

A NOVEL APPROACH FOR PROVING THE EXISTENCE OF WEAK SOLUTIONS TO STATIONARY NAVIER-STOKES EQUATIONS IN THE EXTERIOR DOMAIN

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ABSTRACT. In this note, we prove that when the boundary satisfies the enhanced uniform C^∞ -regularity condition, one can show the existence of the extension solution to the stationary Navier-Stokes equations in the exterior domain, shedding new lights on Leray's invading domains method [1].

1. INTRODUCTION

Consider the following exterior domain problem in the 2-dimensional case,

$$(1) \quad \begin{cases} -\Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{v}(x) \rightarrow \mathbf{v}_\infty = \lambda \mathbf{e}_1 & \text{as } |x| \rightarrow \infty, \end{cases}$$

herein, $\Omega = \mathbb{R}^2 \setminus \overline{\Omega_0}$, Ω_0 is open bounded with sufficiently smooth boundary, i.e., the boundary of Ω satisfies the following condition:

Definition 1 (Enhanced Uniform C^∞ -Regularity Condition). *We say that the domain Ω satisfies the enhanced uniform C^∞ -regularity condition if there exists a locally finite open cover $\{U_j\}_{j=1}^N$ such that $\partial\Omega \subset \bigcup_{j=1}^N U_j$, and a corresponding C^∞ -smooth transformations $\{\Phi_j\}_{j=1}^N$ with Φ_j taking U_j onto the ball $B = \{y \in \mathbb{R}^n : |y| < 1\}$ and having inverse $\Psi_j = \Phi_j^{-1}$, such that:*

- *There exists finite R , the intersection of every collection of $R+1$ of the sets U_j is empty;*
- *$\exists \delta > 0$, s.t. $\Omega_\delta \subset \bigcup_{j=1}^\infty \Psi(\{y \in \mathbb{R}^n : |y| < \frac{1}{2}\})$, where,*

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\},$$

$\operatorname{dist}(\cdot, \cdot)$ denotes the distance between two elements;

- *$\forall j$, $\Phi_j(U_j \cap \Omega) = \{y \in B : y_n > 0\}$;*
- *Denote the components of Φ_j and Ψ_j as $(\phi_{j,1}, \dots, \phi_{j,n})$ and $(\psi_{j,1}, \dots, \psi_{j,n})$, $\exists M$ (M is finite), such that $\forall \alpha$ satisfying $0 < |\alpha| \leq m$, $\forall i, 1 \leq i \leq n$, $\forall j$, one has*

$$\begin{aligned} |D^\alpha \phi_{j,i}(x)| &\leq M, & \forall x \in U_j, \\ |D^\alpha \psi_{j,i}(y)| &\leq M, & \forall y \in B; \end{aligned}$$

- *$\forall x \in \partial\Omega$, there exists a neighbourhood U of x , such that $U \cap \partial\Omega$ can be represented as*

$$x_j = \gamma(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \quad 1 \leq j \leq n,$$

herein, $\gamma \in C^\infty$.

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Besides, $0 \in \Omega_0$. $\lambda \geq 0$ denotes the Reynold number, $\mathbf{e}_1 = (1, 0)$ is the unit vector along x -axis. Equation (1) governs stationary flows around rigid obstacles¹.

In 1933, Leray first proved the existence of weak solutions in two-dimensional unbounded domains for vanishing flux through the boundaries [1]. In [3], Russo proved the case for small fluxes. In [4] (Theorem 2.6), the authors discussed the existence and uniqueness of a steady weak solution to the Navier-Stokes equation in the whole plane. In [5], Guillo et al. proved the existence and uniqueness of the D-solution to equation (1)_{1,2,4} under small forces. In this paper, we explore a new way to establish the existence results of weak solutions to the exterior domain problem (1)_{1,2,3} using the extension-based approach, shedding new lights on Leray's invading domains method [1].

Specifically, denote $\Omega' = \Omega \cap \overline{\mathcal{B}_R}$, \mathcal{B}_R represents the disc in \mathbb{R}^2 centered at 0 with radius $R > 0$, such that $\partial\Omega \cap \{|x| = R\} = \emptyset$. Since $\partial\Omega$ is sufficiently smooth ($\partial\Omega$ satisfies the enhanced uniform C^∞ -regularity condition), both Ω and Ω' satisfy the cone condition.

Now consider the following sub-problem of the exterior domain problem (1) on Ω' .

$$(2) \quad \begin{cases} -\Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = 0 & \text{in } \Omega' = \overline{\Omega \cap \mathcal{B}_R}, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega', \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{v} = \mathbf{v}_\infty & \text{for } |x| = R. \end{cases}$$

Note that, the main difference between system (2) and system (3.1.1) in [6] is that we add a constant boundary condition on the spherical surface $|x| = R$.

Given problem (2), we then present the definition of the weak solution to equation (2)_{1,2,3} on the bounded domain Ω' , which can be stated as follows.

Definition 2 (Weak solution on bounded domains [6]). *Function $\mathbf{v} \in J_0^{1,2}(\Omega')$ is a weak solution to the exterior domain problem (2)_{1,2,3} if \mathbf{v} satisfies the following integral equation*

$$(3) \quad \int_{\Omega'} \nabla \mathbf{v} \cdot \nabla \eta - (\mathbf{v} \cdot \nabla) \eta \cdot \mathbf{v} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega').$$

Due to the Poincaré inequality, $H_0^{1,2}(\Omega') = J_0^{1,2}(\Omega')$. Hence, using Theorem 3.1.2 in [6], we can immediately obtain the existence of velocity \mathbf{v} on bounded domains.

Theorem 1. *There exists at least one weak solution $\mathbf{v} \in J_0^{1,2}(\Omega')$ to equation (2)_{1,2,3} with finite Dirichlet integral.*

Next, we prove the existence of the weak solution to problem (1)_{1,2,3}, which is essentially an extension of solution \mathbf{v} on the bounded domain Ω' . For this, we propose the following extension theorem, which is the main result of our paper.

Theorem 2 (Extension theorem). *For the weak solution $\mathbf{v} \in J_0^{1,2}(\Omega')$ to equation (2)_{1,2,3} satisfying*

$$\int_{\Omega'} \nabla \mathbf{v} \cdot \nabla \eta dx - \int_{\Omega'} (\mathbf{v} \cdot \nabla) \eta \cdot \mathbf{v} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega'),$$

there exists a linear bounded extension operator $P : J_0^{1,2}(\Omega') \rightarrow J_0^{1,2}(\Omega)$ such that $\forall \mathbf{v} \in J_0^{1,2}(\Omega')$:

- $P\mathbf{v} = \mathbf{v}$ a.e. in Ω' ;
- $\|P\mathbf{v}\|_{J_0^{1,2}(\Omega)} \lesssim_{\Omega'} \|\mathbf{v}\|_{J_0^{1,2}(\Omega')}$;
- $P\mathbf{v}|_{\partial\Omega} = \mathbf{0}$,

$P\mathbf{v}$ represents the weak solution to (1)_{1,2,3} on the whole space Ω , satisfying

$$\int_{\Omega} \nabla(P\mathbf{v}) \cdot \nabla \eta dx - \int_{\Omega} (P\mathbf{v} \cdot \nabla) \eta \cdot P\mathbf{v} dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega),$$

which is in the class of the D-solution.

¹According to [2], here we mainly consider the case which describes a flow around the obstacle.

Afterwards, we can prove the existence of the pressure p for the extension solution by De Rham's theory [2].

Organization of the paper. The outline of the paper can be summarized as follows. Section 2 introduces the preliminaries and notations in the paper. The rest of the article is devoted to prove the existence of weak solutions to stationary Navier-Stokes equations in the exterior domain using extension-based methods.

2. NOTATIONS AND PRELIMINARIES

Notations. Let $\Omega \subset \mathbb{R}^2$ be an exterior domain, $C_c^\infty(\Omega)$ represents the infinitely differentiable functions in Ω with compact support. Denote the closure of $C_c^\infty(\Omega)$ in the $W^{k,p}(\Omega)$ norm as $W_0^{k,p}(\Omega)$, for $k \in \mathbb{N}$ and $p \in [1, +\infty]$. Let $J_0^\infty(\Omega) = \{\mathbf{v} \in C_c^\infty(\Omega) : \operatorname{div} \mathbf{v} = 0\}$. Then, for $r \geq 1$, denote $J_0^{1,r}(\Omega)$ as the closure of $J_0^\infty(\Omega)$ in the $W^{1,r}(\Omega)$ norm. Obviously, $J_0^{1,r}(\Omega)$ is a Banach space. In addition, for bounded domains, we define $H_0^{1,r}(\Omega)$ as the closure of $J_0^\infty(\Omega)$ in the $D_0^{1,r}(\Omega)$ norm, i.e.,

$$\|\mathbf{v}\|_{D_0^{1,r}(\Omega)} = \|\nabla \mathbf{v}\|_{L^r(\Omega)} = \left(\int_{\Omega} |\nabla \mathbf{v}(x)|^r dx \right)^{1/r},$$

where $\mathbf{v} = (v_1, v_2)$, $|\nabla \mathbf{v}(x)|^r = \sum_{j=1}^2 \sum_{k=1}^2 \left| \frac{\partial v_j(x)}{\partial x_k} \right|^r$. \hookrightarrow represents the Sobolev imbedding.

The Invading Domains Method. In the seminal work [1], Leray introduced an approach called the *invading domains method* [7]. Specifically, denote \mathbf{v}_k as the solution to the problem

$$(4) \quad \begin{cases} -\Delta \mathbf{v}_k + (\mathbf{v}_k \cdot \nabla) \mathbf{v}_k + \nabla p_k = 0 & \text{in } \Omega \cap \mathcal{B}_{R_k}, \\ \operatorname{div} \mathbf{v}_k = 0 & \text{in } \Omega \cap \mathcal{B}_{R_k}, \\ \mathbf{v}_k = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{v}_k = \mathbf{v}_\infty & \text{for } |x| = R_k, \end{cases}$$

on the intersection of Ω with the disk \mathcal{B}_{R_k} of radius R_k (for all $k = 1, 2, \dots$, $\partial\Omega \cap \partial\mathcal{B}_{R_k} = \emptyset$), whose existence he proved before, satisfying the finite Dirichlet integral condition

$$(5) \quad \int_{\Omega \cap \mathcal{B}_{R_k}} |\nabla \mathbf{v}_k|^2 \leq C,$$

herein, C is a positive constant depending on $\partial\Omega$ and \mathbf{v}_∞ .

Indeed, based on the results in inequality (5), one can extract a subsequence $\{\mathbf{v}_{k_n}\}_{n=1}^\infty$ from $\{\mathbf{v}_k\}_{k=1}^\infty$ such that \mathbf{v}_{k_n} weakly converge (i.e., the convergence is uniform on every bounded set) to a solution \mathbf{v}_L to problem (1) with finite Dirichlet integral [8]. The solution \mathbf{v}_L is the so-called Leray solution [9]. After obtaining \mathbf{v}_L , the pressure p can be solved by De Rham's theory [2].

3. PROOF OF THEOREM 2

Obviously, $\partial\Omega'$ is sufficiently smooth, Ω' is bounded. Besides, we note that $J_0^{1,2} \hookrightarrow W_0^{1,2} \hookrightarrow L^2$. Hence, for any $\mathbf{v} \in J_0^{1,2}$, $\mathbf{v}, \nabla \mathbf{v} \in L^2$ and the imbedding holds for both Ω', Ω .

Then, for any x on $\partial\Omega'$, since $\partial\Omega'$ is not necessarily flat near x . We select a ball U centered at x ($\partial\Omega_0 \cap \partial U = \emptyset$) and denote by $U^+ = U \cap \Omega'$, $U^- = U \cap (\Omega \setminus \Omega')$. Obviously, according to the existence of \mathbf{v} on bounded domain Ω' , we have $\mathbf{v} \in J_0^{1,2}(U^+)$.

Then, one can find a diffeomorphism Φ , such that $\operatorname{div} \mathbf{v} \circ \Phi = 0$ and

$$\begin{aligned} \Phi(U \cap \Omega') &= \{y \in \tilde{U} : y_n \geq 0\}, \\ \Phi(U \cap \partial\Omega') &= \{y \in \tilde{U} : y_n = 0\}, \end{aligned}$$

wherein, \tilde{U} denotes the ball intersecting with the hyperplane $\{y \in \mathbb{R}^2 : y_n = 0\}$.

Denote $\tilde{U}^+ = \tilde{U} \cap \{y \in \mathbb{R}^2 : y_n \geq 0\}$, $\tilde{U}^- = \tilde{U} \cap \{y \in \mathbb{R}^2 : y_n \leq 0\}$. Then, define the extension of $\mathbf{v}(y)$ on \tilde{U} as

$$(6) \quad \bar{\mathbf{v}}(y) = \begin{cases} \mathbf{v}(y) & , y \in \tilde{U}^+, \\ \mathbf{0} & , y \in \tilde{U}^-, \end{cases}$$

satisfying

$$\int_{\tilde{U}} \nabla \bar{\mathbf{v}} \cdot \nabla \eta \, dx - \int_{\tilde{U}} (\bar{\mathbf{v}} \cdot \nabla) \eta \cdot \bar{\mathbf{v}} \, dx = 0, \quad \forall \eta \in J_0^{1,2}(\tilde{U}).$$

Since $\mathbf{v} \in J_0^{1,2}(\tilde{U}^+)$, there exists $\{\mathbf{v}_m\}_{m=1}^\infty \subset J_0^\infty(\tilde{U}^+)$, such that $\mathbf{v}_m \rightarrow \mathbf{v}(J_0^{1,2}(\tilde{U}^+))$.

Therefore, the extension $\bar{\mathbf{v}}_m$ for the sequence \mathbf{v}_m on \tilde{U} defined similar to equation (6) also belongs to $C_c^\infty(\tilde{U})$.

Besides, the norm of $\bar{\mathbf{v}}_m$ in $J_0^{1,2}(\tilde{U})$ is the same as the norm of \mathbf{v}_m in $J_0^{1,2}(\tilde{U}^+)$. Hence, $\bar{\mathbf{v}}_m$ is a Cauchy sequence in $J_0^{1,2}(\tilde{U})$, $\bar{\mathbf{v}}_m \rightarrow \bar{\mathbf{v}}(J_0^{1,2}(\tilde{U}))$. Therefore, we have $\bar{\mathbf{v}}(y) \in J_0^{1,2}(\tilde{U})$.

Meanwhile, $\bar{\mathbf{v}}(y)$ has finite Dirichlet integral, that is,

$$\|\bar{\mathbf{v}}(y)\|_{J_0^{1,2}(\tilde{U})} = \|\bar{\mathbf{v}}(y)\|_{H_0^{1,2}(\tilde{U})} = \|\nabla \bar{\mathbf{v}}(y)\|_{L^2(\tilde{U})} \lesssim \|\mathbf{v}(y)\|_{J_0^{1,2}(\tilde{U}^+)}.$$

Now, we map \tilde{U} back to the original shape using Φ^{-1} , then we have

$$\bar{\mathbf{v}}(x) = \begin{cases} \mathbf{v}(x) & , x \in U^+, \\ \mathbf{0} & , x \in U^-, \end{cases}$$

$\bar{\mathbf{v}} \in J_0^{1,2}(U)$, satisfying

$$\int_U \nabla \bar{\mathbf{v}} \cdot \nabla \eta \, dx - \int_U (\bar{\mathbf{v}} \cdot \nabla) \eta \cdot \bar{\mathbf{v}} \, dx = 0, \quad \forall \eta \in J_0^{1,2}(U)$$

and the finite Dirichlet integral condition

$$(7) \quad \|\bar{\mathbf{v}}(x)\|_{J_0^{1,2}(U)} = \|\nabla \bar{\mathbf{v}}(x)\|_{L^2(U)} \lesssim \|\mathbf{v}(x)\|_{J_0^{1,2}(U^+)} = \|\mathbf{v}(x)\|_{J_0^{1,2}(\Omega')}.$$

Afterwards, since $\partial\Omega'$ is compact, there exists finite points $\{x_i\}_{i=1}^N \in \partial\Omega'$, open sets $\{U_i\}_{i=1}^N$ and extensions $\{\bar{\mathbf{v}}_i\}_{i=1}^N$ of \mathbf{v} on domain U_i , such that $\partial\Omega' \subset \bigcup_{i=1}^N U_i$ and

$$\int_{U_i} \nabla \bar{\mathbf{v}}_i \cdot \nabla \eta \, dx - \int_{U_i} (\bar{\mathbf{v}}_i \cdot \nabla) \eta \cdot \bar{\mathbf{v}}_i \, dx = 0, \quad \forall \eta \in J_0^{1,2}(U_i).$$

Take $U_0 \subset \subset \Omega'$ such that $\Omega' \subset \bigcup_{i=0}^N U_i$ and let $\{\xi_i\}_{i=0}^N$ be an associated partition of unity. Let $\bar{\mathbf{v}} = \sum_{i=0}^N \xi_i \bar{\mathbf{v}}_i$, where $\bar{\mathbf{v}}_0 = \mathbf{v}$. Using inequality (7), we have the following bound

$$(8) \quad \|\bar{\mathbf{v}}\|_{J_0^{1,2}(\Omega)} \lesssim_{\Omega'} \|\mathbf{v}\|_{J_0^{1,2}(\Omega')}.$$

Besides, the integral equation still holds

$$\int_{\Omega} \nabla \bar{\mathbf{v}} \cdot \nabla \eta \, dx - \int_{\Omega} (\bar{\mathbf{v}} \cdot \nabla) \eta \cdot \bar{\mathbf{v}} \, dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega).$$

Finally, take $P\mathbf{v} = \bar{\mathbf{v}}$ and recall that $\mathbf{v} \in C_c^\infty(\Omega')$. For any sequence $\{\mathbf{v}_m\}_{m=1}^\infty \subset C_c^\infty(\Omega')$ converging to \mathbf{v} in $J_0^{1,2}(\Omega')$. Then, using the linearity of E and inequality (8), the following inequality holds,

$$\|P\mathbf{v}_j - P\mathbf{v}_k\|_{J_0^{1,2}(\Omega)} \lesssim \|\mathbf{v}_j - \mathbf{v}_k\|_{J_0^{1,2}(\Omega')}.$$

Hence $\{P\mathbf{v}_m\}_{m=1}^\infty$ is a Cauchy sequence converging to $P\mathbf{v}$, satisfying

$$\int_{\Omega} \nabla (P\mathbf{v}) \cdot \nabla \eta \, dx - \int_{\Omega} (P\mathbf{v} \cdot \nabla) \eta \cdot P\mathbf{v} \, dx = 0, \quad \forall \eta \in J_0^{1,2}(\Omega),$$

which is our desired extension for \mathbf{v} in Ω . In addition, $\|\nabla (P\mathbf{v})\|_{L^2(\Omega)} \leq \|\mathbf{v}\|_{J_0^{1,2}(\Omega')}$, which implies that $P\mathbf{v}$ is in the class of the D-solution.

By the trivial traces theorem, we can immediately obtain that $P\mathbf{v}|_{\partial\Omega} = \mathbf{0}$.

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