Chapter 1

Combinatorics

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> *"I think you're begging the question," said Haydock, "and I can see looming ahead one of those terrible exercises in probability where six men have white hats and six men have black hats and you have to work it out by mathematics how likely it is that the hats will get mixed up and in what proportion. If you start thinking about things like that, you would go round the bend. Let me assure you of that!" (Agatha Christie, The Mirror Crackd)*

This chapter is dedicated to combinatorics, which refers broadly to different ways of counting objects.

1.1 Slots and Choices

Suppose, for example, that we have two slots to be filled, and for the first slot, there are n_1 choices, while there are n_2 choices for the second slot. How many ways are there to fill up both slots? For the first slot, we have n_1 choices. Now for each choice for the first slot, we still have n_2 choices for the second slot, for a total of $n_1 n_2$ choices.

Filling slots with a number of choices for each slot is a way to formalize counting problems.

Example 1 *Suppose we are at a tea stand. We can choose a tea flavour among green tea, black tea and earl grey, and the tea can be either iced or hot. To count the number of possible teas that are available, we represent both the tea flavour and the tea temperature as slots. We thus have two slots:*

$$\underset{\textit{flavour}}{\underline{\hspace{2cm}}}\ \underset{\textit{iced/hot}}{\underline{\hspace{2cm}}}$$

Since there are $n_1 = 3$ choices of flavours, the first slot can be filled with either *green tea, black tea or earl grey, irrespectively of what the temperature will be:*

$$\underline{\textit{green tea}} \quad \underset{\textit{iced/hot}}{\underline{\qquad\quad}}$$
$$\underline{\textit{black tea}} \quad \underset{\textit{iced/hot}}{\underline{\qquad\quad}}$$
$$\underline{\textit{earl grey}} \quad \underset{\textit{iced/hot}}{\underline{\qquad\quad}}$$

Now the temperature can be either iced or hot, so for every choice of flavours, we get two choices of temperature. We will thus get a total of $n_1 n_2 = 3 \cdot 2 = 6$ *choices of teas, listed below:*

$$\underline{\textit{green tea}} \quad \underline{\textit{iced}}$$
$$\underline{\textit{green tea}} \quad \underline{\textit{hot}}$$
$$\underline{\textit{black tea}} \quad \underline{\textit{iced}}$$
$$\underline{\textit{black tea}} \quad \underline{\textit{hot}}$$
$$\underline{\textit{earl grey}} \quad \underline{\textit{iced}}$$
$$\underline{\textit{earl grey}} \quad \underline{\textit{hot}}$$

More generally, we have the following.

> Given k slots, and n_1 choices for the 1st slot, n_2 choices for the second slot, and so on, until n_k choices for the kth slot, we have $n_1 \cdot n_2 \cdots n_k$ total choices.

Example 2 *Suppose we are at a frozen yoghurt stand. We have* $n_1 = 11$ *choices of fruits,* $n_2 = 16$ *choices of crunches, and* $n_3 = 15$ *choices of sauces. How many choices of yoghurts do we have, if we can choose one fruit, one crunch, and one sauce? We have* $k = 3$ *slots: one for fruits, one for crunches, and one for sauces:*

$$\underset{\textit{fruit}}{\underline{\qquad\quad}}\ \underset{\textit{crunch}}{\underline{\qquad\quad}}\ \underset{\textit{sauce}}{\underline{\qquad\quad}}\,.$$

We can pick any of the $n_1 = 11$ *fruits, so we have 11 choices for the first slot. Next, no matter what our choice was for the fruit, we can choose from* $n_2 = 16$ *different crunches. Finally, for each of our choices of a fruit and a crunch, we can choose from* $n_3 = 15$ *different sauces, which gives a total of* $n_1 n_2 n_3 = 11 \cdot 16 \cdot 15$ *choices for our yoghurt.*

Now $n_1 \cdot n_2 \cdots n_k$ is also the cardinality of the cartesian product of k sets, where the set i has n_i elements. The formal definition of cartesian product is given in LINK . For now, we will just give an example.

Example 3 *Suppose we are at a restaurant. After taking a look at the menu, we see that we have three choices for the main course, and two choices for the dessert. How many choices of two-course meals do we have? We have* $k = 2$ *slots: one for the main course and one for the dessert. For the main course,* *we have* $n_1 = 3$ *choices. Now, for main course 1, we have* $n_2 = 2$ *choices for our dessert, for main course 2, we again have 2 choices for our dessert, and for main course 3, we still have 2 choices for our dessert. Thus we have a total of* $6 = 3 \cdot 2$ *choices for our two-course meal. An alternative way to view this question is to explicitly list all the choices:*

main course 1, dessert 1	*main course 1, dessert 2*
main course 2, dessert 1	*main course 2, dessert 2*
main course 3, dessert 1	*main course 3, dessert 2*

This makes a total of $n_1 n_2 = 6$ *two-course meals. Notice that when we list all the options, for every element in the set* { *main course* }, *we list all the elements in the set* { *dessert* }. *We get what is called a* cartesian product of two sets, *the set* { *main course 1, main course 2, main course 3* }, *and the set* { *dessert 1, dessert 2* }, *which by definition is* { *(main course 1, dessert 1),(main course 1, dessert 2),(main course 2, dessert 1),(main course 2, dessert 2),(main course 3, dessert 1),(main course 3, dessert 2)*}.

Given a set A, the power set $P(A)$ is the set of all subsets of A, including the empty set. We will count the number of elements in $P(A)$. Write $A = \{a_1, \dots, a_n\}$. Now list all subsets of A, and to each subset, associate a binary vector of length n, where every coefficient is either 0 or 1: the first coefficient is 1 if a_1 is in the subset, and 0 otherwise, similarly, the second coefficient is 1 if a_2 is in the subset, and 0 otherwise, and so on and so forth. Since every element is in a given subset or not, we do obtain all possible binary vectors of length n, and there are 2^n of them. We will see LINK another derivation using the so-called binomial theorem.

Example 4 *Consider the set* $A = \{1, 2\}$.

$\varnothing$	*00*
$\{1\}$	*10*
$\{2\}$	*01*
A	*11*

Now suppose that there are n elements, to be put in k slots. If elements can be repeated, we are in the scenario we have just seen, and there are n^k choices. Now if elements cannot be repeated, then, we have n choices for the first slot, $n-1$ choices for the second slot, and so on and so forth, until $n-(k-1)$ choices for the last slot. Thus the total number of choices is

$$n(n-1)(n-2)\cdots(n-k+1). \tag{1.1}$$

Example 5 *In how many ways can we award a 1st, a 2nd and a 3rd prize to 8 contestants? We have* $k = 3$ *slots, one for each of the prizes:*

1st *2nd* *3rd*

For the first prize, any of the 8 can be awarded. But then once one contestant gets the first prize, only 7 contestants are left (because one contestant does not get two prizes!) So for the second prize, we have 7 choices. And then for the 3rd prize, only 6 choices are left, so the total is $8 \cdot 7 \cdot 6 = 336$.

1.2 Permutations

A *permutation* of n elements is any rearrangement of the elements.

When considering a permutation of n elements, we think of each element being assigned a slot.

Example 6 *Suppose we have four slots and four elements* A, B, C, *and* D. *The rearrangement* $\underline{\;A\;}\,\underline{\;B\;}\,\underline{\;D\;}\,\underline{\;C\;}$ *is a permutation of* $\underline{\;A\;}\,\underline{\;B\;}\,\underline{\;C\;}\,\underline{\;D\;}$.

To count the number of permutations of n elements (where all n elements are rearranged), see that we have n slots, and we have n ways to assign one element in the first slot. Then we have a second slot, and this time $n-1$ elements to choose from. Then we have a third slot, and now $n-2$ elements to choose from, etc, until we reach the last slot, where the last element is placed. The total number of permutations of n elements is

$$n(n-1)(n-2)\cdots 2\cdot 1.$$

We denote the above expression by $n!$ (we say "n factorial"), that is,

$$\boxed{n! = n(n-1)(n-2)\cdots 2\cdot 1.}$$

Example 7 *To count the number of permutations of* $n=3$ *elements, we have* $n=3$ *slots:*

$$\underline{\;\;\;\;}\,\underline{\;\;\;\;}\,\underline{\;\;\;\;}$$

in which we need to place the elements say $1,2,3$. *We can put any of* $n=3$ *elements* $1,2,3$ *in the first slot:*

$$\begin{array}{c}
\underline{\;1\;}\,\underline{\;\;\;\;}\,\underline{\;\;\;\;}\\
\underline{\;2\;}\,\underline{\;\;\;\;}\,\underline{\;\;\;\;}\\
\underline{\;3\;}\,\underline{\;\;\;\;}\,\underline{\;\;\;\;}
\end{array}$$

but once the first element is fixed, the second element can only be chosen between the remaining $n-1=2$ *elements:*

$$\begin{array}{c}
\underline{\;1\;}\,\underline{\;2\;}\,\underline{\;\;\;\;}\\
\underline{\;1\;}\,\underline{\;3\;}\,\underline{\;\;\;\;}\\
\underline{\;2\;}\,\underline{\;1\;}\,\underline{\;\;\;\;}\\
\underline{\;2\;}\,\underline{\;3\;}\,\underline{\;\;\;\;}\\
\underline{\;3\;}\,\underline{\;1\;}\,\underline{\;\;\;\;}\\
\underline{\;3\;}\,\underline{\;2\;}\,\underline{\;\;\;\;}
\end{array}$$

and there is only $n - 2 = 1$ *choice for the 3rd element:*

$$\begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & 2 \\
2 & 1 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
3 & 2 & 1
\end{array}$$

giving a total of $3 \cdot 2 \cdot 1 = 3! = 6$ *permutations of 3 elements.*

Note that $0!$ (an empty product) is defined to be equal to 1. This leads to an amusing looking equality: $1 = 0!$

Alternatively, we can take the formula (1.1) for $k = n$, by noticing that all the n elements are attributed to the n slots, which gives a permutation of the n elements. This also shows that the number of permutations of n elements is

$$n(n-1)(n-2)\cdots 2 \cdot 1 = n!.$$

Formula (1.1) that counts the number of choices for putting n elements into k slots without repeating elements can be expressed in terms of permutations.

The number of ways we can place n (distinct) elements or objects in k slots without repeating elements is

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Each such arrangement is called a *permutation of n objects taken k at a time*.

Example 8 *We look again at Example 5, where we were asked in how many ways we can award a 1st, a 2nd and a 3rd prize to 8 contestants. We have* $k = 3$ *slots (the 3 prizes) for* $n = 8$ *elements (the 8 candidates) which cannot be repeated. Thus the number of ways is* $P(n,k) = P(8,3) = 8 \cdot 7 \cdot 6 = 336$.

Now consider the case where some of our n elements are indistinguishable from one another.

Example 9 *Consider the case where our* $n = 6$ *elements are the letters of the word "LONDON". Here we have four different types of elements: "L" (type 1, say), "O" (type 2), "N" (type 3), and "D" (type 4). In this case, we have* $r_1 = 1$ *element of type 1,* $r_1 = 2$ *elements of type 2,* $r_3 = 2$ *elements of type 3, and* $r_4 = 1$ *element of type 4. We say that elements of the same type are* indistinguishable.

In general we have r_1 elements of type 1, r_2 elements of type 2, and so on, until r_k elements of type k. Now we are interested in the number of distinguishable arrangements of our n elements.

Example 10 *Let's start with the arrangement*

$$\underset{1}{\mathsf{L}}\ \underset{2}{\mathsf{O}}\ \underset{3}{\mathsf{N}}\ \underset{4}{\mathsf{D}}\ \underset{5}{\mathsf{O}}\ \underset{6}{\mathsf{N}}$$

where the slots have been numbered from 1 *to* 6*. If we swap the contents of slot* 1 *with slot* 5*, we obtain the arrangement*

$$\underset{1}{\mathsf{O}}\ \underset{2}{\mathsf{O}}\ \underset{3}{\mathsf{N}}\ \underset{4}{\mathsf{D}}\ \underset{5}{\mathsf{L}}\ \underset{6}{\mathsf{N}}.$$

This arrangement is distinguishable *from the original arrangement. However, starting from the original arrangement, if we swap the contents of slot* 2 *with slot* 5*, we obtain the arrangement*

$$\underset{1}{\mathsf{L}}\ \underset{2}{\mathsf{O}}\ \underset{3}{\mathsf{N}}\ \underset{4}{\mathsf{D}}\ \underset{5}{\mathsf{O}}\ \underset{6}{\mathsf{N}},$$

which is indistinguishable *from the original arrangement.*

To count the number of distinguishable arrangements of our n elements, we can proceed as follows. First we form n slots. Place r_1 elements in r_1 of the n slots. Note that each of the $r_1!$ possible orderings are indistinguishable. Thus there are $P(n, r_1)/r_1!$ ways for us to place the first r_1 elements. Next we place r_2 elements in $n - r_1$ places. Again the ordering of these r_2 elements does not matter, which means there are $P(n - r_1, r_2)/r_2!$ ways for us to place these r_2 elements. Now we repeat this process until we have placed the last r_k elements in the remaining places. This means that we have

$$\frac{P(n, r_1)}{r_1!}\frac{P(n - r_1, r_2)}{r_2!}\cdots\frac{P(r_k, r_k)}{r_k!} \tag{1.2}$$

total distinguishable permutations of n elements, where there are r_1 elements of type 1, r_2 elements of type 2, until r_k elements of type r. (Note that we must have $n = r_1 + \cdots + r_k$.) We can expand (1.2) as

$$\frac{n!}{r_1!(n - r_1)!}\frac{(n - r_1)!}{r_2!(n - r_1 - r_2)!}\cdots\frac{(n - r_1 - \cdots r_{k-1})!}{r_k!}$$

where we notice that we can cancel out numerator and denominator to finally obtain

$$\frac{n!}{r_1!r_2!\cdots r_r!}.$$

Summarising, we have the following.

The number of distinguishable permutations of n elements, where there are r_1 elements of type 1, r_2 elements of type 2, and so on, until r_k elements of type r can be expressed as

$$\frac{n!}{r_1!r_2!\cdots r_k!}.$$

Example 11 *Suppose we want the number of distinguishable permutations of the four letters of the word "DATA". We have that the letter* D *appears once* ($r_1 = 1$)*, the letter* A *appears* $r_2 = 2$ *times, and the letter* T *appears* $r_3 = 1$ *times. Thus the number of permutations is*

$$\frac{4!}{1!2!1!} = 12.$$

We can check this by enumerating each permutation:

D A T A	D A A T	D T A A
A D T A	A D A T	T D A A
A T D A	A A D T	T A D A
A T A D	A A T D	T A A D

Example 12 *Suppose we want the number of distinguishable permutations of "MISSISSIPPI". We have that the letter* M *appears once* ($r_1 = 1$)*, the letter* I *appears* $r_2 = 4$ *times, the letter* S *appears* $r_3 = 4$ *times, and the letter* P *appears* $r_4 = 2$ *times. Thus the number of permutations is*

$$\frac{11!}{1!4!4!2!}.$$

1.3 Combinations

Suppose we have n elements. In this section we want to count the number of ways that we can select k of these n elements. Such a selection of k elements from n elements is called a *combination* of k elements from n elements.

Example 13 *Start with* $n = 4$ *elements* $\clubsuit$, $\diamondsuit$, $\heartsuit$, $\spadesuit$. *Now we list all combinations of* $k = 3$ *elements of these* $n = 4$ *elements:*

$$\begin{array}{c} \diamondsuit\ \heartsuit\ \spadesuit \\ \clubsuit\ \heartsuit\ \spadesuit \\ \clubsuit\ \diamondsuit\ \spadesuit \\ \clubsuit\ \diamondsuit\ \heartsuit \end{array}$$

Note that we did not list $\heartsuit\ \diamondsuit\ \spadesuit$ *since this selection of elements is considered the same as* $\diamondsuit\ \heartsuit\ \spadesuit$*, which we have in our list.*

In Section 1.2, we learned that the number of ways of placing k distinct objects into n slots is

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

while it becomes

$$\frac{P(n,k)}{k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!},$$

if the k objects are indistinguishable.

We can interpret placing k indistinguishable distinct objects into n slots as choosing k slots where to put the k indistinguishable objects among the n slots, which means selecting k elements from n elements (the slots). This is the definition of combination of k elements from n elements.

Example 14 *Start with* $k = 3$ *indistinguishable elements* $\clubsuit$, $\clubsuit$, $\clubsuit$. *Now list all the (distinguishable) ways of placing these 3 elements into* $n = 4$ *slots:*

$$\underset{1}{\underline{\clubsuit}}\ \underset{2}{\underline{\clubsuit}}\ \underset{3}{\underline{\clubsuit}}\ \underset{4}{\underline{\ \ }}$$
$$\underset{1}{\underline{\clubsuit}}\ \underset{2}{\underline{\clubsuit}}\ \underset{3}{\underline{\ \ }}\ \underset{4}{\underline{\clubsuit}}$$
$$\underset{1}{\underline{\clubsuit}}\ \underset{2}{\underline{\ \ }}\ \underset{3}{\underline{\clubsuit}}\ \underset{4}{\underline{\clubsuit}}$$
$$\underset{1}{\underline{\ \ }}\ \underset{2}{\underline{\clubsuit}}\ \underset{3}{\underline{\clubsuit}}\ \underset{4}{\underline{\clubsuit}}$$

The slots have been indexed from 1 *to* 4. *Now we read off the indices of the slots that contain a* $\clubsuit$ *to obtain:*

$$\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 4 \\ 2 & 3 & 4 \end{matrix}.$$

Each collection of indices is a selection of $k = 3$ *elements from the* $n = 4$ *elements* $1\ 2\ 3\ 4$. *In other words, we obtain all the combinations of* $k = 3$ *elements from* $n = 4$ *elements.*

Summarising, we have the following.

> The number of combinations of k elements from n elements is given by
>
> $$C(n, k) = \frac{n!}{k!(n-k)!}.$$
>
> Combinations are so prevalent in mathematics that we commonly use the notation $\binom{n}{k}$ as a succinct alternative to $C(n, k)$. We read $\binom{n}{k}$ as "n choose k".

Example 15 *From a committee of 8 persons, in how many ways can you choose a chair and a vice-chair (one person cannot hold more than one position)? It is* $P(8, 2) = \frac{8!}{6!} = 8 \cdot 7$. *Indeed, we have* $k = 2$ *slots: one for the chair, one for the vice-chair.*

$$\underset{\textit{chair}}{\underline{\qquad\quad}}\ \underset{\textit{vice-chair}}{\underline{\qquad\quad}}$$

Once the chair is chosen (8 *choices for the first slot), we have 7 choices for the vice-chair.*

Example 16 *From a committee of 8 persons, in how many ways can you choose a subcommittee of 2 persons: it is* $C(8, 2) = \frac{8!}{2!6!} = 28$. *Observe that if we select Person 1 then select Person 2, our subcommittee will consist of Person 1 and Person 2. However, we would arrive at the same subcommittee if we select Person 2 then Person 1. In this case, we can use slots to correspond to each person of the committee.*

$$\underset{\textit{Person 1}}{\underline{\qquad\quad}}\ \underset{\textit{Person 2}}{\underline{\qquad\quad}}\ \underset{\textit{Person 3}}{\underline{\qquad\quad}}\ \underset{\textit{Person 4}}{\underline{\qquad\quad}}\ \underset{\textit{Person 5}}{\underline{\qquad\quad}}\ \underset{\textit{Person 6}}{\underline{\qquad\quad}}\ \underset{\textit{Person 7}}{\underline{\qquad\quad}}\ \underset{\textit{Person 8}}{\underline{\qquad\quad}}$$

Then the number of ways of selecting 2 slots from these 8 slots is the same as the number of combinations of $k = 2$ *elements from* $n = 8$ *elements. This is precisely* $C(8, 2) = 28$.

Note the differences between Example 15 and Example 16 - in Example 15 we are *filling* slots and in Example 16 we are *choosing* slots. In Example 15, we used the slots to represent the different job titles, i.e., chair and vice-chair and in Example 16, we used the slots to represent the 8 persons of the committee.

Example 16 also illustrates that $k!C(n, k) = P(n, k)$. Indeed, we know that $P(8, 2)$ takes into account the ordering, therefore, if say Person 1 is chair and Person 2 is vice-chair, it counts for 1 choice, while it counts for 2 choices for the subcommittee, since Person 1 and Person 2, and person 2 and person 1, represent the same subcommittee.

The terms $C(n, k)$ can be conveniently arranged in a triangle (known as Pascal's triangle) as below. The sole entry of the first row is $C(0, 0) = 1$, the entries of the second row are $C(1, 0) = 1$ and $C(1, 1) = 1$.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

More generally, the entries of row $k+1$ are $C(k, 0), C(k, 1), \ldots, C(k, k)$ for $k \in \mathbb{N}$. To construct the table we can take advantage of the following equation

$$\boxed{\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1},}$$

which holds for all $n \in \mathbb{N}$ and $k = 0, \ldots, n$.

1.4 The pigeonhole principle

We conclude this chapter by discussing the pigeonhole principle and the notion of countable sets. We start with the pigeonhole principle.

> The pigeonhole principle states the following: if we have k pigeonholes, and n pigeons, but the number n of pigeons is more than the number k of pigeonholes, then at least one pigeonhole contains at least two pigeons.

Here is a simple illustration: if we have 4 pigeons and 3 pigeonholes:

1. Put the first pigeon in the first pigeonhole, if the second pigeon is also here, then we are done, we have at least one pigeonhole with at least 2 pigeons.

2. If the second pigeon went into the second pigeonhole, repeat the argument: if the third pigeon is also here, then we are done, we have at least one pigeonhole with at least 2 pigeons.

3. If the third pigeon went into the third pigeonhole, then at this time, we have 3 pigeonholes, each containing one pigeon, therefore no matter where the fourth pigeon will go, we have at least one pigeonhole with at least 2 pigeons!

This principle is attributed to the mathematician Dirichlet, and is actually very powerful.

If you know about sets and functions (if you don't, you may come back to this remark after learning about it), notice that this is a consequence of the fact that a function from a finite set (the number of pigeons) to a smaller finite set (the number of pigeonholes) cannot be one-to-one, meaning that there must be at least two elements (two pigeons) in the domain, that have the same image (the same pigeonhole) in the co-domain!

Example 17 *Consider Thorin and his twelve dwarf companions.*

- *At least two of the dwarves were born on the same day of the week (Monday to Sunday): this is a consequence of the pigeonhole principle. We have 7 days of the week, and more than 7 dwarves, therefore 7 of them at most could be born each on one day of the week, but the 8th one will necessarily have to share the same day of the week as birthday.*
- *They sleep at the Prancing Pony Inn, Thorin gets a room of his own (of course, he is the chief!) but the 12 others got to share 4 rooms. Then at least 3 dwarves sleep in at least one room. This is again a consequence of the pigeonhole principle. Imagine room 1, room 2, room 3 and room 4, and 12 dwarves have to fit. The first 4 dwarves could choose room 1, 2, 3, and 4, and be alone in each room. But then the next 4 dwarves will add up, and we will have 2 dwarves in each room. Then no matter how, at least 3 dwarves will end up in one room!*

1.5 More Examples

We are often advised, when creating passwords, to use long passwords, with symbols and numerical digits. We will compute a few examples of how quickly the number of possible passwords grow using combinatorics. This illustrates how counting techniques can be combined to handle different scenarios.

Example 18 *Suppose that a password is 9 characters long, and each character in the password can be an upper case letter, a lower case letter, a digit, or one of the six special characters* $*$, $<$, $>$, !, $+$, $=$. *How many different passwords are available for this computer system?*

In this case, we have 9 slots, and for each of these slots, we can choose one character. There are 68 different possibilities for each character (26 upper case letters, 26 lower case letters, 10 digits, and 6 special characters), thus the answer is 68^9.

Example 19 *Suppose that a password is 9 characters long, and each character in the password can be an upper case letter, a lower case letter, a digit, or one of the six special characters* $*$, $<$, $>$, !, $+$, $=$. *How many different passwords contain at least one occurrence of at least one of the six special characters?*

To solve this, what we will do is count the total number of passwords, and then remove some passwords, namely those which do not contain any special character. We already know from the above example that the total is 68^9. Now we need to count those which do not contain any special character. So now we have still 9 slots to be filled, but we can use only 62 characters, thus we have 62^9 choices, and the answer is $68^9 - 62^9$.

The above example teaches us that sometimes, it is easier to count by "removing" the elements we are not interested in, rather than to count directly those we are interested in!

Example 20 *Suppose that a password is 9 characters long, and each character in the password can be an upper case letter, a lower case letter, or a digit. How many different passwords are possible if a password must include at least one uppercase letter, one lower-case letter, and one digit?*

To solve this, we will use the technique as in the above example. First, we count the total number of passwords, and then remove those we do not want. To count the total number of passwords, we recall that we have 9 slots, each of them can take 62 values, as already done above, so this makes 62^9. Next we start counting the passwords that are not valid.

A password must include at least one upper case letter, so we have to remove all the passwords made using no upper case letter, and we have 36^9 of them. Then we have to remove the passwords made using only lower-case letter, also 36^9 of them. We further remove those made without any digit, 52^9 of them. This would give

$$62^9 - 36^9 - 36^9 - 52^9.$$

But there is a catch! In Example 19, we only removed one set of invalid passwords. But here, we are actually removing 3 sets, and the problem in doing that, is that we could be counting elements several times! To check whether this is happening, we need to check whether the sets of invalid passwords intersect.

- *All digit strings both have no upper case letter, and no lower case letter, so they were counted twice, and there are 10^9 of them.*
- *All lower case letter strings both have no digit and no upper case letter, so they were counted twice, and there are 26^9 of them.*
- *All upper case letter strings both have no digit and no lower case letter, so they were counted twice, and there are 26^9 of them.*

Note that there is no string that satisfy the three conditions. So the final number of valid passwords is, adding back passwords that were removed twice:

$$62^9 - 36^9 - 36^9 - 52^9 + 10^9 + 26^9 + 26^9.$$

Example 21 *Suppose that a password is 9 characters long. How many different passwords do we have that contain exactly 3 digits and 6 characters that are either upper or lower case?*

In this scenario, we know that we need 3 digits, which can be positioned anywhere. This means we first count $\binom{9}{3}$ ways to position the 3 digits - each slot chosen to contain a digit has 10 choices for the digit. Once the digits are positioned, there is no choice for the position of the 6 characters, and for each of the 6 slots to be filled by these 6 characters, there are 52 possibilities, for a total of $\binom{9}{3}10^3 52^6$ passwords.

Exercises for Chapter 1

Exercise 1 *A set menu proposes 2 choices of starters, 3 choices of main dishes, and 2 choices of desserts. How many possible set menus are available?*

Exercise 2
- *In a race with 30 runners where 8 trophies will be given to the top 8 runners (the trophies are distinct, there is a specific trophy for each place), in how many ways can this be done?*
- *In how many ways can you solve the above problem if a certain person, say Jackson, must be one of the top 3 winners?*

Exercise 3 *In how many ways can you pair up 8 boys and 8 girls?*

Exercise 4 *How many ternary strings of length 4 have zero ones?*

Exercise 5 *How many permutations are there of the word "repetition"?*

Exercise 6 *If you pick five cards from a deck of 52 cards, prove that at least two will be of the same suit.*

Exercise 7 *If you have 10 black socks and 10 white socks, and you are picking socks randomly, you will only need to pick three to find a matching pair.*

Exercise 8
1. *For all* $n \in \mathbb{N}$, *show that*

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

   for $1 \leq k \leq n$.

2. *Prove by mathematical induction that*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

   You will need 1. for this!

3. *Deduce that*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$