Now consider the example of a homogeneous ellipsoid with density ρ, centered at the origin of the coordinate. Suppose its principal axes of x, y, and z axes are $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, and $\boldsymbol{a_3}$ respectively. The gravitational potential at a point within the spheroid is:

$$\boldsymbol{\phi(x) = -\pi G\rho[A_1\left(a_1^2 - x^2\right) + A_2\left(a_2^2 - y^2\right) + A_3^2(a_3^2 - z^2)]}$$

where

$$\boldsymbol{A_i \equiv a_1 a_2 a_3 \int_0^{\infty} \frac{du}{\Delta\left(a_i^2 + u\right)}\ (i = 1, 2, 3)},$$

$$\Delta \equiv \boldsymbol{\left[\left(a_1^2 + u\right)\left(a_2^2 + u\right)\left(a_3^2 + u\right)\right]^{\frac{1}{2}}}.$$

When $a_1 = a_2 > a_3$, the ellipsoid becomes a MacLaurin spheroid. These integral can be evaluated analytically for the MacLaurin spheroid:

$$A_1 = A_2 = \frac{\sqrt{1-e^2}}{e^3}\sin^{-1} e - \frac{1-e^2}{e^2},$$

$$A_3 = \frac{2}{e^2} - \frac{\sqrt{1-e^2}}{e^3}\sin^{-1} e,$$

where e is the ellipticity of the ellipsoid:

$$e = \sqrt{1 - \left(\frac{\boldsymbol{a_3}}{a_1}\right)^2}.$$

The potential inside the MacLaurin spheroid is:

$$\boldsymbol{\phi(x) = -\pi G\rho\left[2A_1 a_1^2 - A_1(x^2 + y^2) + A_3^2\left(a_3^2 - z^2\right)\right]}.$$

The gravitational potential at a point outside the ellipsoid, the potential is given

$$\phi(x) = \pi G \rho a_1 a_2 a_3 \int_{\lambda}^{\infty} \frac{du}{\Delta}\left(1 - \frac{x^2}{a_1^2+u} - \frac{y^2}{a_2^2+u} - \frac{z^2}{a_3^2+u}\right).$$

Where λ is the positive root of the equation:

$$\frac{x^2}{a_1^2+\lambda} + \frac{y^2}{a_2^2+\lambda} + \frac{z^2}{a_3^2+\lambda} = 1.$$

In the case of the MacLaurin spheroid, the first term of the integral can be evaluated as

$$\int_{\boldsymbol{\lambda}}^{\infty} \frac{\boldsymbol{du}}{\Delta}$$
$$= \int_{\lambda}^{\infty} \frac{du}{(a_1^2+u)(a_3^2+u)^{\frac{1}{2}}}$$
$$= \int_{\lambda}^{\infty} \frac{du}{\frac{\boldsymbol{(a_1^2+u)}}{\boldsymbol{(a_3^2+u)}}(a_3^2+u)^{\boldsymbol{\frac{3}{2}}}}.$$

Let

$$s = \sqrt{\frac{a_1^2-a_3^2}{\boldsymbol{a_3^2+u}}},$$

then

$$1+s^2 = \frac{\boldsymbol{a_3^2+u}+a_1^2-a_3^2}{\boldsymbol{a_3^2+u}} = \frac{a_1^2+\boldsymbol{u}}{\boldsymbol{a_3^2+u}},$$

and

$$\frac{ds}{du} = \sqrt{a_1^2-a_3^2}\,\frac{d}{du}(a_3^2+u)^{-\frac{1}{2}}$$
$$= -\frac{1}{2}\sqrt{a_1^2-a_3^2}(a_3^2+u)^{-\frac{3}{2}},$$

or

$$\frac{du}{(a_3^2+u)^{\frac{3}{2}}} = \frac{-2}{\sqrt{a_1^2-a_3^2}}ds.$$

Using

$$a_1 e = \sqrt{a_1^2 - a_3^2},$$

and a constant

$$h = \sqrt{\frac{a_1^2 - a_3^2}{\boldsymbol{a_3^2} + \lambda}} = \frac{a_1 e}{\sqrt{a_3^2 + \lambda}},$$

the first term of the integral becomes

$$\int_h^\infty \frac{-2}{\sqrt{a_1^2 - a_3^2}} \frac{ds}{1 + s^2}$$
$$= \frac{-2}{\sqrt{a_1^2 - a_3^2}} \tan^{-1} h$$
$$= \frac{-2}{a_1 e} \tan^{-1} h$$

The second and the third term of the integral can be combined

$$-\int_{\boldsymbol{\lambda}}^\infty \frac{\boldsymbol{du}}{\Delta} \left(\frac{\boldsymbol{x^2} + \boldsymbol{y^2}}{\boldsymbol{a_1^2} + \boldsymbol{u}} \right)$$
$$= -\int_\lambda^\infty \frac{(\boldsymbol{x^2} + \boldsymbol{y^2}) du}{(\boldsymbol{a_1^2} + \boldsymbol{u})^2 (a_3^2 + u)^{\frac{1}{2}}}$$
$$= -\frac{\boldsymbol{x^2} + \boldsymbol{y^2}}{2(a_1^2 - a_3^2)} \int_\lambda^\infty \frac{2(a_1^2 - a_3^2) du}{(\boldsymbol{a_1^2} + \boldsymbol{u})^2 (a_3^2 + u)^{\frac{1}{2}}}$$
$$= -\frac{\boldsymbol{x^2} + \boldsymbol{y^2}}{2(a_1^2 - a_3^2)} \int_\lambda^\infty \frac{[(a_1^2 + u) + (a_1^2 - 2a_3^2 - u)] du}{(\boldsymbol{a_1^2} + \boldsymbol{u})^2 (a_3^2 + u)^{\frac{1}{2}}}$$
$$= -\frac{\boldsymbol{x^2} + \boldsymbol{y^2}}{2(a_1^2 - a_3^2)} \int_\lambda^\infty \frac{(a_1^2 + u) du}{(\boldsymbol{a_1^2} + \boldsymbol{u})^2 (a_3^2 + u)^{\frac{1}{2}}} - \frac{\boldsymbol{x^2} + \boldsymbol{y^2}}{2(a_1^2 - a_3^2)} \int_\lambda^\infty \frac{(a_1^2 - 2a_3^2 - u) du}{(\boldsymbol{a_1^2} + \boldsymbol{u})^2 (a_3^2 + u)^{\frac{1}{2}}}.$$

Here, the first term can be evaluated as

$$-\frac{\boldsymbol{x^2} + \boldsymbol{y^2}}{2(a_1^2 - a_3^2)} \int_\lambda^\infty \frac{(a_1^2 + u) du}{(\boldsymbol{a_1^2} + \boldsymbol{u})^2 (a_3^2 + u)^{\frac{1}{2}}}$$
$$= -\frac{\boldsymbol{x^2} + \boldsymbol{y^2}}{2(a_1^2 - a_3^2)} \int_\lambda^\infty \frac{du}{(a_1^2 + u)(a_3^2 + u)^{\frac{1}{2}}}$$
$$= -\frac{\boldsymbol{x^2} + \boldsymbol{y^2}}{2(a_1^2 - a_3^2)} \frac{-2}{\sqrt{a_1^2 - a_3^2}} \tan^{-1} h$$

$$= \frac{\boldsymbol{x^2 + y^2}}{(a_1 e)^3} \tan^{-1} h$$

The second term can be evaluated as

$$-\frac{\boldsymbol{x^2 + y^2}}{2(a_1^2 - a_3^2)} \int_\lambda^\infty \frac{(a_1^2 - 2a_3^2 - u)du}{\left(\boldsymbol{a_1^2 + u}\right)^2 (a_3^2 + u)^{\frac{1}{2}}}$$

$$= -\frac{\boldsymbol{x^2 + y^2}}{2(a_1^2 - a_3^2)} \int_\lambda^\infty \frac{[(a_1^2 + u) - 2(a_3^2 - u)]du}{\left(\boldsymbol{a_1^2 + u}\right)^2 (a_3^2 + u)^{\frac{1}{2}}}$$

$$= -\frac{\boldsymbol{x^2 + y^2}}{(a_1^2 - a_3^2)} \int_\lambda^\infty \left(\frac{1}{2(a_1^2 + u)(a_3^2 + u)^{\frac{1}{2}}} - \frac{(a_3^2 + u)^{\frac{1}{2}}}{a_1^2 + u} \right) du$$

$$= -\frac{\boldsymbol{x^2 + y^2}}{(a_1^2 - a_3^2)} \frac{\sqrt{a_3^2 + \lambda}}{a_1^2 + \lambda}$$

$$= -\frac{\boldsymbol{x^2 + y^2}}{(a_1^2 - a_3^2)^{\frac{3}{2}}} \left(\sqrt{a_1^2 - a_3^2} \frac{\sqrt{a_3^2 + \lambda}}{a_1^2 + \lambda} \right)$$

$$= -\frac{\boldsymbol{x^2 + y^2}}{(a_1 e)^3} \frac{h}{1 + h^2},$$

where in the last equality, we use:

$$1 + h^2 = 1 + \frac{a_1^2 - a_3^2}{a_3^2 + \lambda} = \frac{a_1^2 + \lambda}{a_3^2 + \lambda},$$

and

$$\frac{h}{1 + h^2} = \sqrt{\frac{a_1^2 - a_3^2}{\boldsymbol{a_3^2} + \lambda}} \frac{a_3^2 + \lambda}{a_1^2 + \lambda} = \sqrt{a_1^2 - a_3^2} \frac{\sqrt{a_3^2 + \lambda}}{a_1^2 + \lambda}.$$

The third term of the integral

$$-\int_\lambda^\infty \frac{du}{\Delta} \frac{z^2}{a_3^2 + u}$$

$$= -\int_\lambda^\infty \frac{z^2 du}{\left(\boldsymbol{a_1^2 + u}\right)(a_3^2 + u)^{\frac{3}{2}}}$$

$$= -\frac{z^2}{\boldsymbol{2\left(a_1^2 - a_3^2\right)}} \int_\lambda^\infty \frac{\boldsymbol{2\left(a_1^2 - a_3^2\right)} du}{\left(\boldsymbol{a_1^2 + u}\right)(a_3^2 + u)^{\frac{3}{2}}}$$

$$= -\frac{z^2}{\mathbf{2}\left(\boldsymbol{a_1^2}-\boldsymbol{a_3^2}\right)}\int_\lambda^\infty \frac{\mathbf{2}\left(\boldsymbol{a_1^2}+u\right)du}{\left(\boldsymbol{a_1^2}+\boldsymbol{u}\right)(a_3^2+u)^{\frac{3}{2}}} + \frac{z^2}{\mathbf{2}\left(\boldsymbol{a_1^2}-\boldsymbol{a_3^2}\right)}\int_\lambda^\infty \frac{\mathbf{2}\left(\boldsymbol{a_3^2}+u\right)du}{\left(\boldsymbol{a_1^2}+\boldsymbol{u}\right)(a_3^2+u)^{\frac{3}{2}}}$$

$$= \frac{2z^2}{\left(\boldsymbol{a_1^2}-\boldsymbol{a_3^2}\right)}\int_\lambda^\infty \left(-\frac{1}{2(a_3^2+u)^{\frac{3}{2}}}\right)du + \frac{z^2}{\left(\boldsymbol{a_1^2}-\boldsymbol{a_3^2}\right)}\int_\lambda^\infty \frac{du}{\left(\boldsymbol{a_1^2}+\boldsymbol{u}\right)(a_3^2+u)^{\frac{1}{2}}}$$

$$= \frac{2z^2}{\left(\boldsymbol{a_1^2}-\boldsymbol{a_3^2}\right)}\frac{1}{(a_3^2+\lambda)^{\frac{1}{2}}} - \frac{z^2}{\left(\boldsymbol{a_1^2}-\boldsymbol{a_3^2}\right)}\frac{\mathbf{2}}{\sqrt{\boldsymbol{a_1^2}-\boldsymbol{a_3^2}}}\mathbf{tan^{-1}}\,\boldsymbol{h}$$

$$= \frac{2z^2h}{(a_1e)^3} - \frac{\mathbf{2}z^2}{(a_1e)^3}\mathbf{tan^{-1}}\,\boldsymbol{h},$$

where in the last equality, we use:

$$\frac{1}{\left(\boldsymbol{a_1^2}-\boldsymbol{a_3^2}\right)(a_3^2+\lambda)^{\frac{1}{2}}} = \frac{1}{\boldsymbol{a_1^2}-\boldsymbol{a_3^2}}\frac{h}{\sqrt{\boldsymbol{a_1^2}-\boldsymbol{a_3^2}}} = \frac{h}{(a_1e)^3}.$$

The potential outside the MacLaurin spheroid is:

$$\phi(x) = \pi G\rho\boldsymbol{a_1^2}a_3\left(\frac{\mathbf{-2}}{\boldsymbol{a_1e}}\mathbf{tan^{-1}}\,\boldsymbol{h}\right.$$

$$\left.+\frac{1}{(\boldsymbol{a_1e})^3}\left((x^2+y^2)\left(\tan^{-1}h-\frac{\boldsymbol{h}}{\mathbf{1}+\boldsymbol{h^2}}\right)+2z^2(h-\tan^{-1}h)\right)\right)$$

$$= -\frac{2a_3}{e^2}\pi G\rho\left[a_1e\,\mathbf{tan^{-1}}\,h\right.$$

$$\left.-\frac{1}{2a_1e}\left((x^2+y^2)\left(\tan^{-1}h-\frac{\boldsymbol{h}}{\mathbf{1}+\boldsymbol{h^2}}\right)+2z^2(h-\tan^{-1}h)\right)\right]$$