

An Introduction to Discontinuous Galerkin Methods

Module 2: A Simple 1D DG Solver

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Linear Solution Approximation: Definition

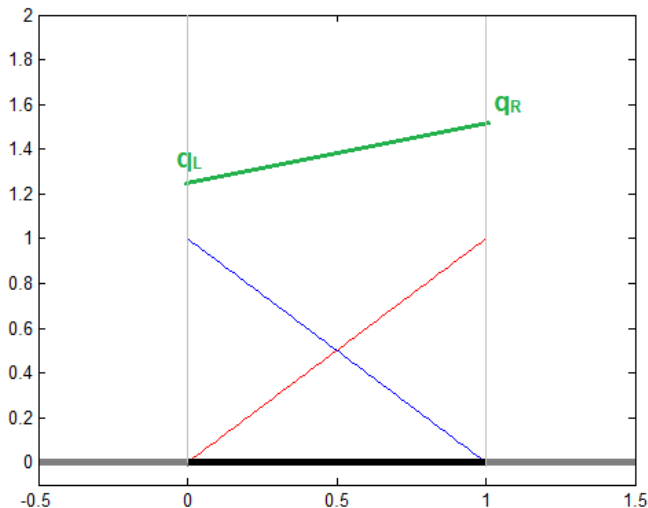
- ▶ The exact sol'n $q(x)$ is not usually obtainable, the best we can do is an approximate sol'n $\tilde{q}(x)$
- ▶ We can think of the approximate solution belonging to some finite vector space $\tilde{q} \in V_h$, this vector space must have a set of basis vectors ψ such that

$$\tilde{q}(x) = \sum_{i=0}^M a_i \psi_i(x) \quad (1)$$

- ▶ We will choose V_h in this simple example to be the space of all polynomials of order 1 and less: $\mathbb{P}^M, M = 1$
- ▶ Because $M = 1$, we must have two basis functions. We shall choose two ramp functions, let's see what they look like...

Bases and Resultant Approximation

- Basis funs (blue) $\psi_0(x) = 1 - x$ and (red) is $\psi_1(x) = x$.
Resultant approximation is linear: $\tilde{q}(x) = a_0(1 - x) + a_1(x)$

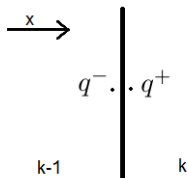


Test Function Choice (Galerkin)

- ▶ We have now defined our approximation space, but what about the weighting function in the weak form: ϕ ?
- ▶ We will discuss this in greater detail, but as in a FEM, DG chooses the space of weighting functions to be the same as the approximation space, $\phi \in V_h$
- ▶ This means our weighting functions are ramp functions as well $\phi_0(x) = \psi_0(x), \phi_1(x) = \psi_1(x)$
- ▶ In order to solve for the approximation, our weak form sol'n must be satisfied for *each* weighting function.

Upwind Flux

- ▶ If our flow is incompressible and the global domain is 1D, we must have a constant flow with velocity c everywhere.
- ▶ Let's assume that $c > 1$, the numerical flux is then $\tilde{c}q^-$ at all boundaries
- ▶ Note that there are two distinct numerical fluxes for element k , on the left it is $cq^-(x_L)$ and the right $cq^-(x_R)$, where $q^-(x_L)$ doesn't necessarily equal $q^-(x_R)$



Mass Matrix

$$\int_k \frac{\partial \tilde{q}}{\partial t} \phi_j dx + [f(\tilde{q})\phi_j] \Big|_{x_L}^{x_R} - \int_k f(\tilde{q}) \frac{d\phi_j}{dx} dx = 0 \quad \text{for all } j \leq N \quad (2)$$

- ▶ From convention for dynamic systems the left integral in the weak form is called the *mass matrix*
- ▶ We now have explicit formula for \tilde{q} and ϕ , substituting

$$\int_k \frac{\partial \tilde{q}}{\partial t} \phi_n dx = \sum_{i=0}^M \frac{da_i(t)}{dt} \int_k \psi_i(x) \phi_j(x) \quad (3)$$

Mass Matrix

$$\int_k \frac{\partial \tilde{q}}{\partial t} \phi_n dx = \sum_{i=0}^M \frac{da_i(t)}{dt} \int_k \psi_i(x) \phi_j(x) dx$$

- ▶ Let's substitute in our expressions for approximation and the first weighting basis and analytically integrate

$$\int_k \psi_i(x) \phi_0(x) dx = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad (4)$$

- ▶ Do this for each weighting basis to get 2x2 matrix: $\mathbf{M}_{j,i}$
- ▶ If the time derivative of each of the approximation coefficients is a vector \mathbf{a}' then

$$\sum_{i=0}^M \frac{da_i(t)}{dt} \int_k \psi_i(x) \phi_j(x) dx = \mathbf{M} \mathbf{a}' \quad (5)$$

Stiffness Matrix

$$\int_k \frac{\partial q}{\partial t} \phi dx + [f(q)\phi] \Big|_{x_L}^{x_R} - \int_k f(q) \frac{d\phi}{dx} dx = 0$$

- Similar to mass matrix, substitute our approximation and bases into the third term.

$$\int_k f(q) \frac{d\phi}{dx} dx = \sum_{i=0}^M ca_i(t) \int_k \psi_i(x) \phi'_j(x) dx \quad (6)$$

- The integral for all weighting bases combine to form the stiffness matrix \mathbf{K} , we can analytically integrate for this simple case

Stiffness Matrix

- ▶ Similar to the mass matrix, we have a vector of our element approximation coefficients \mathbf{a} , such that

$$\sum_{i=0}^M c a_i(t) \int_k \psi_i(x) \phi'_j(x) dx = \mathbf{K} \mathbf{a} \quad (7)$$

- ▶ Let's substitute in our expressions for approximation and the first weighting basis and analytically integrate

$$\int_k \psi_i(x) \phi'_0(x) dx = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad (8)$$

Putting it all Together

- ▶ Just need to calculate the numerical flux vector now.

$$[\hat{f}(q)\phi]\Big|_{x_L}^{x_R} = cq_k(x_R)\phi_j(x_R) - cq_{k-1}(x_R)\phi_j(x_L) \quad (9)$$

- ▶ ϕ_j will be zero sometimes, depending on j
- ▶ So what do we get?
- ▶ $\hat{\mathbf{f}} =$

Semi-discrete system

- ▶ We can now express the semi-discrete system compactly in matrix/vector form

$$\mathbf{M} \mathbf{a}' + \hat{\mathbf{f}} - c\mathbf{K} \mathbf{a} = 0 \quad (10)$$

- ▶ For each time step we have \mathbf{a} , starting with \mathbf{a}_0 from ICs
- ▶ We'd like to calculate the time rate of change of our coefficients to use in our time discretization, so solve for \mathbf{a}'

$$\mathbf{a}' = \left[c\mathbf{K} \mathbf{a} - \hat{\mathbf{f}} \right] \mathbf{M}^{-1} \quad (11)$$

Time Discretization: Forward Euler

- ▶ Now we need to complete the method by discretizing in time the semi-discrete system
- ▶ Let's use a Forward Euler approach; very simple explicit ODE solver

$$\mathbf{a}_{t+1} = \mathbf{a}_t + \mathbf{a}'_t \Delta t \quad (12)$$

- ▶ We can overwrite our previous timestep's coefficients with the new ones to save memory
- ▶ We may want to periodically save \mathbf{a} for plotting
- ▶ We must choose Δt carefully, can you predict why?

Investigation- h-Convergence

- ▶ What do we expect for h-convergence?
- ▶ How coarse can our elements be and still get a "good-enough" answer?
- ▶ What effect does the shape/smoothness of the solution have on convergence?

Investigation- t-Convergence

- ▶ How big can we make Δt ?
- ▶ Do we expect periodic solutions to stay steady over time?
- ▶ How do h and t convergence relate?

Investigation- Stability (CFL)

- ▶ As was seen, too large of a timestep lead to issues, why?
- ▶ What happens if a parcel of quantity moves fast enough to skip elements between timesteps?
- ▶ We need to ensure

$$CFL = \frac{c\Delta t}{\Delta x} \leq C_{max} = 1 \quad (13)$$

for explicit methods. Implicit methods relax this restriction to permit higher values.