

# A short elementary proof of the Mohr-Mascheroni Theorem

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## 1 Introduction

In 1797 Lorenzo Mascheroni surprised the mathematical world with the theorem that every geometric construction that can be carried out by compasses and ruler may be done without ruler (see [1]). It turned out later that Georg Mohr proved this theorem in 1672 already (see [2]). The proofs given by Mohr and Mascheroni are quite complicated. Later easier proofs have been developed (see [3] or [4]). Furthermore the proof could be simplified by means of the circular inversion (see [5] or [6]). Here we give a very short and direct proof for the theorem that does not appeal to inversion.

## 2 The Mohr-Mascheroni Theorem

**Theorem** *Every geometric construction carried out by compasses and ruler can be done without ruler.*

### Proof

We have to prove that the following three fundamental constructions are possible to carry out with compasses alone.

1. Points of intersection of two circles given by its centers and radii.
2. Points of intersection of a circle (given by center and radius) and a straight line (given by two points).
3. Point of intersection of two straight lines each of them given by two points.

There is nothing to prove for the intersection of two circles, so let us consider

### 2.1 Points of intersection of a circle and a straight line

Here we have to distinguish two cases:

1. The straight line misses the center of the circle.
2. The straight line passes through the center of the circle.

The first case is covered by the following construction:

#### Construction 1

If the straight line  $g$  is given by the points  $P_1$  and  $P_2$ , we reflect the center  $M$  of the given circle  $K$  with respect to  $g$  as Figure 1 indicates. Then we find the two points of intersection  $\{X, Y\} = K \cap g$  as the points of intersection of  $K$  and the reflected circle  $K'$ .

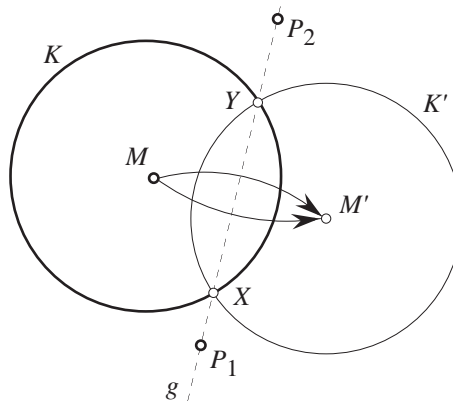


Figure 1

Before we are able to attack the second case, we need to have a construction which allows to bisect a segment  $AB$  without ruler. This can be done as follows:

### Construction 2

Let  $K_1$  be the circle through  $B$  with center  $A$  and  $K_2$  the circle through  $A$  with center  $B$  with  $K_1 \cap K_2 = \{C, D\}$  (see Figure 2). We then find a point  $E$  as the intersection of  $K_2$  and the circle  $K_3$  through  $D$  around  $C$ . Note that  $B$  is the bisection point of  $AE$ . Let  $F$  and  $G$  be the points of intersection of  $K_1$  and the circle  $K_4$  through  $A$  around  $E$ . Then we get the bisection point  $M$  of  $AB$  as intersection of the circles  $K_5$  through  $A$  around  $F$  and  $K_6$  through  $A$  around  $G$ .

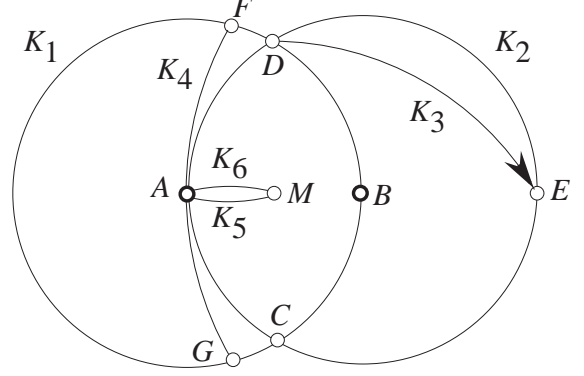


Figure 2

The correctness of the construction is evident: Note that the triangles  $FAM$  and  $EFA$  are similar with proportion  $1 : 2$ .

**Remark 1** Note that  $AE$  has double the length of the segment  $AB$ !

Now we construct the points of intersection  $X$  and  $Y$  of a circle  $K$  with center  $M$  and a straight line  $MP$ :

### Construction 3

Let  $A$  be an arbitrary point on  $K$  and  $K \cap AP = \{A, B\}$  (see Figure 3).  $B$  is constructed according to construction 1. Let  $K_1$  be a circle through  $A$  and  $B$  with radius larger than the radius  $R$  of  $K$  and  $M_1$  the center of  $K_1$ . Now we construct a segment  $CD$  with endpoints on  $K_1$  and length  $2R$  (see Remark 1). Then we obtain  $P'$  as the intersection of  $CD$  and the circle  $K_2$  through  $P$  around  $M_1$  according to construction 1. Let  $M_3$  be the bisection point of  $CD$  (see construction 2) and  $K_3$  the circle around  $M_3$  through  $C$ . Let  $E$  be a point of  $K_3$  with  $P'E = PB$ . Now  $X$  and  $Y$  lie on  $K$  and  $BX = EC$  and  $BY = ED$ .

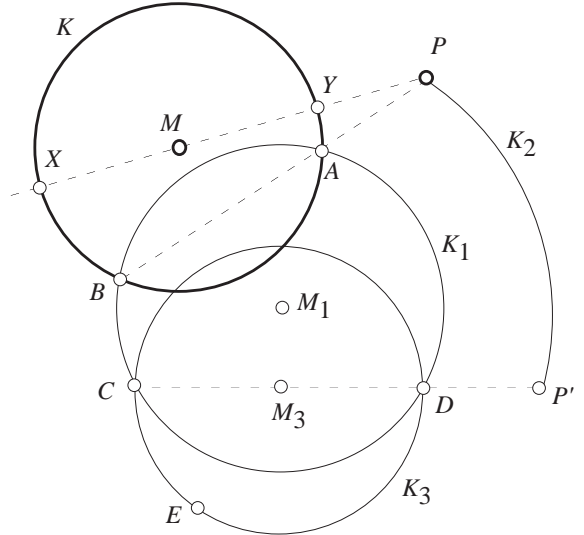


Figure 3

The correctness of the construction can be verified as follows: Note that  $PX \cdot PY = PA \cdot PB = P'C \cdot P'D$  by applying Euler's Theorem on intersecting secants, once for  $K$  and then for  $K_1$ . Hence the sets of points  $P, Y, M, X, B$  and  $P', D, M_3, C, E$  are congruent by construction. Thus in fact  $X$  and  $Y$  are obtained as described.

## 2.2 Point of intersection of two straight lines

Here we first need the following construction with compasses alone which allows to construct the footpoint  $L$  of the perpendicular through a point  $Q$  on a straight line  $P_1P_2$ :

### Construction 4

Just reflect  $Q$  with respect to  $P_1P_2$  (see Figure 4). If  $Q'$  is the reflected point, we find  $L$  as the bisection point of  $QQ'$  by construction 2.

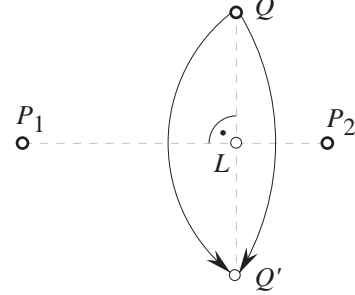


Figure 4

Let us now analyze the situation of two straight lines  $P_1P_2$  and  $Q_1Q_2$  intersecting in  $S$  (see Figure 5):

Let  $L$  be the footpoint of the perpendicular through  $Q_1$  on  $P_1P_2$  and  $N$  be the footpoint of the perpendicular through  $L$  on  $Q_1Q_2$ . Both  $L$  and  $N$  are obtained by construction 4. Hence we have the relation

$$(Q_1L)^2 = Q_1N \cdot Q_1S.$$

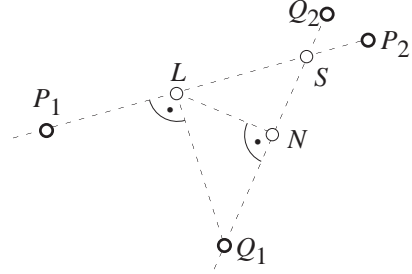


Figure 5

The idea is now to construct the length  $l$  of  $Q_1S$  since then we find  $S$  as intersection of  $Q_1Q_2$  and a circle with center  $Q_1$  and radius  $l$  (see construction 3) and we are through! In fact  $l$  is obtained as follows:

### Construction 5

First we double  $Q_1L$  ( $Q_1C = 2Q_1L$ ) according to Remark 1 (see Figure 6). Let  $K$  be an arbitrary (but large enough) circle through  $Q_1$  and  $C$  and let  $D$  be a point of  $K$  with  $LD = Q_1N$ . Further let  $E$  the intersection of  $LD$  and  $K$  (see construction 1). Then  $LE$  has length  $l$  since we have  $(Q_1L)^2 = Q_1L \cdot LC = LD \cdot LE = Q_1N \cdot LE$  by Euler's theorem on intersecting chords in a circle.

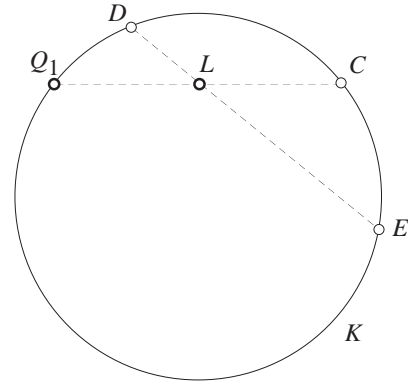


Figure 6

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