Graph II
MST, Shortest Path

Adapted from Y. Huang's slides

Graph Terminology

- Node (vertex)
- Edge (arc)
- Directed graph, undirected graph
- Degree, in-degree, out-degree
- Subgraph
- Simple path
- Cycle
- Directed acyclic graph
- Weighted graph

Graph representation

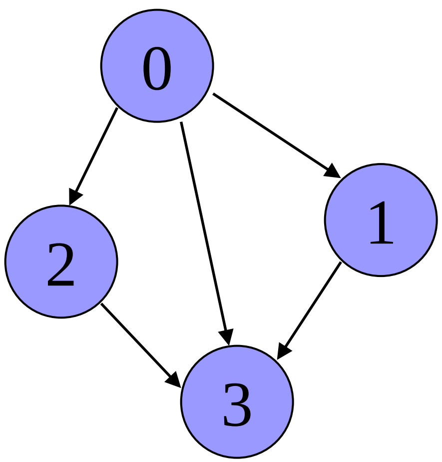
Adjacency Matrix

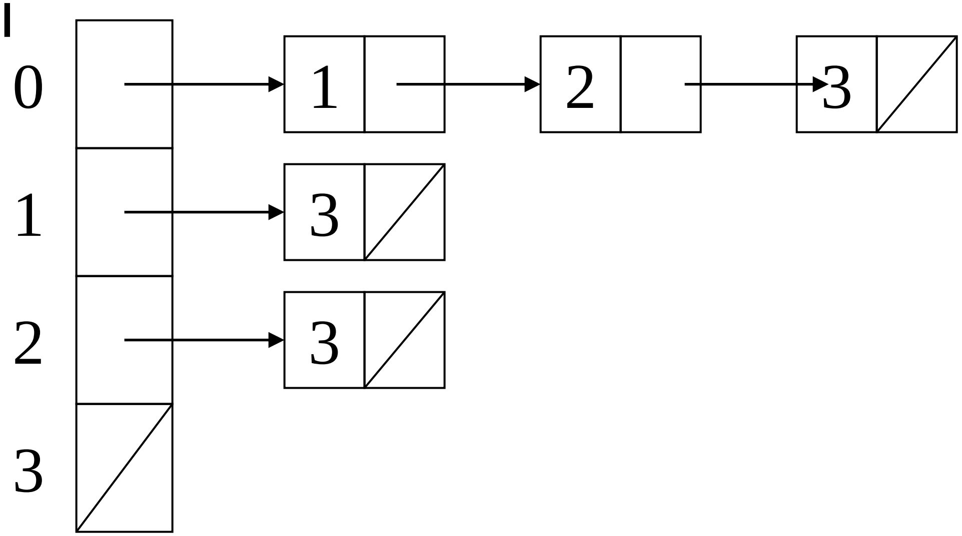
- Assume N nodes in graph
- Use Matrix A[0…N-1][0…N-1]
  - if vertex i and vertex j are adjacent in graph, A[i][j] = 1,
  - otherwise A[i][j] = 0
  - if vertex i has a loop, A[i][i] = 1
  - if vertex i has no loop, A[i][i] = 0

Graph representation

Adjacency List

- An array of list
- the i*th* element of the array is a list of vertices that connect to vertex i





vertex 0 connect to vertex 1, 2 and 3
vertex 1 connects to 3
vertex 2 connects to 3

Graph Traversal

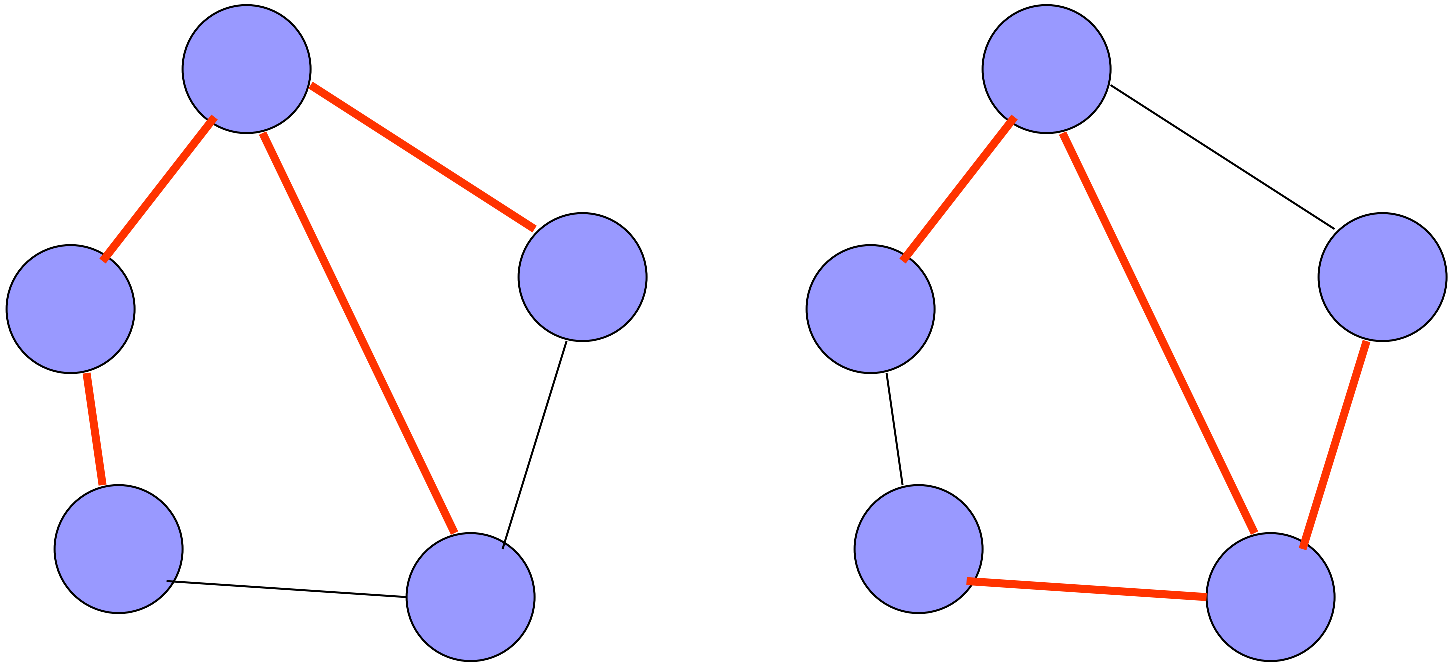
- From one vertex, list out all vertices that can be reached in graph G
- Set of nodes to expand
- Each node has a flag to indicate visited or not
- Depth First Traversal
- Breadth First Traversal

Spanning Tree

- Connected subgraph that includes all vertices of the original connected graph
- Subgraph is a tree
  - If original graph has n vertices, the spanning tree has n vertices and n-1 edges.
  - No circle in this subgraph

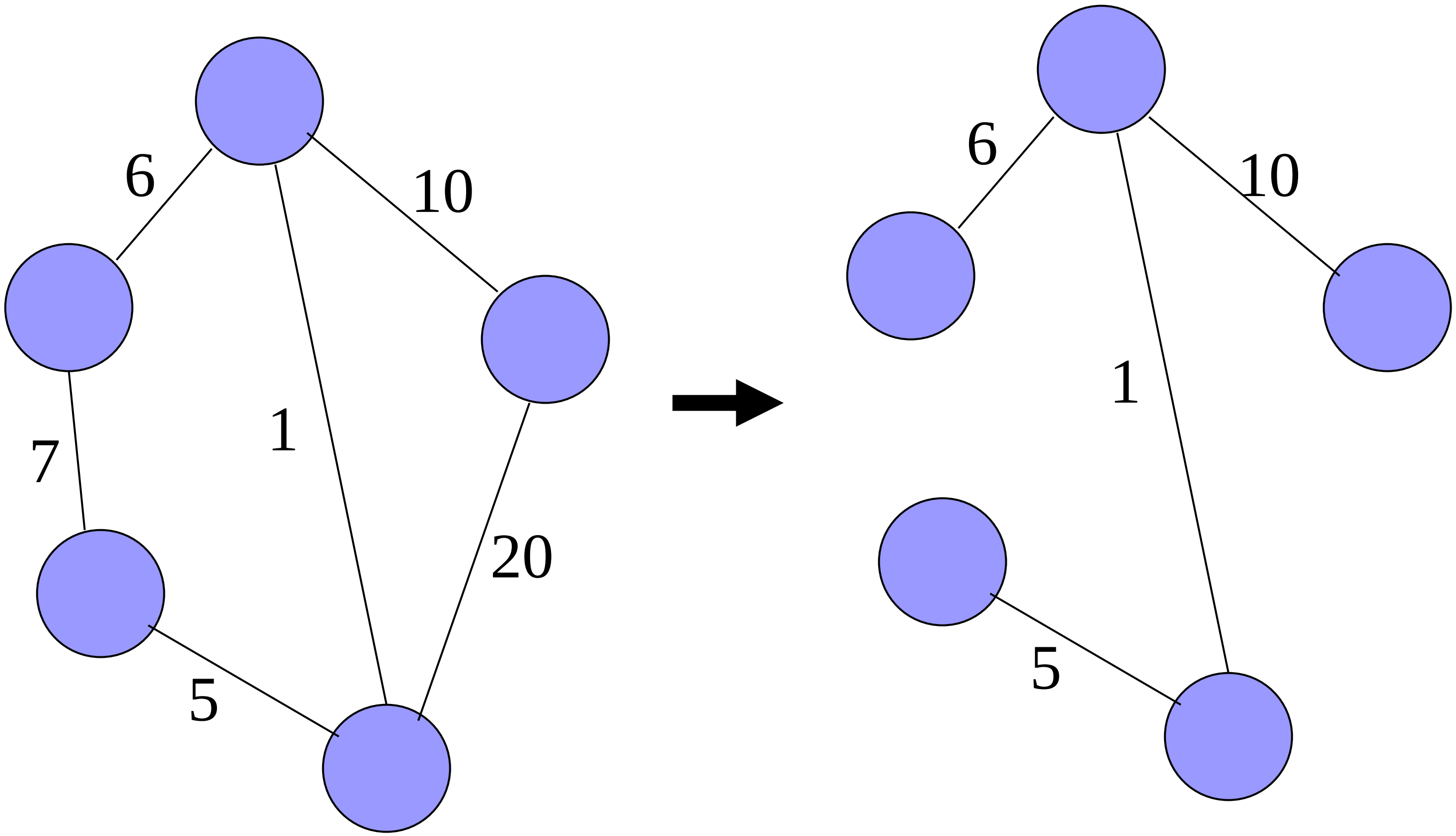
Spanning Tree

- Minimum number of edges to keep it connected
- If N vertices, spanning tree has N-1 edges



Minimum Spanning Tree (MST)

Spanning tree with minimum weight



Prim's Algorithm For Finding MST

1. All nodes are unselected, mark node v selected
2. For each node of graph,

   {

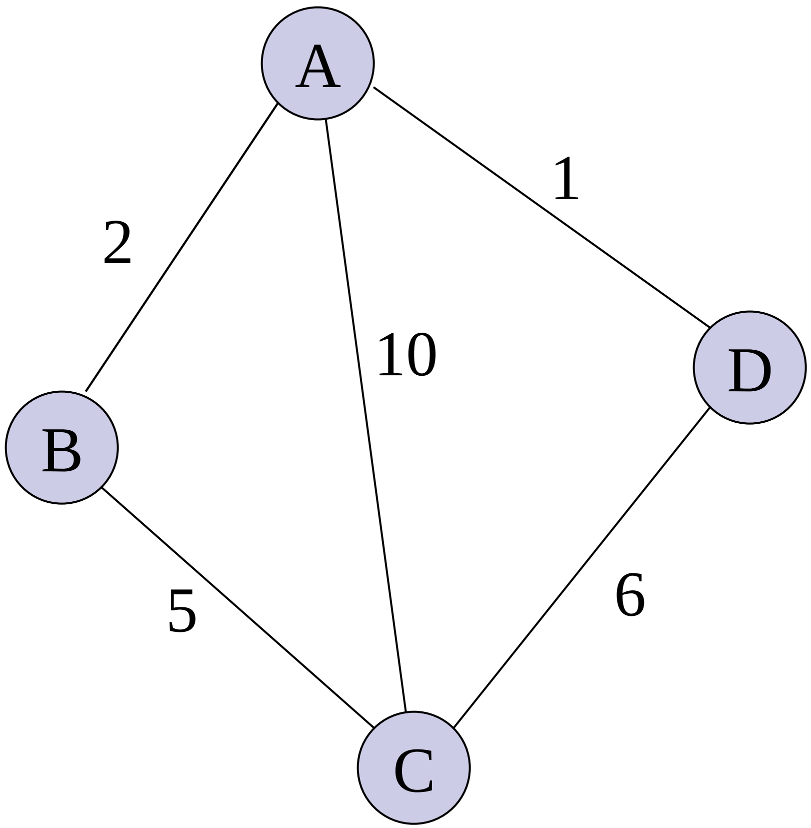
   Find the edge with minimum weight that connects an unselected node with a selected node

   Mark this unselected node as selected

   }

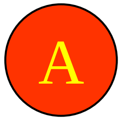
Example

- Find the MST of the following graph



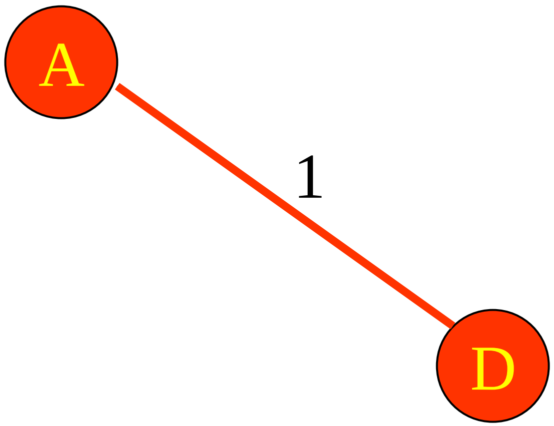
Demos of Prim's Algorithm

- Step 1: mark vertex A as selected



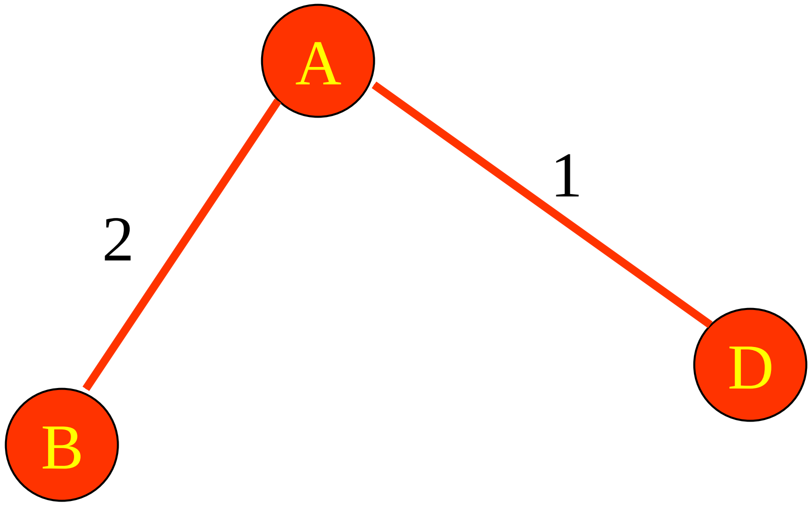
Demos of Prim's Algorithm

- Step 2: find the minimum weighted edge connected to vertex A, and mark the other vertex on this edge as selected.



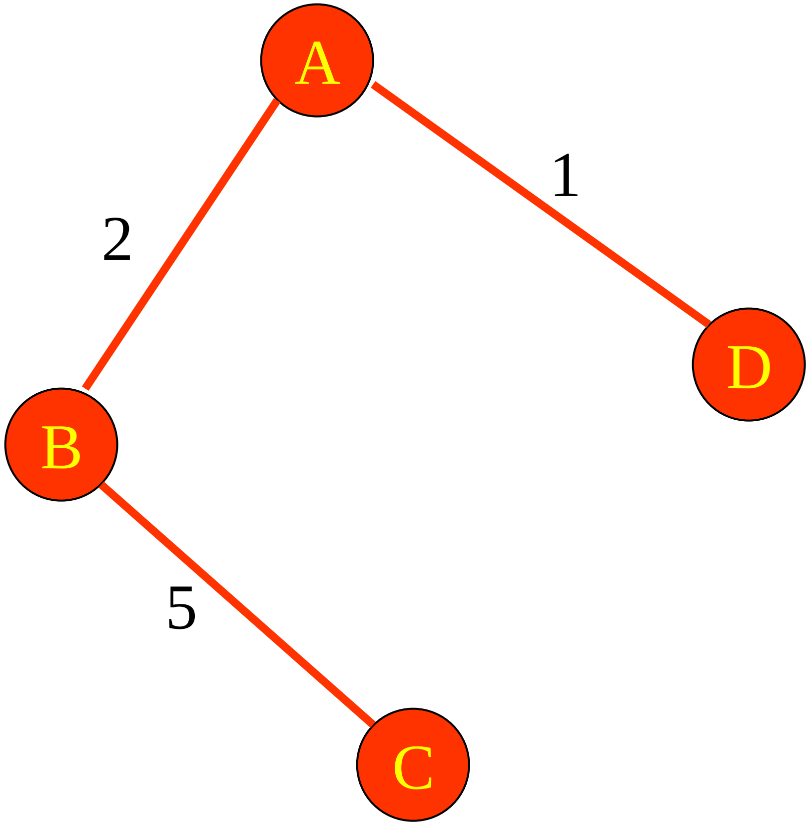
Demos of Prim's Algorithm

- Step 3: find the minimum weighted edge connected to vertices set { A, D } , and mark the other vertex on this edge as selected.



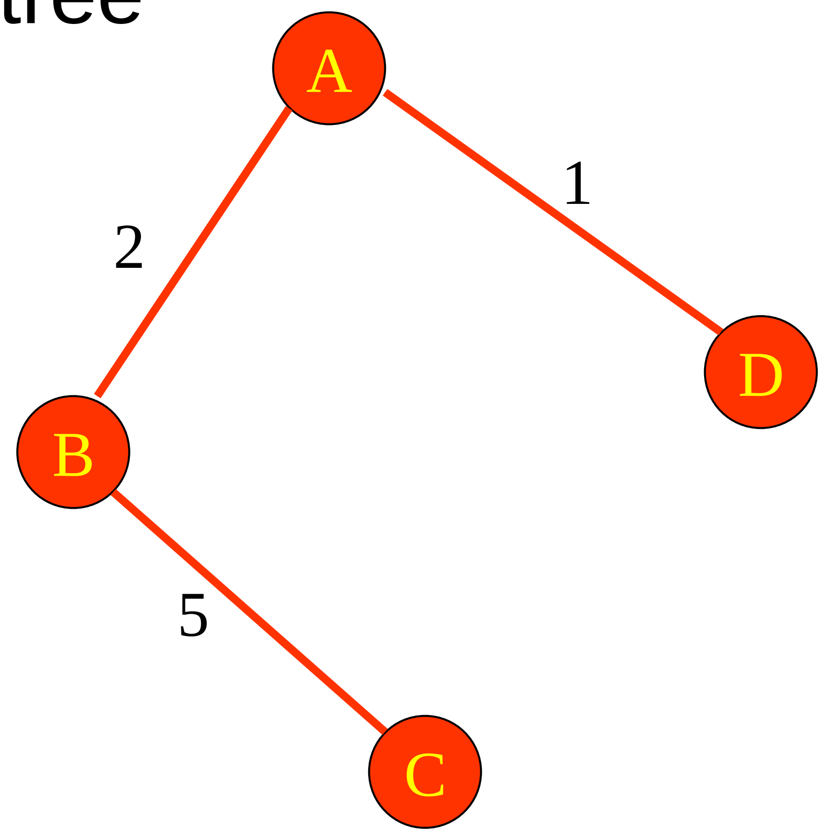
Demos of Prim’s Algorithm

- Step 4: find the minimum weighted edge connected to vertices set { A, D, B} , and mark the other vertex on this edge as selected.



Demos of Prim's Algorithm

- Step 5: All vertex are marked as selected, So we find the minimum spanning tree



Pseudo code for Prim's Alg.

- Minimum-Spanning-Tree-by-Prim(G, weight-function, source)
  - **for each** vertex *u* **in** graph **G**
    - **set** key **of** *u* **to** ∞
    - **set** parent **of** *u* **to** *nil*
  - **set** key **of** source vertex **to** *zero*
  - **enqueue all vertices to Q**
  - **while Q** is not *empty*
    - **extract** vertex *u* from **Q** *// u is the vertex with the lowest key that is in Q*
    - **for each** adjacent vertex *v* **of** *u* **do**
      - **if** (*v* is *still* in **Q**) and (weight-function(u, v) < key of *v*) **then**
        - **set** *u* **to** be parent **of** *v* *// in minimum-spanning-tree*
        - update *v*'s key to equal weight-function(u, v)

Kruskal’s Algorithm For Finding MST

1. All edges are unselected
2. Sort all edges and store them in set S
3. For each edge in S

{

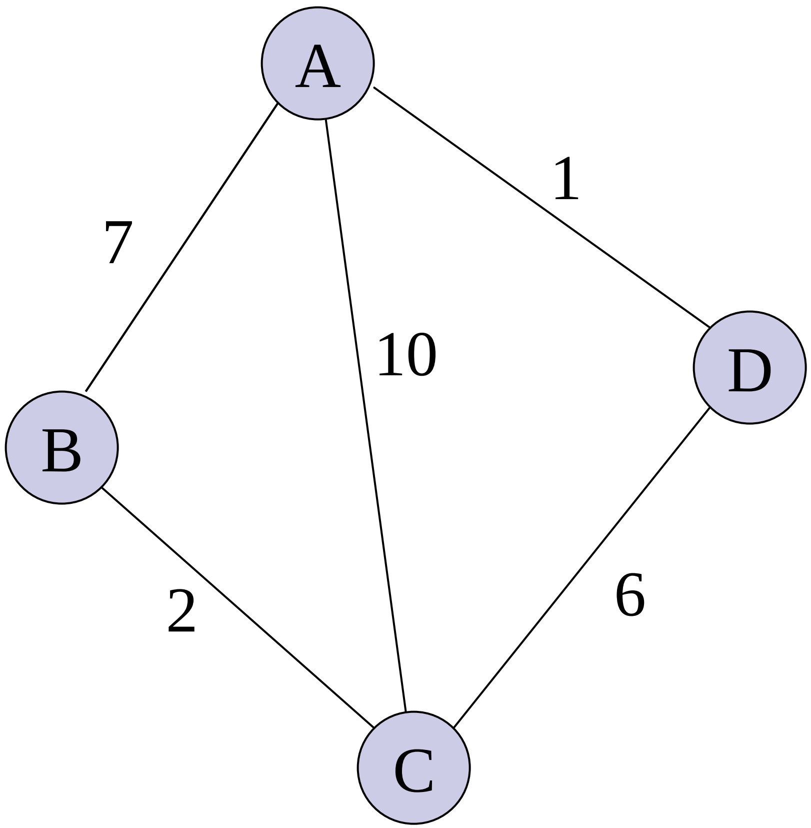
If adding the edge to MST does not form a circle, add this edge to MST;
Delete this edge from S;

If |edges in MST|=|nodes in graph|-1, exit;

}

Example

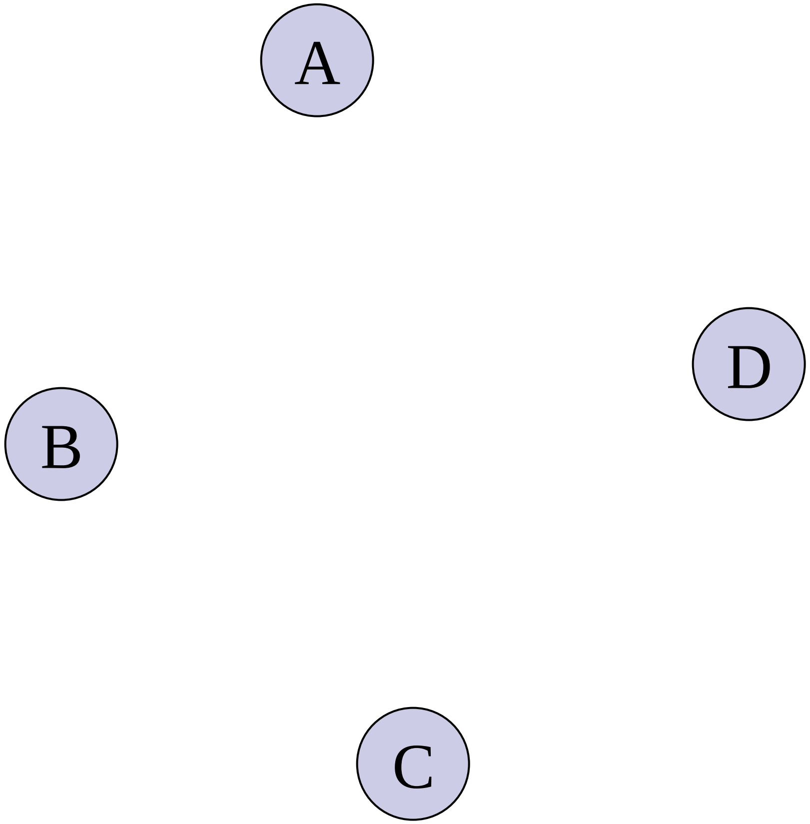
- Find the MST of the following graph



Demos of Kruskal's Algorithm

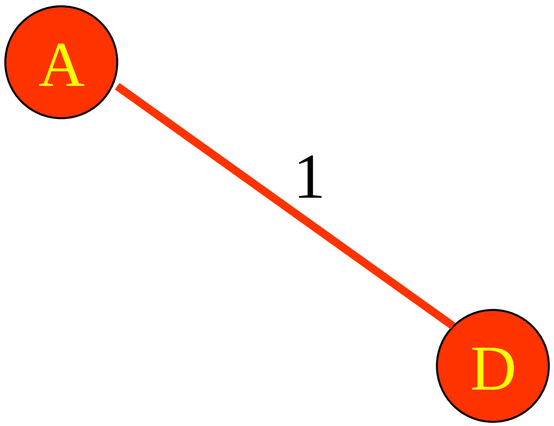
- Step 1: Initialize S and empty MST

S={(A,D)=1, (B,C=)2,
(C,D)=6, (A,B)=7,
(A,C)=10 }



Demos of Kruskal's Algorithm

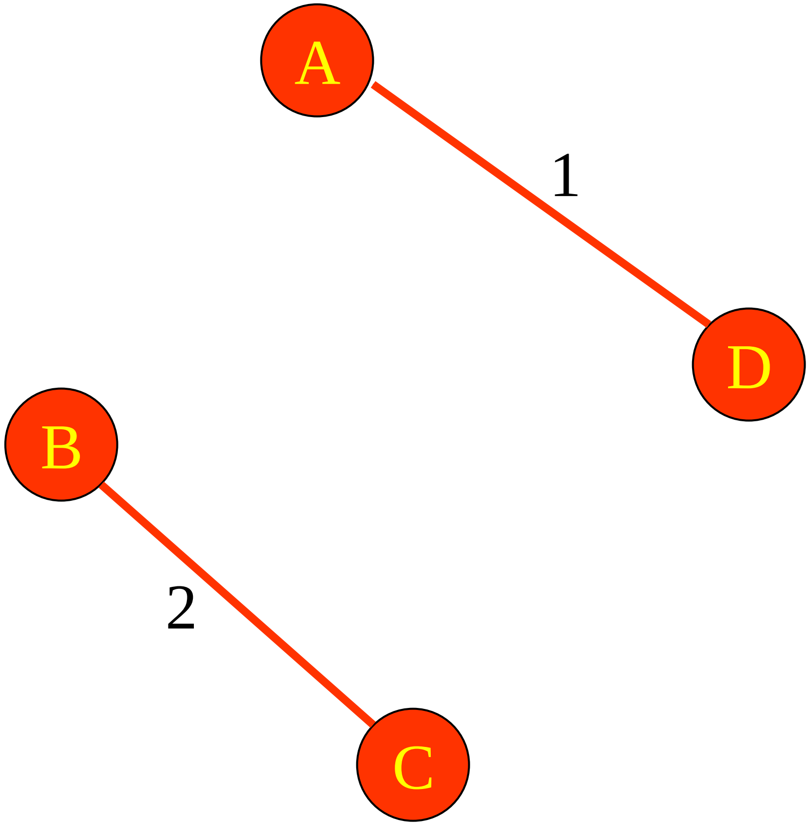
- Step 2: Add AD to MST, and delete it edge from S.



S={(B,C=)2, (C,D)=6, (A,B)=7, (A,C)=10 }

Demos of Kruskal's Algorithm

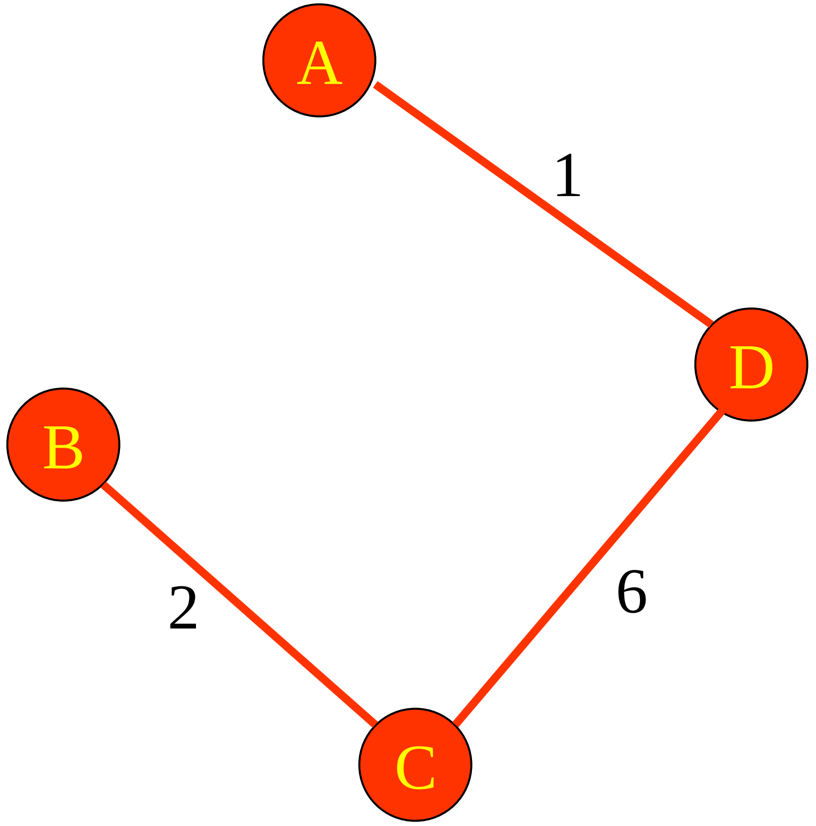
- Step 3: Add BC to MST, and delete it from S.



S={(C,D)=6, (A,B)=7, (A,C)=10 }

Demos of Kruskal's Algorithm

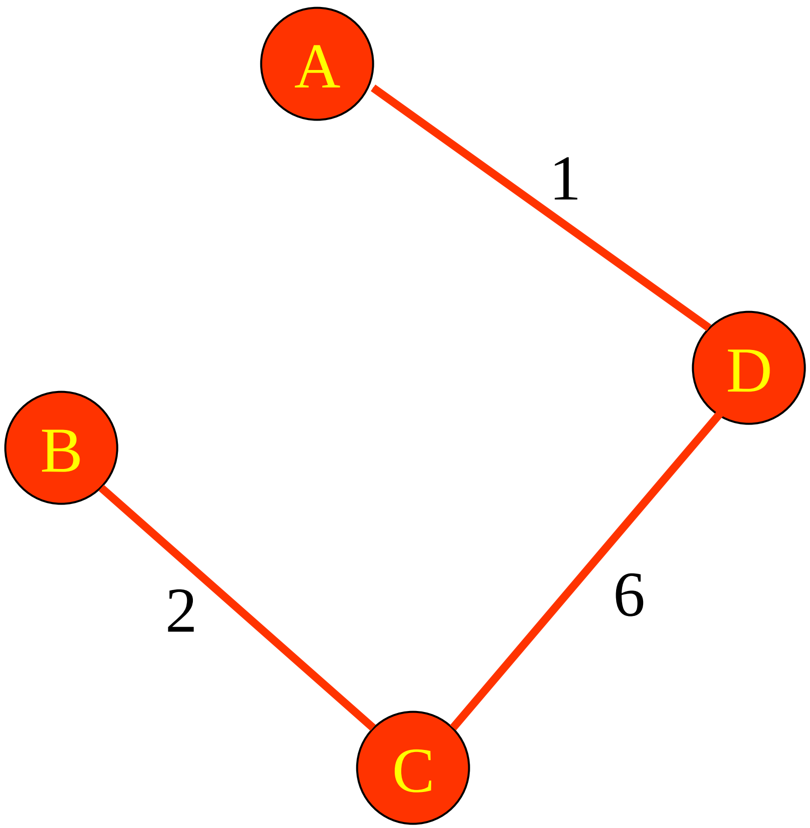
- Step 4: Add CD to MST, and delete it from S



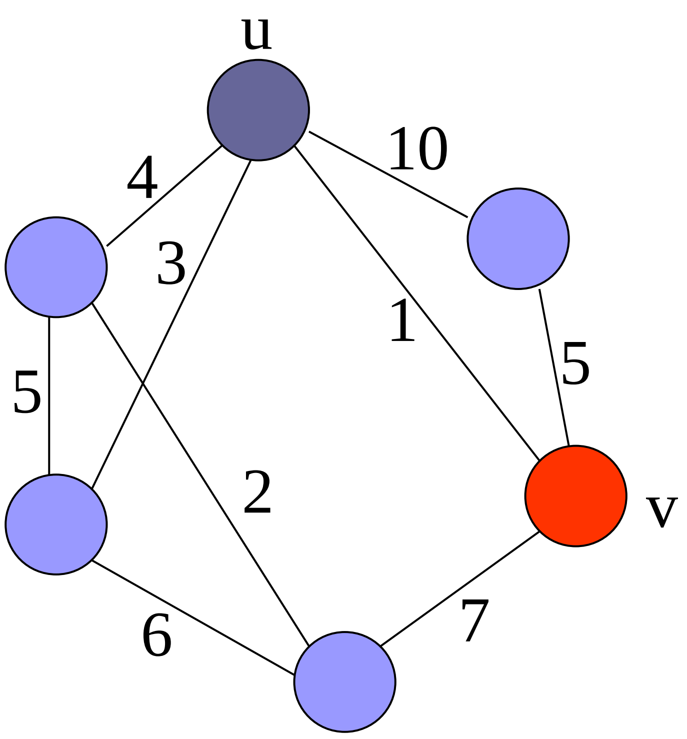
S={(A,B)=7, (A,C)=10}

Demos of Kruskal's Algorithm

- Step 5: Satisfy the exiting condition, So we find the minimum spanning tree



Shortest Path Problem



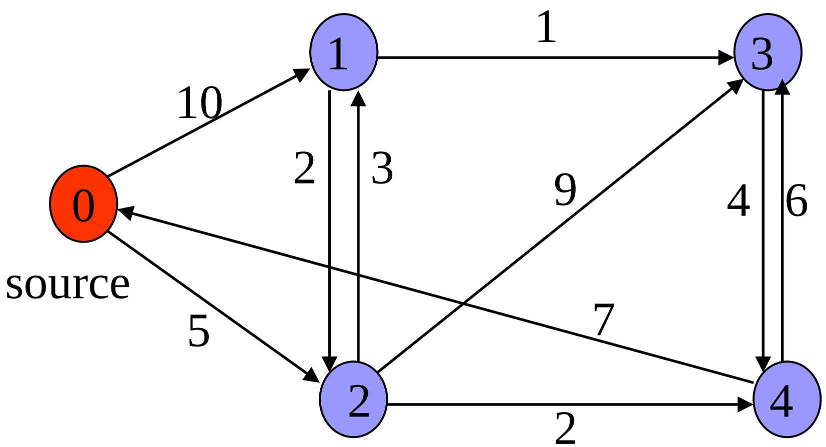
- Weight: cost, distance, travel time, hop …

Single Source Shortest Path Problem

- Single source shortest path problem
  - Find the shortest path to all other nodes
- Dijkstra's shortest path algorithm
  - Find shortest path greedily by updating distance to all other nodes
  - Not applicable for negative weights

Example – Dijkstra Algorithm

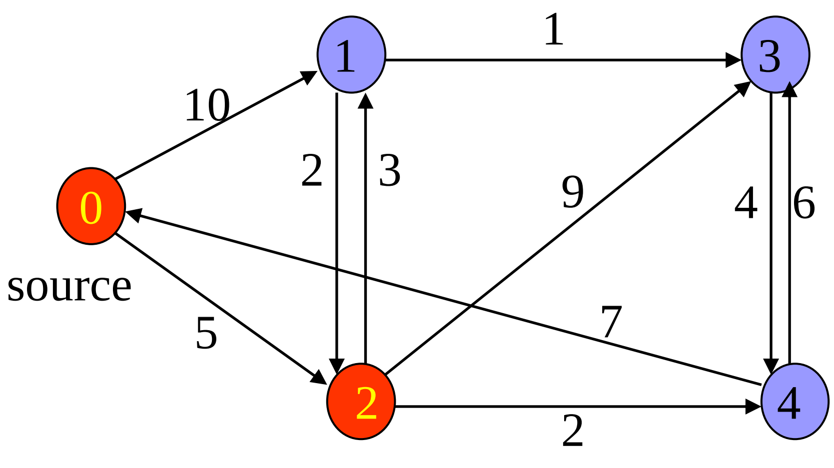
- Greedy Algorithm
- Assume all weight of edge >0



node	from node V_0 to other nodes			
V_1	10 (V_0)			
V_2	5 (V_0)			
V_3	∞ (V_0)			
V_4	∞ (V_0)			
best				

Example – Dijkstra Algorithm

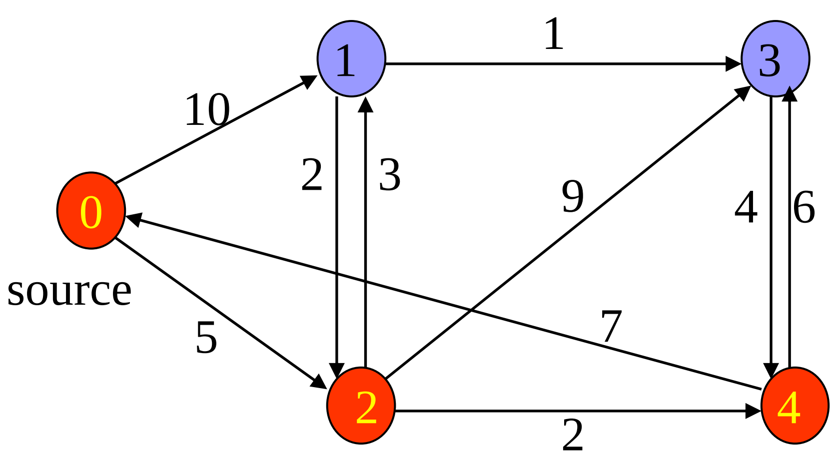
- step 1: find the shortest path to node 0
  - node 2 is selected



node	from node V_0 to other nodes			
V_1	10 (V_0)			
V_2	5 (V_0)			
V_3	∞ (V_0)			
V_4	∞ (V_0)			
best	V_2			

Example – Dijkstra Algorithm

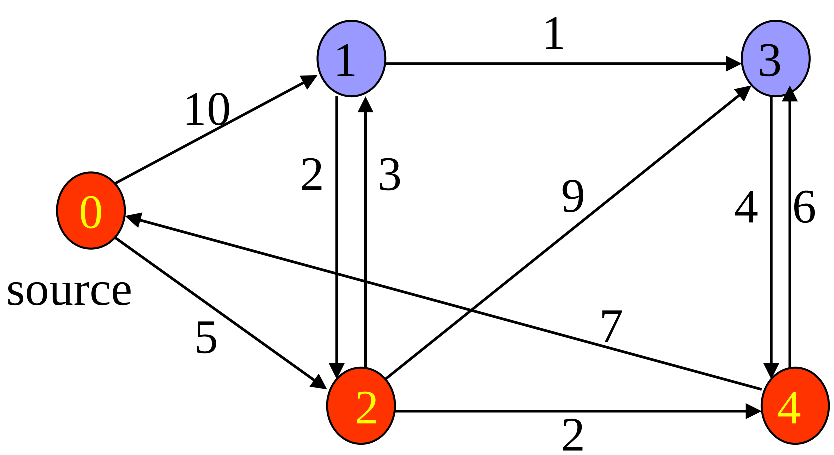
- step 2: recalculate the path to all other nodes
  - find the shortest path to node 0. Node 4 is selected



node	from node V_0 to other nodes			
V_1	10 (V_0)	8 (V_2)		
V_2	5 (V_0)			
V_3	∞ (V_0)	14 (V_2)		
V_4	∞ (V_0)	7 (V_2)		
best	V_2			

Example – Dijkstra Algorithm

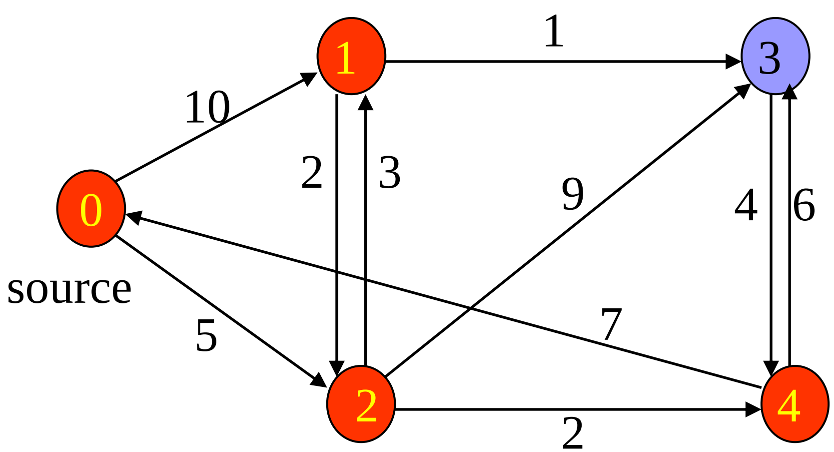
- step 2: recalculate the path to all other nodes
  - find the shortest path to node 0. Node 4 is selected



node	from node V_0 to other nodes			
V_1	10 (V_0)	8 (V_2)		
V_2	5 (V_0)			
V_3	∞ (V_0)	14 (V_2)		
V_4	∞ (V_0)	7 (V_2)		
best	V_2	V_4		

Example – Dijkstra Algorithm

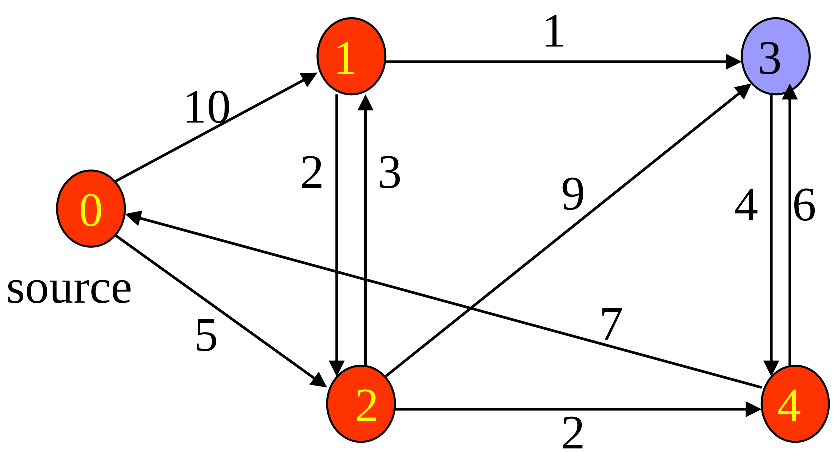
- step 3: recalculate the path to all other nodes
  - find the shortest path to node 0. node 1 is selected



node	from node V_0 to other nodes			
V_1	10 (V_0)	8 (V_2)	8 (V_2)	
V_2	5 (V_0)			
V_3	∞ (V_0)	14 (V_2)	13 (V_4)	
V_4	∞ (V_0)	7 (V_2)		
best	V_2	V_4		

Example – Dijkstra Algorithm

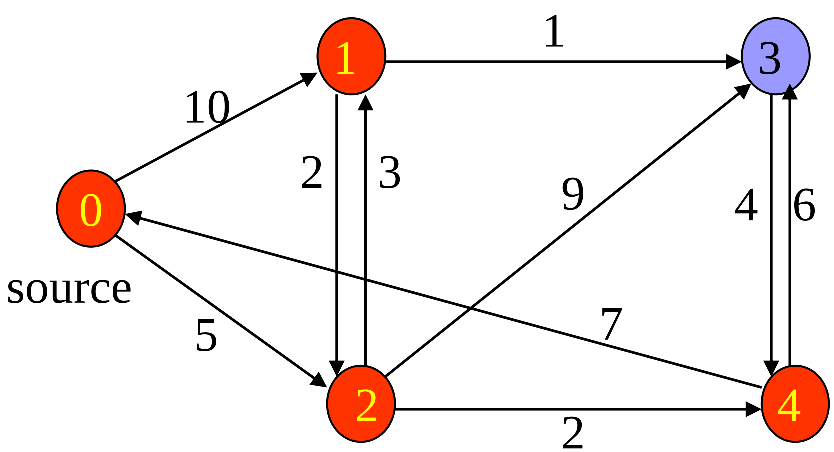
- step 3: recalculate the path to all other nodes
  - find the shortest path to node 0. node 1 is selected



node	from node V_0 to other nodes			
V_1	10 (V_0)	8 (V_2)	8 (V_2)	
V_2	5 (V_0)			
V_3	∞ (V_0)	14 (V_2)	13 (V_4)	
V_4	∞ (V_0)	7 (V_2)		
best	V_2	V_4	V_1	

Example – Dijkstra Algorithm

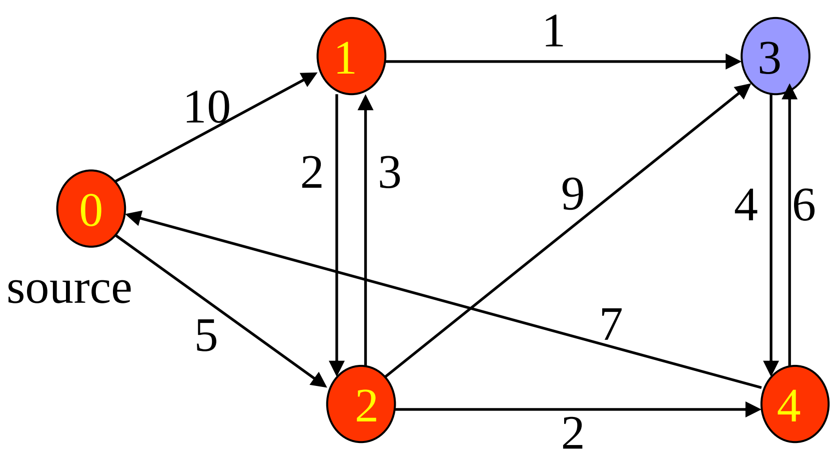
- step 3: recalculate the path to all other nodes
  - find the shortest path to node 0. node 1 is selected



node	from node V_0 to other nodes			
V_1	10 (V_0)	8 (V_2)	8 (V_2)	
V_2	5 (V_0)			
V_3	∞ (V_0)	14 (V_2)	13 (V_4)	9 (V_1)
V_4	∞ (V_0)	7 (V_2)		
best	V_2	V_4	V_1	

Example – Dijkstra Algorithm

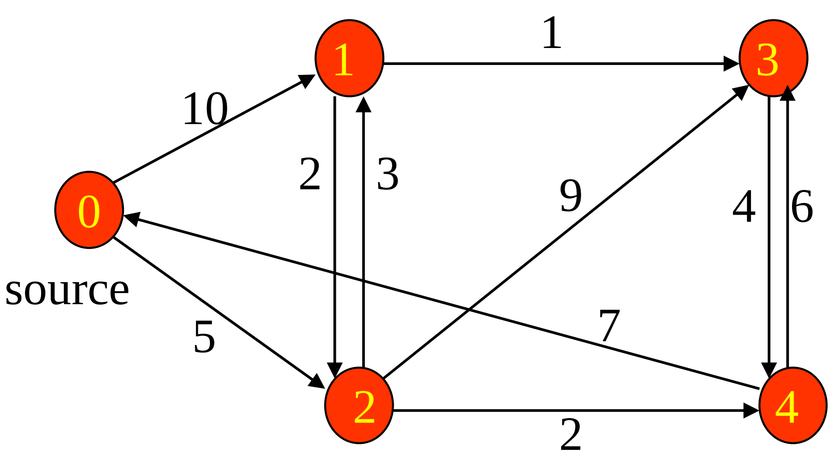
- step 3: recalculate the path to all other nodes
  - find the shortest path to node 0. node 1 is selected



node	from node V_0 to other nodes			
V_1	10 (V_0)	8 (V_2)	8 (V_2)	
V_2	5 (V_0)			
V_3	∞ (V_0)	14 (V_2)	13 (V_4)	9 (V_1)
V_4	∞ (V_0)	7 (V_2)		
best	V_2	V_4	V_1	V_3

Example – Dijkstra Algorithm

- Now we get all shortest paths to each node



node	from node V_0 to other nodes			
V_1	10 (V_0)	8 (V_2)	8 (V_2)	
V_2	5 (V_0)			
V_3	∞ (V_0)	14 (V_2)	13 (V_4)	9 (V_1)
V_4	∞ (V_0)	7 (V_2)		
best	V_2	V_4	V_1	V_3

Dijkstra Algorithm

Mark source node selected

Initialize all distances to Infinite, source node distance to 0.

Make source node the current node.

While (there is unselected node)

{

Expand on current node

Update distance for neighbors of current node

Find an unselected node with smallest distance, and make it current node and mark this node selected

}

Pseudo-code For Dijkstra's Algorithm

```
function Dijkstra(G, w, s)
    for each vertex v in V[G] // Initializations
        d[v] := infinity
        previous[v] := undefined
    d[s] := 0
    S := empty set
    Q := V[G]
    while Q is not an empty set // The algorithm itself
        u := Extract_Min(Q)
        S := S union {u}
        for each edge (u,v) outgoing from u
            if d[u] + w(u,v) < d[v] // Relax (u,v)
                d[v] := d[u] + w(u,v)
                previous[v] := u
```