Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics 1** **Tutorial 2**

Relevant sections for further reading

[S] Calculus by James Stewart Chapter 12 (Sections 12.1 -12.5) or

[T] Thomas' Calculus: Chapter 11 (Sections 11.1 -11.5).

1. Let $\mathbf{u} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.
   Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$, $\mathbf{v} \times \mathbf{u}$, and $\text{proj}_{\mathbf{v}}\mathbf{u}$.

2. For which values of k are $\mathbf{x} = (k, k, 1)$ and $\mathbf{y} = (k, 5, 6)$ in $\mathbb{R}^3$ perpendicular to each other (i.e., orthogonal)?

3. Consider the parallelogram $ABPC$ with adjacent sides AB and AC and vertices $A\,(1,0,0)$, $B\,(0,1,0)$ and $C\,(0,0,1)$.

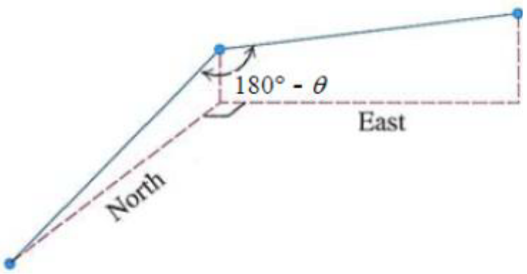
   (a) Find the area of the parallelogram.

   (b) Find the coordinates of the vertex P.

   (c) Find the angle between the diagonals of the parallelogram.

4. Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) along the line from the origin to the point $(1,1)$. (Distance measured in metres).

5. A water main is to be constructed with at 20% grade (i.e., slope $= \dfrac{\text{height}}{\text{horizontal distance}} = 0.2$) in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east (i.e., the angle θ you need to bend the water main).



6. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$.

   (a) Using $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ and some properties of dot products, prove that

$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \pm 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2.$$

   Hence prove that

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2.$$

   (b) Use part (a) to prove that two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$. Also, interpret this geometrically in $\mathbf{R}^2$.

7. (a) Find the vector equation of the line through $A\,(1, 0, 1)$ and $B\,(1, -1, 1)$.
   (b) Find the parametric equation of the line through $P\,(1, 2, -1)$ and $Q\,(-1, 0, 1)$.
   (c) Find the parametric equation of the line through the point $R\,(2, 4, 5)$ and perpendicular to the plane $3x + 7y - 5z = 21$.

8. Consider vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

   (a) Find a unit vector that is perpendicular to vectors $\mathbf{u}$ and $\mathbf{v}$.
   (b) Determine the scalar equation of the plane Π which passes through the point $(1, 1, 0)$ and is parallel to $\mathbf{u}$ and $\mathbf{v}$. What is the distance between planes Π and the plane containing the origin and parallel to $\mathbf{u}$ and $\mathbf{v}$?

9. (a) Find the vector equation and scalar equation of the plane through the point $P\,(1, -1, 3)$ parallel to the plane $3x + y + z = 7$.
   (b) Find the vector equation of the plane through $A\,(1, -2, 1)$ perpendicular to OA.

10. (a) Find the distance from $S\,(3, -1, 4)$ to the line $\ell : x = 4 - t, y = 3 + 2t, z = -5 + 3t$.

    (b) Find the distance from $S\,(2, -3, 4)$ to the plane $x + 2y + 2z = 13$.
    (c) Find the distance between the two planes $x + 2y + 6z = 1$ and $x + 2y + 6z = 10$.

11. Consider four distinct points $A\,(0, 0, 0)$, $B\,(1, 2, 0)$, $C\,(0, -3, 2)$ and $D\,(3, -4, 5)$ where AB, AC and AD are three edges of a parallelepiped.

    (a) Find the volume of the parallelepiped via scalar triple product.
    (b) If A, B and C are three vertices on the base of the parallelepiped, compute the height of the parallelepiped.
    (c) ★Let ℓ_1 be the line through A and B and ℓ_2 the line through D and parallel to AC. What is the distance between the skew lines ℓ_1 and ℓ_2?

Challenging Questions(will not be discussed)

1. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be three non zero, non coplanar vectors.

   (a) Let $\mathbf{v}_1 = \mathbf{x}_1$ and $\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1$. Show that $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_2$.
   (b) Find a vector $\mathbf{v}_3$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are perpendicular to each other.

2. Given any three non collinear points A, B, C. Let O be the circumcenter of ΔABC (i.e., A, B, C are point on the circle with center O and radius $= ||\overrightarrow{OA}|| = ||\overrightarrow{OB}|| = ||\overrightarrow{OC}||$). Let $\overrightarrow{OT} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. Show that T is the orthocenter of ΔABC. (The orthocenter of a triangle ΔABC is a point H such that $AH \perp BC, BH \perp AC$, and $CH \perp AB$.)

Answers

1. $\sqrt{27}$; $\sqrt{6}$; 8; $4\mathbf{i}-9\mathbf{j}-\mathbf{k}$; $-4\mathbf{i}+9\mathbf{j}+\mathbf{k}$; $\frac{8}{6}\left(2\mathbf{i}+\mathbf{j}-\mathbf{k}\right)=\frac{8}{3}\mathbf{i}+\frac{4}{3}\mathbf{j}-\frac{4}{3}\mathbf{k}$
2. $k=-2$ or $k=-3$
   (a) The area is $\sqrt{3}$.
   (b) $(-1,1,1)$
   (c) $\pi/2$
3. $5J$.
4. $\theta=\arccos(\frac{2}{\sqrt{104}\sqrt{101}})$ or $\theta\approx 1.55$ rad, or $88.88°$.
7. (a) $\mathbf{r}=(1,0,1)+t(0,-1,0), t\in\mathbb{R}$.
   (b) $x=1-2t, y=2-2t, z=-1+2t$
   (c) $x=2+3t, y=4+7t, z=5-5t, t\in\mathbb{R}$.
8. (a) $\frac{1}{\sqrt{53}}(-4,6,-1)$
   (b) $-4x+6y-z=2$, and the distance is $\dfrac{2}{\sqrt{53}}$.
9. (a) Vector equation: $\mathbf{r}\cdot(3,1,1)=5$, Scalar equation: $3x+y+z=5$
   (b) Vector equation: $\mathbf{r}\cdot(1,-2,1)=6$.
10. (a) $\frac{9\sqrt{42}}{7}$, (b) 3, (c) $\frac{9}{\sqrt{41}}$
11. (a) 5, (b), Height of the parallelepiped $=\frac{5}{\sqrt{29}}$, (c) $\frac{5}{\sqrt{29}}$.