An Introduction to Discontinuous Galerkin Methods

Module 2: A Simple 1D DG Solver

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Linear Solution Approximation: Definition

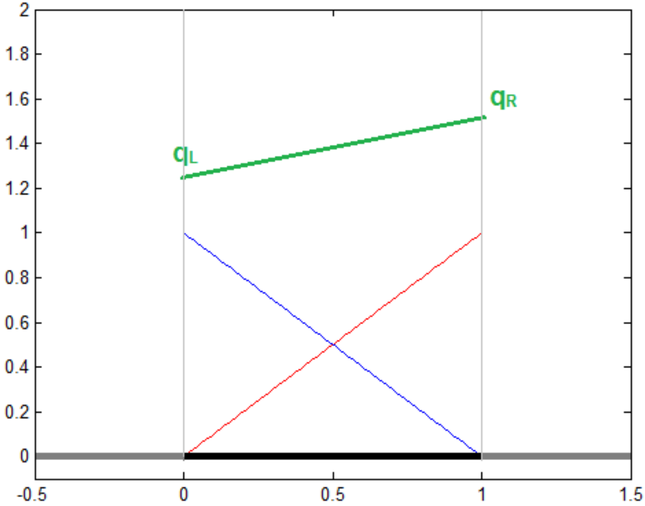
- The exact sol'n $q(x)$ is not usually obtainable, the best we can do is an approximate sol'n $\widetilde{q}(x)$
- We can think of the approximate solution belonging to some finite vector space $\widetilde{q} \in V_h$, this vector space must have a set of basis vectors ψ such that

$$\widetilde{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x) \tag{1}$$

- We will choose V_h in this simple example to be the space of all polynomials of order 1 and less: $\mathbb{P}^M, M = 1$
- Because $M = 1$, we must have two basis functions. We shall choose two ramp functions, let's see what they look like...

Bases and Resultant Approximation

- Basis funs (blue) $\psi_0(x) = 1 - x$ and (red) is $\psi_1(x) = x$. Resultant approximation is linear: $\widetilde{q}(x) = a_0(1 - x) + a_1(x)$

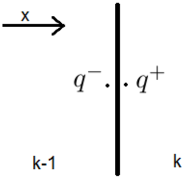


Test Function Choice (Galerkin)

- We have now defined our approximation space, but what about the weighting function in the weak form: ϕ?
- We will discuss this in greater detail, but as in a FEM, DG chooses the space of weighting functions to be the same as the approximation space, $\phi \in V_h$
- This means our weighting functions are ramp functions as well $\phi_0(x) = \psi_0(x), \phi_1(x) = \psi_1(x)$
- In order to solve for the approximation, our weak form sol'n must be satisfied for *each* weighting function.

Upwind Flux

- If our flow is incompressible and the global domain is 1D, we must have a constant flow with velocity c everywhere.
- Let's assume that $c > 1$, the numerical flux is then $c\widetilde{q}^-$ at all boundaries
- Note that there are two distinct numerical fluxes for element k, on the left it is $cq^-(x_L)$ and the right $cq^-(x_R)$, where $q^-(x_L)$ doesn't necessarily equal $q^-(x_R)$



Mass Matrix

$$\int_k \frac{\partial \widetilde{q}}{\partial t}\, \phi_j \, dx + [f(\widetilde{q})\phi_j]\Big|_{x_L}^{x_R} - \int_k f(\widetilde{q})\, \frac{d\phi_j}{dx}\, dx = 0 \quad for\, all\, j \leq N \tag{2}$$

- From convention for dynamic systems the left integral in the weak form is called the *mass matrix*
- We now have explicit formula for $\widetilde{q}$ and ϕ, substituting

$$\int_k \frac{\partial \widetilde{q}}{\partial t}\, \phi_n \, dx = \sum_{i=0}^{M} \frac{da_i(t)}{dt} \int_k \psi_i(x)\phi_j(x) \tag{3}$$

Mass Matrix

$$\int_k \frac{\partial \widetilde{q}}{\partial t}\, \phi_n \, dx = \sum_{i=0}^{M} \frac{d a_i(t)}{dt} \int_k \psi_i(x)\phi_j(x)$$

- Let's substitute in our expressions for approximation and the first weighting basis and analytically integrate

$$\int_k \psi_i(x)\phi_0(x) = \qquad\qquad \Big|_0^1 \tag{4}$$

- Do this for each weighting basis to get 2x2 matrix: $\mathbf{M}_{j,i}$
- If the time derivative of each of the approximation coefficients is a vector $\mathbf{a}$ then

$$\sum_{i=0}^{M} \frac{d a_i(t)}{dt} \int_k \psi_i(x)\phi_j(x) = \mathbf{M}\,\mathbf{a} \tag{5}$$

Stiffness Matrix

$$\int_k \frac{\partial q}{\partial t}\,\phi\,dx + \left[f(q)\phi\right]\Big|_{x_L}^{x_R} - \int_k f(q)\,\frac{d\phi}{dx}\,dx = 0$$

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Putting it all Together

Semi-discrete system

Time Discretization: Forward Euler

Investigation- h-Convergence

Investigation- t-Convergence

Investigation- Stability (CFL)