Wendy
U2122559J Take home assignment 2

Q1) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

a) $\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = (2-\lambda)\left((2-\lambda)^2 - 1\right) - 1\left((2-\lambda)-1\right) + 1\left(1-(2-\lambda)\right)$

$= (2-\lambda)(4 - 4\lambda + \lambda^2 - 1) - 2 + \lambda + 1 + 1 - 2 + \lambda$

$= (2-\lambda)(\lambda^2 - 4\lambda + 3) + 2\lambda - 2$

$= 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + 2\lambda - 2$

$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$ //

b) eigenValues of A

$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda-1)^2(\lambda-4)$ $\quad \lambda = 1$ // $\quad \lambda = 4$ //

c) if $\lambda = 1$ $V = \begin{pmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{Row echelon form}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = -x_2 - x_3$

if $\lambda = 4$ $V = \begin{pmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{Row echelon form}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = x_3$, $x_2 = x_3$

if $\lambda = 1$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3$

if $\lambda = 4$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_3$

for $\lambda = 1$ basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ //

for $\lambda = 4$ basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ //

Q2a) $A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{pmatrix}$ $\quad P = [x_1\, x_2\, x_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad D = \lambda I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$A^5 = PD^5P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & (-1)^5 \end{bmatrix} P^{-1}$

$P^{-1} = (P \mid I) = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_1 = R_1 - R_2]{R_2 \leftarrow -R_2,\ R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$ P^{-1}

$A^5 = PD^5P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

$A^5 = \begin{bmatrix} 32 & 1 & 0 \\ 32 & 0 & -1 \\ 32 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 31 & -31 \\ 33 & 32 & -33 \\ 33 & 31 & -32 \end{bmatrix}$ //

Q2b) Reflection transformation matrix about $y=x$ line $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$(A-\lambda I)=0$

$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$

$\lambda = \pm 1$

if $\lambda = 1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_1 = x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2$

if $\lambda = -1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x_1 = -x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x_2$

if $\lambda = 1$, eigenVector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

if $\lambda = -1$, eigenVector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Q3) i)

$Ax = b$

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

least square solution normal equation $x = (A^TA)^{-1}(A^Tb)$

$A^TA = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix}$

$(A^TA)^{-1} = \frac{1}{27-25}\begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix}$

$A^Tb = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$x = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

ii) $\| b - A\hat{x} \|$ best approximate vector solution

$A^TA = \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix}$

$A^Tb = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$\left(\begin{array}{cc|c} 9 & 5 & 4 \\ 5 & 3 & 2 \end{array}\right) \xrightarrow[R_2 \leftarrow (\frac{9}{2})R_2,\ R_1 \leftarrow R_1/9]{R_2 \leftarrow R_2 - R_1(\frac{5}{9})} \left(\begin{array}{cc|c} 1 & 5/9 & 4/9 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - (\frac{5}{9})R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$ $\hat{x}$

least square solution $= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Q4 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$Ax = \lambda x$

$AAx = A\lambda x$

$A^2x = A\lambda I x$

$A^2 = I_2$

anything time identity matrix end with format $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$

$y = x + Ax$

$Ay = A(x + Ax)$ $A^2 = I_2$

$Ay = Ax + A^2x$

$Ay = Ax + x$

$Ay = (1)y$

$y = 1$

$z = x - Ax$

$Az = Ax - A^2x$

$Az = Ax - x$

$Az = (-1)z$

$z = -1$

Q5 if eigen value $\lambda = 0$.

$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ is true if

$x_1 = -2x_2 - 3x_3 - 4x_4$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$