Q1.

1) since $(0.11111\cdots)_2 = (2^{-1}+2^{-2}+2^{-3}+\cdots)_{10}$

$$= \frac{1}{2}\cdot\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1 \quad \text{as } n\to\infty$$

$$\therefore \frac{1}{3} = (0.11111\cdots)_2 / (11)_2 = (0.010101\cdots)_2$$

since the mantissa start with 1

$$(1.010101\cdots)_2 = 2^0+\frac{1}{3} = \frac{4}{3} = \frac{1}{3}\cdot 2^2$$

$$\therefore \frac{1}{3} = (1.010101\cdots)_2 \cdot 2^{-2}$$

$$\therefore e=-2, \quad e+v=13=(01101)_2$$

$\frac{1}{3}$ can be written as 0 01101 0101010101

b) $(1.0101010101 0)_2\cdot 2^{-2} = (2^0+2^{-2}+2^{-4}+2^{-6}+2^{-8})\cdot 2^{-2}$

$$= 0.333251953125$$

relative error $e = \dfrac{|0.333251953125-\frac{1}{3}|}{\frac{1}{3}} \approx 2.44140625\times 10^{-4}$

c) the machine epsilon $\varepsilon = 2^{-10}$ because of there are 10 digits at most for half-precision explicit for the normalised $b_0=1$

$$\varepsilon = 9.765625\times 10^{-4} > e$$

Q2.

$$fl(fl(y_1)-fl(y_0)) = (y_1(1+\delta_1) - y_0(1+\delta_2))\cdot(1+\delta_3)$$

$$= (y_1-y_0)\cdot\left(1+\frac{y_1\delta_1-y_0\delta_2}{y_1-y_0}\right)(1+\delta_3)$$

$$= (y_1-y_0)\cdot(1+\delta_4)$$

where $|\delta_4| \leq 2\varepsilon\dfrac{\max(|y_1|-|y_0|)}{|y_1-y_0|} + \varepsilon + O(\varepsilon^2)$

Similarly,

$$fl(fl(x_1)-fl(x_0)) = (x_1-x_0)\cdot(1+\delta_5)$$

where $|\delta_5| \leq 2\varepsilon\dfrac{\max(|x_1|-|x_0|)}{|x_1-x_0|} + \varepsilon + O(\varepsilon^2)$

$$fl\left(\frac{a}{b}\right) = \frac{y_1-y_0}{x_1-x_0}\cdot\frac{(1+\delta_4)(1+\delta_6)}{(1+\delta_5)}$$, where $|\delta_6| \leq \varepsilon$

$$\text{relative error} \leq \frac{1+\delta_4+\delta_6+\delta_4\cdot\delta_6}{1+\delta_5} \leq 1+\frac{\delta_4+\varepsilon+O(\varepsilon^2)-\delta_5}{1+\delta_5}$$

$$\leq 1+\frac{2\varepsilon\frac{\max}{|y_1-y_0|}+\varepsilon-2\varepsilon\frac{\max}{|x_1-x_0|}+O(\varepsilon^2)}{1+2\varepsilon\frac{\max}{|x_1-x_0|}+\varepsilon+O'(\varepsilon^2)}$$

Q3.

a) $x_0 = \frac{1}{3}$, $x_1 = \frac{1}{2}$, $x_2 = \frac{2}{3}$

$$L_0(x) = \frac{(x-\frac{1}{2})(x-\frac{2}{3})}{(\frac{1}{3}-\frac{1}{2})(\frac{1}{3}-\frac{2}{3})} = 18(x-\tfrac{1}{2})(x-\tfrac{2}{3}) = 18x^2-21x+6$$

$$L_1(x) = \frac{(x-\frac{1}{3})(x-\frac{2}{3})}{(\frac{1}{2}-\frac{1}{3})(\frac{1}{2}-\frac{2}{3})} = -36(x-\tfrac{1}{3})(x-\tfrac{2}{3}) = -(36x^2-36x+8)$$

$$L_2(x) = \frac{(x-\frac{1}{2})(x-\frac{1}{3})}{(\frac{2}{3}-\frac{1}{2})(\frac{2}{3}-\frac{1}{3})} = 18(x-\tfrac{1}{2})(x-\tfrac{1}{3}) = 18x^2-15x+3$$

$$\int_0^1 f(x)\,dx \approx \int_0^1 L(x)\,dx$$

$$\approx f(\tfrac{1}{3})\int_0^1 (18x^2-21x+6)\,dx + f(\tfrac{1}{2})\int_0^1 (36x^2-36x+8)\,dx + f(\tfrac{2}{3})\int_0^1 (18x^2-15x+3)\,dx$$

$$= \left[6x^3-\tfrac{21}{2}x^2+6x\right]_0^1 \cdot f(\tfrac{1}{3}) + \left[-(12x^3-18x^2+8x)\right]_0^1 f(\tfrac{1}{2}) + \left[6x^3-\tfrac{15}{2}x^2+3x\right]_0^1 \cdot f(\tfrac{2}{3})$$

$$= \tfrac{3}{2} f(\tfrac{1}{3}) - 2f(\tfrac{1}{2}) + \tfrac{3}{2} f(\tfrac{2}{3})$$

b) $$\int_1^\infty \frac{1-e^{-\frac{1}{x}}}{x^2}\,dx = \int_1^\infty \frac{e^{-\frac{1}{x}}(e^{\frac{1}{x}}-1)}{x^2}\,dx$$

$$= e^{-\frac{1}{x}}(e^{\frac{1}{x}}-1)\Big|_1^\infty - \int_1^\infty -\frac{1}{x^2}\,dx$$

$$= 1-e^{-\frac{1}{x}}\Big|_1^\infty + \int_1^\infty \frac{1}{x^2}\,dx$$

$$= 1-(-e^{-1}) - 1$$

$$= \frac{1}{e}$$

Q3.

c) Let $y=\frac{1}{x}$, $dy=-x^{-2}dx$, $\begin{cases} y_0=\frac{2}{3} \\ y_1=\frac{1}{2} \\ y_2=\frac{1}{3} \end{cases}$

$$\int_1^\infty f(x)\,dx = \int_1^0 f(\tfrac{1}{y}) \cdot -x^2 dy,\ y=\frac{1}{x},\ dy=-x^{-2}dx$$

$$= \int_0^1 f(\tfrac{1}{y})\, y^{-2} dy$$

Let $g(y)=f(\frac{1}{y})\cdot y^{-2}$

$$\therefore \int_1^\infty f(x)\,dx = \int_0^1 g(y)\,dy$$

$$= \frac{3}{2} g(y_0) - 2g(y_1) + \frac{3}{2}(y_2)$$

$$= \frac{3}{2} f(\tfrac{1}{y_0})\cdot y_0^{-2} - 2f(\tfrac{1}{y_1})\cdot y_1^{-2} + \frac{3}{2} f(\tfrac{1}{y_2})\cdot y_2^{-2}$$

$$= \frac{27}{8} f(\tfrac{3}{2}) - 8f(2) + \frac{27}{2} f(3)$$

d) Let $f(x)=\frac{1-e^{-\frac{1}{x}}}{x^2}\,dx$

$$\int_1^\infty f(x)\,dx \approx \frac{27}{8}\cdot\frac{1-e^{-\frac{2}{3}}}{(\frac{3}{2})^2} - 8\cdot\frac{1-e^{-\frac{1}{2}}}{2^2} + \frac{27}{2}\cdot\frac{1-e^{-\frac{1}{3}}}{3^2}$$

$$= 1 - \frac{3}{2} e^{-\frac{2}{3}} + 2e^{-\frac{1}{2}} - \frac{3}{2} e^{-\frac{1}{3}}$$

relative error $\varepsilon = \dfrac{\left|1-\frac{3}{2}e^{-\frac{2}{3}}+2e^{-\frac{1}{2}}-\frac{3}{2}e^{-\frac{1}{3}}-e^{-1}\right|}{|e^{-1}|}$

$$= 7.04670648 1\times 10^{-4}$$

Q4.

The chebyshev-polynomials $T_n(x) = \cos(n \cdot \arccos(x))$

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$\therefore T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \approx \sum_{i=0}^{n} C_i f(x_i) = C_0 \cdot T_0(x) + C_1 \cdot T_1(x) + \cdots + C_n \cdot T_n(x)$$

When $n = 0$:

take x_0 to be the root of $T_1(x) = x$

$\therefore x_0 = 0$, for $f(x) = 1$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = C_0 \cdot f(x)$$

$$\arcsin 1 - \arcsin(-1) = C_0$$

$$\therefore C_0 = \pi$$

When $n = 1$: $T_2(x) = 2x^2 - 1$, we take roots $x_0 = -\frac{\sqrt{2}}{2}$, $x_1 = \frac{\sqrt{2}}{2}$

for $f(x) = 1, x$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = C_0 + C_1 = \pi$$

Let $u = 1 - x^2$

$du = -2x\,dx$

$= \int_{-1}^{1} \frac{1}{2} \cdot u^{-\frac{1}{2}}\,du$

$= [-u^{\frac{1}{2}}]_{-1}^{1}$

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{\sqrt{2}}{2} C_0 + \frac{\sqrt{2}}{2} C_1$$

$$= \left[-\sqrt{1-x^2}\right]_{-1}^{1}$$

$$= 0$$

$$\therefore C_0 = C_1 = \frac{\pi}{2}$$

$$\therefore \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \approx \frac{\pi}{2} f\left(-\frac{\sqrt{2}}{2}\right) + \frac{\pi}{2} f\left(\frac{\sqrt{2}}{2}\right)$$