The Logic of Justified Belief

Bachelor's Thesis

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Abstract

Justification Logic is a new field that studies provability, knowledge, and belief via explicit proofs or justifications that are part of the language. There are many justification logics closely related to modal epistemic logics of knowledge and belief. Instead or in addition to modality $\Box$, which has an existential epistemic reading 'there exists a proof of F,' all justification logics use constructs $t : F$, where a justification term t represents a blueprint of a Hilbert-style proof of F. The first justification logic, LP, introduced by Sergei Artemov, was shown to be a justification counterpart of modal logic $\mathsf{S4}$ and serves as a missing link between $\mathsf{S4}$ and Peano arithmetic, thereby solving a long-standing problem of provability semantics for $\mathsf{S4}$ and Int. Both modal logic $\mathsf{S4}$ and its justification counterpart LP have been combined by Artemov and Nogina in a logic of justified knowledge $\mathsf{S4LP}$. In this bachelor's thesis the idea of Artemov and Nogina has been applied to $\mathsf{K4}$ and its justification counterpart $\mathsf{J4}$, leading to the logic of justified belief $\mathsf{K4J4}$. As semantics for this new logic, AF-models for $\mathsf{K4J4}$ are introduced and it is shown that $\mathsf{K4J4}$ is sound and complete with respect to these AF-models. Furthermore, the logic is extended to its multi-agent version $\mathsf{K4_nJ4}$. AF-models are introduced for $\mathsf{K4_nJ4}$ and also soundness, and completeness are shown for the multi-agent version.

Contents

1 Introduction

1.1 Background

This bachelor's thesis introduces a hybrid logic which combines a justification logic and a modal logic. Therefore, we first take a quick look at several standard modal logics and afterwards we discuss what justification logics are.

Modal logics are logics in a language that contains at least one modal operator besides the regular boolean connectives. Usually, these operators are denoted by "$\Box$" with or without indices. For now, we focus only on modal languages with one modality, $\Box$:

$$F ::= S|\bot|F \to F|F \wedge F|F \vee F|\neg F|\Box F$$

In this definition S denotes a sentence variable. Formula $\Box F$, depending on the context, can be read as "formula F is provable" or "formula F is known."

We will define several modal logics by giving their Hilbert-style axioms and rules. The following list contains axioms and rules which we are concerned with in this introduction. Let F and G be any formulas.

K: $\Box(F \to G) \to (\Box F \to \Box G)$ (distribution axiom)

N: $\vdash F$ implies $\vdash \Box F$ (necessitation rule)

T: $\Box F \to F$ (reflexivity axiom)

D: $\Box F \to \neg\Box\neg F$ (seriality axiom)

4: $\Box F \to \Box\Box F$ (positive introspection axiom)

In Table 1 some of the most well-known modal logics are defined, especially the two that we will concentrate our discussion on: $\mathsf{S4}$, which can be used to reason about knowledge, and $\mathsf{K4}$, which is the equivalent of $\mathsf{S4}$ for reasoning about belief. Additionally, all modal logics in Table 1 consist of a finitary axiomatization of classical propositional logic and the rule Modus Ponens.

Artemov introduced (see [2]) an explicit version of modal logic. Here, by explicit version we mean that we consider a richer language where a family of so-called proof polynomials t replace the single $\Box$. Accordingly, statements $\Box F$ are replaced by $t : F$ with the reading "t is a proof of F" or "t justifies F". Proof polynomials are built up from proof constants and variables by

Axioms,Rules \ System	K	D	T	K4	D4	S4
K	*	*	*	*	*	*
N	*	*	*	*	*	*
T			*			*
D		*			*	
4				*	*	*

Table 1: Definitions of some well-known modal logics.

three operators: the binary sum operator $+$, the binary application operator $\cdot$ and the unary proof checker !. The following is the formal definition of proof polynomials where a is a proof constant and x a proof variable:

$$t ::= x|a|!t|t \cdot t|t + t$$

As already mentioned, we use these proof polynomials instead of just the modality $\Box$. Before we list the axioms and rules of Artemov's logic, we define the justification language first:

$$F ::= S|\bot|F \to F|F \wedge F|F \vee F|\neg F|t : F$$

where S is a sentence variable and t a proof polynomial.

The Logic of Proofs LP, introduced by Artemov in [2], has the following axioms and rules. We assume F, G are formulas and s, t are proof polynomials.

P: Any finitary axiomatization of classical logic

LP1: $s : (F \to G) \to (t : F \to (s \cdot t) : G)$ (application)

LP2: $t : F \to !t : t : F$ (inspection)

LP3: $s : F \to (s + t) : F \qquad t : F \to (s + t) : F$ (sum)

LP4: $t : F \to F$ (reflexivity)

R1: Modus Ponens

R2: For any constant c and any axiom A infer $c : A$ (axiom necessitation)

The Logic of Proofs LP provided a solution to the long-sought problem of exact provability semantics for $\mathsf{S4}$, and through it for the intuitionistic logic (see [3, 4]). The justification language in general proved to be a convenient tool in various areas of computer science and epistemology with examples such as logical omniscience problem (see [7]), a new approach to common knowledge (see [1, 5]), and study of self-referential proofs/justifications in modal logics (see [11, 16]).

These and other applications and the desire to study various modal logics by means of the refined justification language lead to the creation of a family of logics, called justification logics, that are close to LP in nature. Some of these logics are subsystems of LP while others use other axioms and rules such as

LP4$_D$: $t : \bot \to \bot$ (consistency axiom)

R2*: For any constant c and any axiom A infer $!^n c :!^{n-1} c : \ldots !^2 c :!^1 c :!^0 c : A$ (We define $!^n t = \ !!^{n-1} t$ and $!^0 t = t$ for $n \in \{0, 1, \ldots\}$ and any proof polynomial t.)

The table 2 gives a small overview of some justification logics. As in the case of modal logics, we only mention the most common justification logics we are interested in:

Axioms,Rules \ System	J	JD	JT	J4	JD4	LP
P	*	*	*	*	*	*
LP1	*	*	*	*	*	*
LP2				*	*	*
LP3	*	*	*	*	*	*
LP4			*			*
LP4$_D$		*			*	
R1	*	*	*	*	*	*
R2				*	*	*
R2*	*	*	*			

Table 2: Definition of some justification logics.

Justification logics enjoy the standard properties such as the deduction theorem or closure under substitution (for additional information consult [4, 6]).

Theorem 1 (Deduction Theorem)**.** *Let* Γ *be a set of formulas and* $\mathsf{J}*$ *one of the justification logics* $\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{JD4}, \mathsf{J4}, \mathsf{LP}$ *then*

$$\Gamma, G \vdash_{\mathsf{J}*} F \textit{ iff } \Gamma \vdash_{\mathsf{J}*} G \to F$$

Theorem 2 (Substitution Property)**.** *Let* Γ *be a set of formulas,* F *a formula and* $\mathsf{J}*$ *one of the justification logics* $\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{JD4}, \mathsf{J4}, \mathsf{LP}$ *then*

$$\Gamma(S,x) \vdash_{\mathsf{J}*} F(S,x) \text{ implies } \Gamma(S/G, x/t) \vdash_{\mathsf{J}*} F(S/G, x/t)$$

for any proof variable x, proof polynomial t and formula G.

One characteristic feature of justification logics is the internalization property. The internalization property means that it is possible to analyze proofs in a logic inside the logic itself. The following statement is due to Artemov [2] and Brezhnev [10].

Lemma 3. *Let $G_1, \ldots, G_n, F$ be any formulas and $\mathsf{J}*$ one of the justification logics $\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{JD4}, \mathsf{J4}, \mathsf{LP}$ then*

$$G_1, \ldots, G_n \vdash_{\mathsf{J}*} F \text{ implies } x_1 : G_1, \ldots, x_n : G_n \vdash_{\mathsf{J}*} t(x_1, \ldots, x_n) : F$$

for some proof polynomial $t(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are fresh distinct proof variables.

Corollary 4 (Internalization)**.** *Whenever $\mathsf{J}*$ is one of the justification logics $\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{JD4}, \mathsf{J4}, \mathsf{LP}$, it holds that*

$$\mathsf{J}* \vdash F \text{ implies } \mathsf{J}* \vdash t : F$$

for some ground proof polynomial t, i.e., for some t that does not contain any proof variables.

In order to explain the exact relationship between $\mathsf{LP}, \mathsf{S4}$ and other justification and modal logics, we define the following function that maps justification formulas into modal formulas:

Definition 5. Let F, G be any justification formula. The forgetful projection of F is F° and can be obtained through $(t : F)^\circ = \Box(F^\circ)$, $(F \vee G)^\circ = F^\circ \vee G^\circ$, $(F \wedge G)^\circ = F^\circ \wedge G^\circ$, $(\neg F)^\circ = \neg(F^\circ)$, $(F \to G)^\circ = F^\circ \to G^\circ$, $S^\circ = S$ and $\bot^\circ = \bot$.

We also need to know what a forgetful projection of a logic is. Since a logic can be identified by a set of theorems (all derivable formulas in the logic), we only need to know how to apply the forgetful projection to a infinite sets of formulas.

Definition 6. Let $\Gamma = \{G_1, G_2, \ldots\}$ be a set of justification formulas. The forgetful projection of Γ is $\Gamma^\circ = \{G_1^\circ, G_2^\circ, \ldots\}$.

Definition 7. A monomodal logic ML is called the forgetful projection of a justification logic $\mathsf{J}*$ if $\mathsf{J}*^\circ = \mathsf{ML}$.

As mentioned earlier LP is an explicit version of a modal logic. More precisely, LP is an explicit version of S4. Artemov and Brezhnev analyzed the relation between the justification logics J, JD, JT, J4, JD4, LP and the modal logics K, D, T, K4, D4, S4. In [2, 10] they proved the realization theorem.

Theorem 8 (Realization Theorem)**.** *Each of the modal logics* K, D, T, K4, D4, S4 *is the forgetful projection of one of the justification logics* J, JD, JT, J4, JD4, LP *according to Table 3.*

Justification Logic $\mathsf{J}*$	J	JD	JT	J4	JD4	LP
Forgetful Projection $\mathsf{J}*^\circ$	K	D	T	K4	D4	S4

Table 3: Some justification logics and their forgetful projection.

As a result of the realization theorem, we realize that there is a strong relationship between justification logics and modal logics. Thus, it seems natural to consider and analyze hybrid logics, which combine justifications and modality, S4LP, was studied by Artemov and Nogina in [8]. As the name suggests, this logic combines modal logic S4 with justification logic LP. S4 is used to describe knowledge whereas LP is its justification counterpart, hence S4LP can be viewed as a logic for justified knowledge. Artemov and Nogina also introduced a semantics for S4LP, called AF-models, and showed the soundness and completeness of S4LP with respect to this semantics.

The goal of this bachelor's thesis is to create a similar hybrid logic for belief. Since the modal logic K4 is commonly used to describe belief, it seems reasonable to combine it with its explicit counterpart J4. We will call the resulting hybrid logic of justified belief K4J4, by analogy with S4LP. We will adapt the AF-models from [8] to our new logic and prove soundness and completeness of K4J4 with respect to this semantics. We will further generalize these results (completeness and soundness with respect to AF-models) to the case of multiple modalities.

1.2 Outline

In this thesis, the logic of belief with justifications K4J4, analogous to the logic of knowledge with justifications S4LP, will be introduced. It will be shown to possess all the standard features of justification logics, including internalization property. The semantics of AF-models will be adapted to this new logic. We will prove soundness and completeness of K4J4 with respect to its AF-models. In fact, the canonical model constructed in the proof will also be an F-model since it will use the same accessibility relation for the modality and for evidence terms. Therefore, soundness and completeness of K4J4 with respect to appropriately defined F-models will be a simple corollary.

Further, we will generalize K4J4 to the case of multiple belief agents, all of whom accept the same justifications for their beliefs. The resulting logic $\mathsf{K4_nJ4}$ will also be shown to be sound and complete with respect to its AF-models.

1.3 Acknowledgments

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2 A Single-Agent Logic of Justified Belief

2.1 The System K4J4

K4J4 is a logic of belief with justification. As its name suggests, it is a hybrid logic which combines K4 and J4. In addition, a connection axiom is introduced to accomplish an interaction between the two logics. The language we use consists of $\neg, \rightarrow, \Box, \wedge, \vee, PP, SV, \bot$ where SV stands for the set of all sentence variables and PP is the set of all proof polynomials. Proof polynomials in K4J4 are the same as in LP i.e. terms built from variables $x, y, z, \ldots$ and constants $a, b, c, \ldots$ with the binary application $\cdot$, binary sum $+$ and unary inspection ! operator.

Formulas in K4J4 are

$$A ::= S|\bot|A \rightarrow A|A \wedge A|A \vee A|\neg A|\Box A|t : A$$

where S is a sentence variable and t a proof polynomial. The formal definition of proof polynomials is

$$t ::= x|a|!t|t \cdot t|t + t$$

where x is a proof variable and a a proof constant. In the following we write Fml for the set of all formulas, F, G are said to be formulas and s, t proof polynomials. As for precedence, we assume '$t : ()$', '$\Box()$' and '$\neg$' bind stronger than '$\vee$' and '$\wedge$,' which bind stronger than '$\rightarrow$'.

Definition 9. The system K4J4 consists of the following axioms and rules:

Classical Propositional Logic

P: Any finitary axiomatization
R1: Modus Ponens

Justification Logic J4

LP1: $s : (F \to G) \to (t : F \to (s \cdot t) : G)$ (application)
LP2: $t : F \to !t : t : F$ (inspection)
LP3: $s : F \to (s + t) : F, t : F \to (s + t) : F$ (sum)
R2: For any constant c and axiom A infer $\vdash c : A$ (axiom necessitation)

Epistemic Logic K4

E1: $\Box(F \to G) \to (\Box F \to \Box G)$
E2: $\Box F \to \Box\Box F$
R3: $\vdash F \Rightarrow \vdash \Box F$

Connection Axiom

C1: $t : F \to \Box F$

In justification logic a formula $t : F$ is usually interpreted as t is a justification for F. Thus, the axiom LP1 (application) states that, when an implication $F \to G$ has a justification s and there is a justification t for the premise F then we can build a justification for the conclusion G, namely $s \cdot t$. LP2 (inspection) represents the property that a justification is verifiable. Whenever a formula F has a justification (for example t) then there is a justification (also called verification) $!t$ for $t : F$. LP3 reflects the fact that whenever additional evidence (for instance s) is added to t, for example, then the new concatenated justification $t + s$ is at least still a justification for everything t was. In other words, by adding evidence we never lose justifiability power. For epistemic logic of beliefs $\Box F$ is interpreted as "we believe F." As a result, axiom E1 states that if we believe $F \to G$ and F then we also believe G. E2 represents the property that if we believe F, then we also believe that we believe F. In order to connect K4 and J4, C1 reflects the principle that if there is a justification for F then we also believe F.

Like in many other logics, deductions in K4J4 are always finite in length. Hence, only a finite subset of the set $\{c : A | c \text{ a proof constant}, A \text{ an axiom}\}$ of all available instances of rule R2 is used in any given K4J4 derivation. Thus, it is sensible to analyze the system K4J4 which has a restricted rule R2. This is what we are going to do next.

Definition 10. Any set $\mathcal{CS} = \{c_1 : A_1, c_2 : A_2, \ldots\}$ where A_i's are axioms and c_i's are proof constants is called a constant specification. When the rule R2 is restricted to a constant specification $\mathcal{CS}$ we write $\mathsf{K4J4}_{\mathcal{CS}}$. When we say R2 is restricted to $\mathcal{CS}$ we mean that R2 is replaced by "$c : A \in \mathcal{CS}$ implies $\vdash c : A$".

Remark 11. With $\mathsf{K4J4}_{\mathcal{CS}}$ R2 can only be applied to combinations of proof constants and axioms which are elements of $\mathcal{CS}$. Thus we can say that R2 is in fact restricted. Notice, $\mathsf{K4J4}$ is the same as $\mathsf{K4J4}_{\mathcal{CS}}$ when $\mathcal{CS} = \{c : A | c$ a proof constant, A an axiom$\}$. This constant specification is called the total constant specification ($\mathcal{TCS}$). Thus, from now on we are only concerned with the more general $\mathsf{K4J4}_{\mathcal{CS}}$.

Definition 12. Let Γ be a set of formulas and G a formula. A deduction of G from Γ in $\mathsf{K4J4}_{\mathcal{CS}}$ (denoted by $\Gamma \vdash_{\mathsf{K4J4}_{\mathcal{CS}}} G$) is a finite sequence of formulas $(F_1, F_2, \ldots, F_n)$ where $F_n \equiv G$ and each F_i $i \leq n$ is either

1. an axiom,
2. an element of Γ,
3. obtainable by R3 from some hypotheses-independent F_k $(k < i)$ or
4. obtainable from $\{F_1, F_2, \ldots, F_{i-1}\}$ by Modus Ponens or R2.

F_i is said to be hypotheses-dependent if F_i is an element of Γ or F_i is obtained by Modus Ponens where at least one of the premises is hypotheses-dependent. Otherwise F_i is said to be hypotheses-independent.

Remark 13. F_n is hypotheses-independent in a derivation from Γ iff there is a derivation of F_n with the empty set of hypotheses

Just like many other logics, $\mathsf{K4J4}_{\mathcal{CS}}$ enjoys the properties of the deduction theorem and closure under substitution.

Theorem 14 (Deduction Theorem)**.** *Let Γ be a set of formulas then*

$$\Gamma, G \vdash_{\mathsf{K4J4}_{\mathcal{CS}}} F \text{ iff } \Gamma \vdash_{\mathsf{K4J4}_{\mathcal{CS}}} G \to F$$

Proof. "$\Leftarrow$" Trivial application of Modus Ponens.
"$\Rightarrow$" Induction on the length of the derivation $(F_1, \ldots, F_n)$. For the base case $F_n \equiv G$ we use the tautology $F_n \to F_n$. For the base case $F_n \in \Gamma$ and where F_n is an axiom we use the tautology $F_n \to (G \to F_n)$.
For the inductive step where F_n is hypotheses-independent we use the tautology $F_n \to (G \to F_n)$. For the case where F_n is hypotheses-dependent and obtained from formula B and $B \to F_n$ by Modus Ponens we use the induction hypothesis and the tautology $(G \to (B \to F_n)) \to ((G \to B) \to (G \to F_n))$. $\square$

Theorem 15 (Substitution Property)**.** *Let Γ be a set of formulas and F a formula then*

$$\Gamma(S, x) \vdash_{\mathsf{K4J4}_{\mathcal{CS}}} F(S, x) \text{ implies } \Gamma(S/G, x/t) \vdash_{\mathsf{K4J4}_{\mathcal{CS}}} F(S/G, x/t)$$

for any proof variable x, proof polynomial t and formula G.

Proof. This holds due to the schema-style formulation of $\mathsf{K4J4}_{\mathcal{CS}}$. □

Furthermore, the internalization lemma is available for $\mathsf{K4J4}$.

Lemma 16. *Let $G_1, \ldots, G_n, F$ be any formulas then*

$$G_1, \ldots, G_n \vdash_{\mathsf{K4J4}} F \text{ implies } x_1 : G_1, \ldots, x_n : G_n \vdash_{\mathsf{K4J4}} t(x_1, \ldots, x_n) : F$$

for some proof polynomial $t(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are fresh distinct proof variables.

Proof. By translating the given derivation into the desired derivation. For any axiom A we can use $c : A$ because of R2. For all $G_1, \ldots, G_n$ we use $x_1 : G_1 \ldots, x_n : G_n$ since this is the hypothesis. Whenever we have $c : A$ we replace it by $!c : c : A$ which can done because of LP2 and Modus Ponens. Every use of the rule Modus Ponens can be replaced as follows

$$\frac{F \to G \quad F}{G} \quad \Rightarrow \quad \frac{s_1 : (F \to G) \quad s_2 : F}{(s_1 \cdot s_2) : G}$$

The right-hand derivation is obtained by an instance of the application axiom (LP1) and by applying the rule Modus Ponens twice. □

Corollary 17 (Internalization Property)**.** *If* $\mathsf{K4J4} \vdash F$ *then* $\mathsf{K4J4} \vdash t : F$ *for some ground proof polynomial t i.e t does not contain any proof variables.*

Proof. This is the case of Lemma 16 where no $G_1, \ldots, G_n$ are used. □

So far all important syntax related definitions were given. The next step is to define what a typical semantics for $\mathsf{K4J4}_{\mathcal{CS}}$ is.

2.2 AF-models

For the semantics, we follow the idea of Artemov and Nogina in [8]. The paper deals with the hybrid logic $\mathsf{S4LP}$ and uses AF-models as a semantic. In the paper, it was shown that $\mathsf{S4LP}$ is sound and complete with respect to AF-models. We will see that AF-models can be adapted to $\mathsf{K4J4}$ too. AF-models are based on F-models, which were introduced by Fitting as a Kripke-style semantics for LP [13, 15]. As a result, AF-models also use admissible evidence functions, which are crucial for soundness and completeness.

Definition 18. An AF-model for $\mathsf{K4J4}$ consists of a tuple $(W, R, R^e, \mathcal{E}, \mathcal{V})$ where

- W is a non-empty set of states (or possible worlds)
- $R \subseteq W \times W$ a transitive accessibility relation
- $R^e \subseteq W \times W$ a transitive evidence accessibility relation which includes R
- $\mathcal{V} : W \to 2^{SV}$ an arbitrary function which maps a world to a set of sentence variables
- $\mathcal{E} : W \times PP \to 2^{Fml}$ an admissible evidence function which has the following properties:

  Monotonicity : $uR^e v$ implies $\mathcal{E}(u,t) \subseteq \mathcal{E}(v,t)$

  Closure Properties :

    - Application: $F \to G \in \mathcal{E}(u,s)$ and $F \in \mathcal{E}(u,t)$ implies $G \in \mathcal{E}(u, s \cdot t)$
    - Inspection: $F \in \mathcal{E}(u,t)$ implies $t : F \in \mathcal{E}(u, !t)$
    - Sum: $\mathcal{E}(u,s) \cup \mathcal{E}(u,t) \subseteq \mathcal{E}(u, s+t)$

- $\Vdash \subseteq W \times Fml$ a binary relation which is defined as follows:
  - $u \nVdash \bot$
  - $u \Vdash S$ iff $S \in \mathcal{V}(u)$
  - $u \Vdash F \vee G$ iff $u \Vdash F$ or $u \Vdash G$
  - $u \Vdash F \wedge G$ iff $u \Vdash F$ and $u \Vdash G$
  - $u \Vdash F \to G$ iff $u \nVdash F$ or $u \Vdash G$
  - $u \Vdash \neg F$ iff $u \nVdash F$
  - $u \Vdash \Box F$ iff $v \Vdash F$ for all v such that uRv
  - $u \Vdash t : F$ iff $F \in \mathcal{E}(u,t)$ and $v \Vdash F$ for all v such that $uR^e v$

Remark 19. In the following $\mathcal{M} = (W, R, R^e, \mathcal{E}, \mathcal{V})$ stands for an arbitrary AF-model and $u, v \in W$ for possible worlds of that AF-model. As for interpretation, $u \Vdash F$ can be read as "F holds at u." Furthermore, we say F is valid in an AF-model $\mathcal{M}$ (denoted as $\mathcal{M} \vDash F$) iff $u \Vdash F$ for all $u \in W$. Lastly, a formula F is called valid iff F is valid in every model.

The same terms can be used for sets of formulas. A set of formulas X holds at u (denoted as $u \Vdash X$) iff $u \Vdash F$ for all $F \in X$. Similarly, X is said to be valid in an AF-model $\mathcal{M}$ (denoted as $\mathcal{M} \vDash X$) iff $u \Vdash X$ for all $u \in W$. Finally, X is called valid iff X is valid in every model.

In the previous chapter we defined what a constant specification is. It turns out that a notion for AF-models is needed which sets AF-models in a relation with constant specifications. This will be crucial for soundness and completeness.

Definition 20. We say an AF-model $\mathcal{M} = (W, R, R^e, \mathcal{E}, \mathcal{V})$ meets $\mathcal{CS}$ iff for all $c : A \in \mathcal{CS}$ and all $u \in W$ it holds that $A \in \mathcal{E}(u, c)$.

Now that we have both, syntax as well as semantics, we are going to analyze soundness and completeness. The first and easier case will be soundness.

2.3 Soundness

Theorem 21 (Soundness)**.** *For all formulas F and all AF-models $\mathcal{M} = (W, R, R^e, \mathcal{E}, \mathcal{V})$ that meet $\mathcal{CS}$ it holds that if $\mathsf{K4J4}_{\mathcal{CS}} \vdash F$ then $\mathcal{M} \vDash F$.*

Proof. The proof is an induction on the derivation of F. For the whole proof u is an arbitrary world, in other words an element of W.

P *Axioms*: $\Vdash$ respects the boolean connectives. Therefore, the propositional axioms hold.

$E1$ *Axiom*: We want to show that $u \Vdash \Box(F \to G)$ and $u \Vdash \Box F$ implies $u \Vdash \Box G$.

By definition of $\Vdash$ we know that $u \Vdash \Box(F \to G)$ and $u \Vdash \Box F$ hold when for all $v \in W$ such that uRv, $v \nVdash F$ or $v \Vdash G$ holds and for all $v \in W$ such that uRv, $v \Vdash F$ holds. It follows that for all $v \in W$ such that uRv, $v \Vdash G$ holds. Thus by definition of $\Vdash$, $u \Vdash \Box F$.

$E2$ *Axiom*: We want to show that $u \Vdash \Box F$ implies $u \Vdash \Box\Box F$.

From $u \Vdash \Box F$ we know by definition of $\Vdash$ that for all $v \in W$ such that uRv, $v \Vdash F$ holds. The transitivity of R implies that for all $v, y \in W$ such that $uRvRy$ also uRy. From the fact that for all $v \in W$ such that uRv, $v \Vdash F$ holds, we know that for all $y, v \in W$ such that $uRvRy$, $y \Vdash F$ holds. By definition of $\Vdash$ we conclude that for all $v \in W$ such that uRv, $v \Vdash \Box F$ holds. This, on the other hand, is by definition of $\Vdash$, $u \Vdash \Box\Box F$.

$LP1$ *Axiom*: We want to show that $u \Vdash s : (F \to G)$ and $u \Vdash t : F$ implies $u \Vdash (s \cdot t) : G$.

By the definition of $\Vdash$ it follows from $u \Vdash s : (F \to G)$ and $u \Vdash t : F$ that $F \to G \in \mathcal{E}(u, s)$, $F \in \mathcal{E}(u, t)$ and for all $v \in W$ such that $uR^e v$, $v \Vdash F \to G$ and $v \Vdash F$ hold. First, from the application closure property, $F \to G \in \mathcal{E}(u, s)$ and $F \in \mathcal{E}(u, t)$ we obtain $G \in \mathcal{E}(u, s \cdot t)$. Secondly, we know that for all $v \in W$ such that $uR^e v$, $v \Vdash G$ holds. Hence, we have $u \Vdash (s \cdot t) : G$.

$LP2$ *Axiom*: We want to show that $u \Vdash t : F$ implies $u \Vdash !t : t : F$.

$u \Vdash t : F$ and the definition of $\Vdash$ implies $F \in \mathcal{E}(u, t)$ and that for all $v \in W$ such that $uR^e v$, $v \Vdash F$ holds. We need to show three facts in order to continue. First, by transitivity of R^e we know that for all $v, y \in W$ such that $uR^e vR^e y$, $uR^e y$ holds. Hence, for all $v, y \in W$ such that $uR^e vR^e y$, $y \Vdash F$ holds. Secondly, from the monotonicity of $\mathcal{E}$ we obtain that for all $v \in W$ such that $uR^e v$, $F \in \mathcal{E}(v, t)$. Thirdly, by the inspection closure property, we get $t : F \in \mathcal{E}(u, !t)$ from $F \in \mathcal{E}(u, t)$. The result of the first and second fact is, by definition of $\Vdash$, that for all $v \in W$ such that $uR^e v$, $v \Vdash t : F$ holds. From this and the third fact we conclude $u \Vdash !t : t : F$.

$LP3$ *Axiom*: We want to show that $u \Vdash s : F$ implies $u \Vdash (s + t) : F$. The second case of this axiom can be proven analogously by applying the appropriate instance of the sum closure property.

From $u \Vdash s : F$ and the definition of $\Vdash$ we know that $F \in \mathcal{E}(u, s)$ and for all $v \in W$ such that $uR^e v$, $v \Vdash F$ holds. Due to the fact that $F \in \mathcal{E}(u, s)$ and that the sum closure property is available, we conclude that $F \in \mathcal{E}(u, s + t)$. Hence, $u \Vdash (s + t) : F$.

$C1$ *Axiom*: We want to show that $u \Vdash t : F$ implies $u \Vdash \Box F$.

Once again by definition of $\Vdash$, we conclude from $u \Vdash t : F$ that $F \in \mathcal{E}(u, t)$ and for all $v \in W$ such that $uR^e v$, $v \Vdash F$ holds. Since $R \subseteq R^e$, it must be that for all $v \in W$ such that uRv, $v \Vdash F$ holds. This is by definition of $\Vdash$ what is required for $u \Vdash \Box F$.

$R1$ *Modus Ponens*: $\Vdash$ respects the boolean connectives. Therefore, Modus Ponens is sound.

$R2$ *Rule*: We want to show that whenever $c : A \in \mathcal{CS}$ then for all $u \in W$, $u \Vdash c : A$ holds.

For all $u \in W$, $u \Vdash A$ holds since A is an axiom and is, therefore, covered by the previous cases P, E1, E2, LP1 - LP3, C1. Additionally, for all $u \in W$, $A \in \mathcal{E}(u, c)$ holds because of the precondition of $\mathcal{M}$ meeting $\mathcal{CS}$. As a result, for all $u \in W$, $u \Vdash c : A$ holds.

R3 Rule: We want to show that whenever $\vdash \Box F$ was derived from $\vdash F$ by R3 then for all $u \in W$, $u \Vdash \Box F$ holds.

From $\vdash F$ we get by induction hypothesis that for all $u \in W$, $u \Vdash F$ holds. Certainly, the less general statement that for all $v \in W$ such that uRv, $v \Vdash F$ holds, is true. This, however, means that for all $u \in W$, $u \Vdash \Box F$ holds.

□

2.4 Completeness

To prove the completeness theorem we work with constructions of maximal consistent sets and we also need a few properties of maximal consistent sets. First, let us start by defining what a consistent set is.

Definition 22. A consistent set for $\mathsf{K4J4}_{\mathcal{CS}}$ is a set of formulas Γ such that for any $\{F_1, \ldots, F_n\} \subseteq \Gamma$ it holds that $\mathsf{K4J4}_{\mathcal{CS}} \nvdash \neg F_1 \vee \ldots \vee \neg F_n$.

As mentioned, we are going to work with maximal consistent sets. We know now what a consistent set is. Thus, we need to define what makes a consistent set maximal.

Definition 23. If Γ is consistent and if for all consistent $\Gamma' \supseteq \Gamma$ it follows that $\Gamma' = \Gamma$, then Γ is called a maximal consistent set.

In order to complete the proof for completeness, we need several properties of maximal consistent sets. Especially, the properties 24.2 and 24.3 will be heavily used in the upcoming proofs.

Lemma 24. *For any maximal consistent set Γ the following properties hold:*

1. *For any formula F either $F \in \Gamma$ or $\neg F \in \Gamma$ but not both.*

2. *If $\mathsf{K4J4}_{\mathcal{CS}} \vdash F$ then $F \in \Gamma$.*

3. *Γ is closed under Modus Ponens i.e. $F \to G \in \Gamma$ and $F \in \Gamma$ then $G \in \Gamma$.*

4. *$F \in \Gamma$ or $G \in \Gamma$ iff $F \vee G \in \Gamma$.*

5. *$F \in \Gamma$ and $G \in \Gamma$ iff $F \wedge G \in \Gamma$.*

Proof. The cases 1. and 2. are proofs by contradiction. The others are straight forward proofs.

1. First, we need to show that $F \in \Gamma$ and $\neg F \in \Gamma$ cannot hold. This is obvious because $\neg F \vee \neg\neg F$ is a tautology. Therefore, $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg F \vee \neg\neg F$, which would make Γ inconsistent. Hence, we have obviously a contradiction.
   Secondly, we show that $F \notin \Gamma$ and $\neg F \notin \Gamma$ cannot hold. If this were the case, then either $\{F\} \cup \Gamma$ or $\{\neg F\} \cup \Gamma$ would be consistent and also be a superset of Γ. Again, this would contradict the maximal consistency of Γ. As a result, it remains to show that either $\{F\} \cup \Gamma$ or $\{\neg F\} \cup \Gamma$ is consistent.
   Assume both are inconsistent, then $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg Y_1 \vee \ldots \vee \neg Y_n \vee \neg F$ and $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg Z_1 \vee \ldots \vee \neg Z_m \vee \neg\neg F$ for some $\{Y_1, \ldots, Y_n\} \subseteq \Gamma$ and $\{Z_1, \ldots, Z_m\} \subseteq \Gamma$. Without loss of generality we can assume $\{Y_1, \ldots, Y_n\} = \{Z_1, \ldots, Z_m\}$. Then, $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg Y_1 \vee \ldots \vee \neg Y_n \vee \neg F$ and $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg Y_1 \vee \ldots \vee \neg Y_n \vee \neg\neg F$. From $(\neg Y_1 \vee \ldots \vee \neg Y_n \vee \neg F) \rightarrow ((\neg Y_1 \vee \ldots \vee \neg Y_n \vee \neg\neg F) \rightarrow (\neg Y_1 \vee \ldots \vee \neg Y_n))$ being a tautology we conclude $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg Y_1 \vee \ldots \vee \neg Y_n$. Hence, Γ is inconsistent, which is clearly a contradiction.

2. Assume $F \notin \Gamma$ then $\{F\} \cup \Gamma \supset \Gamma$ and still be consistent. This, however, would be a contradiction.
   It remains to show that $\{F\} \cup \Gamma$ is consistent. Assume inconsistency, then $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg Y_1 \vee \ldots \vee \neg Y_n \vee \neg F$ for some $\{Y_1, \ldots, Y_n\} \subseteq \Gamma$. With propositional reasoning it follows that $\mathsf{K4J4}_{\mathcal{CS}} \vdash F \rightarrow \neg Y_1 \vee \ldots \vee \neg Y_n$. With this, $\mathsf{K4J4}_{\mathcal{CS}} \vdash F$ and Modus Ponens (rule R1) $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg Y_1 \vee \ldots \vee \neg Y_n$ can be derived. Hence, Γ is inconsistent. However, this cannot be by precondition. Thus, we have a contradiction.

3. $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg(F \rightarrow G) \vee \neg F \vee \neg\neg G$ because the formula is a tautology. From $F \rightarrow G \in \Gamma$ and $F \in \Gamma$ it follows that $\neg G \notin \Gamma$. Otherwise, Γ would be inconsistent. By the first property it follows that $G \in \Gamma$.

4. "$\Rightarrow$" $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$ are tautologies. Therefore, both formulas are elements of Γ by the second property. From this and the fact that $A \in \Gamma$ or $B \in \Gamma$ we conclude by property three that $A \vee B \in \Gamma$.

   "$\Leftarrow$" $\neg A \notin \Gamma$ or $\neg B \notin \Gamma$ otherwise Γ would be inconsistent with use of Lemma 24.3 and the tautology $\neg A \rightarrow (\neg B \rightarrow \neg(A \vee B))$, which is an element of Γ according to Lemma 24.2. Hence, with the first property either $A \in \Gamma$ or $B \in \Gamma$.

5. "$\Rightarrow$" $F \to (G \to F \land G) \in \Gamma$ because of Lemma 24.2 and the fact that the formula is a tautology. Since maximal consistent sets are closed under Modus Ponens, we know that $F \land G \in \Gamma$.

   "$\Leftarrow$" $F \land G \to F \in \Gamma$ and $F \land G \to G \in \Gamma$ because both are tautologies and Lemma 24.2. From $F \land G \in \Gamma$ and the third property we conclude that $F \in \Gamma$ and $G \in \Gamma$.

$\square$

As mentioned at the beginning of this section, we are going to construct maximal consistent sets from any consistent set of formulas. One possible construction was given by Lindenbaum. Nowadays, it is known as the standard Lindenbaum construction. There are several books which discuss the Lindenbaum construction. For more detail consult, for example, [9].

Now, that we have all the necessary definitions and results at our disposal, we are ready for the main theorem in the section, the completeness theorem.

Theorem 25 (Completeness)**.** *For all $\mathcal{CS}$ and all formulas F, if for all AF-models $\mathcal{M} = (W, R, R^e, \mathcal{E}, \mathcal{V})$ that meet $\mathcal{CS}$ it holds that $\mathcal{M} \vDash F$ then $\mathsf{K4J4}_{\mathcal{CS}} \vdash F$.*

Proof. In order to prove the theorem we use the contraposition.
For any $\mathcal{CS}$ and there is an AF-model $\mathcal{M}$ such that for any formula F

$$\mathsf{K4J4}_{\mathcal{CS}} \nvdash F \text{ implies } \mathcal{M} \nvDash F$$

We define $\mathcal{M} = (W, R, R^e, \mathcal{E}, \mathcal{V})$ where W is the collection of all maximal consistent sets. We define $\mathcal{V}$ through "$S \in \mathcal{V}(\Gamma)$ iff $S \in \Gamma$" for every maximal consistent set Γ. For traditional reasons, worlds and maximal consistent set are denoted by capital Greek letters. The accessibility relations are defined as $R = R^e = \{(\Gamma, \Delta) : \Gamma^\sharp \cup \Gamma^\flat \subseteq \Delta\}$, where $\Gamma^\flat = \{t : F, F | t : F \in \Gamma\}$ and $\Gamma^\sharp = \{\Box F, F | \Box F \in \Gamma\}$. Lastly, we define $\mathcal{E}$ by $F \in \mathcal{E}(\Gamma, t)$ iff $t : F \in \Gamma$.

Now, we need to show that $\mathcal{M}$ is in fact an AF-model meeting $\mathcal{CS}$.

$\mathcal{M}$ *meets* $\mathcal{CS}$**:** Our goal is to prove the following. For all $c : A \in \mathcal{CS}$ it holds that for all $\Gamma \in W$, $A \in \mathcal{E}(\Gamma, c)$. We know that $\mathsf{K4J4}_{\mathcal{CS}} \vdash c : A$ by rule R2. With Lemma 24.2, $c : A \in \Gamma$. This implies by definition of $\mathcal{E}$ that $A \in \mathcal{E}(\Gamma, c)$, which is what we wanted to show.

$R \subseteq R^e$**:** This holds due to the definition of $R = R^e$.

$R = R^e$ *transitive*: Assume $\Gamma R \Delta R \Phi$. From the definition of the relations we know $\Gamma^\sharp \cup \Gamma^\flat \subseteq \Delta$ and $\Delta^\sharp \cup \Delta^\flat \subseteq \Phi$. It also follows from the definition of $()^\flat$ and $()^\sharp$ that $\Gamma^\sharp \cup \Gamma^\flat \subseteq \Delta^\sharp \cup \Delta^\flat$. Hence, $\Gamma^\sharp \cup \Gamma^\flat \subseteq \Delta^\sharp \cup \Delta^\flat \subseteq \Phi$. Since the subset relation is transitive, it holds that $\Gamma R \Phi$.

$\mathcal{E}$ *Monotonicity*: Let $\Gamma R^e \Delta$. From $F \in \mathcal{E}(\Gamma, t)$ we conclude by definition of $\mathcal{E}$ that $t : F \in \Gamma$. Then, by definition of $()^\flat$ it holds that $t : F \in \Gamma^\flat$. From the fact that $\Gamma R^e \Delta$ we know $t : F \in \Delta$, which means that $F \in \mathcal{E}(\Delta, t)$.

$\mathcal{E}$ *Inspection*: From $F \in \mathcal{E}(\Gamma, t)$ it follows by definition of $\mathcal{E}$ that $t : F \in \Gamma$. $\mathsf{K4J4}_{\mathcal{CS}} \vdash t : F \rightarrow !t : t : F$ since the formula is an instance of the LP2 axiom. By Lemma 24.2, it follows that $t : F \rightarrow !t : t : F \in \Gamma$. We know from Lemma 24.3 that maximal consistent sets are closed under Modus Ponens. Thus, $!t : t : F \in \Gamma$. Again by definition of $\mathcal{E}$, we have $t : F \in \mathcal{E}(\Gamma, !t)$.

$\mathcal{E}$ *Sum*: Let $F \in \mathcal{E}(\Gamma, t) \cup \mathcal{E}(\Gamma, s)$. Without loss of generality we assume that $F \in \mathcal{E}(\Gamma, s)$. Then, we have $s : F \in \Gamma$ by definition of $\mathcal{E}$. It holds that $s : F \rightarrow (s + t) : F \in \Gamma$ because the formula is an instance of the LP3 axiom and Lemma 24.2. Γ is closed under Modus Ponens (Lemma 24.3), therefore, $(s + t) : F \in \Gamma$. Finally, we get $F \in \mathcal{E}(\Gamma, s + t)$ once more by the definition of $\mathcal{E}$.

$\mathcal{E}$ *Application*: As always we conclude $s : (F \rightarrow G) \in \Gamma$ and $t : F \in \Gamma$ from $F \rightarrow G \in \mathcal{E}(\Gamma, s)$ and $F \in \mathcal{E}(\Gamma, t)$. Furthermore, we have $s : (F \rightarrow G) \rightarrow (t : F \rightarrow s \cdot t : G) \in \Gamma$ by Lemma 24.2 and the fact that the formula is an instance of the LP1 axiom. We conclude that $s \cdot t : G \in \Gamma$ because Γ is closed under Modus Ponens (Lemma 24.3). Hence, $G \in \mathcal{E}(\Gamma, s \cdot t)$.

The following lemma is going to play a crucial role in the proof of the truth lemma, which is the key part in the whole completeness proof.

Lemma 26. *For any formula F and any $\Gamma \in W$ it holds that $F \in \Gamma^\sharp \cup \Gamma^\flat$ implies $\Box F \in \Gamma$*

Proof. By definition of $\Gamma^\sharp$ and $\Gamma^\flat$ we know that F is of the form Y, $\Box Y$ or $t : Y$ for a formula Y. Now, let us consider each case:

$F \equiv \Box Y \in \Gamma^\sharp$ *because* $\Box Y \in \Gamma$: $\mathsf{K4J4}_{\mathcal{CS}} \vdash \Box Y \rightarrow \Box\Box Y$ since it is an instance of axiom E2. By Lemma 24.2, $\Box Y \rightarrow \Box\Box Y \in \Gamma$. Γ is closed under Modus Ponens according to Lemma 24.3. Thus, $\Box\Box Y \in \Gamma$, which means $\Box F \in \Gamma$.

$F \equiv t : Y \in \Gamma^\flat$ *because* $t : Y \in \Gamma$**:** Here we use the instance $t : Y \to !t : t : Y$ of axiom LP2. Hence, $\mathsf{K4J4}_{\mathcal{CS}} \vdash t : Y \to !t : t : Y$. From Lemma 24.2 and 24.3 we know $!t : t : Y \in \Gamma$. Furthermore, it holds that $\mathsf{K4J4}_{\mathcal{CS}} \vdash !t : t : Y \to \Box t : Y$ because the formula is an instance of the axiom C1. Again, Lemma 24.2 and 24.3 imply $\Box t : Y \in \Gamma$, which is the same as $\Box F \in \Gamma$.

$F \equiv Y \in \Gamma^\flat$ *because* $t : Y \in \Gamma$ *for some* t**:** It holds that $\mathsf{K4J4}_{\mathcal{CS}} \vdash t : Y \to \Box Y$ because the formula is an instance of the C1 axiom. According to Lemma 24.2 we can conclude that $t : Y \to \Box Y \in \Gamma$. Lemma 24.3 implies $\Box Y \in \Gamma$. In other words, $\Box F \in \Gamma$.

$F \equiv Y \in \Gamma^\sharp$ *because* $\Box Y \in \Gamma$**:** Since $\Box Y \equiv \Box F$, we do not need to show anything.

$\Box$

Finally, we are about to state, as well as prove, the main lemma in this proof, the so called truth lemma.

Lemma 27 (Truth Lemma)**.** *For all* $\Gamma \in W$ *and all formulas* F *it holds that*

$$\Gamma \Vdash F \textit{ iff } F \in \Gamma$$

Proof. The proof is an induction on F.

$F \equiv S$**:** $\Gamma \Vdash S$ iff $S \in \mathcal{V}(\Gamma)$ and $S \in \mathcal{V}(\Gamma)$ iff $S \in \Gamma$ by definition of $\mathcal{V}$, W and $\Vdash$.

$F \equiv A \vee B$**:** $\Gamma \Vdash A \vee B$ iff $\Gamma \Vdash A$ or $\Gamma \Vdash B$ by definition of $\Vdash$. The induction hypothesis implies that $\Gamma \Vdash A$ or $\Gamma \Vdash B$ iff $A \in \Gamma$ or $B \in \Gamma$. Finally, from Lemma 24.4 we know that $A \in \Gamma$ or $B \in \Gamma$ iff $A \vee B \in \Gamma$.

$F \equiv \neg A$**:** From Lemma 24.1 we know that $\neg A \notin \Gamma$ iff $A \in \Gamma$. The induction hypothesis implies that $A \in \Gamma$ iff $\Gamma \Vdash A$. Finally, it holds that $\Gamma \Vdash A$ iff $\Gamma \nVdash \neg A$ by definition of $\Vdash$.

$F \equiv A \wedge B$**:** $A \wedge B \to \neg(\neg A \vee \neg B)$ and $\neg(\neg A \vee \neg B) \to A \wedge B$ are propositional tautologies. Therefore, $\mathsf{K4J4}_{\mathcal{CS}} \vdash A \wedge B \to \neg(\neg A \vee \neg B)$ and $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg(\neg A \vee \neg B) \to A \wedge B$. Since $\Vdash$ respects the Modus Ponens and maximal consistent sets are closed under Modus Ponens, it remains to show that $\Gamma \Vdash \neg(\neg A \vee \neg B)$ iff $\neg(\neg A \vee \neg B) \in \Gamma$. This, however, holds due to the fact that this lemma holds for $\vee$ and $\neg$.

$F \equiv A \to B$: $\mathsf{K4J4}_{\mathcal{CS}} \vdash (A \to B) \to \neg A \vee B$ and $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg A \vee B \to (A \to B)$ because both formulas are tautologies. We apply the identical argument as in the previous case and conclude that the lemma holds for this case as well.

$F \equiv \Box A$: "$\Rightarrow$" From $\Box A \notin \Gamma$ it follows that $\Gamma^\sharp \cup \Gamma^\flat \cup \{\neg A\}$ is consistent. Let us assume the contrary holds. Then, there are $Y_1, \dots, Y_n \in \Gamma^\sharp \cup \Gamma^\flat$ such that $\mathsf{K4J4}_{\mathcal{CS}} \vdash \neg Y_1 \vee \dots \vee \neg Y_n \vee \neg\neg A$. With propositional logic we can derive $\mathsf{K4J4}_{\mathcal{CS}} \vdash Y_1 \wedge \dots \wedge Y_n \to A$. From this we obtain $\mathsf{K4J4}_{\mathcal{CS}} \vdash \Box(Y_1 \wedge \dots \wedge Y_n \to A)$ by rule R3. By normal modal logic reasoning $\mathsf{K4J4}_{\mathcal{CS}} \vdash \Box Y_1 \wedge \dots \wedge \Box Y_n \to \Box A$ can be deduced. Lemma 24.2 implies that $\Box Y_1 \wedge \dots \wedge \Box Y_n \to \Box A \in \Gamma$. From Lemma 26 and $Y_1, \dots, Y_n \in \Gamma^\sharp \cup \Gamma^\flat$ we know that $\Box Y_1, \dots, \Box Y_n \in \Gamma$. Lemma 24.5 brings us to $\Box Y_1 \wedge \dots \wedge \Box Y_n \in \Gamma$. Finally, $\Box Y_1 \wedge \dots \wedge \Box Y_n \to \Box A \in \Gamma$, $\Box Y_1 \wedge \dots \wedge \Box Y_n \in \Gamma$ and Lemma 24.3 imply that $\Box A \in \Gamma$, which is clearly a contradiction to our assumption that $\Box A \notin \Gamma$.
Now, we know that $\Gamma^\sharp \cup \Gamma^\flat \cup \{\neg A\}$ is consistent. Hence, we can build a maximal consistent set Δ from $\Gamma^\sharp \cup \Gamma^\flat \cup \{\neg A\}$. It holds that $A \notin \Delta$ due to the consistency of Δ. From this, we get to $\Delta \nVdash A$ by induction hypothesis. Since $\Gamma R \Delta$, which is because $\Gamma^\sharp \cup \Gamma^\flat \subseteq \Delta$, we know $\Gamma \nVdash \Box A$.

"$\Leftarrow$" By definition of R, $\Box A \in \Gamma$ implies that for all $\Delta \in W$ such that $\Gamma R \Delta$, $A \in \Delta$ holds. From the induction hypothesis it follows that for all $\Delta \in W$ such that $\Gamma R \Delta$, $\Delta \Vdash A$ holds. Therefore, $\Gamma \Vdash \Box A$.

$F \equiv t : A$: "$\Rightarrow$" From $\Gamma \Vdash t : A$ it follows by definition of $\Vdash$ that $A \in \mathcal{E}(\Gamma, t)$, which is $t : A \in \Gamma$ by definition of $\mathcal{E}$.

"$\Leftarrow$" By definition of R^e it follows from $t : A \in \Gamma$ that for all $\Delta \in W$ such that $\Gamma R^e \Delta$, $A \in \Delta$ holds. The induction hypothesis implies that for all $\Delta \in W$ such that $\Gamma R^e \Delta$, $\Delta \Vdash A$. Furthermore, $A \in \mathcal{E}(\Gamma, t)$ holds since $t : A \in \Gamma$. From these two results we obtain $\Gamma \Vdash t : A$.

$\Box$

Back to the completeness. Assume $\nvdash F$. This implies that $\{\neg F\}$ is consistent. Build a maximal consistent set Γ from $\{\neg F\}$. $F \notin \Gamma$ due to consistency. The truth lemma implies that $\Gamma \nVdash F$

$\Box$

3 A Multi-Agent Logic of Justified Belief

3.1 The System $\mathsf{K4_nJ4}$

A natural extension to $\mathsf{K4J4}$ is $\mathsf{K4_nJ4}$, the multi-agent version of $\mathsf{K4J4}$. Here, we have multiple modalities $\Box_i$, $i \in \{1, \ldots, n\}$. The rest remains the same as before. As a result, formulas in $\mathsf{K4_nJ4}$ are

$$A ::= S|\bot|A \to A|A \wedge A|A \vee A|\neg A|\Box_i A|t : A$$

where S is a sentence variable, t a proof polynomial and $i \in \{1, \ldots, n\}$. Proof polynomials are the same as before,

$$t ::= x|a|!t|t \cdot t|t + t$$

where x is a proof variable and a a proof constant. Also the precedence remains the same. Thus, we assume '$t : ()$', '$\Box_i()$' and '$\neg$' bind stronger than '$\vee$' and '$\wedge$,' which bind stronger than '$\to$'. As for the axioms and rules, for each single-agent axiom or rule we have n multi-agent axioms or rules where any occurrence of $\Box$ is replaced by one distinct $\Box_1, \ldots, \Box_n$. This brings us to the following definition of $\mathsf{K4_nJ4}$.

Definition 28. The system $\mathsf{K4_nJ4}$ consists of the following axioms and rules ($i \in \{1, \ldots, n\}$):

Classical Propositional Logic

P: Any finitary axiomatization
R1: Modus Ponens

Justification Logic $\mathsf{J4}$

LP1: $s : (F \to G) \to (t : F \to (s \cdot t) : G)$ (application)
LP2: $t : F \to !t : t : F$ (inspection)
LP3: $s : F \to (s + t) : F, t : F \to (s + t) : F$ (sum)
R2: For any constant c and axiom A infer $\vdash c : A$ (axiom necessitation)

Multi-Agent Epistemic Logic $\mathsf{K4_n}$

E1: $\Box_i(F \to G) \to (\Box_i F \to \Box_i G)$
E2: $\Box_i F \to \Box_i \Box_i F$
R3: $\vdash F \Rightarrow \vdash \Box_i F$

Connection Axiom

C1: $t : F \to \Box_i F$

Just like we did for our single-agent system $\mathsf{K4J4}$, we can restrict the rule R2 to a subset of $\{c : A | c$ a proof constant, A an axiom$\}$. This results in the following definition of $\mathsf{K4_nJ4}_{\mathcal{CS}}$.

Definition 29. Any set $\mathcal{CS} = \{c_1 : A_1, c_2 : A_2, \ldots\}$ where A_i's are axioms and c_i's are proof constants is called a constant specification. When the rule R2 is restricted to a constant specification $\mathcal{CS}$ we write $\mathsf{K4_nJ4}_{\mathcal{CS}}$. When we say R2 is restricted to $\mathcal{CS}$ we mean that R2 is replaced by "$c : A \in \mathcal{CS}$ implies $\vdash c : A$".

Notice, $\mathsf{K4_nJ4}$ enjoys the deduction theorem and closure under substitution as well. The proofs for $\mathsf{K4_nJ4}$ remain almost identical to the $\mathsf{K4J4}$ cases.

Now, that we have all the corresponding definitions of the single-agent for our multi-agent logic, we are going to define how the single-agent AF-models can be transformed into multi-agent AF-models.

3.2 AF-models

In Artemov's paper [5], he showed already that the following definition of a semantics works well for the hybrid logics $\mathsf{T_nLP}, \mathsf{S4_nLP}$ and $\mathsf{S5_nLP}$ (for more detail consult [5]). We will see that AF-models can be adapted to $\mathsf{K4_nJ4}$.

Definition 30. An AF-model for $\mathsf{K4_nJ4}$ consists of a tuple of the form $(W, R_1, \ldots, R_n, R^e, \mathcal{E}, \mathcal{V})$ where

- W is a non-empty set of states (or possible worlds)
- $R_i \subseteq W \times W$ transitive accessibility relations for $i = 1, \ldots, n$
- $R^e \subseteq W \times W$ a transitive evidence accessibility relation which includes $R_1 \cup \ldots \cup R_n$
- $\mathcal{V} : W \to 2^{SV}$ an arbitrary function which maps a world to a set of sentence variables
- $\mathcal{E} : W \times PP \to 2^{Fml}$ a function which has the following properties:

  Monotonicity : $uR^e v$ implies $\mathcal{E}(u, t) \subseteq \mathcal{E}(v, t)$

  Closure Properties :

  - Application: $F \to G \in \mathcal{E}(u, s)$ and $F \in \mathcal{E}(u, t)$ implies $G \in \mathcal{E}(u, s \cdot t)$
  - Inspection: $F \in \mathcal{E}(u, t)$ implies $t : F \in \mathcal{E}(u, !t)$
  - Sum: $\mathcal{E}(u, s) \cup \mathcal{E}(u, t) \subseteq \mathcal{E}(u, s + t)$

- $\Vdash \subseteq W \times Fml$ a binary relation which is defined as follows:
  - $u \nVdash \bot$
  - $u \Vdash S$ iff $S \in \mathcal{V}(u)$
  - $u \Vdash F \vee G$ iff $u \Vdash F$ or $u \Vdash G$
  - $u \Vdash F \wedge G$ iff $u \Vdash F$ and $u \Vdash G$
  - $u \Vdash F \to G$ iff $u \nVdash F$ or $u \Vdash G$
  - $u \Vdash \neg F$ iff $u \nVdash F$
  - $u \Vdash \Box_i F$ iff $v \Vdash F$ for all v such that $uR_i v$
  - $u \Vdash t : F$ iff $F \in \mathcal{E}(u, t)$ and $v \Vdash F$ for all v such that $uR^e v$

Remark 31. The notions of a formula or a set of formulas being valid or valid in an AF-model are the same as before in the single-agent case.

What remains to be defined is what we mean by a multi-agent AF-model meeting a constant specification $\mathcal{CS}$. In fact, it is the same definition as for single-agent AF-models but it is presented once more for completeness reasons.

Definition 32. We say an AF-model $\mathcal{M} = (W, R_1, \ldots, R_n, R^e, \mathcal{E}, \mathcal{V})$ meets $\mathcal{CS}$ iff for all $c : A \in \mathcal{CS}$ and all $u \in W$ it holds that $A \in \mathcal{E}(u, c)$.

The next step, now, is to prove soundness and completeness for $\mathsf{K4_nJ4}_{\mathcal{CS}}$ with respect to multi-agent AF-models.

3.3 Soundness

Theorem 33 (Soundness)**.** *For all formulas F and all AF-models $\mathcal{M} = (W, R_1, \ldots, R_n, R^e, \mathcal{E}, \mathcal{V})$ that meet $\mathcal{CS}$ it holds that if $\mathsf{K4_nJ4}_{\mathcal{CS}} \vdash F$ then $\mathcal{M} \models F$.*

Proof. The proofs for propositional and justification axioms and rules are exactly the same as for the single-agent soundness. Therefore, only the proofs for E1, E2, R3, C1 are given here.

$E1$ *Axiom*: We want to show that $u \Vdash \Box_i(F \to G)$ and $u \Vdash \Box_i F$ implies $u \Vdash \Box_i G$.

By definition of $\Vdash$ we know that $u \Vdash \Box_i(F \to G)$ and $u \Vdash \Box_i F$ hold when for all $v \in W$ such that uR_iv, $v \nVdash F$ or $v \Vdash G$ holds and for all $v \in W$ such that uR_iv, $v \Vdash F$ holds. It follows that for all $v \in W$ such that uR_iv, $v \Vdash G$ holds. This is by definition of $\Vdash$ nothing else than $u \Vdash \Box_i F$.

$E2$ *Axiom*: We want to show that $u \Vdash \Box_i F$ implies $u \Vdash \Box_i \Box_i F$.

From $u \Vdash \Box_i F$ we know by the definition of $\Vdash$ that for all $v \in W$ such that uR_iv, $v \Vdash F$ holds. The transitivity of R_i implies that for all $v, y \in W$ such that uR_ivR_iy it holds that uR_iy. Hence, from the fact that for all $v \in W$ such that uR_iv, $v \Vdash F$ holds, we obtain that for all $v, y \in W$ such that uR_ivR_iy, $y \Vdash F$ holds. By definition of $\Vdash$ we conclude that for all $v \in W$ such that uR_iv, $v \Vdash \Box_i F$ holds. This, on the other hand, is $u \Vdash \Box_i \Box_i F$ by definition of $\Vdash$.

$C1$ *Axiom*: We want to show that $u \Vdash t : F$ implies $u \Vdash \Box_i F$.

Once again by definition of $\Vdash$, we conclude from $u \Vdash t : F$ that $F \in \mathcal{E}(u, t)$ and for all $v \in W$ such that uR^ev, $v \Vdash F$ holds. Since $R_i \subseteq R^e$, it must be that for all $u \in W$ such that uR_iv, $v \Vdash F$ holds. This is, by definition of $\Vdash$, what is required for $u \Vdash \Box_i F$.

$R3$ *Rule*: We want to show that whenever $\vdash \Box_i F$ was derived from $\vdash F$ by R3 then for all $u \in W$, $u \Vdash \Box_i F$ holds.

From $\vdash F$ we get by induction hypothesis that for all $u \in W$, $u \Vdash F$ holds. Certainly, the less general statement that for all $v \in W$ such that uR_iv, $v \Vdash F$ holds, is true. This, however, states that for all $u \in W$, $u \Vdash \Box_i F$ holds.

□

3.4 Completeness

To prove multi-agent completeness we take the same approach as for single-agent completeness. As a result, we need maximal consistent sets once more. Definition 22, 23 and Lemma 24 can easily be translated to their multi-agent version by replacing all occurrences of $\mathsf{K4J4}_{\mathcal{CS}}$ with $\mathsf{K4_nJ4}_{\mathcal{CS}}$.

Theorem 34 (Completeness)**.** *For all $\mathcal{CS}$ and all formulas F, if for all AF-models $\mathcal{M} = (W, R_1, \ldots, R_n, R^e, \mathcal{E}, \mathcal{V})$ that meet $\mathcal{CS}$ it holds that $\mathcal{M} \vDash F$ then $\mathsf{K4_nJ4}_{\mathcal{CS}} \vdash F$.*

Proof. In order to prove the theorem we again use the contraposition.
For any $\mathcal{CS}$ there is an AF-model $\mathcal{M}$ such that

$$\mathsf{K4_nJ4}_{\mathcal{CS}} \nvdash F \text{ implies } \mathcal{M} \nvDash F$$

We define $\mathcal{M} = (W, R_1, \ldots, R_n, R^e, \mathcal{E}, \mathcal{V})$ where W is the collection of all maximal consistent sets. $\mathcal{V}$ is defined through "$S \in \mathcal{V}(\Gamma)$ iff $S \in \Gamma$" for every $\Gamma \in W$. The accessibility relations are defined as $R_i = \{(\Gamma, \Delta) | \Gamma^{\sharp_i} \subseteq \Delta\}$ and $R^e = \{(\Gamma, \Delta) : \Gamma^\flat \subseteq \Delta\}$, where $\Gamma^\flat = \{t : F, F | t : F \in \Gamma\}$ and $\Gamma^{\sharp_i} = \{\Box_i F, F | \Box_i F \in \Gamma\}$. Lastly, we define $\mathcal{E}$ by $F \in \mathcal{E}(\Gamma, t)$ iff $t : F \in \Gamma$.

Now we need to show that $\mathcal{M}$ is in fact an AF-model meeting $\mathcal{CS}$. For this proof we do not need to show the $\mathcal{E}$ monotonicity and closure properties again. The proofs are the same as for single-agent completeness because $\mathcal{E}$ is defined in the same way.

$\mathcal{M}$ *meets* $\mathcal{CS}$**:** Our goal is to prove the following. For all $c : A \in \mathcal{CS}$ it holds that for all $\Gamma \in W$, $A \in \mathcal{E}(\Gamma, c)$. We know that $\mathsf{K4_nJ4}_{\mathcal{CS}} \vdash c : A$ by rule R2. With the multi-agent version of Lemma 24.2 $c : A \in \Gamma$. This implies by definition of $\mathcal{E}$ that $A \in \mathcal{E}(\Gamma, c)$, which is what we wanted to show.

$R_1 \cup \ldots \cup R_n \subseteq R^e$**:** We only need to show that $R_i \subseteq R^e$ for all $i \in \{1, \ldots n\}$. Thus, the goal is to prove that $\Gamma R_i \Delta$ implies $\Gamma R^e \Delta$. So, when we can show that $\Gamma^\flat \subseteq \Delta$, we are done.
Let $F \in \Gamma^\flat$ then $F \equiv t : Y$ or $F \equiv Y$.
First, let us consider $F \equiv Y$ then $t : Y \in \Gamma$ for some t. By $t : Y \rightarrow \Box_i Y$ (C1), multi-agent Lemma 24.2 and 24.3 it holds that $\Box_i Y \in \Gamma$. Now, this implies that $Y \in \Gamma^{\sharp_i}$. From the precondition $\Gamma^{\sharp_i} \subseteq \Delta$ it follows that $Y \in \Delta$, which is want we wanted to show.
The second case $F \equiv t : Y \in \Gamma$. By $t : Y \rightarrow !t : t : Y$ (LP2), multi-agent Lemma 24.2 and 24.3 we conclude that $!t : t : Y \in \Gamma$. From $!t : t : Y \rightarrow \Box_i t : Y$ (C1) multi-agent Lemma 24.2 and 24.3 we get that $\Box_i t : Y \in \Gamma$. The definition of $\Gamma^{\sharp_i}$ forces $t : Y \in \Gamma^{\sharp_i}$. As before, the precondition brings us to the fact that $t : Y \in \Delta$.

$R_1, \ldots, R_n$ *transitive***:** Assume $\Gamma R_i \Delta R_i \Phi$. From the definition of the relations we know $\Gamma^{\sharp_i} \subseteq \Delta$ and $\Delta^{\sharp_i} \subseteq \Phi$. It also follows from the definition of $()^{\sharp_i}$ that $\Gamma^{\sharp_i} \subseteq \Delta^{\sharp_i}$. Hence, $\Gamma^{\sharp_i} \subseteq \Delta^{\sharp_i} \subseteq \Phi$. Since the subset relation is transitive, it holds that $\Gamma R_i \Phi$.

R^e *transitive* Assume $\Gamma R^e \Delta R^e \Phi$. From the definition of the relations we know $\Gamma^\flat \subseteq \Delta$ and $\Delta^\flat \subseteq \Phi$. It also follows from the definition of $()^\flat$ that $\Gamma^\flat \subseteq \Delta^\flat$. Hence, $\Gamma^\flat \subseteq \Delta^\flat \subseteq \Phi$. Since the subset relation is transitive, it holds that $\Gamma R^e \Phi$.

Lemma 26 was crucial for the truth lemma in the single-agent case. With multi-agent completeness, this is no different. Therefore, we have to prove an analogous fact for the multi-agent case as well.

Lemma 35. *For any formula F, any $\Gamma \in W$ and any $i \in \{1, \ldots, n\}$ it holds that $F \in \Gamma^{\sharp_i}$ implies $\Box_i F \in \Gamma$*

Proof. From $F \in \Gamma^{\sharp_i}$ we know that $F \equiv Y$ or $F \equiv \Box_i Y$ for some formula Y. The first case is trivial. Whenever $Y \in \Gamma^{\sharp_i}$ because $\Box_i Y \in \Gamma$, it is trivial that $\Box_i Y \in \Gamma$.
The second case is that $\Box_i Y \in \Gamma^{\sharp_i}$ because $\Box_i Y \in \Gamma$. The formula $\Box_i Y \to \Box_i \Box_i Y$ (E2) brings us to $\Box_i \Box_i Y \in \Gamma$, which is nothing else than $\Box_i F \in \Gamma$. $\Box$

Now, that all preparations for the proof of the truth lemma were made we are ready to prove the truth lemma for the multi-agent case.

Lemma 36. *For all $\Gamma \in W$ and all formulas F it holds that*

$$\Gamma \Vdash F \text{ iff } F \in \Gamma$$

Proof. Induction on F. This is the identical proof as the single-agent version. Therefore, only the cases where $F \equiv \Box_i A$ and $F \equiv t : A$ differ from the previous proof of the truth lemma:

$F \equiv \Box_i A$**:** "$\Rightarrow$" From $\Box_i A \notin \Gamma$ it follows that $\Gamma^{\sharp_i} \cup \{\neg A\}$ is consistent.
Let us assume the contrary holds. Then, there are $Y_1, \ldots, Y_m \in \Gamma^{\sharp_i}$ such that $\mathsf{K4_nJ4}_{\mathcal{CS}} \vdash \neg Y_1 \vee \ldots \vee \neg Y_m \vee \neg\neg A$. With propositional logic we can derive $\mathsf{K4_nJ4}_{\mathcal{CS}} \vdash Y_1 \wedge \ldots \wedge Y_m \to A$. From this we obtain $\mathsf{K4_nJ4}_{\mathcal{CS}} \vdash \Box_i(Y_1 \wedge \ldots \wedge Y_m \to A)$ by rule R3. By normal modal logic reasoning $\mathsf{K4_nJ4}_{\mathcal{CS}} \vdash \Box_i Y_1 \wedge \ldots \wedge \Box_i Y_m \to \Box_i A$ can be deduced. The multi-agent Lemma 24.2 implies that $\Box_i Y_1 \wedge \ldots \wedge \Box_i Y_m \to \Box_i A \in \Gamma$. From Lemma 35 and $Y_1, \ldots, Y_m \in \Gamma^{\sharp_i}$ we know that $\Box_i Y_1, \ldots, \Box_i Y_m \in \Gamma$. The multi-agent version of Lemma 24.5 brings us to $\Box_i Y_1 \wedge \ldots \wedge \Box_i Y_m \in \Gamma$. Finally, $\Box_i Y_1 \wedge \ldots \wedge \Box_i Y_m \to \Box_i A \in \Gamma$, $\Box_i Y_1 \wedge \ldots \wedge \Box_i Y_m \in \Gamma$ and the multi-agent version of Lemma 24.3 implies that $\Box_i A \in \Gamma$, which is clearly a contradiction to our assumption that $\Box_i A \notin \Gamma$.
Now, we know that $\Gamma^{\sharp_i} \cup \{\neg A\}$ is consistent. Hence, we can build a maximal consistent set Δ from $\Gamma^{\sharp_i} \cup \{\neg A\}$. It holds that $A \notin \Delta$ due to

the consistency of Δ. From this, we obtain $\Delta \not\Vdash A$ from the induction hypothesis. Since $\Gamma R_i \Delta$, which is because $\Gamma^{\sharp_i} \subseteq \Delta$, we know $\Gamma \not\Vdash \Box_i A$.

"$\Leftarrow$" By definition of R_i, $\Box_i A \in \Gamma$ implies that for all $\Delta \in W$ such that $\Gamma R_i \Delta$, $A \in \Delta$ holds. From the induction hypothesis it follows that for all $\Delta \in W$ such that $\Gamma R_i \Delta$, $\Delta \Vdash A$ holds. Therefore, $\Gamma \Vdash \Box_i A$.

$F \equiv t : A$: "$\Rightarrow$" From $\Gamma \Vdash t : A$ it follows by definition of $\Vdash$ that $A \in \mathcal{E}(\Gamma, t)$, which is $t : A \in \Gamma$ by definition of $\mathcal{E}$.

"$\Leftarrow$" By definition of R^e it follows from $t : A \in \Gamma$ that for all $\Delta \in W$ such that $\Gamma R^e \Delta$, $A \in \Delta$ holds. The induction hypothesis implies that for all $\Delta \in W$ such that $\Gamma R^e \Delta$, $\Delta \Vdash A$. Furthermore, $A \in \mathcal{E}(\Gamma, t)$ holds since $t : A \in \Gamma$. From these two results we obtain $\Gamma \Vdash t : A$.

$\Box$

The rest of the proof is again the same as before. Assume $\not\vdash F$. This implies that $\{\neg F\}$ is consistent. Build a maximal consistent set Γ from $\{\neg F\}$. $F \notin \Gamma$ due to consistency. The truth lemma implies that $\Gamma \not\Vdash F$ $\Box$

4 Conclusion and Future Work

We have introduced the logic of justified belief $\mathsf{K4J4}$ and AF-models for $\mathsf{K4J4}$ as a semantics. We accomplished the soundness and completeness theorem for $\mathsf{K4J4}$ with respect to the introduced AF-models. Furthermore, we showed that $\mathsf{K4J4}$ enjoys standard properties such as the deduction theorem, closure under substitution and the internalization property. In addition, the single-agent logic of justified belief $\mathsf{K4J4}$ was extended by multiple belief agents. This resulted in the logic $\mathsf{K4_nJ4}$. Moreover, also AF-models were adapted to qualify as an adequate semantics for $\mathsf{K4_nJ4}$. In other words, $\mathsf{K4_nJ4}$ is sound and complete with respect to AF-models, which was also proven. However, there are still many further problems that need to be addressed.

Among the many open questions to be answered about these new logics are whether they are decidable and, if so, what is their exact complexity. Further logics of justified common belief $\mathsf{K4_n^J}$ can be considered similar to $\mathsf{S4_n^J}$ (see [5]). These logics are obtained by forgetfully projecting $t : F$ in $\mathsf{K4_nJ4}$ onto a new (n+1)th modality J. In addition to the questions of decidability and complexity, the axiomatization needs to be found and its exact correspondence to the traditional logics of common belief (see [12]).

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