Solution Guide OA: Integration

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Qn1

$\frac{d}{dx}\int_{-x^2}^{0} \sin(6t^2)\,dt = \frac{d}{dx} - \int_0^{-x^2} \sin(6t^2)\,dt$

Let $u = -x^2$

$\frac{d}{du} - \int_0^{u} \sin(6t^2)\,dt \cdot \frac{du}{dx} = -\sin(6u^2) \cdot -2x$

$= 2x \cdot \sin(6x^4)$

Qn2

$\int_0^{10} f(x)\,dx = 15$ Find $\int_1^{e^2} \frac{f(5\ln(x))}{x}\,dx$. Let $x = 5\ln(u)$

$\frac{dx}{du} = \frac{5}{u}$, $dx = \frac{5}{u}\,du$

$x=10,\ u = e^2$

$x=0,\ u=1$

By substitution,

$\int_0^{10} f(x)\,dx = \int_1^{e^2} f(5\ln(u)) \cdot \frac{5}{u}\,du$

$15 = 5\int_1^{e^2} \frac{f(5\ln(u))}{u}\,du$

$\therefore \int_1^{e^2} \frac{f(5\ln(u))}{u}\,du = \frac{15}{5}$

$= 3$

Qn3

$\frac{1}{x^2(x-4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$

$1 = A(x)(x-4) + B(x-4) + C(x^2)$

$x=0,\ B = -\frac{1}{4}$ $\quad x = 1,$

$x = 4,\ C = \frac{1}{16}$ $\quad 1 = -3A - 3B + C$

$A = -\frac{1}{16}$

$\int \frac{1}{x^2(x-4)}\,dx = \int -\frac{1}{16x} - \frac{1}{4x^2} + \frac{1}{16(x-4)}\,dx$

$= -\frac{1}{16}\ln|x| + \frac{1}{4x} + \frac{1}{16}\ln|x-4| + C$ Exclude +C in answer!

Qn4

$\int x^5\sin(15+x^3)\,dx$ $\qquad$ Let $u = 15+x^3$

$\frac{dy}{dx} = 3x^2$

$du = 3x^2\,dx$

$\int (u-15)\sin(u)\,du = \int 3x^5 \sin(15+x^3)\,dx$

$\int x^5 \sin(15+x^3)\,dx = \frac{1}{3}\int (u-15)\sin u\,du$

$= \frac{1}{3}\left[(u-15)\int \sin u\,du - \int 1\left(\int \sin u\,du\right)du\right]$

$= \frac{1}{3}\left[(u-15)(-\cos u) - (-\sin u)\right]$

$= \frac{1}{3}\left[(x^3)(-\cos(15+x^3) + \sin(15+x^3)\right] + C$

$\int_0^{a\pi} \cos^4(x)\,dx = \left[\cos^3(x)\sin(x) + \int 3\cos^2(x)\sin(x)(\sin(x))\,dx\right]_0^{a\pi}$

$= \left[\cos^3(x)\sin(x) + \int 3\left(\frac{\sin(2x)}{2}\right)^2 dx\right]_0^{a\pi}$

$= \left[\cos^3(x)\sin(x) + \int \frac{3\sin^2(2x)}{4}\,dx\right]_0^{a\pi}$

$= \left[\cos^3(x)\sin(x) + \int \frac{3(1-\cos(4x))}{8}\,dx\right]_0^{a\pi}$

$= \left[\cos^3(x)\sin(x) + \frac{3}{8}\left(x - \frac{\sin(4x)}{4}\right)\right]_0^{a\pi}$

$12\pi = \cos^3(a\pi)\sin(a\pi) + \frac{3}{8}\left(a\pi - \frac{\sin(4a\pi)}{4}\right)$

$~~12\pi = \cos^2(a\pi)\frac{\sin(2a\pi)}{2} + \frac{3}{8}a\pi~~$

$12\pi = \frac{3}{8}a\pi$

$a = 32$

Qn6

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{15k+n}}\right) = \lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{15k+n}}\right)\cdot\frac{\sqrt{n}}{\sqrt{n}}$

$= \lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{\sqrt{15\frac{k}{n}+1}}\right)$

$= \int_0^1 \frac{1}{\sqrt{15x+1}}\,dx$

$= \left[\frac{\sqrt{15x+1}}{(\frac{1}{2})15}\right]_0^1$

$= \left[\frac{2\sqrt{15x+1}}{15}\right]_0^1$

$= \frac{1}{15}(8-2)$

$= \frac{2}{5}$

Qn7

$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{5n}\right) = \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+4n}\right)$

$= \lim_{n\to\infty}\sum_{k=1}^{4n}\frac{1}{n+k}$

$= \lim_{n\to\infty}\sum_{k=1}^{4n}\frac{1}{n}\cdot\frac{1}{1+\frac{k}{n}}$

$= \int_0^4 \frac{1}{1+x}\,dx$

$= \left[\ln|1+x|\right]_0^4$

$= \ln 5 - \ln 1$

$= \ln(5)$

Qn8.

t (s)		v (m/s)
0	v_0	0
1	1	3
2	2	5
3	3	7
4	4	9
5	5	10
6	6	11

Area under velocity = Distance

$= \frac{\Delta t}{2}\cdot\left[v_0 + v_6 + 2(v_1 + v_2 + \dots + v_5)\right]$

$= \frac{1}{2}\cdot\left(0+11+2(3+5+7+9+10)\right)$

$= \frac{79}{2}$

Qn9.

t (s)		v (m/s)
0	v_0	0
1	1	2
2	2	4
3	3	7
4	4	9
5	5	10
6	6	11

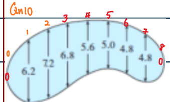
Area under velocity = Distance

$= \frac{\Delta t}{3}\left[v_0 + v_6 + 4(v_1 + v_3 + v_5) + 2(v_2 + v_4)\right]$

$= \frac{1}{3}\left[0+11+4(2+7+10)+2(4+9)\right]$

$= \frac{113}{3}$

Qn10



Width measured at 1m intervals

Area $= \frac{1}{3}\left[0 + 4(6.2+6.8+5.0+4.8) + 2(7.2+5.6+4.8) + 0\right]$

$= \frac{632}{15} \approx 42.13$ (2d.p)

Qn11

$\int_{-8}^{8} \frac{1+10x\cos x}{\pi(64+x^2)}\,dx$.

Let $f(x) = \frac{1}{\pi(64+x^2)}$, $g(x) = 10x$, $h(x) = \cos(x)$

$f(-a) = f(a) \Rightarrow f(x)$ even

$g(-a) = -g(a) \Rightarrow g(x)$ odd

$h(-a) = h(a) \Rightarrow h(x)$ even

$\int_{-8}^{8} \frac{1+10x\cos x}{\pi(64+x^2)}\,dx = \int_{-8}^{8}\frac{1}{\pi(64+x^2)}\,dx + \int_{-8}^{8}\frac{10x\cos x}{\pi(64+x^2)}\,dx$

Let $x = 8u$, $\frac{dx}{du} = 8$, $dx = 8\,du$

$= \frac{1}{\pi}\int_{-8}^{8}\frac{1}{64+x^2}\,dx$

$\frac{1}{\pi}\int_{-8}^{8}\frac{1}{64+x^2}\,dx = \frac{1}{\pi}\int_{-8}^{8}\frac{1}{64(1+u^2)}\cdot 8\,du$

$= \frac{1}{8\pi}\int_{-1}^{1}\frac{1}{1+u^2}\,du$

$= \frac{2}{8\pi}\left[\tan^{-1}(u)\right]_0^1$

$= \frac{2}{8\pi}\left(\frac{\pi}{4}\right) = \frac{1}{16}$

Qn12

$\int_{-\infty}^{\infty}\frac{x^2}{\pi(4+x^6)}\,dx = \frac{2}{\pi}\int_0^{\infty}\frac{x^2}{4+x^6}\,dx$

Let $u = \frac{x^3}{2}$, $\frac{du}{dx} = \frac{3x^2}{2}$, $dx = \frac{2}{3x^2}\,du$

$= \frac{2}{\pi}\int_0^{\infty}\frac{2}{3(4+4u^2)}\,du$

$= \frac{1}{3\pi}\int_0^{\infty}\frac{1}{1+u^2}\,du$

$= \lim_{t\to\infty}\frac{1}{3\pi}\left[\tan^{-1}\left(\frac{x^3}{2}\right)\right]_0^t$

$= \frac{1}{3\pi}\left(\frac{\pi}{2}\right) = \frac{1}{6}$

Qn13

$y^2 = 4x$, $y = 2x-4$ $\qquad \frac{y^2}{4} = \frac{y+4}{2}$

$x = \frac{y^2}{4}$, $x = \frac{y+4}{2}$ $\qquad \frac{1}{4}y^2 - \frac{1}{2}y - 2 = 0$

$y = 4$ or -2

Area of $R = \left|\int_{-2}^{4}\frac{1}{4}y^2 - \frac{1}{2}y - 2\,dy\right|$

$= \left|\left[\frac{1}{12}y^3 - \frac{1}{4}y^2 - 2y\right]_{-2}^{4}\right|$

$= 9$

Qn14

$y = \frac{1}{\pi}\sin(16x^2)$, $x = \frac{\sqrt{\pi}}{4}$. By shell method, Vol $= 2\pi\int_a^b x f(x)\,dx$

Volume of $R = 2\pi\int_0^{\frac{\sqrt{\pi}}{4}} x\cdot\frac{1}{\pi}\sin(16x^2)\,dx$

$= \int_0^{\frac{\sqrt{\pi}}{4}} 2x\sin(16x^2)\,dx$.

Let $u = x^2$, $\frac{du}{dx} = 2x$, $dx = \frac{1}{2x}\,du$

$x=0,\ u=0$

$x = \frac{\sqrt{\pi}}{4},\ u = \frac{\pi}{16}$

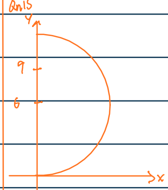
$= \int_0^{\frac{\pi}{16}}\sin(16u)\,du$

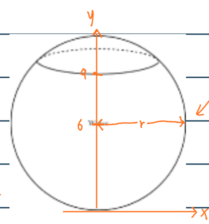
$= \left[-\frac{1}{16}\cos(16u)\right]_0^{\frac{\pi}{16}}$

$= \left[-\frac{1}{16}\cos\pi + \frac{1}{16}\cos 0\right]$

$= \frac{1}{8}$

Qn15





Volume made up of cylinders of radius x, height of dy.

$(y-6)^2 + (x-0)^2 = 6^2$

$x = \sqrt{36-(y-6)^2}$, $x > 0$

Volume $= \pi\int_0^9 r^2\,dy$

$= \pi\int_0^9 36-(y-6)^2\,dy$

$= \pi\int_0^9 -y^2 + 12y\,dy$

$= \pi\left[6y^2 - \frac{1}{3}y^3\right]_0^9$

$= \pi \quad -y^2 + 12y \, dy$

$= \pi \left[6y^2 - \frac{1}{3} y^3 \right]_0^9$

$= 243$pi