MATH3511/6111 Scientific Computing
Semester 1, 2022 — Lindon Roberts (MSI)

Assignment 3 (MATH3511 non-HPO)
Due date: 9am Tuesday 26 April (Week 8)

Please show all relevant working and present your solutions clearly: do not expect full marks for a correct answer without working, or where your reasoning is hard to follow.

Question 1 (6 marks)**.** What is the representation of (the decimal number) $1/3$ in the floating-point system given by half-precision IEEE arithmetic? What is the relative error in this representation? Calculate the associated machine epsilon and confirm your relative error is less than this.

Question 2 (4 marks)**.** Suppose we have a floating-point number system with machine epsilon of ϵ and we wish to calculate the secant slope

$$g = \frac{y_1 - y_0}{x_1 - x_0},$$

for $x_0, x_1, y_0, y_1 \in \mathbb{R}$ with $x_0 \neq x_1$. Under the floating-point model from lectures (i.e. with the $\mathrm{fl}(x)$ function), calculate the relative floating-point error in the evaluation of g where the computation is performed using the following algorithm:

Algorithm Method for calculating the secant slope g.

1: $a = y_1 - y_0$
2: $b = x_1 - x_0$
3: $g = a/b$
4: **return** g

You may assume that $b \neq 0$ even with rounding errors. Your answer should be of the form $C\epsilon + \mathcal{O}(\epsilon^2)$.

Hint: Remember that $\mathrm{fl}(x) = x$ *if* x *is already a machine number.*

Question 3 (6 marks)**.** In this question you will derive a quadrature rule for integration over an infinite domain.

(a) Find the Newton-Cotes formula of the form

$$\int_0^1 f(x)dx \approx a_0 f\left(\frac{1}{3}\right) + a_1 f\left(\frac{1}{2}\right) + a_2 f\left(\frac{2}{3}\right).$$

[2 marks]

(b) Show that

$$\int_1^\infty \frac{1 - e^{-1/x}}{x^2} dx = \frac{1}{e}.$$

[1 mark]

(c) Based on your quadrature rule from (a), derive the quadrature rule

$$\int_1^\infty f(x)dx \approx \frac{27}{8} f\left(\frac{3}{2}\right) - 8f(2) + \frac{27}{2} f(3)$$

Hint: show that the true integral is equal to $\int_0^1 g(y)dy$ *for some function* $g(y)$. [2 marks]

(d) What is the relative error in the quadrature rule approximation from (c) when used to evaluate the integral from (b)? [1 mark]

Question 4 (4 marks)**.** Gauss-Chebyshev quadrature is a form of Gaussian quadrature for estimating integrals of the form

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \approx \sum_{i=0}^{n} c_i f(x_i).$$

Just like Gauss-Legendre quadrature, the process for deriving Gauss-Chebyshev rules is:

1. Choose x_i to be the $n+1$ distinct roots of the Chebyshev polynomial $T_{n+1}(x)$ (see interpolation lectures for the definition).

2. Find c_i using the method of undetermined coefficients (by requiring the rule to be exact for all polynomials of degree $\leq n$).

Using this process, derive the Gauss-Chebyshev rule

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \approx \frac{\pi}{2} f\left(-\frac{1}{\sqrt{2}}\right) + \frac{\pi}{2} f\left(\frac{1}{\sqrt{2}}\right).$$