# 1 Relations

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"Mathematics is concerned only with the enumeration and comparison of relations". (Carl Friedrich Gauss)

As a natural transition from sets into functions, we will consider the concept of a "relation" in this section.

**Relation**

Let $A$ and $B$ two non-empty sets. A relation from $A$ to $B$ is defined to be any subset of $A \times B$.

Relations are usually denoted by $R$. Thus the definition can be rephrased as "$R$ is a relation from $A$ to $B$ if $R \subseteq A \times B$". Thus any relation $R$ consists of some ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. If $(a, b) \in R$, we say $a$ is *related to* $b$ *via* $R$ and we denote this by $aRb$. If $(a, b) \notin R$, then $a$ and $b$ are not related and we denote this by $a \not R b$. If a relation $R$ is defined from $A$ to $A$, we will simply say $R$ *is defined on* $A$.

**Example 1** *Let $A$ and $B$ any non-empty sets. Then we have two trivial relations from $A$ to $B$.*

1. $R_1 = \emptyset$ *is sometimes called the null relation. In this case* $a \not R_1 b$ *for all* $a \in A$ *and* $b \in B$.

2. $R_2 = A \times B$. *In this case we have* $aR_2b$ *for all* $a \in A$ *and* $b \in B$.

**Example 2** *Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. The following are some of the many examples of relations that can be defined from $A$ to $B$:*

1. $R_1 = \{(1, a), (1, b), (1, c)\}$.

2. $R_2 = \{(2, c)\}$.

3. $R_3 = \{(1, a), (2, b), (3, c)\}$.

4. $R_4 = \{(1, c), (2, a)\}$.

5. $R_5 = \{(1, a), (1, b), (1, c), (2, a), (2, b)\}$.

**Example 3** *Let $A = \{1, 2, 3\}$. The following are some of the many examples of relations that can be defined on $A$ (i.e. from $A$ to $A$):*

1. $R_1 = \{(1, 1), (2, 2), (3, 3)\}$.

2. $R_2 = \{(1, 2), (1, 3)\}$.

3. $R_3 = \{(1, 2), (2, 1), (3, 2)\}$.

4. $R_4 = \{(1, 2), (2, 3)\}$.

5. $R_5 = \{(1,1),(2,3),(3,3),(3,2),(3,1)\}$.

**Example 4** *Let $\mathbb{Z}$ be the set of integers. The following are some of the many examples of relations that can be defined on $\mathbb{Z}$:*

1. $xR_1y \Leftrightarrow x = 2y$. *In this case,* $2R_11$, $0R_10$, $4R_12$, *but* $1\not R_11$, $3\not R_12$, $\ldots$.
2. $xR_2y \Leftrightarrow x = y^2$. *In this case,* $1R_21$, $16R_24$, $4R_22$, *but* $3\not R_21$, $-4\not R_22$, $\ldots$.
3. $xR_3y \Leftrightarrow 3x = y$. *In this case,* $1R_33$, $-1R_3(-3)$, $0R_30$, *but* $1\not R_31$, $0\not R_33$, $\ldots$.
4. $xR_4y \Leftrightarrow x = y$. *In this case,* $1R_41$, $-1R_4(-1)$, $0R_40$, *but* $1\not R_42$, $3\not R_41$, $\ldots$.
5. $xR_5y \Leftrightarrow x \geq y$. *In this case,* $1R_51$, $2R_51$, $0R_5(-1)$, *but* $1\not R_52$, $0\not R_51$, $\ldots$.
6. $xR_6y \Leftrightarrow x = y^2+1$. *In this case,* $2R_61$, $5R_6(-2)$, *but* $1\not R_61$, $3\not _64$, $\ldots$.
7. $xR_7y \Leftrightarrow |x| = |y|$. *In this case,* $2R_72$, $5R_7(-5)$, *but* $1\not R_72$, $0\not R_73$, $\ldots$.
8. $xR_8y \Leftrightarrow xy = 1$. *In this case,* $1R_81$, $(-1)R_8(-1)$, *but* $x\not R_8y$ *otherwise.*

**Example 5** *Let $A$ be the set of all people living in the world. The following are some of the many examples of relations that can be defined on $A$:*

1. $xR_1y$ *if and only if $x$ and $y$ speak a common language.*
2. $xR_2y$ *if and only if $x$ is older than $y$.*
3. $xR_3y$ *if and only if $x$ and $y$ have the same eye color.*
4. $xR_4y$ *if and only if $x$ and $y$ are from the same country.*
5. $xR_5y$ *if and only if $x$ and $y$ share a parent.*
6. $xR_6y$ *if and only if $y$ is a child of $x$.*

**Example 6** *Let $A = \mathbb{R}^2$, the points in the real plane. The following are some of the many examples of relations that can be defined on $A$:*

1. $(x_1,y_1)R_1(x_2,y_2) \Leftrightarrow x_1^2+y_1^2 = x_2^2+y_2^2$. *In other words, two points in the plane are related if and only if they have the same distance from the origin. So, for example* $(1,1)R_1(0,\sqrt{2})$, $(3,4)R_1(5,0)$, $(1,2)R_1(2,1)$ *but* $(1,1)\not R_1(2,0)$.
2. $(x_1,y_1)R_2(x_2,y_2) \Leftrightarrow x_1 - 3y_1 = x_2 - 3y_2$. *In this case two points are related if and only if they are on the same line of slope* $1/3$. *So, for example* $(1,1)R_2(1,1)$, $(0,0)R_2(3,1)$, $(x,y)R_2(x,y)$ *but* $(1,1)\not R_2(0,0)$.

**Example 7** *For a positive integer $n$, let $A = \mathbb{P}(\{1,2,\ldots,n\})$, the power set of $\{1,2,\ldots,n\}$. In other words, elements of $A$ are all the subsets of $\{1,2,\ldots,n\}$. The following are some of the many examples of relations that can be defined on $A$:*

1. $SR_1T \Leftrightarrow S \subseteq T$.
2. $SR_2T \Leftrightarrow S \cap T = \emptyset$.
3. $SR_3T \Leftrightarrow S \cup T = \{1,2,\ldots,n\}$.

## 1.1 Types of Relations

We will now consider different properties for relations, which will lead to different definitions. The common setup is that we have a non-empty set $A$ and the relations will all be defined on $A$, i.e., they will be from $A$ to $A$.

**Reflexive**

Let $A$ be a non-empty set and let $R$ be a relation defined on $A$. The relation $R$ is said to be reflexive if $aRa$ for all $a \in A$. Equivalently, $R$ is reflexive if and only if $\{(a, a)|a \in A\} \subseteq R$.

It is important to remember the following equivalent statement to the definition:

For a relation to be reflexive, every element in $A$ has to be related to itself. If there is an $a \in A$ such that $a \not R a$, then $R$ is not reflexive.

**Example 8** 1. *The null relation is not reflexive as* $a \not R a$ *for all* $a \in A$.

2. *Let* $R_1$ *be defined on* $\mathbb{Z}$ *by* $xR_1y \Leftrightarrow x^2 + x = y^2 + y$. *The relation* $R_1$ *is reflexive since* $x^2 + x = x^2 + x$ *for all* $x \in \mathbb{Z}$.

3. *Let* $R_2$ *be defined on* $\mathbb{Z}$ *by* $xR_2y \Leftrightarrow xy \geq 0$. $R_2$ *is reflexive since* $x \cdot x = x^2 \geq 0$ *for all* $x \in \mathbb{Z}$.

4. *Let* $R_3$ *be defined on* $\mathbb{Z}$ *by* $xR_3y \Leftrightarrow xy \geq 1$. $R_3$ *is not reflexive since* $0 \not R_3 0$.

5. *The relations of having the same eye color and speaking a common language, defined on the set of all people in the world are both reflexive, while the relation of being a parent or being taller are not reflexive.*

6. *The subset relation defined on* $\mathbb{P}(\{1, 2, \ldots, n\})$ *is reflexive since* $A \subseteq A$ *for all sets* $A$.

7. *The relation defined on* $\mathbb{R}^2$ *by* $(x_1, y_1)R(x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$ *is also reflexive since* $x_1^2 + y_1^2 = x_1^2 + y_1^2$ *for all* $(x_1, y_1) \in \mathbb{R}^2$.

**Irreflexive**

Let $A$ be a non-empty set and let $R$ be a relation defined on $A$. The relation $R$ is said to be irreflexive if $a \not R a$ for all $a \in A$. Equivalently, $R$ is irreflexive if and only if there does not exist an $a \in A$ such that $(a, a) \in R$.

It is important to note that irreflexive does not mean nog reflexive. A relation can be not reflexive by still not be irreflexive.

**Example 9** 1. *The relation on the set* $\{1,2,3\}$*,* $R = \{(1,1),(2,2),(1,2),2,3)\}$ *is neither irreflexive nor reflexive. It is not reflexive since* $(3,3) \notin R$*. It is not irreflexive since* $(1,1) \in R$*.*

2. *The relation on the set* $\{1,2,3\}$*,* $R = \{(1,2),(2,3),(3,2)\}$ *is irreflexive.*

3. *The relation on the set* $\{1,2,3\}$*,* $R = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ *is reflexive since* $(a,a) \in R$ *for all* $a \in \{1,2,3\}$*.*

4. *The relation* $<$ *defined on* $\mathbb{Z}$ *is irreflexive, since* $a \not< a$ *for all* $a \in \mathbb{Z}$*.*

5. *The relation* $>$ *defined on* $\mathbb{R}$ *is irreflexive, since* $a \not> a$ *for all* $a \in \mathbb{R}$*.*

6. *The relation* $R = \{(a,b)\ |a = 2b\}$ *defined on* $\mathbb{Z}$ *is not irreflexive, since* $(0,0) \in R$*.*

7. *The relation* $\emptyset$ *defined on any non-empty set is irreflexive since* $(a,a) \notin \emptyset$ *for all* $a$ *in the set.*

We now move on to the next definition of a property of relations.

**Symmetric**

Let $A$ be a non-empty set and let $R$ be a relation defined on $A$. The relation $R$ is said to be symmetric if $aRb$ implies $bRa$ for all $a,b \in A$.

Before moving on to examples, we make the following observation.

From the truth of implication, we observe that $R$ will be non-symmetric only if there exists $a,b \in A$ such that $aRb$ but $b\not R a$.

**Example 10** 1. *The null relation is vacuosly symmetric.*

2. *Let* $R_1$ *be defined on* $\mathbb{Z}$ *by* $xR_1y \Leftrightarrow x^2+x = y^2+y$*.* $R_1$ *is symmetric since* $x^2+x = y^2+y$ *implies* $y^2+y = x^2+x$*.*

3. *Let* $R_2$ *be defined on* $\mathbb{Z}$ *by* $xR_2y \Leftrightarrow xy \geq 0$*.* $R_2$ *is symmetric since* $xy \geq 0$ *implies* $yx \geq 0$ *for all* $x,y \in \mathbb{Z}$*.*

4. *The relations of having the same eye color and speaking a common language, defined on the set of all people in the world are both symmetric, while the relation of being a parent or being taller are not symmetric.*

5. *The subset relation defined on* $\mathbb{P}(\{1, 2, \ldots, n\})$ *is not symmetric as for example* $\{1\} \subseteq \{1, 2\}$ *while* $\{1, 2\} \not\subseteq \{1\}$.

6. *The relation defined on* $\mathbb{R}^2$ *by* $(x_1, y_1)R(x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$ *is also symmetric since* $x_1^2 + y_1^2 = x_2^2 + y_2^2$ *implies* $x_2^2 + y_2^2 = x_1^2 + y_1^2$ *for all* $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

7. *Let* $R$ *be defined on* $\mathbb{R}$ *by* $xR_1y \Leftrightarrow x \geq y$. *The relation* $R$ *is not symmetric since for example* $2 \geq 1$ *while* $1 \ngeq 2$.

8. *Let* $R$ *be the relation on the set* $\{1, 2, 3\}$ *defined by* $R = \{(1, 1), (1, 2)(2, 1), (3, 2)\}$. *The relation* $R$ *is not symmetric since* $(3, 2) \in R$ *but* $(2, 3) \notin R$.

9. *Let* $R$ *be the relation on the set* $\{1, 2, 3\}$ *defined by* $R = \{(1, 1)\}$. *The relation* $R$ *is symmetric since there is only one element in* $R$, *namely* $(1, 1)$ *and its symmetric pair is also* $(1, 1)$ *which is in* $R$.

We now define the anti-symmetric relations.

**Anti-Symmetric**

Let $A$ be a non-empty set and let $R$ be a relation defined on $A$. The relation $R$ is said to be anti-symmetric if $aRb$ and $bRa$ implies $a = b$ for all $a, b \in A$.

Before moving on to examples, we make the following observation:

From the truth of implication, we observe that $R$ will be not anti-symmetric only if there exists $a \neq b$ in $A$ such that $aRb$ and $bRa$. If $aRb$ but $b\not{R}a$, then this does not cause any problem in being an anti-symmetric relation.

**Example 11** 1. *The null relation is vacuosly anti-symmetric.*

2. *Let* $A = \{1, 2, 3, 4\}$. *Then the relations* $\{(1, 1), (2, 2)\}$, $\{(1, 2), (2, 2)\}$, $\{(1, 2)\}$ *are all anti-symmetric while* $\{(1, 2), (2, 1)\}$ *is not anti-symmetric.*

3. *Let* $R_1$ *be defined on* $\mathbb{Z}$ *by* $xR_1y \Leftrightarrow x^2+x = y^2+y$. $R_1$ *is not antisymmetric since* $1R_1(-2)$ *and* $(-2)R_11$.

4. *Let* $R_2$ *be defined on* $\mathbb{Z}$ *by* $xR_2y \Leftrightarrow xy \geq 0$. $R_2$ *is not anti-symmetric since* $1R_22$ *and* $2R_21$.

5. *Let* $R$ *be defined on* $\mathbb{R}$ *by* $xRy \Leftrightarrow x \leq y$. *Then* $R$ *is anti-symmetric since* $x \leq y$ *and* $y \leq x$ *implies* $x = y$.

6. *The subset relation defined on* $\mathbb{P}(\{1, 2, \ldots, n\})$ *is anti-symmetric as* $S \subseteq T$ *and* $T \subseteq S$ *implies* $S = T$.

7. *Let* $R$ *be defined on* $\mathbb{N}$ *by* $aRb \Leftrightarrow a|b$. $R$ *is anti-symmetric because for* $a, b \in \mathbb{N}$, $a|b$ *and* $b|a$ *implies* $a = b$.

**Remark 1** *The notions of symmetric and anti-symmetric are not exactly opposite of each other nor are they mutually exclusive. There can be examples of relations that are both symmetric and anti-symmetric; symmetric but not anti-symmetric; anti-symmetric but not symmetric; not symmetric and not anti-symmetric.*

The following example illustrates the point we made in the above remark.

**Example 12** *Let* $A = \{1, 2, 3, 4\}$. *Then*

1. *The relation* $R_1 = \{(1, 1), (2, 2)\}$ *is both symmetric and anti-symmetric,*
2. *The relation* $R_2 = \{(1, 2), (2, 1)\}$ *is symmetric but not anti-symmetric,*
3. *The relation* $R_3 = \{(1, 2)\}$ *is anti-symmetric but not symmetric, and*
4. *The relation* $R_4 = \{(1, 2), (2, 1), (1, 3)\}$ *is neither anti-symmetric nor symmetric.*

However, we have the following theorem that describes all relations that are both symmetric and anti-symmetric:

**Theorem 1** *Let* $A$ *be a non-empty set and* $R$ *be a relation defined on* $A$. *Then* $R$ *is both symmetric and anti-symmetric if and only if* $R \subseteq \{(a, a)|A \in A\}$.

Proof: For the forward direction, assume that $R$ is both symmetric and anti-symmetric. If for $a, b \in A$ with $a \neq b$, we had $aRb$, then symmetric property would force $bRa$, which from the antisymmetric property would lead to $a = b$, which is a contradiction to the assumption. Thus if $aRb$, then we must have $a = b$, which implies $R \subseteq \{(a, a)|a \in A\}$.

Conversely, if $R \subseteq \{(a, a)|a \in A\}$, then it is both symmetric and anti-symmetric as there is no case where $a \neq b$ and $aRb$. $\square$

We now give another definition for a property related to symmetry.

**Aymmetric**

Let $A$ be a non-empty set and let $R$ be a relation defined on $A$. The relation $R$ is said to be asymmetric if $aRb$ implies $b \not{R} b$ $a, b \in A$.

As with irreflexive, this is not saying that the relation is simply not symmetric, but rather that there are no symmetric pairs at all. A relation can be not symmetric and not asymmetric.

**Example 13** 1. *Let* $R$ *be the relation on the set* $\{1,2,3\}$ *defined by* $\{(1,2),(2,1),(2,3)\}$. *The relation is not symmetric since* $(2,3) \in R$ *but* $(3,2) \notin R$. *The relation is not asymmetric since* $(1,2)$ *is in* $R$ *and* $(2,1)$ *is in* $R$ *as well.*

2. *The relation* $<$ *on* $\mathbb{Z}$ *is asymmetric since* $a \not< a$ *for all* $a \in \mathbb{Z}$.

3. *The relation* $>$ *on* $\mathbb{R}$ *is asymmetric since* $a \not> a$ *for all* $a \in \mathbb{R}$.

4. *Let* $R$ *be the relation on the set* $\{1,2,3\}$ *defined by* $\{(1,2)\}$. *This relation is asymmetric since* $(2,1) \notin R$. *The relation is also antisymmetric, since there is no occasion of* $(a,b) \in R$ *and* $(b,a) \in R$.

5. *The relation* $\leq$ *on* $\mathbb{Z}$ *is antisymmetric but it is not asymmetric since* $(a,a) \in R$ *for all* $a \in \mathbb{Z}$ *since* $a \leq a$.

6. *The relation* $=$ *on* $\mathbb{R}$ *is antisymmetric but it is not asymmetric*

**Transitive**

Let $A$ be a non-empty set and let $R$ be a relation defined on $A$. The relation $R$ is said to be transitive if $aRb$ and $bRc$ implies $aRb$ for all $a, b, c \in A$.

Before moving on to examples, we make the following observation:

From the truth of implication, we observe that $R$ will fail to be transitive only if there exists $a, b, c$ in $A$ such that $aRb$, $bRc$ but $a\not{R}c$. In all the other cases the relation is transitive.

**Example 14** 1. *The null relation is vacuosly transitive.*

2. *Let* $A = \{1,2,3,4\}$. *Then the relations* $\{(1,1),(2,2)\}$, $\{(1,2),(2,2)\}$, $\{(1,2)\}$ *are all transitive while* $\{(1,2),(2,1)\}$, $\{(1,2),(2,1),(1,1)\}$, $\{(1,2),(2,3)\}$ *are not transitive.*

3. *Let* $R_1$ *be defined on* $\mathbb{Z}$ *by* $xR_1y \Leftrightarrow x^2 + x = y^2 + y$. $R_1$ *is transitive since* $x^2 + x = y^2 + y$ *and* $y^2 + y = z^2 + z$ *implies* $x^2 + x = z^2 + z$.

4. *Let* $R_2$ *be defined on* $\mathbb{Z}$ *by* $xR_2y \Leftrightarrow xy \geq 0$. $R_2$ *is not transitive since* $2R_20$ *and* $0R_2(-1)$ *but* $2\not{R}_2(-1)$.

5. *Let* $R$ *be defined on* $\mathbb{R}$ *by* $xRy \Leftrightarrow x \leq y$. *Then* $R$ *is transitive since* $x \leq y$ *and* $y \leq z$ *implies* $x \leq z$.

6. *The subset relation defined on* $\mathbb{P}(\{1,2,\ldots,n\})$ *is transitive since* $S \subseteq T$ *and* $T \subseteq U$ *implies* $S \subseteq U$.

7. *Let* $R$ *be defined on* $\mathbb{P}(\{1, 2, \ldots, n\})$ *by* $SRT \Leftrightarrow S \cap T = \emptyset$. $R$ *is not transitive and we can illustrate this by an example. Let* $S = \{1, 2\}$, $T = \{3\}$ *and* $U = \{1\}$. *Then* $S \cap T = \emptyset$ *and* $T \cap U = \emptyset$, *while* $S \cap U = \{1\} \neq \emptyset$.

8. *Let* $R$ *be defined on* $\mathbb{N}$ *by* $aRb \Leftrightarrow a|b$. $R$ *is transitive because for* $a, b, c \in \mathbb{N}$, $a|b$ *and* $b|c$ *implies* $a|c$.

9. *The relation defined on the set of people in the world as speaking a common language is not transitive. For example person* $A$ *could be speaking English and Spanish, person* $B$ *could be speaking Spanish and German and person* $C$ *could be speaking German and French. Then* $A$ *and* $B$ *are related and* $B$ *and* $C$ *are related but* $A$ *and* $C$ *have no common language that they speak.*

10. *The relations of having same eye color, being older, being a descendant defined on the set of all people are transitive.*

11. *The relation defined on* $\mathbb{R}^2$ *by* $(x_1, y_1)R(x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$ *is transitive since* $x_1^2 + y_1^2 = x_2^2 + y_2^2$ *and* $x_2^2 + y_2^2 = x_3^2 + y_3^2$ *implies* $x_1^2 + y_1^2 = x_3^2 + y_3^2$*for all* $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$.

**Example 15** *Consider the following false proof that a symmetric and transitive relation must be reflexive. Assume a relation* $R$ *is symmetric and transitive. Then* $(a, b) \in R$ *implies that* $(b, a) \in R$ *since the relation is symmetric. Then* $(a, b) \in R$ *and* $(b, a) \in R$ *implies* $(a, a) \in R$ *since* $R$ *is transitive. Then if* $(a, a) \in R$ *it must be reflexive.*

*The argument, although somewhat convincing is false. The reason it fails is that a given* $a$ *in the set may not be related to any other element. For example, the relation on the set* $A = \{1, 2, 3\}$ *given by* $R = \{((2, 2), (2, 3), (3, 2), (3, 3)\}$ *is symmetric and transitive but not reflexive since* $(1, 1) \notin R$.

Partial orders are a special type of relations that can be defined on a set, with further properties. These are described in full in another chapter. Here, we give the definition and the three classical examples of partial orders.

**Partial Order**

Let $A$ be a non-empty set and let $R$ be a relation defined on $A$. $R$ is said to be a partial order on $A$ if it is

1. reflexive,
2. anti-symmetric, and
3. transitive.

In this case we call the set $A$, a partially ordered set (POSET).

We start with three classical examples of partial orders and POSETs:

**Example 16** *Let* $R$ *be defined on* $\mathbb{R}$ *by* $xRy \Leftrightarrow x \leq y$*. As we have seen before, this is reflexive, anti-symmetric and transitive. Thus* $\leq$ *is a partial order on* $\mathbb{R}$ *and* $(\mathbb{R}, \leq)$ *is a POSET.*

**Example 17** *Let* $R$ *be defined on* $\mathbb{P}(\{1, 2, \ldots, n\})$ *by* $SRT \Leftrightarrow S \subseteq T$*. As we have seen before, and as can be verified by properties of sets,* $R$ *is a partial order in* $\mathbb{P}(\{1, 2, \ldots, n\})$ *and hence* $(\mathbb{P}(\{1, 2, \ldots, n\}), \subseteq)$ *is a POSET.*

**Example 18** *Let* $R$ *be defined on* $\mathbb{N}$ *by* $aRb \Leftrightarrow a|b$*. Clearly,* $a|a$ *for all* $a \in \mathbb{N}$*.* $a|b$ *and* $b|a$ *implies* $a = b$ *so* $R$ *is reflexive and anti-symmetric.*

*Finally, if* $a|b$ *and* $b|c$*, then this means* $b = ak$ *and* $c = bn$ *for* $k, n \in \mathbb{N}$*, which then leads to* $c = a(kn)$ *and hence* $a|c$*. This shows that* $R$ *is transitive.*

*Thus* $R$ *is a partial order on* $\mathbb{N}$ *and hence* $(\mathbb{N}, |)$ *is a POSET.*

**Example 19** *We shall describe some of the most common relations and which properties they satisfy.*

| | Reflexive | Irreflexive | Symmetric | Anti-symmetric | Asymmetric | Transitive |
|---|---|---|---|---|---|---|
| $(\mathbb{R}, <)$ | No | Yes | No | Yes | Yes | Yes |
| $(\mathbb{Z}, \leq)$ | Yes | No | No | Yes | No | Yes |
| $(A, =)$ | Yes | No | Yes | Yes | No | Yes |
| $(\mathcal{P}(A), \subseteq)$ | Yes | No | No | Yes | No | Yes |

# Exercises for Section 1

**Exercise 1** *Let* $A = \{1, 2, 3, 4\}$*. For each part below, give an example of a relation that satisfies the properties given.*

1. *Reflexive, symmetric, transitive, antisymmetric*
2. *Reflexive, symmetric, transitive, not antisymmetric*
3. *Reflexive, symmetric, not transitive, antisymmetric*
4. *Reflexive, not symmetric, transitive, antisymmetric*
5. *not Reflexive, symmetric, transitive, antisymmetric*
6. *Reflexive, not symmetric, not transitive, antisymmetric*
7. *Reflexive, symmetric, not transitive, not antisymmetric*
8. *Reflexive, not symmetric, transitive, not antisymmetric*
9. *not Reflexive, symmetric, transitive, not antisymmetric*
10. *not Reflexive, symmetric, not transitive, antisymmetric*
11. *not Reflexive, not symmetric, transitive, antisymmetric*
12. *Reflexive, not symmetric, not transitive, not antisymmetric*
13. *not Reflexive, not symmetric, transitive, not antisymmetric*

*14. not Reflexive, symmetric, not transitive, not antisymmetric*

*15. not Reflexive, not symmetric, not transitive, antisymmetric*

*16. not Reflexive, not symmetric, not transitive, not antisymmetric*

**Exercise 2** *Repeat Exercise 2 for the set* $A = \mathbb{Z}$.

**Exercise 3** *For each of the following relations defined on* $\mathbb{Z}$, *write down all the properties of the relation (reflexive, symmetric, transitive, anti-symmetric)*

**a)** $xRy \Leftrightarrow xy \neq 1$

**b)** $xRy \Leftrightarrow x^2 - x + 1 = y^2 - y + 1$

**c)** $xRy \Leftrightarrow x > y$ *or* $x < y$.

**Exercise 4** *Let* $R$ *be a relation defined on* $\mathbb{Z}$ *with* $mRn$ *if and only if* $m$ *and* $n$ *have a common divisor greater than 1.*

**a)** *Prove that 1 is not related to any* $a \in \mathbb{Z}$.

**b)** *Prove that every prime number is related to infinitely many integers.*

**c)** *Prove that for any* $m \in \mathbb{Z}$, $m$ *is either not related to any integer or it is related to infinitely many integers.*

**Exercise 5** *Let* $A = \{1, 2, 3\}$. *Describe all relations on* $A$

**a)** *That are both reflexive and symmetric.*

**b)** *That are reflexive but not symmetric.*

**c)** *That are reflexive, symmetric but not transitive.*

**Exercise 6** *Define the following three relations on* $\mathbb{R}$.

$$xR_1y \Leftrightarrow x \neq 2y,$$
$$xR_2y \Leftrightarrow xy < 0$$
$$xR_3y \Leftrightarrow |x - y| = 0 \text{ or } 1.$$

*Write down the properties of these relations on the following table with Yes and No:*

| | Reflexive | Symmetric | Anti-symmetric | Transitive |
|---|---|---|---|---|
| $R_1$ | | | | |
| $R_2$ | | | | |
| $R_3$ | | | | |

**Exercise 7** *Let* $|A| = m$ *and* $|B| = n$.

**a)** *Find the number of different relations that can be defined from* $A$ *to* $B$.

**b)** *Find the number of reflexive relations that can be defined on* $A$.

**c)** *Find the number of symmetric relations that can be defined on* $A$.

**d)** *Find the number of relations on* $A$ *that are both reflexive and symmetric.*

**e)** *Find the number of relations on* $A$ *that are both symmetric and anti-symmetric.*

## 1.2 Equivalence Relations and Equivalence Classes

Now that we have seen different types of relations in reflexive, symmetric, anti0symmetric and transitive relations we will be combining some of these definitions to introduce new types of relations. We first introduce the concept of an equivalence relation and the related concept of an equivalence class. Equivalence relations and equivalence classes are important concepts that appear in almost every branch of higher mathematics such as Combinatorics, Number Theory, Abstract Algebra, Analysis, etc.

We start with the definition of an equivalence relation:

**Equivalence Relation**

Let $A$ be a non-empty set and let $R$ be a relation defined on $A$. The relation $R$ is said to be an equivalence relation if it is

1. reflexive,
2. symmetric and
3. transitive.

**Example 20** *1. Let $A = \{1, 2, 3, 4\}$. Then the relations*

$$\{(1,1),(2,2),(3,3),(4,4)\},$$

$$\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1)\},$$

$$A \times A$$

*are all equivalence relations.*

2. *Let $R_1$ be defined on $\mathbb{Z}$ by $xR_1y \Leftrightarrow x^2 + x = y^2 + y$. $R_1$ is an equivalence relation as we saw above.*

3. *Let $n > 1$ be an integer and let $R_2$ be defined on $\mathbb{Z}$ by $xR_2y \Leftrightarrow x \equiv y \pmod{n}$. $R_2$ is clearly reflexive and symmetric and we can show that it is transitive. So let $x \equiv y \pmod{n}$ and $x \equiv y \pmod{n}$ be given. This means $x - y = nk$ and $y - z = nm$ for some integers $k, m$. But then we have*
$$(x - z) = (x - y) + (y - z) = nk + nm = n(k + m),$$
*which implies $x \equiv z \pmod{n}$, and hence $R_2$ is transitive. Since it is reflexive, symmetric and transitive, we conclude that it is an equivalence relation.*

4. *Let $R$ be defined on $\mathbb{R}$ by $xRy \Leftrightarrow x \leq y$. Then $R$ is not an equivalence relation since it is not symmetric.*

5. *The relations of having same eye color, being from the same country are equivalence relations defined on the set of all people in the world.*

6. *The relation defined on* $\mathbb{R}^2$ *by* $(x_1, y_1)R(x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$ *is an equivalence relations since we already demonstrated that it is reflexive, symmetric and transitive in the first section.*

An important concept related to equivalence relations is the concept of equivalence classes, which we define below:

**Equivalence Class**

Let $A$ be a non-empty set and let $R$ be an equivalence relation defined on $A$. For an element $a \in A$, by $R(a)$ we denote the equivalence class of $a$, which is defined as the set of elements in $A$ that are related to $a$. In other words

$$R(a) = \{x \in A | xRa\}.$$

Before giving some examples of equivalence classes, we will give some theoretical properties of equivalence classes. We start with the following theorem.

**Theorem 2** *Let* $A$ *be a non-empty set and* $R$ *be an equivalence relation defined on* $A$*. For* $a, b \in A$ *we have*

1. $R(a) = R(b) \Leftrightarrow aRb$
2. *Either* $R(a) = R(b)$ *or* $R(a) \cap R(b) = \emptyset$.

Proof:

1. For the forward implication, we assume $R(a) = R(b)$. Since $R$ is reflexive, we have $a \in R(a)$. Since $R(a) = R(b)$, this implies $a \in R(b)$, which implies $aRb$ by definition.

   For the backward implication, we first assume $x \in R(a)$. This means $xRa$. Since $aRb$, by transitivity, we obtain $xRb$, which implies $x \in R(b)$. This shows $R(a) \subseteq R(b)$.

   Conversely, if $x \in R(b)$, then $xRb$. Since $R$ is symmetric, $aRb$ implies $bRa$. But then, by transitivity again, we obtain $xRa$, which implies $x \in R(a)$ or that $R(b) \subseteq R(a)$. Combining the two set-inclusions, we conclude that

   $$R(a) = R(b).$$

2. We will prove that if $R(a) \cap R(b) \neq \emptyset$, then $R(a) = R(b)$. So assume that $x \in R(a) \cap R(b)$. This means that $xRa$ and $xRb$. But since $R$ is symmetric, we have $aRx$. Now, since $R$ is transitive and $aRx$ and $xRb$, we get $aRb$. But then by (1), we get $R(a) = R(b)$.

□

The theorem implies that an equivalence class can be represented by any element in the class, and also that any two given equivalence classes are either identical or disjoint. This property leads to a very nice structural property about equivalence classes.

**Partition**

A collection $\{A_1, A_2, \ldots, \}$ of subsets of a set $A$ form a partition for $A$ if

1. $A = \bigcup_i A_i$ and
2. $A_i \cap A_j = \emptyset$ for all $i, j$ with $i \neq j$.

We can view a partition of a set like the slices that make up a loaf of bread.

One of the biggest source of examples of partitions comes from equivalence classes:

**Theorem 3** *Let $A$ be a non-empty set and $R$ be an equivalence relation defined on $A$. Then the equivalence classes form a partition of $A$.*

Proof: We already proved in Theorem 2 that the equivalence classes are either identical or disjoint. So, to complete the proof, we only need to prove that the union of equivalence classes is $A$.

We first observe that $R(a) \subseteq A$ for all $a \in A$ by definition, which implies $\cup_a R(a) \subseteq A$.

On the other hand, since $R$ is reflexive, $a \in R(a)$ for each $a \in A$, which implies $A \subseteq \cup_a R(a)$. $\square$

We now give several examples of equivalence classes and see how they form a partition of the ambient set.

**Example 21** *Let $R$ be defined on $\mathbb{Z}$ by $xRy \Leftrightarrow x \equiv y \pmod 3$. $R$ is an equivalence relation as we saw above. We now consider the equivalence classes. Let us start with the equivalence class of $0$:*

$$R(0) = \{x \in \mathbb{Z} | xR0\} = \{x \in \mathbb{Z} | x \equiv 0 \pmod 3\} = \{\ldots, -6, -3, 0, 3, 6, \ldots\} = 3\mathbb{Z}.$$

*Note that $18 \in R(0)$, $-24 \in R(0)$, in short, $R(0)$ covers all multiples of $3$.*

*We note that $1 \notin R(0)$, which means its equivalence class will be different. Similarly, we calculate its equivalence class:*

$$R(1) = \{x \in \mathbb{Z} | xR1\} = \{x \in \mathbb{Z} | x \equiv 1 \pmod 3\} = \{\ldots, -5, -2, 1, 4, 7, \ldots\} = 1+3\mathbb{Z}.$$

*Note that $R(1)$ covers all integers of the form $1 + 3k$. Since $2 \notin R(0)$ and $2 \notin R(1)$, it will have a different equivalence class:*

$$R(2) = \{x \in \mathbb{Z} | xR2\} = \{x \in \mathbb{Z} | x \equiv 2 \pmod 3\} = \{\ldots, -4, -1, 2, 5, 8, \ldots\} = 2+3\mathbb{Z}.$$

*We now observe that* $R(0)$, $R(1)$ *and* $R(2)$ *cover all integers as every integer has to be of the form* $3k$, $3k+1$ *or* $3k+2$. *Thus they do form the partition that we are looking for. Thus we have*

$$\mathbb{Z} = 3\mathbb{Z} \cup (1+3\mathbb{Z}) \cup (2+3\mathbb{Z}) = R(0) \cup R(1) \cup R(2).$$

*Consequently, there are three distinct equivalence classes represented by* $0, 1$ *and* $2$. *We recall that the representatives are not unique while the partition is unique. So, for example we could also write the partition as*

$$\mathbb{Z} = R(-15) \cup R(25) \cup R(101).$$

*This would correspond to exactly the same partition as by Theorem 2,* $R(-15) = R(0)$, $R(25) = R(1)$ *and* $R(101)1 = R(2)$.

**Example 22** *Let* $R$ *be the relation defined on* $\mathbb{R}^2$ *by* $(x_1, y_1)R(x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$. *We already saw before that this is an equivalence relation. We now proceed to discuss the equivalence classes. To get a better understanding of the general picture, we start with some specific points:*

$$R(0,0) = \{(x,y) \in \mathbb{R}^2 | (x,y)R(0,0)\} = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 0\} = \{(0,0)\}.$$

*So, the equivalence class of the origin consists of a single point, that is the origin.*

$$\begin{aligned} R(1,2) &= \{(x,y) \in \mathbb{R}^2 | (x,y)R(1,2)\} = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1^2 + 2^2\} \\ &= \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 5\}, \end{aligned}$$

*which has a nice geometric meaning, that is the circle of radius* $\sqrt{5}$ *centered at the origin.*

*In general, if* $(a, b) \in \mathbb{R}^2$ *with* $a^2 + b^2 = r^2$, *then*

$$\begin{aligned} R(a,b) &= \{(x,y) \in \mathbb{R}^2 | (x,y)R(a,b)\} = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = a^2 + b^2\} \\ &= \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = r^2\}, \end{aligned}$$

*the circle of radius* $r$ *centered at the origin. Thus for every non-negative real number* $r$, *we have a different equivalence class. By* $\mathcal{C}_r$, *we denote the circle*

$$\mathcal{C}_r = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = r^2\},$$

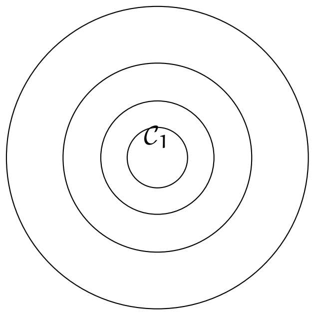
*where* $\mathcal{C}_0$ *is the degenerate circle of radius 0, namely the point* $(0,0)$. *We then have the following partition of* $\mathbb{R}^2$ *into equivalence classes:*

$$\mathbb{R}^2 = \bigcup_{r \geq 0} \mathcal{C}_r.$$

*Some of the equivalence classes are shown in Figure 1.*

**Example 23** *Let* $R$ *be defined on the set of all people of the world as having the same birthday. This is clearly an equivalence relation and the equivalence classes are simply the set of all people who are born on a particular day. Thus there are 366 equivalence classes in total (do not forget February 29).*

Figure 1: The equivalence classes of circles centered at origin




**Example 24** *In our last example, we will see how a familiar set of numbers is in fact a set of equivalence classes:*

*Let* $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ *and define the relation* $R$ *on* $A$ *by*

$$(a, b)R(c, d) \Leftrightarrow ad = bc.$$

*We first show that this is an equivalence relation. Clearly* $ab = ba$ *which means* $(a, b)R(a, b)$ *or that* $R$ *is reflexive.*

*If* $(a, b)R(c, d)$ *then* $ad = bc$ *which can be written in the form* $cb = da$*, which then implies* $(c, d)R(a, b)$*, thus* $R$ *is symmetric.*

*Finally, let* $(a, b)R(c, d)$ *and* $(c, d)R(e, f)$*, with* $b, d, f \neq 0$*. This implies that* $ad = bc$ *and* $cf = de$*. Multiply the second equation by* $b$ *and using the first equation, we get* $adf = bde$*. Since* $d \neq 0$*, we can cancel out* $d$ *from both sides, which would lead to* $af = be$*, which implies* $(a, b)R(e, f)$*. Thus* $R$ *is transitive.*

*So, we have been able to establish that* $R$ *is an equivalence relation. We will now proceed to consider the equivalence classes. For* $b \neq 0$*, we have*

$$R(a, b) = \{(x, y)|(x, y)R(a, b)\} = \{(x, y)|xb = ya\}.$$

*Now assuming that* $a \neq 0$*, and* $GCD(a, b) = 1$*,* $xb = ya$ *implies that* $b|y$ *and* $a|x$ *or that* $R(a, b) = \{k(a, b)|k \in \mathbb{Z} \setminus \{\ 0\}\}$*.*

*If* $GCD(a, b) = d > 1$*, then* $(a, b)R(a/d, b/d)$ *and* $GCD(a/d, b/d) = 1$ *and so* $R(a, b) = R(a/d, b/d)$*.*

*If* $a = 0$*, we have* $(x, y)R(0, b)$ *if and only if* $x = 0$*, and so* $R(0, b) = R(0, 1)$*.*

*In summary, the set of representatives can be given by* $(0, 1)$ *and* $(a, b)$ *where* $(a, b) = 1$*. If we denote* $R(a, b) = \overline{(\frac{a}{b})}$*, then we see that the equivalence classes are basically all rational numbers. Hence the set of all equivalence classes of* $R$ *is precisely the set of all rational numbers. It is clear that in* $\mathbb{Q}$*, we have*

$$\frac{1}{2} = \frac{2}{4} = \frac{5}{10} = \frac{-1}{-2} = \ldots,$$

*which are all equivalences under the relation we have defined.*

**Theorem 4** *Let* $A$ *be a set and let* $\{A_1, a_2, \ldots,\}$ *be a partition of the set. Define the relation* $R = \{(a, b) \mid a, b \in A_i, \text{for some } i\}$*. The relation* $R$ *is an equivalence relation.*

Proof: (Reflexive) If $a \in A$, then $a \in A_i$ for some $i$. Necessarily the element $a$ is only one $A_i$. Therefore, $(a, a) \in R$ and the relation is reflexive.

(Symmetric) If $(a, b) \in R$ then $a, b \in A_i$ for some $i$. This implies $(b, a) \in R$ and the relation is symmetric.

(Transitive) If $(a, b) \in R$ then $a, b \in A_i$ for some $i$. If $(b, c) \in R$ then $b, c \in A_j$ for some $j$. But $b$ can be in only one set in the partition, this implies $i = j$. This gives $a, c \in A_i$ and $(a, c) \in R$ and the relation is transitive. □

**Example 25** *Let* $A = \{1, 2, 3, 4, 5\}$*. Let the partition of* $A$ *be* $\{1, 2\}, \{3, 4\}, \{5\}$*. Then the equivalence relation formed from this partition is:*

$$R = \{(1, 1), (2, 2), (1, 2), (2, 1), (3, 3), (4, 4), (3, 4), (4, 3), (5, 5)\}.$$

Combining Theorem 3 and Theorem 4 we get the following.

**Theorem 5** *A relation is an equivalence relation if and only if the equivalence classes form a partition of the ambient set.*

# Exercises for Section 1.2

**Exercise 8** *Define* $R$ *on* $\mathbb{R}$ *as follows:*

$$xRy \Leftrightarrow x = y = 0 \text{ or } xy > 0.$$

**a)** *Prove that* $R$ *is an equivalence relation.*

**b)** *Describe all equivalence classes. In particular, find the number of different equivalence classes.*

**Exercise 9** *Let* $A = \mathbb{R}$*. Define the relation* $R$ *on* $A$ *such that* $xRy$ *if and only if* $\frac{x^2+1}{x^2+2} = \frac{y^2+1}{y^2+2}$.

**a)** *Prove that* $R$ *is an equivalence relation.*

**b)** *What is* $R(0)$*? What is* $R(-1)$*? Find* $R(a)$ *for any* $a \in \mathbb{R}$.

**Exercise 10** *Define* $R$ *on* $\mathbb{R}$ *by* $xRy$ *if and only if* $x - y \in \mathbb{Z}$.

**a)** *Prove that* $R$ *is an equivalence relation*

**b)** *Describe the equivalence classes. Also describe the partition of* $\mathbb{R}$ *into equivalence classes.*

**Exercise 11** *Let* $R$ *be defined on* $\mathbb{R}^2$ *as*

$$(x, y)R(u, v) \Leftrightarrow x^2 - y = u^2 - v.$$

**(a)** *Prove that* $R$ *is an equivalence relation.*

**(b)** *Find* $R(0, 0)$, $R(1, 2)$*. Give a geometrical description to all the equivalence classes.*

**Exercise 12** *1. Describe an equivalence relation on $\mathbb{N}$ so that the equivalence classes are*

$$\{1, 2, \ldots, 9\}, \{10, 11, \ldots, 99\}, \{100, 101, 102, \ldots, 999\}, \ldots.$$

*2. Describe a relation on $\mathbb{Z}$ such that the equivalence classes are*

$$\{0\}, \{-1, 1\}, \{2, -2\}, \{3, -3\}, \ldots.$$

*3. Desrcibe a relation on $\mathbb{Z}$ such that the equivalence classes are given by $\{0\}, \mathbb{Z}_+$, and $\mathbb{Z}_-$.*

**Exercise 13** *Define a relation $R$ on $\mathbb{R}^2$ by $(x_1, y_1)R(x_2, y_2) \Leftrightarrow 3x_1 - y_1 = 3x_2 - y_2$.*
**a)** *Prove that $R$ is an equivalence relation*
**b)** *Describe the equivalence classes. Can you find a geometric interpretation for the equivalence classes? Also describe the partition of $\mathbb{R}^2$ into equivalence classes.*

**Exercise 14** *Define $R$ on $\mathbb{R}$ by $xRy$ if and only if $2(x - y) \in \mathbb{Z}$.*
**a)** *Prove that $R$ is an equivalence relation*
**b)** *Describe the equivalence classes. Also describe the partition of $\mathbb{R}$ into equivalence classes.*

**Exercise 15** *For $a, b \in \mathbb{R} \setminus \{0\}$, define $a \sim b$ if and only if $a/b \in \mathbb{Q}$.*
**(a)** *Prove that $\sim$ is an equivalence relation*
**(b)** *Find $R(2)$, the equivalence class of 2.*
**(c)** *Show that $R(\sqrt{3}) = R(\sqrt{12})$.*

**Exercise 16** *Suppose $A = \cup_{i=1}^{n} A_i$ is a partition of $A$ into $n$ sets. Define a relation $R$ on $A$ by $xRy$ if and only if $x, y \in A_i$ (in other words, $x$ and $y$ belong to the same set $A_i$) for some $i$.*
**a)** *Prove that $R$ is an equivalence relation.*
**b)** *What are the equivalence classes for this relation?*

## 1.3 Operations on Relations and Related Structures

In this section, we will see some operations on relations and consider some related structures such as matrices and directed graphs. We start with the set operations on relations:

**Operations on Relations**

Let $R$ and $S$ be two relations from the set $A$ to $B$. The union and intersection of $R$ and $S$ are relations that are obtained from the unions and intersections of $R$ and $S$ as sets:

1. $a(R \cup S)b$ if $aRb$ or $aSb$,
2. $a(R \cap S)b$ if $aRb$ and $aSb$.

**Example 26** *Let* $A = \{1, 2, 3\}$ *and* $B = \{a, b, c\}$. *Suppose we have the following relations defined from* $A$ *to* $B$*:*

$$\begin{aligned} R_1 &= \{(1, a), (1, b), (1, c)\}, \\ R_2 &= \{(2, c)\}, \\ R_3 &= \{(1, a), (2, b), (3, c)\}, \\ R_4 &= \{(1, c), (2, a)\}, \\ R_5 &= \{(1, a), (1, b), (1, c), (2, a), (2, b)\}. \end{aligned}$$

*Then we have*

1. $R_1 \cup R_2 = \{(1, a), (1, b), (1, c), (2, c)\}$.
2. $R_1 \cup R_2 \cup R_3 = \{(1, a), (1, b), (1, c), (2, c), (2, b), (3, c)\}$.
3. $R_3 \cap (R_1 \cup R_4) = \{(1, a)\}$.
4. $R_4 \cap R_5 = \{(1, c), (2, a)\}$.
5. $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 = \{(1, a), (1, b), (1, c), (2, a), (2, b), (3, c), (2, c)\}$.
6. $R_2 \cap R_3 = \emptyset$.
7. $R_1 \cap R_4 = \{(1, c)\}$.
8. $(R_1 \cup R_2) \cap (R_3 \cup R_4 \cup R_5) = \{(1, a), (1, b), (1, c)\}$.

**Example 27** *Let* $R$ *and* $S$ *be two relations defined on* $\mathbb{Z}$ *with*

$$xRy \Leftrightarrow xy \geq 0,$$

$$xSy \Leftrightarrow x \leq y.$$

*Then* $x(R \cup S)y$ *if and only if* $x \leq y$ *or* $xy \geq 0$. *Thus for example* $(-3)(R \cup S)2$, $3(R \cup S)2$, *but* $4(R \not\cup S)(-1)$.

*On the other hand* $x(R \cap S)y$ *if and only if* $xy \geq 0$ *and* $x \leq y$. *For example* $3(R \cap S)4$, $2(R \cap S)5$, *but* $3(R \not\cap S)0$, $(-1)(R \not\cap S)(-2)$.

## Compositions of Relations

Let $S$ be a relation from $A$ to $B$ and $R$ be a relation from $B$ to $C$. The composition of $R$ and $S$, denoted by $R \circ S$, is a relation from $A$ to $C$ defined by:

$$a(R \circ S)c \Leftrightarrow \exists b \in B, \ aSb \text{ and } bRc.$$

Note that the composition of two relations might not always be well defined however, if $R$ and $S$ are defined on the same set $A$, then both $R \circ S$ and $S \circ R$ are well defined as relations.

Let us now give a few examples of these.

**Example 28** *Let* $A = \{1,2,3\}$*,* $B = \{a,b,c\}$ *and* $C = \{e,f\}$*. Let* $R$ *from* $A$ *to* $B$ *and* $S$ *from* $B$ *to* $C$ *be defined as*

$$\begin{aligned} R &= \{(1,a),(1,b),(2,c),(3,b),(3,c)\} \\ S &= \{(b,e),(c,f),(a,f)\}. \end{aligned}$$

*Then*

$$S \circ R = \{(1,f),(1,e),(2,f),(3,e),(3,f)\}.$$

**Example 29** *Let* $A = \{1,2,3\}$ *and consider the following relations defined on* $A$*:*

$$\begin{aligned} R &= \{(1,1),(1,2),(2,2),(2,3),(3,3)\} \\ S &= \{(1,2),(2,3),(3,1)\}. \end{aligned}$$

*Then we have*

$$\begin{aligned} R \circ R &= \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}, \\ R \circ S &= \{(1,2),(1,3),(2,3),(3,1),(3,2)\}, \\ S \circ S &= \{(1,3),(2,1),(3,2)\}, \\ S \circ R &= \{(1,2),(1,3),(2,3),(2,1),(3,1)\}. \end{aligned}$$

### Inverse of a Relation

Let $R$ be a relation from $A$ to $B$. Then the inverse of $R$, denoted by $R^{-1}$, is a relation from $B$ to $A$ defined by:

$$bR^{-1}a \Leftrightarrow aRb.$$

In other words we have $R^{-1} = \{(b,a)|(a,b) \in R\}$.

**Example 30** *Let* $A = \{1,2,3\}$ *and consider the following relations defined on* $A$*:*

$$\begin{aligned} R &= \{(1,1),(1,2),(2,2),(2,3),(3,3)\} \\ S &= \{(1,2),(2,3),(3,1)\}. \end{aligned}$$

*Then we have*

$$\begin{aligned} R^{-1} &= \{(1,1),(2,1),(2,2),(3,2),(3,3)\}, \\ S^{-1} &= \{(2,1),(3,2),(1,3\}. \end{aligned}$$

Note that a relation $R$ defined on a set $A$ is symmetric if and only if $R = R^{-1}$.

We now give another operation on relations.

### Complement of a Relation

Let $R$ be a relation from $A$ to $B$. Then the complement of $R$, denoted by $\overline{R}$, is a relation from $A$ to $B$ defined by:

$$a\overline{R}b \Leftrightarrow a \not R b.$$

### Matrix of a Relation

If $R$ is a relation from a finite set $A$ to a finite set $B$, we can construct a matrix $M(R)$ that represents the relation as follows:

1. The rows of the matrix will be labeled by elements of $A$.
2. The columns of the matrix will be labeled by elements of $B$.
3. The entry of $M(R)$ corresponding to $(a, b)$ where $a \in A$ and $b \in B$ will be:
$$M(R)_{a,b} = \begin{cases} 1 & aRb \\ 0 & a \not R b. \end{cases}$$

Note that this matrix carries all the information regarding the relation.

**Example 31** *Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $C = \{e, f\}$. Let $R$ from $A$ to $B$ and $S$ from $B$ to $C$ be defined as*

$$\begin{aligned} R &= \{(1, a), (1, b), (2, c), (3, b), (3, c)\} \\ S &= \{(b, e), (c, f), (a, f)\}. \end{aligned}$$

*Then*

$$M(R) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad M(S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that if the relation is defined from on a finite set $A$ (i.e., the relation is from $A$ to $A$), then the matrix of the relation will be a square matrix.

**Example 32** *Let $A = \{1, 2, 3\}$ and consider the following relations defined on $A$:*

$$\begin{aligned} R &= \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\} \\ S &= \{(1, 2), (2, 3), (3, 1)\}. \end{aligned}$$

*Then we have*

$$M(R) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M(S) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

When $R$ is a relation on a finite set $A$, then we can quickly observe that $R$ is reflexive if and only if the main diagonal of $M(R)$ consists of all 1s. Similarly, we can say that $R$ is symmetric if and only if $M(R)$ is a symmetric matrix.

We can now describe how to get a directed graph from a relation.

**Directed Graphs Associated with Relations**

When $R$ is a relation defined over a finite set $A$, we can construct a directed graph that represents the relation in the following way. The vertices of the directed graph are the elements of $A$. There is a directed edge from the point $a$ to the point $b$ if and only if $aRb$.

Note that the connection between relations and directed graphs is a two-way connection. In a similar way a directed graph can lead to a relation.

**Example 33** *Let $A = \{1, 2, 3, 4\}$ and consider the following relation defined on $A$:*

$$R = \{(1, 2), (3, 2), (3, 1), (4, 1)\}.$$

*Then, the directed graph corresponding to $R$ is given as follows*

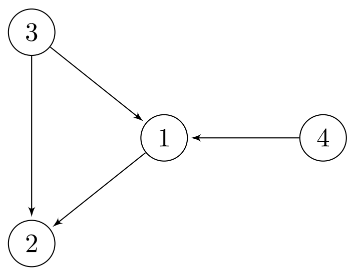



Figure 2: The directed graph corresponding to $R$

**Example 34** *Consider the following directed graphs.*

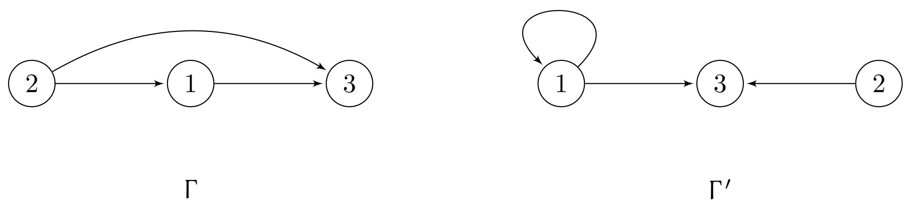



Figure 3: Directed graphs $\Gamma$ and $\Gamma'$

*We can then write the relations $R(\Gamma)$ and $R(\Gamma')$ corresponding to the graphs $\Gamma$ and $\Gamma'$ on the set $A = \{1, 2, 3\}$ as follows:*

$$R(\Gamma) = \{(1,3), (2,1), (2,3)\}$$
$$R(\Gamma') = \{(1,1), (1,3), (2,3)\}.$$

Consider the following operations defined on the set $\{0, 1\}$.

**Operations**

| $\wedge$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $\vee$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

We note that $\wedge$ is just logical **and** where $1$ is True and $0$ is False and $\vee$ is just logical **or** where $1$ is True and $0$ is False. When applying these operations to matrices we apply it coordinate-wise, namely

$$(M \wedge N)_{ij} = M_{ij} \wedge N_{ij}$$

and

$$(M \vee N)_{ij} = M_{ij} \vee N_{ij}.$$

We also have the following operation.

**Operations**

$$\begin{aligned} \overline{0} &= 1 \\ \overline{1} &= 0 \end{aligned}$$

We note that $\overline{a}$ is just logical **not** where $1$ is True and $0$ is False.
When applying this operation to matrices we apply it coordinate-wise, namely

$$\overline{M}_{ij} = \overline{M_{ij}}.$$

Before stating the next theorem, recall that the transpose of a matrix $M$ is given by

$$M^T_{a,b} = M_{b,a}.$$

**Example 35** *Let*

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

*Then*

$$M \wedge N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$M \vee N = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

*and*

$$\overline{M} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Theorem 6** *Let $R$ and $S$ be relations on a set $A$.*

1. $M(R \cap S)_{a,b} = M(R)_{a,b} \wedge M(S)_{a,b}$
2. $M(R \cup S)_{a,b} = M(R)_{a,b} \vee M(S)_{a,b}$
3. $M(R^{-1}) = M^T$
4. $M(\overline{R}) = \overline{M(R)}$.

We now give an operation which will be used to find the composition of two relations.

**Operations**

Let $M$ and $N$ be matrices where the entries are from the set $\{0, 1\}$. The Boolean product of $M$ and $N$ is defined as

$$(M \odot N)_{ij} = (M_{i1} \wedge N_{1j}) \vee (M_{i2} \wedge N_{2j}) \vee (M_{i3} \wedge N_{3j}) \vee \cdots \vee (M_{in} \wedge N_{nj}).$$

**Example 36** *Let*

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} N = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

*Then*

$$M \odot N = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

*and*

$$N \odot M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

*It is immediate that this operation is not commutative.*

**Theorem 7** *Let $R$ and $S$ be relations on the set $A$. Then*

$$M_{R \circ S} = M_S \odot M_R.$$

Proof: There is a 1 in the $i,j$-th location of $M_S \odot M_R$ if and only if there is a 1 in the $i,k$-th location of $M_S$ and the $k,j$-th location of $M_R$ for some $k$. That is there is a $k$ with $(a_i, a_k) \in S$ and $(a_k, a_j) \in R$, which is equivalent to $(a_i, a_j) \in R \circ S$. $\square$

**Example 37** *Let* $R = \{(1,1),(1,3),(2,1),(3,2),(3,1)$ *and let* $S = \{(1,2),(1,3),(2,2),(3,1),(3,2)\}$. *Then*

$$R \circ S = \{(1,1),(1,2),(2,1),(3,1),(3,3)\}$$

*and*

$$S \circ R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,2),(3,3)\}.$$

*We have*

$$M_R = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

*and*

$$M_S = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

*Then*

$$M_{R \circ S} = M_S \odot M_R = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

*and*

$$M_{S \circ R} = M_R \odot M_S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

# Exercises for Section 1.3

**Exercise 17** *For the following relations, give the matrix and the directed graph of the relation.*

1. $A = \{1,2,3,4\}$, $R = \{(1,1),(1,2),(3,2),(3,3),(3,4),(4,2)\}$.
2. $A = \{1,2,3,4\}$, $R = \{(1,1,)(1,4),(2,1),(2,3),(3,1),(3,4),(4,1),(4,3)\}$.
3. $A = \{1,2,3,4\}$, $R = \{(1,1),(2,2),(3,3),(4,4)\}$.

**Exercise 18** *For the following relations find* $R \circ S$ *and* $S \circ R$.

1. $A = \{1,2,3,4\}$, $R = \{(1,1),(1,2),(3,2),(3,3),(3,4),(4,2)\}$, $S = \{(1,3),(2,3),(2,4),(3,1),(3,3),(4,1)\}$.
2. $A = \{1,2,3,4\}$, $R = \{(1,1,)(1,4),(2,1),(2,3),(3,1),(3,4),(4,1),(4,3)\}$, $S = \{(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,2)\}$.
3. $A = \{1,2,3,4\}$, $R = \{(1,1),(2,2),(3,3),(4,4)\}$, $S = \{(1,1),(1,4),(2,3),(3,1),(3,3),(4,1)\}$.

**Exercise 19** *For the following relations find* $M(R \circ S)$ *and* $M(S \circ R)$ *by taking the Boolean product of* $M_R$ *and* $M_S$*. Make sure your answers match the answers in the previous exercise.*

1. $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (1, 2), (3, 2), (3, 3), (3, 4), (4, 2)\}$, $S = \{(1, 3), (2, 3), (2, 4), (3, 1), (3, 3), (4, 1)\}$.
2. $A = \{1, 2, 3, 4\}$, $R = \{(1, 1, )(1, 4), (2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$, $S = \{(2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 2)\}$.
3. $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$, $S = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 3), (4, 1)\}$.

**Exercise 20** *For the following relations find* $R \cup S$ *and* $R \cap S$.

1. $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (1, 2), (3, 2), (3, 3), (3, 4), (4, 2)\}$, $S = \{(1, 3), (2, 3), (2, 4), (3, 1), (3, 3), (4, 1)\}$.
2. $A = \{1, 2, 3, 4\}$, $R = \{(1, 1, )(1, 4), (2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$, $S = \{(2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 2)\}$.
3. $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$, $S = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 3), (4, 1)\}$.

**Exercise 21** *For the following relations* $R$ *find* $\overline{R}$.

1. $(\mathbb{Z}, <)$
2. $(\mathbb{Z}, \geq)$
3. $(\mathbb{R}, =)$
4. $(\mathbb{R}, \neq)$
5. $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 2), (3, 3), (4, 1)\}$

**Exercise 22** *For the following relations* $R$ *find* $R^{-1}$.

1. $(\mathbb{Z}, <)$
2. $(\mathbb{Z}, \geq)$
3. $(\mathbb{R}, =)$
4. $(\mathbb{R}, \neq)$
5. $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 2), (3, 3), (4, 1)\}$