline{|A|}$
17. The determinant of a hermitian matrix is always a real number.
18. If A is an idempotent matrix than $|A| = 0$ (or) 1
19. $|I_n| = 1$
20. The determinant of a skew symmetric matrix of odd order is zero.

Inverse of a square matrix

Minor of an element:

If $A = (a_{ij})$ is a square matrix of order 'n' then the minor of an element a_{ij} in A is the determinant of a square matrix that remains after deleting corresponding the i[th] row and j[th] column of A. It is denoted by M_{ij}.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

the minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$,

the minor of a_{23} is $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{12}a_{31}$,

the minor of a_{32} is $M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}a_{23} - a_{13}a_{21}$.

Cofactor of an element: If $A = (a_{ij})$ is a square matrix of order 'n' then the cofactor of an element a_{ij} is denoted by A_{ij} and defined as $(-1)^{i+j}M_{ij}$ where M_{ij} is a minor of a_{ij}.

$$\therefore A_{ij} = (-1)^{i+j}M_{ij}$$

Ex: If $A=\begin{bmatrix}1 & 2 & -4\\ 3 & -5 & -2\\ 0 & 8 & 9\end{bmatrix}$ then the cofactor of

$$(a_{23}=-2)=A_{23}=(-1)^{2+3}M_{23}=(-1)\begin{vmatrix}1 & 2\\ 0 & 8\end{vmatrix}=-8$$

the cofactor of $a_{31}=0$ is

$$A_{31}=(-1)^{3+1}M_{31}=\begin{vmatrix}2 & -4\\ -5 & -2\end{vmatrix}=-24$$

Note:

1. If $A=(a_{ij})$ is a square matrix of order 'n' then the sum of the products of the elements of any row (column) with their cofactors is always equal to $|A|$ or det (A).

   i.e., $\sum_{j=1}^{n} a_{ij}M_{ij}=|A|$ and $\sum_{i=1}^{n} a_{ij}M_{ij}=|A|$

2. If $A=(a_{ij})$ is a square matrix of order 'n' then the sum of the products of elements of any row (or column) with the cofactors of the corresponding elements of some other row (or column) is zero.

   i.e., $\sum_{j=1}^{n} a_{ij}M_{kj}=0$ & $\sum_{i=1}^{n} a_{ij}M_{ik}=0$

Cofactor matrix:

If $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ is a square matrix of order 3 then the matrix

$B=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$ where A_{ij} is the cofactor of an element a_{ij} is called cofactor matrix of $A_{3\times3}$.

Adjoint matrix:

If $B_{n\times n}$ is a cofactor matrix of matrix $A_{n\times n}$ then the adjoint matrix of $A_{n\times n}$ is denoted by adj(A) and defined as B^T.

$\therefore$ adj(A) $= B^T$.

Singular matrix:

A square matrix A of order 'n' is said to be singular matrix if $|A_{n\times n}|=0$.

Non-singular matrix:

A square matrix A of order 'n' is said to be non-singular matrix if $|A_{n\times n}|\neq 0$.

Note:

If A and B are non-singular matrices of same order then AB is non-singular matrix of same order.

Inverse (or) reciprocal of a square matrix:

If for a non-singular matrix A of order 'n' there exists another non-singular matrix B of order 'n' such that $AB=BA=I_n$ then B is called the inverse of A. It is denoted by A^{-1}.

$\therefore B=A^{-1}$ and $AA^{-1}=A^{-1}A=I_n$.

Note:

If the inverse of a square matrices A exists then the matrix is called invertible matrix.

Inner product:

The inner product of two vectors

$X=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$ and $Y=\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix}$ is denoted by

X · Y and defined as

$$X\cdot Y=X^TY=[x_1\ x_2\ \ldots\ x_n]\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix}$$

$= x_1y_1+x_2y_2+\ldots+x_ny_n$

which is a scalar quantity.

Note:

1. $X^TY=Y^TX$ i.e Inner Product is symmetric
2. $X.Y=0 \Rightarrow$ the vectors X and Y are perpendicular.
3. $X.Y=\pm1 \Rightarrow$ the vectors X and Y are parallel.

Length or norm of a vector:

If $X=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$ is a vector of order n then the positive square root of inner product of X and X i.e X^TX is called length of X and it is denoted by $\|X\|$.

$\therefore \|X\|=\sqrt{X\cdot X}=\sqrt{x_1^2+x_2^2+\ldots+x_n^2}$

Ex: If $X=[1\ \ 2\ \ 3]$

then $\|X\|=\sqrt{1^2+2^2+3^2}=\sqrt{9}=3$

Unit vector or normal vector:

A vector X is said to be a unit vector if $\|X\|=1$

Ex: If $X=[0\ \ -1\ \ 0]$

then $\|X\|=\sqrt{0^2+(-1)^2+1^2}=1$

Orthogonal vectors:

The two column vectors X_1 & X_2 are said to be orthogonal if $X_1\cdot X_2=X_1^TX_2=0$.

Ex: $X_1=\begin{bmatrix}1\\ -2\\ 0\end{bmatrix}$ and $X_2=\begin{bmatrix}2\\ 1\\ 0\end{bmatrix}$

$\Rightarrow X_1\cdot X_2=X_1^TX_2=X_2^TX_1$

$=(1)(2)+(-2)(1)+(0)(0)=0$

$\therefore X_1, X_2$ are orthogonal vectors.

Orthogonal set :

The column vectors $X_1, X_2, \ldots X_n$ of same order are said to be orthogonal if

$X_i\cdot X_j=X_i^TX_j=0$ for all $i\neq j$.

or

A set S of column vectors $X_1, X_2, \ldots, X_n$ of same order 'n' is said to be an orthogonal set it

$X_i^TX_j=0\ \forall\ i\neq j$.

Ex: $X_1=\begin{bmatrix}1\\ -2\\ 0\end{bmatrix}$, $X_2=\begin{bmatrix}2\\ 1\\ 0\end{bmatrix}$, $X_3=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

$\Rightarrow X_1^TX_2=0,\ X_1^TX_3=0$ and $X_2^TX_3=0$

$\therefore X_1, X_2, X_3$ are orthogonal vectors.

Orthonormal vectors/Orthonormal set:

The column vectors $X_1, X_2, \ldots X_n$ of same order are said to be orthonormal if $X_i^TX_j=\begin{cases}0, & \forall i\neq j\\ 1, & \forall i=j\end{cases}$.(or)

A set S of column vectors $X_1, X_2, \ldots X_n$ of same order is said to be an orthonormal set if

$X_i^TX_j=\delta_{ij}=\begin{cases}0, & \forall i\neq j\\ 1, & \forall i=j\end{cases}$.

Ex: $X_1=\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}$, $X_2=\begin{bmatrix}0\\ -1\\ 0\end{bmatrix}$, $X_3=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

$\Rightarrow X_1^TX_2=0, X_1^TX_3=0, X_2^TX_3=0$ and

$X_1^TX_1=1, X_2^TX_2=1, X_3^TX_3=1$

$\therefore X_1, X_2, X_3$ are orthonormal vectors.

Note:

The orthonormal vectors $X_1, X_2, \ldots X_n$ are said to form an orthonormal system.

Orthogonal matrix:

A square matrix of order 'n' is said to be orthogonal if its rows and columns (form an orthonormal system) are orthonormal vectors.

Note:

If a matrix A of order 'n' is an orthogonal matrix then $A^{-1}=A^T$.

Ex: $A=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$

$\Rightarrow X_1=\frac{1}{3}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$, $X_2=\frac{1}{3}\begin{bmatrix}2\\ 1\\ 2\end{bmatrix}$, $X_3=\frac{1}{3}\begin{bmatrix}2\\ -2\\ -1\end{bmatrix}$

$\Rightarrow X_1^TX_2=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)+\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right)=0$

Similarly $X_1^TX_3=0,\ X_2^TX_3=0$

and $X_1^T X_1 = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = \sqrt{\frac{9}{9}} = 1$

Similarly $X_2^T X_2 = 1$, $X_3^T X_3 = 1$

$\therefore X_1, X_2, X_3$ are orthonormal vecters and A is an orthogonal matrix.

Properties of adjoint matrix:

1. If A is a square matrix of order 'n' then $A\,adj(A) = adj(A)\,A = |A| I_n$.
2. If O is a zero matrix of order 'n' then $adj(O) = O$
3. If I_n is a unit matrix of order 'n' then $adj(I_n) = I_n$
4 If D is diagonal matrix of order 'n' then adj (D) is also a diagonal matrix.

Ex: If $A = \begin{bmatrix} \ell & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & n \end{bmatrix}$ then

$$adj(A) = \begin{bmatrix} mn & 0 & 0 \\ 0 & \ell n & 0 \\ 0 & 0 & \ell m \end{bmatrix}.$$

5. If A is a square matrix of order 'n' then $adj(A^T) = (adj(A))^T$
6. If A and B are non-singular matrices then $adj(AB) = adj(B).adj(A)$. (If A or B is a singular matrix then $adj(AB) = adj(B).adj(A)$)
7. If $|A| = 0$ then $|adj(A)| = 0$.
8. If A is a square matrix of order 'n' then $|adj(A)| = |A|^{n-1}$.
9. If A is a non-singular matrix of order 'n' then $adj(adj(A)) = |A|^{n-2} A$.
10. If A is a square matrix of order 'n' then $|adj(adj(A))| = |A|^{(n-1)^2}$.
11. If A is a square matrix then $adj(A^\theta) = (adj(A))^\theta$.
12. If A is a Hermitian matrix then adj(A) is also hermitian.
13. If A is a symmetric matrix then adj (A) is also symmetric.
14. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Properties of inverse of a square matrix:

01. The necessary and sufficient condition for a square matrix A to possess (have) the inverse is that $|A| \neq 0$ (i.e nonsingular matrix).
02. If the inverse of a square matrix A exists then it is unique.
03. If A, B and C are non-singular matrices then (i) $(AB)^{-1} = B^{-1}A^{-1}$ (reversal law)
    (ii) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
04. If A is non-singular matrix of order 'n' then $(A^T)^{-1} = (A^{-1})^T$ and $(A^\theta)^{-1} = (A^{-1})^\theta$.
05. If A is non-singular & symmetric matrix then A^{-1} is also symmetric.
06. If AB = BA for two non-singular matrices then $A^{-1}B^{-1} = B^{-1}A^{-1}$.
07. Cancellation law: If A is a non-singular matrix of order 'n' and B, C are square matrices of same order as A then
    (i) $AB=AC \Rightarrow B = C$ (left cancellation law).
    (ii) $BA=CA \Rightarrow B = C$ (Right cancellation law).
08. If the product of two non-zero square matrices is a zero matrix then both A &B must be singular matrices.
09. If A and B are 'n' rowed square matrices such that AB = O and B is non-singular matrix then A = O.
10. If A is an idempotent matrix and $A \neq I$ then A is singular.
11. If A is an orthogonal matrix then A^T and A^{-1} are also orthogonal.
12. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and k is a scalar then $(kA)^{-1} = \frac{1}{k}A^{-1}$.
13. The determinant of skew-symmetric matrix of odd order is zero.
14. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ such that $|A| \neq 0$ then

$$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}.$$

15. $I_n^{-1} = I_n$.
16. If A is a non-singular matrix of order n then $A^{-1} = \frac{1}{|A|} adj(A)$

Note:

- If $|A| = 0$ then A^{-1} does not exist.
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $A^{-1} = \frac{adj(A)}{|A|}$

$$= \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Examples

01: Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow |A| = 4 - 6 = -2 \neq 0$

$\Rightarrow$ A is non-singular matrix

$\Rightarrow A^{-1}$ exists

$\therefore A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{(-2)}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

02. Find the cofactor matrix and adjoint of

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Cofactor of $a_{11}=0$ is $A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$

Cofactor of $a_{12}=1$ is $A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8$

Cofactor of $a_{13}=2$ is $A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$

Cofactor of $a_{21}=1$ is $A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$

Cofactor of $a_{22}=2$ is $A_{22} = (-1)^{2+2}\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$

Cofactor of $a_{23}=3$ is $A_{23} = (-1)^{2+3}\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3$

Cofactor of $a_{31}=3$ is $A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

Cofactor of $a_{32}=1$ is $A_{32} = (-1)^{3+2}\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$

Cofactor of $a_{33}=1$ is $A_{33} = (-1)^{3+3}\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$

$\therefore$ B = Cofactor matrix of $A = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$

and $adj(A) = B^T = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$.

03. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$\Rightarrow |A| = -2 \neq 0$

$\Rightarrow$ A is non-singular matrix and A^{-1} exists.

Now $adj(A) = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$\therefore A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{(-2)}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$.

1.3. Rank of a Matrix

One of the most important characteristics of a matrix is its rank. A knowledge of the rank of a matrix is important when dealing with the uniqueness of solution of sets of linear equations and can save a lot of tedious work.

Sub-matrix of a matrix: If a matrix is obtained form a given matrix A of order m×n by deleting (or leaving) some rows or columns or both in $A_{m\times n}$ then the resulting matrix is called sub-matrix of $A_{m\times n}$.

Ex: If $A = \begin{bmatrix} 1 & -2 & 3 & 7 \\ 5 & 4 & 8 & 2 \\ 9 & 10 & 4 & 6 \end{bmatrix}$ is a matrix of order 3×4 then $B = [-2]$, $C = [5 \ \ 4 \ \ 8 \ \ 2]$,

$D = \begin{bmatrix} 4 & 8 \\ 10 & 4 \end{bmatrix}$ and $E = \begin{bmatrix} -2 & 3 & 7 \\ 4 & 8 & 2 \\ 10 & 4 & 6 \end{bmatrix}$ are sub-matrices of $A_{3\times4}$.

Here B is obtained by deleting rows 2 & 3 and columns 1, 3 & 4.

C is obtained by deleting rows 1 and 3.

D is obtained by deleting row 1 and columns 1 & 4.

E is obtained by deleting only column 1.

Note:

1. Sub-matrix may or may not be square.
2. The matrix A itself is a sub-matrix of A because it is obtained from A by leaving no rows or columns.
3. If A is square matrix of order 'n' then A itself is a highest order square sub matrix.

Minor of a matrix:

If A is a matrix of order m×n then the determinant of every square sub-matrix of A is called a minor of a matrix A.

Ex: If $A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 7 & 6 & 9 & 8 \\ 3 & 5 & -8 & -2 \end{bmatrix}$ is a matrix of order 3×4 then $|1|$, $|2|$, $|-3|$ etc are 1- rowed minors of A,

$\begin{vmatrix} 1 & 2 \\ 7 & 6 \end{vmatrix}$, $\begin{vmatrix} 2 & -3 \\ 6 & 9 \end{vmatrix}$, $\begin{vmatrix} 9 & 8 \\ -8 & -2 \end{vmatrix}$ etc are 2-rowed minors of A

and $\begin{vmatrix} 1 & 2 & -3 \\ 7 & 6 & 9 \\ 3 & 5 & -8 \end{vmatrix}$, $\begin{vmatrix} 1 & -3 & 4 \\ 7 & 9 & 8 \\ 3 & -8 & -2 \end{vmatrix}$, etc are 3-rowed minors of A or minors of order 3 of matrix A.

Rank of a Matrix

Definition:

A number r (r∈N) is said to be rank of matrix A of order m×n if

(i) there exists at least one non-zero minor of order 'r'

(ii) all minors of order (r+1) if they exist, are zeros.

Rank of a matrix A = The order of any largest non vanishing minor of A.

Ex: $A = \begin{bmatrix} 3 & 2 & 2 \\ 6 & 2 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

Here $|A| = 3(2-8) - 2(6-12) + 2(12-6)$

$= -18 + 12 + 12$

$= 6 \neq 0$

$\Rightarrow$ the highest order square sub matrix is non-singular.

$\therefore \rho(A) = 3$ = order of non-singular square sub-matrix of order 3.

Ex: $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 5 & 7 \\ 4 & 2 & 6 \end{bmatrix}$

Here square sub-matrix of order 3 is singular (ie $|A| = 0$)

$\Rightarrow \rho(A) \neq 3$ (i,e $\rho(A) < 3$)

Now, consider one of the 2[nd] order square sub-matrix $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

$\Rightarrow \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 7 \neq 0$

$\therefore \rho(A) = 2$ = The order of non-singular sub-matrix of order 2.

Ex: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}$

Here square sub-matrix of order 3 is singular and all square sub-matrices of order 2 are singular.

$\Rightarrow \rho(A) \neq 3$ and $\rho(A) \neq 2$

But A is not a null matrix.

$\therefore \rho(A) = 1$.

Note:

01. Rank of a null matrix is zero.
02. If $A \neq O$ then $\rho(A) \geq 1$.
03. If $A_{m\times n} \neq O$ then $\rho(A_{m\times n}) \leq \min\{m,n\}$.
04. If $|A_{n\times n}| \neq 0$ then $\rho(A_{n\times n}) = n$.
05. If $|A_{n\times n}| = 0$ then $\rho(A_{n\times n}) < n$.
06. $\rho(I_n) = n$.
07. $\rho(A) = \rho(A^T)$.
08. $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$.
09. $\rho(A+B) \leq \rho(A) + \rho(B)$.
10. $\rho(A-B) \geq \rho(A) - \rho(B)$.
11. The rank of a diagonal matrix is equal to the number of non-zero diagonal elements.

Equivalent matrices:

The two matrices A and B are said to be equivalent matrices if a matrix B of order m×n is obtained from an m×n matrix A by applying a finite number of elementary transformations.

If A and B equivalent matrices then we write A ~B.

Here the symbol '~' is used for equivalence.

Elementary matrices:

If a matrix A is obtained from a unit matrix I by applying a single elementary row (or column) operation on I then the matrix A is called an elementary matrix (or E-matrix).

Ex: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

which is an elementary matrix.

Echelon form:

A matrix A of order m×n is said to be in row echelon form if

(i) zero rows (if any occur) then they must be below the non-zero rows

(ii) the number of zeros before the first non-zero element in each row is less than the number of such zeros in the next non zero row.

Note:

If a matrix A of order m×n is in row echelon form then $\rho(A_{m\times n})$ = number of non-zero rows in A

Ex: $A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}_{4\times3}$

It is in row echelon form

$\therefore \rho(A_{4\times3}) = 3$

Ex: $A = \begin{bmatrix} 3 & 9 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is in row echelon form

$\therefore \rho(A) = 3$

Ex: $A = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ is in an echelon form.

$\therefore \rho(A) = 2$

Ex: $A = \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -1 \\ 0 & 0 & -4 \end{bmatrix}$ is not in an echelon form, because the number of zeros in 2[nd] row is not less than the number of zeros in the 3[rd] row. But, we can reduce A into its echelon form by applying some elementary row operations on it.

By applying $R_3 \to R_3 - 4R_2$ on A, we obtain A

$\sim \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ which is in an echelon form.

$\therefore \rho(A) = 2$ = the number of non-zero rows.

Elementary operations do not change the order or rank of a matrix.

Linear combination of vectors: If $X_1, X_2, \ldots X_r$ are 'r' vectors of order 'n' and $k_1, k_2, \ldots k_r$ are 'r' scalars then the expression of the form $k_1X_1 + k_2X_2 + \ldots + k_rX_r$ is also a vector and it is called linear combination of the vectors $X_1, X_2, \ldots X_r$.

Linearly dependent vectors: The vectors X_1, X_2,.....X_r of same order 'n' are said to be linearly dependent if there exist scalars (or numbers) $k_1, k_2, \ldots k_r$ not all zero such that $k_1X_1+k_2X_2+\ldots\ldots+k_rX_r = O$ where O denotes the zero vector of order n.

Linearly independent vectors

The vectors $X_1, X_2, \ldots\ldots X_r$ of same order n are said to be linearly independent vectors if every relation of the type

$$k_1X_1+k_2X_2+\ldots+k_rX_r = 0$$

$$\Rightarrow k_1 = k_2 = \ldots\ldots = k_r = 0$$

Note:

If $X_1, X_2, \ldots\ldots\ldots, X_r$ are linearly dependent vectors then at least one of the vectors can be expressed as a linear combination of other vectors.

A necessary and sufficient condition that the vectors $X_1 = [x_{11}\ x_{12} \ldots x_{1m}]$, $X_2 = [x_{21}\ x_{22} \ldots x_{2m}]$, ... $X_n = [x_{n1}\ x_{n2} \ldots x_{2m}]$ of order m are

(i) linearly independent is that $\rho(A) = n$

(ii) linearly dependent is that $\rho(A) \neq n$

$$\text{where } A = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1m} \\ x_{21} & x_{22} & \ldots & x_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ x_{n1} & x_{n2} & \ldots & x_{nm} \end{bmatrix}_{n\times m} \text{ and}$$

n = number of given vectors.

If A is a square matrix of order n and $|A| = 0$ then the rows and columns are linearly dependent.

If A is a square matrix of order 'n' and $|A| \neq 0$ then the rows and columns are linearly independent.

If A is a matrix with rank 'r' then the number of linearly independent rows (or columns) of A is 'r'.

i.e., Rank(A) = number of linearly independent rows (or columns) of A

Any subset of a linearly independent set is itself linearly independent set.

If a set of vectors includes a zero vector then the set of vectors is linearly dependent set.

Examples

01. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Applying $R_2 \to R_2-2R_1$, $R_3 \to R_3-3R_1$ and $R_4 \to R_4-6R_1$, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in an echelon form

$\therefore \rho(A) = 3$

02. Test whether the following vectors are linearly dependent or independent $X_1 = [1\ \ 2\ \ 2]$, $X_2 = [2\ \ 1\ \ -2]$ and $X_3 = [2\ \ -2\ \ 1]$

Sol: Consider the relation

$X_1 = a\,X_2 + BX_3$

$\Rightarrow [1\ 2\ 2] = a[2\ 1\ -2] + b[2\ -2\ 1]$

$\Rightarrow 2a + 2b = 1$ ---- (1)

$a - 2b = 2$ ---- (2)

$-2a + b = 2$ ---- (3)

Solving (1) and (2), we get $a = 1$, $b = -\frac{1}{2}$

If we substitute these values in equation (3), we get $-\frac{5}{2} = 2$

$\because$ (3) is not satisfied

$\therefore$ The vectors are linearly independent

03. Show that the vectors $X_1 = (1\ 2\ 3)$, $X_2 = (3\ -2\ 11)$, $X_3 = (1\ -6\ 5)$ form a linearly dependent set.

Sol: Consider the relation

$X_1 = a\,X_2 + bX_2$

$\Rightarrow [1\ 2\ 3] = a[3\ -2\ 11] + b[1\ -6\ 5]$

$\Rightarrow 1 = 3a + b$ ---- (1)

$2 = -2a - 6b$ ---- (2)

$3 = 11a + 5b$ ---- (3)

solving (1) and (2),

we get $a = \frac{1}{2}$ and $b = -\frac{1}{2}$

If we substitute these values in equation (3), we get $3 = 3$

$\therefore$ The vectors are linearly dependent

1.4. System of linear equations

Linear non-homogeneous equation:

An equation of the form $a_1x_1+a_2x_2+\ldots+a_nx_n = b_1$ where $x_1, x_2, \ldots x_n$ are unknowns and $a_1, a_2, \ldots a_n$, & $b_1(\neq 0)$ are constants is called a non-homogeneous linear equation in 'n' unknowns.

Ex: $2x+4y-3 = 0$ is non-homogeneous linear equation in 2 variables.

Linear homogeneous equation:

An equation of the form $a_1x_1+a_2x_2+\ldots+a_nx_n = 0$ where $x_1, x_2, \ldots\ldots x_n$ are unknowns and $a_1, a_2, \ldots\ldots a_n$ are constants is called a homogeneous linear equation in 'n' variables (or unknowns).

Ex: $3x - 4y = 0$ is a homogeneous linear equation in 2 variables.

System of linear equations:

The set (or collection or group) of linear equations in 'n' variables is called system of linear equations.

Non-homogeneous system of linear equations:

If the system of 'm' non-homogeneous linear equation in 'n' variables $x_1, x_2, \ldots x_n$ is given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \text{-----(1)}$$

then the set of these equations can be written in matrix form as

$$AX = B \text{ ------------------ (2)}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$ is called the coefficient matrix,

$X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}_{n\times 1}$ is the matrix of unknowns,

and $B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}_{m\times 1}$ is a constant matrix of the system (1).

Solution of the linear equations:

A set of values of the variables $x_1, x_2, \ldots x_n$ which satisfies all the given 'm' equations of (1) simultaneously is called a solution of the system.

Ex: Consider the system of 2 non-homogeneous linear equations

$$3x+y = 3,\ 2x+y = 1.$$

Here $x = 2$, $y = -3$ is a solution of the above system of linear equations.

Consistent system:
If the system of equations (1) has one or more solutions then system (1) is called consistent system of equations.

If a system (1) has unique solution then system (1) is said to be determinate.

If a system (1) has more solutions than one solution then the system (1) is called indeterminate system.

Ex: (1) Consider $2x + y = 1$
$x - y = 2$.

This is a non-homogeneous system of two equations in two unknowns(variables).This system has only one solution namely $x = 1$, $y = -1$. Therefore the system is consistent and has unique solution.

Ex: (2) Consider $x + y = 2$
$2x + 2y = 4$

This is a non-homogeneous system of two equations in two unknowns. The second equation reduces to the first equations on dividing by 2. i.e., $x + y = 2$.
This is a set of one equations in two unknowns x and y.

Now $x + y = 2 \Rightarrow y = 2 - x$
If we choose $x = k$ where k is an arbitrary constant then we get $y = 2 - k$.
$\therefore$ The solution of the system is $x = k$,
$y = 2 - k$.

For different values of k we will get the corresponding values y.

Thus we can have infinite set of values of x and y corresponding to different values of k which will satisfy the given system of equations.

Hence the system is consistent and has infinitely many solutions.

Inconsistent system:
If the system of equations (1) has no solution then the system (1) is said to be inconsistent system.

Ex:
Consider $x + y = 2$
$2x + 2y = 6$

This is a non-homogeneous system of two equations in two unknowns. From the 2[nd] equation we get $x + y = 3$ where as from the first equation $x + y = 2$. Here we can not get a set of values of x and y which satisfy the above equations simultaneously. Hence the solution of the system does not exist and the system is inconsistent.

Augmented matrix:
For the system $AX = B$
The matrix obtained by writing the elements of B as the last column or $(n+1)^{th}$ column in the coefficient matrix A then the resulting matrix is called augmented matrix and it is denoted by [A:B] or [A|B].
From (1), we have

$$[A\,|\,B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{bmatrix}_{m\times(n+1)}$$

Note:

Condition for consistency:
A system of 'm' linear equations in 'n' unknowns given by $AX = B$ is consistent if and only if the coefficient matrix A and the augmented matrix [A | B] have the same rank i.e., $\rho(A) = \rho([A\,|\,B])$

If A is an n-rowed non-singular matrix, X is an $n\times1$ matrix and B is an $n\times1$ matrix then the system of equations $AX = B$ has a unique solution.

Working rule for finding the solution of equations $AX = B$

Let $AX = B$ be the given system of 'm' non-homogeneous linear equations in 'n' variables $x_1, x_2, \ldots x_n$.

Steps:

1. Write the given system of linear equations in matrix from $AX = B$
2. Write the augmented matrix [A|B] and reduce it to echelon form by applying only row operations.
3. Find the $\rho(A)$, ρ (A|B) and n = number of variables .

4. (i) If $\rho(A) = \rho(A\,|\,B) = n$ then the system is consistent and has a unique solution.

   (ii) If $\rho(A) = \rho(A\,|\,B) < n$ then the system is consistent and has an infinite number of solutions.

   (Here (n-r) variables can be assigned arbitrary values where $\rho(A) = r$ and remaining variables can be expressed in terms of these arbitrary values)

   (iii) If $\rho(A) \neq \rho(A\,|\,B)$ then the system is inconsistent and has no solution.

5. If solution exists then re-write the system of equations from the echelon form of [A | B] and find the solution by backward substitution.

Homogeneous system of linear equations :
If the system of 'm' homogeneous linear equations in 'n' variables $x_1, x_2, \ldots x_n$ is given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \text{------ (1)}$$

then the set of these equations can be written in matrix form as $AX = B$ --- (2)

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$ is called the coefficient matrix,

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times1}$ is the matrix of unknowns,

and $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times1}$ = O is null matrix of the (1).

Note:

1. For every homogeneous system $AX = O$, $\rho(A)$ and $\rho(A\,|O)$ are always same. Therefore $AX = O$ is always consistent and solution exists.
2. $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = O$ is always a solution of every homogeneous system $AX = O$ and it is called a trivial solution or zero solution or unique solution.
3. If any solution exists other than trivial solution for $AX = O$ then such a solution is called non-zero solution or non-trivial solution.
4. If $\rho(A) = r$ and n = number of variables then the number of linearly independent solutions of $AX = O$ is (n-r).
5. If the number of equations is less than the number of variables in $AX = O$ then $AX = O$ will have non-zero solution.
6. If A is non-singular matrix in $AX = 0$ then $AX = O$ will have only trivial solution.
7. If A is singular matrix in $AX = O$ then $AX = O$ will also have non-trivial solution.

Procedure to find the solution of $AX = O$

Let $AX = O$ be the given system of m – homogeneous linear equations in n - variables $x_1, x_2, \ldots x_n$.

Steps:

1. Write the given system of linear equation in matrix form as $AX = O$.
2. Write the coefficient matrix A and reduce it to echelon form by applying only row operations.
3. Find $\rho(A)$ and n= number of variables
4. (i) If $\rho(A) = n$ then $AX = O$ has only trivial solution.

(ii) If $\rho(A) \neq n$ then $AX = O$ also has non-trivial solution or infinite number of solutions. Here we have to assign arbitrary values for $(n-r)$ variables and the remaining variables can be expressed in terms of these arbitrary values.

5. If the non-trivial solution exists then rewrite the system of equations from the echelon form of the matrix A and find the solution by backward substitution.

Examples

01. Solve the following equations
$x + y + z = 6,\ y + 3z = 11,\ x + z = 2y$

Sol: Given $\left.\begin{aligned} x + y + z &= 6 \\ y + 3z &= 11 \\ x - 2y + z &= 0 \end{aligned}\right\}$ --- (1)

The above system (1) can be written as $AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and

$B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

Consider $[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 1 & -2 & 1 & 0 \end{array}\right]$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & -3 & 0 & -6 \end{array}\right]$$

$R_3 \to R_3 + 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 9 & 27 \end{array}\right] \text{------- (2)}$$

The above matrix is in an echelon form

$\therefore\ \rho(A) = 3,\ \rho(A \mid B) = 3$ and $n = 3$

Here $\rho(A) = \rho(A \mid B) = 3$ and $n = 3$

$\therefore$ The system is consistent and has unique solution.

From (2), we have

$$x + y + z = 6$$
$$y + 3z = 11$$
$$9z = 27$$
$$\therefore z = 3,\ y = 2,\ x = 1$$

02. Solve the system of equations
$x-y+2z+t=2,\ 3x+2y+t=1,\ 4x+y+2z+2t = 3$

Sol: The above set of equations can be written in matrix form as $AX = B$ where

$$[A \mid B] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 4 & 1 & 2 & 2 & 3 \end{array}\right]$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, we get

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 0 & 5 & -6 & -2 & -5 \\ 0 & 5 & -6 & -2 & -5 \end{array}\right]$$

Applying $R_3 \to R_3 - R_2$, we get

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 0 & 5 & -6 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is in an echelon form

$\therefore\ \rho(A) = 2,\ \rho(A|B) = 2$ and $n = 4$

Hence $\rho(A) = 2 = \rho(A|B) < n = 4$

$\therefore$ The system is consistent and has an infinite number of solutions

From the above echelon form, the reduced system is $x-y+2z+t = 2$
$5y-6z-2t = -5$

To find these solutions, we have to assign arbitrary values for $p = n-r = 4-2 = 2$ variables and remaining 2 variables shall be found in terms of these arbitrary values.

Let $z = k_1$ and $t = k_2$ where k_1 and k_2 are arbitrary constants. Then

$$y = \frac{6}{5}k_1 + \frac{2}{5}k_2 - 1 \text{ and } x = 1 - \frac{4}{5}k_1 - \frac{3}{5}k_2$$

$\therefore$ The system has infinite number of solutions for different values of k_1 and k_2.

03. Test the following system for consistency
$x+2y+2z = 5,\ 2x+y+3z = 6,\ 3x-y+2z = 4,$
$x+y+z = -1$

Sol: The above system (1) can be written in matrix form as $AX = B$ where

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 2 & 1 & 3 & 6 \\ 3 & -1 & 2 & 4 \\ 1 & 1 & 1 & -1 \end{array}\right]$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$,

$R_4 \to R_4 - R_1$, we get

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & -7 & -4 & -11 \\ 0 & -1 & -1 & -6 \end{array}\right]$$

$R_3 \to 3R_3 - 7R_2$
$R_4 \to 3R_4 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -2 & -14 \end{array}\right]$$

$R_4 \to 5R_4 - 2R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -60 \end{array}\right]$$

The above matrix is in an echelon form.
$\therefore \rho(A) = 3$ and $\rho(A|B) = 4$
But $\rho(A) = 3 \neq \rho(A|B) = 4$
$\therefore$ The given system is inconsistent and has no solution.

04. Solve the system $x+y+w = 0,\ y+z = 0,$
$x+y+z+w = 0,\ x+y+2z = 0$

Sol: The above system can be written in matrix in form as $AX = O$

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{----- (2)}$$

The above matrix is in an echelon form
$\therefore \rho(A) = 4$ and $n = 4$
Hence $\rho(A) = n = 4$
$\therefore$ The given system has only trivial solution
$x = 0,\ y = 0,\ z = 0,\ w = 0$

05. Solve the system $3x-y-z+2w = 0,$
$3x+y-z-2w = 0,\ 12x+3y-4z-6w = 0.$

Sol: The above system can be written in matrix form as $AX = O$ where

$$A = \begin{bmatrix} 3 & -1 & -1 & 2 \\ 3 & 1 & -1 & -2 \\ 12 & 3 & -4 & -6 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 7 & 0 & -14 \end{bmatrix}$$

$R_3 \to 2R_3 - 7R_2$

$$\sim \begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{----- (2)}$$

The above matrix is in an echelon form

$\therefore \rho(A) = 2$ and $n = 4$

But $\rho(A) = 2 < n = 3$

$\therefore$ The given system has infinite number solutions. So $p = n - r = 4 - 2 = 2$ variables are considered as arbitrary constants and remaining two variables shall be uniquely determined in terms of these arbitrary choosen values.

From (2), the reduced system is

$3x - y - z + 2w = 0$

$2y - 4w = 0$

Let $z = k$ and $w = k_2$ when k_1, k_2 are arbitrary constant. Then $y = 2k_2$ and $x = \frac{k_1}{3}$.

The system has non-trivial solution for different values of k_1 and k_2.

1.5. Eigen Values, Eigen vectors & Cayley - Hamilton Theorem

If $A = (a_{ij})$ is a matrix of order 'n', λ is a scalar and I is unit matrix of order 'n' then the matrix

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

is called eigen matrix or characteristic matrix of $A_{n \times n}$

- $|A - \lambda I|$ is called characteristic polynomial of $A_{n \times n}$
- $|A - \lambda I| = 0$ is called characteristic equation of $A_{n \times n}$.
- the roots of a characteristics equation $|A - \lambda I| = 0$ are called eigen values or characteristics roots or latent rots or proper values of the matrix $A_{n \times n}$.

Ex: If $A = \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$ then

(i) $A - \lambda I = \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2-\lambda & 6 \\ 3 & 8-\lambda \end{bmatrix}$ is a characteristic matrix of A.

(ii) $|A - \lambda I|$ (or) $\begin{vmatrix} 2-\lambda & 6 \\ 3 & 8-\lambda \end{vmatrix}$ or $\lambda^2 - 10\lambda - 2$ is a characteristic polynomial of A.

(iii) $|A - \lambda I| = 0$ or $\lambda^2 - 10\lambda - 2 = 0$ is a characteristic equation of A.

(iv) the roots $\lambda = 5+3\sqrt{3}, 5-3\sqrt{3}$ of $\lambda^2 - 10\lambda - 2 = 0$ are eigen values of matrix A of order 2.

Note:
The number of eigen values of a square matrix A of order 'n' is equal to order of the matrix. Because the characteristic equation of matrix A of order 'n' is an arbitrary polynomial in terms of unknown scalar λ of degree 'n'.

Spectrum of matrix A:
The set of eigen values of matrix A of order 'n' is called spectrum of A.

Ex: If 1, 2, 3 are eigen values of $A = \begin{bmatrix} 2 & 0 & 6 \\ 4 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

then spectrum of A = {1,2,3}.

Eigen Vector:

If for a given square matrix A of order 'n' there exists a non-zero column vector X of order n such that $AX = \lambda X$ for some scalar λ (or eigen value) then the non-zero column vector X is called an

eigen vector or characteristic vector or proper vector or latent vector of a matrix of order 'n'.

Ex: Consider $A = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}$ & $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq O$

$$AX = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda X$$

Here the matrix equation $AX = \lambda X$ is satisfied for a matrix A and a vector X.

$\therefore$ The vector $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector of matrix A.

Ex: Consider $A = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}$ & $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq O$

$$AX = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 12 \end{bmatrix} \neq \lambda X$$

The matrix equation $AX = \lambda X$ is not satisfied for a matrix A and a vector X.

$\therefore$ The vector $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigen vector of matrix A.

Note:
If X is an eigen vector of A of order 'n' corresponding to eigen value λ then by definition of eigen vector of A, we have

$AX = \lambda X$

$\Rightarrow (A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

-------------(1)

The above equation (1) is a homogeneous system of n equations in n unknown $x_1, x_2, \ldots x_n$

Here, the non-trivial solution X is called an eigen vector.

Algebraic multiplicity (A.M) of an eigen value:
If an eigen value λ of a square matrix A of order 'n' is repeated 'm' times then the number 'm' is called algebraic multiplicity (A.M) of an eigen value λ.

Ex: If 2,5,5,4,4,4 are eigen values of matrix of order 6×6 then

(i) the A.M of $\lambda = 2$ is 1
(ii) the A.M of $\lambda = 5$ is 2
(iii) the A.M of $\lambda = 4$ is 3

Geometric multiplicity (G.M) of an eigen value:
If the number 'p' is the number of linearly independent eigen vectors of matrix A of order 'n' corresponding to an eigen value λ then 'p' is called geometric multiplicity of λ and it is given by

$p = n - r$ = (number of variables) $- \rho(A - \lambda I)$.

Examples:

01. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ and hence find the normalized eigen vectors of A.

Sol: Given matrix is $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Eigen values: To find eigen values of A of order 2×2 consider the characteristic equation $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda = 1, 5$$

The eigen values of A are 1 and 5

Eigen vectors: To find the eigen vector of A order 2×2.

Consider the homogeneous system $(A - \lambda I)X = O$ or $AX = \lambda X$.

$$\Rightarrow \begin{bmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{----- (1)}$$

Case (i) Put $\lambda = 1$ in (1), we get

i.e $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 = 0$

Let $x_1 = k_1$ where k_1 is an arbitrary constant. Then $x_2 = -3k_1$.

$\therefore$ The eigen vector of A corresponding to an eigen value $\lambda_1 = 1$ is

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -3k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ where } k_1 \neq 0$$

Here we get infinite number of eigen vectors by giving different values for k_1. The simplest eigen vector is $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ for $k_1 = 1$

$\Rightarrow$ the length of the vector X_1 is

$$\|X_1\| = \sqrt{(1)^2 + (-3)^2} = \sqrt{10}$$

$\therefore$ The normalized eigen vectors of X_1 is

$$\frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

Case (ii): Put $\lambda = 5$ in (1), we get

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_2 = 0$$

Let $x_2 = k_2$. Then $x_1 = k_2$ where k_2 is an arbitrary constant.

$\therefore$ The eigen vector of A corresponding to an eigen value $\lambda_2 = 5$ is

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } k_2 \neq 0$$

The simplest eigen vector is

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } k_2 = 1$$

$\Rightarrow$ the length of the vector X_2 is

$$\|X_2\| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$\therefore$ The normalized eigen vectors is

$$\frac{X_2}{\|X_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

02. Find the geometric multiplicity of eigen value $\lambda = 2$ of a matrix $A = \begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$ when $-1, 2, 2$ are eigen values of A.

Sol: Given $A = \begin{bmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$ and the eigen value $\lambda = 2$

Consider $(A - \lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 0-\lambda & -2 & -2 \\ -1 & 1-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & -2 \\ -1 & -1 & 2 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } \lambda = 2$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & -2 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & -2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $\rho(A - \lambda I) = 2$, $n = 3$

$p = n - r = n - \rho(A - \lambda I) = 3 - 2 = 1$

$\therefore$ A.M = 2 and G.M = 1

Here G.M < A.M

Note:

The geometric multiplicity of an eigen value cannot exceed its algebraic multiplicity i.e G.M≤A.M

Properties of eigen values

- The eigen values of a diagonal matrix, scalar matrix, unit matrix, upper or lower triangular matrix are just its diagonal elements.

Ex: $A = \begin{bmatrix} 2 & 5 & -6 \\ 0 & 8 & 0 \\ 0 & 0 & -9 \end{bmatrix}$

$\therefore \lambda = 2, 8, -9$ are eigen values of A.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 8 & 0 \\ 0 & -9 & 3 \end{bmatrix}$

$\therefore \lambda = 1, 8, 3$ are eigen values of A.

Ex: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \lambda = 2, 4, 0$ are eigen values of A.

Ex: $A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

$\therefore \lambda = -4, -4, -4$ are eigen values of A.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \lambda = 1, 1, 1$ are eigen values of A.

- If $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are eigen values of matrix A of order n then

(i) $\lambda_1 + \lambda_2 + \ldots\ldots + \lambda_n = tr(A) = a_{11} + a_{22} + \ldots\ldots + a_{nn}$ where $a_{11}, a_{22}, \ldots\ldots a_{nn}$ are diagonal elements of A of order n.

(ii) $\lambda_1 \cdot \lambda_2 \ldots\ldots \cdot \lambda_n = |A|$

Note:

- 0 is an eigen value of matrix A if and only if A is singular
- If all the eigen values of A are non-zero then A is non-singular.

Ex: If $A = \begin{bmatrix} 2 & -9 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ then the eigen values of A are $\lambda_1 = 2$, $\lambda_2 = 4$ and $\lambda_3 = -4$

Here $\lambda_1 + \lambda_2 + \lambda_3 = tr(A) = 2 + 4 + (-4) = 2$

and $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A| = -32$

- If λ is an eigen value of a matrix A and k is a scalar then

(i) λ^m is eigen value of matrix A^m ($\because m \in N$)

(ii) $k\lambda$ is an eigen value of matrix kA.

(iii) $\lambda + k$ is an eigen value of matrix A+kI

(iv) $\lambda - k$ is an eigen value of matrix A−kI

(v) $a_0 + a_1\lambda + a_2\lambda^2$ is an eigen value of matrix $a_0 I + a_1 A + a_2 A^2$.

Ex: If 1,3,2 are eigen values of

$$A = \begin{bmatrix} 1 & 8 & 9 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ then}$$

(i) $1^4, 3^4, 2^4$ are eigen values of matrix A^4

(ii) 3(1), 3(3), 3(2) are eigen values of matrix 3A.

(iii) 1+4, 3+4, 2+4 are eigen values of matrix A+4I

(iv) 1-7, 3-7, 2-7 are eigen values of matrix A-7I

- If λ is an eigen value of a non-singular matrix A then

(i) $\frac{1}{\lambda}$ is an eigen value of A^{-1} and

(ii) $\frac{|A|}{\lambda}$ is eigen value of adj(A).

Ex: If $\lambda = 2, -3, 7$ are eigen values of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -3 & 0 \\ 9 & 4 & 7 \end{bmatrix} \text{ then}$$

$\frac{1}{2}, \frac{-1}{3}, \frac{1}{7}$ are eigen values of A^{-1}

$\frac{-42}{2}, \frac{-42}{-3}, \frac{-42}{7}$ are eigen values of adj(A) when $|A| = -42$.

The eigen values of A and A^T are same

Ex: If $A = \begin{bmatrix} 2 & -3 & -4 \\ 0 & 8 & 2 \\ 0 & 0 & 9 \end{bmatrix}$ then

$A^T = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 8 & 0 \\ -4 & 2 & -9 \end{bmatrix}$

$\Rightarrow$ the eigen value of A are 2,8,–9 and the eigen values of A^T are 2, 8, –9

$\therefore$ The matrices A and A^T have same eigen values.

If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also another eigen value of same matrix A.

Ex: If 1, 1, –1 are eigen value of an orthogonal matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ then eigen values 1 and 1 are reciprocals.

Ex: If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix then its eigen values are $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$.
Here the eigen value $\cos\theta + i\sin\theta$ is a reciprocal of other eigen value of $\cos\theta - i\sin\theta$.

➢ If $a + \sqrt{b}$ is an eigen value of a real matrix A then $a - \sqrt{b}$ is also other eigen value of matrix A.

Ex: If $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 4 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ then its eigen values are $-3+\sqrt{5}, 3-\sqrt{5}$.
Here $3+\sqrt{5}$ and $3-\sqrt{5}$ are pair of conjugate surds.

➢ If a+ib is an eigen value of a real matrix A then a–ib is also other eigen value of A.

Ex: If $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$ then its eigen value are $0, i\sqrt{10}, -i\sqrt{10}$

Here $0+i\sqrt{10}$ and $0-i\sqrt{10}$ are pair of complex conjugates.

➢ The eigen values of a real symmetric (or Hermitian) matrix are always real and the eigen values of a real skew-symmetric (or Skew-Hermitian) matrix are either zero or purely imaginary.

Ex: $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is a real symmetric matrix

$\therefore$ The eigen value of A are $\lambda = 2, 2, 4$ which are real.

Ex: $A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$ is a real skew-symmetric matrix.

$\therefore$ The eigen values of A are $\lambda = 0, 0 + i\sqrt{14}, 0 - i\sqrt{14}$. Here one of the eigen values is zero and other eigen values are purely imaginary.

➢ The eigen values of an orthogonal (or unitary) matrix are of unit modulus.

Ex: $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

$\Rightarrow \lambda_1 = \frac{1-i\sqrt{3}}{2}, \lambda_2 = \frac{1+i\sqrt{3}}{2}$

Here $|\lambda_1| = \left|\frac{1-i\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

Similarly

$|\lambda_2| = \left|\frac{1+i\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

Properties of eigen vectors:

➢ 'λ' is an eigen value of a matrix A if and only if there exists a non-zero vector X such that AX= λX.

➢ If X is an eigen vector of matrix A corresponding to the eigen value 'λ' then kX for every non-zero scalar k, is also an eigen vector of A corresponding to the same eigen value 'λ'.

Ex: If $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigen vector of

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 7 \\ 0 & 0 & 7 \end{bmatrix}$ corresponding to an eigen

value $\lambda = 1$ then $kX_1 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ is also an eigen

vector of A for every non-zero scalar 'k'. Therefore, for each eigen value of a matrix there are infinitely many eigen vectors.

➢ If X is an eigen vector of a matrix A then X cannot correspond to more than one eigen value of A. i.e We cannot get same eigen vector for two different eigen values of matrix A.

Ex: If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then the eigen values and eigen vectors of A are 1,6 and $\begin{bmatrix} k \\ -k \end{bmatrix}, \begin{bmatrix} 4k_1 \\ k_1 \end{bmatrix}$ respectively.

Here the two eigen vectors $X = \begin{bmatrix} k \\ -k \end{bmatrix}$ and $Y = \begin{bmatrix} 4k_1 \\ -k_1 \end{bmatrix}$ never be same for any scalars k and k_1.

➢ If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigen values of A square matrix A of order 'n' then the corresponding eigen vectors $X_1, X_2, \ldots, X_n$ of matrix A are linearly independent.

Ex: $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to $\lambda = 1$ and

$X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ corresponding to $\lambda = 6$

$\therefore X_1$ and X_2 are linearly independent vectors corresponding to different eigen values $\lambda_1 = 1$ and $\lambda_2 = 6$.

➢ The eigen vectors of A and A^m are same

➢ The eigen vectors of A and A^{-1} are same

➢ The eigen vectors of A and

$P(A) = a_0I + a_1A + a_2A^2$ are same

➢ The eigen vectors of A and kA are same

➢ The eigen vectors of A and A^T are not same.

➢ If some eigen values of matrix A are repeated then eigen vectors of A may or may not be linearly independent.

Cayley-Hamilton theorem

Theorem:
Every square matrix satisfies its own characteristic equation.

Ex: If $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ is a characteristic equation of matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ then

by Cayley-Hamilton theorem, we have

$A^3 - 12A^2 + 36A - 32I = O$

Applications of Cayley-Hamilton theorem:

The important applications of Cayley-Hamilton theorem are

(i) To find higher powers of matrix A

(ii) To find the inverse of matrix A.

(i) **Higher powers of matrix A**

Let A be the square matrix of order n, Then by Cayley-Hamilton theorem, we have

$a_0I + a_1A + a_2A^2 + \ldots + a_{n-1}A^{n-1} + a_nA^n = 0$ -- (1)

Let k be positive integer such that $k \geq n$. Then to find the A^k of A, pre-multiply both sides of (1) by A^{k-n}.

$$A^{k-n}(a_0I+a_1A+a_2A^2+..+a_{n-1}A^{n-1}+a_nA^n) = O$$

$$a_0A^{k-n}+a_1A^{k-n+1}+.......+a_{n-1}A^{k-1}+a_nA^k = O$$

$$\therefore A^k = \frac{-1}{a_n}\left[a_{n-1}A^{k-1}+.......+a_1A^{k-n+1}+a_0A^{k-n}\right]$$

The above relation can be used to express any positive integral powers of A i.e A^k ($\because k \geq n$) linearly in terms of lower order terms.

(ii) Inverse of matrix:

If A is non-singular matrix of order n then A^{-1} exists. To find A^{-1}, pre-multiply both sides of (1) by A^{-1}.

$$A^{-1}[a_0I+a_1A+a_2A^2+...+a_{n-1}A^{n-1}+a_nA^n] = O$$

$$a_0A^{-1}+a_1I+a_2A+....+a_{n-1}A^{n-2}+a_nA^{n-1} = O$$

$$\therefore A^{-1} = \frac{-1}{a_0}\left[a_1I+a_2A+...+a_{n-1}A^{n-2}+a_nA^{n-1}\right]$$

Examples

01. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ then find A^9.

Sol: Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow \lambda^2 - 0\lambda + (-5) = 0 \Rightarrow \lambda^2 - 5 = 0$

By Cayley-Hamilton theorem, we have

$A^2 - 5I = 0$

$\therefore A^2 = 5I$

Now $A^9 = AA^8 = A(A^2)^4 = A(5I)^4 = A625I$

$\therefore A^9 = 625\,A$

02. If $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$ is a characteristic equations of $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ then find A^{-1} in terms of A and A^2.

Sol: The characteristic equation of matrix A of order 3 is given by $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

$\Rightarrow A^3 - 6A^2 + 7A + 2I = 0$

Multiply both sides of above with A^{-1}

i,e $A^{-1}(A^3 - 6A^2 + 7A + 2I) = A^{-1}\,0$

$\Rightarrow A^2 - 6A + 7I + 2A^{-1} = 0$

$\therefore A^{-1} = \frac{1}{2}\left[6A - 7I - A^2\right]$

Diagonalizable matrix:

If for a given square matrix A of order n, there exists a non-singular matrix P such that $P^{-1}AP = D$ (or) $AP = PD$ where D is a diagonal matrix then A is said to be diagonalizable matrix.

Note:

If X_1, X_2, X_3 are linearly independent eigen vectors of $A_{3\times3}$ corresponding to eigen values λ_1, λ_2, λ_3 then P can be found such that $P^{-1}AP = D$ (or) $AP = PD$ where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \text{ and}$$

$$P = [X_1 \quad X_2 \quad X_3].$$

LEVEL – 1 Questions

01. If AB = BA then which of the following need not be true (n is a +ve integer)?
(a) $AB^n = B^nA$
(b) $(AB)^n = A^nB^n$
(c) $(A + B)(A - B) = A^2 - B^2$
(d) A = I or B = I

02. If a diagonal matrix is commutative with every matrix of the same order then it is necessarily a
(a) a zero matrix (b) a unit matrix
(c) a scalar matrix (d) a symmetric matrix

03. If A is any m x n matrix such that AB and BA are both defined then B is a matrix of order
(a) n x n (b) m x m
(c) m x n (d) n x m

04. If $A_{m\times n}$ and $B_{n\times p}$ are matrices, then the number of multiplications and additions in computing AB are
(a) m p n, mp(n – 1)
(b) (m – 1) pn, mp(n – 1)
(c) m p n, (m – 1)(n – 1)
(d) mn, nm

05. The number of terms in the expansion of the determinant of $A_{n\times n}$ is
(a) n^2 (b) 2^n (c) n! (d) n

06. If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & 1 & -3 \end{pmatrix}$ and

$\text{Adj } A = \begin{pmatrix} -3 & 4 & k \\ -3 & -1 & 4 \\ -3 & 1 & 1 \end{pmatrix}$ then k =
(a) –2 (b) 4 (c) –5 (d) 6

07. Rank of unit matrix I_n is
(a) one (b) zero
(c) n (d) not defined

08. Rank of a non – singular matrix $A_{n\times n}$ is
(a) one (b) zero
(c) n (d) not defined

09. Rank of a singular matrix $A_{n\times n}$ is
(a) n (b) zero (c) <n (d) ≥n

10. Rank of a diagonal matrix $A_{n\times n}$ is
(a) n
(b) no. of zeros in the diagonal
(c) no. of non zero elements in the diagonal
(d) zero

11. If $A_{m\times 1}$ is non zero column matrix and $B_{1\times n}$ is a non zero row matrix then $\rho(AB)$ =
(a) m (b) n (c) 1 (d) zero

12. If $A_{n\times n}$ is a non singular matrix and $B_{n\times n}$ is a matrix then $\rho(AB)$ =
(a) Rank of A (b) ρ(B)
(c) 0 (d) 1

13. If $\rho(A_{n\times n})$ is equal to (n – 2) then $\rho(\text{Adj } A)$ =
(a) (n – 1) (b) (n – 2)
(c) (n – 3) (d) zero

14. $\rho(A_{3\times3}) = 2$ then | A | =
(a) 1 (b) 0
(c) Any non zero number (d) 2

15. If $\rho(A) = n$ then which of the following is false?
(a) There exists at least one non zero minor of order 'n'.
(b) All the minors of A of order greater than 'n' vanish
(c) All the minors of A of order 'n' are not zero.
(d) If A is nxn matrix then $|A| \neq 0$.

16. If $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ then $\rho(A)$ =
(a) 2 (b) 1
(c) 0 (d) does not exist

17. If $A = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$ then $\rho(A)$ =
(a) 2 (b) 1
(c) 0 (d) does not exist

18. If $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 6 & 1 \end{pmatrix}$ then $\rho(A)$ =
(a) 0 (b) 1 (c) 2 (d) 3

19. If $A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 3 & -3 \end{bmatrix}$ then $\rho(A)$ =
(a) 0 (b) 1 (c) 2 (d) 3

20. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \\ 2 & 3 & -1 \end{bmatrix}$ then $\rho(A)$ =
(a) 0 (b) 1 (c) 2 (d) 3

21. If $A = \begin{bmatrix} 2 & -3 & -1 \\ -4 & 6 & 2 \\ 6 & -9 & -3 \end{bmatrix}$ then $\rho(A)$ =
(a) 0 (b) 1 (c) 2 (d) 3

Let k be positive integer such that $k \geq n$. Then to find the A^k of A, pre-multiply both sides of (1) by A^{k-n}.

$$A^{k-n}(a_0I+a_1A+a_2A^2+..+a_{n-1}A^{n-1}+a_nA^n) = O$$

$$a_0A^{k-n}+a_1A^{k-n+1}+\ldots\ldots+a_{n-1}A^{k-1}+a_nA^k = O$$

$$\therefore A^k = \frac{-1}{a_n}\left[a_{n-1}A^{k-1}+\ldots\ldots+a_1k^{k-n+1}+a_0A^{k-n}\right]$$

The above relation can be used to express any positive integral powers of A i.e A^k ($\because k \geq n$) linearly in terms of lower order terms.

(ii) **Inverse of matrix:**

If A is non-singular matrix of order n then A^{-1} exists. To find A^{-1}, pre-multiply both sides of (1) by A^{-1}.

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$$a_0A^{-1}+a_1I+a_2A+\ldots+a_{n-1}A^{n-2}+a_nA^{n-1} = O$$

$$\therefore A^{-1} = \frac{-1}{a_0}\left[a_1I+a_2A+\ldots+a_{n-1}A^{n-2}+a_nA^{n-1}\right]$$

Examples

01. If $A = \begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$ then find A^9.

Sol: Given $A = \begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

$\Rightarrow |A-\lambda I| = 0$

$\Rightarrow \lambda^2-0\lambda+(-5) = 0 \Rightarrow \lambda^2-5 = 0$

By Cayley-Hamilton theorem, we have

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Sol: The characteristic equation of matrix A of order 3 is given by $\lambda^3-6\lambda+7\lambda+2 = 0$

$\Rightarrow A^3-6A^2+7A+2I = 0$

Multiply both sides of above with A^{-1}

i.e $A^{-1}(A^3-6A^2+7A+2I) = A^{-1}\,0$

$\Rightarrow A^2-6A+7I+2A^{-1} = 0$

$$\therefore A^{-1} = \frac{1}{2}\left[6A-7I-A^2\right]$$

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If X_1, X_2, X_3 are linearly independent eigen vectors of $A_{3\times3}$ corresponding to eigen values λ_1, λ_2, λ_3 then P can be found such that $P^{-1}AP = D$ (or) $AP = PD$ where

$$D = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \text{ and}$$

$$P = [X_1 \;\; X_2 \;\; X_3].$$

LEVEL – 1 Questions

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(b) no. of zeros in the diagonal
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(d) zero

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(c) (n – 3) (d) zero

14. $\rho(A_{3\times3}) = 2$ then | A | =
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15. If ρ(A) = n then which of the following is false?
(a) There exists at least one non zero minor of order 'n'.
(b) All the minors of A of order greater than 'n' vanish
(c) All the minors of A of order 'n' are not zero.
(d) If A is nxn matrix then $|A| \neq 0$.

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(a) 2 (b) 1
(c) 0 (d) does not exist

18. If $A = \begin{pmatrix}2 & 3 & -1\\ 4 & 6 & 1\end{pmatrix}$ then ρ(A) =
(a) 0 (b) 1 (c) 2 (d) 3

19. If $A = \begin{bmatrix}2 & -2\\ -1 & 1\\ 3 & -3\end{bmatrix}$ then ρ(A) =
(a) 0 (b) 1 (c) 2 (d) 3

20. If $A = \begin{bmatrix}1 & 0 & 0\\ 3 & 2 & 4\\ 2 & 3 & -1\end{bmatrix}$ then ρ(A) =
(a) 0 (b) 1 (c) 2 (d) 3

21. If $A = \begin{bmatrix}2 & -3 & -1\\ -4 & 6 & 2\\ 6 & -9 & -3\end{bmatrix}$ then ρ(A) =
(a) 0 (b) 1 (c) 2 (d) 3

22. If $A = \begin{bmatrix} 2 & -3 & 4 \\ 3 & -2 & 5 \\ 1 & 1 & 1 \end{bmatrix}$ then $\rho(A) =$

(a) 0 (b) 1 (c) 2 (d) 3

23. Which of the statements is false?
Rank of the matrix is equal to
(a) The no. of its linearly independent rows
(b) The no. of its linearly independent columns
(c) The no. of non – zero rows
(d) The order of its largest non – vanishing minor

24. The system AX = B has no solution if
(a) $\rho(A) = \rho(A/B)$ (b) $\rho(A) < \rho(A/B)$
(c) $\rho(A) > \rho(A/B)$ (d) $\rho(A) \geq \rho(A/B)$

25. The system AX = O in n variables has infinitely many solutions if
(a) $\rho(A) = n$ (b) $\rho(A) > n$
(c) $\rho(A) < n$ (d) $\rho(A) \geq n$

26. If A is a square matrix then the system AX = O has non zero solutions when
(a) $|A| \neq 0$
(b) $|A| = 0$
(c) $\rho(A)$ = no. of variables
(d) $\rho(A) \geq$ no. of variables

27. If $|A| \neq 0$ for the system AX = O then which of the following is false?
(a) The system has unique solution
(b) The system has a zero solution
(c) The system has a trivial solution
(d) The system has a non - zero solution

28. The system AX = B has a unique solution if
(a) $\rho(A)$ = no. of variables in the system
(n) = $\rho(A/B)$
(b) $\rho(A) < n$
(c) $\rho(A) > n$
(d) $\rho(A) = \rho(A/B) < n$

29. If $\rho(A) = r$ and number of variables = n, then the number of linearly independent solutions of the system AX = O is

(a) n (b) r (c) n – r (d) n + r

30. The system
$2x + 3y + 4z = 1$
$3y - z = 2$
$-6y + 2z = 3$ has
(a) no solution
(b) unique solution
(c) infinitely many solutions
(d) two linearly independent solutions

31. If the system $x + y + x = 0$,
$(\lambda + 1) y + (\lambda + 1) z = 0$,
$(\lambda^2 - 1) z = 0$ has two linearly independent solutions, then $\lambda =$
(a) 1 (b) –1 (c) 0 (d) 3

32. The system given in example 31 has only one independent solution when $\lambda =$
(a) 1 (b) –1 (c) 0 (d) 3

33. The system given in example 31 has no linearly independent solutions when $\lambda =$
(a) 1 (b) –1 (c) 0 (d) ± 1

34. If $\rho(A) = 1$ and number of variables = 3, then the system AX = 0 has
(a) Two linearly independent solutions
(b) 3 linearly independent solutions
(c) 1 linearly independent solutions
(d) no independent solutions

35. The rank of the matrix, every element of which is unity is =
(a) one (b) zero
(c) order of the matrix (d) >1

36. If A is a skew symmetric matrix then which of the following is false
(a) $\rho(A) = 1$ (b) $\rho(A) = 0$
(c) $\rho(A) = 2$ (d) $\rho(A) = 3$

37. Rank of an elementary matrix =
(a) order of the matrix (b) one
(c) zero (d) two

38. When elementary transformations are applied on a given matrix then its rank
(a) will remain same
(b) reduced by one
(c) becomes one
(d) is equal to the order of matrix

39. If $\rho(A_{m \times n}) = r$ then
(a) $r \leq m, r \leq n$ (b) $r > m, r \leq n$
(c) $r > m, r > n$ (d) $r \leq m, r > n$

40. The rank of matrix $\begin{bmatrix} 2 & 3 & -1 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ is =__

(a) 1 (b) 2 (c) 3 (d) 4

41. If A and B are any two matrices then which of the following is false?
(a) $\rho(AB) < \rho(A)$ (b) $\rho(AB) \leq \rho(A)$
(c) $\rho(AB) < \rho(B)$ (d) $\rho(AB) > \rho(B)$

42. If a square matrix A is orthogonal then
(a) $A = A^1$ (b) $A^1 = A^{-1}$
(c) $A = \overline{A}$ (d) $A^1 = A$

43. If $(\overline{A^1}) = A$ then A is
(a) orthogonal (b) symmetric
(c) hermitian (d) unitary

44. If A is skew - hermitian then iA is
(a) hermitian (b) symmetric
(c) unitary (d) skew – symmetric

45. If a matrix is in row echelon form then its rank is =
(a) no. of non – zero rows of the matrix
(b) no. of zero rows of the matrix
(c) order of the matrix
(d) one

46. If $A_{n \times n}$ is a non singular matrix then which of the following is false?
(a) $\rho(A) = \rho(A^1)$ (b) $\rho(A) = \rho(A^{-1})$
(c) $\rho(A) = \rho(I_n)$ (d) $\rho(A) = 1$

47. If $\rho(A_{n \times n}) = n$ then $\rho(\text{Adj } A) =$
(a) n (b) n – 1 (c) 1 (d) 0

48. If $\rho(A_{n \times n}) = n - 1$ then $\rho(\text{Adj } A) =$
(a) n (b) n – 1 (c) 1 (d) zero

49. The system $3x + 4y - 2z = 4$
$6x + 8y - 4z = 10$ has
(a) no solution
(b) infinitely many solutions
(c) unique solution
(d) one linearly independent solution

50. Rank of a non zero matrix is
(a) equal to order of the matrix
(b) equal to the number of rows of the matrix
(c) < 1
(d) ≥1

51. Eigen values of a matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ are
(a) 2, 3 (b) -2, -3
(c) 1, 6 (d) -1, -6

52. Eigen values of the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ are
(a) 1, 2, 3 (b) 1, 1, 1
(c) 2, 2, 2 (d) 0, 1, 2

53. Eigen values of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ are
(a) 3, 2, -1 (b) 3, 2, 1
(c) 3, 3, -1 (d) 4, -1, 0

54. If '0' and '3' are eigen values of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then the third eigen value is
(a) 7 (b) 8 (c) 15 (d) 0

55. If $-1 + \sqrt{3}$ is eigen value of matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ then the other two eigen values are
(a) $-1-\sqrt{3}$, 2 (b) $-1+\sqrt{3}$, 1
(c) $-1-\sqrt{3}$, -2 (d) $-1+\sqrt{3}$, -2

56. If 3 is the eigen value of the singular matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then the other eigen values of matrix are
(a) 0, 15 (b) 8, 15
(c) 7, 15 (d) 0, 3, 2

57. Which of the following is an eigen vector of the matrix $\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$?
(a) (1, 0, 0) (b) (1, 0, 1)
(c) (0, 0, 1) (d) (1, 1, 1)

58. Which of the following is an eigen vector of the matrix $\begin{bmatrix}5&4\\1&2\end{bmatrix}$ corresponding to eigen value $\lambda = 6$?
(a) (4, -1) (b) (1, -4)
(c) (-1, -4) (d) (4, 1)

59. If 2, 2, 8 are eigen values of a matrix $\begin{bmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{bmatrix}$ then the matrix is
(a) singular
(b) non – singular
(c) skew symmetric
(d) triangular

60. If 2, 2, 8 are eigen values of the matrix $\begin{bmatrix}6&-2&2\\-2&3&-1\\2&-1&k\end{bmatrix}$ then k
(a) 0 (b) 1 (c) 2 (d) 3

Common Data for Questions 61 to 65

For the matrix $A = \begin{bmatrix}3&-2&5\\0&5&6\\0&0&-4\end{bmatrix}$ answer the following

61. The eigen values of A^3 are
(a) 3, 5, -4 (b) 9, 25, 16
(c) 27, 125, -64 (d) 9, 15, -12

62. The eigen values of A^{-1} are
(a) 1/3, 1/5, -1/4 (b) 1/3, 1/5, 1/4
(c) -3, -5, -4 (d) -1/3, -1/5, 1/4

63. The eigen values of 9A are
(a) 27, 45, -36 (b) 12, 14, 5
(c) 9, 15, -12 (d) $3^9, 5^9, 4^9$

64. The eigen values of Adj(A) are
(a) 3, 5, -4 (b) 9, 25, 16
(c) 20, 12, -15 (d) -20, -12, 15

65. The eigen values of A^1 are
(a) 3, 5, -4 (b) -3, -5, 4
(c) 1/3, 1/5, -1/4 (d) does not exist

66. For the matrix $\begin{bmatrix}2&-3&5\\0&1&2\\0&0&0\end{bmatrix}$ the eigen values of A^{-1} are
(a) 2, 1, 0 (b) 1/2, 1, $\propto$
(c) -2, -1, 0 (d) does not exist

67. The number of linearly independent eigen vectors corresponding to any distinct eigen value of the matrix $A_{3\times3}$ =
(a) 1 (b) 2
(c) 3 (d) cannot be determined

68. If an eigen value λ is repeated two times for a matrix $A_{3\times3}$ then the no. of linearly independent eigen vectors for λ are
(a) 2 (b) < 2 (c) ≤ 2 (d) > 2

69. For the matrix $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$ which of the following is an eigen vector?
(a) (1, 0, 1) (b) (1, 1, 0)
(c) (1, 0, 1) (d) (1, 1, 1)

70. For the matrix $\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$ which of the following is not an eigen vector?
(a) (1, 0, 0) (b) (0, 1, 0)
(c) (0, 0, 1) (d) (0, 0, 0)

71. If $A = \begin{bmatrix}1&2\\2&-1\end{bmatrix}$ then A^8 =
(a) 5I (b) 25I
(c) 625I (d) 3125I

72. If $A = \begin{bmatrix}1&2&3&4\\0&-1&5&6\\0&0&i&7\\0&0&0&-i\end{bmatrix}$ then A^4 =
(a) I_4 (b) $4I_4$ (c) $16I_4$ (d) $64I_4$

73. If $\lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + \ldots\ldots + K_n = 0$ is the characteristic equation of a matrix A then A^{-1} exists if
(a) $K_1 = 0$ (b) $K_1 \ne 0$
(c) $K_n = 0$ (d) $K_n \ne 0$

74. The sum and product of the eigen values of the matrix $\begin{bmatrix}2&2&1\\1&3&1\\1&2&2\end{bmatrix}$ are ___ and ___
(a) 7, 12 (b) 7, 5
(c) 12, 5 (d) 7, 9

75. The eigen values of a triangular matrix are
(a) diagonal elements of the matrix
(b) zero
(c) non – diagonal elements
(d) ±1

76. The eigen values of a real skew symmetric matrix are
(a) real
(b) ±1
(c) purely imaginary or zero
(d) does not exist

77. The eigen values of an orthogonal matrix are
(a) real
(b) ±1
(c) purely imaginary
(d) of unit modulus

78. For each eigen value of the matrix $A_{3\times3}$, the number of eigen vectors =
(a) 1 (b) 2 (c) 3 (d) $\propto$

79. If a matrix $A_{3\times3}$ has 3 distinct eigen values then the no. of linearly independent eigen vectors for A =
(a) 1 (b) 2 (c) 3 (d) α

80. If zero is eigen value of a square matrix A, then A is
(a) singular (b) non singular
(c) orthogonal (d) symmetric

81. If 2 is eigen value of a scalar matrix $A_{3\times3}$, then an eigen value of Adj A is
(a) 2 (b) 3 (c) 4 (d) 8

82. Which of the following is not an eigen vector of a 2 x 2 unit matrix?
(a) (0, 0) (b) (1, 1) (c) (1, 0) (d) (0, 1)

83. The solution of the system $x + 2y + 3z = 6$, $2x + 4y + z = 7$, $3x + 2y + 9z = 14$ is
(a) 1, –1, 2 (b) 1, 1, 1
(c) 1, 0, 2 (d) None

84. Find the values of k for which the system of equations $x - ky + z = 0$, $kx + 3y - kz = 0$, $3x + y - z = 0$ has only trivial solution
(a) $k \ne 2$ and $k \ne -3$ (b) $k = 2$ and $k = -3$
(c) $k \ne 2$ and $k = -3$ (d) $k = 2$ and $k \ne -3$

85. For what values of a and b, the system of equations $x + 2y + z = 6$, $x + 4y + 3z = 10$, $x + 4y + az = b$ has no solution
(a) $a = 3, b \ne 10$ (b) $a = 3, b = 10$
(c) $a \ne 3, b = 10$ (d) $a \ne 3, b \ne 10$

KEY for LEVEL – 1

01. (d)	02. (c)	03. (d)	04. (a)	05. (c)
06. (c)	07. (c)	08. (c)	09. (c)	10. (c)
11. (c)	12. (b)	13. (d)	14. (b)	15. (c)
16. (a)	17. (b)	18. (c)	19. (b)	20. (d)
21. (b)	22. (c)	23. (c)	24. (b)	25. (c)
26. (b)	27. (d)	28. (a)	29. (c)	30. (a)

31. (b)	32. (a)	33. (c)	34. (a)	35. (a)
36. (a)	37. (a)	38. (a)	39. (a)	40. (d)
41. (d)	42. (b)	43. (c)	44. (a)	45. (a)
46. (d)	47. (a)	48. (c)	49. (a)	50. (d)
51. (c)	52. (c)	53. (b)	54. (c)	55. (a)
56. (a)	57. (a)	58. (d)	59. (b)	60. (d)
61. (c)	62. (a)	63. (a)	64. (d)	65. (a)
66. (d)	67. (a)	68. (c)	69. (b)	70. (d)
71. (c)	72. (a)	73. (d)	74. (b)	75. (a)
76. (c)	77. (d)	78. (d)	79. (c)	80. (a)
81. (c)	82. (a)	83. (b)	84. (a)	85. (a)

LEVEL – 2 Questions

Matrix algebra and determinants

01. If $A=\begin{bmatrix}2 & -1 & 0\\ 0 & 3 & -1\\ 4 & 2 & 5\end{bmatrix}$ then det (A) = ____

(a) 9 (b) 38 (c) 12 (d) None

02. If $A=\begin{bmatrix}1 & 4 & 1 & 0\\ 0 & 2 & 0 & 1\\ 0 & 1 & 0 & 0\\ 2 & 0 & -1 & 2\end{bmatrix}$ then det (A) =

(a) – 3 (b) 3 (c) 0 (d) None

03. If A is a 3 x 3 matrix with |A| = 5 & B= 4A then |B| =

(a) 20 (b) 100 (c) 320 (d) 1600

04. The value of $\begin{vmatrix}x & a & a & a\\ a & x & a & a\\ a & a & x & a\\ a & a & a & x\end{vmatrix}$ is ____

(a) (x+3a) (b) $(x-a)^3$
(c) $(x+3a)(x-a)^3$ (d) $(x+3a)(x-a)$

05. If $A = (a_{ij})_{n\times n}$ for $n \ge 3$ is defined by $a_{ij} = 1$ for $i \ne j$ and $a_{ij} = 0$ for $i = j$ then $|A| =$

(a) 0 (b) – 1
(c) n(n – 1) (d) $(-1)^{n-1}(n-1)$

06. If $A=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0 & 2 & 0\\ 0 & 0 & 1 & 2 & 0 & 0\\ 0 & 0 & 2 & 1 & 0 & 0\\ 0 & 2 & 0 & 0 & 1 & 0\\ 2 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$ then | A | = ____

(a) 0 (b) –27 (c) 27 (d) None

07. If $A_{3\times4}$, $B_{4\times5}$ and $C_{5\times3}$ are three real matrices then the minimum number of multiplication operations needed to find the matrix ABC is

(a) 95 (b) 96 (c) 105 (d) 106

08. If $A_{3\times3}$ is a matrix with | A | = 3 then Adj (Adj A) =

(a) 3A (b) 9A (c) 27A (d) 91A

09. If $A=\begin{bmatrix}2 & -1 & 0\\ 0 & 3 & -1\\ 1 & 2 & 0\end{bmatrix}$ then | 2 Adj (AdjA) | =

(a) 1000 (b) 6250 (c) 1250 (d) 5000

Inverse of a Square Matrix:

10. The inverse of $A=\begin{bmatrix}2+3i & i\\ -i & 2-3i\end{bmatrix}$ is

(a) $\begin{bmatrix}2+3i & -i\\ i & 2-3i\end{bmatrix}$ (b) $\frac{1}{12}\begin{bmatrix}2-3i & -i\\ i & 2+3i\end{bmatrix}$

(c) $\begin{bmatrix}2-3i & -i\\ i & 2+3i\end{bmatrix}$ (d) Does not exist

11. The inverse of $A=\begin{bmatrix}-1 & 2 & 1\\ 3 & 1 & -2\\ 4 & 5 & 6\end{bmatrix}$ is ___

(a) $\frac{1}{57}\begin{bmatrix}2 & -1 & 2\\ 1 & -3 & 2\\ 4 & 5 & 6\end{bmatrix}$ (b) $\begin{bmatrix}1 & -2 & -1\\ -3 & -1 & 2\\ -4 & -5 & -6\end{bmatrix}$

(c) does not exist (d) None

12. If $A=\begin{bmatrix}1 & 4 & 1 & 0\\ 0 & 2 & 0 & 1\\ 0 & 1 & 0 & 0\\ 2 & 0 & -1 & 2\end{bmatrix}$ then $\det[(2A)^{-1}] =$

(a) –1/2 (b) –1/48 (c) 1 (d) –1/3

13. If $A = [a_{ij}]$ is defined by $a_{ij} = i^4 - j^4$ $\forall$ i, j where $1 \le i, j \le n$ then A^{-1} for n=5 is ____

(a) $\begin{bmatrix}0 & -5 & 9 & 0 & 7\\ 5 & 0 & 3 & 0 & 0\\ -9 & 3 & 0 & 0 & 0\\ 0 & 0 & 0 & 6 & 0\\ 7 & 0 & 0 & 6 & 0\end{bmatrix}$ (b) $\begin{bmatrix}2 & 5 & 0 & 0 & 1\\ 2 & 5 & 8 & 4 & 1\\ 0 & 0 & 1 & 4 & 2\\ 1 & 2 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 1\end{bmatrix}$

(c) Inverse does not exist (d) None

Rank of a Matrix

14. If $A=\begin{bmatrix}1 & -2 & 0 & 4\\ 2 & -1 & 1 & -2\\ -1 & 4 & 2 & 0\end{bmatrix}$ then ρ (A) =

(a) 4 (b) 3 (c) 2 (d) 1

15. The Rank of matrix $\begin{bmatrix}0 & 0 & -3\\ 9 & 3 & 5\\ 3 & 1 & 1\end{bmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

16. If every minor of order 'r' of a matrix A is zero then rank of A is

(a) > r (b) < r (c) r + 1 (d) r – 1

17. If $A=\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 2\end{bmatrix}$ then ρ (A – I) where I is the 4 × 4 unit matrix, is ____

(a) 4 (b) 3 (c) 2 (d) 1

18. If $X = [a_1\ a_2 \cdots a_n]^T$ is a non-zero vector then the rank of matrices XX^T and X^TX are

(a) 0, 0 (b) 1, 1 (c) n, 1 (d) None

19. If A= [a_{ij}] is defined by $a_{ij} = i + j$ $\forall$ i, j where $1\le i \le 4$ & $1\le j\le 5$ then ρ (A) = ___

(a) 0 (b) 1 (c) 2 (d) None

20. If the Rank of matrix $\begin{bmatrix}\mu & -1 & 0\\ 0 & \mu & -1\\ -1 & 0 & \mu\end{bmatrix}$ is 2 then μ =

(a) any row number (b) 3
(c) 1 (d) 2

21. Check whether the following vectors are linearly independent or dependent

(i) X_1 = [1 -2 4], X_2 = [2 0 -1], X_3 = [-1 1 -1]

(ii) X_1 = [2 5 10], X_2 = [4 5 -2], X_3 = [1 -4 0], X_4 = [10 5 4]

System of linear equations

22. The system 3x + 4y + 5z = a, 4x + 5y + 6z = b, 5x + 6y + 7z = c is consistent when

(a) a + b =2c (b) b + c = 2a
(c) a + c = 2b (d) a + 2b = c

23. If a 4 ×4 matrix has rank 2 then AX = O has
(a) only trivial solution.
(b) one independent solution.
(c) two independent solutions.
(d) three independent solutions.

24. For what values of k the system of equations x + y + 2z = 1, x+2y+4z=5, x+3y+kz=7 has unique solution?

(a) all values of k (b) 2
(c) all values of k except k=6 (d) 6

25. For what values of 'α' the system of linear equations $\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\\ 1 & 4 & 10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ \alpha\\ \alpha^2\end{bmatrix}$ have infinite number of solutions ?

(a) 1, -2 (b) 1, 2 (c) -1, 2 (d) -1, -2

26. Consider the following system of linear equations :

$$\begin{bmatrix}1 & 2 & -8\\ 4 & 3 & -12\\ 2 & -1 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}7\\ 5\\ d\end{bmatrix}$$

For how many values of d does this system of equations has no solution?

(a) 0 (b) 1 (c) 2 (d) ∞

27. If $A=\begin{bmatrix}1 & 4 & 6 & a & 12\\ 0 & 1 & 6 & 9 & b\\ 0 & 0 & 1 & 11 & 15\end{bmatrix}$ where a & b are real numbers then which of the following is correct?

(a) There exists values of a & b for which $\rho(A) = 2$.
(b) There exists values of a & b for which AX = O has X = O as the only solution.
(c) There exists values of a & b for which the columns of A are linearly independent.
(d) For all values of a & b the rows of A are linearly independent.

28. A is 5 × 4 matrix with real entries such that AX = O if and only if X = O where X is a 4×1 vector and O is null vector. Then $\rho(A)$ is__

(a) 4 (b) 5 (c) 2 (d) 1

29. The system $\begin{bmatrix}1 & 1 & 3\\ 4 & 3 & k\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$ has a non zero solution when k =

(a) 8 (b) 5 (c) –6 (d) – 4

30. Consider the system of equations
x+y+3z=2, x+2y+4z=3, 2x+3y+az=b.
Then for what values of a & b the system has infinitely many solutions?

(a) a = 7, b = 5 (b) a = 5, b = 5
(c) a = 7, b = 4 (d) a = 5, b = 4

31. If $A_{n\times n}$ is a matrix in which all the elements of A are 1, then the number of linearly independent solutions in the system AX = O is _____

(a) 1 (b) n (c) n–1 (d) 0

32. The system x+y+az = 0, 2x+3y+bz = 0, 3x+4y+cz =0 has a unique solution if

(a) a+b-c =0 (b) b+c-a = 0
(c) c+a-b = 0 (d) a+b+c =0

Eigen Values, Eigen Vectors and Cayley Hamilton Theorem:

33. The eigen value of $\begin{bmatrix}0 & i & i\\ i & 0 & i\\ i & i & 0\end{bmatrix}$ is

(a) -i, -i, 2i (b) i, i, -2i
(c) i, -1, 2i (d) i, -i, -2i

34. The eigen value of the matrix $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -3 & 3\end{bmatrix}$ are

(a) 1, -1, 1 (b) –1, 0, 1
(c) 1, 2, 3 (d) 1, 1, 1

35. If $A=\begin{bmatrix}2 & -1 & 0 & 0\\ 0 & 4 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & -1 & 3\end{bmatrix}$ then eigen values of A are

(a) 2, 4, -2, 3 (b) 0, 3, 4, 0
(c) 2, 2, 2, 1 (d) 1, 1, 1, 4

36. The number of distinct eigen values of the matrix $A=\begin{bmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix}_{n\times n}$

(a) n (b) n –1 (c) 1 (d) 2

37. If the Eigen values of the matrix $M=\begin{bmatrix}2 & 6 & 0\\ 1 & \alpha & 0\\ 0 & 0 & 3\end{bmatrix}$ are 4, - 1 and β, then the values of α, β are

(a) 4, 0 (b) 2, 2
(c) 1, 3 (d) – 1, 4

38. If $A=\begin{bmatrix}1 & 0\\ 2 & 4\end{bmatrix}$ then the eigen values of $4A^{-1}+3A+2I$ are

(a) 6, 15 (b) 9, 12 (c) 9, 15 (d) 7, 15

39. If -2 is an eigen value of the matrix $\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$ then the corresponding eigen vector is

(a) $(1\ \ -1\ \ 0)^T$ (b) $(0\ \ 1\ \ -1)^T$
(c) $(1\ \ 0\ \ -1)^T$ (d) $(1\ \ -1\ \ 1)^T$

40. Which of the following is an eigen vector of the matrix $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -3 & 3\end{bmatrix}$?

(a) (1, 0, 1) (b) (0, 1, 1)
(c) (1, 1, 0) (d) (1, 1, 1)

41. Let $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$. Then the number of Linearly independent Eigen vectors of A is

(a) 0 (b) 1 (c) 2 (d) 3

42. The number of linearly independent eigen vectors of $A=\begin{bmatrix}0 & 1 & 2 & 3\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

43. If –2 and 5 are eigen values and $[-4\ \ 3]^T$ & $[1\ \ 1]^T$ are the corresponding eigen vectors of a matrix A then matrix A = ____

(a) $\begin{pmatrix}1 & -4\\ -3 & 2\end{pmatrix}$ (b) $\begin{pmatrix}2 & 4\\ 3 & 1\end{pmatrix}$

(c) $\begin{pmatrix}-1 & 4\\ 3 & -2\end{pmatrix}$ (d) $\begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}$

44. If $A=\begin{bmatrix}3 & 1\\ 1 & -3\end{bmatrix}$ then A^9 = ____

(a) 10000 A (b) 100 A
(c) 10000 (d) None

45. If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ then which of the following matrix is inverse of A?

(a) A – 5I (b) A + 5I
(c) $\frac{1}{2}(A-5I))$ (d) $\frac{1}{2}(A+5I)$

KEY for LEVEL – 2

01. (b)	02. (a)	03. (c)	04. (c)	05. (d)
06. (b)	07. (b)	08. (a)	09. (d)	10. (b)
11. (d)	12. (b)	13. (c)	14. (b)	15. (c)
16. (b)	17. (b)	18. (b)	19. (c)	20. (c)
21. (i) L.I.	(ii) L.D.		22. (c)	23. (c)
24. (c)	25. (b)	26. (d)	27. (d)	28. (a)
29. (a)	30. (a)	31. (c)	32. (a)	33. (a)
34. (d)	35. (a)	36. (d)	37. (c)	38. (c)
39. (c)	40. (d)	41. (b)	42. (b)	43. (d)
44. (a)	45. (c)			

2 Calculus

Sir Isaac Newton
(1643 – 1727)

G. W. Von Leibnitz
(1646 – 1716)

"I was like a boy playing on the sea shore, and diverting myself now and then finding a smoother pebble or prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me".

- Isaac Newton

"It is unworthy of excellent men to lose hours like slaves in the labor of calculation which could safely be relegated to a anyone else if machines were used".

- Leibnitz

2.0. Introduction

Calculus carried physics and Chemistry and their cognate fields to numberless victories including the motion of machines, mechanisms, automobiles, air-crafts and rockets, the development of electricity and radio, spectral analysis and weather forecasting.

In a word, almost everything around us is indebted to calculus.

2.1. Limit

Function: A relation $f: A \to B$ is said to be a function if $\forall\ x \in A$ there exists a unique $y \in B \ni f(x) = y$ or $(x, f(x))$ or (x,y).

Types of functions:

Explicit function:
A function expressed as
$z = f(x_1, x_2, \ldots\ldots x_n)$ where $x_1, x_2, \ldots\ x_n$ are independent variables and z is a dependent variable is defined as an explicit function.
Ex: $y = x^2 - 2x + 2$

Implicit function:
A function expressed as $\phi(z, x_1, x_2, \ldots\ldots x_n) = 0$ is defined as an implicit function.
Ex: $x^3 + y^3 - 3xy = 0$

Composite function:
If $z = f(x,y)$ where $x = \phi(t)$ & $y = \psi(t)$ then the function z is called composite function of 't' (or) function of function.
Ex: If $z = f(x,y) = x^2 + y$
where $x = \phi(t) = t+1$ & $y = \psi(t) = 1-t$
$\therefore z = f(x,y) = (t+1)^2 + (1-t)$ is a composite function of 't'.

Some Special Functions:

Polynomial function:
A function of the form
$f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$
where $a_0, a_1, a_2, \ldots, a_n$ are real numbers, $a_n \neq 0$ & $n \in W$, is called a polynomial function of degree 'n'.

Ex: $f(x) = 2 + 3x + 4x^2$ is a polynomial function of 'x' with degree 2.

Note:
A polynomial function of degree '0' is called a constant polynomial function (or) simply constant function.

Rational function:
If h(x) and g(x) are two polynomial functions then the function of the form

$$f(x) = \frac{h(x)}{g(x)}\ (g(x) \neq 0) \text{ is called rational function.}$$

Ex: $f(x) = \dfrac{h(x)}{g(x)} = \dfrac{2x^3 - 2x + 2}{3x^2 + 2x + 2}$ is a rational function.

Algebraic function:
If a relation arises due to performing a finite number of fundamental operations additions, subtraction, multiplication, division, root extraction etc. on polynomial functions then such a relation is called Algebraic function.

Ex: (1) $f(x) = (2x^2 + 3x + 2)\left(\sqrt{x^2+1}\right)$

(2) $f(x) = \sqrt{2x^2 + 3x}$

(3) $f(x) = \dfrac{2x^3 + 4}{\sqrt{x^2+1}}$

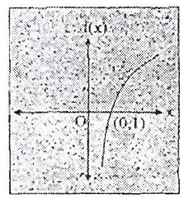
Issac Newton and Leibnitz independently developed calculus which leads to the development of differential and integral equations of mathematical physics

Note:
All polynomial functions are algebraic but not the converse.
A function that is not algebraic is called transcendental function.

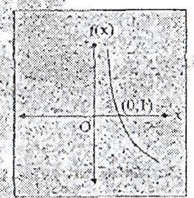
Even function:
A function f(x) is said to be an even function if $f(-x) = f(x)$.
Ex: $\cos x, |x|, x^2 \ldots\ldots\ldots$

Odd Function:
A function f(x) is said to be an odd function if $f(-x) = -f(x)$.
Ex: $\sin x, x, x^3, \ldots\ldots\ldots\ldots$

Logarithmic function:
A function of the form $f(x) = \log_a x$ where $a \in R \ni a > 0, a \neq 1$ & $x > 0$ is called Logarithmic function.

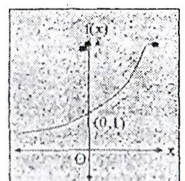


Case – I
For a > 1

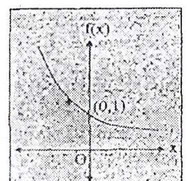


Case – II
For 0 < a < 1

Exponential function:
A function of the form $f(x) = a^x = e^{x\log a}$
$(a > 0, a \neq 1$ & $x \in R)$ is called an Exponential function.
Graph of exponential function can be as follows:



Case – I
For a > 1

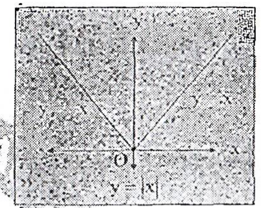


Case – II
For 0 < a < 1

Modulus function:
A function of the form $f(x) = |x|$

$$= \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is called Modulus function (or) Absolute Value function.

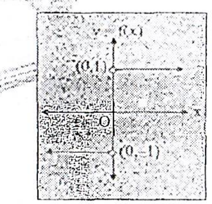


Note:
1. $|x|$ is a continuous function for every $x \in R$.
2. $|x|$ is not differentiable function at $x = 0$.
3. $|x+a|$ is not differentiable function at $x = -a$ where $a \in R$.

Signum function:
A function of the form $f(x) = \text{sgn}(x)$

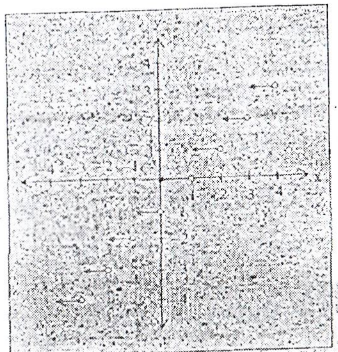
$$= \begin{cases} \dfrac{|x|}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases} = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases} \text{ is}$$

called Signum function.



Note: $\text{sgn}\, f(x) = \begin{cases} \dfrac{|f(x)|}{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$

Step function or Greatest integer function or Bracket function:
A function of the form $f(x) = [x] = n$ such that $n \leq x < n+1\ \forall n \in z$ & $x \in R$ is called greatest integer function where [x] is equal to greatest integer less than or equal to x.



Ex:

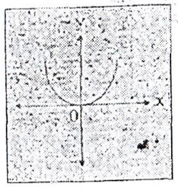
01. [7.2] = 7	01. [x] = −1 for $-1 \le x < 0$
02. [7] = 7	02. [x] = 0 for $0 \le x < 1$
03. [−1.2] = −2	03. [x] = 1 for $1 \le x < 2$
04. [0] = 0	04. [x] = 2 for $2 \le x < 3$

Note:
Step function is discontinuous at every integer point.

Symmetric properties of a curve:

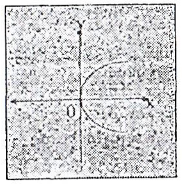
1. If the given equation of curve contains only even powers of 'x' (i.e. $f(-x, y) = f(x,y)$) then it is symmetric about y-axis

Ex: $x^2 = 4ay$



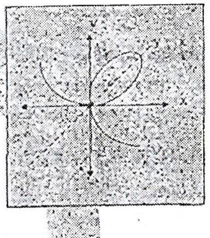
2. If the given equation of curve contains only even powers of 'y' (i.e. $f(x,-y) = f(x, y)$) then it is symmetric about x – axis.

Ex: $y^2 = 4ax$



03. If $f(x, y) = f(y,x)$ then it is symmetric about the line $y = x$.

Ex: $x^3+y^3-3axy = 0$



Intervals:

Closed interval:
If a & b are two given real numbers such that $a < b$ then the set of all real numbers 'x' such that $a \le x \le b$ is called a closed interval and is denoted by [a,b].

$\therefore [a,b] = \{x \in R: a \le x \le b\}$

Ex: $[1,2] = \{x \in R: 1 \le x \le 2\}$

Open Interval:
If a & b are two given real numbers such that $a < b$ then set of all real numbers 'x' such that $a < x < b$ is called an open interval and is denoted by (a,b).

$\therefore (a,b) = \{x \in R: a < x < b\}$

Ex: $(1,2) = \{x \in R: 1 < x < 2\}$

Neighbourhood (nbd) of a point:
If 'a' is a real number & 'δ' is positive real number then the set of all real numbers lying between a–δ and a+δ is called the neighbourhood of a point 'a' of radius 'δ' and it is denoted by $N_\delta(a)$.

$\therefore N_\delta(a) = (a-\delta, a+\delta)$
$= \{x \in R: a-\delta < x < a+\delta\}$

Deleted Neighbourhood of a point 'a':
If $N_\delta(a) = (a-\delta, a+\delta)$ is a nbd of 'a' then $N_\delta(a) - \{a\}$ (or) $(a-\delta, a) \cup (a, a+\delta)$ is called deleted nbd of a number 'a' of radius δ.

Note:

01. $x \in (a-\delta, a+\delta) \Leftrightarrow a-\delta < x < a+\delta$
$\Leftrightarrow |x-a| < \delta$

02. $x \in (a-\delta, a) \cup (a, a+\delta)$
$\Rightarrow a-\delta < x < a+\delta \;\&\; x \ne a$
$\Leftrightarrow 0 < |x-a| < \delta$

03. The set $(a-\delta, a)$ is called the left nbd of 'a' & the set $(a, a+\delta)$ is known as the right nbd of 'a'.

Limit of a function:
Let f(x) be defined over a deleted neighbourhood of $a \in R$ (i.e., $(a-\delta, a+\delta) - \{a\}$). Then $l \in R$ is said to be limit of f(x) as x approaches 'a' if for a given $\epsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $|x-a| < \delta$.

It is denoted by $\underset{x\to a}{\text{Lt}} f(x) = l$

One sided limits:

Left limit:
If $x < a$ and $x \to a$ then we write $\underset{x\to a^-}{\text{Lt}} f(x)$ and if it exists, it is said to be left limit of f(x) at x = a. It is evaluated as $\underset{x\to a^-}{\text{Lt}} f(x) = \underset{h\to 0}{\text{Lt}} f(a-h)$

Right Limit:
If $x > a$ and $x \to a$ then we write $\underset{x\to a^+}{\text{Lt}} f(x)$ and if it exists then it is said to be right limit of f(x) at x = a. It is evaluated as $\underset{x\to a^+}{\text{Lt}} f(x) = \underset{h\to 0}{\text{Lt}} f(a+h)$

Condition for existence of limit:
$\underset{x\to a}{\text{Lt}} f(x)$ exists $\Leftrightarrow \underset{x\to a^-}{\text{Lt}} f(x) = \underset{x\to a^+}{\text{Lt}} f(x)$

Note:
(1) If $\underset{x\to a}{\text{Lt}} f(x)$ exists then it is unique
(2) If f(x) is a polynomial function then $\underset{x\to a}{\text{Lt}}(x) = f(a)$

2.2. Continuity

Continuity of a function at a point:
A function f(x) is said to be continuous at x = a if it satisfies the following conditions
(i) f(a) is defined
(ii) $\underset{x\to a}{\text{Lt}} f(x)$ exists i.e $\underset{x\to a^-}{\text{Lt}} f(x) = \underset{x\to a^+}{\text{Lt}} f(x)$
(iii) $\underset{x\to a}{\text{Lt}} f(x) = f(a)$

Left continuous (or) continuity from the left at a point:
A function f(x) is said to be continuous from the left (or) left continuous at x = a if (i) f(a) is defined
(ii) $\underset{x\to a^-}{\text{Lt}} f(x) = f(a)$.

Right continuous (or) continuity from the right at a point:
A function f(x) is said to be continuous from the right (or) right continuous at x = a if (i) f(a) is defined
(ii) $\underset{x\to a^+}{\text{Lt}} f(x) = f(a)$

Continuity of a function in an open interval:
A function f(x) is said to be continuous in an open interval (a,b) if f(x) is continuous $\forall\, x \in (a,b)$ (or) $\underset{x\to c}{\text{Lt}} f(x) = f(c)\ \forall\, c \in (a,b)$.

Continuity of a function on closed interval:
A function f(x) is said to be continuous on closed interval [a,b] if
(i) f(x) is continuous $\forall\, x \in (a,b)$
(ii) $\underset{x\to a^+}{\text{Lt}} f(x) = f(a)$
(iii) $\underset{x\to b^-}{\text{Lt}} f(x) = f(b)$

Note:
(1) If f(x) and g(x) are two continuous functions then $f(x) + g(x)$, $f(x) - g(x)$, $f(x)\cdot g(x)$ and $\frac{f(x)}{g(x)}$ $(g(x) \ne 0)$ are also continuous.
(2) Polynomial function, exponential function, sine and cosine functions, modulus function are continuous every where.
(3) Logarithmic functions are continuous in $(0,\infty)$

Examples:

01. Let $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2 \\ 0 & \text{for } x = 2 \end{cases}$

Discuss the continuity at x = 2.

Sol: $f(2) = 0$

$$\underset{x\to 2}{\text{Lt}}\, f(x) = \underset{x\to 2}{\text{Lt}}\, \frac{x^2-4}{x-2}$$

$$= 2(2)^{2-1} \quad (\because \text{Standard limit})$$

$$= 4 \neq f(2)$$

$\therefore f(x)$ is discontinuous at $x = 2$

02. If $f(x) = \begin{cases} -x^2, & x \le 0 \\ 5x-4, & 0 < x \le 1 \\ 4x^2-3x, & 1 < x < 2 \\ 3x+4, & x \ge 2 \end{cases}$

then f(x) is discontinuous at x = ______

(a) 0 (b) 1 (c) 2 (d) 3

Sol: $f(0) = 0$

$\underset{x\to 0^-}{\text{Lt}}\, f(x) = 0$, $\underset{x\to 0^+}{\text{Lt}}\, f(x) = 5(0) - 4 = -4$

$\therefore \underset{x\to 0^-}{\text{Lt}}\, f(x) \neq \underset{x\to 0^+}{\text{lt}}\, f(x)$

$\Rightarrow \underset{x\to 0}{\text{Lt}}\, f(x)$ does not exist

$\therefore f(x)$ is discontinuous at $x = 0$

2.3. Differentiability

Derivative of a function at a point:

If a function f(x) is defined on a neighbourhood of a real number 'a' and $\underset{x\to a}{\text{Lt}}\, \dfrac{f(x)-f(a)}{x-a}$ exists and finite then the finite limit is called derivative or differential coefficient of f(x) at a point 'a' and it is denoted by $f^1(a)$.

$$\therefore \underset{x\to a}{\text{Lt}}\, \frac{f(x)-f(a)}{x-a} = f^1(a).$$

Note:

- $\underset{x\to a}{\text{Lt}}\, \dfrac{f(x)-f(a)}{x-a}$ exists finitely

$$\Leftrightarrow \underset{x\to a^-}{\text{Lt}}\, \frac{f(x)-f(a)}{x-a} = \underset{x\to a^+}{\text{Lt}}\, \frac{f(x)-f(a)}{x-a}$$

$\underset{x\to a^-}{\text{Lt}}\, \dfrac{f(x)-f(a)}{x-a} = f^1(a^-)$ is called left hand derivative (L·H·D) of f(x) at x = a.

$$\therefore f^1(a^-) = Lf^1(a) = \underset{x\to a^-}{\text{Lt}}\, \frac{f(x)-f(a)}{x-a}$$

$$= \underset{h\to 0}{\text{Lt}}\, \frac{f(a-h)-f(a)}{-h}.$$

$\underset{x\to a^+}{\text{Lt}}\, \dfrac{f(x)-f(a)}{x-a} = f^1(a^+)$ is called right hand derivative (R·H·D) of f(x) at x = a.

$$\therefore f^1(a^+) = Rf^1(a) = \underset{x\to a^+}{\text{Lt}}\, \frac{f(x)-f(a)}{x-a}$$

$$= \underset{h\to 0}{\text{Lt}}\, \frac{f(a+h)-f(a)}{h}$$

- If the derivative of f(x) exists at x = a then the function f(x) is said to be differentiable function at x = a
- $f^1(a)$ exists at $x = a \Leftrightarrow Lf^1(a) = Rf^1(a)$.
- If f(x) and g(x) are two differentiable functions then f(x)+g(x), f(x)−g(x), f(x)·g(x) and $\dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$ are also differentiable.
- Polynomial functions, exponential functions, sine and cosine functions are differentiable every where.
- Every differentiable function is continuous but a continuous function need not be differentiable.

Derivability of a function in an open interval:

A function f(x) is said to be derivable (or) differentiable in an open interval (a,b) if $f^1(c)$ exists $\forall c \in (a,b)$.

Derivability of a function on closed interval:

A function f(x) is said to be derivable (or) differentiable on closed interval [a,b] if

(i) $f^1(c)$ exists $\forall c \in (a,b)$

(ii) $Rf^1(a)$ exists

(iii) $Lf^1(b)$ exists.

Examples:

01. Let f(x) = |x| where x∈R. Discuss the continuity and differentiability of f(x) at x = 0

Sol: Given $f(x) = |x| = \begin{cases} x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -x & \text{for } x < 0 \end{cases}$

$f(0) = 0$,

$L{\cdot}H{\cdot}L = \underset{x\to 0^-}{\text{Lt}}\, f(x) = 0$

$R{\cdot}H{\cdot}L = \underset{x\to 0^+}{\text{Lt}}\, f(x) = 0$

$\therefore f(x)$ is continuous at $x = 0$

$$L{\cdot}H{\cdot}D = \underset{h\to 0}{\text{Lt}}\, \frac{f(0-h)-f(0)}{-h}$$

$$= \underset{h\to 0}{\text{Lt}}\, \frac{|-h|-0}{-h} = \underset{h\to 0}{\text{Lt}}\, \frac{h}{-h} = -1$$

$$R{\cdot}H{\cdot}D = \underset{h\to 0}{\text{Lt}}\, \frac{f(0+h)-f(0)}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\, \frac{|h|-0}{h} = \underset{h\to 0}{\text{Lt}}\, \frac{h}{h} = 1$$

$L{\cdot}H{\cdot}D \neq R{\cdot}H{\cdot}D$

$\therefore f(x)$ is not differentiable at $x = 0$

- f(x) = |x − a| is continuous every where and f(x) = |x − a| is differentiable everywhere except at x = a

Standard limits:

01. $\underset{x\to a}{\text{Lt}}\, \dfrac{x^n-a^n}{x-a} = na^{n-1}$, $a \neq 0$ & $n \in Q$

02. $\underset{x\to 0}{\text{Lt}}\, \dfrac{e^{mx}-1}{x} = m$

03. $\underset{x\to 0}{\text{Lt}}\, \dfrac{a^x-1}{x} = \log_e a$

04. $\underset{x\to 0}{\text{Lt}}\, [1+ax]^{\frac{1}{x}} = e^a$

05. $\underset{x\to \infty}{\text{Lt}}\, \left[1+\dfrac{a}{x}\right]^x = e^a$

06. $\underset{x\to 0}{\text{Lt}}\, \dfrac{\sin x}{x} = \underset{x\to 0}{\text{Lt}}\, \dfrac{\tan x}{x} = 1$

07. $\underset{x\to \infty}{\text{Lt}}\, \dfrac{\sin x}{x} = 0$

08. $\underset{x\to 0}{\text{Lt}}\, \left[\dfrac{a^x+b^x}{2}\right]^{\frac{1}{x}} = \sqrt{ab}$

09. $\underset{x\to 0}{\text{Lt}}\, [\cos x + a \sin bx]^{1/x} = e^{ab}$

10. $\underset{x\to 0}{\text{Lt}}\, \dfrac{1-\cos ax}{x^2} = \dfrac{a^2}{2}$

Standard Derivatives:

01 $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

02. $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{v\left(\dfrac{du}{dx}\right) - u\left(\dfrac{dv}{dx}\right)}{v^2}$

03. $\dfrac{du}{dx} = \dfrac{du}{dy}\cdot\dfrac{dy}{dx}$ (Chain Rule)

04. $\dfrac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}.a$

05. $\dfrac{d}{dx}(e^x) = e^x$

06. $\dfrac{d}{dx}(a^x) = a^x \log_e a$

07. $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

08 $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x}\log_e a$

09. $\dfrac{d}{dx}(\sin x) = \cos x$

10. $\dfrac{d}{dx}(\cos x) = -\sin x$

11. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

12. $\dfrac{d}{dx}(\cot x) = -\text{cosec}^2 x$

13. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

14. $\dfrac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

15. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

16. $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$

17. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

18. $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$

19. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{(x^2-1)}}$

20. $\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right) = \frac{-1}{x\sqrt{(x^2-1)}}$

21. $\frac{d}{dx}(\sinh x) = \cosh x$

22. $\frac{d}{dx}(\cosh x) = \sinh x$

Indeterminate forms:

$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$

L-Hospital's Rule:

If $\underset{x\to a}{\text{Lt}} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ (or) $\frac{\infty}{\infty}$

then $\underset{x\to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x\to a}{\text{Lt}} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$

when $f'(a) = g'(a) \neq 0$

Examples:

Evaluate the following limits:

01. $\underset{x\to 0}{\text{Lt}} \frac{\sin x^0}{x}$

Sol: $180^0 = \pi^c \Rightarrow x^0 = \frac{\pi}{180}x^c$

$\underset{x\to 0}{\text{Lt}} \frac{\sin x^0}{x} = \underset{x\to 0}{\text{Lt}} \frac{\sin\frac{\pi}{180}x}{x}$

$= \frac{\pi}{180} \quad \left[\because \underset{x\to 0}{\text{Lt}} \frac{\sin mx}{x} = m\right]$

02. $\underset{x\to 0}{\text{Lt}} \frac{\log x}{\cot x}$

Sol: It is in $\frac{\infty}{\infty}$ form,

Using L-Hospital's Rule,

$\underset{x\to 0}{\text{Lt}} \frac{\log x}{\cot x} = \underset{x\to 0}{\text{Lt}} \frac{1/x}{-\operatorname{cosec}^2 x}$

$= \underset{x\to 0}{\text{Lt}} -\frac{\sin x}{x} \times \sin x = -1\times 0 = 0$

03. $\underset{x\to \frac{\pi}{2}}{\text{Lt}} (1-\sin x)\tan x$

Sol: It is in the form of $0\times\infty$

$\underset{x\to \frac{\pi}{2}}{\text{Lt}} (1-\sin x)\tan x = \underset{x\to \frac{\pi}{2}}{\text{Lt}} \frac{1-\sin x}{\cot x}\left(\frac{0}{0}\right)$

Using L-Hospital's Rule

$= \underset{x\to \frac{\pi}{2}}{\text{Lt}} \frac{-\cos x}{-\operatorname{cosec}^2 x}$

$= \frac{0}{1} = 0$

04. $\underset{x\to 0}{\text{Lt}} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$

Sol: It is in the form of $\infty - \infty$

Then simplify the given function as

$\underset{x\to 0}{\text{Lt}} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right] = \underset{x\to 0}{\text{Lt}} \left[\frac{\sin^2 x - x^2}{x^2.\sin^2 x}\right]\left(\frac{0}{0}\right)$

$= \underset{x\to 0}{\text{Lt}} \left[\frac{\sin^2 x - x^2}{x^4 \frac{\sin^2 x}{x^2}}\right]$

$= \underset{x\to 0}{\text{Lt}} \left[\frac{\sin^2 x - x^2}{x^4}\right]$

Using L-Hospital's Rule for two times

$= \underset{x\to 0}{\text{Lt}} \left[\frac{2\cos 2x - 2}{12x^2}\right]$

$= \frac{-2}{12} \underset{x\to 0}{\text{Lt}} \left[\frac{1-\cos 2x}{x^2}\right]$

$= \frac{-2}{12} \times \frac{(2)^2}{2} = \frac{-1}{3}$

05. $\underset{x\to 0}{\text{Lt}} x^x$

Sol: It is in 0^0 form,

For 0^0, 1^∞ and ∞^0 forms limit is evaluated using logarithms.

Let $f(x) = x^x$

Then $\log_e f(x) = \log_e x^x = x \log x$

$\Rightarrow \underset{x\to 0}{\text{Lt}} \log f(x) = \underset{x\to 0}{\text{Lt}} x \log x$ [$0\times\infty$ form]

$\Rightarrow \log\left[\underset{x\to 0}{\text{Lt}} f(x)\right] = \underset{x\to 0}{\text{Lt}} \left[\frac{\log x}{1/x}\right]\left(\frac{\infty}{\infty}\text{ form}\right) = 0$

$\therefore \underset{x\to 0}{\text{Lt}} x^x = e^0 = 1$

Note:

1. If f(x) tends to 0 as x tends a then $\underset{x\to a}{\text{Lt}} [f(x)]^{f(x)} = 1$

2. If $\underset{x\to a}{\text{Lt}} [f(x)]^{g(x)}$ is of the form 1^∞

then $\underset{x\to a}{\text{Lt}} [f(x)]^{g(x)} = e^{\underset{x\to a}{\text{Lt}} g(x)[f(x)-1]}$

06. $\underset{x\to 0}{\text{Lt}} [3\cos x + 2\sin 3x]^{\frac{1}{x}}$

Sol: $\underset{x\to 0}{\text{Lt}} 3^{\frac{1}{x}}\left[\cos x + \frac{2}{3}\sin 3x\right]^{\frac{1}{x}}$

[This is in 1^∞ form using standard limit,

this limit $= e^{\frac{2}{3}3}$

$\therefore$ Left limit $= 0\times e^2 = 0$

Right limit $= \infty \times e^2 = \infty$

$\therefore$ Limit does not exist

2.4. Mean Value Theorems

Rolle's Theorem:

Let f(x) be defined in [a,b] such that

(i) f(x) is continuous in [a,b]

(ii) f(x) is differentiable in (a,b)

(iii) f(a) = f(b)

Then there exists atleast one point $c\in(a,b)$ such that $f^1(c) = 0$.

Lagrange's mean value Theorem

Let f(x) be defined in [a,b] such that

(i) f(x) is continuous in [a,b]

(ii) f(x) is differentiable in (a,b)

Then there exist atleast one point $c\in(a,b)$ such that

$f^1(c) = \frac{f(b)-f(a)}{b-a}$.

Cauchy's Mean value theorem:

Let f(x) and g(x) be defined in [a,b] such that

(i) f(x) and g(x) are continuous in [a,b]

(ii) f(x) and g(x) are differentiable in (a,b)

(iii) $g^1(x) \neq 0$ for all x in (a,b)

Then there exist atleast one point $c\in(a,b)$ such that

$\frac{f^1(c)}{g^1(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Examples:

01. Find 'c' of Rolle's theorem for $f(x) = e^x \sin x$ in $[0,\pi]$

Sol: f(x) is continuous in $[0, \pi]$

$f^1(x) = e^x[\sin x+\cos x]$ exists and finite in $(0,\pi)$

$f(0) = 0$, $f(\pi) = 0$

$\therefore$ By Rolle's Theorem $\exists$ atleast one point $c\in(0,\pi)$ such that $f^1(c) = 0$

$\Rightarrow e^c[\sin c+\cos c] = 0$

$\Rightarrow \sin c = -\cos c$

$\therefore c = \frac{3\pi}{4} \in (0,\pi)$.

02. Find 'c' of Lagrange's mean value theorem for $f(x) = x^3-6x^2+11x-6$ in [0,4]

Sol: f(x) is continuous in [0,4]

$f^1(x) = 3x^2-12x+11$ exists and finite in (0,4)

$f(0) = -6$, $f(4) = 6$

By L.M.V.T $\exists$ at least one point $c\in(0,4)$ such that $f^1(c) = \frac{f(4)-f(0)}{4-0}$

$\Rightarrow 3c^2-12c+11 = \frac{6-(-6)}{4}$

$\therefore c = 2\pm\frac{2}{\sqrt{3}} \in (0,4)$

03. Find 'c' of Cauchy's mean value theorem for $f(x) = \sin x$ and $g(x) = \cos x$ in $\left[-\frac{\pi}{2},0\right]$

Sol: sinx and cosx are continuous in $\left[-\frac{\pi}{2},0\right]$

$f^1(x) = \cos x$, $g^1(x) = -\sin x$ exist and finite in $\left(-\frac{\pi}{2},0\right)$ and

$g^1(x) \neq 0\ \forall x\in\left(-\frac{\pi}{2},0\right)$

$\therefore$ By C.M.V.T $\exists$ at least one point $c\in\left(-\frac{\pi}{2},0\right)$ such that

$\frac{f^1(c)}{g^1(c)} = \frac{f(0)-f\left(\frac{-\pi}{2}\right)}{g(0)-g\left(\frac{-\pi}{2}\right)}$

$\Rightarrow \frac{\cos c}{-\sin c} = \frac{0-(-1)}{1-0}$

$\Rightarrow \cot c = -1$

$\therefore c = -\frac{\pi}{4} \in \left(-\frac{\pi}{2},0\right)$

2.5. Definite Integrals

First fundamental theorem of Integral Calculus:
If f(x) is continuous on [a,b] then the function

$$F(x) = \int_a^x f(t)\,dt$$

(i) is continuous on [a,b]
(ii) is differentiable in (a,b) and
(iii) $\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$ (or)

$F^1(x) = f(x), \forall x \in [a,b]$.

Second Fundamental theorem of Integral Calculus (Newton-Leibnitz Formula):
If f(x) is continuous on [a,b] and F(x) is an anti-derivative of f(x) on [a,b]
i.e. $\int f(x)\,dx = F(x)$) then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Corollary:
If f(x) is continuous on [a,b] then there exists a function F(x) whose derivative on [a,b] is f(x).

Theorem:
If f(x) is continuous on [a,b] and U(x) and V(x) are differentiable functions of x whose values lie in [a,b] then

$$\frac{d}{dx}\int_{U(x)}^{V(x)} f(t)\,dt = f(V(x))\frac{dV}{dx} - f(U(x))\frac{dU}{dx}$$

Standard Integrals:

01. $\int dx = x$
02. $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ $(n \neq -1)$
03. $\int \frac{1}{x}\,dx = \log_e x$
04. $\int e^x\,dx = e^x$
05. $\int \log x\,dx = x(\log x - 1)$
06. $\int a^x\,dx = \frac{a^x}{\log_e a}$
07. $\int \sin x\,dx = -\cos x$
08. $\int \cos x\,dx = \sin x$
09. $\int \tan x\,dx = -\log\cos x = \log\sec x$
10. $\int \cot x\,dx = \log\sin x$
11. $\int \sec x\,dx = \log(\sec x + \tan x) = \log\left(\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)$
12. $\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x)$
$= \log\left(\tan\left(\frac{x}{2}\right)\right)$
13. $\int \sec^2 x\,dx = \tan x$
14. $\int \operatorname{cosec}^2 x\,dx = -\cot x$
15. $\int \sec x \tan x\,dx = \sec x$
16. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$
17. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) = \frac{-1}{a}\cot^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$
19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$
20. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) = -\cos^{-1}\left(\frac{x}{a}\right)$
21. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) = \log\left|x+\sqrt{a^2+x^2}\right|$
22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \log\left|x+\sqrt{x^2-a^2}\right|$
23. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x) = -\operatorname{cosec}^{-1}(x)$
24. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{(a^2-x^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
25. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{(a^2+x^2)} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$
$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|$
26. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$
$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|$

27. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$
28. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$
29. $\int \sinh x\,dx = \cosh x$
30. $\int \cosh x\,dx = \sinh x$
31. $\int u(x)v(x)\,dx = u(x)\int v(x)\,dx - \int\left[u^1(x)\int v(x)\,dx\right]dx$

Properties of Definite Integrals:

01. $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$
02. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
03. If $a < c < b$ then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
04. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
05. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ and
$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$
06. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if f(x) is even function
$= 0$ if f(x) is odd function
07. $\int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx$ if $f(a-x) = f(x)$
$= 0$ if $f(a-x) = -f(x)$
08. $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(x+a) = f(x)$
i.e f(x) is a periodic function with period 'a'.
09. $\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$ if $f(a-x) = f(x)$
10. $\int_a^b x f(x)\,dx = \frac{b-a}{2}\int_a^b f(x)\,dx$ if $f(a+b-x) = f(x)$
11. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx =$
$$= \begin{cases} \left(\frac{(n-1)}{n}\right)\left(\frac{(n-3)}{(n-2)}\right)\left(\frac{(n-5)}{(n-4)}\right)\cdots\cdots\left(\frac{2}{3}\right) & \text{if } n \text{ is odd} \\ \left(\frac{(n-1)}{n}\right)\left(\frac{(n-3)}{(n-2)}\right)\left(\frac{(n-5)}{(n-4)}\right)\cdots\cdots\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) & \text{if } n \text{ is even} \end{cases}$$
12. $\int_0^{\pi/2} \sin^m x.\cos^n x\,dx =$
$$\frac{\{(m-1)(m-3)(m-5)\ldots.(2 \text{ or } 1)\}\{(n-1)(n-3)\ldots(2 \text{ or } 1)\}}{(m+n)(m+n-2)(m+n-4)\ldots.(2 \text{ or } 1)}k$$
where $k = \begin{cases} \frac{\pi}{2} & \text{if } m \,\&\, n \text{ are even} \\ 1 & \text{otherwise} \end{cases}$
13. $\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx$
14. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Examples:

01. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$

Sol: Let $f(x) = \sin x$

Then $f\left(0+\frac{\pi}{2}-x\right) = \cos x$

$$\int_0^{\frac{\pi}{2}} \frac{f(x)}{f(x)+f\left(0+\frac{\pi}{2}-x\right)}\,dx = \frac{\frac{\pi}{2}-0}{2} = \frac{\pi}{4}$$

02. $\int_0^4 (|x|+|3-x|)\,dx$

Sol: $\int_0^4 x\,dx + \int_0^3 (3-x)\,dx + \int_3^4 -(3-x)\,dx$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[3x-\frac{x^2}{2}\right]_0^3 - \left[3x-\frac{x^2}{2}\right]_3^4 = 13$$

03. $\int_0^1 x(1-x)^5 dx$

Sol: Using $\int_0^a f(x)d(x) = \int_0^a f(a-x)dx$

$$\int_0^1 x(1-x)^5 dx = \int_0^1 (1-x)x^5 dx = \left[\frac{x^6}{6} - \frac{x^7}{7}\right]_0^1 = \frac{1}{42}$$

04. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{[1+\sin x \cos x]} dx$

Sol: Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cdot \cos x} dx$ ---------- (1)

Then $I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\cos x \cdot \sin x} dx$ ---------- (2)

$\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I + I = 0$ (equations (1) + (2))

$\therefore I = 0$

05. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Sol: We have $\int_0^a x f(x) dx = \frac{a}{2}\int_0^a f(x)\, dx$

if $f(a-x) = f(x)$

Let $f(x) = \frac{\sin x}{1+\cos^2 x}$

Then $f(\pi-x) = \frac{\sin x}{1+\cos^2 x} = f(x)$

$$\therefore \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$= \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2} = \frac{\pi^2}{4}$$

06. $\int_0^{2\pi} \sin^4 x \cdot \cos^6 x\, dx$

Sol: Using $\int_0^{2a} f(x)\, d(x) = 2\int_0^a f(x)\, dx$

if $f(2a-x) = f(x)$

Here $2a = 2\pi$ & $f(x) = \sin^4 x \cdot \cos^6 x$

$f(2a-x) = f(2\pi - x) = \sin^4 x \cos^6 x = f(x)$

$$\therefore \int_0^{2\pi} \sin^4 x \cos^6 x\, dx = 2\int_0^{\pi} \sin^4 x \cos^6 x\, dx$$

$$= 2 \cdot 2\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x\, dx$$

(Again using the same property)

$$= 4\left[\frac{(3\cdot 1)(5\cdot 3\cdot 1)}{10\cdot 8\cdot 6\cdot 4\cdot 2}\times\frac{\pi}{2}\right]$$

$$= \frac{3\pi}{128}$$

2.6. Improper Integrals

Improper integrals are classified into two kinds.

Improper Integral of first kind:

$\int_a^b f(x)\, dx$ is said to be improper integral of first kind if $a = -\infty$ (or) $b = \infty$ (or) both

Ex: (1) $\int_{-\infty}^{0} e^x dx$ (2) $\int_0^{\infty} e^{-x} dx$ (3) $\int_{-\infty}^{\infty} e^{-x^2} dx$

Improper Integral of second kind:

$\int_a^b f(x)\, dx$ is said to be improper integral of second kind if a and b are finite but f(x) is infinite for some $x \in [a,b]$

Ex: (1) $\int_0^1 \log(1-x)\, dx$ (2) $\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} dx$

(3) $\int_{-1}^{1} \frac{1}{x} dx$ (4) $\int_1^2 \frac{1}{(x-1)(x-2)} dx$

Convergence:

01. If $\int_a^b f(x)\, dx$ is finite then it said to be a convergent improper integral.

02. If $\int_a^b f(x)\, dx$ is not finite then it is said to be a divergent improper integral.

Examples:

01. Find the convergence of $\int_0^{\infty} e^{-ax} \cos px\, dx$

Sol: we have

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$$

$$\int_0^{\infty} e^{-ax} \cos px\, dx = \left.\frac{e^{-ax}}{(-a)^2+p^2}[-a \cos px + p \sin px]\right]_0^{\infty}$$

$$= \frac{a}{a^2+p^2} = \text{finite}$$

$\therefore$ The given integral is convergent.

02. Find the convergence of $\int_0^3 \frac{1}{x^2+3x+2} dx$

Sol: The given function is discontinuous at x = 1 and 2, so split the integral into three parts as

$$\int_0^3 \frac{1}{x^2+3x+2} dx = \int_0^1 \frac{1}{(x-1)(x-2)} dx + \int_1^2 \frac{1}{(x-1)(x-2)} dx + \int_2^3 \frac{1}{(x-1)(x-2)} dx$$

$$= \log\left(\frac{x-2}{x-1}\right)\Big]_0^1 + \log\left(\frac{x-2}{x-1}\right)\Big]_1^2 + \log\left(\frac{x-2}{x-1}\right)\Big]_2^3$$

$= \infty$

$\therefore$ The given integral is divergent

Comparison Test:

Method – I:

(i) If $0 \le f(x) \le g(x)$ for all x in [a,b] and $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ also converges.

(ii) If $0 \le f(x) \le g(x)$ for all x in [a,b] and $\int_a^b f(x)dx$ diverges then $\int_a^b g(x)dx$ also diverges.

Method – II: [Limit Form]

For first kind of Improper integrals:

If f(x) and g(x) are two positive functions such that $\underset{x\to\infty}{Lt}\frac{f(x)}{g(x)} = \ell$ (finite and non-zero) then $\int_a^{\infty} f(x)dx$ and $\int_a^{\infty} g(x)dx$ both converge (or) diverge together.

For second kind of Improper integrals:

If f(x) and g(x) are two positive functions and

(i) $f(x) \to \infty$ as $x \to a$ such that $\underset{x\to a}{Lt}\frac{f(x)}{g(x)} = \ell$ (finite and non-zero) (or)

(ii) $f(x) \to \infty$ as $x \to b$ such that $\underset{x\to b}{Lt}\frac{f(x)}{g(x)} = \ell$ (finite and non-zero)

then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both converge (or) diverge together.

Some useful comparison integrals:

(1) $\int_a^{\infty}\frac{dx}{x^p}$, (a>0) converges to $\frac{1}{(p-1)a^{p-1}}$ when p>1 and diverges to ∞ when $p \le 1$.

(2) $\int_a^{\infty} e^{-px}dx$ and $\int_{-\infty}^{b} e^{px}dx$ converge for any constant p>0 and diverge for $p \le 0$.

(3) $\int_a^b \frac{dx}{(b-x)^p}$ converges to $\frac{1}{(1-p)(b-a)^{p-1}}$ when p<1 and diverges to ∞ then $p \ge 1$.

(4) $\int_a^b \frac{dx}{(x-a)^p}$ converges to $\frac{1}{(1-p)(b-a)^{p-1}}$ when p<1 and diverges to ∞ when $p \ge 1$.

Examples:

01. Find the convergence of $\int_1^{\infty}\frac{dx}{x^2(1+e^x)}$

Sol: $\frac{1}{x^2(1+e^x)} \le \frac{1}{x^2}$ for all x >1

$\int_1^{\infty}\frac{1}{x^2}dx$ is convergent

$\therefore \int_1^{\infty}\frac{1}{x^2(1+e^x)}dx$ also converges

02. Find the convergence of $\int_1^{\infty}\frac{x\,\mathrm{Tan}^{-1}x}{\sqrt{4+x^3}}dx$

Sol: Let $f(x) = \frac{x\,\mathrm{Tan}^{-1}x}{\sqrt{4+x^3}} = \frac{\mathrm{Tan}^{-1}x}{\sqrt{x}\cdot\sqrt{\frac{4}{x^3}+1}}$

Let $g(x) = \frac{1}{\sqrt{x}}$

$$\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{\sqrt{x}}dx \text{ is divergent}$$

$$\therefore \underset{x\to\infty}{Lt} \frac{f(x)}{g(x)} = \underset{x\to\infty}{Lt} \frac{Tan^{-1}x}{\sqrt{\frac{4}{x^3}+1}} = \frac{\pi}{2}$$

$= \text{finite}$

$\therefore$ By limit form of comparison test, the given integral also diverges.

03. Find the convergence of $\int_1^3 \frac{1}{(x-1)^2}dx$

Sol: $\int_a^b \frac{dx}{(x-a)^p}$ is convergent for $p < 1$ and divergent for $p \geq 1$.

By comparing given integral

$$\int_1^3 \frac{1}{(x-1)^2}dx \text{ with } \int_a^b \frac{dx}{(x-a)^p},$$

we have a=1 and $p = 2 > 1$.

$\therefore$ The given integral $\int_1^3 \frac{1}{(x-1)^2}dx$

diverges for $p = 2 > 1$.

04. Find the convergence of $\int_1^\infty \frac{1}{\sqrt{x}}dx$.

Sol: $\int_a^\infty \frac{1}{x^p}dx$ converges for $p > 1$ and diverges $p \leq 1$.

By comparing given integral $\int_1^\infty \frac{1}{\sqrt{x}}dx$ with

$\int_a^\infty \frac{1}{x^p}dx$, we have a=1 and p=1/2

$\therefore$ The given integral $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges for

$p = 1/2 < 1$.

2.7. Partial and total Derivatives

Partial Derivatives:

If $u = f(x,y)$ then

$$\frac{\partial u}{\partial x} = \underset{h\to 0}{Lt} \frac{f(x+h,y)-f(x,y)}{h} = u_x$$

$$\frac{\partial u}{\partial y} = \underset{k\to 0}{Lt} \frac{f(x,y+k)-f(x,y)}{k} = u_y$$

Similarly higher order derivatives

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x} \text{ can be obtained.}$$

In general $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ (i.e $u_{xy} = u_{yx}$)

Homogeneous function:

A function is said to be homogeneous if degree of each term in the function is same.

Ex:

(1) $x^2yz - 3yz^3$ is a homogeneous function with degree 4

(2) $\frac{x^2y + xy^2}{2x - y}$ is a homogeneous function with degree 2

(3) $\cos^{-1}\left(\frac{x+y}{x-y}\right)$ is a homogeneous function with degree 0

(4) $\log\left(\frac{x^2}{y}\right)$ is non-homogeneous function

- If $f(kx,ky) = k^n f(x,y)$ then $f(x,y)$ is a homogeneous function with degree 'n'.
- If $f(x,y)$ is a homogeneous function with degree 'n' then $f(x,y) = \begin{cases} x^n\ \phi\left(\frac{y}{x}\right) \\ y^n\ \psi\left(\frac{x}{y}\right) \end{cases}$

Euler's Theorem:

If $u = f(x,y)$ is a homogeneous function with degree 'n' then

(i) $x.u_x + y.u_y = n.u$

(ii) $x^2.u_{xx} + 2xyu_{xy} + y^2.u_{yy} = n(n-1)u$.

Corollary-1:

If $u = f(x,y,z)$ is a homogeneous function with degree 'n' then

(i) $x.u_x + y.u_y + z.u_z = n.u$

(ii) $x^2.u_{xx} + y^2.u_{yy} + z^2.u_{zz} + 2xyu_{xy} + 2yzu_{yz} + 2x.zu_{xz} = n(n-1)u$

Corollary-2:

If $u = u(x,y)$ is not a homogeneous function but f(u) is a homogeneous function of degree 'n' then

(i) $x.u_x + y.u_y = n\frac{f(u)}{f^1(u)} = G(u)$

(ii) $x^2.u_{xx} + 2xy.u_{xy} + y^2.u_{yy} = G(u)[G'(u)-1]$

Corollary-3:

If $u = f(x,y) + g(x,y) + h(x,y)$ where f,g and h are homogeneous functions of degrees m,n and p respectively then

(i) $x.u_x + y.u_y = mf + ng + ph$

(ii) $x^2.u_{xx} + 2xy.u_{xy} + y^2.u_{yy} = m(m-1)f + n(n-1)g + p(p-1)h$

Total derivative:

If $u = f(x,y)$ where x and y are functions of 't' then the total derivative of u with respect to 't' is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

- The above formula can be used to find the total derivative of $u = f(x,y)$ with respective to x (or) y when x and y are connected by some relation.

  If $u = f(x,y,z)$ where $x = x(t)$, $y = y(t)$ and $z = z(t)$ then

  $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

Total differential :

If $u = f(x,y)$ is function of two variables x and y then the total differential of u is denoted by du or df and defined as

$$du = df = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

If $u = f(x,y,z)$ then

$$du = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

If $f(x,y) = c$ is an implicit function of x and y then $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

Chain Rule for Partial Differentiation:

If $u = f(x,y)$ where $x = g(r,s)$ & $y = h(r,s)$ then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

and $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

Examples:

01. If $u = \frac{xy}{\sqrt{x}+\sqrt{y}}$ then find

(i) $x\cdot u_x + y\cdot u_y$

(ii) $x^2\cdot u_{xx} + 2xyu_{xy} + y^2\cdot u_{yy}$

Sol: u is a homogeneous function with degree

$n = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore x\cdot u_x + y\cdot u_y = n\cdot u = \frac{3u}{2}$ and

$x^2\cdot u_{xx} + 2xyu_{xy} + y^2\cdot u_{yy} = n(n-1)u$

$= \frac{3}{2}\left(\frac{3}{2}-1\right)u = \frac{3}{4}u$

02. If $\log u = \frac{x^3+y^3}{3x-y}$ then find $x^2\cdot u_{xx} + 2xyu_{xy} + y^2\cdot u_{yy}$

Sol: $u = e^{\left(\frac{x^3+y^3}{3x-y}\right)}$ is not a homogeneous function

But $f(u) = \log u = \frac{x^3+y^3}{3x-y}$ is a homogeneous function with degree '2'.

$\therefore x\cdot u_x + y\cdot u_y = n\frac{f(u)}{f^1(u)} = 2\frac{\log u}{1/u}$

$= 2\cdot u\cdot \log u = G(u)$ and

$x^2\cdot u_{xx} + 2xyu_{xy} + y^2\cdot u_{yy} = G(u)\,[\,G'(u)-1]$

$= 2u \log u\,[\,2+2\log u - 1]$

$= 2u \log u\,[\,2\log u + 1]$

03. If $z = x^n f\left(\frac{y}{x}\right) + y^{-n} g\left(\frac{x}{y}\right)$ then find $x\cdot z_x + y\cdot z_y + x^2\cdot z_{xx} + 2xyz_{xy} + y^2z_{yy}$

Sol: z is a sum of two homogeneous functions with degrees n and $-n$.

Let $\phi(x,y) = x^n\ f\left(\frac{y}{x}\right)$ and $\psi(x,y) = y^{-n}\ g\left(\frac{x}{y}\right)$

$\therefore x\cdot z_x + y\cdot z_y + x^2\cdot z_{xx} + 2xyz_{xy} + y^2z_{yy}$

$= [n\phi + (-n)\psi] + [n(n-1)\phi + (-n)(-n-1)\psi]$

$= n\phi - n\psi + n^2\phi - n\phi + n^2\psi + n\psi$

$= n^2[\phi + \psi] = n^2 z$

04. Find the total derivative of x^3y^2 with respect to 'x' when x and y are connected by the relation $x^2+xy+y^2 = 1$.

Sol: Let $u = x^3y^2$ & $f(x,y) = x^2 + xy + y^2 - 1 = 0$

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= 3x^2y^2(1) + 2x^3y^2\left[-\frac{f_x}{f_y}\right]$$

$$= 3x^2y^2 - 2x^3y^2\left[\frac{2x+y}{x+2y}\right]$$

05. If $U = f(x-y, y-z, z-x)$ then find $U_x + U_y + U_z$

Sol: Let $r = x - y$, $s = y - z$, $t = z - x$

Then $U = f(r, s, t)$

$$\therefore U_x = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x}$$

$$\Rightarrow U_x = f_r.(1) + f_s.(0) + f_t(-1)$$

$$\Rightarrow U_x = f_r - f_t$$

Similarly $U_y = -f_r + f_s$ and

$U_z = -f_s + f_t$

$\therefore U_x + U_y + U_z = 0$

2.8. Maxima and Minima

Maxima and minima for functions of one variable:-

Local or relative maximum:
A function f (x) is said to have a Maximum at x = c if there exists $\delta > 0$ such that $|x - c| < \delta \Rightarrow f(x) \le f(c)$.

Local or relative minimum:
A function f (x) is said to have a minimum at x = c if there exists $\delta > 0$ such that $|x - c| < \delta \Rightarrow f(x) \ge f(c)$.

Stationary points:
The values of x for which $f^1(x) = 0$ are called stationary points or turning points.

Stationary values:
A function f (x) is said to be stationary at x = a if $f^1(a) = 0$ and f (a) is a stationary value.

Extreme point:
The point at which the function has a maximum or a minimum is called extreme point.

Extreme values:
The values of the function at extreme points are called extreme values (Extrema).

Point of inflection:
The stationary point at which the function f(x) has neither maximum nor minimum is called point of inflection.

Note:

- A necessary condition for a function to have an extreme value at x = a is $f^1(a) = 0$
- $f^1(a) = 0$ is only a necessary condition but not sufficient condition for f(a) to be an extreme value of f (x).

Ex: For the function $f(x) = x^3$, f (0) is not an extreme even though $f^1(a) = 0$

Ex: f (0) is a minimum value of f (x) = |x| even though $f^1(0)$ does not exist.

- Every extreme point is a stationary point but every stationary point need not be an extreme point.

Absolute or Global maximum / minimum:
The absolute maximum/minimum values of the function f(x) in the closed interval [a,b] are given by

01. Absolute maximum value = max{f(a), f(b), all local maximum values of 'f' }
= greatest value of f(x) in [a, b].

02. Absolute minimum value = min{f(a), f(b), all local minimum values of 'f' }
= least value of f(x) in [a, b].

Working Rule to find-maxima and minima:
Let f (x) be the given function

Step 1: Find $f^1(x)$

Step 2: Equate $f^1(x)$ to zero to obtain the stationary points.

Step 3: Find $f^{11}(x)$ at each stationary point.

(i) If $f^{11}(x_0) > 0$ then f(x) has a minimum at $x = x_0$

(ii) If $f^{11}(x_0) < 0$ then f(x) has a maximum at $x = x_0$

(iii) If $f^{11}(x_0) = 0$ then f(x) may (or) may not have extremum.

In this case check for maxima and minima using the changes in sign of $f^1(x)$ as given below.

(a) For $x < x_0$ if $f^1(x) < 0$ and $x > x_0$ if $f^1x) > 0$ then $f(x_0)$ is a minimum value of f (x).

(b) For $x < x_0$ if $f^1(x) > 0$ and $x > x_0$ if $f^1(x) < 0$ then $f(x_0)$ is a maximum value of f(x).

(c) For $x < x_0$ and $x > x_0$ if $f^1(x) > 0$ (or) $f^1(x) < 0$ then $f(x_0)$ is not an extremum.

Examples:

01. The function $f(x) = 2x^3 - 3x^2 - 36x + 10$ has a maximum value at x = ——

Sol: $f^1(x) = 6x^2 - 6x - 36 = 0$

$\Rightarrow x^2 - x - 6 = 0$

$\therefore x = 3, -2$

$f^{11}(x) = 12x - 6$

$f^{11}(3) = 30 > 0$

$\Rightarrow$ f (x) has a minimum value at x = 3.

$f^{11}(-2) = -30 < 0$

$\Rightarrow$ f(x) has a maximum value at x = –2.

02. Find the maximum and minimum values of $f(x) = 2x^3 - 24x + 107$ in [1, 3].

Sol: $f^1(x) = 6x^2 - 24 = 0 \Rightarrow x = \pm 2$.

But $x = -2 \notin [1, 3]$.

$\therefore$ x = 2 is the only stationary point.

f(1) = 85, f(2) = 75, f(3) = 89.

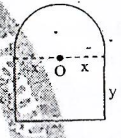
$\therefore$ Maximum value of f (x) is 89 at x = 3 and minimum value is 75 at x = 2.

Some geometric applications of maxima & minima:

01. A window has the form of a rectangle surmounted by a semi-circle. If the perimeter is 40 ft. then find its dimensions so that the greatest amount of light may be admitted.

Sol: The greatest amount of light may be admitted means that the area of the window may be maximum.

Let 'x' ft. be the radius of the semi-circle so that one side of the rectangle is 2x ft. Let the other side of the rectangle 'y' ft.



Then the perimeter of the whole figure
$= \pi x + 2x + 2y = 40$ -------- (1).

and the area $A = \frac{1}{2}\pi x^2 + 2xy$ -------- (2)

Here A is a function of two variables x and y. To express A in terms of one variable x (say), we substitute the value of 'y' from (1) in (2).

$$\therefore A = \frac{1}{2}\pi x^2 + x[40 - (\pi + 2)x]$$

$$= 40x - \left(\frac{\pi}{2} + 2\right)x^2$$

Then $\frac{dA}{dx} = 40 - (\pi + 4)x$

For A to be maximum or minimum, we must have $\frac{dA}{dx} = 0$.

i.e., $40 - (\pi+4)x = 0$

or $x = \frac{40}{(\pi + 4)}$

$\therefore$ From (1),

$$y = \frac{1}{2}[40 - (\pi + 2)x]$$

$$= \frac{1}{2}\left[40 - \frac{(\pi+2)40}{(\pi+4)}\right] = \frac{40}{(\pi+4)}$$

i.e., x = y

Also $\frac{d^2A}{dx^2} = -(\pi + 4)$, which is negative

Thus the area of the window is maximum when the radius of the semi circle is equals to the height of the rectangle.

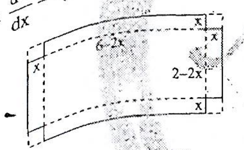
02. A rectangular sheet of metal of length 6 meters and width 2 meters is given. Four equal squares are removed from the corners. The sides of this sheet are now turned up to form an open rectangular box. Find approximately, the height of the box, such that the volume of the box is maximum.

Sol: Let the side of each of the squares cut off be 'x' m so that the height of the box is 'x' m and the sides the base are (6–2x) m, (2–2x) m.

$\therefore$ Volume V of the box

$= x(6-2x)(2-2x) = 4(x^3-4x^2+3x)$

Then $\frac{dV}{dx} = 4(3x^2 - 8x + 3)$



For V to be maximum or minimum, we must have $\frac{dV}{dx} = 0$ i.e., $3x^2 - 8x + 3$

$\therefore x = \frac{8 \pm \sqrt{[64 - 4\times3\times3]}}{6} = 2.2$ or 0.45 m

The value $x = 2.2$ m is inadmissible, as no box is possible for this value.

Also $\frac{d^2V}{dx^2} = 4(6x-8)$, which is negative for $x = 0.45$ m

Hence the volume of the box is maximum when its height is 45 cm.

Maxima and minima for functions of two variables:

Let $z = f(x, y)$ be the function of two variables for which maxima or minima is to be obtained.

Let $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

Working Rule:

Step1: Find p, q, r, s and t

Step2: Equate p and q to zero for obtaining stationary points.

Step3: Find r, s and t at each stationary point.

(a) If $rt - s^2 > 0$ and $r > 0$ then f (x, y) has a minimum at that stationary point.

(b) If $rt - s^2 > 0$ and $r < 0$ then f (x, y) has a maximum at that stationary point.

(c) If $rt - s^2 < 0$ then f (x, y) has no extreme at that stationary point and such points are called saddle points.

(d) If $rt - s^2 = 0$ then the case is undecided.

Constrained Maxima and Minima

Some times it is required to find the extremum of a function subject to some other conditions involving the variables. Such problems are called constrained maxima and minima problems and they can be solved by using Lagrange's method of undetermined multipliers.

Working rule for Lagrange's Method of Undetermined multipliers:

Let f (x, y, z) be the given function subject to the condition ϕ (x, y, z) = 0 ---- (1) for which extremum is to be found.

Step1: Form Lagrangian function

F (x, y, z) = f (x, y, z) + λ ϕ (x, y, z) where λ is the Lagrange's multiplier which is to be determined by the following conditions.

Step2: The necessary condition to have extreme value are

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda . \frac{\partial \phi}{\partial x} = 0 \text{ ---- (2)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda . \frac{\partial \phi}{\partial y} = 0 \text{ ---- (3)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda . \frac{\partial \phi}{\partial z} = 0 \text{ ----- (4)}$$

- Equations (2), (3) and (4) are called Lagrange's equations.

Step3: Solving equations (1), (2), (3) and (4) we obtain the values of x, y, z and λ.

The values of x, y, z so obtained will give the stationary points at which f(x, y, z) has extremum.

Examples:

01. Find the maxima (or) minima of $f(x, y) = x^2 + y^2 + 6x + 10$

Sol: $p = 2x + 6 = 0 \Rightarrow x = -3$

$q = 2y = 0 \Rightarrow y = 0$

$r = 2, s = 0, t = 2.$

$\therefore$ (–3, 0) is a stationary point.

At (–3, 0), $rt - s^2 = 4 > 0$ & $r = 2 > 0$

$\Rightarrow$ f (x, y) has minimum value at (–3, 0)

$\therefore$ Minimum value = f (–3, 0) = 1.

02. Find the maximum and minimum values of x + y + z subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Sol: Let f (x, y, z) = x + y + z

$$\phi(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 \quad \text{--- (1)}$$

$$F(x, y, z) = x + y + z + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$$

Lagrange's equations are

$$\frac{\partial F}{\partial x} = 1 + \lambda\left(-\frac{1}{x^2}\right) = 0 \Rightarrow \lambda = x^2 \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial y} = 1 + \lambda\left(-\frac{1}{y^2}\right) = 0 \Rightarrow \lambda = y^2 \quad \text{--- (3)}$$

$$\frac{\partial F}{\partial z} = 1 + \lambda\left(-\frac{1}{z^2}\right) = 0 \Rightarrow \lambda = z^2 \quad \text{--- (4)}$$

Solving equations (1) to (4) we get

$$x^2 = y^2 = z^2 = \lambda$$

From equation (1), we get

$$\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} = 1$$

$\Rightarrow \lambda = 9$

$\Rightarrow x = y = z = \pm 3$

$\Rightarrow$ Stationary points are (3, 3, 3) & (–3, –3, –3,)

$\therefore$ Maximum value = f (3, 3, 3) = 9 and Minimum value = f (–3, –3, –3,) = –9.

2.9. Multiple Integrals

Double Integral:

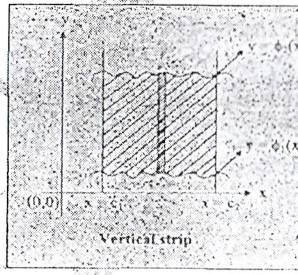
Let f(x,y) be defined at each point in a region 'R'. Let the region 'R' be divided into 'n' sub-regions each of area $\delta A_1, \delta A_2, \ldots \delta A_n$. Let (x_i, y_i) be an arbitrary point in a sub-region with area δA_i.

$$\text{Then } \underset{n\to\infty}{\text{Lt}}\left[\sum_{i=1}^{n} f(x_i, y_i)\delta A_i\right] = \iint_R f(x, y)\,dx\,dy$$

Method of Evaluation:

Case (i) : If the limits of integration are $y = \phi_1(x)$ to $y = \phi_2(x)$ and $x = c_1$ to $x = c_2$

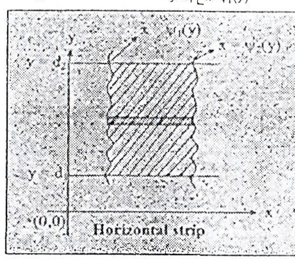
$$\text{then } \iint_R f(x, y)\,dx\,dy = \int_{x=c_1}^{c_2}\left[\int_{y=\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy\right]dx$$



Vertical strip

Case (ii): If the limits of integration are $x = \psi_1(y)$ to $x = \psi_2(y)$ and $y = d_1$ to $y = d_2$

$$\text{then } \iint_R f(x, y)\,dx\,dy = \int_{y=d_1}^{d_2}\left[\int_{x=\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx\right]dy$$

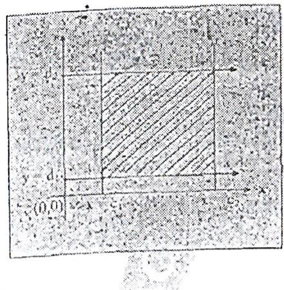


Horizontal strip

Case (iii): If the limits of integration are $x = c_1$ to $x = c_2$ and $y = d_1$ to $y = d_2$

then $\iint_R f(x,y)\,dx\,dy = \int_{x=c_1}^{c_2}\left[\int_{y=d_1}^{d_2} f(x,y)\,dy\right]dx$

$= \int_{y=d_1}^{d_2}\left[\int_{x=c_1}^{c_2} f(x,y)\,dx\right]dy$



Note:

01. A double integral is evaluated first with respect to the variable which has variable limits and then with respect to the variable which has constant limits.
02. If both variables have constant limits then the order of integration will not be a matter.

Examples:

01. Evaluate $\int_0^2\int_0^x y\,dy\,dx$

Sol: $\int_0^2\left[\int_0^x y\,dy\right]dx = \int_0^2\left[\frac{y^2}{2}\right]_0^x dx$

$= \int_0^2 \frac{x^2}{2}dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$

02. $\int_0^a\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$

Sol: $I = \int_0^a\left[\int_0^{\sqrt{a^2-y^2}} \sqrt{\left(\sqrt{a^2-y^2}\right)^2 - x^2}\,dx\right]dy$

$= \int_0^a\left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{(a^2-y^2)}{2}\sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy$

$= \int_0^a\left(\frac{a^2-y^2}{2}\right)\left(\frac{\pi}{2}\right)dy$

$= \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a = \frac{\pi a^3}{6}$

Change of order of Integration:

The order of integration is responsible for the description of the region and accordingly the limits of integration. In a double integral with variable limits, the change of order of integration changes the limits of integration. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

If the region of integration consists of a vertical strip and slide along x-axis then in the changed order a horizontal strip and slide along y- axis are to be considered and vice-versa.

Example:

01. Change the order of integration and evaluate

$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$

Sol: Given limits are

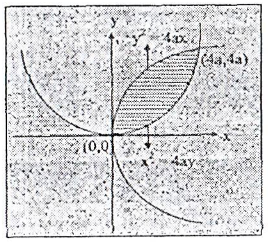
$y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$

i.e $x^2 = 4ay$ i.e $y^2 = 4ax$

and $x = 0$ to $x = 4a$

First shade the region of integration using a vertical strip and slide along x- axis.

Then consider a horizontal strip and slide along y-axis for the same region of integration to get the new limits as



$x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$

and $y = 0$ to $y = 4a$

$\therefore \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right]dy$

$= \int_0^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right]dy$

$= \left[2\sqrt{a}\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$

$= \frac{16a^2}{3}$

Double Integrals in polar co-ordinates:

A double integral can be in terms of polar coordinates also. Let 'R' be the region of integration bounded by the curves $r = f_1(\theta)$ and $r = f_2(\theta)$ between the straight lines $\theta = \theta_1$ and $\theta = \theta_2$.

Then $\iint_R F(r,\theta)\,dR = \int_{\theta=\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\,dr\,d\theta$

Here we first integrate with respect to 'r' between the limits $r = f_1(\theta)$ and $r = f_2(\theta)$ keeping θ fixed and then the resulting expression is integrated with respect to 'θ' from θ_1 to θ_2.

Example:

01. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Sol: $I = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$

$= \int_0^{\pi}\frac{a^2\sin\theta}{2}d\theta$

$= \frac{a^2}{2}\int_0^{\pi}\frac{1-\cos 2\theta}{2}d\theta$

$= \frac{a^2}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi} = \frac{a^2\pi}{4}$

Triple Integral:

Let f(x,y,z) be a function defined over a 3 – dimensional finite region V. Divide the region V' into elementary volumes δV_1, $\delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point in the r^{th} sub-division δV_r. Then

$$\operatorname*{Lt}_{n\to\infty}\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r = \iiint_V f(x,y,z)\,dV$$

Method of Evaluation:

Let the given region be bounded by the limits as

$z = f_1(x,y)$ to $z = f_2(x,y)$

$y = g_1(x)$ to $y = g_2(x)$ and

$x = c_1$ to $x = c_2$

Then

$$\iiint_V f(x,y,z)\,dV = \int_{x=c_1}^{c_2}\left[\int_{y=g_1(x)}^{g_2(x)}\left\{\int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\right\}dy\right]dx$$

Note:

If all the limits are constants then the triple integral can be evaluated in any order.

Example:

01. $\int_2^4\int_0^x\int_0^{\sqrt{x+y}} z\,dz\,dy\,dx$

Sol: $I = \int_2^4\int_0^x\left(\frac{z^2}{2}\right)_0^{\sqrt{x+y}} dy\,dx$

$= \int_2^4\int_0^x \frac{x+y}{2}dy\,dx$

$= \frac{1}{2}\int_2^4\left[xy + \frac{y^2}{2}\right]_0^x dx$

$= \frac{1}{2}\int_2^4 \frac{3x^2}{2}dx = 14$

Change of variables:

By the choice of an appropriate coordinate system, a given integral can be transformed into a simpler integral involving the new variables.

01. Double integral:

Let the variables x, y be changed to the new variables u, v by the transformation $x = \phi(u, v)$ and $y = \psi(u, v)$ where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R^1_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R^1_{uv}} f[\phi(u,v),\psi(u,v)]\,|J|\,du\,dv$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} (\neq 0)$ is the Jacobian of transformation from (x, y) to (u, v) co-ordinates.

Particular Case:

To change Cartesian co-ordinates (x, y) to polar co-ordinates (r,θ), put $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = |J|\,dr\,d\theta$ where

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

$$\therefore \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R^1_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

02. Triple integral:

Let the variables x, y, z be changed to the new variables u, v, w by the transformation $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ where $x(u,v,w)$, $y(u,v,w)$ and $z(u,v,w)$ are continuous and have continuous first order derivatives in some region R^1_{uvw} in the uvw-space which corresponds to the R_{xyz} in the xyz-space.

$$\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_{R^1_{uvw}} f[x(u,v,w), y(u,v,w), z(u,v,w)]\,|J|\,du\,dv\,dw$$

where $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} (\neq 0)$ is the Jacobian of transformation from (x, y, z) to (u, v, w) co- ordinates.

Particular Cases:

i) To change rectangular co-ordinates (x, y, z) to cylindrical co-ordinates (ρ, ϕ, z), put $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ and $dx\,dy\,dz = |J|\,d\rho\,d\phi\,dz$ where

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$$

Then $\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz$

$$= \iiint_{R^1_{\rho\phi z}} f(\rho\cos\phi, \rho\sin\phi, z)\,\rho\,d\rho\,d\phi\,dz$$

ii) To change rectangular co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$ where

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

Then $\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz$

$$= \iiint_{R^1_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

Examples:

01. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$ by transforming into polar coordinates.

Sol: Given limits are $y = 0$ to $y = \sqrt{a^2-x^2}$ (or) $x^2+y^2 = a^2$ and $x = 0$ to $x = a$.

$\therefore$ The given region is positive quadrant of the circle $x^2+y^2 = a^2$

Let $x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$ then

Limits: $r = 0$ to a and $\theta = 0$ to $\dfrac{\pi}{2}$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^a (r\sin\theta)\,r\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \frac{a^4}{4}[-\cos\theta]_0^{\pi/2} = \frac{a^4}{4}$$

02. Evaluate $\iiint_T z\,dx\,dy\,dz$ where T is the hemisphere of radius 'a' $x^2+y^2+z^2 = a^2, z \geq 0$.

Sol: To change cartesian to spherical coordinates, put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, and $|J| = r^2\sin\theta$

The limits of integration are $0 \leq r \leq a$

$0 \leq \phi \leq 2\pi$

$0 \leq \theta \leq \pi/2$

$$\therefore \iiint_T z\,dx\,dy\,dz$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\cos\theta\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \sin\theta\cos\theta\left[\frac{r^4}{4}\right]_0^a d\theta\,d\phi$$

$$= \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi/2} \frac{\sin 2\theta}{2}\,d\theta\,d\phi$$

$$= \frac{a^4}{8}\int_0^{2\pi}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} d\phi$$

$$= \frac{a^4}{8}\int_0^{2\pi} 1\,d\phi = \frac{\pi a^4}{4}$$

Area enclosed by a plane curve:

The area of the region R bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ in the XY plane is

$$\text{Area} = \iint_R dx\,dy \text{ (or) } \iint_R dy\,dx = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} dy\,dx$$

Similarly, if the region is represented through polar coordinates then the area is

$$\text{Area} = \iint_R r\,dr\,d\theta.$$

Length of an arc of a curve:

The length of an arc of a curve $y = f(x)$ between $x = x_1$ and $x = x_2$ is

$$\text{Length} = \int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

In Polar coordinates,

$$\text{Length} = \int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Volume of solids of revolution:

01. The volume of the solid generated by revolving the are bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ about x – axis is

$$V = \int_a^b \pi y^2\,dx$$

02. The volume of the solid generated by revolving the area bounded by the curve $x = g(y)$, y–axis and the lines $y = a$ and $y = b$ about y–axis is

$$V = \int_a^b \pi x^2\,dy$$

03. In polar coordinates, the volume of solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$

(i) about the initial line, $\theta = 0$ is

$$V = \int_\alpha^\beta \frac{2\pi}{3} r^3\sin\theta\,d\theta$$

(ii) about the line, $\theta = \dfrac{\pi}{2}$ is

$$V = \int_\alpha^\beta \frac{2\pi}{3} r^3\cos\theta\,d\theta$$

Examples:

01. Find the length of the curve $3x^2 = y^3$ between $y = 0$ and $y = 1$.

Sol: Given curve is $3x^2 = y^3 \Rightarrow \dfrac{dx}{dy} = \dfrac{\sqrt{3y}}{2}$

$$\therefore \text{Length} = \int_0^1\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_0^1\sqrt{1+\frac{3y}{4}}\,dy = \frac{1}{2}\int_0^1\sqrt{4+3y}\,dy$$

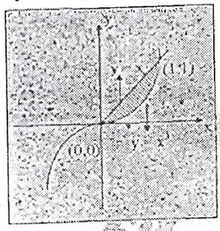
$$= \frac{1}{2}\left[\frac{(4+3y)^{3/2}}{\frac{3}{2}\times 3}\right]_0^1$$

$$= \frac{1}{9}\left(7\sqrt{7}-8\right)$$

02. Find the area bounded by $y = x^3$ and $y = x$ in the first quadrant.

Sol: Area $= \int_0^1 (x - x^3)dx$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

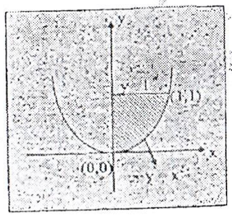


03. Find the volume of solid generated by revolving the area bounded by the curve $y = x^2$, $x = 0$ and $y = 1$ about y – axis.

Sol: Required volume $= \int_0^1 \pi x^2 dy$

$= \int_0^1 \pi y\, dy$

$= \pi\left[\frac{y^2}{2}\right]_0^1 = \frac{\pi}{2}$



LEVEL – I Questions

01. $\underset{x\to 0}{\text{Lt}}\ 2^{1/x} =$
(a) ∞ (b) 0
(c) indeterminate (d) does not exist

02. $\underset{x\to 0}{\text{Lt}}\ \text{Sin}(1/x) =$
(a) ∞ (b) 0
(c) 1 (d) does not exist

03. $\underset{x\to 0}{\text{Lt}}\ x.\sin(1/x) =$
(a) 0 (b) 1
(c) ∞ (d) does not exist

04. $\underset{x\to \infty}{\text{Lt}}\ x.\sin(1/x) =$
(a) 0 (b) 1
(c) ∞ (d) does not exist

05. $\underset{x\to 0}{\text{Lt}}\ |x|/x =$
(a) 1 (b) – 1
(c) indeterminate (d) does not exist

06. If $\underset{x\to a}{\text{Lt}}\ [x]$ does not exist;
where [x] = greatest integer less than or equal to x then which of the following is true?
(a) a is any real number
(b) a is any rational number
(c) a is any integer
(d) a is a complex number

07. $\underset{x\to 2}{\text{Lt}}\ \sqrt{4 - x^2} =$
(a) 0 (b) imaginary
(c) does not exist (d) indeterminate

08. $\underset{x\to \pi/2}{\text{Lt}}\ \tan x =$
(a) ∞ (b) - ∞
(c) does not exist (d) 0

09. $\underset{x\to 0^+}{\text{Lt}}\ \log x =$
(a) ∞ (b) – ∞
(c) 0 (d) does not exist

10. $\underset{x\to 0}{\text{Lt}}\ \frac{e^{1/x} - 1}{e^{1/x} + 1} =$
(a) – 1 (b) 1
(c) ∞ (d) does not exist

11. $\underset{x\to \infty}{\text{Lt}}\ \left[\frac{x-1}{x+1}\right]^x =$
(a) e (b) e^2 (c) e^{-2} (d) ∞

12. If $\underset{x\to 0}{\text{Lt}}\ (\text{Sin}2x + a\,\text{Sin}x)/x^3$ is finite, then a =
(a) 0 (b) 2 (c) – 2 (d) 1/2

13. The value of the limit given in the previous example is
(a) 0 (b) – 1
(c) 1 (d) does not exist

14. If $\underset{x\to 0}{\text{Lt}}\ \left\{\frac{x(1 + a\cos x) - b\sin x}{x^3}\right\} = 1$
then (a, b) =
(a) –5/2, 3/2 (b) 5/2, -3/2
(c) 5/2, 3/2 (d) –5/2, -3/2

15. $\underset{x\to \pi/2}{\text{Lt}}\ (\text{Sec}\,x / \text{Sec}\,3x) =$
(a) 3 (b) – 3 (c) 1/3 (d) – 1/3

16. $\underset{x\to 0}{\text{Lt}}\ \{(1/x^2) - (1/\text{Sin}^2 x)\} =$
(a) 0
(b) 1/3
(c) – 1/3
(d) cannot be determined

17. $\underset{x\to 0}{\text{Lt}}\ (\cos x)^{1/x^2} =$
(a) 1 (b) e^{-2} (c) $e^{-1/2}$ (d) e

18. $\underset{x\to 0}{\text{Lt}}\ (\log x / \log \text{cosec}\, x) =$
(a) 1 (b) – 1
(c) 0 (d) does not exist

19. $\underset{x\to \infty}{\text{Lt}}\ (x^n / e^x) =$
(a) 0 (b) n! (c) 1 (d) ∞

20. $\underset{x\to \infty}{\text{Lt}}\ (\log x / x^n) =$
(a) 0 (b) 1/n (c) 1 (d) –1/n

21. Let $f(x) = (1 + 3x)^{1/x}$, $x \neq 0$
$= k$, $x = 0$
If f(x) is continuous at x = 0 then k =
(a) e^3 (b) 3 (c) $e^{1/3}$ (d) $e^{-1/3}$

22. If $f(x) = \{\text{Sin}^2(ax)/x^2\}$, $x \neq 0$
$= 1$, $x = 0$
is continuous then a =
(a) 0, 1 (b) -1, 1
(c) 0, -1 (d) none of these

23. If f(x) = 1 when x < 0
= 1 + Sinx when $0 \le x \le \pi/2$
$= 2 + (x - \pi/2)^2$ when $x > \pi/2$ then f(x) is
(a) continuous at x = 0 but discontinuous at x = π/2
(b) continuous at x = π/2 but discontinuous at x = 0
(c) continuous for all values of x
(d) discontinuous at x = 0 and at x = π/2

24. If f(x) $= -x^2$, $x \le 0$
$= 5x - 4$, $0 < x \le 1$
$= 4x^2 - 3x$, $1 < x < 2$
$= 3x + 4$, $x \ge 2$
then f(x) is discontinuous at x =
(a) 0 (b) 1 (c) 2 (d) 3

25. The function f(x) = {x . Sin(1/x)} is
(a) continuous at x = 0 but not differentiable at x = 0
(b) discontinuous at x = 0 but differentiable at x = 0
(c) continuous and differentiable at x = 0
(d) neither continuous nor differentiable at x=0

26. The function f(x) = |x| is
(a) continuous at x = 0 but not differentiable at x = 0
(b) differentiable at x = 0 but not continuous at x = 0
(c) continuous and differentiable at x = 0
(d) neither continuous nor differentiable at x = 0

27. If $f(x) = x^2\sin(1/x)$ then which of the following is true?
(a) $f^1(0)$ exists but $f^{11}(0)$ does not exist
(b) both $f^1(0)$ and $f^{11}(0)$ does not exist
(c) neither $f^1(0)$ nor $f^{11}(0)$ does not exist
(d) $f^1(0)$ does not exist but $f^{11}(0)$ exists

28. If $f(x) = x\,(1+(1/3)\,\text{Sin}\,(\log x))$ then f(x) is
(a) continuous at x = 0 but not differentiable at x = 0
(b) differentiable at x=0 but not continuous at x = 0
(c) continuous and differentiable at x = 0
(d) neither continuous nor differentiable at x = 0

29. The function $f(x) = |x| + |x+1| + |x-2|$ is differentiable at x =
(a) 1 (b) – 1 (c) 0 (d) 2

30. If $f(x) = 2 + x$ when $x \geq 0$
$= 2 - x$ when $x < 0$
then f(x) at x = 0 is
(a) continuous but not differentiable
(b) continuous and differentiable
(c) neither continuous nor differentiable
(d) differentiable at x = 0 but not continuous

31. If $f(x) = 1$ when $x < 0$
$= 1 + \sin x$ when $0 \leq x < \pi/2$
$= 2 + (x - \pi/2)^2$ when $x \geq \pi/2$
then at $x = \pi/2$, f(x) is
(a) continuous but not differentiable
(b) differentiable but not continuous
(c) continuous and differentiable
(d) neither continuous nor differentiable

32. Find C of the Rolle's theorem for
$f(x) = x\,(x-1)\,(x-2)$ in [1, 2]
(a) 1.5 (b) $1 - (1/\sqrt{3})$
(c) $1 + (1/\sqrt{3})$ (d) 1.25

33. Find C of the Rolle's theorem for
$f(x) = e^x \sin x$ in $[0, \pi]$
(a) $\pi/4$ (b) $\pi/2$
(c) $3\pi/4$ (d) does not exist

34. Find C of Rolle's theorem for
$f(x) = (x+2)^3\,(x-3)^4$ in [-2, 3]
(a) 1/7 (b) 2/7
(c) 1/2 (d) 3/2

35. Find C of Rolle's theorem for
$f(x) = e^x\,(\sin x - \cos x)$ in $[\pi/4, 5\pi/4]$
(a) $\pi/2$ (b) $3\pi/4$
(c) π (d) does not exist

36. Find C of Rolle's theorem for
$f(x) = x(x+3)\,e^{-x/2}$ in [-3, 0]
(a) –1 (b) –2 (c) –0.5 (d) 0.5

37. Find C of Rolle's theorem for
$f(x) = \log[(x^2 + ab) / (a+b)x]$ in [a,b]
(a) (a+b)/2 (b) $\sqrt{ab}$
(c) 2ab / (a + b) (d) (b – a)/2

38. Rolle's theorem cannot be applied for the function $f(x) = |x|$ in [-2, 2] because
(a) f(x) is not continuous in [-2, 2]
(b) f(x) is not differentiable in (-2, 2)
(c) $f(-2) \neq f(2)$
(d) none of the above

39. Rolle's theorem cannot be applied for the function $f(x) = |x+2|$ in [-2, 0] because
(a) f(x) is not continuous in [-2, 0]
(b) f(x) is not differentiable in (-2, 0)
(c) $f(-2) \neq f(0)$
(d) none of these

40. Find C of Lagrange's mean value theorem for
$f(x) = (x-1)\,(x-2)\,(x-3)$ in [1, 2]
(a) $2 - 1/\sqrt{3}$ (b) $2 + (1/\sqrt{3})$
(c) $1 + (1/\sqrt{3})$ (d) $1 - (1/\sqrt{3})$

41. Find C of Lagrange's mean value theorem for
$f(x) = \log x$ in [1, e]
(a) e – 2 (b) e – 1
(c) (e + 1) / 2 (d) (e – 1) / 2

42. Find C of Lagrange's mean value theorem for
$f(x) = lx^2 + mx + n$ in [a, b]
(a) (a + b) / 2 (b) $\sqrt{ab}$
(c) 2ab / (a + b) (d) (b – a)/2

43. Find C of Lagrange's theorem mean value theorem for $f(x) = 7x^2 - 13x - 19$ in [-11/7, 13/7]
(a) 1/7 (b) 2/7
(c) 3/7 (d) 4/7

44. Find C of Lagrange's mean value theorem for
$f(x) = e^x$ in [0, 1]
(a) 0.5
(b) log(e – 1)
(c) log(e + 1)
(d) log[(e + 1)/ (e – 1)]

45. Lagrange's mean value theorem cannot be applied for the function $f(x) = x^{1/3}$ in [-1, 1] because
(a) f(x) is not continuous in [-1, 1]
(b) f(x) is not differentiable in (-1,1)
(c) f(x) is neither continuous not differentiable in [-1, 1]
(d) none of the above

46. Find C of Cauchy's mean value theorem for
$f(x) = e^x$ and $g(x) = e^{-x}$ in [a, b]
(a) (a + b) /2 (b) $\sqrt{ab}$
(c) 2ab / (a + b) (d) (b – a) / 2

47. Find C of Cauchy's mean value theorem for
$f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$ in [a, b]
(a) (a + b) / 2 (b) $\sqrt{ab}$
(c) 2ab / (a + b) (d) (b – a) / 2

48. Find C of Cauchy's mean value theorem for the functions 1/x and $1/x^2$ in [a, b]
(a) (a + b)/2 (b) $\sqrt{ab}$
(c) 2ab / (a + b) (d) (b – a)/2

49. Find C of Cauchy's mean value theorem for the functions sinx and cosx in $(-\pi/2, 0)$
(a) $-\pi/3$ (b) $-\pi/4$
(c) $-\pi/6$ (d) $-\pi/8$

50. $\int_0^2 |1-x|\,dx =$
(a) 1 (b) – 1 (c) 2 (d) 3/2

51. $\int_0^1 x(1-x)^5\,dx =$
(a) 1/42 (b) 1/48
(c) 1/12 (d) 1/56

52. $\int_{-1}^{1} \frac{x^2.\sin x}{x^4+1}\,dx =$
(a) 0 (b) π (c) 2 (d) $\pi/2$

53. $\int_{-\pi/2}^{\pi/2} x^{10} \log|(1 + \text{Sin}x) / (1 - \text{Sin}x)|\,dx =$
(a) 0 (b) π (c) 2 (d) $\pi/2$

54. $\int_{-1}^{1} |x|\,dx =$
(a) 0 (b) 1 (c) 2 (d) 4

55. $\int_0^{\pi} \sin^3 x\,dx =$
(a) 2/3 (b) 4/3 (c) 0 (d) $\pi/3$

56. $\int_0^{\pi/2} [(\sin x - \cos x) / (1 + \sin x \cos x)]\,dx =$
(a) 0 (b) π (c) $\pi/2$ (d) $\pi/4$

57. $\int_0^{\pi/2} dx / \left(1 + \sqrt{\cot x}\right) =$
(a) 0 (b) π (c) $\pi/2$ (d) $\pi/4$

58. $\int_0^{\pi/2} (\sin 2x\,.\log(\tan x))\,dx =$
(a) 0 (b) π (c) $\pi/2$ (d) $\pi/4$

59. $\int_0^{\pi/4} \log(1 + \tan x)\,dx =$
(a) 0 (b) $(\pi/2)\log 2$
(c) $(\pi/8)\log 2$ (d) $(-\pi/4)\log 2$

60. $\int_0^{\pi} [(x \sin x) / (1 + \cos^2 x)]\,dx =$
(a) π^2 (b) $\pi^2/2$ (c) $\pi^2/4$ (d) $\pi^2/8$

61. $\int_0^{\pi} dx / \left(a^2 \cos^2 x + b^2 \sin^2 x\right) =$
(a) 0 (b) πab
(c) π/ab (d) $\pi / (a^2 + b^2)$

62. $\int_0^{\pi} x.\text{Sin}^6 x.\text{Cos}^4 x\,dx =$
(a) $3\pi^2 / 512$ (b) $5\pi^2/256$
(c) $3\pi^2/128$ (d) $5\pi^2/128$

63. $\int_0^{\pi} [(x \tan x) / (\sec x + \tan x)]\,dx =$
(a) 0 (b) $\pi(\pi - 2)/4$
(c) $\pi(\pi - 2)/2$ (d) π

64. $\int_0^{\pi^2/4} \sin\sqrt{x}\, dx =$

(a) 0 (b) 1 (c) 2 (d) $\pi/2$

65. $\int_2^3 \{\sqrt{x}/(\sqrt{x} + \sqrt{5-x})\}\, dx =$

(a) 1 (b) 2.5 (c) 0.5 (d) 1.5

66. $\int_{-\pi}^{\pi} \sin^4 x\, dx =$

(a) $\pi/4$ (b) $\pi/2$ (c) $3\pi/4$ (d) 0

67. $\int_0^{\pi} \sin^4 x \cos^5 x\, dx =$

(a) 0 (b) $3\pi/256$
(c) $3\pi/128$ (d) $5\pi/128$

68. $\int_0^{2\pi} \sin^4 x \cos^6 x\, dx =$

(a) $3\pi/128$ (b) $3\pi/256$
(c) $3\pi/64$ (d) 0

69. $\int_0^{2\pi} \sin^4 x \cos^5 x\, dx =$

(a) 0 (b) $3\pi/128$
(c) $5\pi/128$ (d) $3\pi/256$

70. Which of the following improper integrals is divergent?

(a) $\int_0^{\infty} [1/(1+x^2)]dx$ (b) $\int_0^{1} [x/\sqrt{(1-x^2)}]dx$

(c) $\int_0^{1} \log x\, dx$ (d) $\int_0^{\infty} x \sin x\, dx$

71. Which of the following improper integrals is divergent?

(a) $\int_{-\infty}^{\infty} [1/(1+x^2)]dx$ (b) $\int_0^{\infty} e^{-x}\, dx$

(c) $\int_1^{\infty} dx/x^5$ (d) $\int_0^{2} 1/x^3\, dx$

72. Which of the following improper integrals is divergent ?

(a) $\int_0^{1} [dx/x^{1/3}]dx$ (b) $\int_1^{\infty} [dx/x\sqrt{(x^2-1)}]$

(c) $\int_1^{\infty} [1/x^2]dx$ (d) $\int_1^{\infty} (1/\sqrt{x})dx$

73. Which of the following integrals is divergent ?

(a) $\int_{-\infty}^{-1} 1/x^4\, dx$

(b) $\int_0^{1} [1/x^2]dx$

(c) $\int_0^{1} [1/\sqrt{(1-x^2)}]dx$

(d) $\int_0^{\infty} [1/(x^2+2x+2)]\, dx$

74. Which of the following improper integrals is divergent ?

(a) $\int_0^{\infty} x$ [illegible] dx (b) $\int_0^{\infty} [\log x/x^3]dx$

(c) $\int_0^{1} x \log x\, dx$ (d) $\int_{-1}^{1} dx/(x.x^{1/3})$

75. Consider the integrals

$I_1 = \int_1^{\infty} dx/[x^2(1+e^x)]$ and

$I_2 = \int_1^{\infty} [(x+1)/x\sqrt{x}]\, dx$

Which of the following is true ?
(a) I_1 is convergent and I_2 is divergent
(b) I_1 is divergent and I_2 is convergent
(c) I_1 and I_2 are convergent
(d) I_1 and I_2 are divergent

76. Which of the following integrals is divergent ?

(a) $\int_0^{1} [1/\sqrt{(1-x)}]\, dx$ (b) $\int_1^{\infty} (\sin x/x^3)dx$

(c) $\int_{-1}^{1} (1/x^2)dx$ (d) $\int_0^{2} \left(1/\sqrt{|(1-x^2)|}\right) dx$

77. Which of the following integrals is divergent ?

(a) $\int_0^{1} [1/\sqrt{(x+4x^3)}]\, dx$ (b) $\int_{e^2}^{\infty} dx/[x(\log x)^3]$

(c) $\int_0^{\pi/2} \sec x\, dx$ (d) $\int_1^{e} [dx/x(\log x)^{1/3}]$

78. Which of the following improper integrals is convergent?

(a) $\int_1^{\infty} [(x+\sqrt{(x+1)})/(x^2+2(x^4+1)^{1/5})]dx$

(b) $\int_2^{\infty} [(3+2x^2)^{1/7}/(x^3-1)^{1/5}]\, dx$

(c) $\int_3^{\infty} dx/\sqrt{x(x-1)(x-2)}$

(d) $\int_1^{\infty} [1/x^{1/3}(1+x)^{1/2}]dx$

79. Which of the following improper integrals is divergent ?

(a) $\int_1^{\infty} [x/(1+x)^3]dx$

(b) $\int_1^{\infty} dx/[(1+x)\sqrt{x}]$

(c) $\int_0^{1} dx/[x^{1/3}(1+x^2)]$

(d) $\int_0^{1} dx/[x^2(1+x)^2]$

80. Which of the following improper integrals is convergent ?

(a) $\int_0^{1} [1/\{(x-1)^6(x-2)^{1/5}\}]dx$

(b) $\int_0^{1} [x^{a-1}/x+1]dx$ (a>0)

(c) $\int_0^{1} dx/[x^2(1+x)^2]$

(d) $\int_{1/2}^{1} dx/[x^{1/3}(1-x)^{1/3}]$

81. The total derivative of x^2y with respect to x, when x and y are connected by the relation $x^2+xy+y^2=1$ is

(a) $2xy - x^2\left(\frac{2x+y}{x+2y}\right)$ (b) $\frac{x^2y+xy^2}{x+2y}$

(c) $2xy \div y^2\left(\frac{x+2y}{2x+y}\right)$ (d) $\frac{x^2y-xy^2}{y+2x}$

82. If $u = x\cdot\log(xy)$ where $x^3+y^3+3xy=1$ then $(du/dx) =$
(a) 0
(b) $1+\log(xy)-[x(x^2+y)/y(y^2+x)]$
(c) $1+\log(y/x)-[(x^2+y)/(x+y^2)]$
(d) $\log(x/y)+[(x^2+y)/(x+y^2)]$

83. If $u = \sin(x^2+y^2)$ where $a^2x^2+b^2y^2=c^2$ then $(du/dx) =$
(a) $2(1-a^2/b^2)x \cos(x^2+y^2)$
(b) $2x(a^2+b^2)/b^2 \cos(x^2+y^2)$
(c) $2x(a^2-b^2)/a^2 \cos(x^2+y^2)$
(d) $2x(a^2-b^2)/b^2 \cos(x^2+y^2)$

84. If $u = \sin^{-1}(x/y) + \cos(y/x)$ then $u_x/u_y =$
(a) x/y (b) y/x
(c) $-x/y$ (d) $-y/x$

85. Let $r^2 = x^2+y^2+z^2$ and $v = r^n$. Then $v_{xx}+v_{yy}+v_{zz} =$
(a) 0 (b) $n(n+1)\, r^{n-2}$
(c) $n(n-1)\, r^{n-2}$ (d) $n(n+2)\, r^{n-2}$

86. If $u = (y/z)+(z/x)$ then $xu_x + y\,u_y + zu_z =$
(a) 0 (b) xy/z^2
(c) yz/x^2 (d) zx/y^2

87. If $v = (x^2+y^2+z^2)^{-2}$ then $v_{xx}+v_{yy}+v_{zz} =$
(a) 0 (b) $(x^2+y^2+z^2)$
(c) $12(x^2+y^2+z^2)^{-3}$ (d) $(x^2+y^2+z^2)^{-2}$

88. If $u = f(r)$ where $r^2 = x^2+y^2+z^2$ then $u_{xx}+u_{yy}+u_{zz} =$
(a) $f''(r) + (2/r)\, f'(r)$
(b) $f''(r) + (1/r^2)\, f'(r)$
(c) $f''(r) + (3/r)\, f'(r)$
(d) $f''(r) - (2/r)\, f'(r)$

89. If $u = f(r)$ where $x = r\cos\theta$ and $y = r\sin\theta$ then $u_{xx} + u_{yy} =$
(a) $f''(r) + (1/r) f'(r)$ (b) $f''(r) + (2/r) f'(r)$
(c) $f''(r) - (1/r) f'(r)$ (d) $f''(r) - (2/r) f'(r)$

90. If $\sin u = [(x + 2y + 3z) / (x^8 + y^8 + z^8)]$ then $x u_x + y u_y + z u_z =$
(a) $(1/7)\tan u$ (b) $-7\tan u$
(c) $(1/7)\sec u$ (d) $(-1/20)\tan u$

91. If $u = \log[(x^4 + y^4) / (x - y)]$ then $x^2 u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} =$
(a) 0 (b) 3 (c) -3 (d) 1/3

92. If $u = \operatorname{cosec}^{-1}[(x^{1/4} + y^{1/4}) / (x^{1/5} - y^{1/5})]$ then $x u_x + y u_y =$
(a) $(1/20)\cot u$ (b) $(-1/20)\cot u$
(c) $(1/20)\tan u$ (d) $(-1/20)\tan u$

93. If $u = [(x^3 + y^3) / (x - y)] + x\sin(x/y)$ then $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} =$
(a) 0
(b) $2[(x^3 + y^3) / (x - y)]$
(c) $[(x^3 + y^3) / (x - y)] - \sin(x/y)$
(d) $[(x^3 + y^3) / (x - y)] - \cos(x/y)$

94. If $z = x^n f_1(y/x) + y^{-n} f_2(x/y)$ then $x(\partial z/\partial x) + y(\partial z/\partial y) + x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} =$
(a) 0 (b) $n(n + 1)z$
(c) $n^2 z$ (d) $(n^2 - n)z$

95. If $u = f(r, s)$ where $r = x + y$ and $s = x - y$ then $u_x + u_y =$
(a) $2u_r$ (b) $2u_s$
(c) $-2u_r$ (d) $-2u_s$

96. If $u = f(x - y, y - z, z - x)$ then $u_x + u_y + u_z =$
(a) 0 (b) u (c) 2u (d) 3u

97. If $z = f(x, y)$ where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$ then $z_u - z_v =$
(a) $x z_x - y z_y$ (b) $x z_x + y z_y$
(c) $x z_y + y z_x$ (d) $x z_y - y z_x$

98. If $u = f(x + cy) + g(x - cy)$ then $u_{xx} / u_{yy} =$
(a) c^2 (b) c^{-2} (c) $-c^2$ (d) $-c^{-2}$

99. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then $u_x + u_y + u_z =$
(a) $-3 / (x + y + z)$ (b) $3 / (x + y + z)$
(c) $9 / (xy + yz + zx)$ (d) $-9 / (xy + yz + zx)$

100. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then $(\partial/\partial x + \partial/\partial y + \partial/\partial z)^2 u =$
(a) $3 / (x + y + z)^2$ (b) $-3/(x + y + z)^2$
(c) $9 / (x + y + z)^2$ (d) $-9 / (x + y + z)^2$

101. If $u = e^{ax + by} f(ax - by)$ then $b u_x + a u_y =$
(a) 0 (b) 2abu
(c) $2(a + b)u$ (d) $2(a - b)u$

102. The function $f(x) = 2x^3 - 3x^2 - 36x + 10$ has a maximum at $x =$
(a) 3 (b) 2 (c) -3 (d) -2

103. The minimum value of $f(x) = 2x^3 - 3x^2 - 36x + 10$ is
(a) 0 (b) -13 (c) -17 (d) -71

104. A maximum value of $f(x) = (\log x / x)$ is
(a) e (b) e^{-1} (c) $e - 1$ (d) $e + 1$

105. The function $f(x) = x^x$ has a minimum at $x =$
(a) e (b) e^{-1} (c) 0 (d) $e + 1$

106. The minimum value of $f(x) = x \cdot \log x$ is
(a) e (b) e^{-1} (c) $-e$ (d) $-e^{-1}$

107. The maximum value of $x \cdot e^{-x}$ is
(a) e (b) e^{-1} (c) 1 (d) $-e$

108. The maximum value of $f(x) = \sin x + \cos 2x$ in the interval $[0, \pi]$ is
(a) 2 (b) 1.5 (c) 5/7 (d) 9/8

109. $f(x, y) = x^3 + y^3 - 3xy$ has
(a) a maximum at (1, 1)
(b) a minimum at (1, 1)
(c) a saddle point at (1, 1)
(d) neither maximum nor minimum

110. At (a, a), $f(x, y) = xy + a^3/x + a^3/y$ has
(a) a maximum
(b) a minimum
(c) a maximum if $a > 0$
(d) neither maximum nor minimum

111. At $(\sqrt{2}, -\sqrt{2})$, $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ has
(a) a minimum
(b) a maximum
(c) a saddle point
(d) neither maximum nor minimum

112. A rectangular box open at the top is to have a volume 32 C.C. Find the dimensions of the box requiring least material for its construction
(a) 4 cm,4 cm,2 cm
(b) 2 cm,2 cm,8 cm
(c) 16 cm,1 cm,1cm
(d) 8 cm,8 cm,1/2 cm

113. If $f'(x) = (x + 2)(x - 1)^2(2x - 1)(x - 3)$ then at $x = ½$, f(x) has
(a) a maximum
(b) a minimum
(c) neither maximum nor minimum
(d) no stationary point

114. Find the minimum value of $x^2 + y^2 + z^2$ so that $xyz = 8$
(a) 8 (b) 12 (c) 21 (d) 27

115. Find the maximum value of $x^2 + y^2 + z^2$ so that $x + y + z = 1$
(a) 1 (b) 1/2 (c) 1/3 (d) 1/4

116. Divide 24 into three parts x, y, z so that xy^2z^3 is a maximum
(a) 8, 8, 8 (b) 4, 8, 12
(c) 6, 9, 9 (d) 6, 8, 10

117. Let $T = 400\, xyz^2$. Then find the maximum value of T so that $x^2 + y^2 + z^2 = 1$
(a) 50 (b) 100 (c) 200 (d) 800

118. Find the volume of the greatest rectangular parellelopiped that can be inscribed in the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$
(a) $abc / 3\sqrt{3}$ (b) $4abc / 3\sqrt{3}$
(c) $8abc / 3\sqrt{3}$ (d) $2abc / 3\sqrt{3}$

119. Change the order of integration in the integral

$$I = \int_{-a}^{a} \int_{0}^{\sqrt{(a^2-y^2)}} f(x,y)\,dx\,dy$$

(a) $I = \int_{0}^{a} \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dy\,dx$

(b) $I = \int_{-a}^{a} \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dy\,dx$

(c) $I = \int_{0}^{a} \int_{-a}^{a} f(x,y)\,dy\,dx$

(d) None

120. Calculate $\int\int r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$
(a) 2.25 (b) 2.25π (c) 0 (d) None.

121. $\int_{1}^{2}\int_{1}^{3} xy^2\, dx\, dy =$
(a) 10 (b) 11 (c) 13 (d) 12

122. $\int_{0}^{1}\int_{x}^{\sqrt{x}} (x^2 + y^2)\, dx\, dy =$
(a) $\frac{2}{35}$ (b) $-\frac{3}{35}$ (c) $\frac{3}{35}$ (d) $-\frac{2}{35}$

123. $\int_{0}^{1}\int_{4y}^{4} e^{x^2}\, dx\, dy =$
(a) $\frac{(e^{16} - 1)}{8}$ (b) $-\frac{(e^{16} + 1)}{8}$
(c) 0 (d) $-\frac{(e^{16} - 1)}{8}$

124. $\int_{0}^{4}\int_{0}^{x^2} e^{y/x}\, dy\, dx =$
(a) $3e^4$ (b) $3e^4 + 7$
(c) $-3e^4$ (d) $3e^4 - 7$

125. $\int_{0}^{1}\int_{0}^{\sqrt{(1+x^2)}} \left(\frac{1}{1 + x^2 + y^2}\right) dy\,dx =$
(a) $\left(\frac{1}{4}\right)\pi\log(1 + \sqrt{2})$
(b) $\left(\frac{1}{4}\right)\pi\log(1 - \sqrt{2})$
(c) $\left(\frac{1}{4}\right)\pi\log(1 + \sqrt{3})$
(d) $\left(\frac{1}{4}\right)\pi\log(1 - \sqrt{3})$

126. $\int\int xy\, dx\, dy =$, over the positive quadrant of the circle $x^2 + y^2 = a^2$

(a) $\frac{a^4}{4}$ (b) $-\frac{a^4}{4}$

(c) $-\frac{a^4}{8}$ (d) $\frac{a^4}{8}$

127. $\int\int xy(x + y)\, dx\, dy =$, over the area between $y = x^2$ and $y = x$.

(a) $\frac{6}{56}$ (b) 0 (c) $\frac{3}{56}$ (b) $-\frac{3}{56}$

128. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} y^2\, dxdy =$

(a) $-\frac{\pi}{16}$ (b) $\frac{\pi}{16}$ (c) $\frac{\pi}{8}$ (d) $-\frac{\pi}{8}$

129. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y)dx\, dy =$

(a) $-\frac{241}{60}$ (b) $\frac{241}{60}$ (c) $-\frac{241}{61}$ (d) 0

130. $\int_0^\infty \int_x^\infty \left(\frac{e^{-y}}{y}\right)dy\, dx =$

(a) 0 (b) 2 (c) 3 (d) 1

131. $\int_0^\infty \int_0^x xe^{-\frac{x^2}{y}}\, dy\, dx =$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) None

132. $\int\int r \sin\theta\, dr\, d\theta =$, over the cardioid $r = a(1 - \cos\theta)$ above the initial line.

(a) 0 (b) -4 (c) $\frac{4a^3}{3}$ (d) 2

133. $\iint_R r^2 \sin\theta\, dr\, d\theta =$, where R is the semi-circle $r = 2a \cos\theta$ above the initial line

(a) $\frac{2a^3}{3}$ (b) $-\frac{2a^3}{3}$ (c) $\frac{2a^3}{5}$ (d) $-\frac{2a^3}{5}$

134. $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x + y + z)dx\, dy\, dz =$

(a) 1 (b) 2 (c) 3 (d) 0

135. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\, dxdydz =$

(a) $\frac{1}{24}$ (b) $-\frac{1}{24}$ (c) $\frac{1}{48}$ (d) $-\frac{1}{48}$

136. $\int_0^1 \int_0^2 \int_1^2 x^2yz\, dx\, dy\, dz =$

(a) $-\frac{7}{3}$ (b) $\frac{7}{3}$ (c) $\frac{7}{2}$ (d) $-\frac{7}{2}$

137. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{(4z-x^2)}} dy\, dx\, dz =$

(a) 8π (b) 4π (c) 2π (d) 0

138. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

(a) $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$

(b) $\frac{1}{8}e^{4a} + \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$

(c) $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} - e^a - \frac{3}{8}$

(d) $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a + \frac{3}{8}$

139. Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis.

(a) $-\frac{8\pi a^3}{3}$ (b) $\frac{8\pi a^3}{3}$

(c) $\frac{8\pi a^3}{5}$ (d) $-\frac{8\pi a^3}{5}$

140. $\iint_R (x + y)^2\, dx\, dy =$, where R is the parallelogram in the XY-plane with vertices (1, 0) (3, 1) (2, 2) (0, 1) using the transformation $u = x + y$ and $v = x - 2y$.

(a) 21 (b) 3 (c) 9 (d) 0

141. By changing to polar coordinates, evaluate

$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy =$,

(a) $\left(\frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{3}\right)$

(c) $-\sqrt{\left(\frac{\pi}{2}\right)}$ (d) $\sqrt{\left(\frac{\pi}{2}\right)}$

KEY for LEVEL - 1

01. (d)	02. (d)	03. (a)	04. (b)	05. (d)
06. (c)	07. (c)	08. (c)	09. (b)	10. (d)
11. (c)	12. (c)	13. (b)	14. (d)	15. (b)
16. (c)	17. (c)	18. (b)	19. (a)	20. (a)
21. (a)	22. (b)	23. (c)	24. (a)	25. (a)
26. (a)	27. (a)	28. (a)	29. (a)	30. (a)
31. (c)	32. (c)	33. (c)	34. (a)	35. (c)
36. (b)	37. (b)	38. (b)	39. (c)	40. (a)
41. (b)	42. (a)	43. (a)	44. (b)	45. (b)
46. (a)	47. (b)	48. (c)	49. (b)	50. (a)
51. (a)	52. (a)	53. (a)	54. (b)	55. (b)
56. (a)	57. (d)	58. (a)	59. (c)	60. (c)
61. (c)	62. (a)	63. (c)	64. (c)	65. (c)
66. (c)	67. (a)	68. (a)	69. (a)	70. (d)
71. (d)	72. (d)	73. (b)	74. (d)	75. (a)
76. (c)	77. (c)	78. (c)	79. (d)	80. (b)
81. (a)	82.(b)	83. (a)	84. (d)	85. (b)
86. (a)	87. (a)	88. (a)	89. (a)	90. (b)
91. (c)	92. (d)	93. (b)	94. (c)	95. (a)
96. (a)	97. (a)	98. (b)	99. (b)	100.(d)
101.(b)	102.(d)	103.(d)	104.(b)	105.(b)
106.(d)	107.(b)	108.(d)	109.(b)	110.(b)
111.(a)	112.(a)	113.(a)	114.(b)	115. (c)
116.(b)	117.(a)	118.(c)	119. (a)	120. (b)
121. (c)	122. (c)	123. (a)	124. (d)	125. (a)
126. (d)	127. (c)	128. (b)	129. (b)	130. (d)
131. (a)	132.(c)	133.(a)	134.(d)	135.(c)
136.(b)	137. (a)	138. (a)	139. (b)	140.(a)
141. (d)				

LEVEL - 2 Questions

Limit

01. $\underset{x\to 0}{\text{Lt}} \frac{1-\cos x}{x \sin x} =$

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) does not exist

02. If $\underset{x\to 0}{\text{Lt}} \frac{\sin 2x + a \sin x}{x^3} = b$ where b is finite value then find a and b

(a) $a = -2, b = -1$ (b) $a = 2, b = -1$

(c) $a = 0, b = 3$ (d) $a = -2, b = 1$

03. $\underset{x\to 0}{\text{Lim}} \frac{\log(\sin 2x)}{\log(\sin x)} =$ ________

04. $\underset{x\to a}{\text{Lt}} (a - x) \tan\left(\frac{\pi x}{2a}\right) =$ ________

05. $\underset{x\to 0}{\text{Lt}} \left[\frac{1}{x} - \frac{1}{e^x - 1}\right] =$

(a) $\frac{1}{2}$ (b) 1 (c) 0 (d) 2

06. $\underset{x\to 0}{\text{Lt}}\, x^{\sin x} =$ ________

07. $\underset{x\to 0}{\text{Lt}} \left(\frac{1}{x}\right)^{\tan x} =$ ________

08. $\underset{x\to \pi}{\text{Lt}} \cot(x) =$

(a) 0 (b) 1

(c) -1 (d) does not exist

09. $\underset{x\to a}{\text{Lt}} [x] =$, where [x] is step function and 'a' is an integer

(a) 2 (b) 4 (c) a^2 (d) does not exist

10. $\underset{x\to\infty}{Lt}\ x\tan\left(\frac{1}{x}\right)=$ ________

Continuity

11. If $f(x)=(x+1)^{\cot x}$ is continuous at $x=0$ then $f(0)=$ _____
(a) 0 (b) 1 (c) e (d) none of these

12. Let $f(x)=\begin{cases} 0 & \text{for } x=0 \\ \frac{1}{2}-x & \text{for } 0<x<\frac{1}{2} \\ \frac{1}{2} & \text{for } x=\frac{1}{2} \\ \frac{3}{2}-x & \text{for } \frac{1}{2}<x<1 \\ 1 & \text{for } x\ge 1 \end{cases}$ then which of the following is true?
(a) $f(x)$ is right continuous at $x=0$
(b) $f(x)$ is discontinuous at $x=\frac{1}{2}$
(c) $f(x)$ is continuous at $x=1$
(d) All are true

Differentiability

13. If $f(x)=3+x$ when $x\ge 0$
$=3-x$ when $x<0$
then $f(x)$ at $x=0$ is
(a) continuous but not differentiable
(b) continuous and differentiable
(c) neither continuous nor differentiable
(d) differentiable but not continuous

14. If $(f)=x|x|$ then $f(x)$ at $x=0$ is
(a) continuous and differentiable
(b) continuous but not differentiable
(c) differentiable but not continuous
(d) neither continuous nor differentiable

15. The function $f(x)=|x+1|$ on the interval $[-2,0]$ is
(a) continuous and differentiable
(b) continuous on the interval but not differentiable
(c) neither continuous nor differentiable
(d) differentiable but not continuous

16. The function $f(x)=\frac{x\,e^{1/x}}{1+e^{1/x}},\ x\ne 0$
$=0,\quad x=0$ is
(a) differentiable but not continuous at $x=0$
(b) not differentiable at $x=0$
(c) differentiable and continuous at $x=0$
(d) not continuous at $x=0$

Mean Value Theorems

17. Find 'C' of Rolle's Theorem for $f(x)=e^x(\sin x-\cos x)$ in $[\pi/4, 5\pi/4]$
(a) $\pi/2$ (b) $3\pi/4$
(c) π (d) does not exist

18. The value of ξ of $f(b)-f(a)=(b-a)\,f^1(\xi)$ for the function $f(x)=Ax^2+Bx+c$ in the interval $[a,b]$ is ____
(a) $b-a$ (b) $b+a$
(c) $\frac{b-a}{2}$ (d) $\frac{b+a}{2}$

19. The mean value C of Lagrange's Theorem for the function $f(x)=3x^2+5x+8$ in $\left[\frac{11}{2},\frac{13}{2}\right]$ is ______

20. If $f(x)=\frac{1}{x}, g(x)=\frac{1}{x^2}$ in $[1,2]$ then the mean value C of Cauchy's mean value theorem is
(a) $\frac{4}{3}$ (b) $\frac{5}{4}$ (c) $\frac{5}{3}$ (d) none of these

21. How many of the following functions satisfy Lagrange's mean value theorem in the given interval?
$f(x)=|x+2|$ in $[-2, 0]$
$g(x)=2+(4-x)^{1/3}$ in $[1, 6]$
$h(x)=\log(1+x^3)$ in $[0, 3]$
$p(x)=\begin{cases} 1+x^2 & \text{in } 0\le x<1 \\ 1 & \text{in } x=1 \end{cases}$
(a) 0 (b) 1 (c) 2 (d) 3

22. Let $f(x)$ and $g(x)$ be differentiable for $0\le x\le 2$ such that $f(0)=4$, $f(2)=8$, $g(0)=0$ and $f^1(x)=g^1(x)$ for all x in $[0, 2]$ then the value of $g(2)$ must be____

23. If $f^1(x)=\frac{1}{5-x^2}$ and $f(0)=1$ then the lower and the upper bounds of $f(1)$ estimated by mean value theorem are
(a) $\frac{6}{5}$ and $\frac{5}{4}$ (b) $\frac{5}{6},\frac{1}{10}$
(c) $\frac{3}{4}$ and $\frac{4}{3}$ (d) none of these

Definite Integrals

24. $\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos x\,\log\left(\frac{1+x}{1-x}\right)dx=$ ______
(a) 0 (b) 1 (c) $\frac{\pi}{2}$ (d) none of these

25. $\int_{-\pi}^{\pi}\sin^4 x.\,dx=$
(a) π (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) 0

26. $\int_0^{\pi}x\,\sin^4 x\,\cos^6 x\,dx$
(a) $3\pi^2/512$ (b) $5\pi^2/256$
(c) $3\pi^2/128$ (d) none of these

27. $\int_{-1}^{2}\frac{|x|}{x}\,dx=$ ------

28. $\int_0^{\pi}|\cos x|\,dx=$

29. $\int_0^{n}[x]\,dx$ = ____ , where $[x]$ is a step function and 'n' is an integer.
(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n-1)}{2}$
(c) $\frac{n}{2}$ (d) $\frac{n+1}{2}$

30. $\int_0^{\frac{\pi}{2}}\log(\tan x)\,dx=$
(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

31. Let $\frac{d}{dx}[F(x)]=\frac{e^{\sin x}}{x},\ x>0.$
If $\int_1^4\left(\frac{2e^{\sin x^2}}{x}\right)dx=F(k)-F(1)$
then $k=$ ________

Improper Integrals

32. The integral $\underset{a\to\alpha}{Lt}\int_1^a x^{-4}\,dx$
(a) diverges
(b) converges to 1/3
(c) converges to $-1/a^3$
(d) converges to 0

33. $\int_{-\infty}^{0}\sin hx\,dx$
(a) diverges to $-\infty$ (b) converges to 3
(c) diverges to ∞ (d) converges to 0

34. The improper definite integral $\int_{-\infty}^{\infty}xe^{-x^2}\,dx$
(a) converges to 1 (b) converges to 0
(c) converges to 2 (d) diverges

35. $\int_1^3\left(\frac{\sqrt{1+x}}{(x-1)^2}\right)dx$
(a) divergent (b) -2
(c) 2 (d) 0

36. $\int_1^2\left(\frac{x^3+1}{\sqrt{2-x}}\right)dx$
(a) convergent
(b) diverges to ∞
(c) diverges to $-\infty$
(d) none of the above

37. Consider the integrals
$I_1=\int_0^1\frac{dx}{x^{1/3}},\ I_2=\int_{-1}^{1}\frac{dx}{x^2},\ I_3=\int_0^1 x\log x\,dx$ Then
(a) I_1 & I_3 are convergent
(b) I_2 & I_3 are convergent
(c) I_1 & I_2 are convergent
(d) only I_1 is convergent

38. $\int_1^{\infty} \frac{e^{-x}}{x^2}\, dx =$ ______

(a) diverges to ∞ (b) diverges to $-\infty$
(c) 24 (d) convergent

39. $\int_2^3 \frac{dx}{(x-2)^{1/4}(3-x)^2} =$

(a) convergent (b) divergent
(c) log 2 (d) none

Partial Derivatives

40. If $\mu = \log\left(\frac{x^2}{y}\right)$ then $x\frac{\partial \mu}{\partial x} + y\frac{\partial \mu}{\partial y} =$

(a) μ (b) 2μ (c) 0 (d) 1

41. If $\mu = \frac{x^2 y}{x^{\frac{5}{2}} + y^{\frac{5}{2}}}$

then $x^2 \mu_{xx} + 2xy\, \mu_{xy} + y^2 \mu_{yy} =$ ______

(a) $\frac{1}{4}\mu$ (b) $-\frac{1}{4}\mu$
(c) $\frac{3}{4}\mu$ (d) $-\frac{3}{4}\mu$

42. If $z = \sin^{-1}\left[\frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{6}} - y^{\frac{1}{6}}}\right]$

then $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} =$ ______

(a) $\frac{1}{144}\tan z(\tan^2 z - 11)$
(b) $\frac{1}{12}\tan z$
(c) $\frac{\tan z}{144}(\sec^2 z - 11)$
(d) none of these

43. If $u(x,y) = x^2 \tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, $x>0$, $y>0$ then
$x^2(\partial^2 u/\partial x^2) + 2xy(\partial^2 u/\partial x\partial y) + y^2(\partial^2 u/\partial y^2) =$

(a) u (b) – u (c) 2 u (d) 3 u

44. If $x^y + y^x = c$ then $\frac{dy}{dx}$ at(1,1) is

(a) 1 (b) –1 (c) 0 (d) 2

45. If $u = x\, e^y z$ where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$ then find $\frac{du}{dx}$ at (0,1,1) is ___

(a) e (b) – e (c) e^{-1} (d) 2e

46. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then $6\, u_x + 4\, u_y =$

(a) $-2\, u_z$ (b) $8\, u_z$
(c) $-3\, u_z$ (d) $3\, u_z$

Maxima & Minima

47. The function $f(x) = 3x^4 - 4x^3 + 10$ has a minimum value at x = ______

48. The maximum value of the function $f(x) = x^3 - 9x^2 + 24x + 5$ in [1,6] is

49. If $f(x) = a \log x + bx^2 - x$ has its extreme values at x = -1 and x = 2 then

(a) a = 2, b = -1 (b) a = 2, b = -1 / 2
(c) a = -2, b = 2 (d) a = -2, b = 1 / 2

50. The function $f(x, y) = x^3 - 3x^2 + 4y^2 - 10$ at (2,0) has

(a) a maximum (b) a minimum
(c) a saddle point (d) both (a) & (b)

51. The function $f(x, y) = x^2y - 3xy + 2y + x$ has

(a) no local extremum
(b) one local minimum but no local maximum
(c) one local maximum but no local minimum
(d) one local minimum and one local maximum

52. If $f(x,y) = xy + (x-y)$ then the saddle point of f(x, y) is

(a) (1, – 1) (b) (–1, 1)
(c) (–1,–1) (d) (1,1)

53. The distance between origin and a point nearest to it on the surface $z^2 = 1 + xy$ is

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 1 (d) none of these

54. A rectangular box open at the top is to have a volume of 32 ft^3. Find the dimensions of the box requiring least material for its construction are

(a) 4, 4, 2 (b) $3, \frac{2}{3}, 16$
(c) 1, 4, 8 (d) 1, 2, 16

Multiple Integrals

55. $\int_{y=0}^{2} \int_{x=0}^{3} xy\, dx\, dy =$ ______

(a) 9 (b) 18 (c) 27 (d) 6

56. $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx =$ ______

(a) $4e^3 - 8$ (b) $3e^4 - 7$
(c) $3e^4 + 7$ (d) $3e^4 - 9$

57. The value of $\iint_R xy\, dx\, dy$ where 'R' is the region bounded by x-axis, ordinate x = 2a and the curve $x^2 = 4ay$, is

(a) $\frac{a^3}{4}$ (b) $\frac{a^4}{3}$
(c) $\frac{a^4}{6}$ (d) $\frac{a^4}{8}$

58. The value $\iint_R r^2 \sin\theta\; dr\; d\theta$, where 'R' is the semi-circle $r = 2\, a \cos\theta$ above the initial line, is

(a) $\frac{a^3}{3}$ (b) $\frac{a^3}{6}$
(c) $\frac{2a^3}{3}$ (d) $\frac{a^4}{6}$

59. The value of $\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
(c) 1 (d) –1

60. By changing the order of integration, the double integral $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} f(x,y)\, dy\, dx$ can be expressed as $\int_p^q \int_r^s f(x, y)\, dx\, dy$ then $q \times r =$

(a) y (b) y^2 (c) 0 (d) $\sqrt{y}$

61. By changing into polar coordinates, the double integral $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{x^2 + y^2}\, dy\, dx$ may be represented as

(a) $\int_0^{\pi/4} \int_0^{\sqrt{2}} \frac{1}{r} \cos\theta\; dr\; d\theta$ (b) $\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \frac{1}{r} \cos\theta\; dr\; d\theta$
(c) $\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \cos\theta\; dr\; d\theta$ (d) $\int_0^{\pi/4} \int_0^{\sqrt{2}} \cos\theta\; dr\; d\theta$

62. The value of $\int_0^1 \int_y^1 \int_0^{1-x} x\, dz\, dx\, dy$ is

(a) $\frac{1}{12}$ (b) $\frac{1}{16}$
(c) 12 (d) $\frac{1}{21}$

63. The value of $\int_0^a \int_0^x \int_0^y xyz\; dz\, dy\, dx$ is

(a) $\frac{a^4}{16}$ (b) $\frac{a^4}{12}$
(c) $\frac{a^6}{48}$ (d) $\frac{a^4}{4}$

64. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1 - x^2 - y^2 - z^2}} =$ ______

(a) $\frac{\pi^2}{2}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi}{12}$ (d) $\frac{\pi^2}{8}$

65. The area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(a) $\frac{\pi ab}{2}$ (b) $\frac{\pi ab}{4}$
(c) πab (d) $2\pi ab$

66. The length of the curve $y=\frac{2}{3}x^{3/2}$ from $x=0$ to $x=3$ is ____

(a) 121 (b) $121\frac{1}{3}$
(c) $121\frac{2}{3}$ (d) None

67. The volume of the solid obtained by revolving the region bounded by the curve $y=x^3$, the y-axis and the line $y=1$ about y-axis is

(a) $\frac{2\pi}{3}$ (b) $\frac{3\pi}{5}$
(c) $\frac{\pi}{5}$ (d) $\frac{4\pi}{3}$

KEY for LEVEL – 2

01. (c)	02. (a)	03. (1)	04. 2a/π	05. (a)
06. (1)	07. (1)	08. (d)	09. (d)	10. (1)
11. (c)	12. (b)	13. (a)	14. (a)	15. (b)
16. (b)	17. (c)	18. (d)	19. (6)	20. (a)
21. (c)	22. (4)	23. (a)	24. (a)	25. (c)
26. (a)	27. (1)	28. (2)	29. (b)	30. (a)
31. 16	32. (b)	33. (a)	34. (b)	35. (a)
36. (a)	37. (a)	38. (d)	39. (b)	40. (d)
41. (b)	42. (a)	43. (c)	44. (b)	45. (a)
46. (c)	47. (1)	48. 41	49. (d)	50. (b)
51. (a)	52. (b)	53. (c)	54. (a)	55. (a)
56. (b)	57. (b)	58. (c)	59. (c)	60. (b)
61. (c)	62. (a)	63. (c)	64. (d)	65. (b)
66. (d)	67. (b)			

3 Vector Calculus

Carl Friedrich Gauss
(1777 – 1855)

"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment".
- Carl Friedrich Gauss

3.0. Introduction:

Vector calculus is a branch of mathematics concerned with differentiation and integration of vector fields, primarily in 3 dimensional Euclidean space.

Vector calculus plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

Vector Differentiation

Scalar function:

A scalar function f(x,y,z) is a function defined at each point in a certain domain D in space.

Vector function:

A function $\overline{F}(x,y,z)=F_1\overline{i}+F_2\overline{j}+F_3\overline{k}$ where F_1, F_2 and F_3 are functions of x,y and z, defined at each point $P \in D$ is called a vector function.

Position vector:

If a point O is fixed as the origin in space or in plane and P is any point in space then $\overline{OP}$ is called the position vector of a point P with respect to origin O and it is denoted by $\overline{r}$.

$\therefore \overline{OP}=\overline{r}=x\overline{i}+y\overline{j}+z\overline{k}$ and it's magnitude is $r=|\overline{r}|=\sqrt{x^2+y^2+z^2}$.

Note:
In parametric form the position vector $\overline{r}$ is
$$\overline{r}(t)=x(t)\overline{i}+y(t)\overline{j}+z(t)\overline{k}$$

Parametric representation of curves:
The curve 'C' in two dimensional plane can be parameterized by $x=x(t)$, $y=y(t)$ where $a \le t \le b$.
Then the position vector of a point P on the curve 'C' is written as $\overline{r}(t)=x(t)\overline{i}+y(t)\overline{j}$.
Therefore, the position vector of a point on a curve defines a vector function i.e., $\overline{r}=x\overline{i}+y\overline{j}$.
Similarly, a three-dimensional curve can be parameterized as $\overline{r}(t)=x(t)\overline{i}+y(t)\overline{j}+z(t)\overline{k}$ where $a \le t \le b$.

Differentiability:
A vector function $\overline{F}(t)$ is said to be differentiable function at a point 't' if $\underset{\delta t\to 0}{Lt}\frac{\overline{F}(t+\delta t)-\overline{F}(t)}{\delta t}$ exists and finite.

If $\overline{F}(t)=\overline{r}(t)=x(t)\overline{i}+y(t)\overline{j}+z(t)\overline{k}$ is a parametric representation of a curve 'C' then $\frac{d\overline{r}}{dt}$ represents the tangent vector to the curve 'C'.

Point function:
If the value of the function depends on the position of the point in space but not on any particular coordinate system being used then the function is called point function.

Types of point functions:

(i) **Scalar point function:**
If for each point P(x,y,z) of a region 'R' in space, a unique scalar or a number $\phi(x,y,z)$ or $\phi(P)$ is associated by some function ϕ then the function $\phi(x,y,z)$ is called "scalar point function".
The set of all points of the region R together with the function values $\phi(P)$ is called a scalar field over region R.

Ex: Temperature (T) of a heated body in steady state is different at different points so that T is a scalar point function.

Sometimes referred to as the "the Prince of Mathematicians" and "greatest mathematician since antiquity", Gauss had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians.

(ii) Vector point function:
If for every point P(x,y,z) in a region 'R' of space, a unique vector $\bar{f}(x,y,z)$ or $\bar{f}(P)$ is associated by a function $\bar{f}$ then the function $\bar{f}(x,y,z)$ or $\bar{f}(P)$ is called "vector point function" or vector function of position.

The set of all points of the region R together with the function values $\bar{f}(P)$ is called a vector field over region R.

Ex: The velocity of a moving fluid at any time is a vector point function.

Level surface:
If $\phi(x,y,z)$ is a scalar point function then the set of all points P(x,y,z) satisfying $\phi(x,y,z) = c$ where 'c' is an arbitrary constant, is called a level surface of ϕ at level c.

Note:
For different values of 'c' we get different level surfaces and the set of all level surfaces is known as family of level surfaces.

Vector differential operator:
The vector differential operator is denoted by the symbol ∇ (read as nabla) and defined as

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

3.1. Gradient

Gradient of a scalar point function:
If $\phi(x,y,z)$ is a scalar point function defined and differentiable at each point in a region of space then the gradient of ϕ is denoted by gradϕ (or) $\nabla\phi$ and defined as

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}.$$

The Physical Interpretation of $\nabla\phi$:
The gradient of a scalar function $\phi(x,y,z)$ i.e., $\nabla\phi$ at a point P(x,y,z) is a vector along the normal to the level surface $\phi(x,y,z) = c$ at P and is in increasing direction.

Note:

- $\nabla\phi$ is always a vector function whose components ϕ_x, ϕ_y, ϕ_z are functions of x,y,z.
- Gradient of constant scalar point function is a zero vector.
- $\frac{\nabla\phi}{|\nabla\phi|}$ is a unit vector normal to the level surface $\phi(x,y,z) = c$

Directional Derivative(D.D):
If $\phi(x,y,z)$ is a differentiable scalar function then the rate of change of ϕ at a point P in the direction of a given vector $\bar{a}$ is called directional derivative of ϕ.
It is given by $(\nabla\phi)_p \cdot \frac{\bar{a}}{|\bar{a}|}$

$$\therefore \text{D.D} = (\nabla\phi)_p \cdot \frac{\bar{a}}{|\bar{a}|}$$

Note:

- The directional derivative of a scalar function $\phi(x,y,z)$ at a point P(x,y,z) in the direction of a unit vector $\bar{e}$ is $(\nabla\phi).\bar{e}$
- D.D of ϕ in the direction of x-axis is $\frac{\partial\phi}{\partial x} = (\nabla\phi).\bar{i}$
- D.D of ϕ in the direction of y-axis is $\frac{\partial\phi}{\partial y} = (\nabla\phi).\bar{j}$
- D.D of ϕ in the direction of z-axis is $\frac{\partial\phi}{\partial z} = (\nabla\phi).\bar{k}$
- If P is any point of the surface $\phi = c$ then the greatest rate of change of ϕ occurs in the direction of normal to the surface $\phi(x,y,z) = c$ at P.
- The greatest rate of increase (or) maximum value of directional derivative of $\phi(x,y,z)$ at a point P is $|\nabla\phi|$ at P.

Angle between two surfaces:
If $\phi_1(x,y,z) = c_1$ and $\phi_2(x,y,z) = c_2$ are two surfaces and θ is the angle between the two surfaces at their point of intersection P then

$$\theta = \cos^{-1}\left[\frac{(\nabla\phi_1).(\nabla\phi_2)}{|\nabla\phi_1|\,|\nabla\phi_2|}\right].$$

Note:

- If $\phi(x,y,z) = c$ is level surface and P(x,y,z) is any point on the surface such that $(\nabla\phi)_p = a\bar{i} + b\bar{j} + c\bar{k}$ then
  - (i) the equation of tangent plane to the surface is
    $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
  - (ii) the equation of normal line to the surface is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.
- If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ & $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$ then $\nabla[f(r)] = \bar{r}\frac{f^1(r)}{r}$.
- If $\bar{F}(t)$ is a vector with constant magnitude then $\bar{F}.\frac{d\bar{F}}{dt} = 0$.
- If $\bar{F}(t)$ is a vector with constant direction then $\bar{F} \times \frac{d\bar{F}}{dt} = \bar{0}$.

Examples:

01. Find a unit normal vector to the given surface $x^2y + 2xz = 4$ at the point (2,-2,3)

Sol: Let $\phi(x,y,z) = x^2y + 2xz - 4$ and
P = (2,-2,3)
Then $\nabla\phi = (2xy + 2z)\bar{i} + x^2\bar{j} + 2x\bar{k}$
$\Rightarrow (\nabla\phi)_p = -2\bar{i} + 4\bar{j} + 4\bar{k} = \bar{a}$
$\therefore$ The unit vector normal to the given surface is $\frac{\bar{a}}{|\bar{a}|} = \frac{-2\bar{i} + 4\bar{j} + 4\bar{k}}{\sqrt{4 + 16 + 16}}$
$= \frac{1}{3}(-\bar{i} + 2\bar{j} + 2\bar{k})$.

02. Find the directional derivative of $f = x^2 - y^2 + 2z$ at P(1,2,3) in the direction of the line PQ where Q = (5,0,4).

Sol: Given $f = x^2 - y^2 + 2z$, P = (1,2,3) and Q=(5,0,4)
$\Rightarrow \overline{PQ} = 4\bar{i} - 2\bar{j} + \bar{k}$ and $\nabla f = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$
$\Rightarrow (\nabla f)_p = 2\bar{i} - 4\bar{j} + 12\bar{k}$
$\therefore$ The directional derivative of f at P along $\overline{PQ}$ is

$$\text{D.D} = (\nabla f)_p \cdot \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{8 + 8 + 12}{\sqrt{16 + 4 + 1}} = \frac{28}{\sqrt{21}}.$$

03. In what direction from the point (−1,1,2) is the directional derivative of $\phi = xy^2z^3$ a maximum? What is the magnitude of this maximum.

Sol: Given $\phi = xy^2z^3$ & P = (−1,1,2)
The directional derivative of $\phi(x,y,z)$ is maximum in the direction of normal to ϕ.
$\Rightarrow \nabla\phi = (y^2z^3)\bar{i} + (2xyz^3)\bar{j} + (3xy^2z^2)\bar{k}$
$\Rightarrow (\nabla\phi)_p = 8\bar{i} - 16\bar{j} - 12\bar{k}$
$\therefore$ Maximum value of directional derivative is $|\nabla\phi| = \sqrt{64 + 256 + 144} = \sqrt{464}$.

04. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then find $\nabla(\cos r)$.

Sol: Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$
$\Rightarrow r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$
Now $\nabla(f(r)) = \bar{r}\frac{f^1(r)}{r}$
$\therefore \nabla(\cos r) = \bar{r}\left(\frac{-\sin r}{r}\right)$.

3.2. Divergence

Divergence of a vector function:
If a vector point function $\bar{F}(x,y,z) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ is defined and differentiable at each point in some region of space then the divergence of $\bar{F}$ is denoted by div$\bar{F}$ or $\nabla.\bar{F}$ and defined as

$$\text{div}\bar{F} = \nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Physical Interpretation:
Let $\bar{F}(x,y,z)$ be the velocity of a fluid at a point P(x,y,z). Consider a small rectangular box within the fluid. Then div$\bar{F}$ measures the rate per unit volume at which the fluid flows out at any given time. i.e., divergence measures the outward flow (or) expansion of the fluid from their point at any time.

Harmonic function:

If $\phi(x,y,z)$ is a scalar point function such that $\nabla^2\phi = 0$ then the function ϕ is called harmonic function and equation $\nabla^2\phi = 0$ is called Laplace's equation where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ is Laplacian operator.}$$

Note:

- $(\nabla.\bar{F}) \neq (\bar{F}.\nabla)$
- $(\nabla.\bar{F})$ is a scalar function
- $\nabla.(\nabla\phi) = \nabla^2\phi$
- If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ & $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ then $\nabla^2[f(r)] = f''(r) + \frac{2}{r}f'(r)$.

Solenoidal vector:

A vector point function $\bar{F}$ is said to be solenoidal vector if $\nabla.\bar{F} = 0$.

3.3. Curl

Curl of a vector:

If a vector point function $\bar{F}(x,y,z) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ is defined and differentiable at each point in some region of space then curl of $\bar{F}$ is denoted by curl$\bar{F}$ or $\nabla\times\bar{F}$ and defined as curl$\bar{F} = \nabla\times\bar{F}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

Physical Interpretation:

If $\bar{\omega}$ is an angular velocity of a rigid body rotating about a fixed axis and $\bar{V}$ is the velocity of any point P(x,y,z) on the body then $\bar{\omega} = \frac{1}{2}\text{curl}\,\bar{V}$.

Note:

- $(\nabla\times\bar{F}) \neq (\bar{F}\times\nabla)$
- curl$\bar{F}$ is a vector function
- $\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$
- If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ & $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ then $\nabla\times(\bar{r}f(r)) = \bar{0}$ and $\nabla\times(\bar{r}) = \bar{0}$

Irrotational vector:

A vector point function $\bar{F}$ is said to be an a irrotational vector if curl$\bar{F} = \bar{0}$

Scalar potential function:

If for a given an irrotational vector $\bar{F}$ there exists a scalar point function $\phi(x,y,z)$ such that $\bar{F} = \nabla\phi$ then $\phi(x,y,z)$ is called scalar potential function of $\bar{F}$.

3.4. Vector Identities:

If $\bar{A}$ and $\bar{B}$ are differentiable vector functions and f and g are differentiable scalar functions of position (x, y, z) then

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
3. $\nabla.(\bar{A}+\bar{B}) = (\nabla.\bar{A}) + (\nabla.\bar{B})$
4. $\nabla\times(\bar{A}+\bar{B}) = (\nabla\times\bar{A}) + (\nabla\times\bar{B})$
5. $\nabla.(f\bar{A}) = (\nabla f).\bar{A} + f(\nabla.\bar{A})$
6. $\nabla\times(f\bar{A}) = (\nabla f)\times\bar{A} + f(\nabla\times\bar{A})$
7. $\nabla.(\bar{A}\times\bar{B}) = \bar{B}.(\nabla\times\bar{A}) - \bar{A}.(\nabla\times\bar{B})$
8. $\nabla\times(\nabla\phi) = \bar{0}$ (or) $\text{curl}(\text{grad}\phi) = \bar{0}$ i.e., grad ϕ is always an irrotational vector.
9. $\nabla.(\nabla\times\bar{A}) = 0$ (or) $\text{div}(\text{curl}\bar{A}) = 0$ i.e., curl$\bar{A}$ is always a solenoidal vector
10. $\nabla\times(\nabla\times\bar{A}) = \nabla(\nabla.\bar{A}) - \nabla^2\bar{A}$ (or) $\text{curl}(\text{curl}\bar{A}) = \nabla(\nabla.\bar{A}) - \nabla^2\bar{A}$

Examples:

01. If $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ then find div$\bar{F}$ at (1,–1,1).

Sol: Given $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$$\text{div}\bar{F} = \nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= y^2 + 2x^2z - 6yz$$

$\therefore$ At (1,–1,1), div$\bar{F} = 9$

02. If $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ then find curl$\bar{F}$ at (1,–1,1)

Sol: Given $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$$\text{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}(-3z^2 - 2x^2y) - \bar{j}(0-0) - \bar{k}(4xyz - 2xy)$$

$\therefore$ At (1,–1,1), curl$\bar{F} = -\bar{i} - 2\bar{k}$

03. Show that vector $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - xz)\bar{j} + (z^2 - xy)\bar{k}$ is irrotational and find it's scalar potential.

Sol: $\text{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - xz & z^2 - xy \end{vmatrix} = \bar{0}$

$\therefore \bar{F}$ is irrotational.

Let $\bar{F} = \nabla\phi$ where ϕ is scalar potential.

$$\Rightarrow F_1\bar{i} + F_2\bar{j} + F_3\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = F_1,\ \frac{\partial\phi}{\partial y} = F_2,\ \frac{\partial\phi}{\partial z} = F_3$$

Consider $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$= F_1\,dx + F_2\,dy + F_3\,dz$

$\Rightarrow d\phi = (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz$

$\Rightarrow d\phi = x^2dx + y^2dy + z^2dz - d(xyz)$

$$\therefore \phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$$

is a scalar potential function

04. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then find $\nabla^2(\log r)$

Sol: Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

But $\nabla^2(f(r)) = f''(r) + \frac{2}{r}f'(r)$

$$\therefore \nabla^2(\log r) = \frac{-1}{r^2} + \frac{2}{r}\left(\frac{1}{r}\right) = \frac{1}{r^2}$$

05. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then find $\nabla\times(\bar{r}\log r)$

Sol: $\nabla\times(\phi\bar{A}) = (\nabla\phi\times\bar{A}) + \phi(\nabla\times\bar{A})$

where $\phi = \log r$ and $\bar{A} = \bar{r}$

$\nabla\times(\log r\,\bar{r}) = (\nabla\log r)\times\bar{r} + \log r(\nabla\times\bar{r})$

$$= \frac{1}{r}\bar{r}\times\bar{r} + \bar{0} = \frac{1}{r}(\bar{r}\times\bar{r}) = \bar{0}$$

Vector Integration

3.5. Line Integral:

In general, any integral which is to be evaluated along a curve is called a line integral.

Let $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$ be the position vector of any point P (x, y, z) on a curve C joining the points P_1 and P_2.

We assume that C is composed of a finite number of curves for each of which $\bar{r}$ has a continuous derivative.

Let $\bar{A}(x, y, z) = A_1\,\bar{i} + A_2\,\bar{j} + A_3\,\bar{k}$ be a differentiable vector function. Then the integral of tangential component of $\bar{A}$ along C from P_1 to P_2 is

$$\int_{P_1}^{P_2} \bar{A}.d\bar{r} = \int_{P_1}^{P_2} A_1\,dx + A_2\,dy + A_3\,dz$$

Circulation:

If C is a simple closed curve then the line integral of $\bar{A}$ along a closed curve C is denoted by $\oint_C \bar{A}.d\bar{r}$.

In Aerodynamics and fluid mechanics $\oint_C \bar{A}.d\bar{r}$ is called the circulation of $\bar{A}$ around C where $\bar{A}$ represents the fluid velocity.

Work done by force:

If $\bar{A}$ is a force acting on a particle which moves from a point P_1 to a point P_2 along a curve C then the line integral $\int_{P_1}^{P_2} \bar{A}.d\bar{r}$ gives the total work done by force $\bar{A}$.

Note:

- If $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$ and $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\int_C \bar{A}\,.\,d\bar{r} = \int_C (A_1dx + A_2dy + A_3dz)$ which is a line integral in cartesian form.
- The value of the line integral of a vector point function depends (upon) on the path joining A & B (unless the vector function is an irrotational).
- If $\bar{A}$ is a conservative field or an irrotational vector (i.e., $\nabla\times\bar{A} = \bar{0}$) in a region R of space then
  - (i) the line integral $\int_{P_1}^{P_2} \bar{A}.d\bar{r}$ is independent of path C joining P_1 and P_2 in R and $\int_{P_1}^{P_2} \bar{A}\,.\,d\bar{r} = \int_{P_1}^{P_2} \nabla\phi.d\bar{r} = \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1)$ where $\phi(x,y,z)$ is a scalar potential function.
  - (ii) $\oint_C \bar{A}\,.\,d\bar{r} = 0$ around any closed curve C in a region R.

3.6. Surface Integral:

Suppose S is a piece wise smooth surface and $\bar{F}(x, y, z)$ is a differentiable vector function over S. Let P be any point on S and let $\bar{n}$ be the unit vector at P in the direction of outward drawn normal to the surface S at P then $\iint_S (\bar{F}.\bar{n})\,dS$ is an example of surface integral.

Method of evaluation:

- If R_1 is the projection of 'S' on xy-plane then $\iint_S (\bar{F}.\bar{n})\,ds = \iint_{R_1} (\bar{F}.\bar{n})\dfrac{dx\,dy}{|\bar{n}.\bar{k}|}$.
- If R_2 is the projection of 'S' on yz-plane then $\iint_S (\bar{F}.\bar{n})\,ds = \iint_{R_2} (\bar{F}.\bar{n})\dfrac{dy\,dz}{|\bar{n}.\bar{i}|}$.
- If R_3 is the projection of 'S' on xz-plane then $\iint_S (\bar{F}.\bar{n})\,ds = \iint_{R_3} (\bar{F}.\bar{n})\dfrac{dx\,dz}{|\bar{n}.\bar{j}|}$.

3.7. Volume Integral:

If S is a closed surface enclosing a volume region V then $\iiint_V \bar{A}\,dV$ and $\iiint_V \phi\,dV$ are examples of volume integrals.

Examples:

01. Find $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = x^2y^2\bar{i} + y\bar{j}$ and C is the curve $y^2 = 4x$ in the XY-plane from (0, 0) to (4, 4).

Sol: $\int_C \bar{F}.d\bar{r} = \int_C x^2y^2\,dx + y\,dy$

Given $y^2 = 4x$

$\Rightarrow 2\,y\,dy = 4\,dx$

$\Rightarrow y\,dy = 2\,dx$

$$\int_C \bar{F}.d\bar{r} = \int_0^4 x^2 4x\,dx + 2\,dx$$

$$= \left[4\,\frac{x^4}{4} + 2x\right]_0^4 = 264$$

02. Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\,\bar{k}$ along the straight line joining the points (0, 0, 0) and (2, 1, 3).

Sol: Equation of straight line is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$\Rightarrow x = 2t,\ y = t,\ z = 3t$

$\Rightarrow dx = 2dt,\ dy = dt,\ dz = 3dt$

$$\text{Workdone} = \int_C \bar{F}.d\bar{r}$$

$$= \int_C 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

$$= \int_0^1 [3(2t)^2(2dt) + (2(2t)(3t) - t)\,dt + (3t)3dt]$$

$$= \int_0^1 [36t^2 + 8t]\,dt = 16$$

03. Evaluate $\int_S \bar{F}.\bar{N}\,ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and 'S' is the part of the surface of the plane 2x+3y+6z = 12 located in the first octant.

Sol: Let the equation of given surface be

$\phi = 2x + 3y + 6z - 12$

Then normal to the surface is

$\nabla\phi = 2\bar{i} + 3\bar{j} + 6\bar{k}$

$$\Rightarrow \bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i} + 3\bar{j} + 6\bar{k}}{7}$$

$$\therefore \bar{F}.\bar{N} = \frac{6}{7}[6z - 6 + 3y]$$

Let 'R' be projection of 'S' onto XY plane.

$$\text{Then} \int_S \bar{F}.\bar{N}\,ds = \iint_R \bar{F}.\bar{N}\frac{dx\,dy}{|\bar{N}.\bar{K}|}$$

$$= \iint_R \frac{6}{7}(6z - 6 + 3y)\frac{dx\,dy}{6/7}$$

$$= \iint_R (6z - 6 + 3y)\,dx\,dy$$

From the equation of plane, we have 6z = 12-2x-3y

$$= \iint_R (6 - 2x)\,dx\,dy$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (6 - 2x)\,dy\,dx$$

$$= \int_0^6 (6 - 2x)\frac{(12 - 2x)}{3}\,dx$$

$$= \frac{4}{3}\int_0^6 (18 - 9x + x^2)\,dx = 24$$

04. If $\bar{F} = (2x^2 - 3z)\,\bar{i} - 2xy\,\bar{j} - 4x\,\bar{k}$ then evaluate $\int_V \nabla.\bar{F}\,dV$ where V is the closed region bonded by x=0, y=0, z=0 and 2x+2y+z = 4.

Sol: $\nabla.\bar{F} = 4x - 2x - 0 = 2x$

$$\therefore \int_V \nabla.\bar{F}\,dV = \int_V 2x\,dV$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x(4 - 2x - 2y)\,dy\,dx$$

$$= \int_0^2 \left[4(2x - x^2)y - 4x\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= 4\int_0^2 \left[(2x - x^2)(2 - x) - \frac{x}{2}(2 - x)^2\right] dx$$

$$= 2\int_0^2 (x^3 - 4x^2 + 4x)\,dx = -8$$

3.8. Green's Theorem:

If R is a closed region of the xy plane bounded by a simple closed curve C and if M(x,y), N(x,y), $\dfrac{\partial N}{\partial x}, \dfrac{\partial M}{\partial y}$ are continuous functions of x and y in R then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in positive direction.

3.9. Gauss Divergence Theorem:

If V is the volume of the region bounded by a closed surface S and $\bar{A}(x,y,z)$ is a differentiable vector function over S then

$$\oiint_S (\bar{A}.\bar{n})\, dS = \iiint_V (\nabla.\bar{A})\, dV$$

where $\bar{n}$ is outward drawn unit normal to the surface S in positive direction.

3.10. Stoke's Theorem:

If S is an open, two–sided surface bounded by a simple closed curve C and $\bar{A}(x, y, z)$ is a differentiable vector function then

$$\oint_C \bar{A}.d\bar{r} = \iint_S ((\nabla\times\bar{A}).\bar{n})\, dS$$

where C is traversed in the positive direction and $\bar{n}$ is outward drawn unit normal to the surface S in positive direction.

Examples:

01. Find $\int_C (2xy - x^2)dx - (x^2 + y^2)dy$ where 'C' is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

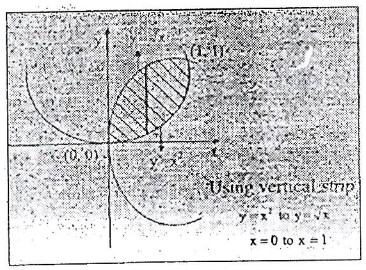
Sol: Here $M = 2xy - x^2$, $N = -(x^2 + y^2)$

$$\frac{\partial M}{\partial y} = 2x,\ \frac{\partial N}{\partial x} = -2x$$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4x$$

∴ By Grun's theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$



$$\oint_C M\,dx + N\,dy = \int_0^1 \int_{x^2}^{\sqrt{x}} -4x\,dy\,dx$$

$$= \int_0^1 -4x[y]_{x^2}^{\sqrt{x}}\,dx$$

$$= -4\left[\frac{x^{5/2}}{(5/2)} - \frac{x^4}{4}\right]_0^1$$

$$= -4\left[\frac{x^{5/2}}{(5/2)} - \frac{x^4}{4}\right]_0^1$$

$$= -4\left[\frac{2}{5} - \frac{1}{4}\right] = -\frac{3}{5}$$

02. Evaluate $\iint_S (x+z)dydz + (y+z)dzdx + (x+y)dxdy$, where 'S' is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

Sol: Since 'S' is the surface of the sphere $x^2 + y^2 + z^2 = 4$, it is a closed surface. So it can be reduced to volume integral using Gauss Divergence theorem.

Here $\text{div}\,\bar{F} = 1 + 1 + 0 = 2$

$$\therefore \int_S F_1 dydz + F_2 dzdx + F_3 dxdy = \int_V \text{div}\bar{F}\,dV$$

$$= \int_V 2dV = 2V$$

where V is the volume of the sphere

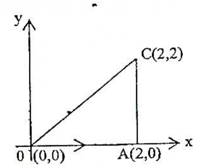
$$= 2 \times \frac{4}{3}\pi r^3$$

$$= \frac{8}{3}\pi(2)^3$$

$$= \frac{64\pi}{3}$$

03. Evaluate $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = 2y^2\bar{i} + 3x^2\bar{j} - (2x+z)\bar{k}$ and 'C' is the boundary of the triangle whose vertices are (0,0,0), (2,0,0) and (2,2,0)

Sol: Since z - coordinate of each vertex of the triangle is zero, the triangle lies in xy-plane. As 'C' is a closed curve in xy-plane, the line integral can be transformed to a surface integral using stokes theorem.



$$\therefore \text{Curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & -2x - z \end{vmatrix} = 2\bar{j} + (6x - 4y)\bar{k}$$

Here $\bar{N} = \bar{k}$.

∴ By stokes theorem, we have

$$\oint_C \bar{F}.d\bar{r} = \int_S (\text{curl}\bar{F}.\bar{N})\,ds = \int (6x - 4y)ds$$

$$= \int_0^2 \int_0^x (6x - 4y)dydx = \frac{32}{3}$$

LEVEL – 1 Questions

01. Find a unit normal to the level surface $x^2 - y^2 + z = 2$ at (1, -1, 2)

(a) $\pm\left(\frac{2\bar{i} + 2\bar{j} + \bar{k}}{3}\right)$ (b) $\left(\frac{2\bar{i} + 2\bar{j} + \bar{k}}{3}\right)$

(c) $-\left(\frac{2\bar{i} + 2\bar{j} + \bar{k}}{3}\right)$ (d) none

02. Find the directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at (1, -2, -1) along $2\bar{i} - \bar{j} - 2\bar{k}$.

(a) 37 (b) $\frac{37}{5}$ (c) $\frac{37}{3}$ (d) none

03. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at P(1, 2, 3) in the direction of $\overline{PQ}$ where Q = (5, 0, 4).

(a) $\frac{28}{21}$ (b) $\frac{28}{\sqrt{21}}$ (c) $\frac{\sqrt{21}}{28}$ (d) $\frac{21}{28}$

04. Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at P(1, 2, 3) along z – axis.

(a) 21 (b) 24 (c) 42 (d) 12

05. What is the directional derivative of $\phi = xy^2 + yz^3$ at P(2, −1, 1) in the direction of a normal to the surface $x \log z - y^2 = -4$ at A(-1, 2, 1) ?

(a) $\frac{\sqrt{17}}{15}$ (b) $\frac{15}{17}$ (c) $\frac{17}{15}$ (d) $\frac{15}{\sqrt{17}}$

06. Find the directional derivative of $f = y/(x^2 + y^2)$ at P(0, 1) along the line which makes an angle 30° with positive x – axis.

(a) −1/2 (b) 1/2 (c) 1/3 (d) none

07. In what direction from the point (1, 1, -1) is the directional derivative of $f = x^2 - 2y^2 + 4z^2$ is a maximum ?

(a) $2\bar{i} + 4\bar{j} - 8\bar{k}$ (b) $2\bar{i} - 4\bar{j} + 8\bar{k}$

(c) $2\bar{i} - 4\bar{j} - 8\bar{k}$ (d) $2\bar{i} + 4\bar{j} + 8\bar{k}$

08. What is the greatest rate of increase of $\phi = xyz^2$ at the point (1, 0, 3) ?

(a) 9 (b) 8 (c) 10 (d) 7

09. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then $\nabla f(r) =$

(a) $\frac{f'(r)}{r}\bar{r}$ (b) $\frac{f'(r)}{r}$

(c) $f'(r)$ (d) $\frac{1}{r}$

10. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then $\nabla(r^n) =$

(a) $n r^{n-2}$ (b) $n r^{n-2}\bar{r}$

(c) $r^{n-2}\bar{r}$ (d) none

11. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then $\nabla(\log r) =$

(a) $\left(\frac{1}{r}\right)\bar{r}$ (b) $\left(\frac{1}{r^2}\right)$ (c) $\left(\frac{1}{2}\right)\bar{r}$ (d) $\left(\frac{1}{r^2}\right)\bar{r}$

12. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then $\nabla(r) =$

(a) $\frac{1}{r}$ (b) $\frac{\bar{r}}{r}$ (c) $\bar{r}$ (d) $-\frac{\bar{r}}{r}$

13. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then $\nabla\left(\frac{1}{r}\right) =$

(a) $\frac{-\bar{r}}{r^3}$ (b) $\frac{\bar{r}}{r^3}$ (c) $\frac{-\bar{r}}{r}$ (d) $\frac{-\bar{r}}{r^2}$

14. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, −1, 2)

(a) $\cos^{-1}\left[\frac{4}{3\sqrt{21}}\right]$ (b) $\cos^{-1}\left[\frac{12}{3\sqrt{21}}\right]$

(c) $\cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$ (d) $\cos^{-1}\left[\frac{8}{3\sqrt{12}}\right]$

15. Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point (1, −1, 2).

(a) $7x + 3y + 8z = 26$ (b) $7x - 3y - 8z = 26$
(c) $7x + 3y - 8z = 26$ (d) $7x - 3y + 8z = 26$

16. If ϕ is a scalar function, and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla\phi . d\bar{r} =$

(a) $-d\phi$ (b) $d\phi$ (c) $d\bar{r}$ (d) dr

17. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\text{grad}(\bar{r}.\bar{a}) =$

(a) b (b) a (c) −a (d) a^2

18. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\text{grad}[\bar{r}.(\bar{a}\times\bar{b})] =$

(a) $\bar{a}\times\bar{b}$ (b) $\bar{b}\times\bar{a}$ (c) $\bar{b}.\bar{a}$ (d) $\bar{a}.\bar{b}$

19. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla . \bar{r} =$

(a) −3 (b) 4 (c) 3 (d) 2

20. If $\bar{A} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$ then find $\nabla . \bar{A}$ at (2, −1, 3)

(a) −6 (b) 6 (c) 12 (d) −12

21. Determine the constant a so that the vector $\bar{V} = (x + 3y)\,\bar{i} + (y - 2z)\,\bar{j} + (x + az)\,\bar{k}$ is solenoidal

(a) 2 (b) 3 (c) −2 (d) −3

22. If $\phi = x^2 y^3 z^4$ then find $\nabla^2\phi$ at (1, −1, 1)

(a) 20 (b) −20 (c) 2 (d) −2

23. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla^2 f(r) =$

(a) $f^{11}(r) - \left(\frac{2}{r}\right) f^{1}(r)$ (b) $f^{11}(r)$

(c) $f^{11}(r) + \left(\frac{2}{r}\right) f^{1}(r)$ (d) $\left(\frac{2}{r}\right) f^{1}(r)$

24. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla^2 (r^n) =$

(a) $n(n + 1)\, r^{n-2}$ (b) $n(n - 1)\, r^{n-2}$
(c) $n(n + 1)$ (d) $n(n + 1)^{n-2}$

25. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla^2(1/r) =$

(a) 1 (b) 2 (c) 0 (d) 3

26. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla^2(r^3) =$

(a) 4 r (b) 8 r (c) 16 r (d) 12 r

27. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla^2(\log r) =$

(a) $\frac{1}{r^2}$ (b) $\frac{1}{r}$ (c) 1 (d) 0

28. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla . (\bar{r}\times\bar{a}) =$

(a) 0 (b) 10 (c) − 10 (d) none

29. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla\times\bar{r} =$

(a) zero (b) $\bar{0}$ (c) 2 (d) 4

30. If $\bar{A} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$ then find Curl(Curl $\bar{A}$) at (−1, 1, 2)

(a) $2\bar{j}$ (b) $3\bar{j}$ (c) $4\bar{j}$ (d) $6\bar{j}$

31. Find constants a, b, c so that $\bar{V} = (x + 2y + az)\,\bar{i} + (bx - 3y - z)\,\bar{j} + (4x + cy + 2z)\,\bar{k}$ is irrotational

(a) a = 4, b = 2, c = −1
(b) a = −4, b = −2, c = −1
(c) a = 1, b = 4, c = −1
(d) a = −1, b = −4, c = −1

32. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla\times(\bar{r}\times\bar{a}) =$

(a) $2\bar{a}$ (b) $-2\bar{a}$ (c) $-3\bar{a}$ (d) $\bar{a}$

33. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla . (r^n\,\bar{r}) =$

(a) $(n - 3)\, r^n$ (b) $(n + 3)\, r^n$
(c) $(n + 3)$ (d) r^n

34. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla.\left(\frac{\bar{r}}{r^3}\right) =$

(a) 1 (b) 2 (c) 3 (d) 0

35. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla.\left(\frac{\bar{r}}{r}\right) =$

(a) $\frac{2}{r}$ (b) 2 (c) $\frac{2}{r^2}$ (d) none

36. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla\times(r^n\,\bar{r}) =$

(a) $\bar{0}$ (b) 0 (c) 2 (d) 4

37. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla\times\left(\frac{\bar{r}}{r^3}\right)$

(a) 1 (b) 2 (c) $\bar{0}$ (d) 4

38. If $\bar{A}$ and $\bar{B}$ are irrotational then $\bar{A}\times\bar{B}$ is

(a) solenoidal (b) irrotational
(c) both (a) and (b) (d) none

39. Curl (grad ϕ) =

(a) 1 (b) 2 (c) $\bar{0}$ (d) 4

40. If f and g are differentiable scalar functions then $(\nabla f\times\nabla g)$ is

(a) solenoidal (b) irrotational
(c) both (a) and (b) (d) none

41. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla.[\bar{a}\times(\bar{r}\times\bar{a})]$

(a) $2a^2$ (c) $3a^2$ (c) 2^2 (d) none

42. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $\bar{A} = \frac{\bar{r}}{r}$ then grad(div $\bar{A}$) = ...

(a) $2r^{-3}\,\bar{r}$ (b) $-2r^{-3}\,\bar{r}$
(c) $2r^3\,\bar{r}$ (d) $-2r^{-3}$

43. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla.\left[r\nabla\left(\frac{1}{r^3}\right)\right] =$

(a) $-3r^{-3}$ (b) $-3r^3$
(c) $3r^{-4}$ (d) $3r^{-3}$

44. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla^2\left[\nabla.\left(\frac{\bar{r}}{r^2}\right)\right] =$

(a) $-2r^{-3}$ (b) $-2r^3$ (c) $2r^{-4}$ (d) $2r^{-3}$

45. If $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$ then evaluate $\int_C \bar{F}.\,d\bar{r}$ along C where C is the curve in the xy-plane $y = 2x^2$ from (0, 0) to (1, 2).

(a) $\frac{7}{6}$ (b) $-\frac{7}{6}$ (c) $\frac{6}{7}$ (d) $-\frac{6}{7}$

46. If $\bar{F} = y\,\bar{i} - x\,\bar{j}$ then evaluate $\int_C \bar{F}.\,d\bar{r}$ from (0, 0) to (1, 1) along the following paths C

(i) $y = x^2$
(ii) straight line joining (0, 0) and (1, 1)
(iii) the straight lines from (0, 0) to (1, 0) and then to (1, 1)

(a) $-\frac{1}{3}$, 0, −1 (b) $-\frac{1}{3}$, 0, 1
(c) $\frac{1}{3}$, 0, −1 (d) $\frac{1}{3}$, 0, 1

47. Find the total work done in moving a particle in a force field given by $\bar{F} = (3x^2 + 6y)\,\bar{i} - 14yz\,\bar{j} + 20xz^2\,\bar{k}$ along the straight line joining (0, 0, 0) to (1, 1, 1)

(a) $\frac{13}{3}$ (b) $-\frac{13}{3}$ (c) $\frac{3}{13}$ (d) $-\frac{3}{13}$

48. If $\bar{F} = (2xy + z^3)\,\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k}$ is irrotational then find the scalar function ϕ such that $\bar{F} = \nabla\phi$.

(a) $\phi = xy + xz^3$ (b) $\phi = x^2y - xz^3$
(c) $\phi = x^2y + xz^3$ (d) $\phi = -x^2y - xz^3$

49. In the previous example, find the work done by F in moving an object in this field from (1, −2, 1) to (3, 1, 4).

(a) 102 (b) 202 (c) 302 (d) 402

50. Evaluate $\iint_S (\bar{A}.\bar{n})\, dS$ where $\bar{A} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$ and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.

(a) $-\frac{4}{8}$ (b) $\frac{2}{8}$ (c) $-\frac{6}{8}$ (d) $\frac{3}{8}$

51. Evaluate $\iint_S (\bar{A}.\bar{n})\, dS$ where $\bar{A} = z\bar{i} + x\bar{j} - 3y^2z\,\bar{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.
(a) 50 (b) 70 (c) 90 (d) 80

52. Evaluate $\int_C (xy + y^2)dx + x^2\, dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.
(a) $-\frac{1}{20}$ (b) $\frac{1}{20}$ (c) $-\frac{1}{16}$ (d) $\frac{1}{16}$

53. Evaluate $\int_C (3x + 4y)dx + (2x - 3y)dy$ where C is a circle of radius 2 with center at origin in the xy – plane, is traversed in the positive sense.
(a) 8π (b) 2π (c) $-\pi$ (d) -8π

54. Evaluate $\int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$.
(a) 5/3 (b) 3/5 (c) 5/8 (d) 8/5

55. Evaluate $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = (x^2 + y^2)\bar{i} - 2xy\bar{j}$ and C is the rectangle in the xy – plane bounded by $x = 0, x = 2, y = 0, y = 3$.
(a) –63 (b) 36 (c) –36 (d) none

56. Evaluate $\int_C \bar{A}.d\bar{r}$ where $\bar{A} = (x - y)\bar{i} + (x + y)\bar{j}$ and C is the boundary of the region bounded by $y = x^2$ and $x = y^2$.
(a) 3/2 (b) 3/5 (c) 5/3 (d) 2/3

57. Evaluate $\iint_S (\bar{r}.\bar{n})dS$ where S is the surface of the unit sphere $x^2 + y^2 + z^2 = 1$
(a) π (b) 4π (c) 8π (d) none

58. Evaluate $\iint_S (\bar{A}.\bar{n})\, dS$ where $\bar{A} = 3x\bar{i} - 4y\bar{j} + 8z\bar{k}$ and S is the surface bounded by $x = 0, x = 3, y = 0, y = 2$ and $z = 0$ and $z = 1$.
(a) 42 (b) 24 (c) 28 (d) 82

59. Evaluate. $\iint_S (\bar{F}.\bar{n})\, dS$ where $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
(a) $\frac{7}{6}$ (b) $-\frac{7}{6}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

60. Evaluate $\iint_S (\bar{A}.\bar{n})\, dS$ where $\bar{A} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ taken over the region bounded by $x^2+y^2 = 4$, $z = 0$ and $z = 3$.
(a) 8π (b) 84π (c) -84π (d) -8π

61. Evaluate $\iint_S (\nabla \times \bar{A}).\bar{n}\, ds$ where $\bar{A} = y\bar{i} - z\bar{j} + x\bar{k}$ and S is the sphere $x^2 + y^2 + z^2 = 1$
(a) 0 (b) 1 (b) 2 (d) 3

62. $\int_C \bar{r}.d\bar{r}$ where C is the curve $x^2 + y^2 = 4$.
(a) 0 (b) 1 (b) 2 (d) 3

63. $\int_C yz\, dx + zx\, dy + xy\, dz$ where C is the curve $x^2 + y^2 = 1, z = y^2$.
(a) 0 (b) 1 (b) 2 (d) 3

64. Evaluate $\int_C \bar{A}.d\bar{r}$ where $\bar{A} = (2x - y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ and C is the boundary of the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$
(a) 2π (b) $-\pi$ (c) π (d) -2π

65. Evaluate $\int_C \bar{A}.d\bar{r}$ where $\bar{A} = (y - z + 2)\bar{i} + (yz + 4)\bar{j} - xz\bar{k}$ and C is boundary of the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy – plane.
(a) 4 (b) –4 (c) 5 (d) –5

KEY for LEVEL – 1

01. (a)	02. (c)	03. (b)	04. (d)	05. (d)
06. (a)	07. (c)	08. (a)	09. (a)	10. (b)
11. (d)	12. (b)	13. (a)	14. (c)	15. (d)
16. (b)	17. (b)	18. (a)	19. (c)	20. (a)
21. (c)	22. (b)	23. (c)	24. (a)	25. (c)
26. (d)	27. (a)	28. (a)	29. (b)	30. (c)
31. (a)	32. (b)	33. (b)	34. (d)	35. (a)
36. (a)	37. (c)	38. (a)	39. (c)	40. (a)
41. (a)	42. (b)	43. (c)	44. (c)	45. (b)
46. (a)	47. (a)	48. (c)	49. (b)	50. (d)
51. (c)	52. (a)	53. (d)	54. (a)	55. (c)
56. (d)	57. (b)	58. (a)	59. (c)	60. (b)
61. (a)	62. (a)	63. (a)	64. (c)	65. (b)

LEVEL – 2 Questions

Gradient

01. Find the gradient of the function $\phi = x^2 - 2xy + z^2$ at the point (2, –1, 1).
(a) $6\bar{i} - 4\bar{j} + 2\bar{k}$ (b) $4\bar{i} - 6\bar{j} + 2\bar{k}$
(c) $6\bar{i} + 4\bar{j} - 2\bar{k}$ (d) $6\bar{i} - 4\bar{j} - 2\bar{k}$

02. A unit normal to the surface $z = 2xy$ at the point (2, 1, 4) is
(a) 2i + 4j – k
(b) 2j + 4j + k
(c) $(1/\sqrt{21})$ (2i + 4j – k)
(d) none

03. If the temperature at any point in space is given by T = xy + yz + zx, then the directional derivative of T in the direction of the vector $3\bar{i} - 4\bar{k}$ at (1, 1, 1) is
(a) $-\frac{2}{5}$ (b) $\frac{1}{5}$ (c) $\frac{5}{2}$ (d) $-\frac{5}{2}$

04. If f(x, y, z) = x + y then the magnitude of the maximum directional derivative of f(x, y, z) is
(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{2} - 1$ (d) $1/\sqrt{2}$

05. In what direction from (3, 1, –2), the directional derivative of $\varphi(x, y, z) = x^2y^2z^4$ is maximum ?
(a) $96(\bar{i} + 3\bar{j} - 3\bar{k})$ (b) $96(\bar{i} - 3\bar{j} + 3\bar{k})$
(c) $96(\bar{i} - 3\bar{j} - 3\bar{k})$ (d) $48(\bar{i} - 3\bar{j} + 3\bar{k})$

Divergence

06. The divergence of $x^2z\,\bar{i} - 2y^3z^2\,\bar{j} + xy^2z\,\bar{k}$ at the point (1, 1, -1) is
(a) –8 (b) 3 (c) $\sqrt{3}$ (d) 2

07. The value of p for which the vector field $\bar{V} = (2x + y)\,\bar{i} + (3x - 2z)\,\bar{j} + (x + pz)\,\bar{k}$ is solenodial
(a) 0 (b) 2 (c) – 2 (d) 1

Curl

08. If the velocity vector in a two-dimensional flow fluid is given by $\bar{V} = 2xy\,\bar{i} + (2y^2 - x^2)\,\bar{j}$ then the curl $\bar{V}$ will be
(a) $2y\bar{j}$ (b) $6y\bar{k}$
(c) zero (d) $-4x\bar{k}$

09. The values of a, b, c for which $\bar{V} = (x+y+az)\,\bar{i} + (bx+3y-z)\,\bar{j} + (3x+cy+z)\,\bar{k}$ is an irrational.
(a) a=1, b=3, c=1 (b) a=1, b=1, c=3
(c) a=3, b=1, c=–1 (d) None

10. A rigid body is rotating with constant angular velocity ω about a fixed axis. If V is the velocity of a point of the body then curl V is equal to
(a) ω (b) ω^2 (c) 2ω (d) $2\omega^2$

Vector Identities

11. The expression Curl (grad f) =
(a) $\nabla^2 f$
(b) div(grad f)
(c) scalar of zero magnitude
(d) vector of zero magnitude

12. If $\bar{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$ then $\nabla.(\nabla \times \bar{F}) =$
(a) zero (b) $\bar{F}$
(c) 2 (d) none

13. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then grad(1/r) is
(a) $\frac{\bar{r}}{r^2}$ (b) $\frac{\bar{r}}{r^3}$ (c) $-\frac{\bar{r}}{r^2}$ (d) $-\frac{\bar{r}}{r^3}$

14. If $\bar{a} = \bar{i} - 2\bar{j} + 2\bar{k}$ then $\text{div}(\bar{a} \times (\bar{r} \times \bar{a})) =$
(a) 4 (b) 8 (c) 10 (d) 14

15. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then $\text{div}\left(\frac{\bar{r}}{r^3}\right) =$
(a) 0 (b) 1 (c) –1 (d) 2

16. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then $\text{curl}(r^n\ \bar{r}) =$
(a) 0 (b) –1 (c) r (d) $\bar{r}$

17. If $\bar{A}$ & $\bar{B}$ are irrotational vectors then the divergence of $\bar{A} \times \bar{B}$ is ______
(a) –1 (b) 0 (c) –2 (d) –3

Line Integral

18. The value of the line integral $\int \bar{F}.d\bar{r}$, where $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ from (0,1,–1) to (1,2,0) is
(a) -1 (b) 3 (c) 0 (d) None

19. The work done in moving a particle in the force field $\bar{f} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$ along the parabola $y^2 = x$ from (0, 0) to (1, 1) is
(a) 24 (b) 18 (c) 3/2 (d) –2/3

20. Evaluate $\int_C \bar{f}.d\bar{r}$ where $\bar{f} = yz\bar{i} + xz\bar{j} + xy\bar{k}$ from (0, 0, 0) to (2, 1, 3).
(a) 6 (b) –6 (c) 0 (d) 8

Green's Theorem

21. The value of the line integral $\int_c (y^2\,dx + x^2\,dy)$ where C is the boundary of the square bounded by x = 0, x = a, y = 0, y = a is
(a) 0 (b) 2(x + y)
(c) 4 (d) 4/3

22. The value of $\int_C \bar{A}.d\bar{r}$ where $\bar{A} = (2y - y^2)\ \bar{i} + (2x + y)\ \bar{j}$ and C is the region bounded by $y = x^2$ and $x = y^2$, is
(a) 1 / 2 (b) 1 / 6
(c) 3 / 4 (d) 4 / 5

23. Find the value of $\int (x\,dy - y\,dx)$ around the circle $x^2 + y^2 = 1$
(a) π (b) 2π (c) 0 (d) -π

Surface Integral

24. Evaluate $\int_S \bar{f}.\bar{n}\,ds$ or $\iint_S \bar{f}.\bar{n}\,ds$ over the surface of the cylinder $x^2 + y^2 = 9$ included in the first octant between z = 0 and z = 5 where $\bar{f} = z\bar{i} + x\bar{j} - yz\bar{k}$.
(a) 40 (b) 60 (c) 80 (d) 100

Volume Integral

25. Find $\iiint_V \phi\,dV$ where $\phi = xyz$ and 'V' is the volume of the region bounded by x = 0, y = 0, y = 6, $z = x^2$, z = 4.
(a) 162 (b) 172
(c) 182 (d) 192

Gauss – Divergence Theorem

26. Evaluate $\int_S \{(x^3 - yz)\bar{i} - 2x^2y\bar{j} + 2\bar{k}\}.\bar{n}\,ds$ where S denotes the surface of the reactangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ is
(a) $\frac{a^3bc}{3}$ (b) $\frac{abc}{3}$
(c) $\frac{ab^3c}{3}$ (d) $\frac{abc^3}{3}$

27. If $\bar{f} = ax\bar{i} + by\bar{j} + cz\bar{k}$ where a,b,c are constants and 's' is the surface of a unit sphere then $\iint_S \bar{f}.ds =$
(a) (4/3) π (a+b+c) (b) 0
(c) (4/3) π$(a+b+c)^2$ (d) none

28. Evaluate $\int_S \bar{f}.\bar{n}\,ds$ where S is the surface of tetrahedron bounded by x = 0, y = 0, z = 0 and the plane x + y + z = 1 and $\bar{f} = xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$.
(a) $\frac{3}{8}$ (b) $\frac{1}{8}$ (c) $\frac{8}{2}$ (d) $\frac{3}{5}$

Stoke's Theorem

29. Evaluate $\oint_C (e^x dx + 2y\,dy - dz)$ where 'C' is the curve $x^2 + y^2 = 4$ and z = 2.
(a) 0 (b) 1 (b) 2 (d) 3

30. Evaluate $\oint_C \bar{f}.d\bar{r}$ where $\bar{f} = (2x - y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ & 'C' is the boundary of the upper half of surface of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane.
(a) 0 (b) 1 (c) 2 (d) none

KEY for LEVEL – 2

01. (a)	02. (c)	03. (a)	04. (b)
05. (a)	06. (a)	07. (c)	08. (d)
09. (c)	10. (c)	11. (d)	12. (a)
13. (d)	14. (d)	15. (a)	16. (a)
17. (b)	18. (d)	19. (d)	20. (a)
21. (a)	22. (b)	23. (b)	24. (b)
25. (d)	26. (a)	27. (a)	28. (b)
29. (a)	30. (d)		