Twist Triviality of Canonical Seifert Surfaces

Masterarbeit

der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

eingereicht von
Michael Pfeuti
2014

Betreut durch

Prof. Dr. Sebastian Baader

Mathematisches Institut
der Universität Bern

Contents

1 Introduction

The goal of this master thesis is threefold. Firstly, we introduce the notion of twist trivial surfaces which is based on an operation called ribbon twist. We call a surface twist trivial if the surface can be changed into a standard n-fold punctured torus with ribbon twists. Secondly, we present a proof for the fact that canonical Seifert surfaces, which are produced by the Seifert algorithm, are twist trivial. Thirdly, we introduce the untwisting number of a twist trivial surface which is the minimal number of ribbon twists required to change a surface into a standard n-fold punctured torus. We also derive an upper and lower bound for the untwisting number of a twist trivial surface.

A *knot* is an embedded circle in $\mathbb{R}^3$ and a *link* with $n \in \mathbb{N}$ components is a disjoint union of n knots. Two links L_1, L_2 are called *equivalent*, denoted as $L_1 \cong L_2$, if there exists an ambient isotopy $H : [0,1]\times\mathbb{R}^3 \to \mathbb{R}^3$ such that $H(0,\cdot) = id_{\mathbb{R}^3}$ and $H(1, L_1) = L_2$. One aspect of knot theory is to tell two knots or links apart. This can be achieved by link invariants. A *link invariant* is a mathematical object $I(L)$ (number, group, ...) such that if $L_1 \cong L_2$ then $I(L_1) = I(L_2)$. It is often helpful to analyze a link not in $\mathbb{R}^3$ but in a projection onto a plane. Not all projections are well suited because the projection should not hide any crossings of the knot nor should the crossings be ambiguous. The admissible projections are injective except at finitely many crossing points where the preimage consists of only two points and the two line segments at a crossing intersect transversally. These projections are called *regular projections*. Clearly, in the projection we lose information about the crossings because we do not know which of the strands of a crossing lies on top of which. By endowing this information about each crossing, we obtain a 2D representation of a link which is called *regular diagram*. It contains the most relevant information about the link.

One of the first properties that comes to mind when thinking about knots is: "Can this knot be unknotted?" A knot K is an embedded circle in $\mathbb{R}^3$ and is therefore closed. In other words, K does not have any lose ends. So, in general, K cannot be unknotted without cutting the knot open. To formalize the unknotting sensibly, we consider a regular diagram of a knot and allow crossing changes, which means that at a crossing the strand underneath becomes the overcrossing strand and vice versa (Figure 1). The minimal number of such crossing changes over all regular projections required to turn a knot K into a knot which is equivalent to the unknot is known as the *unknotting number* $u(K)$. This is one of the most fundamental and straightforward knot invariants. The unknotting number can also be defined analogously for links and it is also a link invariant. It is easy to find bounds for $u(L)$ but it is in general very difficult to find the exact value ([Ble84, Wen37]).

Another prevalent method for studying a link L is to analyze compact, connected, orientable surfaces Σ that have the link as their boundaries $\partial\Sigma = L$. Such a surface is known as a *Seifert surface* of the link L. The well known fact that every link has a Seifert surface was shown by Seifert in [Sei35] and will be discussed in detail in Section 2.

It is true that every link L can be unknotted by $u(L)$ crossing changes (Chapter 3.1 in [Ada94]). In this master thesis we are interested in how a crossing change of L affects a Seifert surface Σ of this link. To understand how the surface changes, it suffices to look at the effect of a crossing change locally because crossing changes are local. A crossing

(a) Trefoil knot (b) Unknotted trefoil knot

Figure 1: The trefoil knot can be unknotted with one crossing change.

change involves two strands. Now, there are two cases. In the first case, the surface forms a ribbon between the two strands. This ribbon features a half twist caused by the crossing of the two strands (Figure 2a). The effect of a crossing change on the surface is a full twist such that the half twist twists in the opposite direction (Figure 2b). Notice that the resulting surface is still a Seifert surface. In the second case, the surface in the vicinity of the crossing is disconnected. Therefore, there are two surface patches, one for each strand (Figure 2c). Here, a crossing change causes the surface patches to be swapped. In general, this causes the surface to be self-intersecting and the resulting surface ceases to be a Seifert surface (Figure 2d).

(a) Ribbon surface (b) Crossing change changes twist direction. (c) Surface patches (d) Crossing change causes self-intersection.

Figure 2: Crossing changes can cause the Seifert surface to remain a Seifert surface (a, b) or it can lead to a self-intersection (c, d).

As the latter of the two cases does not lead to a Seifert surface, we want to exclude such crossing changes. Therefore, we generalize the crossing changes of the first case to a notion that does not depend on the projection. We allow an operation of inserting full twists into any ribbon. We call this operation a *ribbon twist*. Then, we pose the question if a link along with its Seifert surface can be unknotted by isotopies and ribbon twists. Foremost, we would like to know if the link can be unknotted with isotopies and ribbon twists such that the resulting surface is a standard n-fold punctured torus. Notice, it suffice to consider if the surface can be transformed into a standard n-fold punctured torus since the associated link is automatically unknotted in the process. We focus on the standard n-fold punctured torus because this surface is in a sense unknotted. This is also the reason why we call surfaces that can be brought into this form by ribbon twists *twist trivial*.

The well known surface classification theorem states that a connected, compact, orientable surface is homeomorphic to a standard n-fold torus (Theorem 77.5 in [Mun00]). The standard n-fold torus is punctured if the surface has boundary components. The number of holes equals the number of boundary components and the number of tori is the genus of the surface. So, if we allowed any homeomorphism instead of just isotopies and ribbon twists then the answer to our question is positive. However, by allowing only isotopies and ribbon twists we cannot expect to obtain a similar result for any surface with boundaries.

In this master thesis we provide a necessary condition for surfaces to be twist trivial in Theorem 11. Moreover, the main result of our work is Theorem 13, which states that any canonical Seifert surface of any link is twist trivial. In analogy to the unknotting number of links, we define the untwisting number of twist trivial surfaces which is the minimal number of ribbon twists required to change a surface into a standard n-fold torus. As a corollary of Theorem 13 we obtain an upper bound for the untwisting number of canonical Seifert surfaces.

This thesis is structured in four sections. Section 2 is concerned with Seifert surfaces. We present the well known Seifert algorithm ([Sei35]). Furthermore, a recent result about Seifert surface simplification is shown ([Aal14]). This result is the basis for the proof of the main theorem of this thesis, Theorem 13. Section 3 introduces Seifert graphs and discusses two lemmata which also play an essential role in the proof of Theorem 13. Section 4 introduces the notion of twist triviality. A necessary condition for twist triviality is presented and the proof of the main theorem is carried out. In the last section we introduce the untwisting number of twist trivial surfaces. As an application of Theorem 13, we derive an upper bound for untwisting number of a canonical Seifert surface. Moreover, a lower bound of the untwisting number of any twist trivial surface is given.

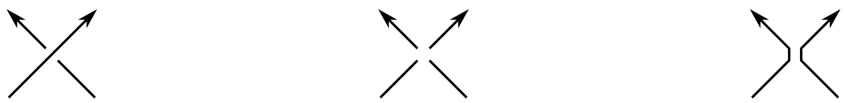
I would like to thank Prof. Dr. Sebastian Baader for offering me this topic as a master thesis and for all the insightful discussions and ideas. Furthermore, I would like to thank Filip Misev, Livio Liechti and Luca Studer for their helpful advice and discussions.

2 Seifert Surfaces

In knot theory, one of the main aspects is the classification of knots, one dimensional submanifolds in $\mathbb{R}^3$. It turned out that Seifert surfaces, two dimensional submanifolds in $\mathbb{R}^3$ whose boundaries coincide with the knots, are very essential for studying the knots themselves. One of the most fundamental theorems was proven by Seifert in 1935 ([Sei35]). He presented an algorithm, now known as the Seifert algorithm, which allows the construction of a Seifert surface for any link (Theorem 1). It laid the foundation for numerous methods for studying knots. For instance, a Seifert surface is required to define the associated Seifert matrix, which is then required for the definition of important knot invariants such as the knot signature or the Alexander polynomial ([Ale28, Mur65, Cro04]).

Theorem 1 ([Sei35])**.** *Every link has a Seifert surface.*

Proof (Seifert Algorithm). First, if the given link is not oriented choose an orientation. Then, choose a regular projection of the link. In the corresponding regular diagram, remove each crossing by deleting the crossing and then reconnect the loose ends such that the orientations are preserved and the ends are exchanged (Figure 3).



(a) Initial crossing (b) Deleted crossing (c) Reconnected strands

Figure 3: Remove a crossing by exchanging ends.

After removing every crossing, the diagram consists only of a set of disjoint closed curves. These curves are known as *Seifert circles.* There are two types of Seifert circles. A *type I Seifert circle* does not enclose nor is it enclosed in any other Seifert circle. A *type II Seifert circle* does enclose or is enclosed in at least one other Seifert circle.

After removing all crossings, each Seifert circle bounds a disc embedded in the projection plane. Furthermore, each Seifert circle has either a clockwise or counterclockwise orientation. Therefore, it is possible to assign each disc which is bounded by a clockwise oriented Seifert circle the label "front" (illustrated as light gray) and for the counterclockwise case the label "back" (illustrated as dark gray).

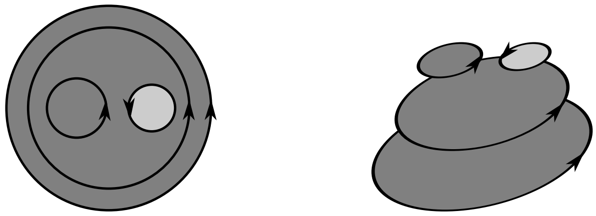
For type II Seifert circles, the discs are considered to be stacked in $\mathbb{R}^3$. In other words, the deeper the circles are nested the higher they are (Figure 4).

The final step is to connect the discs by half twisted bands where there used to be a crossing. The half twists of the bands must be chosen such that their boundaries agree with the initial crossing signs (Figure 5).

After the insertion of all half twisted bands, the surface might still have more than one connected component. In this case, simply join the N components by adding $N-1$ unknotted tubes that agree with the orientation.

The surface obtained in this way has the given link as its boundary and is orientable. Figure 6 illustrates the algorithm. $\square$

Definition 2. A *canonical Seifert surface* is a Seifert surface that can be obtained by the Seifert algorithm.



(a) Discs from type II Seifert circles.

(b) The same discs in $\mathbb{R}^3$ as a stack.

Figure 4: Disc stack from nested (type II) Seifert circles.

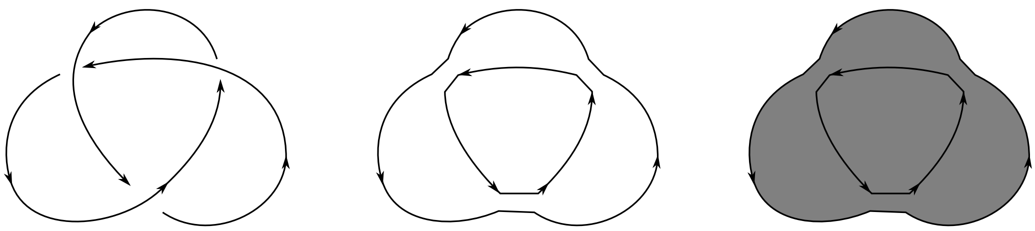
(a) Positive crossing

(b) Half twisted band for positive crossing.

(c) Negative crossing

(d) Half twisted band for negative crossing.

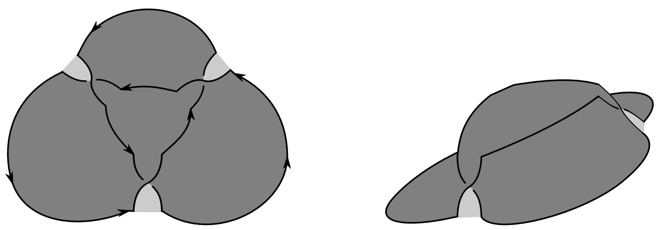
Figure 5: Crossing signs and the corresponding half twisted bands.



(a) Trefoil knot

(b) Seifert circles

(c) Oriented discs from Seifert circles.



(d) Inserted half twisted bands.

(e) Perspective view of the resulting canonical Seifert surface.

Figure 6: Seifert algorithm executed on the trefoil knot.

It is well known that the canonical Seifert surfaces of a link need not be homeomorphic. This follows from the fact that the genus of a canonical Seifert surface can be computed by $1+\frac{1}{2}(c-s-b)$ where c is the number of crossings, s the number of Seifert circles and b the number of link components (Corollary 5.1.3 in [Cro04]). The canonical Seifert surface depends on the orientation of the link, on the chosen projection for the regular diagram on which the Seifert algorithm operates and on how the components are connected by unknotted tubes. So we can only speak about "a" canonical Seifert surface and not about "the" canonical Seifert surface of a link.

The Seifert algorithm gives us a canonical Seifert surface for a link but not all Seifert surfaces of a link are canonical. In order to present a Seifert surface that is not canonical we first state a well known fact. Namely, if Σ is a canonical Seifert surface of a link L then the fundamental group $\pi_1(\mathbb{R}^3 \setminus \Sigma)$ is free ([Kaw96]). We see that we can construct an example by presenting a Seifert surface whose fundamental group of the surface complement is not free. We discuss an example in Section 4 (Figure 13).

In the following we call the discs that are bounded by Seifert circles *Seifert discs*. Likewise, if the Seifert disc is bounded by a type I Seifert circle (respectively type II) we call it *type I Seifert disc* (respectively *type II*). Similarly, a half twisted band is *positive* (respectively *negative*) if it was obtained from a positive (respectively negative) crossing. We saw that a canonical Seifert surface is constructed from Seifert discs, half twisted bands and unknotted tubes. So, we usually interpret a canonical Seifert surface Σ as a set of Seifert disc $\{D_1, \ldots, D_n\}$, a set of half twisted bands $\{B_1, \ldots, B_m\}$ and a set of unknotted tubes $\{T_1, \ldots, T_o\}$. We write $\Sigma = \{D_1, \ldots, D_n; B_1, \ldots, B_m; T_1, \ldots, T_o\}$ for a canonical Seifert surface Σ. The number of half twisted bands attached to a Seifert disc D_i is called the *degree of* D_i and is denoted as $deg(D_i)$. We write $B(D_i)$ for the set of half twisted bands attached to disc D_i. If the unknotted tubes $\{T_1, \ldots, T_o\}$ are removed from a canonical Seifert surface we are left with $o+1$ connected components. We refer to such a connected component simply as a *component* of the canonical Seifert surface. In the following we derive certain results which are only true for individual components of a canonical Seifert surface. In this case we write $\Sigma = \{D_1, \ldots, D_n; B_1, \ldots, B_m; -\}$, which indicates that there are no unknotted tubes and therefore the surface consists of only one component. Lastly, two half twisted bands B_k, B_l are called *adjacent* if B_k and B_l connect the same Seifert discs, $B_k, B_l \in B(D_i) \cap B(D_j)$, and there does not exist any band between B_k and B_l. Figure 7 shows an example.

(a) Adjacent bands (b) Non-adjacent bands

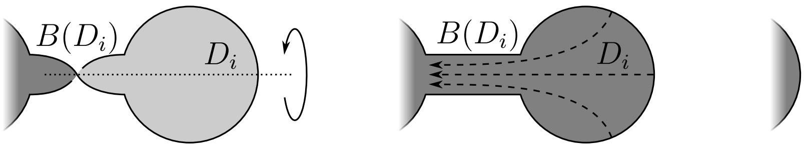
Figure 7: Example of adjacent and non-adjacent bands.

The Seifert algorithm is a brute force method to construct a Seifert surface. It is often obvious that the canonical Seifert surface is isotopic to a simpler Seifert surface which is also canonical with respect to the same projection, as in the following lemma.

Lemma 3. *Let $\Sigma = \{D_1, \ldots, D_n; B_1, \ldots, B_m; T_1, \ldots, T_o\}$ be a canonical Seifert surface and $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $deg(D_{i_1}) = \ldots = deg(D_{i_k}) = 1$. Then,*

$$\Sigma \text{ is isotopic to } \Sigma \setminus \{D_{i_1}, \ldots, D_{i_k}; B(D_{i_1}), \ldots, B(D_{i_k})\}\,.$$

Proof. Let D_i be a Seifert disc of $deg(D_i) = 1$. The disc D_i can be rotated (by an isotopy) around the principal axis of the attached band $B(D_i)$ either clockwise or counterclockwise such that the half twist is removed. Then D_i and its neighboring disc lay in the same plane. D_i and the attached band $B(D_i)$ can be shrunk into the neighboring disc of D_i by an isotopy. Figure 8 illustrates the process. □



(a) Rotate D_i around the axis of $B(D_i)$. (b) Merge D_i and $B(D_i)$ into the neighboring disc along the dashed arrows. (c) Both D_i and $B(D_i)$ are removed after the isotopy.

Figure 8: Isotopy to remove Seifert disc of degree one.

An important observation we can make from the Seifert algorithm is that if there are no type II Seifert circles the resulting canonical Seifert surface is *almost planar*. This means that the surface can be embedded in a plane except for the crossing regions in the half twisted bands and the unknotted tubes. In the following we will make use of this observation several times. Therefore, it is beneficial to us to have type II free canonical Seifert surfaces. Kauffman showed in Proposition 7.3 in [Kau87] that a link diagram with type II Seifert circles can be rid of all type II Seifert circles by adding one additional link component per type II Seifert circle. Aaltonen improved this result by showing that the type II Seifert circles of a link L can be removed by an isotopy instead of adding link components. Moreover, the canonical Seifert surface of L, obtained from a given projection, is also deformed by the isotopy applied to L. The resulting Seifert surface remains canonical for the isotoped link with respect to the same projection.

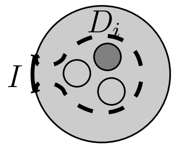
Theorem 4 ([Aal14])**.** *Let Σ be a canonical Seifert surface with type II Seifert discs. Then, Σ is isotopic to a type II free canonical Seifert surface Σ'.*

Proof. We present the proof from [Aal14] in a less technical form than the proof in the publication but the idea is the same. The proof is an induction over the number of type II Seifert discs. It suffices to show that any type II Seifert disc can be removed by an isotopy. The description of the following isotopies is with respect to the projection from which Σ was obtained. Yet, the isotopies are executed in $\mathbb{R}^3$.

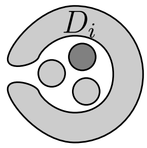
Let D_i be a type II Seifert disc. The bands in $B(D_i)$ which connect to discs inside D_i with respect to the projection are called inward bands. Outward bands are the ones which connect to discs outside D_i. Choose an interval I on ∂D_i such that it does not intersect with any band in $B(D_i)$. Drag the interval I into the interior of D_i such that I traces along $\partial D_i \setminus I$ so closely that the dragged interval does not intersect with any interior disc in the projection (Figure 9a and 9b). Now, for every inward band B_j drag an interval $J \subset I$ below B_j to the exterior of D_i. The dragging is performed around $B_j \cap D_i$ in a rotational motion and in the direction that J never intersects with B_j (Figure 9c). This gives rise to two new bands and two new Seifert discs. After applying such an isotopy for all inward bands in $B(D_i)$ the disc D_i itself is replaced with type I Seifert discs and half twisted bands. Notice that no special treatment is necessary for outward bands and that the new discs are the Seifert discs of the isotoped link.

In conclusion, stretching the interval I into D_i except for inward bands where the interval is outside of D_i removes the type II Seifert disc D_i at the cost of adding two new bands and two new type I Seifert discs for every inward band. Furthermore, the resulting Seifert surface is still canonical with respect to the same projection.

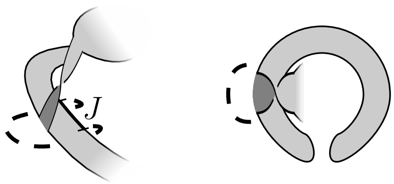
Figure 10 illustrates the algorithm with an example type II Seifert disc containing only one other Seifert disc. □



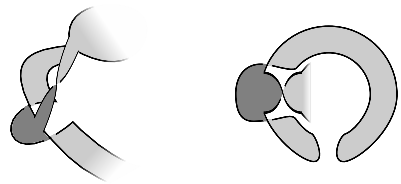
(a) Drag interval $I \subset \partial D_i$ to the dashed line.



(b) The disc D_i becomes a strip which does not intersect the interior discs.

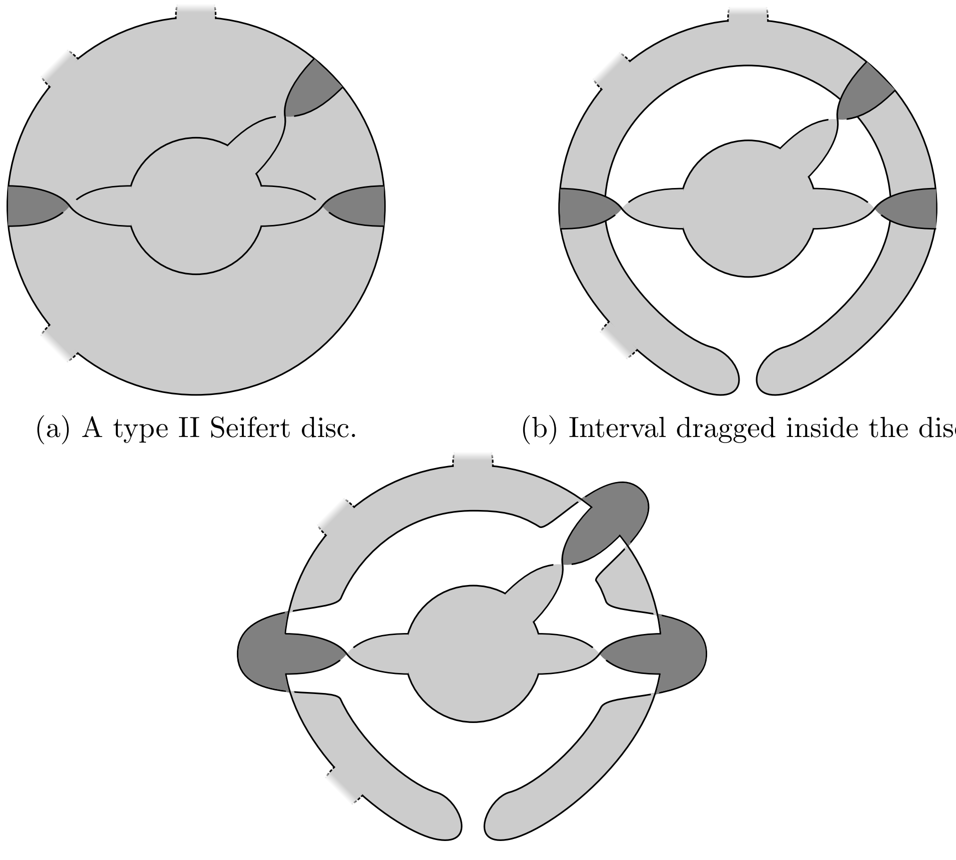


(c) Drag interval J in a rotational motion to the dashed line. Left: Perspective view, Right: Projected view.



(d) The resulting surface has two new bands and discs. Left: Perspective view, Right: Projected view.

Figure 9: Isotopy to remove type II Seifert disc.



(a) A type II Seifert disc.

(b) Interval dragged inside the disc.

(c) Type II Seifert disc removed.

Figure 10: Example for a type II Seifert disc removal.

3 Seifert Graphs

We already mentioned that the Seifert surface is the basis for many important concepts in knot theory. A further concept based on the Seifert surface is the representation of a canonical Seifert surface as a graph. We saw that the Seifert algorithm produces a set of disjoint discs that are eventually connected by half twisted bands and possibly unknotted tubes. For every component of a canonical Seifert surface we can construct a planar graph from the discs and their connecting half twisted bands. This graph representation is helpful because we can apply well known results from graph theory to gain information about the surface.

Definition 5. Let $\Sigma = \{D_1, \ldots, D_n; B_1, \ldots, B_m; -\}$ be a canonical Seifert surface. Then, the *Seifert graph* of Σ is the graph (V, E) where $V = \{D_1, \ldots, D_n\}$ is the set of vertices and $E = \{B_1, \ldots, B_m\}$ the set of edges.

The Seifert graph of a given canonical Seifert surface is unique but it is not unique for the link. This is clear because the canonical Seifert surface is already dependent of the orientation and the projection chosen for the Seifert algorithm. The Seifert graph of a canonical Seifert surface has several useful properties which allow us to proof Lemma 7. Lemma 7 plays an essential role in the proof of the main result of this thesis, namely Theorem 13.

Lemma 6. *A Seifert graph is planar, bipartite and has no loops.*

Proof. We proof that a Seifert graph is planar by an induction over the number of type II Seifert discs.

Base case Without any type II discs there are only type I discs and the Seifert surface is almost planar. This means that the discs and half twisted bands can be embedded in a plane except for small regions in the half twisted bands. However, as each half twisted band becomes an edge in the graph, the graph is planar.

Inductive step Since there is at least one type II disc, there exists a type II disc whose corresponding Seifert circle is not contained in any other Seifert circle. Such an outermost type II circle gives rise to a cut vertex in the Seifert graph. The two subgraphs that arise from removing the cut vertex are planar by the induction hypothesis. These two subgraphs can be reconnected via this cut vertex in a planar way.

In conclusion every Seifert graph is planar.

The Seifert graph is bipartite because every circuit in the graph must have even length. Otherwise, the canonical Seifert surface from which the graph was created would not be orientable as every circuit of uneven length stems from an embedded Möbius-strip, possibly with multiple twists.

It is true that every band connects two different Seifert circles. Hence, every edge in the corresponding Seifert graph connects two different vertices and so has no loops. □

Lemma 7. *Let $n > 1$ and $\Sigma = \{D_1, \ldots, D_n; B_1, \ldots, B_m; -\}$ be a canonical Seifert surface without type II Seifert discs. Then, either*

$$\exists k, l \in \{1, \ldots, m\} \text{ such that } B_k, B_l \text{ are adjacent}$$

or

$$\exists i \in \{1, \ldots, n\} \textit{ such that } deg(D_i) = 1, 2 \textit{ or } 3.$$

Proof. Assume to the contrary that the type II free canonical Seifert surface has no adjacent bands and every Seifert disc has degree at least four.

From Lemma 6 we know that the Seifert graph is planar, bipartite and has no loops. As the graph is planar it can be embedded in the sphere S^2. Such an embedding divides S^2 into a finite number of faces. Let V be the number of vertices, E the number of edges and F the number of faces. It is a well known fact for the Euler characteristic (Theorem 5.A.1 in [Rol76]) that

$$\chi(S^2) = V - E + F = 2. \tag{1}$$

In the following we require that the faces stem from the Seifert graph diagram which is directly induced by the canonical Seifert surface. Furthermore, it is essential to have a canonical Seifert surface without type II Seifert discs. The reason is that without type II Seifert discs, the surface can be reconstructed from the vertices, edges and faces. Each vertex gives rise to a Seifert disc and each edge to a half twisted band (the sign is not important here). Note that this is not possible for type II discs because we lose the information about how the type II discs are connected by the half twisted bands. So, by showing that the directly induced Seifert graph diagram leads to a contradiction we can conclude that such a canonical Seifert surface could not have existed.

Continuing with the proof, from the fact that there are no loops in the Seifert graph we conclude that every face has more than one edge. The assumption that there are no adjacent bands is equivalent with the property that every face is bounded by more than two edges. Furthermore, every face is bounded by an even number of edges because otherwise there would be a circuit of uneven length, which would violate the bipartiteness of the graph. In conclusion, every face is bounded by at least four edges. From these observations it follows that

$$4F \leq \sum_{i=1}^{F} d(f_i) = 2E \tag{2}$$

where $\{f_1, \ldots, f_F\}$ is an enumeration of all faces and $d(f_i)$ is the number of edges bounding face f_i. The reason why the sum equals twice the edge count is the following. Each edge belongs to the boundary of two faces. Therefore, by summing over all faces each edge is counted twice.

From (1) and (2) follows

$$E \leq 2V - 4. \tag{3}$$

Because every Seifert disc is of degree at least four, every vertex has degree at least four. Therefore,

$$4V \leq \sum_{i=1}^{V} d(v_i) = 2E \tag{4}$$

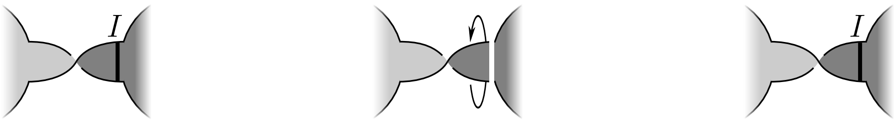
where $\{v_1, \ldots, v_V\}$ is an enumeration of all vertices and $d(v_i)$ is the degree of vertex v_i. The reason why the sum equals twice the edge count is that each edge is incident to two vertices. Therefore, by summing over all vertices each edge is counted twice.

The inequalities (3) and (4) yield the contradiction

$$E \leq 2V - 4 \leq E - 4.$$ □

4 Twist Equivalence and Twist Triviality

Definition 8. A *ribbon twist* is a cut and glue operation on a Seifert surface Σ. Let $I = \varphi([0,1])$ be an embedded interval such that $\varphi(0), \varphi(1) \in \partial\Sigma$ and $\varphi((0,1)) \in \Sigma \setminus \partial\Sigma$. Cut along I, insert a full twist on one side and glue both sides back together along I (Figure 11).



(a) Cut along interval I. (b) Insert a full twist. (c) Glue together along I.

Figure 11: Ribbon twist along a half twisted band.

Based on ribbon twists, we can define a new equivalence relation for surfaces with boundaries.

Definition 9. Let Σ_1 and Σ_2 be two Seifert surfaces. The surfaces Σ_1 and Σ_2 are called *twist equivalent* if $\exists R_1, \ldots, R_n$ ribbon twists such that $R_n \circ \ldots \circ R_1(\Sigma_1)$ is isotopic to Σ_2.

Definition 10. A Seifert surface is called *twist trivial* if it is twist equivalent to a standardly embedded n-fold punctured torus, i.e. a subsurface of the boundary of an unknotted handlebody in $\mathbb{R}^3$ (Figure 12).

It is a fact that not every Seifert surface is twist trivial. A necessary condition can be found by studying the fundamental group of the surface complement. Notice that applying a ribbon twist preserves the fundamental group of the surface complement. Therefore, if two surfaces Σ_1, Σ_2 are twist equivalent then the fundamental groups of their surface complements are isomorphic, $\pi_1(\mathbb{R}^3 \setminus \Sigma_1) \simeq \pi_1(\mathbb{R}^3 \setminus \Sigma_2)$.

Theorem 11. *If a Seifert surface Σ is twist trivial then $\pi_1(\mathbb{R}^3 \setminus \Sigma)$ is a free group.*

Proof. Let Σ' be the standard n-fold torus with m holes to which Σ is twist equivalent. From the fact that neither a ribbon twist nor an isotopy changes the fundamental group $\pi_1(\mathbb{R}^3 \setminus \Sigma)$ it follows that $\pi_1(\mathbb{R}^3 \setminus \Sigma) \simeq \pi_1(\mathbb{R}^3 \setminus \Sigma')$. The Seifert-van Kampen theorem (§70 in [Mun00] or 1.2 in [Hat01]) implies that $\pi_1(\mathbb{R}^3 \setminus \Sigma') \simeq F_{2n+m-1}$, the free group of $2n + m - 1$ generators (Figure 12). $\square$

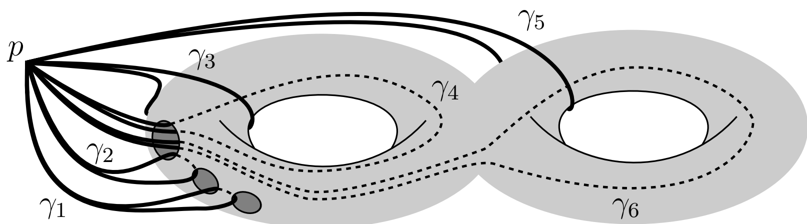


Figure 12: If Σ is the 2-fold torus with 3 holes then $\pi_1(\mathbb{R}^3 \setminus \Sigma, p) = F_6$. There are two generators for each genus hole (left: γ_3, γ_4 right: γ_5, γ_6) and two generators for the three boundary holes (γ_1, γ_2).

From this necessary condition it is straightforward to construct a surface that cannot be twist trivial. Many satellite knots and links ([Cro04] or [Ada94]) serve as counterexamples. For instance, take $L = L_1 \cup L_2$ to be the cable link of the trivial two component link with the trefoil knot as its companion (Figure 13). A possible, however not canonical, Seifert surface Σ is shown in Figure 13. Clearly, the fundamental group of its complement $\pi_1(\mathbb{R}^3 \setminus \Sigma)$ is isomorphic to the knot group of the trefoil knot, which is known to be non-free (Chapter 3 Section B in [Rol76]). Thus, the surface Σ cannot be twist trivial. This is also intuitively obvious for this example because a ribbon twist can only add full twists to the band. However, this will obviously never unknot the surface and so it cannot be twist trivial.

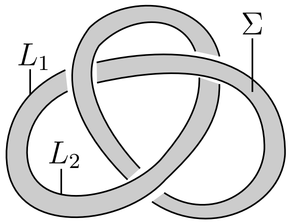


Figure 13: The Seifert surface Σ of the link $L_1 \cup L_2$ is not twist trivial.

Baader and Dehornoy showed that a canonical Seifert surface of a positive braid knot is twist trivial ([BD13]). This implies that a canonical Seifert surface of any braid knot is twist trivial. The reason is that a negative crossing can be changed into a positive crossing by a single ribbon twist. Therefore, we first make the braid knot positive by ribbon twists. Then, by Baader and Dehornoy's result we know that the braid knot is twist trivial. In this master thesis we generalize this result in Theorem 13 to canonical Seifert surfaces of any link. The remainder of this section is concerned with the proof of Theorem 13 and for that we need the following lemma.

Lemma 12. *Let Σ be a Seifert surface, $H : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ an ambient isotopy and R a ribbon twist on Σ. Then, there exists a ribbon twist R' on $H(1,\Sigma)$ such that*

$$H(1, R(\Sigma)) = R'(H(1, \Sigma)).$$

Proof. The ribbon twist R is a local cut and glue operation along an interval $I \subset \Sigma$. The interval I is deformed along with the surface Σ by the isotopy H. So, applying a ribbon twist R' to $H(1,\Sigma)$ along the interval $H(1,I)$ is the same as applying the ribbon twist R along I before the isotopy and then applying the isotopy H (Figure 14). $\square$

The previous lemma helps us to show that two surfaces Σ, Σ' are twist equivalent. Now, we can make use of isotopies and ribbon twists in any order. After we found a sequence of isotopies and ribbon twists which maps Σ to Σ' we can apply the lemma to reorder the sequence such that all ribbon twists are applied first. The resulting surface of the ribbon twists is isotopic to Σ' by the remaining isotopies of the sequence. Hence, Σ and Σ' are twist equivalent.

The fact that we can deform the surface by isotopies between ribbon twists in order to show twist equivalence between surfaces leads to a procedure we call *non-entangling band sliding.* Let $\Sigma = \{D_1, \ldots, D_n; B_1, \ldots, B_m; -\}$ be a type II free canonical Seifert

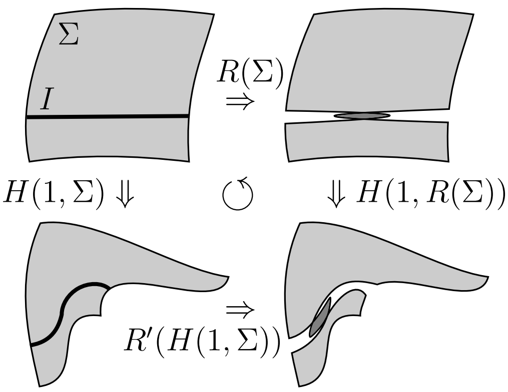


Figure 14: This diagram of the isotopy H, ribbon twist R along I and ribbon twist R' along $H(1,I)$ commutes for a surface Σ, $H(1,R(\Sigma)) = R'(H(1,\Sigma))$.

surface. As observed earlier, Σ is almost planar. Hence, it is possible to slide one end of a band $B_i \in \Sigma$ along $\partial\Sigma$ such that it does not entangle with any other bands. While sliding the end of B_i along the boundary, we can either keep the band above or below the plane in which we can embed Σ. We might need to apply ribbon twists to the bands over which the end is slid (Figure 15). This procedure consists of successive ribbon twists and isotopies. By Lemma 12, the surface after the non-entangling band sliding is twist equivalent to Σ .

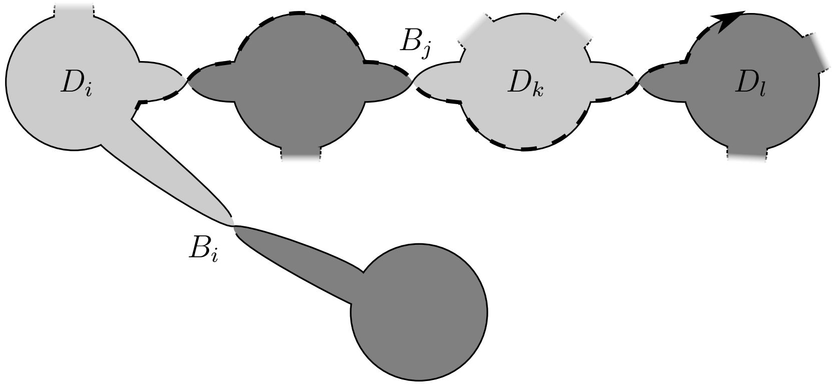


Figure 15: If the D_i-end of B_i is slid along the dashed arrow to D_l then B_i would entangle around D_k because B_j has an unsuitable sign. Applying a ribbon twist to change the sign of B_j produces a path along the dashed arrow which keeps B_i on top of all discs. Hence, non-entangling band sliding is possible.

Theorem 13. *If Σ is a canonical Seifert surface then Σ is twist trivial.*

Proof. The proof strategy is an induction over the number of half twisted bands. In the process of removing bands, the Seifert discs may become punctured or receive handles (Figure 16). These holes and handles can be considered infinitesimally small during the iteration steps. Therefore, they do not interact with any successive band removal.

The canonical Seifert surface may consist of N components which are connected by $N-1$ unknotted tubes. By interpreting each component as a vertex and each tube as an edge, the components and unknotted tubes form a graph. In particular, they do not form

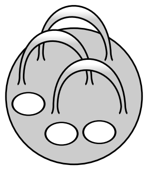


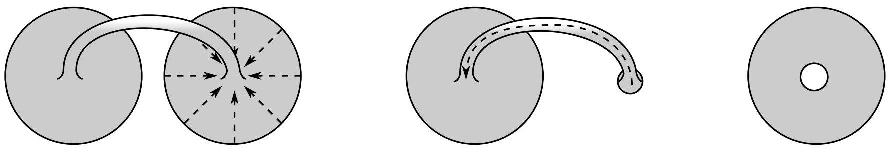
Figure 16: Seifert disc with holes and handles.

any circuit. This implies that the graph is a tree. Due to band removals, the canonical Seifert surface may split into new components. We assure that a new component is connected to the previous components by one unknotted tube. This implies that there are N unknotted tubes connecting $N+1$ components, which means that the induced graph remains a tree.

The tree structure of the components and the unknotted tubes implies that we may consider all unknotted tubes and all but one components to be infinitesimally small during the iteration steps. Therefore, while removing bands from one component, all other components and unknotted tubes are infinitesimally small and do not interact with the band removal on this component.

Before removing any bands, remove all type II Seifert discs according to Theorem 4. We obtain a canonical Seifert surface $\Sigma' = \{D_1, \ldots, D_n; B_1, \ldots, B_m; T_1, \ldots, T_o\}$ where $D_1, \ldots, D_n$ are of type I. The surfaces Σ and Σ' are isotopic and therefore twist equivalent. The induction is now carried out on the type II free canonical Seifert surface Σ'.

Base case Let $\Sigma' = \{D_1, \ldots, D_n; -; T_1, \ldots, T_{n-1}\}$ where the discs $\{D_1, \ldots, D_n\}$ are the components connected by $\{T_1, \ldots, T_{n-1}\}$. Every hole and handle can be moved to disc D_1 over the connecting tubes by an isotopy. As already mentioned, the induced graph of the components and unknotted tubes is a tree without multi-edges. In particular, D_1 can be interpreted as the root of the tree. Successively, the tree can be reduced from the leaves. A leaf is a disc with only one connecting tube. By retracting the tube, the leaf disc becomes a hole in its neighboring disc (Figure 17). This new hole can again be moved to D_1. Repeating these steps leads to a single disc with holes and handles. Thus, Σ' is twist trivial.



(a) Shrink disc to the tube edge. (b) Retract the tube into the neighboring disc. (c) A disc with one additional hole remains.

Figure 17: Isotopy to remove leaf discs.

Inductive step Let $\Sigma' = \{D_1, \ldots, D_n; B_1, \ldots, B_m; T_1, \ldots, T_o\}$. There must be at least one component which consists of at least two Seifert discs and at least one half twisted band. As mentioned we consider all other components and unknotted tubes to be infinitesimally small. This allows the application of Lemma 7 to this component. Therefore, at least one of the following cases occurs.

1. $\exists k, l \in \{1, \ldots, m\}$ such that B_k, B_l are adjacent.
   If two half twisted bands are adjacent then they connect the same discs $D_i, D_j \in \Sigma'$ and there does not exist any other band between B_k, B_l (Figure 7). If the signs of the adjacent bands B_k, B_l are the same then the sign of B_k or B_l can be changed by a ribbon twist. This results in B_k and B_l having different signs. Subsequently, B_k and B_l can be removed by an isotopy, which is in this case a Reidemeister II move (Figure 18a). This results in an unknotted tube T which connects D_i, D_j (Figure 18b).

(a) Apply Reidemeister II move.

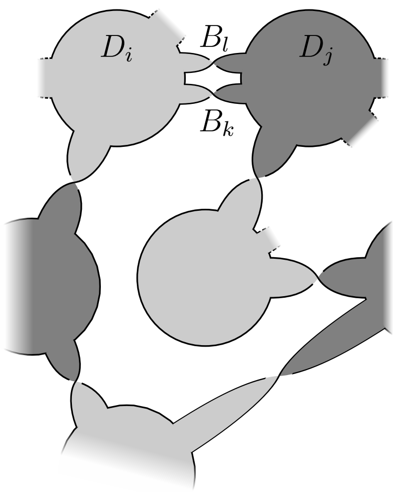
(b) Resulting tube T

Figure 18: Adjacent bands B_k, B_l become a tube T by a Reidemeister II move.

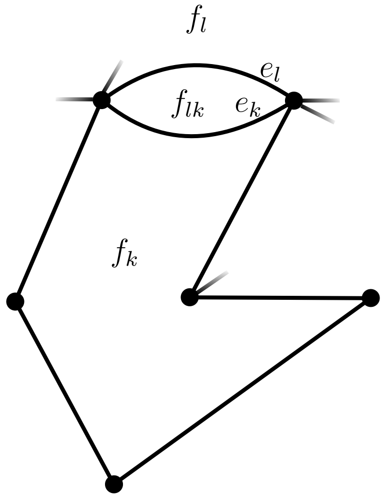
   Next, there are two cases. The first case is when the new tube T is the only connection between otherwise disconnected components. This is the case where one new component arises. Notice that this new component is connected only via T. Hence, $\{D_1, \ldots, D_n; B_1, \ldots, B_m; T_1, \ldots, T_o, T\} \setminus \{B_k, B_l\}$ is the resulting surface with now $o + 2$ components.

   The second case is when the current component is still connected even without the tube T. This means that there exists a path along the remaining half twisted bands in the component such that one end of the tube T can be moved to its other end. Now, our goal is to show that we can even move one end of T to its other end without entanglement. For this, consider the Seifert surface and its Seifert graph prior to the Reidemeister II move (Figure 19a). The bands B_l, B_k induce two edges e_l, e_k in the Seifert graph. The edges e_l, e_k bound a face f_{kl} because B_l, B_k are adjacent. Additionally, each e_l and e_k belong to the boundary of further two faces f_l, f_k. Note that $f_l \neq f_k$ because the current component is connected even without T and so even without B_l, B_k (Figure 19b). Now, consider the boundary ∂f_l or ∂f_k. Without loss of generality we continue with ∂f_k. There exist half twisted bands which induce the path $\partial f_k \setminus e_k$. These half twisted bands form a path from D_i to D_j in the current component and are the reason why the component is still connected even without T. In other words, these bands form a path from one end of the tube to its other end. Since we found this path through the boundary of the face f_k, the tube T and these bands bound an area which does not contain any other disc, tube or band (shaded area in Figure 19c). We can ensure by ribbon twists that the half twisted bands form a path with alternating signs such that one end of the tube T can be moved to its other end without entanglement. The result is a handle on one of the discs D_i, D_j (Figure 19d).

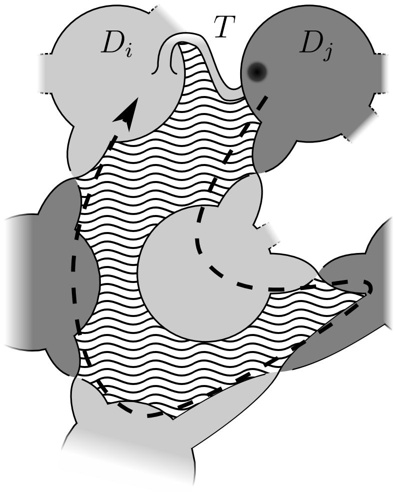
2. $\exists i \in \{1, \ldots, n\}$ such that $deg(D_i) = 1$
   Lemma 3 can be applied to remove the band and the disc.



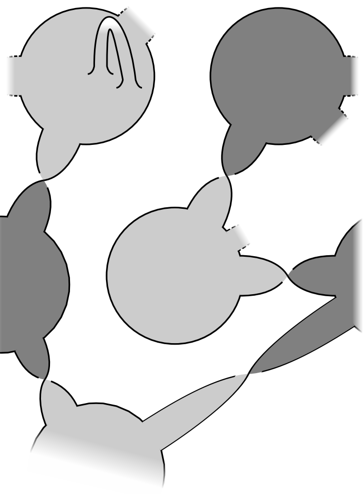
(a) Adjacent bands prior to the Reidemeister II move.



(b) The induced Seifert graph.



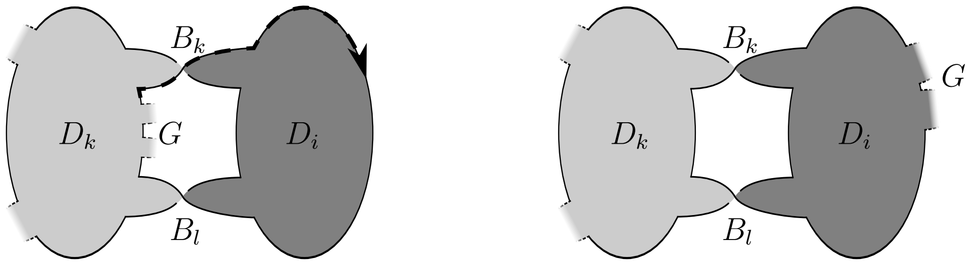
(c) Move end of tube along bands, which have been ribbon twisted such that the tube does not entangle.



(d) Resulting handle after isotopy.

Figure 19: Isotopy to obtain a handle from a connecting tube after suitable ribbon twists.

3. $\exists i \in \{1, \ldots, n\}$ such that $deg(D_i) = 2$
   Let $B(D_i) = \{B_k, B_l\}$ be the two attached bands. In this case there are two subcases. The first case is when both bands are connected to the same disc D_k. It might be that D_k has a group G of bands and discs between B_k, B_l making B_k, B_l non-adjacent. However, G can be slid away over B_k or B_l such that B_k, B_l become adjacent (Figure 20). Then the adjacent bands can be removed as in case 1.



(a) Slide group G of bands and discs along the dashed arrow over B_k.

(b) B_k and B_l are now adjacent.

Figure 20: Group sliding

   The second case is when B_k, B_l connect to two different discs D_k, D_l. By a ribbon twist we can assure that B_k, B_l have the same sign. After a rotation of D_i, it is possible to merge D_i, D_k, D_l, B_k, B_l by an isotopy into one disc (Figure 21).

(a) If B_k, B_l have different signs then the disc D_i can be rotated.

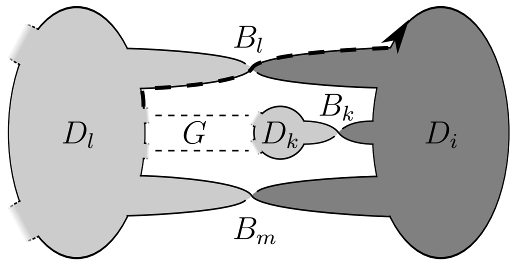
(b) All D_i, D_k, D_l, B_k, B_l can be merged after the rotation.

Figure 21: Isotopy to remove half twisted bands of a Seifert disc of degree two.

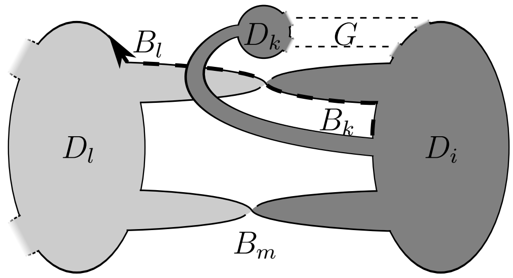
4. $\exists i \in \{1, \ldots, n\}$ such that $deg(D_i) = 3$
   Let $B(D_i) = \{B_k, B_l, B_m\}$ be the three attached bands. In this case there are three subcases. The first case is where B_k, B_l, B_m are connected to the same disc. So the situation is analog to the first subcase of case 3 except that there is one additional band. However, this does not prevent us from removing a pair of bands in the same way as in case 3. Therefore, if a pair of bands is adjacent they can be removed. If both pairs are non adjacent either one of them can be made adjacent as in case 3 by sliding away the group of bands and discs making the pair non-adjacent (Figure 20).

   The second case is when the B_k, B_l, B_m are connected to two different discs D_k, D_l. Without loss of generality let B_k be connected to D_k and B_l, B_m to D_l. If B_l, B_m are adjacent then we can follow case 1. If B_l, B_m are non-adjacent due to a group of bands and discs which are only attached to either D_i or

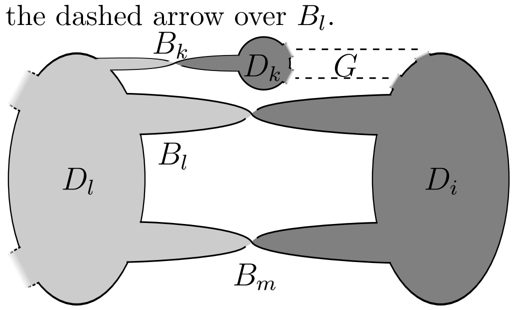
D_l (but not both) we can slide the group away as in case 3. If B_l, B_m are non-adjacent due to a group G of bands and discs which are attached to both D_i and D_l we can exploit that G is attached to D_i only via B_k (Figure 22a). Consider G to be a rigid plane except for B_k. First, slide G over B_l. Then apply a ribbon twist to B_l in order to avoid entanglement in the next slide (Figure 22b). Then slide the D_i-end of B_k over B_l. After the sliding, B_k might be twisted one and a half times but by a ribbon twist we obtain a half twisted band. Eventually, B_l, B_m become a pair of adjacent bands, which can be removed as in case 1 (Figure 22c).



(a) Slide group G of bands and discs along the dashed arrow over B_l.



(b) Change the crossing of B_l and slide B_k along the dashed arrow over B_l.



(c) The bands B_l and B_m are now adjacent.

Figure 22: The bands B_l, B_m can be made adjacent by ribbon twists and isotopies.

The third case is when B_k, B_l, B_m are connected to three different discs D_k, D_l, D_m. In order to determine how to reduce the number of half twisted bands, slide the D_k-end of B_k along the boundary in the same direction until the end of the band reaches one of the three positions a) - c) as shown in Figure 23. The reason that exactly one of a) - c) is reached is that we can continue sliding the end of B_k until the other end of B_k blocks the path. In other words, position b) or c) is reached. If neither b) nor c) is reached, we can continue sliding forever. This can only happen if we reach the point where we started. So, in this case position a) is reached.

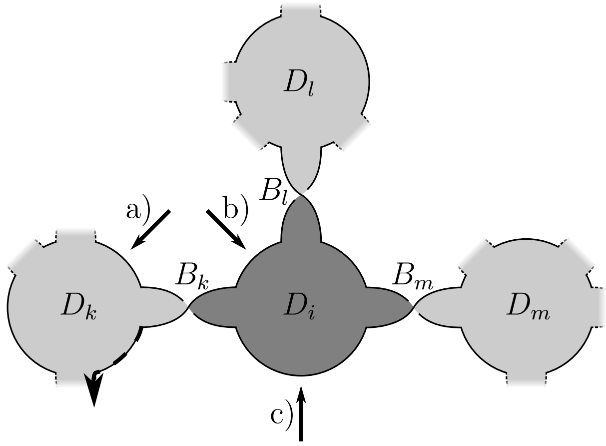
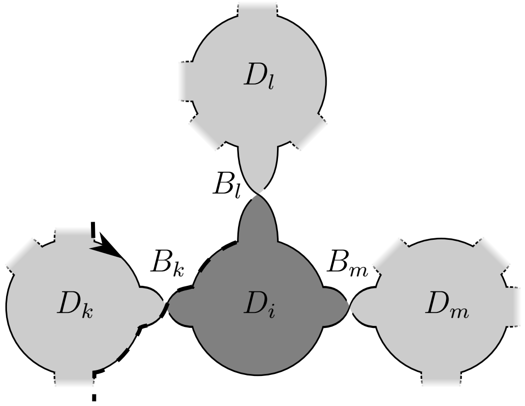


Figure 23: Slide D_k-end of B_k along the dashed arrow until it reaches one of the three only possible positions a) - c).

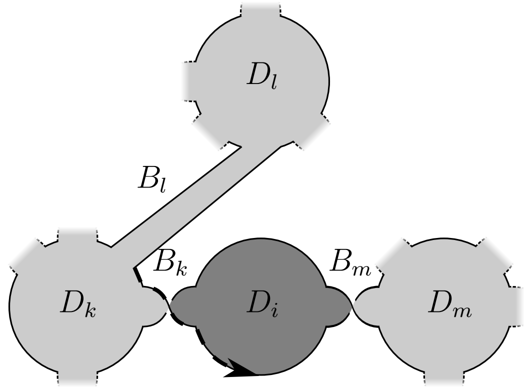
If a) is reached then there is a path from the D_i-end of B_l to a) (Figure 24a). So, by crossing B_k a second time there is even a path from b) to c). By applying ribbon twists and isotopies, the D_i-end of B_l can be slid to position c) without entanglement (Figure 24b). Now B_l is again attached to D_i but to the opposite side of its starting position (Figure 24c). The band B_l might have several full twists (at most as many as the number of crossings B_l crossed) but these can be removed by ribbon twists. With at most two ribbon twists we can change the crossing signs of B_k, B_m such that the entire disc D_i can be rotated into a convenient position (Figure 24d). From this position B_k, B_l, B_m and D_i, D_k, D_l, D_m can be merged into one disc.

If b) or c) is reached then the D_k-end of B_k can be slid to b) or c) without entanglement. This results in one additional hole in D_i (Figure 25). □

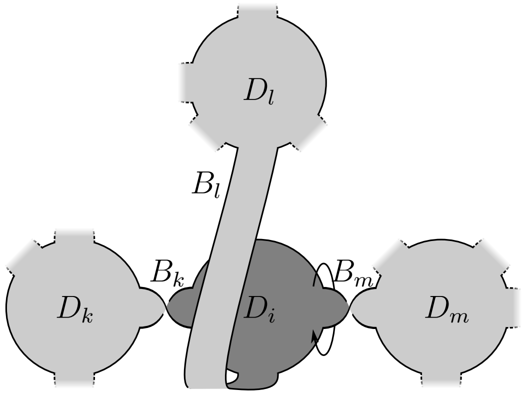
This theorem provides a large class of Seifert surfaces which are twist trivial. However, we do not yet know if this class can be enlarged. For instance, are all fibered Seifert surfaces ([Sta78, Rol76]) twist trivial? Baader and Dehornoy answered this question partially. They showed that fibered Seifert surfaces of positive braid knots are twist trivial ([BD13]). Yet, the most general question to ask is if the necessary condition shown in Theorem 11 is also a sufficient condition. In other words, is every Seifert surface with a free fundamental group of its complement twist trivial? This question could be addressed to continue this thesis's results.



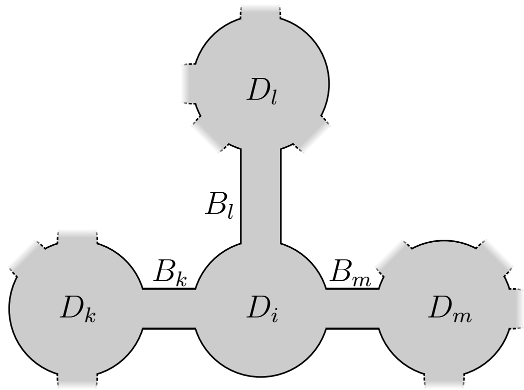
(a) Slide the end of B_l along the dashed arrow to reach position a). Apply ribbon twists when necessary to avoid entanglement.



(b) Slide the end of B_l over B_k to reach position c).



(c) Apply ribbon twists to B_k, B_m to allow the suitable rotation of D_i.



(d) All $B_k, B_l, B_m, D_i, D_k, D_l$, and D_m can be merged into one disc.

Figure 24: If position a) in Figure 23 is reachable then $B_k, B_l, B_m, D_i, D_k, D_l$, and D_m can be merged into one disc by ribbon twists and isotopies as illustrated.

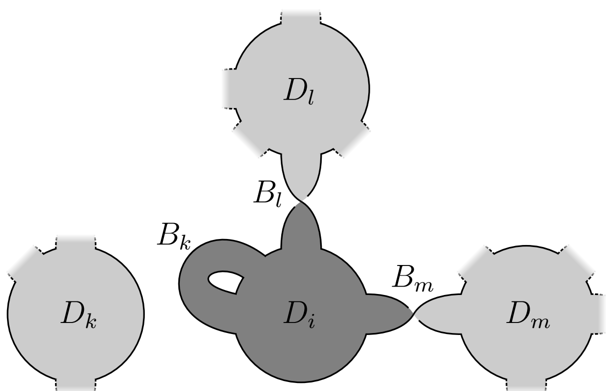


Figure 25: If position b) in Figure 23 is reachable then B_k can be removed by ribbon twists and isotopies. This gives rise to a hole in D_i.

5 Untwisting Number

Definition 14. Let Σ be a twist trivial surface. The *untwisting number* of Σ is $ut(\Sigma)$ the minimal number of ribbon twists required to untwist the surface, that is

$$ut(\Sigma) = \min\{n \in \mathbb{N} | R_n \circ \cdots \circ R_1(\Sigma) \text{ isotopic to standard n-fold punctured torus}\}$$

where $R_1, \ldots, R_n$ are ribbon twists.

The untwisting number of a surface is invariant under isotopy. This follows directly from Lemma 12. This observation is formulated in the next corollary.

Corollary 15. *Let Σ_1, Σ_2 be two twist trivial surfaces. Then,*

$$\Sigma_1 \textit{ isotopic to } \Sigma_2 \implies ut(\Sigma_1) = ut(\Sigma_2).$$

The untwisting number is, just like the unknotting number, very difficult to compute. Nevertheless, a lower bound can be given.

Corollary 16. *Let Σ be twist trivial. Then,*

$$u(\partial\Sigma) \leq ut(\Sigma).$$

Proof. Assume to the contrary that $u(\partial\Sigma) > ut(\Sigma) = n$. Note that the unknotting number can be defined with crossing changes in $\mathbb{R}^3$ instead of regular diagrams. It is known that the two definitions are equivalent (Chapter 3.1 in [Ada94]). Now, observe that a ribbon twist is equivalent to a crossing change in $\mathbb{R}^3$. After applying the n ribbon twists to untwist Σ, the boundary $\partial\Sigma$ is also unknotted. Hence, the n ribbon twists realize an unknotting of $\partial\Sigma$ with less crossing changes than $u(\partial\Sigma)$, which is a contradiction. □

This lower bound is optimal in the sense that there exist twist trivial surfaces for which the equality holds. Baader and Dehornoy showed that a canonical Seifert surface Σ of a positive braid knot can be untwisted by g ribbon twists where g is the genus of the surface ([BD13]). Furthermore, Rudolph showed that the unknotting number of positive braid knots is not smaller than g ([Rud98]). Thus, we know $ut(\Sigma) = g$ from Baader and Dehornoy, $u(\partial\Sigma) \geq g$ from Rudolph and $u(\partial\Sigma) \leq ut(\Sigma)$ from Corollary 16. This implies that $u(\partial\Sigma) = ut(\Sigma) = g$.

The proof of Theorem 13 is basically an algorithm which untwists any canonical Seifert surface. Therefore, we can derive an upper bound of the untwisting number for canonical Seifert surfaces.

Corollary 17. *Let $\Sigma = \{D_1, \ldots, D_n; B_1, \ldots, B_m; T_1, \ldots, T_o\}$ be a canonical Seifert surface. Then,*

$$ut(\Sigma) \leq 2 \cdot \sum_{i=1}^{m-1} i.$$

Proof. From the proof of Theorem 13 we see that we need to analyze four cases.

1. $\exists k, l \in \{1, \dots, m\}$ such that B_k, B_l are adjacent.
   At most, one ribbon twist is required to make the signs of B_k, B_l unequal. Then, if the Reidemeister II move leads to a new component we are done. In the other case, we need at most $m-2$ ribbon twists to move one end of the new tube to its other end (Figure 19).

   This results in at most $m-1$ ribbon twists in order to remove two bands.

2. $\exists i \in \{1, \dots, n\}$ such that $deg(D_i) = 1$
   No ribbon twist is required to remove one band.

3. $\exists i \in \{1, \dots, n\}$ such that $deg(D_i) = 2$
   If both bands connect to the same two discs then this is equivalent to case 1 because we remove two adjacent bands.

   If each band connects to a different disc, then we can remove two bands with at most one ribbon twist (Figure 21).

   This results in $m-1$ ribbon twists in order to remove two bands in the worst case.

4. $\exists i \in \{1, \dots, n\}$ such that $deg(D_i) = 3$
   Let $B(D_i) = \{B_k, B_l, B_m\}$ be the three attached bands. If all bands connect to the same disc then this is equivalent to case 1 because we remove two adjacent bands. Hence, we require at most $m-1$ ribbon twists to remove two bands.

   If the bands connect two discs then we may need to slide away a group of bands (Figure 22). This may require two ribbon twists. Then, we are in the case of adjacent bands again. Therefore, we need at most $m+1$ ribbon twists in order to remove two bands.

   If each band connects to a separate disc we can remove either one or three bands. If only one band can be removed (Figure 25) then we need at most $2(m-1)$ ribbon twists. This is because we need to change the sign of a band twice if the end of B_k is slid twice over this band. If all three bands can be removed (Figure 24) we need at most $2(m-2)$ ribbon twists to slide B_l to the opposite side of D_i. This is because we never cross B_l itself nor B_m and we may cross bands twice. After B_l is slid to the opposite side of D_i it may be twisted at most $m-2$ times. To untwist B_l we need $m-2$ ribbon twists. Finally, we need at most two ribbon twists to enable the rotation of D_i. This adds up to at most $3(m-2)+2$ ribbon twists to remove three bands.

   Therefore, the case where all bands connect to a different disc and only one band can be removed is the worst, with $2(m-1)$ ribbon twists for one band.

If there are $m \geq 3$ bands then the overall worst case is where $2(m-1)$ ribbon twists are needed to remove one band. If there are only $m = 2$ we need obviously at most one ribbon twist. Hence, $2(m-1)$ is also an upper bound. In conclusion,

$$ut(\Sigma) \leq \sum_{i=1}^{m-1} 2(m-i) = \sum_{i=1}^{m-1} 2i.$$

□

This upper bound is a very rough bound. We are convinced that it can be improved because it is derived from the presented untwisting algorithm. Obviously, this algorithm does not untwist a surface with the least possible number of ribbon twists. Therefore, further analysis needs to be conducted in order to improve the upper bound.

References

[Aal14] M. Aaltonen. Note on the canonical genus of a knot. *arXiv:1411.1536*, November 2014.

[Ada94] C. C. Adams. *The Knot Book*. American Mathematical Society, 1994.

[Ale28] J. W. Alexander. Topological invariants of knots and links. *Transactions of the American Mathematical Society*, 30(2):275–306, April 1928.

[BD13] S. Baader and P. Dehornoy. Trefoil plumbing. *arXiv:1308.5866*, August 2013.

[Ble84] S. A. Bleiler. A note on unknotting number. *Mathematical Proceedings of the Cambridge Philosophical Society*, 96:469–471, November 1984.

[Cro04] P. Cromwell. *Knot and Links*. Cambridge University Press, 2004.

[Hat01] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 1 edition, 2001.

[Kau87] L. H. Kauffman. *On Knots*. Annals of mathematics studies. Princeton University Press, 1987.

[Kaw96] A. Kawauchi. *A Survey of Knot Theory*. Birkhauser, 1996.

[Mun00] J. R. Munkres. *Topology*. Prentice Hall, 2 edition, 2000.

[Mur65] K. Murasugi. On a certain numerical invariant of link types. *Transactions of the American Mathematical Society*, 117:387–422, May 1965.

[Rol76] D. Rolfsen. *Knots and Links*, volume 346. Amer Mathematical Society, 1976.

[Rud98] L. Rudolph. Quasipositive plumbing (constructions of quasipositive knots and links, v). *Proceedings of the American Mathematical Society*, 126(1):257–267, January 1998.

[Sei35] H. Seifert. Über das Geschlecht von Knoten. *Mathematische Annalen*, 110(1):571–592, 1935.

[Sta78] J. R. Stallings. Constructions of fibred knots and links. *Proceedings of Symposia in Pure Mathematics*, 32:55–60, 1978.

[Wen37] H. Wendt. Die gordische Auflösung von Knoten. *Mathematische Zeitschrift*, 42:680–696, 1937.