MH1810 Mathematics I
AY2021-22 Semester I
Mock Examination (2 hours)

QUESTION 1. **(10 marks)**

(a) Determine the roots of the equation $z^7 - 5z^5 + 3z^3 - z = 1$, and locate their positions on an Argand diagram.

(b) Find the maximum and minimum values of $z^{\frac{1}{z}}$ in the region $|z - 1 - i| \leq \frac{1}{\sqrt{2}}$.

QUESTION 2. **(10 marks)**

The planes $2x + 3y + az = 1$ and $bx + 2y + z = 3$, where $a, b \in \mathbb{R}$ are constants, meet at an angle of 45°. The length of the line $\frac{x-1}{2} = y - 1 = \frac{z-2}{c}$ between the points of intersection of itself and the two planes is 8 units. Determine the values of a, b and c.

Calculate the area of the triangle whose vertices are the two points of intersection and the origin.

QUESTION 3. **(10 marks)**

Consider the matrix $\begin{pmatrix} -1 & a & 1 \\ a & 1 & -1 \\ 1 & -1 & a \end{pmatrix}$, where a is a real constant.

(i) For what value(s) of a is the above matrix invertible?

(ii) At these value(s), determine the inverse of the above matrix.

(iii) Find a matrix that is similar to but distinct from the above matrix.
[Hint: If square matrices A and B are similar, then $A = PBP^{-1}$ for some square matrix P.]

QUESTION 4. **(15 marks)**

(a) Without using L′Hopital's rule, evaluate the following limits or show that they do not exist.

(i) $\lim_{x \to 0} (\frac{\tan x}{x})^{\frac{1}{x^3}}$, (ii) $\lim_{x \to \infty} (\frac{1}{\sinh x})^{\frac{1}{\tanh x}}$.

(b) Determine the values of a and k for which $f(x) = \begin{cases} x^2 - 1 & \text{for } x \leq a \\ x^3 + kx & \text{for } x > a \end{cases}$ is

(i) continuous on $\mathbb{R}$, (ii) differentiable on $\mathbb{R}$.

QUESTION 5. **(15 marks)**

Let $f(x) = \frac{e^x}{\sin x}$.

(a) Sketch clearly the graph of f on the domain $[-8, 8]$.

(b) Find an expression for the nth derivative of f.

QUESTION 6. **(15 marks)**

Consider the two curves $xy = c$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b, c \neq 0$. Given that these curves do not intersect, determine the conditions for which the two curves are defined, and find the shortest distance between them.

Suppose that each particle is moving counterclockwise along either of the two curves. If the two particles are initially at the closest distance between each other, and that they move at a constant speed, determine whether they will be at the next local minimum distance from each other after some time. Justify your answer.

QUESTION 7. **(10 marks)**

Evaluate the following integrals.

(i) $\int \frac{1}{2+\cos x}\, dx$,

(ii) $\int \tan^4 x \sec^5 x \, dx$.

QUESTION 8. **(15 marks)**

A proposed device, when drafted on a blank paper, is modelled by a region enclosed by the curves $y^2 = 3x + 5$ and $y^2 = 7 - 2x$. Given that this device is formed by rotating this region completely about the line $y = x$, calculate its volume.

Determine whether it is possible to choose another axis of rotation such that the draft piece, when rotated about the new axis, has a completely smooth surface.