An Introduction to Discontinuous Galerkin Methods

Module 1: What is DG?

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Module 1: What is DG?

Overall Content Structure: Assumed Prerequisite Knowledge

- It is assumed the interested viewer is an advanced undergrad or graduate student with the typical STEM background of Calculus, Linear Algebra, and ODEs/PDEs.
- Additionally, it is assumed the viewer has at least a basic background in a programming language of their choice (Matlab etc.)
- Finally it is assumed the viewer has taken a general Numerical Methods course as well a Solution of PDEs course.
- Not intended to teach common underlying techniques (interpolation etc.), but we may [Recall] important features of them.

Numerical Methods Prerequisites

- Linear algebra
  - Vector spaces, bases, properties, etc.
  - Orthogonality
  - maybe some useful spaces (Hilbert, square integrable, etc)?
- Polynomial interpolation(1 and 2D)
  - Lagrange, Hermite
  - monomial basis (and ill-conditioned nature)
  - Orthogonal basis (Legendre, Chebyshev)
  - L2 projection
  - choice of interpolation points (equispaced, GL, LGL, etc)
  - Runge phenomenon
  - Vandermonde matrix (transformation from modal to nodal spaces)
- Quadrature (1 and 2D)
  - Newton-Cotes
  - Gauss/Hermite(Legendre)
  - relation to interpolation
- Solution of ODEs
  - Forward Euler
  - RK4
  - Implicit schemes (e.g. Backward Euler)
  - Stability, Convergence

Solution of PDEs Prerequisites

- Domain representation
  - meshing
  - BCs (Neumann and Dirichlet)
- Finite difference methods (FDM)
  - Pointwise spatial derivatives
  - Computational vs Physical domains
  - Basic mapping (bilinear)
- Finite volume methods (FVM)
  - Flux functions
  - Artificial viscosity
  - Linear vs nonlinear fluxes
- Finite element methods (FEM)
  - Weak and strong form formulation
  - Piecewise linear solution approximation
  - Galerkin style test functions
  - local support

Lecture Goals

- Understand DG spatial discretization (advective)
  - DG weak form (test function to minimize residual or test function orthogonal)
  - solution approximation (and initial conditions)
  - mapping physical to computation domain (for curvilinear domains)
  - DG Galerkin formulation
  - Integration by parts $\rightarrow$ flux functions (solution smoothness requirements): differences from FEM
  - linear vs non-linear flux: ramifications for semi-discrete system
  - hyperbolic vs parabolic
  - applying BCs (include periodic BCs)
- Understand time discretization
  - Method of lines style semi-discrete form
  - Types: e.g. Forward Euler, RK4
  - CFL condition and stability
- Learn how to apply DG to arbitrary PDEs and realm of applicability
  - intuitive understanding of methodology
  - conceptualization of process (not tied down to specific examples)
  - understand pros/cons
  - understand how DG "simplifies" to FVM and FEM
- Generate runnable code of your own
  - Self-contained set of knowledge and algorithms to be able to write a full solver

Topics Layout

Module 1: What is DG?
DG motivation (why vs FEM, FVM, FDM)
Scalar conservation law (linear) PDE
Weak form derivation
Global domain vs local element
Multiple-valued element boundaries
Recall: Flux functions

Topics Layout(cont.)

Module 2: A Simple 1D DG Solver

Linear solution approximation
Test function choice (Galerkin)
Upwind flux
Mass Matrix
Stiffness Matrix
Putting it all together (linear system)
Semi-discrete system
Forward Euler
Investigate h-convergence
Investigate t-convergence
Investigate stability (CFL)

Topics Layout(cont.)

Module 3: To Higher-Orders (nodal)
3A: Sol'n Approximation

Revisit weak form

-Approx. space

-L2 Projection minimizes residual norm

-Test space $\rightarrow$ orthogonal

Monomial basis?

Ill-conditioning of monomials

Recall: Lagrange interpolation (code)

Derive Lagrange spatial approximation

Equispaced interp points?

Runge phenomenon

Why: Bernstein/Markov inequality

Roots of Leg instead

Topics Layout(cont.)

3B: Discrete System
Numerical Quadrature (Gauss)
Hermite interpolation (2N+1 quad)
Truncation error/exact quadrature
GL Lagrange orthogonality
Local Mapping Fun
Mass Integral -¿diagonal/inversion
Log differentiation
Flux interpolation
Stiffness Integral
Numerical Flux (interpolated)
Assembly of system
RK4 time discretization
Investigate p-convergence (smoothness reqs)

A Pedagogical Comment

- Take advantage of format: replay, pause, speed up, slow down
- Each section may have subsections, but the overall section is intended to be a self-contained concept. The first slide of a new section has the title format **Section**: Subsection
- Easy to "zone-out", before the start of a new section try and put what you learned into action. Make a code snippet to test your understanding or verify a claimed result etc.
- Each Module has a larger self-contained concept. You should be able to put together a script that accomplishes something substantial.

DG Motivation: Why DG?

Example PDE: Scalar Conservation Law

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The Weak Form of the PDE

Domain Decomposition: Global vs Local

Element Boundaries: Multiply Defined?

[Recall] Flux Functions