Nanyang Technological University
SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Math 1** **Take Home Test**

Version J

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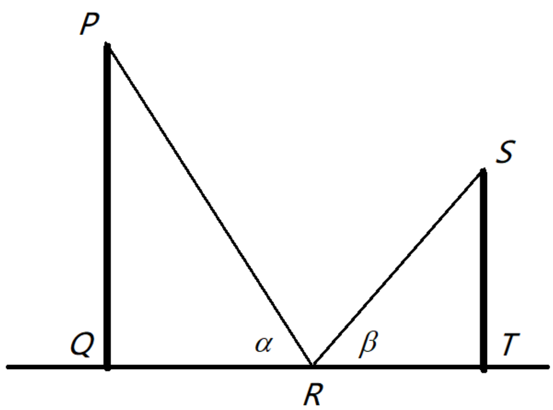
All questions carry the same marks. Answer ALL questions.

1. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the two poles and then to the top of the second pole as shown in the figure. Show that the shortest length of such a rope occurs when $\alpha = \beta$, where $\alpha = \angle PRQ$ and $\beta = \angle SRT$.



2. Let $f(x) = \sqrt{1 + \dfrac{1}{x}}$. Use the *definition of deivatives* to show that

$$f'(x) = -\frac{1}{2x^2\sqrt{\frac{1}{x} + 1}}.$$

3. Express the following as a definite integral $\int_0^1 f(x)\,dx$ and find its exact value.

$$\lim_{n\to\infty}\left(\sqrt[3]{\frac{1}{n^4}}+\sqrt[3]{\frac{2}{n^4}}+\sqrt[3]{\frac{3}{n^4}}+\cdots+\sqrt[3]{\frac{n}{n^4}}\right).$$

4. Show that

   (a) $\displaystyle\int_0^{\pi/2} e^{-x}\cos 2x dx = a\left(e^b+1\right)$, where the numbers a, b are to be determined.

   (b) $\displaystyle\int_0^1 \frac{3^x}{3^x+4^x}dx = \frac{\ln A}{\ln B}$, where the numbers A, B are to be determined.

5. Let R be the region bounded by the curve $y = \dfrac{x}{1+3x^2+x^3}$, $x = 1$, $x = 0$ and $y = 0$. Find the volume when R is rotated 2π radians about the the line $x = -2$. Express your answer in terms of π.

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Take Home Test

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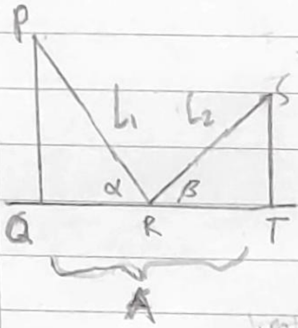


the rope equation $L = L_1 + L_2 = \sqrt{PQ^2 + QR^2} + \sqrt{RT^2 + ST^2}$

find the minimum rope $L' = 0$

takes $QR = x$ and $RT = A - x$

$\cos\alpha = \frac{QR}{L_1}$ so $L_1 = \frac{QR}{\cos\alpha} = \frac{x}{\cos\alpha}$

$\cos\beta = \frac{RT}{L_2}$ so $L_2 = \frac{RT}{\cos\beta} = \frac{A-x}{\cos\beta}$

constant

$L = L_1 + L_2 = \frac{x}{\cos\alpha} + \frac{A-x}{\cos\beta} = \frac{1}{\cos\alpha}x + \frac{A}{\cos\beta} - \frac{1}{\cos\beta}x$

$L' = 0 = \frac{1}{\cos\alpha} - \frac{1}{\cos\beta}\,dx$

$\frac{1}{\cos\alpha} = \frac{1}{\cos\beta} \Rightarrow \cos\alpha = \cos\beta$

$\alpha = \beta$ (proven) //

2 $f(x) = \sqrt{1+\frac{1}{x}}$

let $u = 1 + \frac{1}{x} = 1 + x^{-1}$

$du = -1x^{-2}\,dx$

$= -\frac{1}{x^2}\,dx$

$f(x) = (u)^{\frac{1}{2}}$

$f(x) = \frac{1}{2}(u)^{\left(\frac{1}{2} - \frac{2}{2}\right)}$

$f'(x) = \frac{1}{2}(u)^{-\frac{1}{2}}\,du$

$f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+\frac{1}{x}}} - \frac{1}{x^2}\,dx$

$f'(x) = -\frac{1}{2x^2\sqrt{1+\frac{1}{x}}}$ proven //

3 $\lim_{n\to\infty}\left(\sqrt[3]{\frac{1}{n^4}} + \sqrt[3]{\frac{2}{n^4}} + \sqrt[3]{\frac{3}{n^4}} + \ldots + \sqrt[3]{\frac{n}{n^4}}\right)$

$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n^4}\right)^{\frac{1}{3}}$

$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^{\frac{1}{3}} =$ $a = 0$, $b = 1$ $f(x) = x^{\frac{1}{3}}$ by Riemann Sum

$\hookrightarrow \int_a^b f(x)\,dx = \int_0^1 (x)^{\frac{1}{3}} = \left[\frac{3}{4}(x)^{\frac{4}{3}}\right]_0^1 = \frac{3}{4}$ //

4 a) $\int_0^{\pi/2} e^{-x}\cos 2x\,dx = a(e^b + 1)$

$v' = e^{-x}$ $u = \cos 2x$

$v = -e^{-x}$ $u' = -2\sin 2x$

$\int u.v' = u.v - \int v.u' = \left[-e^{-x}\cos 2x - \int 2\sin 2x\, e^{-x}\right]_0^{\pi/2}$

$v' = e^{-x}$ $u = 2\sin 2x$

$v = -e^{-x}$ $u' = 4\cos 2x$

$\int u.v' = u.v - \int v.u' = \left[-e^{-x}\cos 2x - \left(-2\sin 2x\, e^{-x} - \int -4\cos 2x\, e^{-x}\right]_0^{\pi/2}\right.$

$\left[\int e^{-x}\cos 2x = -e^{-x}\cos 2x + 2e^{-x}\sin 2x - 4\int e^{-x}\cos 2x\right]_0^{\pi/2}$ $a(e^b + 1)$

$\left[5\int e^{-x}\cos 2x = -e^{-x}\cos 2x + 2e^{-x}\sin 2x\right]_0^{\pi/2}$

$\int e^{x}\cos 2x = \left[\frac{-e^{-x}\cos 2x + 2e^{-x}\sin 2x}{5}\right]_0^{\pi/2} = \left(\frac{1}{5}e^{-\pi/2}\right) - \left(-\frac{1}{5}\right) = \frac{1}{5}(e^{-\pi/2} + 1)$ //

$a = \frac{1}{5}$ $b = -\pi/2$

4 b) $f(x) = \int_0^1 \frac{3^x}{3^x+4^x}\,dx = \frac{\ln A}{\ln B}$

$f(x) = \int_0^1 \frac{1}{\left(\frac{3^x+4^x}{3^x}\right)}\,dx = \int_0^1 \frac{1}{1+\left(\frac{4}{3}\right)^x}$

let $u = \left(\frac{4}{3}\right)^x$ $\qquad dx = \frac{1}{u\ln\left(\frac{4}{3}\right)}\,du \qquad \begin{matrix} x=1 & u=4/3 \\ x=0 & u=0 \end{matrix}$

$du = \left(\frac{4}{3}\right)^x \ln\left(\frac{4}{3}\right) dx = u\ln\left(\frac{4}{3}\right) dx$

$f(x) = \int_0^{4/3} \frac{1}{1+u}\,\frac{1}{u\ln\left(\frac{4}{3}\right)}\,du = \frac{1}{\ln\left(\frac{4}{3}\right)} \int_0^{4/3} \frac{1}{u(1+u)}$

$f(x) = \frac{1}{\ln(4/3)} \int_0^{4/3} \frac{1}{u(1+u)} = \frac{a}{u} + \frac{b}{1+u} \Rightarrow 1 = a+u+bu \quad \begin{matrix} a=1 \\ b+1=0 \end{matrix} \Rightarrow \begin{matrix} a=1 \\ b=-1 \end{matrix}$

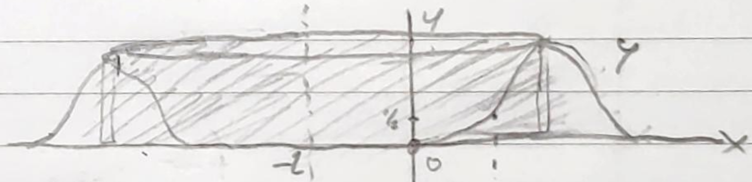
$= \frac{1}{\ln(4/3)} \int_0^{4/3} \frac{1}{u} + \left(-\frac{1}{1+u}\right)$

$= \frac{1}{\ln(4/3)} \Big[\ln|u| - \ln|1+u|\Big]_1^{4/3} = \frac{1}{\ln(4/3)} \cdot \left[\ln 4/3 - \ln 7/3 - \ln_1 + \ln_2\right]$

$= \frac{\ln(8/7)}{\ln(4/3)} = \frac{\ln A}{\ln B}$ with $A = 8/7$ // $B = 4/3$ //

bounded

5) $y = \frac{x}{1+3x^2+x^3}$



$x = 1 \quad +2 \xrightarrow{\text{shift}} = 3$

$x = 0 \quad +2 = 2$

$y = 0$

due to shift sub x to $x-2$

$y = \frac{x-2}{1+3x^2+x^3} = \frac{x-2}{1+3(x-2)^2+(x-2)^3} = \frac{x-2}{x^3-3x^2+5}$

Volume $= \int_2^3 2\pi x \cdot y\,dx$

$= \int_2^3 2\pi x \cdot \frac{x-2}{x^3-3x^2+5}\,dx = 2\pi \int_2^3 \frac{x^2-2x}{x^3-3x^2+5} =$

let $u = x^3-3x^2+5$

$du = 3x^2-6x\,dx$

Volume $= 2\pi \int_2^3 \frac{1}{3}\left(\frac{3x^2-6x}{x^3-3x^2+5}\right) \rightarrow \frac{f'(x)}{f(x)}$

V

$= \frac{2\pi}{3}\Big[\ln|x^3-3x^2+5|\Big]_2^3 = \frac{2\pi}{3}(\ln_5 - \ln_1)$

$= \left(\frac{2\pi}{3}\ln_5\right)$ //