Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics I** **Tutorial 4**

Reference for Limits: [S] Chapter 2, Section 2.1 - 2.3, 2.5 - 2.6. OR [T] Chapter 2, Section 2.1 - 2.2, 2.4 - 2.6.

1. For each of the following matrices, find its matrix of cofactors $C = (C_{ij})$.[C_{ij} is the (i, j)-cofactor of A].

   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

2. Evaluate the following determinant without using cofactor expansion.

   (a) $\begin{vmatrix} 3 & -17 & -3 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{vmatrix}$ (b) $\begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ -8 & \sqrt{2} & 0 & 0 \\ 7 & 0 & -1 & 0 \\ 9 & 5 & 1 & 6 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & -4 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 9 & 0 & -7 & 0 \\ -11 & 3 & 0 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & 7 & 9 \\ \sqrt{2} & \pi & e \\ 1 & 7 & 9 \end{vmatrix}$

3. Let $A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

   (a) Find
   (i) C_{21} (ii) C_{23} (iii) C_{44} (iv) C_{13}

   (b) Evaluate the determinant of A by cofactor expansion along
   (i) the first column, (ii) the third row.

4. Solve for all real numbers x which satisfies the following equation.

$$\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$$

5. The matrix $R = \begin{bmatrix} \cos\theta & -\sin\,\theta & 0 \\ \sin\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of rotation of points in $\mathbb{R}^3$, it rotates points about the z-axis by θ radians in counter-clockwise direction.

   Show that the matrix R is invertible for all values of θ and find the inverse R^{-1} of R.

6. Solve the linear system by Cramer's rule, if it applies.

$$\begin{array}{rcrcrcl} 4x & + & 5y & & & = & 2 \\ 11x & + & y & + & 2z & = & 3 \\ x & + & 5y & + & 2z & = & 1 \end{array}$$

7. Solve for x, y and z.

$$\begin{aligned} \tfrac{1}{x} &+ \tfrac{2}{y} + \tfrac{1}{z} = 1 \\ \tfrac{3}{x} &+ \tfrac{4}{y} + \tfrac{1}{z} = 5 \\ \tfrac{8}{x} &+ \tfrac{6}{y} + \tfrac{7}{z} = 0 \end{aligned}$$

8. (AY 2012/13 Semester 1) Consider the following system of linear equations

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & c & = & 1 \\ -a & + & 4b & + & 2c & = & 0 \\ a & + & rb & - & c & = & -1 \end{array}$$

   (i) Find the values of r at which Cramer's rule is applicable.

   (ii) For $r = 1$, use Cramer's Rule to determine the unknown b.

9. Consider the function $f : [-3, 5] \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} 2 - x & \text{if } -3 \leq x < 1 \\ 0 & \text{if } x = 1 \\ \sqrt{x} & \text{if } 1 < x < 3 \\ (x-1)^2 & \text{if } 3 \leq x \leq 5. \end{cases}$$

   (a) Sketch the graph $y = f(x)$ for $-3 \leq x \leq 5$. From your sketch, write down the range of f, i.e., the set of values where $f(x)$ assumes for $-3 \leq x \leq 5$.

   (b) From your graph, determine each of the following limits if it exists:
   (i) $\lim\limits_{x\to 0} f(x)$ (ii) $\lim\limits_{x\to 2} f(x)$ (iii) $\lim\limits_{x\to 4} f(x)$ (iv) $\lim\limits_{x\to 1^-} f(x)$ (v) $\lim\limits_{x\to 1^+} f(x)$
   (vi) $\lim\limits_{x\to 1} f(x)$ (vii) $\lim\limits_{x\to 3} f(x)$

10. Does the following limit exist? If it does, what is its value? If it is an infinite limit, determine whether it is $+\infty$ and $-\infty$.

   (a) $\lim\limits_{x\to 5^+} \dfrac{6}{x-5}$ (b) $\lim\limits_{x\to \pi^-} \csc x$

   [$\csc\theta = \frac{1}{\sin\theta}$]

11. (a) Sketch graphs of exponential functions $y = a^x$, where $0 < a < 1$ and $a > 1$.

   (b) Use the graphs in part (a) to write down each of the following limits.
   (i) $\lim\limits_{x\to\infty} (1.001)^x$ (ii) $\lim\limits_{x\to-\infty} \pi^x$ (iii) $\lim\limits_{x\to\infty} 0.37^x$ (iv) $\lim\limits_{x\to-\infty} 181^x$

12. Sketch the graph of $y = \ln(2-x)$ and use it to determine each of the following limits.

   (a) $\lim\limits_{x\to 2^-} \ln(2-x)$ (b) $\lim\limits_{x\to 1^-} \ln(2-x)$ (c) $\lim\limits_{x\to 3^+} \ln(2-x)$ (d) $\lim\limits_{x\to -3} \ln(2-x)$ (e) $\lim\limits_{x\to-\infty} \ln(2-x)$.

Challenging Problems (will not be discussed in tutorial).

1. The *adjoint* of a matrix A is the transpose of the cofactor matrix C of A . It is denoted by adj A, i.e.,

$$\text{adj}\, A = C^T,$$

where $C = (C_{ij})$ is the cofactor matrix. Use the cofactors matrices found in Question 1 of the tutorial to find adj A for

$$A = \begin{pmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Verify that the matrix product $A\,\text{adj}\, A$ is a diagonal matrix and hence find the inverse of A. Based on the above observation, propose a way to find the inverse of a nonsingluar matrix.

2. Determine whether there is a nonsingular matrix A such that

$$A^3 = ABA + A^2,$$

where

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

3. A matrix A is *skew symmetric* if $A^T = -A$. Prove that an $n \times n$ skew symmetric matrix, where n is odd, is singular.

4. A matrix A is *othogonal* if $A^T = A^{-1}$.

   (a) Prove that if A is orthogonal, then $\det A = \pm 1$.

   (b) Give a characterization of 2x2 othogonal matrices.

Answers

1. (a) $\begin{pmatrix} 2 & -3 \\ -2 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 0 & 4 & -8 \\ -5 & 1 & 3 \\ 5 & -1 & 17 \end{pmatrix}$

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2. (a) -30 (upper triangular matrix)

   (b) -12 (lower triangular matrix)

   (c) 0 (zero row)

   (d) 0 (Identical rows)

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(a) (i) $C_{21} = -\begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 6$ (ii) $C_{23} = -\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = -12$

(iii) $C_{44} = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 0$ (iv) $C_{13} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 6$

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3. $x = \dfrac{3 \pm \sqrt{33}}{4}$.

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6. $x = \frac{3}{11}, y = \frac{2}{11}, z = -\frac{1}{11}$.

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7. $x = y = 1, z = -1/2$.

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8. (i) $r \neq -\frac{1}{3}$
   (ii) $b = -1$.

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9. (a) the range of f is $\{0\} \cup (1, 16]$
   (b) (i) $\lim_{x\to 0} f(x) = 2$
   (ii) $\lim_{x\to 2} f(x) = \sqrt{2}$
   (iii) $\lim_{x\to 4} f(x) = 9$
   (iv) $\lim_{x\to 1^-} f(x) = 1$
   (v) $\lim_{x\to 1^+} f(x) = 1$
   (vi) $\lim_{x\to 1} f(x) = 1$.
   (vii) $\lim_{x\to 3} f(x)$ does not exist .

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10. (a) $+\infty$ (b) $+\infty$.

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11. (b) (i) $\lim_{x\to\infty} (1.001)^x = +\infty$
   (ii) $\lim_{x\to-\infty} \pi^x = 0$
   (iii) $\lim_{x\to\infty} 0.37^x = 0$
   (iv) $\lim_{x\to-\infty} 181^x = 0$

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12. (a) $\lim_{x\to 2^-} \ln(2-x) = -\infty$
   (b) $\lim_{x\to 1^-} \ln(2-x) = 0$
   (c) $\lim_{x\to 3^+} \ln(2-x)$ is not defined
   (d) $\lim_{x\to -3} \ln(2-x) = \ln 5$
   (e) $\lim_{x\to-\infty} \ln(2-x) = \infty$