Solution Guide OA: Diff(2)

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Qn 1

$x^2y + (x+y)^3 = 27 + 9x$

when $x=0$,
$y^3 = 27$
$y = 3$

$2xy + x^2\frac{dy}{dx} + 3(x+y)^2\left(1+\frac{dy}{dx}\right) = 9$

$\frac{dy}{dx}\left(x^2+3(x+y)^2\right) = 9 - 2xy - 3(x+y)^2$

$\frac{dy}{dx} = \frac{9-2xy-3(x+y)^2}{x^2+3(x+y)^2}$

$\frac{dy}{dx}$ at $x=0$: $\frac{9-2(0)(3)-3(3)^2}{3(3)^2} = \frac{9-27}{27}$

$= -\frac{2}{3}$

Qn 2

$f(x) = \frac{x^3}{x^2+48}, \quad x>0$

$f'(x) = \frac{3x^2(x^2+48)-2x(x^3)}{(x^2+48)^2} = \frac{x^4+144x^2}{(x^2+48)^2}$

$f''(x) = \frac{(4x^3+288x^2)(x^2+48)^2 - 2(x^2+48)(2x)(x^4+144x^2)}{(x^2+48)^4}$

$= \frac{96x(-x^2+144)}{(x^2+48)^3}$, Just simplify online...

Concave upward : $f''(x)>0$ on (a,b)

$96x(-x^2+144) > 0, \quad x>0$

$96x>0 \qquad -x^2+144>0$

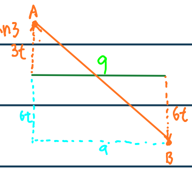
$x>0 \qquad x^2<144$

$x<12$ or $x>-12$ (rej, $x>0$)

$\therefore 0<x<12$

$a=0 \quad b=12$

Qn 3



$AB = \sqrt{(9t)^2+9^2}$

$= \sqrt{81t^2+81}$

$= 9(t^2+1)^{1/2}$

$AB' = \frac{9}{2}(t^2+1)^{-1/2}(2t)$

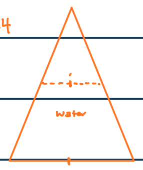
$= \frac{9t}{\sqrt{t^2+1}}$

When $t=9$,

$AB' = \frac{9(9)}{\sqrt{9^2+1}}$

$= \frac{81}{\sqrt{82}}$

Qn 4



$\therefore r = \frac{h}{2}$, similar triangles

Volume of water, $V = \frac{1}{3}\pi(4)^2(8) - \frac{1}{3}\pi r^2 h$

$= \frac{128}{3}\pi - \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$= \frac{128}{3}\pi - \frac{1}{12}\pi h^3$

$\frac{dV}{dt} = -\frac{1}{4}\pi h^2\left(\frac{dh}{dt}\right)$

$-\pi = -\frac{1}{4}\pi h^2 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{h^2}$

When depth of water $= 4m$, $h = 8-4$
$= 4$

$\therefore \frac{dh}{dt} = \frac{4}{16}$. $\frac{dh}{dt}$ is rate of increase of h

$= \frac{1}{4}$ which = rate of decrease of depth.

Qn 5

$x+y=9, \quad x, y \geq 0$

$y = 9-x$

Let $f(x) = x^2y$

$f(x) = x^2(9-x)$

$= 9x^2 - x^3$

$f'(x) = 18x - 3x^2$

$f''(x) = 18 - 6x$

At maximum point, $f'(x) = 0$, $f''(x) < 0$

$18x - 3x^2 = 0$

$3x(6-x) = 0$

$x = 0$ or $x = 6$

$f''(0) = 18 > 0$ (rej) $\qquad f''(6) = -18 < 0$

$f(6) = 9(6)^2 - 6^3$

$= 108$

Qn 6

$\lim_{x\to 0^+}(4x)^{\sin(8x)} = \lim_{x\to 0^+} e^{\sin(8x)\ln(4x)}$

$= e^{\lim_{x\to 0^+}\sin(8x)\ln(4x)}$

$\lim_{x\to 0^+}\sin(8x)\ln(4x) = \lim_{x\to 0^+}\frac{\ln 4x}{\left(\frac{1}{\sin 8x}\right)}$

$= \lim_{x\to 0^+}\frac{\left(\frac{1}{x}\right)}{\frac{-8\cos 8x}{(\sin 8x)^2}}$

$= \lim_{x\to 0^+} -\frac{(\sin 8x)^2}{8x\cos 8x}$

$= \lim_{x\to 0^+} -\frac{16\sin 8x\cos 8x}{8\cos 8x - 64x\sin 8x}$

$= -\frac{0}{8} = 0$

$e^{\lim_{x\to 0^+}\sin(8x)\ln(4x)} = e^0$

$= 1$

Qn 7

$\lim_{x\to 0}\frac{x\ln(1-x)\cos(4x)}{1-e^{x^2}} = \lim_{x\to 0}\cos 4x \cdot \lim_{x\to 0}\frac{x\ln(1-x)}{1-e^{x^2}}$

$= 1 \cdot \lim_{x\to 0}\frac{x\ln(1-x)}{1-e^{x^2}}$

$= \lim_{x\to 0}\frac{\ln(1-x) - \frac{x}{1-x}}{-2xe^{x^2}}$

$= \lim_{x\to 0}\frac{1}{e^{x^2}} \cdot \lim_{x\to 0}\frac{(1-x)\ln(1-x)-x}{-2x(1-x)}$

$= \lim_{x\to 0}\frac{-\ln(1-x) - \frac{1-x}{1-x} - 1}{-2(1-x)+(-1)(-2x)}$

$= \lim_{x\to 0}\frac{-\ln(1-x)-2}{-2+4x}$

$= \frac{-2}{-2} = 1$

Qn 8

$f(x) = ae^x + b$. $\quad \lim_{x\to 3}\frac{f(x)}{\ln x - \ln 3} = e^3$

When $x=3$, denominator $= 0$, but limit not ∞.

$\therefore \frac{0}{0}$ indeterminate form

$\lim_{x\to 3}\frac{f(x)}{\ln x - \ln 3} = \lim_{x\to 3}\frac{ae^x}{\frac{1}{x}}$

$= \frac{ae^3}{\left(\frac{1}{3}\right)} = e^3$

$\frac{a}{\left(\frac{1}{3}\right)} = 1, \quad \therefore a = \frac{1}{3}$

Since $f(x)=0$ when $x=3$.

$\frac{1}{3}e^3 + b = 0, \quad b = -\frac{1}{3}e^3$

Qn 9

$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0}\frac{f(x)}{x}$

$= \lim_{x\to 0}\frac{\ln(1+x^2)\sin\left(6+\frac{2019}{x}\right)}{x}$

$-1 \leq \sin(y) \leq 1$

$-\frac{\ln(1+x^2)}{x} \leq \frac{\ln(1+x^2)}{x}\sin(y) \leq \frac{\ln(1+x^2)}{x}$

$\lim_{x\to 0} -\frac{\ln(1+x^2)}{x} = \lim_{x\to 0} -\frac{2x}{1+x^2} = 0$, $\quad \lim_{x\to 0}\frac{\ln(1+x^2)}{x} = \lim_{x\to 0}\frac{2x}{1+x^2} = 0$

$\therefore$ By squeeze theorem,

$f'(0) = \lim_{x\to 0}\frac{\ln(1+x^2)}{x}\sin\left(6+\frac{2019}{x}\right)$

$= 0$

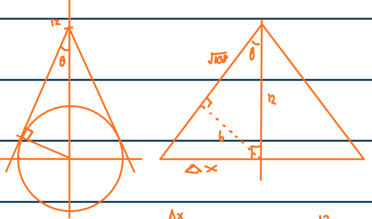
Eqn of tangent

$y - 0 = 0(x-0)$

$y = 0$

Qn 10

$x^2+y^2 = 36$, radius $= \sqrt{36} = 6$



$\tan\theta = \frac{\Delta x}{12}$. Gradient, $m_1 = \frac{12}{\Delta x}$ $\qquad m_2 = -m_1$

$= \frac{6}{\sqrt{108}} \qquad = \frac{1}{\tan\theta} \qquad = -\sqrt{3}$

$= \frac{\sqrt{108}}{6} \qquad \therefore m_2 m_1 = -3$

$= \sqrt{3}$