An Introduction to Discontinuous Galerkin Methods

Module 3B: To Higher-Orders - Discrete System

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- Numerical Quadrature (Gauss)
- Hermite Interpolation (and quadrature)
- Truncation error/exact quadrature
- GL Lagrange Orthogonality
- Local Mapping Function (Jacobian)
- Mass Matrix- Diagonalization
- Log differentiation
- Flux interpolation
- Stiffness Integral
- Numerical Flux (Extrapolated)
- Assembly of System
- RK4 Time iscretization
- Investigate p-Convergence

Numerical Quadrature (Gauss)

- We now have a method for generating a robust arbitrary order solution approximation, but unlike before it isn't practical to analytically pre-calculate all the integrals.
- We can use a numerical quadrature technique to do this instead, able to integrate arbitrary functions
- If we call the interpolation of a function Inf then we assume that for a sufficiently accurate interpolation, we can use the interpolation of the function wherever we could use the function itself before. This is in general M-1 order accurate.

$$f \approx Inf = \sum_{i=0}^{M} f(x_i) L_i(x) \tag{1}$$

$$\int f \approx \int Inf = \int \sum_{i=0}^{M} f(x_i) L_i(x) = \sum_{i=0}^{M} f(x_i) \int L_i(x) = \sum_{i=0}^{M} f(x_i) w_i \tag{2}$$

Hermite Interpolation (and quadrature)

- Recall: Hermite interpolation includes derivatives of interpolated function as well
- Consider a Hermite interpolation polynomial that includes first derivatives as well, the interpolation would be $2M-1$ accurate. The quadrature using this polynomial would look like:

$$\int f\,dx \approx \sum_{i=0}^{M} f(x_i)w_i + \sum_{i=0}^{M}\left[f'(x_i)\int (x-x_i)L_i^2(x)\,dx\right] \tag{3}$$

- It turns out if we choose our quadrature/interpolation points to be the Legendre roots, the integral for the second term is zero. Thus no first derivative terms are needed for the Hermite quadrature, even though it is $2M-1$ order accurate.

Truncation error/exact quadrature

- It is important to consider error sources from the approximation to the solution and integrals, these can affect convergence
- Three main error sources in quadrature: aliasing, truncation, and inexact quadrature
- Aliasing occurs if the function is not sampled frequently enough, it is assumed that the sol'n is sufficiently smooth and the discretization suitably fine to avoid this in most cases
- Truncation is unavoidable except where the exact function is of equal or lesser order than the interpolation/quadrature. Higher order terms present in the exact function are left off.
- Inexact quadrature occurs when the total polynomial order of the product of the interpolated functions undergoing quadrature exceeds the exactness of the quadrature. For Gauss-Legendre quadrature this isn't a problem for one and even two functions in the integrand. Each function is of order M-1 and the quadrature is exact for 2M-1, so the quadrature is able to exactly integrate the interpolation

GL Lagrange Orthogonality

- A final useful property of the Lagrange basis with Legendre interpolation points is orthogonality
- The product of two $M-1$ order Lagrange bases can be rearranged to be a Legendre poly of order M and a remainder polynomial of order $M-2$
- The remainder polynomial can be expressed as a linear combination of Legendre polys all of order $< M$, all are orthogonal to the order M Legendre, so

$$\int_{-1}^{1} L_i(x) L_j(x)\, dx = \delta_{ij} w_i \tag{4}$$

Local Mapping Function (Jacobian)

- a

Mass Matrix- Diagonalization

- a

Log differentiation

- a

Flux interpolation

- a

Stiffness Integral

- a

Numerical Flux (Extrapolated)

- Lobatto alternative

Assembly of System

- a

RK4 Time iscretization

- a

Investigate p-Convergence

- smoothness reqs, maybe inequality?