MATH3511/6111 Scientific Computing
Semester 1, 2022 — Lindon Roberts (MSI)

Assignment 2 (MATH3511 non-HPO)
Due date: 9am Monday 28 March (Week 6)

Please show all relevant working and present your solutions clearly: do not expect full marks for a correct answer without working, or where your reasoning is hard to follow.

Question 1 (4 marks)**.** Suppose we are given the interpolation data

x_i	0	1	2	4
y_i	6	7	32	250

Find the unique interpolating cubic polynomial for this data, written in the monomial, Lagrange and Newton forms. Verify that all three forms correspond to the same polynomial.

Question 2 (4 marks)**.** Using the same data and interpolating polynomial as in Question 1 above:

(a) Suppose we changed our data so $y_0 = 5$ instead of $y_0 = 6$. Write the new interpolating polynomial in any form you wish. [2 marks]

(b) Suppose we added a new point $x_4 = 3$, $y_4 = 81$. Write the new interpolating polynomial in any form you wish. Use the original value $y_0 = 6$. [2 marks]

Question 3 (4 marks)**.** Suppose we find the unique quadratic function $p(x)$ interpolating $f(x) = 2x^4 + x^2$ at the nodes $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$. Using suitable results from the lectures, find an upper bound on the maximum interpolation error $|p(x) - f(x)|$ for any $x \in [-1, 1]$.

Question 4 (4 marks)**.** A clamped cubic spline $S(x)$ for a function $f(x)$ is defined on $[1, 3]$ by

$$S(x) = \begin{cases} 3(x-1) + 2(x-1)^2 - (x-1)^3, & 1 \le x < 2, \\ a + b(x-2) + c(x-2)^2 + d(x-2)^3, & 2 \le x \le 3. \end{cases}$$

Given that $f'(1) = f'(3)$, find the values of a, b, c and d.

Question 5 (6 marks)**.** Derive a finite difference formula of the form

$$f''(x) \approx af(x) + bf(x+h) + cf(x+2h),$$

for some values a and b (depending on h). Derive the accuracy order of this approximation. Explain the similarities between this approximation and the standard approximation for $f''(x+h)$ on slide 9 of the differentiation lectures.
Hint: for the last part, consider how you would derive such formulae using the method of undetermined coefficients.

Question 6 (4 marks)**.** Consider the forward difference approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

for some $h > 0$. Suppose we have some code that evaluates $f(x)$ but introduces some errors; that is, we can only evaluate the function

$$f_{\text{comp}}(x) = f(x) + e(x),$$

where the error $e(x)$ satisfies $|e(x)| \le \epsilon$ for all x, but we know nothing else about it (e.g. $e(x)$ may not be continuous or differentiable).[1] This gives us a new approximation

$$f'(x) \approx \frac{f_{\text{comp}}(x+h) - f_{\text{comp}}(x)}{h}.$$

Suppose that $|f''(x)| \le M$ for all x. Prove that the absolute error in this new approximation is at most $\frac{Mh}{2} + \frac{2\epsilon}{h}$. Use this to justify the choice $h \sim \sqrt{\epsilon}$ as an appropriate choice of step size.

[1] In practice, this happens due to rounding errors, for example.