Q1.

1) $$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 32 \\ 250 \end{bmatrix}$$

$\therefore a_0 = 6,\ a_1 = -3,\ a_2 = 0,\ a_3 = 4$

$\therefore P_3(x) = 6 - 3x + 4x^3$

2) $L_j(x) = \prod_{\substack{i=0 \\ i\neq j}}^{3} \frac{x - x_i}{x_i - x_j}$,

$P_3(x) = \sum_{i=0}^{3} y_i L_i(x)$

$$L_0(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)} = \frac{(x-1)(x-2)(x-4)}{-8}$$

$$L_1(x) = \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)} = \frac{x\cdot(x-2)(x-4)}{3}$$

$$L_2(x) = \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)} = \frac{x(x-1)(x-4)}{-4}$$

$$L_3(x) = \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)} = \frac{x\cdot(x-1)(x-2)}{24}$$

$$P_3(x) = -\frac{3}{4}(x-1)(x-2)(x-4) + \frac{7}{3}x\cdot(x-2)(x-4) - 8\cdot x(x-1)(x-4) + \frac{125}{12}x(x-1)(x-2)$$

$$= 4x^3 + 0x^2 - 3x + 6 = 4x^3 - 3x + 6$$

3) $P_0(x) = y_0 = 6$

$P_1(x) = P_0(x) + c_1(x-0) = 6 + c_1 \cdot x$

$\because P_1(1) = 7$

$\therefore P_1(x) = 6 + x$

$P_2(x) = 6 + x + c_2\cdot(x-0)(x-1)$

$P_2(2) = 8 + 2c_2 = 32$, $c_2 = 12$

$\therefore P_2(x) = 6 + x + 12(x^2 - x)$

$P_3(x) = 12x^2 - 11x + 6 + c_3\cdot(x-0)(x-1)(x-2)$

$P_3(4) = 12\cdot 16 - 44 + 6 + 24\cdot c_3 = 250$, $c_3 = 4$

$\therefore P_3(x) = 12x^2 - 11x + 6 + 4(x^3 - 3x^2 + 2x) = 4x^3 - 3x + 6$

Q2.

a)

x_i	0	1	2	4
y_i	5	7	32	250

in monomial form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 32 \\ 250 \end{bmatrix}$$

$\therefore a_0 = 5,\ a_1 = -\frac{5}{4},\ a_2 = -\frac{7}{8},\ a_3 = \frac{33}{8}$

$\therefore P_3(x) = 5 - \frac{5}{4}x - \frac{7}{8}x^2 + \frac{33}{8}x^3$

b)

x_i	0	1	2	4	3
y_i	6	7	32	250	81

in monomial form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 3 & 9 & 27 & 81 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 32 \\ 250 \\ 81 \end{bmatrix}$$

$\therefore a_0 = 6,\ a_1 = -35,\ a_2 = 56,\ a_3 = -24,\ a_4 = 4$

$\therefore P_4(x) = 6 - 35x + 56x^2 - 24x^3 + 4x^4$

Q3 $f(x)=2x^4+x^2$

x_i	-1	0	1
y_i	3	0	3

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

$\therefore a_0 = a_1 = 0,\ a_2 = 3$

$\therefore P_2(x) = 3x^2$

the interpolation error $\varepsilon = |f(x) - p(x)| = \left| \dfrac{f'''(\xi)}{3!} \cdot (x+1)(x-1) \cdot x \right|$

$$\varepsilon = \left| \frac{48\xi}{6} \cdot (x^3 - x) \right|$$

$$= 8 \left| \xi \cdot (x^3 - x) \right|$$

for $\xi \in [-1,1]$, its maximum is at $\xi = 1$

for $h(x) = x^3 - x$, its maximum is at $x = -\frac{\sqrt{3}}{3}$, which is $\frac{2}{9}\sqrt{3}$

by solving $h'(x) = 3x^2 - 1 = 0$

$\therefore$ the maximum interpolation error is $\frac{16}{9}\sqrt{3}$

Q4. $S(x) = \begin{cases} 3(x-1)+2(x-1)^2-(x-1)^3, & 1 \le x < 2 \\ a+b(x-2)+c(x-2)^2+d(x-2)^3, & 2 \le x < 3 \end{cases}$

interpolation conditions:

$S_0(2) = S_1(2) \Rightarrow 3+2-1 = a, \quad a = 4$

first direvative continuity:

$S_0'(2) = S_1'(2)$

$S_0'(x) = 3+4(x-1)-3(x-1)^2 = -3x^2+10x-4$

$S_1'(x) = b+2c(x-2)+3d(x-2)^2 = 3dx^2+(2c-12d)x+(12d-4c+b)$

$\therefore S_0'(2) = 4 = S_1'(2) = b$

$\therefore b = 4$

second direvative continuity:

$S_0''(2) = S_1''(2)$

$S_0''(x) = -6x+10, \quad S_1''(x) = 2c \cdot x + 6d(x-2)$

$\therefore c = -1$

since its a clamped spline:

$S_0'(1) = f'(1), \quad S_1'(3) = f'(3)$

$\therefore S_0'(1) = S_1'(3)$

$S_0'(1) = 3 = S_1'(3) = 4-2+3d$

$\therefore d = \frac{1}{3}$

$\therefore a = 4, \; b = 4, \; c = -1, \; d = \frac{1}{3}$

Q5. $f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}\cdot h^2 + \frac{f'''(x)}{6}\cdot h^3 + o(h^4)$

$f(x+2h) = f(x) + f'(x)\cdot 2h + f''(x)\cdot 2h^2 + \frac{4}{3}f'''(x)\cdot h^3 + o(h^4)$

② − 2·① : $f(x+2h) - 2f(x+h) = -f(x) + f''(x)h^2 + f'''(x)h^3 + o(h^4)$

$$f''(x) = \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} + o(h) \quad ①$$

$\therefore\ a = \frac{1}{h^2},\ b = -\frac{2}{h^2},\ c = \frac{1}{h^2}$ and is a first-order method

the standard approximation of $f''(x+h)$ is

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + o(h^2) \quad ②$$

Despite the error term,

Two $f''(x)$ are similar as ① is only one step (h) forward than ②, ② can be considered as central difference and ① as forward difference, this is also why the error term of ① is $o(h)$ and that of ② is $o(h^2)$,

the truncation error of central difference decreases faster than that of forward difference,

when we derive both formulae by method of undetermined coefficients, the step of ① will be 0, 1, 2 and the step of ② will be −1, 0, 1

Q6.

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h), \text{ where } O(h) = \left|\frac{f''(x)}{2} h\right|$$

$$= \frac{f_{comp}(x+h) - e(x+h) - f_{comp}(x) + e(x)}{h} + O(h)$$

$$= \frac{f_{comp}(x+h) - f_{comp}(x)}{h} + \frac{e(x) - e(x+h)}{h} + O(h)$$

the error term can be written as

$$E = \frac{e(x) - e(x+h)}{h} + O(h)$$

since $e(x)$ and $e(x+h)$ are all error terms that should be positive on computation

$$\therefore E = \frac{|e(x)| + |e(x+h)|}{h} + \left|\frac{f''(x)}{2} \cdot h\right|$$

as $\forall x, |e(x)| \leq \varepsilon$ and $|f''(x)| \leq M$

E has an upper boundry of $\frac{\varepsilon + \varepsilon}{h} + \frac{M}{2} \cdot h = \frac{M \cdot h}{2} + \frac{2\varepsilon}{h}$

$$E = \frac{M \cdot h}{2} + \frac{2\varepsilon}{h} \geq 2\sqrt{M\varepsilon}$$

since both $\frac{Mh}{2} + \frac{2\varepsilon}{h}$ are positive

$$E = 2\sqrt{M\varepsilon} \text{ iff } \frac{Mh}{2} = \frac{2\varepsilon}{h} \Rightarrow h = \frac{2}{\sqrt{M}} \sqrt{\varepsilon}$$

since we need to make the upper boundry of error to be as small as it can, the step size h should be $\frac{2}{\sqrt{M}} \sqrt{\varepsilon}$