REMARKS ON THE EXISTENCE PROBLEM OF WEAK SOLUTIONS TO STATIONARY NAVIER-STOKES EQUATIONS IN THE EXTERIOR DOMAIN

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Abstract. In this note, we prove that when the boundary is C^∞-smooth and satisfies the strong local Lipschitz condition, one can show the existence of the extension solution to the stationary Navier-Stokes equations in the exterior domain, shedding new lights on Leray's invading domains method [1].

1. Introduction

Consider the following exterior domain problem in the 2-dimensional case,

$$\begin{cases} -\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 \text{ in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega, \\ \mathbf{v}(x) \to \mathbf{v}_\infty = \lambda \mathbf{e_1} \quad \text{as } |x| \to \infty, \end{cases} \tag{1}$$

herein, $\Omega = \mathbb{R}^2 \backslash \overline{\Omega_0}$, Ω_0 is open bounded with sufficiently smooth boundary (that is, $\partial\Omega$ is C^∞-smooth and satisfies the strong local Lipschitz condition) and $0 \in \Omega_0$. $\lambda \geq 0$ denotes the Reynold number, $\mathbf{e}_1 = (1, 0)$ is the unit vector along x-axis. The above equation governs stationary flows around rigid obstacles[1].

In 1933, Leray first proved the existence of weak solutions in two-dimensional unbounded domains for vanishing flux through the boundaries [1]. In [3], Russo proved the case for small fluxes. In [4] (Theorem 2.6), the authors discussed the existence and uniqueness of a steady weak solution to the Navier-Stokes equation in the whole plane. In [5], Guillod et al. proved the existence and uniqueness of the D-solution to equation $(1)_{1,2,4}$ under small forces. In this paper, we explore a new way to establish the existence results of weak solutions to the exterior domain problem (1) using the extension-based approach, shedding new lights on Leray's invading domains method [1].

Specifically, denote $\Omega' = \overline{\Omega \cap \mathcal{B}_R}$, $\mathcal{B}_R$ represents the disc in $\mathbb{R}^2$ centered at 0 with radius $R > 0$, such that $\partial\Omega \bigcap \{|x| = R\} = \varnothing$. Since $\partial\Omega$ is sufficiently smooth ($\partial\Omega$ is C^∞-smooth and satisfies the strong local Lipschitz condition), both Ω and Ω' satisfy the cone condition.

Now consider the following sub-problem of the exterior domain problem (1) on Ω'.

$$\begin{cases} -\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0 \text{ in } \Omega' = \overline{\Omega \bigcap \mathcal{B}_R}, \\ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega'. \\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega, \\ \mathbf{v} = \mathbf{v}_\infty \quad \text{for } |x| = R. \end{cases} \tag{2}$$

Note that, the main difference between system (2) and system (3.1.1) in [6] is that we add a constant boundary condition on the spherical surface $|x| = R$.

Given problem (2), we then present the definition of the weak solution to equation $(2)_{1,2,3}$ on the bounded domain Ω', which can be stated as follows.

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[1]According to [2], here we mainly consider the case which describes a flow around the obstacle.

Definition 1 (Weak solution on bounded domains [6]). *Function* $\mathbf{v} \in J_0^{1,2}(\Omega')$ *is a weak solution to the exterior domain problem* $(2)_{1,2,3}$ *if* $\mathbf{v}$ *satisfies the following integral equation*

$$\int_{\Omega'} \nabla \mathbf{v} \cdot \nabla \eta - (\mathbf{v} \cdot \nabla)\eta \cdot \mathbf{v} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega'). \tag{3}$$

Due to the Poincaré inequality, $H_0^{1,2}(\Omega') = J_0^{1,2}(\Omega')$. Hence, using Theorem 3.1.2 in [6], we can immediately obtain the existence of velocity $\mathbf{v}$ on bounded domains.

Theorem 1. *There exists at least one weak solution* $\mathbf{v} \in J_0^{1,2}(\Omega')$ *to equation* $(2)_{1,2,3}$ *with finite Dirichlet integral.*

Next, we prove the existence of the weak solution to problem $(1)_{1,2,3}$, which is essentially an extension of solution $\mathbf{v}$ on the bounded domain Ω'. For this, we propose the following extension theorem, which is the main result of our paper.

Theorem 2 (Extension theorem). *For the weak solution* $\mathbf{v} \in J_0^{1,2}(\Omega')$ *to equation* $(2)_{1,2,3}$ *satisfying*

$$\int_{\Omega'} \nabla \mathbf{v} \cdot \nabla \eta dx - \int_{\Omega'} (\mathbf{v} \cdot \nabla)\eta \cdot \mathbf{v} dx = 0, \qquad \forall \eta \in J_0^{1,2}(\Omega'),$$

there exists a linear bounded extension operator $P : J_0^{1,2}(\Omega') \to J_0^{1,2}(\Omega)$ *such that* $\forall \mathbf{v} \in J_0^{1,2}(\Omega')$:

- $P\mathbf{v} = \mathbf{v}$ *a.e. in* Ω';
- $||P\mathbf{v}||_{J_0^{1,2}(\Omega)} \lesssim_{\Omega'} ||\mathbf{v}||_{J_0^{1,2}(\Omega')}$;
- $P\mathbf{v}|_{\partial\Omega} = \mathbf{0}$,

$P\mathbf{v}$ *represents the weak solution to* $(1)_{1,2,3}$ *on the whole space* Ω, *satisfying*

$$\int_{\Omega} \nabla(P\mathbf{v}) \cdot \nabla \eta dx - \int_{\Omega} (P\mathbf{v} \cdot \nabla)\eta \cdot P\mathbf{v} dx = 0, \qquad \forall \eta \in J_0^{1,2}(\Omega),$$

which is in the class of the D-solution.

Afterwards, we can prove the existence of the pressure p for the extension solution by De Rham's theory [2].

Organization of the paper. The outline of the paper can be summarized as follows. Section 2 introduces the preliminaries and notations in the paper. The rest of the article is devoted to prove the existence of weak solutions to stationary Navier-Stokes equations in the exterior domain using extension-based methods.

2. Notations and Preliminaries

Notations. Let $\Omega \subset \mathbb{R}^2$ be an exterior domain, $C_c^\infty(\Omega)$ represents the infinitely differentiable functions in Ω with compact support. Denote the closure of $C_c^\infty(\Omega)$ in the $W^{k,p}(\Omega)$ norm as $W_0^{k,p}(\Omega)$, for $k \in \mathbb{N}$ and $p \in [1, +\infty]$. Let $J_0^\infty(\Omega) = \{\mathbf{v} \in C_c^\infty(\Omega) : \text{div } \mathbf{v} = 0\}$. Then, for $r \geq 1$, denote $J_0^{1,r}(\Omega)$ as the closure of $J_0^\infty(\Omega)$ in the $W^{1,r}(\Omega)$ norm. Obviously, $J_0^{1,r}(\Omega)$ is a Banach space. In addition, for bounded domains, we define $H_0^{1,r}(\Omega)$ as the closure of $J_0^\infty(\Omega)$ in the $D_0^{1,r}(\Omega)$ norm, i.e.,

$$||\mathbf{v}||_{D_0^{1,r}(\Omega)} = ||\nabla \mathbf{v}||_{L^r(\Omega)} = \left(\int_{\Omega} |\nabla \mathbf{v}(x)|^r dx \right)^{1/r},$$

where $\mathbf{v} = (v_1, v_2)$, $|\nabla \mathbf{v}(x)|^r = \sum\limits_{j=1}^{2} \sum\limits_{k=1}^{2} |\frac{\partial v_j(x)}{\partial x_k}|^r$. $\hookrightarrow$ represents the Sobolev imbedding.

The Invading Domains Method. In the seminal work [1], Leray introduced an approach called the *invading domains method* [7]. Specifically, denote $\mathbf{v}_k$ as the solution to the problem

$$\begin{cases} -\Delta \mathbf{v}_k + (\mathbf{v}_k \cdot \nabla)\mathbf{v}_k + \nabla p_k = 0 \text{ in } \Omega \bigcap \mathcal{B}_{R_k}, \\ \text{div } \mathbf{v}_k = 0 \text{ in } \Omega \bigcap \mathcal{B}_{R_k}, \\ \mathbf{v}_k = \mathbf{0} \text{ on } \partial\Omega, \\ \mathbf{v}_k = \mathbf{v}_\infty \quad \text{for } |x| = R_k, \end{cases} \tag{4}$$

on the intersection of Ω with the disk $\mathcal{B}_{R_k}$ of radius R_k (for all $k = 1, 2, \cdots$, $\partial\Omega \bigcap \partial\mathcal{B}_{R_k} = \varnothing$), whose existence he proved before, satisfying the finite Dirichlet integral condition

$$\int_{\Omega \bigcap \mathcal{B}_{R_k}} |\nabla \mathbf{v}_k|^2 \leq C, \tag{5}$$

herein, C is a positive constant depending on $\partial\Omega$ and $\mathbf{v}_\infty$.

Indeed, based on the results in inequality (5), one can extract a subsequence $\{\mathbf{v}_{k_n}\}_{n=1}^{\infty}$ from $\{\mathbf{v}_k\}_{k=1}^{\infty}$ such that $\mathbf{v}_{k_n}$ weakly converge (i.e., the convergence is uniform on every bounded set) to a solution $\mathbf{v}_L$ to problem (1) with finite Dirichlet integral [8]. The solution $\mathbf{v}_L$ is the so-called Leray solution [9]. After obtaining $\mathbf{v}_L$, the pressure p can be solved by De Rham's theory [2].

3. Proof of Theorem 2

Obviously, $\partial\Omega'$ is sufficiently smooth, Ω' is bounded. Besides, we note that $J_0^{1,2} \hookrightarrow W_0^{1,2} \hookrightarrow L^2$. Hence, for any $\mathbf{v} \in J_0^{1,2}$, $\mathbf{v}, \nabla\mathbf{v} \in L^2$ and the imbedding holds for both Ω', Ω.

Then, for any x on $\partial\Omega'$, since $\partial\Omega'$ is not necessarily flat near x. We select a ball U centered at x ($\partial\Omega_0 \bigcap \partial U = \varnothing$) and denote by $U^+ = U \bigcap \Omega'$, $U^- = U \bigcap (\Omega \backslash \Omega')$. Obviously, according to the existence of $\mathbf{v}$ on bounded domain Ω', we have $\mathbf{v} \in J_0^{1,2}(U^+)$.

Then, one can find a diffeomorphism Φ, such that div $\mathbf{v} \circ \Phi = 0$ and

$$\Phi(U \bigcap \Omega') = \{y \in \tilde{U} : y_n \geq 0\},$$
$$\Phi(U \bigcap \partial\Omega') = \{y \in \tilde{U} : y_n = 0\},$$

wherein, $\tilde{U}$ denotes the ball intersecting with the hyperplane $\{y \in \mathbb{R}^2 : y_n = 0\}$.

Denote $\tilde{U}^+ = \tilde{U} \bigcap \{y \in \mathbb{R}^2 : y_n \geq 0\}$, $\tilde{U}^- = \tilde{U} \bigcap \{y \in \mathbb{R}^2 : y_n \leq 0\}$. Then, define the extension of $\mathbf{v}(y)$ on $\tilde{U}$ as

$$\overline{\mathbf{v}}(y) = \begin{cases} \mathbf{v}(y) & , y \in \tilde{U}^+, \\ \mathbf{0} & , y \in \tilde{U}^-, \end{cases} \tag{6}$$

satisfying

$$\int_{\tilde{U}} \nabla\overline{\mathbf{v}} \cdot \nabla\eta \, dx - \int_{\tilde{U}} (\overline{\mathbf{v}} \cdot \nabla)\eta \cdot \overline{\mathbf{v}} \, dx = 0, \qquad \forall \eta \in J_0^{1,2}(\tilde{U}).$$

Since $\mathbf{v} \in J_0^{1,2}(\tilde{U}^+)$, there exists $\{\mathbf{v}_m\}_{m=1}^{\infty} \subset J_0^{\infty}(\tilde{U}^+)$, such that $\mathbf{v}_m \to \mathbf{v}(J_0^{1,2}(\tilde{U}^+))$.

Therefore, the extension $\bar{\mathbf{v}}_m$ for the sequence $\mathbf{v}_m$ on $\tilde{U}$ defined similar to equation (6) also belongs to $C_c^{\infty}(\tilde{U})$.

Besides, the norm of $\bar{\mathbf{v}}_m$ in $J_0^{1,2}(\tilde{U})$ is the same as the norm of $\mathbf{v}_m$ in $J_0^{1,2}(\tilde{U}^+)$. Hence, $\bar{\mathbf{v}}_m$ is a Cauchy sequence in $J_0^{1,2}(\tilde{U})$, $\bar{\mathbf{v}}_m \to \bar{\mathbf{v}}(J_0^{1,2}(\tilde{U}))$. Therefore, we have $\overline{\mathbf{v}}(y) \in J_0^{1,2}(\tilde{U})$.

Meanwhile, $\overline{\mathbf{v}}(y)$ has finite Dirichlet integral, that is,

$$||\overline{\mathbf{v}}(y)||_{J_0^{1,2}(\tilde{U})} = ||\overline{\mathbf{v}}(y)||_{H_0^{1,2}(\tilde{U})} = ||\nabla\overline{\mathbf{v}}(y)||_{L^2(\tilde{U})} \lesssim ||\mathbf{v}(y)||_{J_0^{1,2}(\tilde{U}^+)}.$$

Now, we map $\tilde{U}$ back to the original shape using Φ^{-1}, then we have

$$\overline{\mathbf{v}}(x) = \begin{cases} \mathbf{v}(x) & , x \in U^+, \\ \mathbf{0} & , x \in U^-, \end{cases}$$

$\overline{\mathbf{v}} \in J_0^{1,2}(U)$, satisfying

$$\int_U \nabla\overline{\mathbf{v}} \cdot \nabla\eta \, dx - \int_U (\overline{\mathbf{v}} \cdot \nabla)\eta \cdot \overline{\mathbf{v}} \, dx = 0, \qquad \forall \eta \in J_0^{1,2}(U)$$

and the finte Dirichlet integral condition

$$||\overline{\mathbf{v}}(x)||_{J_0^{1,2}(U)} = ||\nabla\overline{\mathbf{v}}(x)||_{L^2(U)} \lesssim ||\mathbf{v}(x)||_{J_0^{1,2}(U^+)} = ||\mathbf{v}(x)||_{J_0^{1,2}(\Omega')}. \tag{7}$$

Afterwards, since $\partial\Omega'$ is compact, there exists finite points $\{x_i\}_{i=1}^N \in \partial\Omega'$, open sets $\{U_i\}_{i=1}^N$ and extensions $\{\overline{\mathbf{v}}_i\}_{i=1}^N$ of $\mathbf{v}$ on domain U_i, such that $\partial\Omega' \subset \bigcup_{i=1}^N U_i$ and

$$\int_{U_i} \nabla\overline{\mathbf{v}}_i \cdot \nabla\eta dx - \int_{U_i} (\overline{\mathbf{v}}_i \cdot \nabla)\eta \cdot \overline{\mathbf{v}} dx = 0, \qquad \forall \eta \in J_0^{1,2}(U_i).$$

Take $U_0 \subset\subset \Omega'$ such that $\Omega' \subset \bigcup_{i=0}^N U_i$ and let $\{\xi_i\}_{i=0}^N$ be an associated partition of unity. Let $\overline{\mathbf{v}} = \sum_{i=0}^N \xi_i \overline{\mathbf{v}}_i$, where $\overline{\mathbf{v}}_0 = \mathbf{v}$. Using inequality (7), we have the following bound

$$||\overline{\mathbf{v}}||_{J_0^{1,2}(\Omega)} \lesssim_{\Omega'} ||\mathbf{v}||_{J_0^{1,2}(\Omega')}. \tag{8}$$

Besides, the integral equation still holds

$$\int_\Omega \nabla\overline{\mathbf{v}} \cdot \nabla\eta dx - \int_\Omega (\overline{\mathbf{v}} \cdot \nabla)\eta \cdot \overline{\mathbf{v}} dx = 0, \qquad \forall \eta \in J_0^{1,2}(\Omega).$$

Finally, take $P\mathbf{v} = \overline{\mathbf{v}}$ and recall that $\mathbf{v} \in C_c^\infty(\Omega')$. For any sequence $\{\mathbf{v}_m\}_{m=1}^\infty \subset C_c^\infty(\Omega')$ converging to $\mathbf{v}$ in $J_0^{1,2}(\Omega')$. Then, using the linearity of E and inequality (8), the following inequality holds,

$$||P\mathbf{v}_j - P\mathbf{v}_k||_{J_0^{1,2}(\Omega)} \lesssim ||\mathbf{v}_j - \mathbf{v}_k||_{J_0^{1,2}(\Omega')}.$$

Hence $\{P\mathbf{v}_m\}_{m=1}^\infty$ is a Cauchy sequence converging to $P\mathbf{v}$, satisfying

$$\int_\Omega \nabla(P\mathbf{v}) \cdot \nabla\eta dx - \int_\Omega (P\mathbf{v} \cdot \nabla)\eta \cdot P\mathbf{v} dx = 0, \qquad \forall \eta \in J_0^{1,2}(\Omega),$$

which is our desired extension for $\mathbf{v}$ in Ω. In addition, $||\nabla(P\mathbf{v})||_{L^2(\Omega)} \leq ||\mathbf{v}||_{J_0^{1,2}(\Omega')}$, which implies that $P\mathbf{v}$ is in the class of the D-solution.

By the trivial traces theorem, we can immediately obtain that $P\mathbf{v}|_{\partial\Omega} = \mathbf{0}$.

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