Twist Triviality of Canonical Seifert Surfaces

Michael Pfeuti

Mathematical Institute
University of Bern

Supervisor:
Prof. Dr. Sebastian Baader

23 February 2015

Seifert Surfaces

Seifert Surface

Definition

A compact, connected, orientable surface Σ that has a link as its boundary $\partial\Sigma = L$ is called a *Seifert surface* of the link L.

Theorem (Seifert Algorithm, 1935)

Every link has a Seifert surface.

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A *canonical Seifert surface* is a Seifert surface that can be obtained by the Seifert algorithm.

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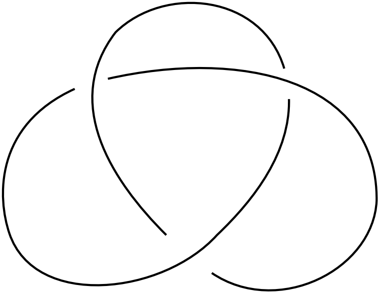
Seifert Algorithm

1. Choose a projection
2. Orient the link
3. Remove crossings
4. Assign each circle a disk
5. Insert half-twisted bands



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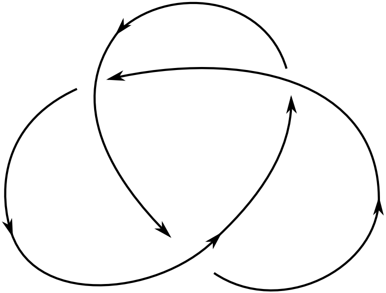

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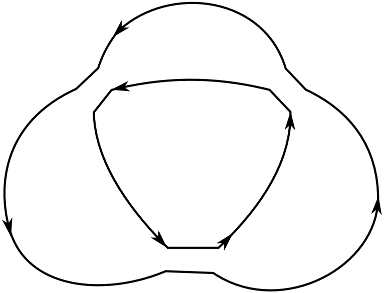

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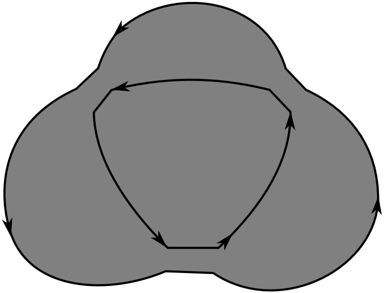

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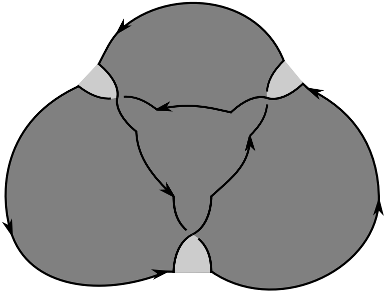

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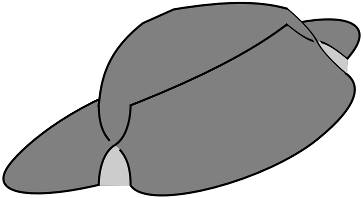

Canonical Seifert Surface

Observe

The Seifert algorithm produces a set of disjoint disks (possibly stacked) and half-twisted bands.

Remark

- The resulting canonical Seifert surface depends on the chosen orientation and projection.
- Canonical Seifert surfaces of the same link need not be homeomorphic.

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There exist non canonical Seifert surfaces.

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Twist Triviality

Ribbon Twist

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A *ribbon twist* is a cut and glue operation on a Seifert surface Σ. Let $I = \varphi([0,1])$ be an embedded interval such that $\varphi(0), \varphi(1) \in \partial\Sigma$ and $\varphi((0,1)) \in \Sigma \setminus \partial\Sigma$. Cut along I, insert a full twist on one side and glue both sides back together along I.

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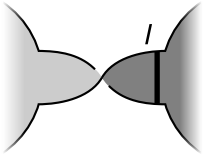

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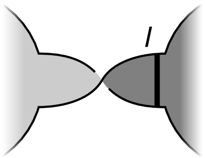

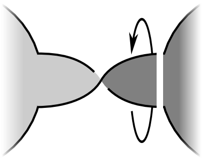

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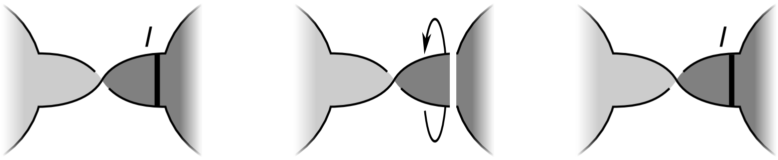

Twist Triviality

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Let Σ_1 and Σ_2 be two Seifert surfaces. The surfaces Σ_1 and Σ_2 are called *twist equivalent* if $\exists F_1, \ldots, F_n$ ribbon twists and isotopies such that $F_n \circ ... \circ F_1(\Sigma_1) = \Sigma_2$.

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Examples...

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Theorem

If Σ is a canonical Seifert surface then Σ is twist trivial.

Proof.

Induction over the number of half-twisted bands.

There exists an isotopy that turns stacked disks into non-stacked disks (Aaltonen, 2014).

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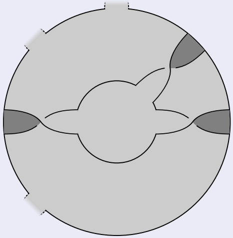

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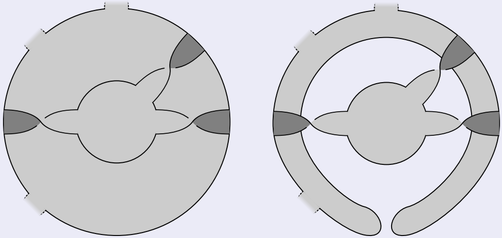

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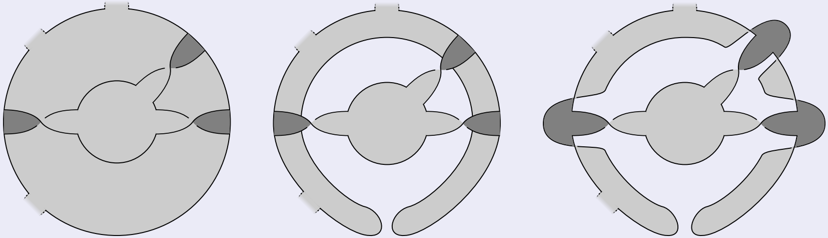

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Proof (cont.)

Now, the Seifert surface (non-stacked discs and half-twisted bands) is "almost planar".

Therefore, at least one of the following can be found:

- Disc with one bands
- Two adjacent bands
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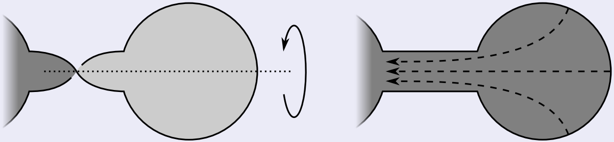
Case 1: Disc with one band



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Case 2: Adjacent Bands

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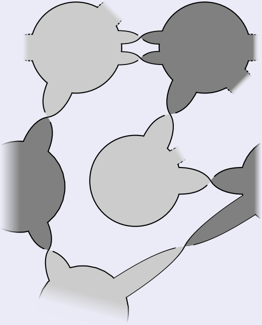

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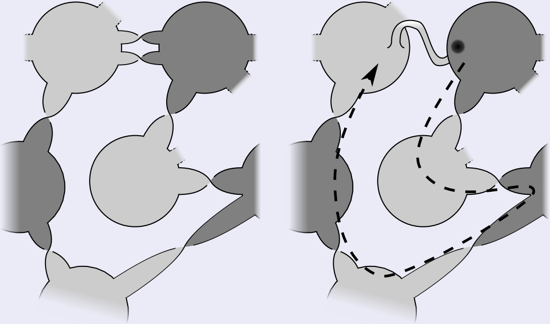

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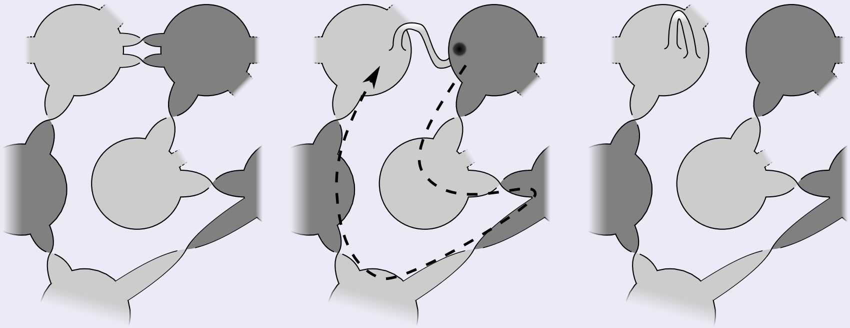

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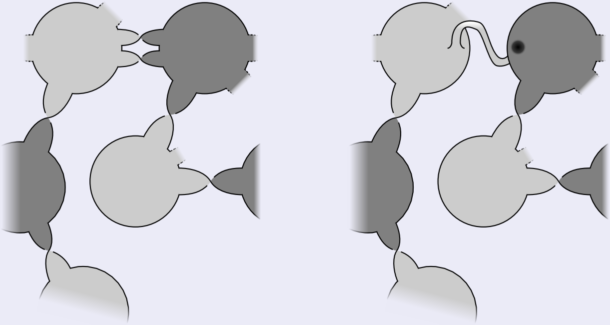

We obtain connected (by half-twisted bands) components which are joined by tubes.

Case 3 and 4: Disc with two and with three bands. □

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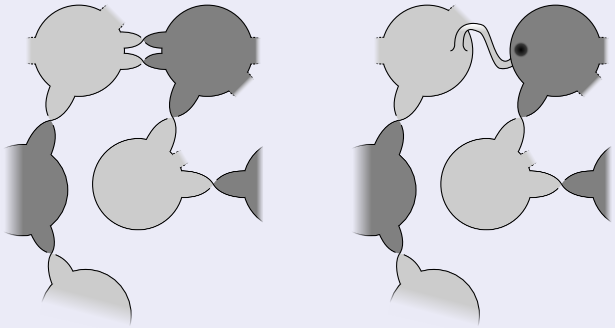

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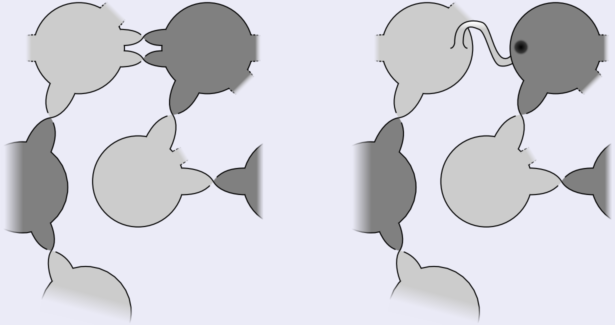

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Thank You For Your Attention.

Questions?

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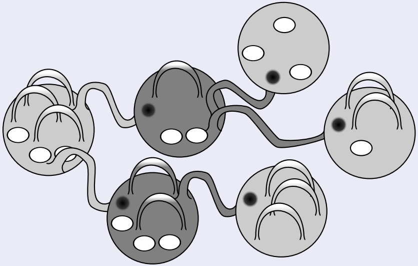
In the process the disks become punctured, receive unknotted handles, and are connected by unknotted tubes such that no circuit is formed ($\Rightarrow$ tree).



Retract disk from the leaves of the tree.

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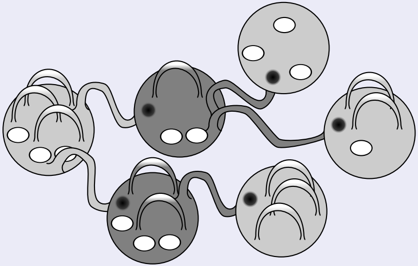

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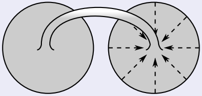

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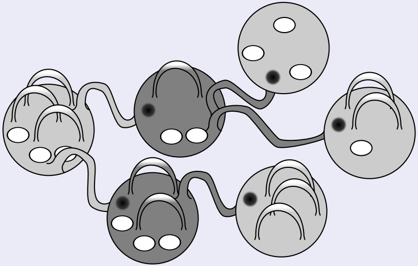

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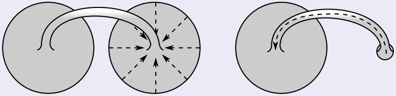

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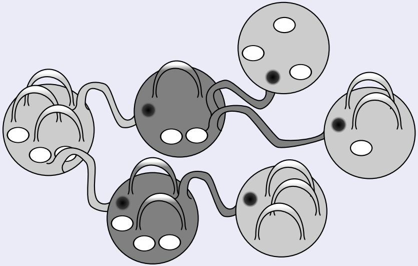

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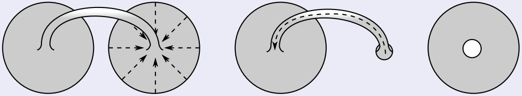

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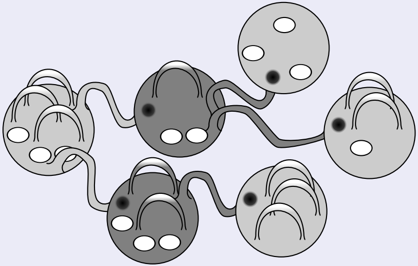

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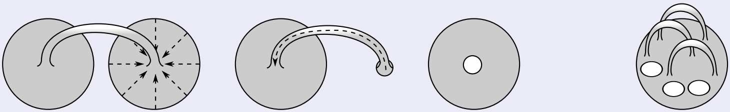

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