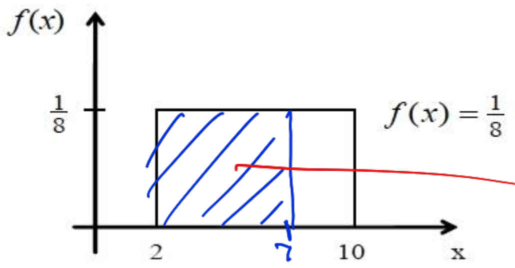
1. The figure below shows the graph of the uniform continuous distribution of a random variable that takes on values on the interval from 2 to 10. Find:

$X \sim U(a=2, b=10)$



In general,
$\underbrace{P(X<x)}_{\text{CDF}} = \int_{-\infty}^{x} f(x)\,dx$

area

(a) $P(X < 7)$
(b) $E(X)$
(c) $Var(X)$

(a) $P(X<x) = \int_{-\infty}^{x} f(x)\,dx = \int_2^7 \frac{1}{8}\,dx = (7-2)\cdot\frac{1}{8}$

(b) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_2^{10} \frac{x}{8}\,dx = \frac{2+10}{2} = 6$

OR Using the formula for $E[X] = \frac{a+b}{2}$

(c) $Var[X] = E[X^2] - E[X]^2$ where $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_2^{10} \frac{x^2}{8}\,dx - 6^2 = \frac{1}{8}\left[\frac{x^3}{3}\right]_2^{10} - 36 = \frac{16}{3}$

OR Using formula for $Var[X] = \frac{(a-b)^2}{12}$

2. The waiting time (r.v. X) for one to be served in a queueing system is a random variable having an exponential distribution with an average of 4 minutes.

p.d.f. $f(x) = \lambda e^{-\lambda x}$, $\lambda = \frac{1}{4}$

(a) Determine the variance of the waiting time.
(b) What is the probability that one has to wait for at least 10 minutes before being served?

$X \sim Exp(\lambda) \Rightarrow$ mean, $E[X] = \frac{1}{\lambda} = 4$

(a) $\therefore Var[X] = E[X]^2 = 16$

(b) $P(X>10) = \int_{10}^{\infty} \frac{1}{4} e^{-\frac{x}{4}}\,dx$

$= \left[-e^{-\frac{x}{4}}\right]_{10}^{\infty} = e^{-\frac{10}{4}}$

3. The cumulative distribution function of the r.v. X is given below:

$$P(X < x) = \int_{-\infty}^{x} f(x)\,dx$$

CDF

$$\text{CDF} \quad F(x) = \begin{cases} 0, & x < 1 \\ 1 - x^{-3}, & x \geq 1 \end{cases}$$

$$\text{p.d.f } f(x) = \frac{d}{dx}\,\text{CDF}$$

(a) Determine the probability density function of X.

(b) Calculate E[X] and var[X].

(a) p.d.f. $f(x) = \begin{cases} 0 & x < 1 \\ \underbrace{\frac{d}{dx}(1 - x^{-3})}_{=3x^{-4}} & x \geq 1 \end{cases}$

(b) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_1^{\infty} \underbrace{x \cdot 3x^{-4}}_{=3x^{-3}}\,dx = \left[-\frac{3}{2}x^{-2}\right]_1^{\infty} = \frac{3}{2}$

$$\text{Var}[X] = \underbrace{E[X^2]}_{\int_1^{\infty} x^2 \cdot 3x^{-4}\,dx \;=\; \left[-3x^{-1}\right]_1^{\infty} = 3} - E[X]^2$$

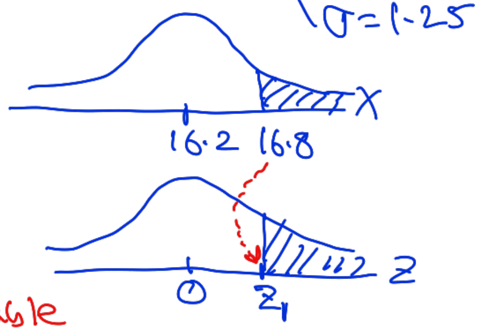
$$= \left(\frac{3}{2}\right)^2$$

4. Given a r.v. X having the normal distribution with $\mu = 16.2$ and $\sigma^2 = 1.5625$, find the probabilities that it will take on a value (use the standard normal distribution table)

$\sigma = 1.25$

(a) greater than 16.8

(b) between 13.6 and 18.8



(a) $P(X > 16.8) = P(Z > Z_1) = P(Z > 0.48)$

$= 1 - \underbrace{P(Z < 0.48)}_{0.6844}$ ← from Z-table

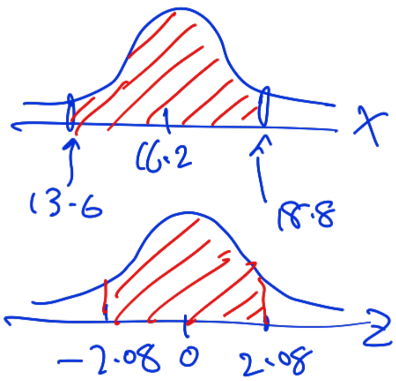
$$Z_1 = \frac{16.8 - 16.2}{1.25} = 0.48$$

(b) $P(13.6 < X < 18.8)$

$= P\left(\frac{13.6 - 16.2}{1.25} < Z < \frac{18.8 - 16.2}{1.25}\right)$

$= P(-2.08 < Z < 2.08)$

$= P(Z < 2.08) - \underbrace{P(Z < -2.08)}_{1 - \underbrace{P(Z < 2.08)}_{\text{obtain this from Z-table}}}$



5. Studies have shown that $\overset{p}{22\%}$ of all patients taking a certain antibiotic will get a headache. Use the normal approximation to the binomial distribution to find the probability that among $\overset{n}{50}$ patients taking this antibiotic

(a) at least 10 will get a headache

(b) at most 15 will get a headache

let X = no. of patients get headache

$\therefore X \sim B(50, 0.22) \Rightarrow np = 11 > 5,\ nq = 39 > 5$

$npq = 8.58$

Use normal approximation to binomial dist.

$\therefore Y \sim N(\mu, \sigma^2)$ where $\mu = np = 11$ and $\sigma^2 = npq = 8.58$

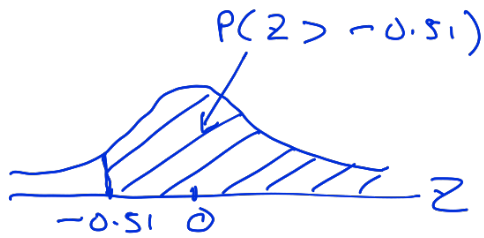
For accurate result, we should apply continuity correction

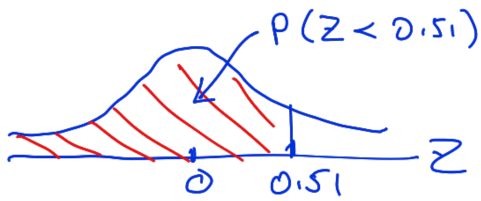
(a) $P(X \geq 10) = P(Y > 9.5)$

$= P\left(Z > \underbrace{\dfrac{9.5 - 11}{\sqrt{8.58}}}_{-0.51}\right)$

$= P(Z > -0.51)$

$= \underbrace{P(Z < 0.51)}_{\text{obtain this from z-table}}$





(b) $P(X \leq 15)$

$= P(Y < 15.5)$

$= P\left(Z < \underbrace{\dfrac{15.5 - 11}{\sqrt{8.58}}}_{1.54}\right)$

$= 0.938 \leftarrow$ from z-table