The Logic of Justified Belief

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Goal of this Presentation

Introduction of the logic of justified belief which is called K4J4.

$$\mathsf{K4J4} = \underbrace{\mathsf{K4}}_{\text{Modal Logic}} + \underbrace{\mathsf{J4}}_{\text{Justification Logic}}$$

Goal

Modal Logic

Justification Logic

Hybrid Logics

Formulas in a Modal Logic

Definition

Formulas in monomodal logics are:

$$F ::= S \mid \bot \mid F \to F \mid F \wedge F \mid F \vee F \mid \neg F \mid \Box F$$

where S is a sentence variable.

Modal Logic S4

Definition

Let F and G be any formulas. S4 consists of

- P: Any axiomatization of classical propositional logic
- R1: Modus Ponens
- K: $\Box(F \to G) \to (\Box F \to \Box G)$ (distribution axiom)
- N: $\vdash F$ implies $\vdash \Box F$ (necessitation rule)
- T: $\Box F \to F$ (reflexivity axiom)
- 4: $\Box F \to \Box\Box F$ (positive introspection axiom)

Often used as a formal system for reasoning about knowledge.

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Goal

Modal Logic

Justification Logic

Hybrid Logics

Justification Logic - The basic Idea

Artemov introduced an "explicit" version of modal logic.
In these new logics we have a richer language,

proof polynomials instead of $\Box$.

Example

$\Box F$ is replaced by $t : F$ (t is a proof polynomial)

$t : F$ can be read as "t justifies/proves F"

Proof Polynomials

The ! operator is called proof checker.
The $+$ operator is called the sum operator.
The $\cdot$ operator is called the application operator.

Definition

$$t ::= x \mid a \mid\ !t \mid t \cdot t \mid t + t$$

where x is a proof variable and a a proof constant.

Formulas in a Justification Logic

Formulas in **modal logic** were defined as

$$F ::= S \mid \bot \mid F \rightarrow F \mid F \wedge F \mid F \vee F \mid \neg F \mid \Box F$$

Definition

Formulas in a **justification logic** is

$$F ::= S \mid \bot \mid F \rightarrow F \mid F \wedge F \mid F \vee F \mid \neg F \mid t : F$$

where S is a sentence variable and t a proof polynomial.

The Logic of Proofs

The Logic of Proofs LP, introduced by Artemov, has the following axioms and rules. We assume F, G are formulas and s, t are proof polynomials.

- P: Any finitary axiomatization of classical logic
- R1: Modus Ponens
- LP1: $s : (F \rightarrow G) \rightarrow (t : F \rightarrow (s \cdot t) : G)$ (application)
- LP2: $t : F \rightarrow !t : t : F$ (inspection)
- LP3: $s : F \rightarrow (s + t) : F \qquad t : F \rightarrow (s + t) : F$ (sum)
- LP4: $t : F \rightarrow F$ (reflexivity)
- R2: For any constant c and any axiom A infer $c : A$ (axiom necessitation)

Justification Logic J4

J4 has the following axioms and rules. We assume F, G are formulas and s, t are proof polynomials.

- P: Any finitary axiomatization of classical logic
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Connection between Modal and Justification Logics

In order to show the relationship between modal and justification Logic we need forgetful projection.

Definition

Let F, G be any justification formula. The forgetful projection of F is F° and can be obtained through

- $(t : F)^\circ = \Box(F^\circ)$
- $S^\circ = S$
- $\bot^\circ = \bot$
- the forgetful projection commutes with Boolean connectives

Connection between Modal and Justification Logics (cont.)

Definition

Let Γ be a set of justification formulas. The forgetful projection of Γ is $\Gamma^\circ = \{G^\circ \mid G \in \Gamma\}$.

Definition

A monomodal logic ML is called the forgetful projection of a justification logic $\mathsf{J}*$ if $\mathsf{J}*^\circ = \mathsf{ML}$.

Connection between Modal and Justification Logics (cont.)

Theorem (Realization Theorem)

- S4 *is the forgetful projection of* LP *(Artemov).*

in other words:

- $\mathsf{LP} \vdash F \Rightarrow \mathsf{S4} \vdash F^\circ$

for each justification formula F. Moreover,

- $\mathsf{S4} \vdash G \Rightarrow \exists G^r$ such that $\mathsf{LP} \vdash G^r$ and $(G^r)^\circ = G$

for each modal formula G.

$\Rightarrow$ *Artemov and Nogina introduced* S4LP

Connection between Modal and Justification Logics (cont.)

Theorem (Realization Theorem)

- K4 *is the forgetful projection of* J4 *(Brezhnev).*

in other words:

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Goal

Modal Logic

Justification Logic

Hybrid Logics

Formulas in Hybrid Logics

Formulas in K4J4 and S4LP are

$$A ::= S \mid \perp \mid A \rightarrow A \mid A \wedge A \mid A \vee A \mid \neg A \mid \Box A \mid t : A$$

where S is a sentence variable and t a proof polynomial.

The Logic of Justified Knowledge S4LP

Definition (S4LP)

Classical Propositional Logic

P: Any finitary axiomatization

R1: Modus Ponens

Justification Logic LP

LP1: $s : (F \to G) \to (t : F \to (s \cdot t) : G)$

LP2: $t : F \to !t : t : F$

LP3: $s : F \to (s + t) : F, t : F \to (s + t) : F$

LP4: $t : F \to F$

R2: For any constant c and axiom A infer $\vdash c : A$

Epistemic Logic S4

E1: $\Box(F \to G) \to (\Box F \to \Box G)$

E2: $\Box F \to \Box\Box F$

E3: $\Box F \to F$

R3: $\vdash F \Rightarrow \quad \vdash \Box F$

Connection Axiom

C1: $t : F \to \Box F$

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The Logic of Justified Belief K4J4

Definition (K4J4)

Classical Propositional Logic

P: Any finitary axiomatization

R1: Modus Ponens

Justification Logic J4

LP1: $s : (F \to G) \to (t : F \to (s \cdot t) : G)$

LP2: $t : F \to !t : t : F$

LP3: $s : F \to (s + t) : F, t : F \to (s + t) : F$

~~LP4: $t : F \to F$~~

R2: For any constant c and axiom A infer $\vdash c : A$

Epistemic Logic K4

E1: $\Box(F \to G) \to (\Box F \to \Box G)$

E2: $\Box F \to \Box\Box F$

~~E3: $\Box F \to F$~~

R3: $\vdash F \Rightarrow \ \vdash \Box F$

Connection Axiom

C1: $t : F \to \Box F$

AF-Models for K4J4

Definition

An AF-model for K4J4 consists of a tuple $(W, R, R^e, \mathcal{E}, \mathcal{V})$ where

- W is a non-empty set of states (or possible worlds)
- $R \subseteq W \times W$ a transitive accessibility relation
- $\mathcal{V} : W \rightarrow 2^{SV}$ an arbitrary function which maps a world to a set of sentence variables
- $R^e \subseteq W \times W$ a transitive evidence accessibility relation which includes R

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AF-Models for K4J4 (cont.)

- $\mathcal{E} : W \times PP \to 2^{Fml}$ an admissible evidence function which has the following properties (exp: $F \in \mathcal{E}(u,t)$):

  Monotonicity : $uR^e v$ implies $\mathcal{E}(u,t) \subseteq \mathcal{E}(v,t)$

  Closure Properties :

  - Application: $F \to G \in \mathcal{E}(u,s)$ and $F \in \mathcal{E}(u,t)$ implies $G \in \mathcal{E}(u, s \cdot t)$
  - Inspection: $F \in \mathcal{E}(u,t)$ implies $t : F \in \mathcal{E}(u, !t)$
  - Sum: $\mathcal{E}(u,s) \cup \mathcal{E}(u,t) \subseteq \mathcal{E}(u, s+t)$

  If c is a proof constant and A an axiom then it holds that $A \in \mathcal{E}(u,c)$ for all $u \in W$

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AF-Models for K4J4 (cont.)

- $\Vdash \subseteq W \times Fml$ a binary relation which is defined as follows:
  - $u \nVdash \bot$
  - $u \Vdash S$ iff $S \in \mathcal{V}(u)$
  - $u \Vdash \neg F$ iff $u \nVdash F$
  - $u \Vdash F \vee G$ iff $u \Vdash F$ or $u \Vdash G$
  - other boolean connectives are similarly classical
  - $u \Vdash \Box F$ iff $v \Vdash F$ for all v such that uRv
  - $u \Vdash t : F$ iff $F \in \mathcal{E}(u, t)$ and $v \Vdash F$ for all v such that $uR^e v$

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Soundness

Theorem (Soundness)

For all formulas F and all AF-models $\mathcal{M} = (W, R, R^e, \mathcal{E}, \mathcal{V})$ it holds that

$$\mathsf{K4J4} \vdash F \text{ implies } \mathcal{M} \vDash F.$$

Proof.

The proof is an induction on the derivation of F. $\square$

Completeness

Theorem (Completeness)

For all formulas F, if for all AF-models $\mathcal{M} = (W, R, R^e, \mathcal{E}, \mathcal{V})$ it holds that

$$\mathcal{M} \models F \text{ then } \mathsf{K4J4} \vdash F.$$

Proof.

Standard canonical model construction. □

Thank you for your attention!

Questions?

Sketch of Completeness Proof

Proof

In order to prove the theorem we use the contraposition. There is an AF-model $\mathcal{M}$ such that for any formula F

$$\mathsf{K4J4} \nvdash F \text{ implies } \mathcal{M} \not\models F$$

We define $\mathcal{M} = (W, R, R^e, \mathcal{E}, \mathcal{V})$ where

W is the collection of all maximal consistent sets.

We define $\mathcal{V}$ through

$$S \in \mathcal{V}(\Gamma) \text{ iff } S \in \Gamma$$

for every maximal consistent set Γ. For traditional reasons, worlds and maximal consistent set are denoted by capital Greek letters.

Sketch of Completeness Proof (cont.)

The accessibility relations are defined as

$$R = R^e = \{(\Gamma, \Delta) : \Gamma^\sharp \cup \Gamma^\flat \subseteq \Delta\},$$

where $\Gamma^\flat = \{t : F, F | t : F \in \Gamma\}$ and $\Gamma^\sharp = \{\Box F, F | \Box F \in \Gamma\}$.

Lastly, we define $\mathcal{E}$ by

$$F \in \mathcal{E}(\Gamma, t) \text{ iff } t : F \in \Gamma$$

It can be shown that $\mathcal{M}$ is in fact an AF-model (R, R^e are transitive and $\mathcal{E}$ satisfies the monotonicity and the closure properties).

Sketch of Completeness Proof (cont.)

The following is the main lemma in this proof, the so called truth lemma.

Lemma (Truth Lemma)

For all $\Gamma \in W$ *and all formulas* F *it holds that*

$$\Gamma \Vdash F \textit{ iff } F \in \Gamma$$

Proof.

The proof is an induction on F. □

Back to the completeness. Assume $\nvdash F$. This implies that $\{\neg F\}$ is consistent. $\{\neg F\}$ can be extended to a maximal consistent set Γ. $F \notin \Gamma$ due to consistency. The truth lemma implies that $\Gamma \nVdash F$

□