Nonlinear Partial Differential Equations.

- Exterior Domain Problem.

$$\begin{cases} -\Delta w + (w\cdot\nabla) w + \nabla p = 0 & \text{in } \Omega \subseteq \mathbb{R}^2 \\ \nabla\cdot w = 0 & \text{in } \Omega \\ w\big|_{\partial\Omega} = 0 \\ w(z) \to w_\infty = \lambda e_1 & \text{as } |z| \to \infty \end{cases}$$

- Incompressible Navier–Stokes equations in three-dimensions.

$$\begin{cases} \partial_t v - \Delta v + (v\cdot\nabla) v + \nabla p = 0 & \text{in } D \subseteq \mathbb{R}^3 \times \mathbb{R} \\ \nabla\cdot v = 0 & \text{in } D. \end{cases}$$

- Monge–Ampère Equation.

$$\det D^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^n$$

for an unknown function $u: \Omega \to \mathbb{R}$

Main Problems
- Existence results for solutions.
- Localization theorems
- Geometry of sublevel sets
- Regularity results in the interior and at the boundary
- Classification of Global solutions
- Spectral theory.

- Sine–Gordon equation

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = 0$$

One of the first PDEs found to admit a family of breathers., which is given explicitly by

$$u^\omega(x,t) = 4\arctan\left(\frac{m}{\omega}\,\frac{\sin(\omega t)}{\cosh(mx)}\right), \quad m,\omega > 0,\ m^2+\omega^2 = 1.$$

- Klein–Gordon equations.

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0,$$

where the nonlinearity f satisfies

$f(u)$ is a real-analytic odd function and $f(u) = O(u^5)$ near 0.

* K–G equation, general form: $(\partial_t^2 v - \partial_x^2)\, v + F(v) = 0$, where $F(v)$

is a smooth real valued odd function $F(v)$ with $F'(0) > 0$ and $F'''(0) < 0$.

- Burgers Equation

$$u_t = uu_x = D u_{xx}.$$

The above equation is nonlinear due to the product of uu_x. This elleptic equation is a simple case of the liquid flow

- Diffusion-Reaction Equation (Fisher Equation).

$$u_t = D u_{xx} + f(u),$$

wherein $f(u)$ is a polynomial of u, which denotes the reaction part of the equation, the diffusion part is $D u_{xx}$.

- Compressible Navier-Stokes equations

The compressible fluid dynamics equations describe a fluid flow whose expression is given as:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad \rho(\partial_t v + v \cdot \nabla v) = -\nabla p + \eta \Delta v + (\zeta + \eta/3)\nabla(\nabla \cdot v),$$
$$\partial_t(\varepsilon + \rho v^2/2) + \nabla \cdot [(p + \varepsilon + \rho v^2/2) v - v \cdot \sigma'] = 0,$$

where ρ is the mass density, v is the fluid velocity, p is the gas pressure, ε is an internal energy describle by the equation of state, σ' is the viscous stress tensor, η and ζ are shear and bulk viscosity, respectively.

This equation can be applied to describe aerodynamics around airplane wings and interstellar gas dynamics.

- Shallow-water Equations

$$\partial_t h + \nabla h u = 0, \quad \partial_t h u + \nabla \left(u^2 h + \frac{1}{2} g_r h^2\right) = -g_r h \nabla b.$$

wherein, $u = u, v$ being the velocities in the horizontal and vertical direction, h describing the water depth, and b describing a spatially varying bathymetry.