COMP4680/COMP8650: Advanced Topics in Machine Learning

Assignment #2: Convex Sets and Convex Functions

Due: 11:55pm on Friday 19 August, 2022.
Submit solutions (with working) as a single PDF file or Jupyter notebook via Wattle.
Don't forget to include your name and student ID at the top of the submitted solutions.

THIS IS A HURDLE ASSESSMENT—YOU MUST PASS THIS ASSESSMENT TO PASS THE COURSE.

1. **Norm Balls (10 marks).** Consider the set $B_p = \{x \in \mathbb{R}^n \mid \|x - a\|_p \leq 1\}$ for fixed $a \in \mathbb{R}^n$ and $p \geq 1$.
   (a) Show that for $n = 1$ the set B_p defines a line segment and that the line segment is the same for any p. Give the end-points of the line segment in terms of a.
   (b) Show that for $p = \infty$ the set B_p defines an intersection of half-spaces. List the half-spaces.
   (c) Prove that B_p is a convex set.

2. **Polyhedron (10 marks).** Consider the polyhedron in $\mathbb{R}^2$ defined as the convex hull over the following set of points
$$\{(-4, 2), (1, -3), (2, 4), (5, 7), (-2, -1), (0, 3)\}.$$
   Express the polyhedron in the form $Ax \preceq b$. Is your solution unique?

3. **Convex Sets (15 marks).** Show that the following sets can be written as the intersection of half-spaces and then argue that they are all convex sets.
   (a) A *slab*, i.e., a set of the form $\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$.
   (b) A *wedge*, i.e., a set of the form $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.
   (c) A *disk*, i..e, a set of the form $\{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1\}$.

   Which of the following sets are also convex and why?

   (d) The set of points closer to a given point than a given set, i.e.,
$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\},$$
   where $S \subseteq \mathbb{R}^n$.
   (e) The set of points closer to one set than another, i.e.,
$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$
   where $S, T \subseteq \mathbb{R}^n$, and
$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$$

(f) The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

4. **Sets of Probability Distributions (15 marks).** Let x be a real-valued random variable that takes on finitely many values, with $\mathbf{prob}(x = a_i) = p_i$ for $i = 1, \ldots n$, where $a_1 < a_2 < \ldots < a_n$. Of course $p \in \mathbb{R}^n$ lies in the standard probability simplex $\Delta = \{p \mid \mathbf{1}^T p = 1,\, p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in \Delta$ that satisfy the condition a convex set?) Justify your answer.

   (a) $\alpha \leq \mathbf{E}x \leq \beta$, where $\mathbf{E}x$ is the expected value of x, i.e., $\mathbf{E}x = \sum_{i=1}^n p_i a_i$.

   (b) $\mathbf{prob}(x \geq \alpha) \leq \beta$.

   (c) $\mathbf{E}x^2 \leq \alpha$.

   (d) $\mathbf{var}(x) \leq \alpha$, where $\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E}x)^2$ is the variance of x.

   (e) $\mathbf{median}(x) \leq \beta$, where $\mathbf{median}(x)$ is the value m such that $\mathbf{prob}(x \geq m) \geq \frac{1}{2}$ and $\mathbf{prob}(x \leq m) \geq \frac{1}{2}$.

5. **Saddle Points (10 marks).** We say the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is *convex-concave* if $f(x, z)$ is a concave function of z, for each fixed x, and a convex function of x, for each fixed z. We also require its domain to have the product form $\mathbf{dom} f = A \times B$ where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are convex.

   (a) Give a second-order condition for a twice differentiable function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ to be convex-concave, in terms of its Hessian $\nabla^2 f(x, z)$.

   (b) Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex-concave and differentiable, with $\nabla f(\tilde{x}, \tilde{z}) = 0$. Show that the *saddle-point property* holds: for all x, z, we have

$$f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z}) \leq f(x, \tilde{z})$$

   Hint: Think of first- and second-order conditions for optimality.

   (c) With f defined as above, consider the function $g(z) = \inf_x f(x, z)$. Argue that $\sup_z g(z) \geq g(\tilde{z})$ and so $\sup_z \inf_x f(x, z) \geq f(\tilde{x}, \tilde{z})$.

   (d) Likewise, consider the function $h(x) = \sup_z f(x, z)$. Argue that $\inf_x h(x) \leq h(\tilde{x})$ and so $\inf_x \sup_z f(x, z) \leq f(\tilde{x}, \tilde{z})$.

   (e) Prove that $\sup_z g(z) \leq \inf_x h(x)$.

   (f) Use the results from above to show that f satisfies the *strong max-min property:*

$$\sup_z \inf_x f(x, z) = \inf_x \sup_z f(x, z)$$

   and that their common value is $f(\tilde{x}, \tilde{z})$.

6. **Quasiconvexity (20 marks).** For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

   (a) $f(x) = e^x - 1$ on $\mathbb{R}$.

   (b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_+$.

   (c) $f(x_1, x_2) = x_1^2 / x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

   (d) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 < \alpha < 1$, on $\mathbb{R}^2_{++}$.

7. **Conjugate Functions (15 marks).** Derive the conjugates of the following functions:

   (a) *Max function.* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbb{R}^n$.

   (b) *Sum of largest elements.* $f(x) = \sum_{i=1}^r x_{[i]}$ on $\mathbb{R}^n$.

   (c) *Negative geometric mean.* $f(x) = -\left(\prod_{i=1}^n x_i\right)^{1/n}$ on $\mathbb{R}^n_{++}$.

8. **Optimisation (5 marks).** Solve the following optimisation problem over $x \in \mathbb{R}^2$,

$$\begin{array}{rl} \text{minimize} & \frac{1}{2}x^T P x + q^T x + r \\ \text{subject to} & a^T x = b \end{array}$$

where $P = I$, $q = (-1, 0)$, $r = 5$, $a = (1, -2)$ and $b = 1$.

How does your solution change if we add a further constraint that $x \succeq \mathbf{1}$?