Q1.

a) $x_0 = \int_0^1 t^0 (t+7)^{-1} dt$

$= \int_0^1 (t+7)^{-1} dt$

Let $u = t+7$

$du = dt$

$x_0 = \int \frac{1}{u} du = [\ln|t+7|]_0^1$

$= \ln 8 - \ln 7 = \ln \frac{8}{7}$

for $n=1$, $x_1 = \int_0^1 t^1 (t+7)^{-1} dt$

$= \int_0^1 \frac{t}{t+7} dt = \int_0^1 (1 - \frac{7}{t+7}) dt = 1 - 7x_0$

$x_n = \frac{1}{n} - 7x_{n-1} = \int_0^1 t^n (t+7)^{-1} dt$

$\frac{1}{n} = \int_0^1 \frac{t^n}{t+7} dt + 7\int_0^1 \frac{t^{n-1}}{t+7} dt$

$RHS = \int_0^1 \frac{t^n + 7t^{n-1}}{t+7} dt$

$= \int_0^1 \frac{t^{n-1}(t+7)}{t+7} dt = \int_0^1 t^{n-1} dt = [\frac{t^n}{n}]_0^1 = \frac{1}{n}$

$= LHS$

$\therefore$ proved

Q1 b)

In [1]:
```
import numpy as np
import scipy.integrate
import matplotlib.pyplot as plt
```

In [2]:
```
def f(x):
    return  pow(x,18) / (x + 7)
a = 0; b = 1
```

In [3]:
```
x18, err = scipy.integrate.quad(f, a, b)
print("The approximation of x18 is ", x18, ", the error is ", e
```

```
The approximation of x18 is  0.006620563824579688 , the error is
7.350302394048756e-17
```

In [4]:
```
def fb(x,n):
    return (1/n - x) / 7
def ff(x,n):
    return 1/n - 7*x
```

In [5]:
```
xb = np.zeros(19)
xb[18] = x18
for i in range(len(xb)-2,-1,-1):
    xb[i] = fb(xb[i+1],i+1)
print(f"xn by backward recurrence formulas:\n{xb}")
```

```
xn by backward recurrence formulas:
[0.13353139 0.06528025 0.04303824 0.03206566 0.02554036 0.02121749
 0.01814422 0.01584763 0.01406657 0.01264509 0.01148435 0.01051866
 0.0097027  0.00900417 0.00839942 0.00787076 0.00740469 0.00699071
 0.00662056]
```

In [6]:
```
xf = np.zeros(19)
xf[0] = np.log(8/7)
for i in range(1,len(xf)):
    xf[i] = ff(xf[i-1],i)
print(f"xn by forward recurrence formulas:\n{xf}")
```

```
xn by forward recurrence formulas:
[ 0.13353139  0.06528025  0.04303824  0.03206566  0.02554036  0.02
121749
  0.01814422  0.01584763  0.01406657  0.0126451   0.01148433  0.01
051875
  0.00970209  0.00900842  0.00836967  0.008079    0.00594698  0.01
719466
 -0.06480705]
```

The result given by backwards recurrence formulas is much more accurate than that of forward recurrence formulas.

In [7]:

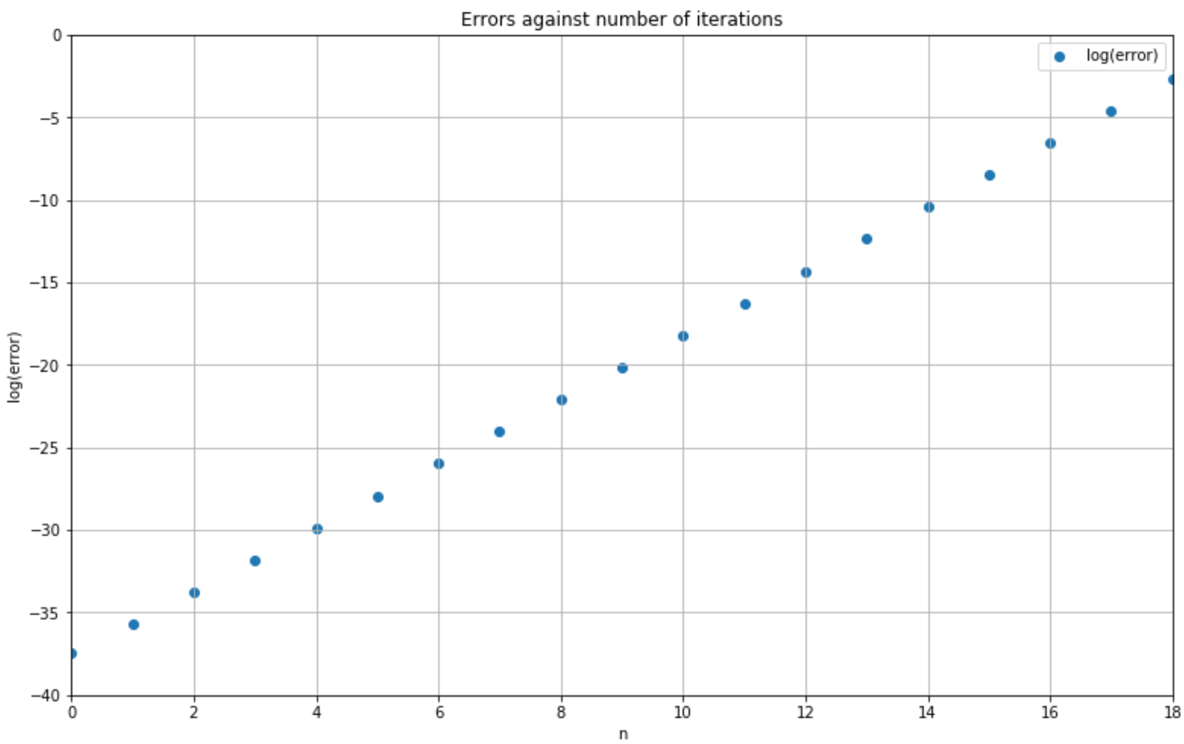
```
float_info = np.finfo(float)
x0err = xf[0] - xb[0]
x18err = xf[18]-xb[18]
print(f" the difference between x0 times 7^18 {x0err*pow(7,18)}
```

```
the difference between x0 times 7^18 -0.09039511350064361
the difference between x18 -0.0714276134172489
```

In [8]:

```
error = abs(xf-xb)
print(f"absolute value of errors are \n{error}")
n = np.linspace(0,18,19)
plt. figure(figsize= (13,8))
# plt.axes(yscale = "log")
plt.scatter(n,np.log(error),label='log(error)')
plt.title('Errors against number of iterations')
plt. xlabel('n')
plt.ylabel('log(error)')
plt. xlim(0,18)
plt.ylim(-40,0)
plt.legend(loc='best')
plt.grid()
plt.show()
```

```
absolute value of errors are
[5.55111512e-17 3.05311332e-16 2.15105711e-15 1.50435220e-14
 1.05315062e-13 7.37212374e-13 5.16047621e-12 3.61233231e-11
 2.52863263e-10 1.77004285e-09 1.23902999e-08 8.67320995e-08
 6.07124696e-07 4.24987288e-06 2.97491101e-05 2.08243771e-04
 1.45770640e-03 1.02039448e-02 7.14276134e-02]
```



In [9]:
```
float_info = np.finfo(float)
x0err = xf[0] - xb[0]
x18err = xf[18]-xb[18]
print(f" the difference between x0 times 7^18 {x0err*pow(7,18)}
```

```
the difference between x0 times 7^18 -0.090395113500064361
the difference between x18 -0.0714276134172489
```

The errors clearly forms a line in log scale, this means the error is multiplied by a certain number when we using the reccurence formulae to find the result. We can clearly find that when applying forwards and backwards recurrence formulas, together with the value of xn, the error is multiplies by 7 and 1/7 in each iteration respectively. Therefore after a number of iterations, the error by forwards reccurence formula is multiplied by 7^18 which is obviously a large number, while the error in backwards reccurence formula is multiplied by (1/7)^18, which makes the starting error extermly small. The above results proves the error is roughly multiplied by 7^18 after 18 iterations.

Q2.

a) $$x^{-1} = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ 0 & (D-CA^{-1}B)^{-1} \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1} CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1} CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$

Let $(D-CA^{-1}B) = P$, $(D-CA^{-1}B)^{-1} = P^{-1}$

$$\therefore X \cdot X^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} A^{-1} + A^{-1}BP^{-1}CA^{-1} & -A^{-1}BP^{-1} \\ -P^{-1} \cdot CA^{-1} & P^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I + BP^{-1}CA^{-1} - BP^{-1}CA^{-1} & -BP^{-1} + BP^{-1} \\ CA^{-1} + CA^{-1}BP^{-1}CA^{-1} - DP^{-1}CA^{-1} & -CA^{-1}BP^{-1} + DP^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ CA^{-1} + \underbrace{(CA^{-1}B - D)}_{-P} P^{-1}CA^{-1} & (D-CA^{-1}B)P^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Since we only need to compute the inverse of 2 smaller square matrices instead of a bigger one, the calculation is much easier.

Q2.

b) $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $C=\begin{bmatrix}1&0\\0&0\end{bmatrix}$

$D=\begin{bmatrix}1&2\\0&1\end{bmatrix}$

$\therefore A^{-1}=I^{-1}=I,\ A^{-1}B=B,\ CA^{-1}=C$

$P=D-CA^{-1}B = D-CB=\begin{bmatrix}1&2\\0&1\end{bmatrix}-\begin{bmatrix}1&0\\0&0\end{bmatrix}\cdot\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$=\begin{bmatrix}1&2\\0&1\end{bmatrix}-\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}$

$P^{-1}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$

$X^{-1}=\begin{bmatrix}I&-B\\0&I\end{bmatrix}\begin{bmatrix}I&0\\0&P^{-1}\end{bmatrix}\cdot\begin{bmatrix}I&0\\-C&I\end{bmatrix}$

$=\begin{bmatrix}I&-BP^{-1}\\0&P^{-1}\end{bmatrix}\cdot\begin{bmatrix}I&0\\-C&I\end{bmatrix}=\begin{bmatrix}I+BP^{-1}C&-BP^{-1}\\-P^{-1}C&P^{-1}\end{bmatrix}$

$BP^{-1}C=\begin{bmatrix}0&1\\1&0\end{bmatrix}\cdot\begin{bmatrix}1&-1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}0&1\\1&-1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}$

$BP^{-1}=\begin{bmatrix}0&1\\1&-1\end{bmatrix}$, $P^{-1}C=\begin{bmatrix}1&-1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}$

$\therefore X^{-1}=\left[\begin{array}{cc|cc}1&0&0&-1\\1&1&-1&1\\\hline -1&0&1&-1\\0&0&0&1\end{array}\right]$

Q3.

a) The condition number of a matrix A

is given by $\kappa(A) = \|A\| \|A^{-1}\|$

$\|A\|_1 = \|A\|_\infty = \max(|\alpha|+1, 2)$ $\quad A^{-1} = \frac{1}{\alpha - 1} \begin{bmatrix} 1 & -1 \\ -1 & \alpha \end{bmatrix}$

$\therefore \|A^{-1}\|_1 = \|A^{-1}\|_\infty = \frac{\max(|\alpha|+1, 2)}{\alpha - 1}$

$\therefore \kappa(A) = \frac{(\max(|\alpha|+1, 2))^2}{\alpha - 1}$, for $\kappa(A)$ to be large,

$\alpha - 1$ should be close to 0

but $\alpha - 1$ comes form A^{-1},

which means when A is closer to singular,

$\kappa(A)$ is more likely to be bigger

and $\kappa(A)$ is infinity when A is singular

where $\alpha = 1$

Q3.

b) $\|A\| = \max\left\{\frac{\|Ax\|}{\|x\|}, x \in \mathbb{R}^n, x \neq 0\right\}$, when $x = 0$

$\theta \geq 0$ exists

Since $\|Ax\| \geq \theta\|x\|$ $\forall x$ with $\theta > 0$

$$\forall x \neq 0, \quad \|A\| = \frac{\|Ax\|}{\|x\|} \geq \theta$$

a singular matrix A can turn $Ax = y$ to zero but cant bring x back by $A^{-1}y$ because A^{-1} don't exist.

$\therefore$ if A is singular $\min \frac{\|Ax\|}{\|x\|} = 0$ instead of some $\theta > 0$,

$\therefore$ A is nonsingular

$$\therefore \quad \min \frac{\|Ax\|}{\|x\|} = \min \frac{\|y\|}{\|A^{-1}y\|} = \frac{1}{\max \frac{\|A^{-1}y\|}{\|y\|}}$$

since $\|Ax\| \leq \|A\|\,\|x\|$,

similarly $\|A^{-1}y\| \leq \|A^{-1}\|\,\|y\|$

$$\therefore \max \frac{\|A^{-1}y\|}{\|y\|} = \|A^{-1}\|$$

$$\therefore \min \frac{\|Ax\|}{\|x\|} = \frac{1}{\|A^{-1}\|}, \text{ since } \frac{\|Ax\|}{\|x\|} \geq \theta$$

$$\therefore \frac{1}{\|A^{-1}\|} \geq \theta, \quad \|A^{-1}\| \leq \theta^{-1}$$

Q4、

b
0.100
0.100
0.100
0.100
0.100
0.100
0.100
0.100
0.100

A								
-1.732	1	0	0	0	0	0	0	0
1	-1.732	1	0	0	0	0	0	0
0	1	-1.732	1	0	0	0	0	0
0	0	1	-1.732	1	0	0	0	0
0	0	0	1	-1.732	1	0	0	0
0	0	0	0	1	-1.732	1	0	0
0	0	0	0	0	1	-1.732	1	0
0	0	0	0	0	0	1	-1.732	1
0	0	0	0	0	0	0	1	-1.732

LU								
-1.732	1	0	0	0	0	0	0	0
1.0000568	-1.7324	1	0	0	0	0	0	0
0	1.0003455	-1.732	1	0	0	0	0	0
0	0	1.0001145	-1.7322	1	0	0	0	0
0	0	0	1.0002876	-1.7323901	1	0	0	0
0	0	0	0	1.000236	-1	1	0	0
0	0	0	0	0	0.9998613	-1.7316093	1	0
0	0	0	0	0	0	0.9998256	-1.7316	1
0	0	0	0	0	0	0	0.9995948	-1.732

P-1LU								
-1.732	1	0	0	0	0	0	0	0
1.0000568	-1.7324	1	0	0	0	0	0	0
0	1.0003455	-1.732	0.9999388	5.84E-05	-2.791E-05	-1.48E-05	4.16E-05	0
0	0	1	-1.732	1	0	0	0	0
0	0	0	1	-1.732	1	0	0	0
0	0	0	0	1	-1.732	1	0	0
0	0	0	0	0	1	-1.732	1	0
0	0	0	0	0	0	1	-1.732	1
0	0	0	0	0	0	0	1	-1.732

L								
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-0.5774	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	-0.8661	1.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-1.155	1.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	-1.733	1.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	1640	1.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	-0.0006093	1.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000	-0.5776	1.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.8662	1.000

U								
-1.732	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	-1.155	1.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-0.8659	1.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	-0.5772	1.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.0006099	1.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	-1641	1.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000	-1.731	1.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.154	1.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.8658

	P								
LUPFactorP	1	0	0	0	0	0	0	0	0
	0	1	0	0	0	0	0	0	0
	0	0	0	1	0	0	0	0	0
	0	0	0	0	1	0	0	0	0
	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	0	1	0
	0	0	0	0	0	0	0	0	1
	0	0	1	0	0	0	0	0	0

	L in PLU Decomposition								
LUPFactorL	1	0	0	0	0	0	0	0	0
	-0.5774	1	0	0	0	0	0	0	0
	0	0	1	0	0	0	0	0	0
	0	0	0	1	0	0	0	0	0
	0	0	0	0	1	0	0	0	0
	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	0	1	0
	0	-0.8661	-0.8659	-0.4998	0.0003048	0.5003	0.8662	1	1

	U in PLU Decomposition								
LUPFactorU	-1.732	1	0	0	0	0	0	0	0
	0	-1.155	1	0	0	0	0	0	0
	0	0	1	-1.732	1	0	0	0	0
	0	0	0	1	-1.732	1	0	0	0
	0	0	0	0	1	-1.732	1	0	0
	0	0	0	0	0	1	-1.732	1	0
	0	0	0	0	0	0	1	-1.732	1
	0	0	0	0	0	0	0	1	-1.732
	0	0	0	0	0	0	0	0	0.8658

Ly = b	Ux = y		
y	xhat	x	xhat-x
0.1	-0.3890475	0.15766089	-0.546708382
0.15774	-0.5738302	0.37306867	-0.946898918
0.23661861	-0.5050339	0.58849404	-1.093527978
0.3732945	-0.2006903	0.74620301	-0.946893281
0.74691937	0.25745607	0.80392957	-0.546473497
-1224.8478	0.74676234	0.74620301	0.000559336
-0.6462997	0.58924551	0.58849404	0.000751472
-0.2733027	0.37368424	0.37306867	0.000615571
-0.1367348	0.15792888	0.15766089	0.000267987
L2 norm of (xhat - x)			
1.893819567			

Ly = Pb	Ux = y		
y	xhat	x	xhat-x
0.1	0.15758769	0.15766089	-0.000073200
0.15774	0.37294189	0.37306867	-0.000126783
0.1	0.58848788	0.58849404	-0.000006162
0.1	0.746203	0.74620301	-0.000000004
0.1	0.80393573	0.80392957	0.000006155
0.1	0.74621367	0.74620301	0.000010665
0.1	0.58850636	0.58849404	0.000012316
0.1	0.37307933	0.37306867	0.000010667
0.13650813	0.15766705	0.15766089	0.000006159
L2 norm of (xhat - x)			
0.000148071			

Q4.

$$\begin{bmatrix} -1.732 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1.732 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.732 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.732 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1.732 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1.732 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1.732 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.732 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.732 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0.100 \\ 0.100 \\ 0.100 \\ 0.100 \\ 0.100 \\ 0.100 \\ 0.100 \\ 0.100 \\ 0.100 \end{bmatrix}$$

a) the green colored matrices

b) $\|\hat{x} - x\|_2 \approx 1.894$

c) $\|\hat{x} - x\|_2$ of LUP decomposition is 0.0001481, much smaller than LU factorisation