Homework 3 - Introduction to Data Processing and Representation

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Dry part

Question 1 - On Circulant Matrices

1. Let us consider

$$J = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

   We can write it as $J_{i,j} = \delta_{i-1(mod\ n),j}$

   And so, $[J^2]_{i,j} = \sum_{m=0}^{n} J_{i,m}J_{m,j} = \sum_{m=0}^{n} \delta_{i-1(mod\ n),m}\, \delta_{m-1(mod\ n),j} = \delta_{i-2(mod\ n),j}$

   Thus, J^k is acquired

$$[J^k]_{i,j} = \sum_{m=0}^{n} J_{i,m}J_{m,j}^{k-1} = \sum_{m=0}^{n} \delta_{i-1(mod\ n),m}\, \delta_{m-(k-1)(mod\ n),j} = \delta_{i-k(mod\ n),j}$$

   And so,

$$J_{i,j}^{n} = \delta_{i-n(mod\ n),j} = \delta_{i(mod\ n),j} = \delta_{i,j}$$

   And so,

$$J^n = I$$

2. The eigenvalues of J are given by the solutions of

$$Jv = \gamma v$$

   Which also solve

$$v = J^n v = \gamma^n v$$
$$\gamma^n = 1$$

   For which the complex roots are:

$$\gamma_k = e^{-\frac{2\pi ik(n-1)}{n}} \forall k \in [0, \dots, n-1]$$

3. Since J has n different eigenvalues and is a nxn matrix, it is diagonalizable.
   We will show that it's unitary and thus unitarily diagonalizable:

$$[JJ^*]_{i,j} = [JJ^t]_{i,j} = \sum_{m=0}^{n} J_{i,m}J_{m,j}^{t} = \sum_{m=0}^{n} J_{i,m}J_{j,m} = \sum_{m=0}^{n} \delta_{i-1(mod\ n),m}\delta_{j-1(mod\ n),m} = \delta_{i,j}$$

   Thus,

$$JJ^* = I$$

   And J is unitary.
   We'll now show that it is diagonalizable by the DFT matrix as an eigen basis

$$\forall k \in [0, \dots, n-1], Jv = e^{-\frac{2\pi i k}{n}} v$$

The k-th column of the DFT is $\frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ e^{-\frac{2\pi i k}{n}} \\ \dots \\ e^{-\frac{2\pi i k(n-1)}{n}} \end{pmatrix}$.

Since for each $j \in [0, \dots, n-1]$, $e^{-\frac{2\pi i k(n-1)}{n}} e^{-\frac{2\pi i k(j+1)}{n}} = e^{-\frac{2\pi i k(j+n)}{n}} = e^{-\frac{2\pi i k j}{n}}$

$$J\frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ e^{-\frac{2\pi i k}{n}} \\ \dots \\ e^{-\frac{2\pi i k(n-1)}{n}} \end{pmatrix} = e^{-\frac{2\pi i k(n-1)}{n}}\frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ e^{-\frac{2\pi i k}{n}} \\ \dots \\ e^{-\frac{2\pi i k(n-1)}{n}} \end{pmatrix}$$

Thus we have n eigen vectors corresponding to n eigen values of J, and since DFT is unitary, a unitary diagonalization is achieved:

$$J = DFT^* * \Gamma * DFT$$

Where $\Gamma = \begin{pmatrix} \gamma_0 & 0 & \dots & 0 & 0 \\ 0 & \gamma_1 & 0 & \dots & 0 \\ \dots & 0 & \gamma_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \gamma_{n-1} \end{pmatrix}$

4. For

$$H = \begin{pmatrix} h_0 & h_{n-1} & \dots & h_2 & h_1 \\ h_1 & h_0 & h_{n-1} & \dots & h_2 \\ h_2 & h_1 & h_0 & \dots & \dots \\ \dots & \dots & \dots & \dots & h_{n-1} \\ h_{n-1} & \dots & h_2 & h_1 & h_0 \end{pmatrix}$$

We notice that

$$H = h_0 I + h_1 J + \dots + h_{n-1} J^{n-1} = h_1 J + \dots + h_{n-1} J^{n-1} + h_0 J^n$$

5. We notice that since DFT is unitary

$$J^k = (DFT * \Gamma * DFT^*)^k = DFT * \Gamma^{\mathrm{k}} * DFT^*$$

And thus,

$$\begin{aligned} H &= h_1 DFT^* * \Gamma * DFT + \dots + h_{n-1} DFT^* * \Gamma^{\mathrm{n-1}} * DFT + h_0 DFT^* * \Gamma^{\mathrm{n}} * DFT \\ &= DFT^* * h_1\Gamma * DFT + \dots + DFT^* * h_{n-1}\Gamma^{\mathrm{n-1}} * DFT + DFT^* * h_0\Gamma^{\mathrm{n}} * DFT \\ &= DFT^* * (h_1\Gamma + \dots + h_{n-1}\Gamma^{\mathrm{n-1}} + h_0\Gamma^{\mathrm{n}}) * DFT \end{aligned}$$

Since Γ is diagonal, Γ^{k} is also diagonal for each k, and so $h_1\Gamma + \dots + h_{n-1}\Gamma^{\mathrm{n-1}} + h_0\Gamma^{\mathrm{n}}$ is diagonal. Thus H is diagonalizable in a unitary basis.

6. Since $[DFT]_{k,l} = \frac{1}{\sqrt{n}} e^{-\frac{2\pi i k l}{n}} = \frac{1}{\sqrt{n}} e^{-\frac{2\pi i k l - 2\pi i n n - 4\pi i n l - 4\pi i k n}{n}} = \frac{1}{\sqrt{n}} e^{\frac{2\pi i (n-k)(n-l)}{n}} =$ $[DFT^*]_{n-k (mod\ n), n-l (mod\ n)} = \begin{cases} [DFT^*]_{0,0}, k = l = 0 \\ [DFT^*]_{n-k,n-l}, o.w. \end{cases}$ (*)

Since DFT and DFT^* are symmetric along the main diagonal, we use the anticircular matrix

$$AC = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ \dots & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} = [\delta_{i+j(mod\ n),0}]$$

For the k-th row of the DFT matrix:

$$\begin{aligned} [DFT]_l AC &= ([DFT]_{l,0} \quad [DFT]_{l,1} \quad \dots \quad [DFT]_{l,n-1})AC \\ &= ([DFT]_{l,0} \quad [DFT]_{l,n-1} \quad \dots \quad [DFT]_{l,1}) \end{aligned}$$

Thus, since DFT is symmetric along the main diagonal and from (*)

$$DFT * AC = DFT^*$$

And deriving the same way from the columns,

$AC * DFT = DFT^*$ (lemma 1)

Also, we notice that

$$AC * AC = I$$

Thus,

$$DFT = AC * AC * DFT = AC * DFT^*$$

Lastly, we need to show that for a diagonal matrix D, $AC * D * AC$ is diagonal.

Let

$$D = \begin{pmatrix} \gamma_0 & 0 & \dots & 0 & 0 \\ 0 & \gamma_1 & 0 & \dots & 0 \\ \dots & 0 & \gamma_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \gamma_{n-1} \end{pmatrix}$$

$$\begin{aligned} AC * D * AC &= AC * \begin{pmatrix} \gamma_0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & \gamma_1 \\ \dots & \dots & 0 & \gamma_2 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \gamma_{n-1} & 0 & \dots & 0 \end{pmatrix} \\ &= AC * \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ \dots & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \gamma_0 & 0 & \dots & 0 & 0 \\ 0 & \gamma_{n-1} & 0 & \dots & 0 \\ \dots & 0 & \gamma_{n-2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \gamma_1 \end{pmatrix} \\ &= I \begin{pmatrix} \gamma_0 & 0 & \dots & 0 & 0 \\ 0 & \gamma_{n-1} & 0 & \dots & 0 \\ \dots & 0 & \gamma_{n-2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \gamma_1 \end{pmatrix} \end{aligned}$$

And so $AC * D * AC$ is a diagonal marix.

And so, the following diagonalizations are gained with DFT^* as their base:

$$J = DFT^* * \Gamma * DFT = DFT * AC * \Gamma * AC * DFT^*$$

$$\begin{aligned} H = DFT^* * (h_1\Gamma + \dots + h_{n-1}\Gamma^{\mathrm{n-1}} + h_0\Gamma^{\mathrm{n}}) * DFT = \\ = DFT * AC * (h_1\Gamma + \dots + h_{n-1}\Gamma^{\mathrm{n-1}} + h_0\Gamma^{\mathrm{n}}) * \mathrm{AC} * DFT^* \end{aligned}$$

7. From

$$\mathrm{H} = DFT * (h_1\Gamma + \dots + h_{n-1}\Gamma^{\mathrm{n-1}} + h_0\Gamma^{\mathrm{n}}) * DFT^*$$

We get that the eigenvalues of h are along the diagonal of $h_1\Gamma + \dots + h_{n-1}\Gamma^{\mathrm{n-1}} + h_0\Gamma^{\mathrm{n}}$.

Since

$$\Gamma = \begin{pmatrix} \gamma_0 & 0 & \dots & 0 & 0 \\ 0 & \gamma_1 & 0 & \dots & 0 \\ \dots & 0 & \gamma_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \gamma_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{n}} & 0 & \dots & 0 \\ \dots & 0 & e^{-\frac{2\pi i 2}{n}} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & e^{-\frac{2\pi i(n-1)}{n}} \end{pmatrix}$$

Let the eigen values of H be $\{\lambda_i\}_{i=0}^{n-1}$.

$$\begin{pmatrix} \lambda_0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_1 & 0 & \dots & 0 \\ \dots & 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \lambda_{n-1} \end{pmatrix}$$

$$= h_1 \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & e^{-\frac{2\pi i(n-1)}{n}} & 0 & \dots & 0 \\ \dots & 0 & e^{-\frac{2\pi i(n-2)}{n}} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix} + \cdots$$

$$+ h_{n-1} \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & e^{-\frac{2\pi i(n-1)(n-1)}{n}} & 0 & \dots & 0 \\ \dots & 0 & e^{-\frac{2\pi i(n-2)(n-1)}{n}} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & e^{-\frac{2\pi i(n-1)}{n}} \end{pmatrix}$$

$$+ h_0 \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & e^{-\frac{2\pi i(n-1)(n)}{n}} & 0 & \dots & 0 \\ \dots & 0 & e^{-\frac{2\pi i(n-2)(n)}{n}} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & e^{-\frac{2\pi i(n)}{n}} \end{pmatrix}$$

Thus, $\forall k \in [0, \dots, n-1]$,

$$\lambda_k = h_0 e^{-\frac{2\pi i(n-k)(n)}{n}} + \sum_{m=0}^{n-1} h_m e^{-\frac{2\pi i(n-k)(m)}{n}} = h_0 + \sum_{m=0}^{n-1} h_m e^{-\frac{2\pi i(n-k)(m)}{n}}$$

$$= \sqrt{n}[DFT]_k \begin{pmatrix} h_0 \\ h_{n-1} \\ \dots \\ h_1 \end{pmatrix}$$

And so,

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \dots \\ \lambda_{n-1} \end{pmatrix} = \sqrt{n} DFT \begin{pmatrix} h_0 \\ h_{n-1} \\ \dots \\ h_1 \end{pmatrix}$$

8. For any two circulant matrices H_1, H_2

$$H_1 = h_1^{(1)} J + \cdots + h_{n-1}^{(1)} J^{n-1} + h_0^{(1)} J^n$$

$$H_2 = h_1^{(2)} J + \cdots + h_{n-1}^{(2)} J^{n-1} + h_0^{(2)} J^n$$

And so,

$$H_1 H_2 = \left(\sum_{i+j=1} h_i^{(1)} h_j^{(2)} \right) J + \cdots + \left(\sum_{i+j=2n-2} h_i^{(1)} h_j^{(2)} \right) J^{2n-2} + \left(\sum_{i+j=2n-1} h_i^{(1)} h_j^{(2)} \right) J^{2n-1}$$
$$= H_2 H_1$$

H_1, H_2 commute, and $H_1 H_2$ is a polynomial of J, so it is circular, since it is a linear combination of circulant matrices.

9. Let us compute DFT^2

$$[DFT * DFT]_{k,l} = \sum_{m=0}^{n-1} [DFT]_{k,m} [DFT]_{m,l} = \sum_{m=0}^{n-1} \frac{1}{\sqrt{n}} e^{-\frac{2\pi ikm}{n}} \frac{1}{\sqrt{n}} e^{-\frac{2\pi iml}{n}} = \frac{1}{n} \sum_{m=0}^{n-1} e^{-\frac{2\pi im(k+l)}{n}}$$

If $k + l (mod\ n) = 0$, $e^{-\frac{2\pi im(k+l)}{n}} = 1$, and so $[DFT * DFT]_{k,l} = \frac{n}{n} = 1$

If $k + l (mod\ n) \neq 0$,

$$[DFT * DFT]_{k,l} = \frac{1}{n} \sum_{m=0}^{n-1} e^{-\frac{2\pi im(k+l)}{n}} = \frac{1}{n} \sum_{m=0}^{n-1} e^{-\frac{2\pi im(k+l)}{n}} = \frac{1}{n} \sum_{m=0}^{n-1} \left(e^{-\frac{2\pi i(k+l)}{n}} \right)^m$$
$$= \frac{\frac{1}{n} \left(e^{-\frac{2\pi i(k+l)^n}{n}} - 1 \right)}{e^{-\frac{2\pi i(k+l)}{n}} - 1} = \frac{\frac{1}{n} \left(e^{-\frac{2\pi i(k+l)n}{n}} - 1 \right)}{e^{-\frac{2\pi i(k+l)}{n}} - 1} = \frac{\frac{1}{n}(1-1)}{e^{-\frac{2\pi i(k+l)}{n}} - 1} = 0$$

Thus, $DFT * DFT = \left[\delta_{i+j(mod\ n),0} \right] = AC$

And so, for $k(mod\ n) = 0$:

$$DFT^k = DFT^{4m} = (DFT * DFT * DFT * DFT)^m = (AC * AC)^m = I^m = I$$

$k(mod\ n) = 1$:

$$DFT^k = DFT^{4m+1} = I * DFT = DFT$$

$k(mod\ n) = 2$:

$$DFT^k = DFT^{4m+2} = I * DFT * DFT = AC$$

$k(mod\ n) = 3$:

$$DFT^k = DFT^{4m+3} = I * DFT * DFT * DFT = AC * DFT =^{(lemma\ 1)} DFT^*$$

10. For $z, x, y \in \mathbb{R}^N$ we'll show that if $z = x \otimes y$ then $DFT * z = (DFT * x) \odot (DFT * y)$

Since

$$z = x \otimes y = \begin{pmatrix} x_0 & x_{n-1} & \cdots & x_2 & x_1 \\ x_1 & x_0 & x_{n-1} & \cdots & x_2 \\ x_2 & x_1 & x_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x_{n-1} \\ x_{n-1} & \cdots & x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \cdots \\ y_{n-1} \end{pmatrix}$$

Since the above is a circular matrix, it can be diagonalized by the DFT matrix,

$$\begin{pmatrix} x_0 & x_{n-1} & \cdots & x_2 & x_1 \\ x_1 & x_0 & x_{n-1} & \cdots & x_2 \\ x_2 & x_1 & x_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x_{n-1} \\ x_{n-1} & \cdots & x_2 & x_1 & x_0 \end{pmatrix} = DFT^* * \Lambda * \mathrm{DFT}$$

And so

$$DFT * z = \Lambda * DFT * y$$

Which is the same as

$$DFT * z = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \cdots \\ \lambda_{n-1} \end{pmatrix} \odot (DFT * y)$$

And from subclause g,

$$DFT * z = \sqrt{n}(DFT * x) \odot (DFT * y)$$

Question 2 - Fourier Transform

a. We'll mark $\tilde{f}(t) = f(t-1), \tilde{g}(t) = g(t+1)$.
Using the definition of convolution, we get:

$$f(t-1) * g(t+1) = \tilde{f}(t) * \tilde{g}(t) = \int_{-\infty}^{\infty} \tilde{f}(\xi) \cdot \tilde{g}(t-\xi) d\xi = \int_{-\infty}^{\infty} f(\xi - 1) \cdot g(t - \xi + 1) d\xi =$$

Marking $\tilde{\xi} = \xi - 1 \Rightarrow d\tilde{\xi} = d\xi$,

$$= \int_{-\infty}^{\infty} f\left(\tilde{\xi}\right) \cdot g\left(t - \tilde{\xi}\right) d\tilde{\xi} = \boxed{f(t) * g(t)}$$

b. By the definition of the Fourier transform:

$$\mathcal{F}(u) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i t u} dt \ , \mathcal{G}(u) = \int_{-\infty}^{\infty} g(x) \cdot e^{-2\pi i x u} dx$$

So, we get:

$$\int_{-\infty}^{\infty} \mathcal{F}(u) \cdot \mathcal{G}(u) du =$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i t u} dt \right) \cdot \left(\int_{-\infty}^{\infty} g(x) \cdot e^{-2\pi i x u} dx \right) du =$$

We'll replace $x = -v \Rightarrow dx = -dv$:

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i t u} dt \right) \cdot \left(- \int_{\infty}^{-\infty} g(-v) \cdot e^{2\pi i v u} dv \right) du =$$

$$= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(-v) \cdot f(t) \cdot e^{2\pi i v u} \cdot e^{-2\pi i t u} dv dt \, du =$$

$$= - \int_{-\infty}^{\infty} \int_{\infty}^{-\infty} \int_{-\infty}^{\infty} \left(e^{2\pi i v u} \cdot e^{-2\pi i t u} du \right) \cdot g(-v) \cdot f(t) dv dt =$$

The Fourier basis is orthonormal, so:

$$\int_{-\infty}^{\infty} e^{2\pi i v u} \cdot e^{-2\pi i t u} du = \delta_{v,t} \Rightarrow$$

$$= - \int_{-\infty}^{\infty} \int_{\infty}^{-\infty} \delta_{v,t} \cdot g(-v) \cdot f(t) dv dt =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_{v,t} \cdot g(-v) \cdot f(t) dv dt = \boxed{\int_{-\infty}^{\infty} g(-t) \cdot f(t) dt}$$

Question 3 - Discrete Fourier Transform

We will use the normalized version of DFT: $DFT_{j,k} = \frac{1}{\sqrt{n}} \cdot e^{\frac{-2\pi ijk}{n}}$.

a. The DFT of ϕ:

$$\phi^F = DFT(\phi) \in \mathbb{R}^{2N}$$

The signal ϕ is given by:

$$\forall l \in \{0,1,\dots,2N-1\}: \phi_l = \begin{cases} 1, & l = 0 \\ \frac{1}{2}, & l = 1 \lor l = 2N-1 \\ 0, & else \end{cases}$$

For every $k \in \{0,1,\dots,2N-1\}$:

$$\phi_k^F = \frac{1}{\sqrt{2N}} \sum_{l=0}^{2N-1} e^{\frac{-2\pi ikl}{2N}} \cdot \phi_l = \frac{1}{\sqrt{2N}} \left(1 \cdot 1 + e^{\frac{-2\pi ik}{2N}} \cdot \frac{1}{2} + e^{\frac{-2\pi ik(2N-1)}{2N}} \cdot \frac{1}{2}\right) =$$
$$= \frac{1}{\sqrt{2N}} \left(1 + \frac{1}{2}\left(e^{\frac{-2\pi ik}{2N}} + e^{\frac{-2\pi ik(2N-1)}{2N}}\right)\right) = \frac{1}{\sqrt{2N}} \left(1 + \frac{1}{2}\left(e^{\frac{-2\pi ik}{2N}} + e^{\frac{2\pi ik}{2N}}\right)\right)$$

In summary, the ϕ^F vector is a linear combination of a vector of ones, column 1 of the DFT matrix and column 2N-1 of the DFT matrix:

$$\boxed{\phi^F = \frac{1}{\sqrt{2N}} \mathbb{1}_{2N\times 1} + \frac{1}{2\sqrt{2N}} (DFT_1 + DFT_{2N-1})}$$

b. First, we'll write the DFT of ψ:

$$\psi^F = DFT(\psi) \in \mathbb{R}^N$$
$$\forall k \in \{0,1,\dots,N-1\}: \psi_k^F = \frac{1}{\sqrt{N}} \sum_{l=0}^{N} e^{\frac{-2\pi ikl}{N}} \cdot \psi_l$$

The DFT of γ in terms of ψ^F:

$$\gamma^F = DFT(\gamma) \in \mathbb{R}^{2N}$$

The signal γ is given by:

$$\gamma_l = \begin{cases} \psi_{\frac{l}{2}}, & l \text{ } is \text{ } even \\ 0, & l \text{ } is \text{ } odd \end{cases}$$

For every $k \in \{0,1,\dots,2N-1\}$:

$$\gamma_k^F = \frac{1}{\sqrt{2N}} \sum_{l=0}^{2N-1} e^{\frac{-2\pi ikl}{2N}} \cdot \gamma_l$$

Changing $p = \frac{l}{2}$ and using the term for ψ_k^F:

$$\gamma_k^F = \frac{1}{\sqrt{2N}} \sum_{p=0}^{N} e^{\frac{-2\pi ikp}{N}} \cdot \gamma_{2p} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{N}} \sum_{p=0}^{N} e^{\frac{-2\pi ikp}{N}} \cdot \psi_p = \frac{1}{\sqrt{2}} \psi_{k \bmod N}^F$$

In summary:

$$\boxed{\gamma^F = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi^F \\ \psi^F \end{pmatrix}}$$

c. Proof:

By commutativity of convolution: $h = \gamma * \phi = \phi * \gamma$

For every $k \in \{0,1,\dots,2N-1\}$, using the definition of ϕ_l:

$$h_k = (\phi * \gamma)_k = \sum_{l=-\infty}^{\infty} \phi_l \cdot \gamma_{(k-l) mod\ 2N} = \sum_{l=0}^{2N-1} \phi_l \cdot \gamma_{(k-l) mod\ 2N} =$$

$$= 1 \cdot \gamma_{(k-0) mod\ 2N} + \frac{1}{2} \cdot \gamma_{(k-1) mod\ 2N} + \frac{1}{2} \cdot \gamma_{(k-2N+1) mod\ 2N} =$$

$$= \gamma_k + \frac{1}{2} \cdot \left(\gamma_{(k-1) mod\ 2N} + \gamma_{(k+1) mod\ 2N}\right)$$

When k is even, using the definition of γ_k:

$$h_{k,even} = \psi_{\frac{k}{2}} + \frac{1}{2} \cdot (0+0) = \psi_{\frac{k}{2}}$$

When k is odd:

$$h_{k,odd} = 0 + \frac{1}{2} \cdot \left(\psi_{\frac{(k-1) mod 2N}{2}} + \psi_{\frac{(k+1) mod 2N}{2}}\right)$$

We get:

$$h_k = \begin{cases} \psi_{\frac{k}{2}}, & k \text{ is even} \\ 0.5\left(\psi_{\frac{(k-1) mod 2N}{2}} + \psi_{\frac{(k+1) mod 2N}{2}}\right), & k \text{ is odd} \end{cases}$$

Or in a vector form:

$$h = \begin{pmatrix} \psi_0 \\ 0.5(\psi_0 + \psi_1) \\ \psi_1 \\ 0.5(\psi_1 + \psi_2) \\ \vdots \\ \psi_{N-1} \\ 0.5(\psi_{N-1} + \psi_0) \end{pmatrix}$$

d. The DFT of h in terms of ψ^F:

$$h^F = DFT(h)$$

As proved in Q1.j:

$$h = \gamma * \phi \Rightarrow DFT(h) = \sqrt{2N}\, DFT(\gamma) \odot DFT(\phi)$$

Where $\odot$ denotes the Hadamard product.

We get that for every $k \in \{0,1,\dots,2N-1\}$:

$$h_k^F = \sqrt{2N}\, \gamma_k^F \cdot \phi_k^F =$$

$$= \sqrt{2N} \cdot \frac{1}{\sqrt{2}} \psi_{k\ mod\ N}^F \cdot \frac{1}{\sqrt{2N}} \left(1 + \frac{1}{2}\left(e^{\frac{\pi i k}{N}} + e^{\frac{\pi i k(2N-1)}{N}}\right)\right) =$$

$$= \frac{1}{\sqrt{2}} \psi_{k\ mod\ N}^F \left(1 + \frac{1}{2}\left(e^{\frac{\pi i k}{N}} + e^{\frac{\pi i k(2N-1)}{N}}\right)\right)$$

Or in a vector form:

$$h^F = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi^F \\ \psi^F \end{pmatrix} \odot \left(\frac{1}{\sqrt{2N}} \mathbb{1}_{2N \times 1} + \frac{1}{2\sqrt{2N}} (DFT_1 + DFT_{2N-1})\right)$$

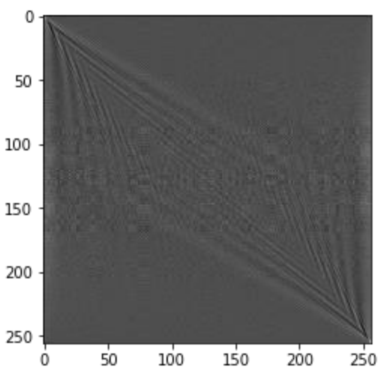
Implementation part

Question 1 - Image example

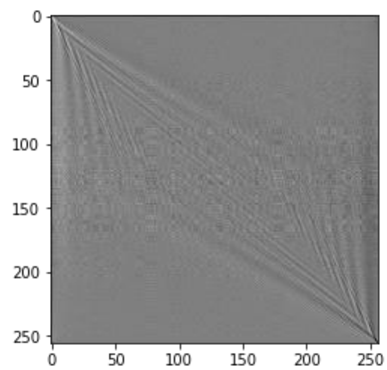
a. Done in code. Matrix A has the coefficients of the original image, where $a_{r,i} = \alpha_{r,i}$, and for matrix B respectively $b_{r,i} = \beta_{r,i}$.

b. First, we showed that A is of full rank because its singular values are non-zero.
Second, we solved the least squares problem by pseudoinverse to get matrix C:

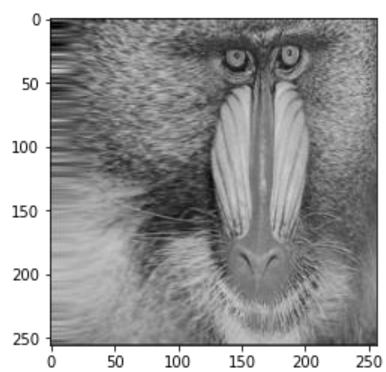
C real values:



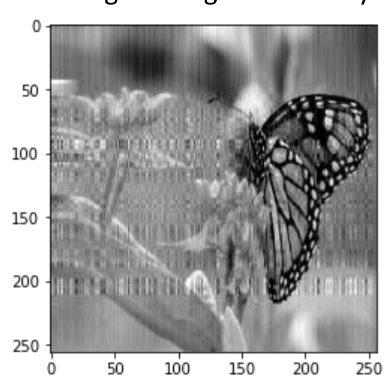
C imaginary values:



c. Distorting the original mandril image using our matrix C, we get this image:



d. Distorting the original butterfly image using our matrix C, we get this image:



Question 2 - Audio example

a. Estimating an approximate functional map:
   Done in code, we marked (as in previous question) the approximation as matrix C.
b. Denoising the distorted Totoro audio:
   Done in code, we first checked that C is invertible.
c. Comparing the reconstructed result with the true audio signal:

d. The MSE error:
   Calculated in code, the MSE is 54417.5830384738.
   It is a high error, but this is because of the chosen library to open WAV files, in our opinion.
   For comparison, we also calculated the MSE between the original signal and the distorted signal and got MSE of 2271256.16729.
   So, the MSE of original_vs_reconstructed is significantly lower than the MSE of original_vs_distorted, and the reconstructed audio sounds really close to the original one.