

# A NOVEL APPROACH FOR PROVING THE EXISTENCE OF WEAK SOLUTIONS TO STATIONARY NAVIER-STOKES EQUATIONS IN THE EXTERIOR DOMAIN

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**ABSTRACT.** In this note, we prove that when the boundary satisfies uniform  $C^\infty$ -regularity condition, one can show the existence of the extension solution to the stationary Navier-Stokes equations in the exterior domain, shedding new lights on Leray's invading domains method [1].

## 1. INTRODUCTION

Consider the following exterior domain problem in the 2-dimensional case,

$$(1) \quad \begin{cases} -\Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{v}(x) \rightarrow \mathbf{v}_\infty = \lambda \mathbf{e}_1 & \text{as } |x| \rightarrow \infty, \end{cases}$$

herein,  $\Omega = \mathbb{R}^2 \setminus \overline{\Omega_0}$ ,  $\Omega_0$  is open bounded with sufficiently smooth boundary (that is,  $\partial\Omega$  satisfies uniform  $C^\infty$ -regularity condition) and  $0 \in \Omega_0$ .  $\lambda \geq 0$  denotes the Reynold number,  $\mathbf{e}_1 = (1, 0)$  is the unit vector along  $x$ -axis. The above equation governs stationary flows around rigid obstacles<sup>1</sup>.

In 1933, Leray first proved the existence of weak solutions in two-dimensional unbounded domains for vanishing flux through the boundaries [1]. In [3], Russo proved the case for small fluxes. In [4] (Theorem 2.6), the authors discussed the existence and uniqueness of a steady weak solution to the Navier-Stokes equation in the whole plane. In [5], Guillod et al. proved the existence and uniqueness of the D-solution to equation (1)<sub>1,2,4</sub> under small forces. In this paper, we explore a new way to establish the existence results of weak solutions to the exterior domain problem (1)<sub>1,2,3</sub> using the extension-based approach, shedding new lights on Leray's invading domains method [1].

Specifically, denote  $\Omega' = \overline{\Omega} \cap \mathcal{B}_R$ ,  $\mathcal{B}_R$  represents the disc in  $\mathbb{R}^2$  centered at 0 with radius  $R > 0$ , such that  $\partial\Omega \cap \{|x| = R\} = \emptyset$ . Since  $\partial\Omega$  is sufficiently smooth ( $\partial\Omega$  satisfies uniform  $C^\infty$ -regularity condition), both  $\Omega$  and  $\Omega'$  satisfy the cone condition.

Now consider the following sub-problem of the exterior domain problem (1) on  $\Omega'$ .

$$(2) \quad \begin{cases} -\Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0 & \text{in } \Omega' = \overline{\Omega \cap \mathcal{B}_R}, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega'. \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{v} = \mathbf{v}_\infty & \text{for } |x| = R. \end{cases}$$

Note that, the main difference between system (2) and system (3.1.1) in [6] is that we add a constant boundary condition on the spherical surface  $|x| = R$ .

Given problem (2), we then present the definition of the weak solution to equation (2)<sub>1,2,3</sub> on the bounded domain  $\Omega'$ , which can be stated as follows.

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2020 *Mathematics Subject Classification.* 76D03.

*Key words and phrases.* Stationary Navier-Stokes equations, two-dimensional exterior domains, existence.

<sup>1</sup>According to [2], here we mainly consider the case which describes a flow around the obstacle.

**Definition 1** (Weak solution on bounded domains [6]). *Function  $\mathbf{v} \in J_0^{1,2}(\Omega')$  is a weak solution to the exterior domain problem (2)<sub>1,2,3</sub> if  $\mathbf{v}$  satisfies the following integral equation*

$$(3) \quad \int_{\Omega'} \nabla \mathbf{v} \cdot \nabla \eta - (\mathbf{v} \cdot \nabla) \eta \cdot \mathbf{v} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega').$$

Due to the Poincaré inequality,  $H_0^{1,2}(\Omega') = J_0^{1,2}(\Omega')$ . Hence, using Theorem 3.1.2 in [6], we can immediately obtain the existence of velocity  $\mathbf{v}$  on bounded domains.

**Theorem 1.** *There exists at least one weak solution  $\mathbf{v} \in J_0^{1,2}(\Omega')$  to equation (2)<sub>1,2,3</sub> with finite Dirichlet integral.*

Next, we prove the existence of the weak solution to problem (1)<sub>1,2,3</sub>, which is essentially an extension of solution  $\mathbf{v}$  on the bounded domain  $\Omega'$ . For this, we propose the following extension theorem, which is the main result of our paper.

**Theorem 2** (Extension theorem). *For the weak solution  $\mathbf{v} \in J_0^{1,2}(\Omega')$  to equation (2)<sub>1,2,3</sub> satisfying*

$$\int_{\Omega'} \nabla \mathbf{v} \cdot \nabla \eta dx - \int_{\Omega'} (\mathbf{v} \cdot \nabla) \eta \cdot \mathbf{v} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega'),$$

*there exists a linear bounded extension operator  $P : J_0^{1,2}(\Omega') \rightarrow J_0^{1,2}(\Omega)$  such that  $\forall \mathbf{v} \in J_0^{1,2}(\Omega')$ :*

- $P\mathbf{v} = \mathbf{v}$  a.e. in  $\Omega'$ ;
- $\|P\mathbf{v}\|_{J_0^{1,2}(\Omega)} \lesssim_{\Omega'} \|\mathbf{v}\|_{J_0^{1,2}(\Omega')}$ ;
- $P\mathbf{v}|_{\partial\Omega} = \mathbf{0}$ ,

$P\mathbf{v}$  represents the weak solution to (1)<sub>1,2,3</sub> on the whole space  $\Omega$ , satisfying

$$\int_{\Omega} \nabla(P\mathbf{v}) \cdot \nabla \eta dx - \int_{\Omega} (P\mathbf{v} \cdot \nabla) \eta \cdot P\mathbf{v} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega),$$

which is in the class of the D-solution.

Afterwards, we can prove the existence of the pressure  $p$  for the extension solution by De Rham's theory [2].

**Organization of the paper.** The outline of the paper can be summarized as follows. Section 2 introduces the preliminaries and notations in the paper. The rest of the article is devoted to prove the existence of weak solutions to stationary Navier-Stokes equations in the exterior domain using extension-based methods.

## 2. NOTATIONS AND PRELIMINARIES

**Notations.** Let  $\Omega \subset \mathbb{R}^2$  be an exterior domain,  $C_c^\infty(\Omega)$  represents the infinitely differentiable functions in  $\Omega$  with compact support. Denote the closure of  $C_c^\infty(\Omega)$  in the  $W^{k,p}(\Omega)$  norm as  $W_0^{k,p}(\Omega)$ , for  $k \in \mathbb{N}$  and  $p \in [1, +\infty]$ . Let  $J_0^\infty(\Omega) = \{\mathbf{v} \in C_c^\infty(\Omega) : \operatorname{div} \mathbf{v} = 0\}$ . Then, for  $r \geq 1$ , denote  $J_0^{1,r}(\Omega)$  as the closure of  $J_0^\infty(\Omega)$  in the  $W^{1,r}(\Omega)$  norm. Obviously,  $J_0^{1,r}(\Omega)$  is a Banach space. In addition, for bounded domains, we define  $H_0^{1,r}(\Omega)$  as the closure of  $J_0^\infty(\Omega)$  in the  $D_0^{1,r}(\Omega)$  norm, i.e.,

$$\|\mathbf{v}\|_{D_0^{1,r}(\Omega)} = \|\nabla \mathbf{v}\|_{L^r(\Omega)} = \left( \int_{\Omega} |\nabla \mathbf{v}(x)|^r dx \right)^{1/r},$$

where  $\mathbf{v} = (v_1, v_2)$ ,  $|\nabla \mathbf{v}(x)|^r = \sum_{j=1}^2 \sum_{k=1}^2 |\frac{\partial v_j(x)}{\partial x_k}|^r$ .  $\hookrightarrow$  represents the Sobolev imbedding.

**The Invading Domains Method.** In the seminal work [1], Leray introduced an approach called the *invading domains method* [7]. Specifically, denote  $\mathbf{v}_k$  as the solution to the problem

$$(4) \quad \begin{cases} -\Delta \mathbf{v}_k + (\mathbf{v}_k \cdot \nabla) \mathbf{v}_k + \nabla p_k = 0 & \text{in } \Omega \cap \mathcal{B}_{R_k}, \\ \operatorname{div} \mathbf{v}_k = 0 & \text{in } \Omega \cap \mathcal{B}_{R_k}, \\ \mathbf{v}_k = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{v}_k = \mathbf{v}_\infty & \text{for } |x| = R_k, \end{cases}$$

on the intersection of  $\Omega$  with the disk  $\mathcal{B}_{R_k}$  of radius  $R_k$  (for all  $k = 1, 2, \dots$ ,  $\partial\Omega \cap \partial\mathcal{B}_{R_k} = \emptyset$ ), whose existence he proved before, satisfying the finite Dirichlet integral condition

$$(5) \quad \int_{\Omega \cap \mathcal{B}_{R_k}} |\nabla \mathbf{v}_k|^2 \leq C,$$

herein,  $C$  is a positive constant depending on  $\partial\Omega$  and  $\mathbf{v}_\infty$ .

Indeed, based on the results in inequality (5), one can extract a subsequence  $\{\mathbf{v}_{k_n}\}_{n=1}^\infty$  from  $\{\mathbf{v}_k\}_{k=1}^\infty$  such that  $\mathbf{v}_{k_n}$  weakly converge (i.e., the convergence is uniform on every bounded set) to a solution  $\mathbf{v}_L$  to problem (1) with finite Dirichlet integral [8]. The solution  $\mathbf{v}_L$  is the so-called Leray solution [9]. After obtaining  $\mathbf{v}_L$ , the pressure  $p$  can be solved by De Rham's theory [2].

### 3. PROOF OF THEOREM 2

Obviously,  $\partial\Omega'$  is sufficiently smooth,  $\Omega'$  is bounded. Besides, we note that  $J_0^{1,2} \hookrightarrow W_0^{1,2} \hookrightarrow L^2$ . Hence, for any  $\mathbf{v} \in J_0^{1,2}$ ,  $\mathbf{v}, \nabla \mathbf{v} \in L^2$  and the imbedding holds for both  $\Omega', \Omega$ .

Then, for any  $x$  on  $\partial\Omega'$ , since  $\partial\Omega'$  is not necessarily flat near  $x$ . We select a ball  $U$  centered at  $x$  ( $\partial\Omega_0 \cap \partial U = \emptyset$ ) and denote by  $U^+ = U \cap \Omega'$ ,  $U^- = U \cap (\Omega \setminus \Omega')$ . Obviously, according to the existence of  $\mathbf{v}$  on bounded domain  $\Omega'$ , we have  $\mathbf{v} \in J_0^{1,2}(U^+)$ .

Then, one can find a diffeomorphism  $\Phi$ , such that  $\operatorname{div} \mathbf{v} \circ \Phi = 0$  and

$$\begin{aligned} \Phi(U \cap \Omega') &= \{y \in \tilde{U} : y_n \geq 0\}, \\ \Phi(U \cap \partial\Omega') &= \{y \in \tilde{U} : y_n = 0\}, \end{aligned}$$

wherein,  $\tilde{U}$  denotes the ball intersecting with the hyperplane  $\{y \in \mathbb{R}^2 : y_n = 0\}$ .

Denote  $\tilde{U}^+ = \tilde{U} \cap \{y \in \mathbb{R}^2 : y_n \geq 0\}$ ,  $\tilde{U}^- = \tilde{U} \cap \{y \in \mathbb{R}^2 : y_n \leq 0\}$ . Then, define the extension of  $\mathbf{v}(y)$  on  $\tilde{U}$  as

$$(6) \quad \bar{\mathbf{v}}(y) = \begin{cases} \mathbf{v}(y) & , y \in \tilde{U}^+, \\ \mathbf{0} & , y \in \tilde{U}^-, \end{cases}$$

satisfying

$$\int_{\tilde{U}} \nabla \bar{\mathbf{v}} \cdot \nabla \eta \, dx - \int_{\tilde{U}} (\bar{\mathbf{v}} \cdot \nabla) \eta \cdot \bar{\mathbf{v}} \, dx = 0, \quad \forall \eta \in J_0^{1,2}(\tilde{U}).$$

Since  $\mathbf{v} \in J_0^{1,2}(\tilde{U}^+)$ , there exists  $\{\mathbf{v}_m\}_{m=1}^\infty \subset J_0^\infty(\tilde{U}^+)$ , such that  $\mathbf{v}_m \rightarrow \mathbf{v}(J_0^{1,2}(\tilde{U}^+))$ .

Therefore, the extension  $\bar{\mathbf{v}}_m$  for the sequence  $\mathbf{v}_m$  on  $\tilde{U}$  defined similar to equation (6) also belongs to  $C_c^\infty(\tilde{U})$ .

Besides, the norm of  $\bar{\mathbf{v}}_m$  in  $J_0^{1,2}(\tilde{U})$  is the same as the norm of  $\mathbf{v}_m$  in  $J_0^{1,2}(\tilde{U}^+)$ . Hence,  $\bar{\mathbf{v}}_m$  is a Cauchy sequence in  $J_0^{1,2}(\tilde{U})$ ,  $\bar{\mathbf{v}}_m \rightarrow \bar{\mathbf{v}}(J_0^{1,2}(\tilde{U}))$ . Therefore, we have  $\bar{\mathbf{v}}(y) \in J_0^{1,2}(\tilde{U})$ .

Meanwhile,  $\bar{\mathbf{v}}(y)$  has finite Dirichlet integral, that is,

$$\|\bar{\mathbf{v}}(y)\|_{J_0^{1,2}(\tilde{U})} = \|\bar{\mathbf{v}}(y)\|_{H_0^{1,2}(\tilde{U})} = \|\nabla \bar{\mathbf{v}}(y)\|_{L^2(\tilde{U})} \lesssim \|\mathbf{v}(y)\|_{J_0^{1,2}(\tilde{U}^+)}.$$

Now, we map  $\tilde{U}$  back to the original shape using  $\Phi^{-1}$ , then we have

$$\bar{\mathbf{v}}(x) = \begin{cases} \mathbf{v}(x) & , x \in U^+, \\ \mathbf{0} & , x \in U^-, \end{cases}$$

$\bar{\mathbf{v}} \in J_0^{1,2}(U)$ , satisfying

$$\int_U \nabla \bar{\mathbf{v}} \cdot \nabla \eta \, dx - \int_U (\bar{\mathbf{v}} \cdot \nabla) \eta \cdot \bar{\mathbf{v}} \, dx = 0, \quad \forall \eta \in J_0^{1,2}(U)$$

and the finite Dirichlet integral condition

$$(7) \quad \|\bar{\mathbf{v}}(x)\|_{J_0^{1,2}(U)} = \|\nabla \bar{\mathbf{v}}(x)\|_{L^2(U)} \lesssim \|\mathbf{v}(x)\|_{J_0^{1,2}(U^+)} = \|\mathbf{v}(x)\|_{J_0^{1,2}(\Omega')}.$$

Afterwards, since  $\partial\Omega'$  is compact, there exists finite points  $\{x_i\}_{i=1}^N \in \partial\Omega'$ , open sets  $\{U_i\}_{i=1}^N$  and extensions  $\{\bar{\mathbf{v}}_i\}_{i=1}^N$  of  $\mathbf{v}$  on domain  $U_i$ , such that  $\partial\Omega' \subset \bigcup_{i=1}^N U_i$  and

$$\int_{U_i} \nabla \bar{\mathbf{v}}_i \cdot \nabla \eta dx - \int_{U_i} (\bar{\mathbf{v}}_i \cdot \nabla) \eta \cdot \bar{\mathbf{v}} dx = 0, \quad \forall \eta \in J_0^{1,2}(U_i).$$

Take  $U_0 \subset\subset \Omega'$  such that  $\Omega' \subset \bigcup_{i=0}^N U_i$  and let  $\{\xi_i\}_{i=0}^N$  be an associated partition of unity. Let  $\bar{\mathbf{v}} = \sum_{i=0}^N \xi_i \bar{\mathbf{v}}_i$ , where  $\bar{\mathbf{v}}_0 = \mathbf{v}$ . Using inequality (7), we have the following bound

$$(8) \quad \|\bar{\mathbf{v}}\|_{J_0^{1,2}(\Omega)} \lesssim_{\Omega'} \|\mathbf{v}\|_{J_0^{1,2}(\Omega')}.$$

Besides, the integral equation still holds

$$\int_{\Omega} \nabla \bar{\mathbf{v}} \cdot \nabla \eta dx - \int_{\Omega} (\bar{\mathbf{v}} \cdot \nabla) \eta \cdot \bar{\mathbf{v}} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega).$$

Finally, take  $P\mathbf{v} = \bar{\mathbf{v}}$  and recall that  $\mathbf{v} \in C_c^\infty(\Omega')$ . For any sequence  $\{\mathbf{v}_m\}_{m=1}^\infty \subset C_c^\infty(\Omega')$  converging to  $\mathbf{v}$  in  $J_0^{1,2}(\Omega')$ . Then, using the linearity of  $E$  and inequality (8), the following inequality holds,

$$\|P\mathbf{v}_j - P\mathbf{v}_k\|_{J_0^{1,2}(\Omega)} \lesssim \|\mathbf{v}_j - \mathbf{v}_k\|_{J_0^{1,2}(\Omega')}.$$

Hence  $\{P\mathbf{v}_m\}_{m=1}^\infty$  is a Cauchy sequence converging to  $P\mathbf{v}$ , satisfying

$$\int_{\Omega} \nabla(P\mathbf{v}) \cdot \nabla \eta dx - \int_{\Omega} (P\mathbf{v} \cdot \nabla) \eta \cdot P\mathbf{v} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega),$$

which is our desired extension for  $\mathbf{v}$  in  $\Omega$ . In addition,  $\|\nabla(P\mathbf{v})\|_{L^2(\Omega)} \leq \|\mathbf{v}\|_{J_0^{1,2}(\Omega')}$ , which implies that  $P\mathbf{v}$  is in the class of the D-solution.

By the trivial traces theorem, we can immediately obtain that  $P\mathbf{v}|_{\partial\Omega} = \mathbf{0}$ .

## REFERENCES

- [1] Jean Leray. Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. *Journal de mathématiques pures et appliquées*, 12:1–82, 1933.
- [2] Matthieu Hillairet and Peter Wittner. On the existence of solutions to the planar exterior navier stokes system. *Journal of Differential Equations*, 255(10):2996–3019, 2013.
- [3] Antonio Russo. A note on the exterior two-dimensional steady-state navier–stokes problem. *Journal of Mathematical Fluid Mechanics*, 11:407–414, 2009.
- [4] Julien Guillod and Peter Wittner. Existence and uniqueness of steady weak solutions to the navier–stokes equations in  $r^2$ . *Proceedings of the American Mathematical Society*, 146(10):4429–4445, 2018.
- [5] Julien Guillod, Mikhail Korobkov, and Xiao Ren. Existence and uniqueness for plane stationary navier–stokes flows with compactly supported force. *Communications in Mathematical Physics*, 397(2):729–762, 2023.
- [6] Mikhail Korobkov, Konstantin Pileckas, and Remigio Russo. *The Steady Navier-Stokes System*. Springer, 2024.
- [7] Mikhail Korobkov and Xiao Ren. Stationary solutions to the navier–stokes system in an exterior plane domain: 90 years of search, mysteries and insights. *Journal of Mathematical Fluid Mechanics*, 25(3):55, 2023.
- [8] Mikhail V Korobkov, Konstantin Pileckas, and Remigio Russo. On the steady navier–stokes equations in 2d exterior domains. *Journal of Differential Equations*, 269(3):1796–1828, 2020.
- [9] D Gilbarg and HF Weinberger. Asymptotic properties of leray's solution of the stationary two-dimensional navier–stokes equations. *Russian Mathematical Surveys*, 29(2):109, 1974.