NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2015–2016
MH1810 – Mathematics 1

NOVEMBER 2015 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **SEVENTEEN (17)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the attachments.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Questions	Marks
1 (15)	
2 (10)	
3 (10)	
4 (10)	

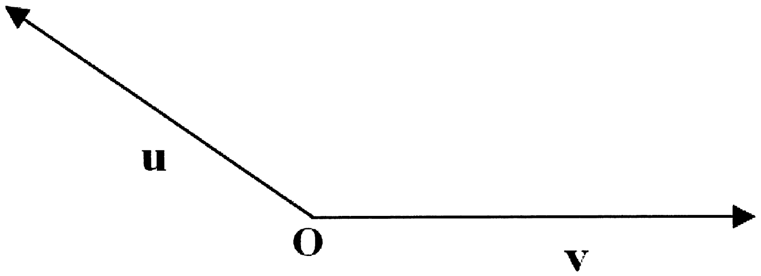
Questions	Marks
5 (15)	
6 (15)	
7 (25)	

Total (100)	

QUESTION 1. **(15 Marks)**

(a) The diagram below shows two vectors $\mathbf{u}$ and $\mathbf{v}$. Draw **on the same diagram** the vectors

(a) **(i)** $\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}$
(ii) $\mathbf{u}-(\mathbf{u}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}$



(b) Given that the shortest distance from the point $A\,(1,2,\alpha)$ to the x-axis is 5 units. Find the possible values of α.

Question 1 continues on Page 3.

(c) Find the determinant of the following matrix by cofactor expansion via the first row. Express your answer in terms of a.

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ a & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

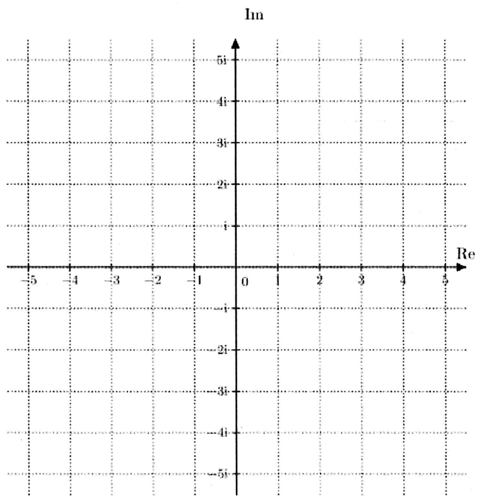
Hence, determine the value(s) of a if the matrix is singular.

QUESTION 2. **(10 Marks)**

(a) Given that $w = \dfrac{3+9i}{1-2i}$.

(i) Find $|w|$ and $\arg(w)$. Express w as $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$.

(ii) Plot the points w and $-iw$ in the diagram below.



Question 2 continues on Page 5.

(b) Solve the equation

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Express the answers in the form $re^{i\theta}$, $r > 0, -\pi < \theta \leq \pi$.

QUESTION 3. **(10 Marks)**

Evaluate the following limits.

(a) $\displaystyle \lim_{x\to-\infty} \frac{x^3}{\sqrt{x^6+x+e^{-x}}}$

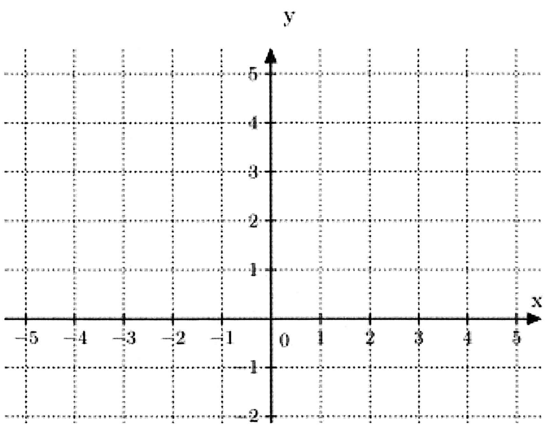
(b) $\displaystyle \lim_{x\to\infty} \frac{\sin\left(e^x\right)}{1+x^2}$

QUESTION 4. **(10 Marks)**

Consider the function f defined as follows:

$$f(x) = \begin{cases} 2x - 1 & \text{if } 1 \leq x \leq 3 \\ |x| & \text{if } -1 < x < 1 \\ 1 & \text{if } -5 \leq x \leq -1 \end{cases}$$

(a) Sketch the graph of $y = f(x)$ for $-5 \leq x \leq 3$ in the diagram below.



(b) Evaluate the integral

$$\int_{-2}^{2} f(x)\ dx.$$

QUESTION 5 **(15 Marks)**

(a) Use the definition of derivative to find $f'(x)$, where $f(x) = x\,|x|$.

Question 5 continues on Page 9.

(b) Find the global maximum and minimum value of $f(x) = x^{1/3}(8-x)$ on the interval $[0,8]$.

QUESTION 6. **(15 Marks)**

(a) Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes 8 hours to melt down to $\frac{1}{8}$ of its orignal volume, how much time does it take for the snowball to melt completely?

[Volume of a sphere: $V = \frac{4}{3}\pi r^3$; Surface area of a sphere: $A = 4\pi r^2$.]

Question 6 continues on Page 11.

(b) Apply Newton's method to the equation $x^2 - a = 0$ to derive the following square-root algorithm to approximate $\sqrt{a}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Hence, starting with $x_0 = 1$, find the approximation x_2 to $\sqrt{2}$.

QUESTION 7. **(25 Marks)**

(a) Evaluate the following integrals.

(i) $\displaystyle\int \tan^{-1}\frac{1}{x}dx$

(ii) $\displaystyle\int \frac{x}{x^2+6x+10}dx$

Question 7 continues on Page 13.

(b) (i) If a and b are positive numbers, show that

$$\int_0^1 x^a (1-x)^b \, dx = \int_0^1 x^b (1-x)^a \, dx.$$

(ii) Prove that

$$\int_0^1 x (1-x)^{1/3} \, dx = \frac{9}{28}.$$

Question 7 continues on Page 14.

(c) A wedge is cut out of a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The other plane intersects the first plane at an angle 60° along a daimeter of the cylinder. Find the volume of the wedge.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x - a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{sech } x) = -\text{sech} x \tanh x \qquad \frac{d}{dx}(\text{csch } x) = -\text{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1 \qquad \int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\int \tan x \sec x \, dx = \sec x + C \qquad \int \cot x \csc x \, dx = -\csc x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C \qquad \int \cot x \, dx = \ln|\sin x| + C$$

$$\int e^x \, dx = e^x + C \qquad \int a^x \, dx = \frac{a^x}{\ln a} + C,\ a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ |x| < |a| \qquad \int \frac{1}{x^2+a^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}} \, dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}} \, dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

MH1810 MATHEMATICS 1

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.