An Introduction to Discontinuous Galerkin Methods

Module 2: A Simple 1D DG Solver

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Linear Solution Approximation: Definition

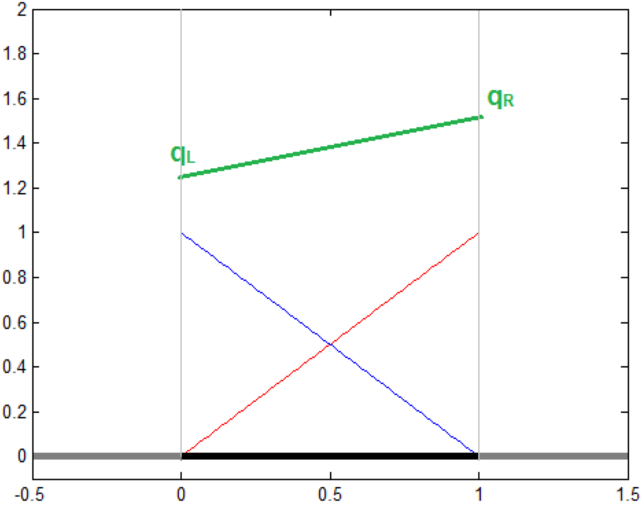
- The exact sol'n $q(x)$ is not usually obtainable, the best we can do is an approximate sol'n $\widetilde{q}(x)$
- We can think of the approximate solution belonging to some finite vector space $\widetilde{q} \in V_h$, this vector space must have a set of basis vectors ψ such that

$$\widetilde{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x) \tag{1}$$

- We will choose V_h in this simple example to be the space of all polynomials of order 1 and less: $\mathbb{P}^M, M = 1$
- Because $M = 1$, we must have two basis functions. We shall choose two ramp functions, let's see what they look like...

Bases and Resultant Approximation

- Basis funs (blue) $\psi_0(x) = 1 - x$ and (red) is $\psi_1(x) = x$. Resultant approximation is linear: $\widetilde{q}(x) = a_0(1-x) + a_1(x)$

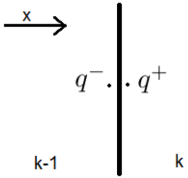


Test Function Choice (Galerkin)

- We have now defined our approximation space, but what about the weighting function in the weak form: ϕ?
- We will discuss this in greater detail, but as in a FEM, DG chooses the space of weighting functions to be the same as the approximation space, $\phi \in V_h$
- This means our weighting functions are ramp functions as well $\phi_0(x) = \psi_0(x), \phi_1(x) = \psi_1(x)$
- In order to solve for the approximation, our weak form sol'n must be satisfied for *each* weighting function.

Upwind Flux

- If our flow is incompressible and the global domain is 1D, we must have a constant flow with velocity c everywhere.
- Let's assume that $c > 1$, the numerical flux is then $c\tilde{q}^{-}$ at all boundaries
- Note that there are two distinct numerical fluxes for element k, on the left it is $cq^{-}(x_L)$ and the right $cq^{-}(x_R)$, where $q^{-}(x_L)$ doesn't necessarily equal $q^{-}(x_R)$



Mass Matrix

$$\int_k \frac{\partial \widetilde{q}}{\partial t} \phi_j \, dx + [f(\widetilde{q})\phi_j]\Big|_{x_L}^{x_R} - \int_k f(\widetilde{q}) \frac{d\phi_j}{dx} \, dx = 0 \quad for\, all\, j \leq N \tag{2}$$

- From convention for dynamic systems the left integral in the weak form is called the *mass matrix*
- We now have explicit formula for $\widetilde{q}$ and ϕ, substituting

$$\int_k \frac{\partial \widetilde{q}}{\partial t} \phi_n \, dx = \sum_{i=0}^{M} \frac{da_i(t)}{dt} \int_k \psi_i(x)\phi_j(x) \tag{3}$$

Mass Matrix

$$\int_k \frac{\partial \tilde{q}}{\partial t}\,\phi_n\,dx = \sum_{i=0}^{M} \frac{da_i(t)}{dt} \int_k \psi_i(x)\phi_j(x)\,dx$$

- Let's substitute in our expressions for approximation and the first weighting basis and analytically integrate

$$\int_k \psi_i(x)\phi_0(x)\,dx = \qquad \Big|_0^1 \tag{4}$$

- Do this for each weighting basis to get 2x2 matrix: $\mathbf{M}_{j,i}$
- If the time derivative of each of the approximation coefficients is a vector $\mathbf{a}'$ then

$$\sum_{i=0}^{M} \frac{da_i(t)}{dt} \int_k \psi_i(x)\phi_j(x)\,dx = \mathbf{M}\,\mathbf{a}' \tag{5}$$

Stiffness Matrix

$$\int_k \frac{\partial q}{\partial t}\,\phi\,dx + \left[f(q)\phi\right]\Big|_{x_L}^{x_R} - \int_k f(q)\,\frac{d\phi}{dx}\,dx = 0$$

- Similar to mass matrix, substitute our approximation and bases into the third term.

$$\int_k f(q)\,\frac{d\phi}{dx}\,dx = \sum_{i=0}^{M} ca_i(t)\int_k \psi_i(x)\phi_j'(x)\,dx \qquad (6)$$

- The integral for all weighting bases combine to form the stiffness matrix $\mathbf{K}$, we can analytically integrate for this simple case

Stiffness Matrix

- Similar to the mass matrix, we have a vector of our element approximation coefficients $\mathbf{a}$, such that

$$\sum_{i=0}^{M} c a_i(t) \int_k \psi_i(x) \phi_j'(x)\, dx = \mathbf{K}\,\mathbf{a} \tag{7}$$

- Let's substitute in our expressions for approximation and the first weighting basis and analytically integrate

$$\int_k \psi_i(x) \phi_0'(x)\, dx = \qquad\qquad\qquad \Big|_0^1 \tag{8}$$

Putting it all Together

- Just need to calculate the numerical flux vector now.

$$\left[\hat{f}(q)\phi\right]\Big|_{x_L}^{x_R} = cq_k(x_R)\phi_j(x_R) - cq_{k-1}(x_R)\phi_j(x_L) \qquad (9)$$

- ϕ_j will be zero sometimes, depending on j
- So what do we get?
- $\hat{\mathbf{f}} =$

Semi-discrete system

- We can now express the semi-discrete system compactly in matrix/vector form

$$\mathbf{M}\,\mathbf{a}' + \hat{\mathbf{f}} - c\mathbf{K}\,\mathbf{a} = 0 \tag{10}$$

- For each time step we have $\mathbf{a}$, starting with $\mathbf{a}_0$ from ICs
- We'd like to calculate the time rate of change of our coefficients to use in our time discretization, so solve for $\mathbf{a}'$

$$\mathbf{a}' = \left[c\mathbf{K}\,\mathbf{a} - \hat{\mathbf{f}}\right]\mathbf{M}^{-1} \tag{11}$$

Time Discretization: Forward Euler

- Now we need to complete the method by discretizing in time the semi-discrete system
- Let's use a Forward Euler approach; very simple explicit ODE solver

$$\mathbf{a}_{t+1} = \mathbf{a}_t + \mathbf{a}_t' \Delta t \tag{12}$$

- We can overwrite our previous timestep's coefficients with the new ones to save memory
- We may want to periodically save $\mathbf{a}$ for plotting
- We must choose Δt carefully, can you predict why?

Investigation- h-Convergence

- What do we expect for h-convergence?
- How coarse can our elements be and still get a "good-enough" answer?
- What effect does the shape/smoothness of the solution have on convergence?

Investigation- t-Convergence

- How big can we make Δt?
- Do we expect periodic solutions to stay steady over time?
- How do h and t convergence relate?

Investigation- Stability (CFL)

- As was seen, too large of a timestep lead to issues, why?
- What happens if a parcel of quantity moves fast enough to skip elements between timesteps?
- We need to ensure

$$CFL = \frac{c\Delta t}{\Delta x} \leq C_{max} = 1 \tag{13}$$

for explicit methods. Implicit methods relax this restriction to permit higher values.