An Introduction to Discontinuous Galerkin Methods

Module 3A: To Higher-Orders - Solution Approximation

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Weak Form- Revisited: Approximation Space

$$\int_{\Omega} \frac{\partial q}{\partial t} \phi_j \, dx + \int_{\Omega} \frac{\partial f(q)}{\partial x} \phi_j \, dx = 0 \tag{1}$$

- Let's revisit the weak form in greater detail, we glossed over some important details.
- To get a flexible method, we'd like to be able to generate an arbitrary order solution approximation. How do we do this in code?
- We glossed over the construction of the bases, we can think in terms of nodal or modal

$$\widetilde{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x) \tag{2}$$

L^2 Projection: Minimize norm

- Where does the weak form come from? Approximate sol'n doesn't satisfy the PDE exactly, left with some residual

$$\frac{\partial \widetilde{q}}{\partial t}\,dx + \frac{\partial f(\widetilde{q})}{\partial x}\,dx = R(x) \tag{3}$$

- A projection finds the "closest" fit on a vector space compared to the original. An orthogonal projection does just this.
- Minimizes the L^2 norm wrt the space projected upon

The Galerkin Condition

- An orthogonal projection means that the difference between the original and the projection should be orthogonal to the space projected upon, so for each basis $\phi \in V_h$

$$\int (f - f_h)\,\phi\,dx = 0 = \int R(x)\phi\,dx \qquad (4)$$

- The projected space doesn't have to be the same space as the solution approximation, but if we do choose them to be the same we have the Bubnov-Galerkin approach
- We could choose the two spaces to be different, this is a Petrov-Galerkin approach. As an example, a weight space of delta functions yields a collocation method

Arbitrary Order Basis: Monomials?

- Let's continue with the choice of polynomials for our basis functions.
- The easy and obvious choice is the monomials $\psi_i(x) = a_i x^i$ so the arbitrary order solution approximation is

$$\widetilde{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x) = \sum_{i=0}^{M} a_i x^i \tag{5}$$

- Can you think of why this a poor choice for a vector basis? Remember a basis needs to be complete and linearly independent. Let's see two separate ways why this is a poor choice.

Linear Dependence

- Normalize the basis wrt to the L^2 norm $|g| = \sqrt{\int g^2}$ so $\bar{\psi}_i = \sqrt{2i+1}x^i$
- Consider the inner product of two adjacent basis functions

$$\int \bar{\psi}_{i-1}\bar{\psi}_i \, dx = \sqrt{1 - \frac{1}{4i^2}} \tag{6}$$

- Linearly independent vectors are orthogonal, that is their inner product is zero. But for our monomials, even moderate values of i give an inner product of close to 1
- Very nearly dependent basis vectors mean small perturbations can lead to trouble finding the correct solution

The ill-conditioned Hilbert matrix

- The alternate way to consider this is to look at the mass matrix for a monomial basis
- 1
- x
- x^2
- x^3
- When we integrate from 0 to 1 it is easy to see the pattern of values, this is a Hilbert matrix
- It is ill-conditioned, let's see why this is a problem in Matlab

[Recall] Lagrange Interpolation

- Instead of a monomial basis, what if we were to use a Lagrange basis (commonly used for interpolation)
- Recall, the Lagrange basis is zero at all interp nodes except the basis node, which it is 1 at
- It has the form

$$L_q(x) = \prod_{p=1,p\neq q}^{Np} \frac{x - x_p}{x_q - x_p} \tag{7}$$

- We can compactly represent this as

```
nn=elim(nd(1:N)',nd(1:N)',[1 3 2]);
Lag= @(x,nv)
prod(bsxfun(@rdivide,bsxfun(@minus,x,nn(nv,:,:)),...
bsxfun(@minus,nd(nv),nn(nv,:,:))),3);
```

Lagrange Spatial Approximation

- We can now represent our arbitrary order sol'n approximation

$$\widetilde{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x) = \sum_{i=0}^{M} q(x_i) L_i(x) \tag{8}$$

- Where we must choose a collection of points to use as our interpolation nodes

Interpolation Points: Equispaced?

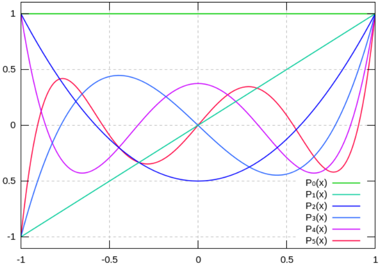
- It would seem logical that equispaced points are a good choice. They evenly cover the domain and are simple to calculate positions for.
- Indeed these points work well for low Np, so one would expect the accuracy to increase as Np increases
- Let's see how it behaves at large Np

Runge's Phenomenon

- The divergence of the interpolation is a result of Runge's phenomenon
- Why this fails is explainable in terms of two relations: the Bernstein and Markov Inequalities. For a polynomial $P^N \in [-1,1]$
$$|P'(x)| \leq \frac{N}{\sqrt{1-x^2}} \|P\|_\infty \tag{9}$$
- The singularity in the Bernstein inequality can be bounded by the Markov inequality
$$\|P'\|\infty \leq N^2 \|P\|_\infty \tag{10}$$
- With the overall being
$$|P'(x)| \leq \min\left(\frac{N}{\sqrt{1-x^2}}, N^2\right) \|P\|_\infty \tag{11}$$

Interpolation Points: Legendre Roots

- The overall inequality implies we'd like quadratic spacing near the element boundaries.
- Though it may seem arbitrary, the roots of the Legendre polynomials have a favorable spacing that matches out needs
- They avoid Runge's phenomenon, and have several other very nice features we shall see in the next module



http://en.wikipedia.org/wiki/File:Legendrepolynomials6.svg