Q1. $h(x) = f(x) - g(x) = e^{x-2} - \frac{1}{(x+1)^2}$

Step	a	b	m	h(m)
1	0.5	1	0.75	-0.040026
2	0.75	1	0.875	0.040208
3	0.75	0.875	0.8125	5.8324×10^{-4}
4	0.75	0.8125	0.78125	-0.19574
5	0.78125	0.8125	0.796875	-9.461997×10^{-3}
6	0.796875	0.8125	0.8046875	-4.4313×10^{-3}
7	0.8046875	0.8125	0.80859375	-1.92209×10^{-3}
8	0.80859375	0.8125	0.810546875	-6.68939×10^{-4}

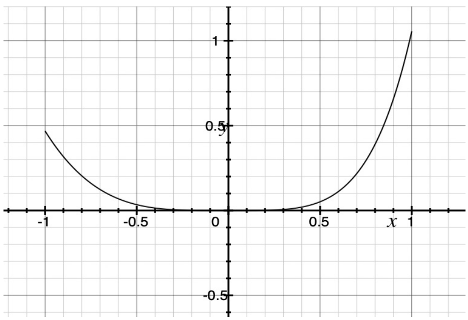
Q2

(a) $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

$f(x) \approx e^0 + 2e^0 \cdot x + \frac{4e^0}{2} \cdot x^2 + \frac{8e^0}{6} \cdot x^3$

$\approx \frac{4}{3}x^3 + 2x^2 + 2x + 1$

(b) $e_3(x) = |f(x) - P(x)| = |e^{2x} - (\frac{4}{3}x^3 + 2x^2 + 2x + 1)|$



We can find the maxima at $x=1$ in $x \in [-1,1]$

$\therefore$ the upper bound of $e_3(x)$ is $e_3(1) \approx 1.056$

(c) $e_k(x) = \frac{f^{k+1}(\xi)}{(k+1)!} \cdot x^{k+1} = \frac{2^{k+1}}{(k+1)!} \cdot e^{2\xi} \cdot x^{k+1}$

on $x \in [-1,1]$, x^{k+1} always has an upperbound at $x=1$

on $\xi \in [-1,1]$, $e^{2\xi}$ has upper bound at $\xi = 1$

$\therefore e_k(x) = \frac{2^{k+1}}{(k+1)!} \cdot e^2$

$\therefore$ we can solve $\frac{2^{k+1}}{(k+1)!} \cdot e^2 = 10^{-5}$ and get $k = 11.985$

$\therefore$ k should be 12 such that the Taylor error is at most 10^{-5}

Q3.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_{n+1}) = f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)$$

$$= f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(x_n)}{2!}(x_{n+1} - x_n)^2$$

$$= f(x_n) + f'(x_n)\left(-\frac{f(x_n)}{f'(x_n)}\right) + \frac{f''(x_n)}{2}\cdot\left(\frac{f(x_n)}{f'(x_n)}\right)^2$$

$$= \frac{f''(x_n)\, f(x_n)^2}{2\cdot f'(x_n)^2}$$

when x_n is close to r

$$\frac{f''(x_n)}{2f'(x_n)^2}\cdot f(x_n)^2 \approx \frac{f''(r)}{2f'(r)^2}\cdot f(x_n)^2 = c\cdot f(x_n)^2$$

$$= f(x_{n+1})$$

$\therefore$ $f(x_n)$ converges to zero quadratically

Q4.

Bisection 2:
$$d_1 = \frac{d_0}{4}, \quad d_2 = \frac{d_0}{4^2}, \quad \cdots, \quad d_n = \frac{d_0}{4^n} = \frac{d_0}{2^{2n}}$$

$$\therefore e_{n+1} \le d_{n+1} = \frac{1}{4} d_n = \frac{1}{4} e_n \quad \text{where } C = \frac{1}{4} < 1$$

$\therefore$ the order of convergence is linear

Newton 2:
$$e_{n+.5} = r - x_{n+.5} = -\frac{f''(\xi_n)}{2f'(x_n)} \cdot e_n^2$$

$$e_{n+1} = -\frac{f''(\xi_{n+.5})}{2f'(x_{n+.5})} \cdot \left(-\frac{f''(\xi_n)}{2f'(x_n)} e_n^2\right)$$

$$= -\frac{f''(\xi_{n+.5})}{2f'(x_{n+.5})} \cdot \left(\frac{f''(\xi_n)}{2f'(x_n)}\right)^2 \cdot e_n^4$$

as x is close to r, ξ is also close to r

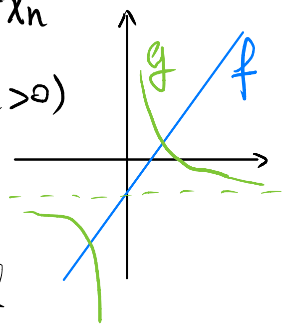
$$\therefore e_{n+1} = C \cdot e_n^4$$

$\therefore$ the order of convergence is 4

Q5. Let $f(x) = cx - 1$, $g(x) = \frac{1}{x} - c$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{cx_n - 1}{c} = \frac{1}{c}$$

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{\frac{1}{x_n} - c}{-\frac{1}{x_n^2}} = 2x_n - cx_n^2$$



since we want to estimate $\frac{1}{c}$ by c ($c>0$)

by $f(x)$ we get $x_{n+1} = \frac{1}{c}$, we can really approximate $\frac{1}{c}$ because $f(x)$ has a fixed $f'(x)$ for $x \in \mathbb{R}$,

by $g(x)$ we can evaluate $\frac{1}{c}$ by iteration $x_{n+1} = 2x_n - cx_n^2$ as long as $x>0$, we only need multiplication here and each iteration will make x_n more accurate and closer to $\frac{1}{c}$.

$\therefore$ the second formula is more useful