Optimisation of tensegrity structures

Håvard Fossdal, Henning Hegstad, Eivind Jensløkken and Elias Solberg Ovesen

Abstract

Tensegrity structures consisting of cables, bars and nodes with mass are investigated and modelled. We have investigated the existence of local and global minima in total energy, consisting of gravitational and elastic energy, given a starting configuration. Optimisation techniques are used to find these numerically, primarily the BFGS algorithm with strong Wolfe line search. Various cases are tested, including only cables, with or without mass of the bars, and both cables and bars. Different constraints are also imposed, namely fixed nodes and ground profile. The methods used have been sufficient to converge to minimisers in all cases, and we have shown that for cables only, we get a global minimiser. However, with bars and cables, there are local minimisers, illustrated by stable above-ground structures, which reflects non-convexity.

Introduction

We want to study different tensegrity structures and consider the problem of finding a stable resting form. Tensegrity structures are mechanical structures that consist of elastic bars and cables together with nodes in the joints between them. These structures can for instance be used in architecture, as seen in the world's largest tensegrity hybrid bridge, Kurilpa Bridge in Australia [2]. They have also been researched within the field of robotics, e.g. the space exploration rover "Super Ball Bot" researched by NASA [1]. The structures are particularly interesting as the shape can be altered by adjusting different cable lengths.

The stable resting position of the structure is attained when the total potential energy E is minimal. Thus, we want to consider an optimisation problem to determine the minimum, given some constraints such that the problem physically makes sense. We will formulate different optimisation-based problems and show some mathematical properties of these. Additionally, numerical methods are implemented to solve the problems.

Notation and definitions

The structures are modelled by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of vertices and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges between them. The vertices, or nodes, represent the joints in the structure and can be loaded with a weight $m_i,\ i = 1, \dots, N$. Moreover, the edge between node i and j is denoted as $e_{ij} = (i, j)$ where $i < j$, and the edge is either a cable or a bar. The cables and bars are denoted $\mathcal{C}, \mathcal{B} \subset \mathcal{E}$ respectively. Each node has a given position $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \in \mathbb{R}^3$, and all positions are then stored in $X = (x^{(1)}, \dots, x^{(N)}) \in \mathbb{R}^{3N}$.

Structural Bars

The bars can be compressed or stretched from their resting length $\ell_{ij} > 0$. This leads to elastic energy which a linear elasticity model models as

$$E_{\text{elast}}^{\text{bar}}(e_{ij}) = \frac{c}{2\ell_{ij}^2} \left(\|x^{(i)} - x^{(j)}\| - \ell ij \right)^2,$$

where $c > 0$ is a material parameter. Additionally, the bars have a gravitational force in the x_3 component (z direction) such that the potential energy is given by

$$E_{\text{grav}}^{\text{bar}}(e_{ij}) = \frac{\rho g \ell_{ij}}{2} \left(x_3^{(i)} + x_3^{(j)} \right),$$

where ρ is the line density of the bar and g is the earth's gravitational acceleration.

Structural cables

The cables are quite similar to the bars, but they simply bend by compression. Thus, the elastic energy is zero when bent, and the following model is considered

$$E_{\text{elast}}^{\text{cable}}(e_{ij}) = \begin{cases} \frac{k}{2\ell_{ij}^2} \left(\|x^{(i)} - x^{(j)}\| - \ell_{ij} \right)^2 & \text{if } \|x^{(i)} - x^{(j)}\| > \ell_{ij}, \\ 0 & \text{if } \|x^{(i)} - x^{(j)}\| < \ell_{ij}. \end{cases}$$

We assume that both the cables and the bars are made of the same material and have the same thickness. However, the mass of the cables is considered negligible compared to the bars. Thus, there is no gravitational contribution to the total energy from the cables.

External forces and total energy

Furthermore, a node i can be loaded with a weight m_i, which yields an additional (external) potential energy to the structure

$$E_{\text{ext}}(X) = \sum_{i=1}^{N} m_i g x_3^{(i)}.$$

By adding the different energy contributions, the total energy is given by

$$E(X) = \sum_{e_{ij}\in\mathcal{B}} \left(E_{\text{elast}}^{\text{bar}}(e_{ij}) + E_{\text{grav}}^{\text{bar}}(e_{ij})\right) + \sum_{e_{ij}\in\mathcal{C}} \left(E_{\text{elast}}^{\text{cable}}(e_{ij})\right) + E_{\text{ext}}(X). \tag{1}$$

Constraints

Note that the total energy E is unbounded below as there are gravitational forces present. Thus, (1) is minimised when $x_3^{(i)} \to -\infty$ for all i. To avoid this, additional constraints are imposed. We will consider two different types of constraints. Consider

$$x^{(i)} = p^i, \quad \text{for } i = 1, \dots, M, \tag{2}$$

for some given $p^i \in \mathbb{R}^3$ and $1 \leq M < N$. In other words, the first M nodes are fixed (constant), which effectively allows for a lower dimensional optimisation problem.

The other constraint ensures that the structure is above ground, and is given by the inequality

$$x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}), \quad \text{for all } i = 1, \dots, N, \tag{3}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$, $f \in C^1$, models the height profile of the ground.

Existence of solutions

To continue studying the problem of minimising the total energy given by (1), we formulate some initial results regarding the solvability of the system.
Consider the case where we impose fixed nodes as a constraint to the optimisation problem.

Theorem 1. *The problem of minimising* (1) *with fixed constraints given by* (2) *admits a solution provided that the graph $\mathcal{G}$ is connected.*

Proof. The problem admits a solution if it is both lower semi continuous and coercive.

Lower semi-continuous: A continuous function is by definition lower semi-continuous.
All components of the sum are clearly continuous, and thus the total energy $E(X)$ is also continuous.

Coercivity (inductive argument): Assume that there is one fixed node, and one free node ($k = 1$) with a cable or bar between them.
Consider the case where the free node tends to infinity in any direction. Then, it is clear that the total energy $E(X)$ also tends to infinity, as the (positive) quadratic term in the cable/bar-energy dominates the linear term in the external gravitational energy. Thus, the total energy is coercive.

Furthermore, we assume that the energy is coercive for $k = n$ free nodes. We need to show that the assumption holds for $k = n + 1$. Assume without loss of generality, that each node is connected in a row via only one cable/bar to the previous and next node (we do not lose generality as more cables/bars only provide more energy when stretched/compressed).
If we then add another node to the existing graph, it will effect all the previous nodes in the graph. However, the total energy is still a sum of quadratic terms, and will thus dominate the linear gravitational energy asymptotically. Thus the total energy tends to infinity when at least one node tends to infinity.
Thus, we can conclude that (1) is coercive.

□

Furthermore, consider the case of inequality constraints where the height profile is modeled as in (3).

Theorem 2. *The problem of minimising* (1) *with inequality constraint given by* (3) *admits a solution if $f \in C^1(\mathbb{R}^2)$ is coercive.*

Proof. Clearly, (1) is continuous in the feasible set with this constraint. We will show that the objective function is still coercive under this constraint. We need to consider several cases. First, we note that f is assumed to be continuously differentiable and coercive. This implies that f admits a minimum and does not approach negative infinity anywhere. This means that the mass energy terms of the objective function cannot approach negative infinity.

The first case to consider is where we have no mass. In this case, moving the tensegrity structure in the x_3-direction does not affect the energy. The objective function is not generally coercive without mass, because if the structure is simply translated until the nodes approach infinity, the total energy does not change. So, if there is no mass, we cannot know for certain that the energy tends to infinity when the nodes tend to infinity.

Let us now assume there is mass. We know that if a given configuration X is feasible, if we let any node approach infinity in any direction, due to the coercivity of f, the x_3-coordinate will also tend to infinity for the constraint (3) to hold. This means that the energy will approach infinity when at least one node approaches infinity, assuming this node has mass.
If we instead have the case where we fix one or more nodes at a point above f, and let a mass-less node approach infinity, then the elastic energy necessarily approaches infinity.

Thus, we can conclude that the objective function is coercive if we assume there is some mass in the structure. Since it is also continuous, we can then conclude that (1) admits a global minimum with the constraint given by (3). If there is no mass, it is necessary to fix a node to achieve coerciveness, as in this case the elastic energy will tend to infinity when one or more free nodes tend to infinity. This means that subject to the additional constraint of at least one fixed node, the objective function is coercive. Feasible configurations of nodes with at least one fixed node is a subset of all feasible configurations, and so there exists at least a local minimum in the feasible set. □

In the case where $f(x_1, x_2) = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$, even if you have mass, you can move the structure infinitely in the x_1 and x_2 directions and not increase the energy. This means the objective function is not generally coercive. So in this case, it is necessary to fix one of the nodes for the function to be coercive, as the elastic energy will tend to infinity if one or more nodes tend to infinity (it is only necessary to fix the x_1 and x_2 coordinates if we have mass). So under this additional constraint, the objective function is coercive and admits a minimum. Since there exists a solution in this subset of configurations of nodes, there is at least a local minimum in the set of all configurations.

Cable nets

Theoretical background

We want to consider the simpler case where the tensegrity structure only consists of cables, with constraints given in (2). The problem of minimising the total energy is then given by

$$\min_X E(X) = \sum_{e_{ij} \in \mathcal{E}} \left(E_{\text{elast}}^{\text{cable}}(e_{ij}) \right) + E_{\text{ext}}(X), \quad \text{s.t. } x^{(i)} = p^{(i)},\ i = 1, \ldots, M. \tag{4}$$

Initially, we formulate some properties of the total energy E.

Lemma 1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function and $g : \mathbb{R} \to \mathbb{R}$ be a convex monotonically increasing function, then $h = g \circ f$ is convex.*

Proof. Let $x, y \in \mathbb{R}^d$ and $\lambda \in (0, 1)$. Then

$$\begin{aligned} h(\lambda x + (1-\lambda)y) = g(f(\lambda x + (1-\lambda)y)) &\underset{g \text{ monotone}}{\overset{f \text{ convex}}{\leq}} g(\lambda f(x) + (1-\lambda) f(y)) \\ &\overset{g \text{ convex}}{\leq} \lambda g(f(x)) + (1-\lambda) g(f(y)) = \lambda h(x) + (1-\lambda) h(y). \end{aligned}$$

□

Theorem 3. *$E(X)$ defined in (4) is C^1 and convex.*

Proof. Assume without loss of generality that the cable e_{ij} is stretched, that the corresponding node i has a weight m_i and that $i > M$. Then the i'th partial derivative of $E(X)$ is

$$\begin{aligned}\partial_{x^{(i)}} E(X) &= \partial_{x^{(i)}} \left(\sum_{e_{ij} \in \mathcal{E}} \frac{k}{2\ell_{ij}^2} \left(\|x^{(i)} - x^{(j)}\| - \ell_{ij} \right)^2 + \sum_{i=1}^{N} m_i g x_3^{(i)} \right) \\ &= \sum_{e_{ij} \in \mathcal{E}_i} \frac{k}{\ell_{ij}^2} \left(\|x^{(i)} - x^{(j)}\| - \ell_{ij} \right) \frac{x^{(i)} - x^{(j)}}{\|x^{(i)} - x^{(j)}\|} + m_i g \hat{x}_3 =: \sum_{e_{ij} \in \mathcal{E}_i} \xi_{ij} + m_i g \hat{x}_3,\end{aligned}$$

where we define $\mathcal{E}_i \subseteq \mathcal{E}$ as the set of cables that is connected to the node i, and $\hat{x}_3$ as the unit vector $(0,0,1)^T \in \mathbb{R}^3$. The external force term $m_i g \hat{x}_3$ is a constant and thus continuous, i.e. we need to show that ξ_{ij} is continuous. That is, we need $\xi_{ij} \to 0$ as $\|x^{(i)} - x^{(j)}\| \to \ell_{ij}$, which holds trivially. Thus $E(X) \in C^1$.

For convexity, we know that a (positive) sum of convex functions is convex. The gravitational energy is linear in the x_3 component and thus clearly convex.
For the cable energy, let $f(X) := \|x^{(i)} - x^{(j)}\|$ which is a convex function (not strict) by the triangle inequality of the norm. Furthermore, define

$$g(y) := \begin{cases} C(y - \ell_{ij})^2, & \text{if } y > \ell_{ij} \\ 0, & \text{if } y \leq \ell_{ij}, \end{cases}$$

which is constant zero for $y \leq \ell_{ij}$ and then behaves as y^2 else. Thus, g is clearly convex and monotonically increasing. Thus, by Lemma 1, the composition $h = g \circ f$ is also convex. By defining $C := \frac{k}{2\ell_{ij}^2} > 0$ and $y = \|x^{(i)} - x^{(j)}\|$, we see that $E_{\text{elast}}^{\text{cable}} = h$ is convex. Thus, the total energy $E(X)$ is convex, as it is a sum of convex functions.
Additionally, note that $g(y)$ is strictly convex if $y > \ell_{ij}$. Moreover, a sum of strictly convex functions and convex functions is strictly convex. Thus, if at least one of the cables are stretched, $E(X)$ is strictly convex. □

From the proof of Theorem 3, we directly obtain the definition of the gradient of $E(X)$.

Corollary 3.1. *The i'th component of the gradient of $E(X)$ defined in (4) is*

$$(\nabla E(X))_i = \sum_{e_{ij} \in \mathcal{E}_i} \frac{k}{\ell_{ij}^2} \left(\|x^{(i)} - x^{(j)}\| - \ell_{ij} \right) \frac{x^{(i)} - x^{(j)}}{\|x^{(i)} - x^{(j)}\|} + m_i g \hat{x}_3.$$

Proposition 1. *$E(X)$ is not typically C^2.*

Proof. We consider the second partial derivative

$$\begin{aligned}\partial_{x^{(i)}}^2 E(X) &= \partial_{x^{(i)}} \left(\sum_{e_{ij} \in \mathcal{E}_i} \frac{k}{\ell_{ij}^2} \left(\|x^{(i)} - x^{(j)}\| - \ell_{ij} \right) \frac{x^{(i)} - x^{(j)}}{\|x^{(i)} - x^{(j)}\|} + m_i g \hat{x}_3 \right) \\ &= \frac{k}{\ell_{ij}^2} \left(1 - \ell_{ij} \left(\frac{1}{\|x^{(i)} - x^{(j)}\|} - \frac{1}{\|x^{(i)} - x^{(j)}\|} \right) \right) = \frac{k}{\ell_{ij}^2}.\end{aligned}$$

By taking the limit we see that $\partial_{x^{(i)}}^2 E(X) \neq 0$ as $\|x^{(i)} - x^{(j)}\| \to \ell_{ij}$. Thus, $E(X) \notin C^2$ generally. However, if all cables are compressed, then $E(X) \in C^2$. □

Furthermore, by Theorem 3, we obtain the optimality conditions for our problem.

Corollary 3.2. *The necessary and sufficient optimality conditions for (4) is*

$$\nabla E(X) = 0.$$

Proof. This follows directly from Theorem 3, as the optimisation problem is C^1 and convex. □

Theorem 4. *The optimisation problem (4) has a unique solution if there exists a free node i with mass $m_i > 0$.*

Proof. The optimisation problem has a unique solution if the objective function is strictly convex. Moreover, we know from the proof of Theorem 3 that $E(X)$ is strictly convex if and only if at least one cable is stretched. Now, as the free node has mass, it will minimise its gravitational potential energy. The cable net is connected and has at least one fixed node, which ensures that the node does not go to minus infinity.
The node will decrease its height until it is stopped by a cable, which necessarily has to stretch until the sum of the forces is zero. Hence, the objective function is strictly convex in this case.

□

Numerical results

To solve (4) numerically, we have chosen to implement BFGS as it is a fast and interesting method. To pair with BFGS, we have chosen the strong Wolfe line search which guarantees convergence of the method. For the implementation, we need the gradient of $E(X)$, which is defined in Corollary 3.1. As each component of the gradient consists of points, it can be represented both as a matrix and as a flattened vector. We have used both representations in our code as they can be useful for different situations.

The number of iterations will be dependent on the choice of parameters (and initial configuration, naturally). By some trial and error, we have found that the coefficients in the Wolfe line search $c_1 = 10^{-3}$ and $c_2 = 0.1$ work well. Moreover, we have used a tolerance in the BFGS of 10^{-8}. This is larger than machine accuracy, but a lower tolerance might lead to zero division errors in the creation of the H matrix.
To test the code initially, we have used a configuration with eight nodes, where four of them are fixed. The configuration is as follows

$$k = 3,\ \ell_{ij} = 3 \text{ for all edges } (i,j),\ m_i g = \frac{1}{6} \text{ for } i = 5,6,7,8.$$
$$p^{(1)} = (5,5,0),\ p^{(2)} = (-5,5,0),\ p^{(3)} = (-5,-5,0),\ p^{(4)} = (5,-5,0),$$

where the p-nodes are fixed nodes. This configuration has an analytic solution for the free nodes

$$x^{(5)} = \left(2,2,-\frac{3}{2}\right),\ x^{(6)} = \left(-2,2,-\frac{3}{2}\right),\ x^{(7)} = \left(-2,-2,-\frac{3}{2}\right),\ x^{(8)} = \left(2,-2,-\frac{3}{2}\right).$$

Our numerical scheme converges to this solution in about 20 BFGS iterations, and the solution is shown in figure 1. Note the different scaling on the axis.

Tensegrity-domes

Theoretical background

We want to extend the problem of minimising the cable nets by adding structural bars. The resulting system is sometimes referred to as tensegrity-domes. We still consider the case of fixed constraints and the resulting optimisation problem is

$$\begin{aligned} \min_X E(X) = \sum_{e_{ij}\in\mathcal{B}} \left(E^{\text{bar}}_{\text{elast}}(e_{ij}) + E^{\text{bar}}_{\text{grav}}(e_{ij})\right) + \sum_{e_{ij}\in\mathcal{C}} \left(E^{\text{cable}}_{\text{elast}}(e_{ij})\right) + E_{\text{ext}}(X) \\ \text{s.t. } x^{(i)} = p^{(i)},\ i = 1,\dots,M. \end{aligned} \tag{5}$$

Proposition 2. *$E(X)$ is not differentiable everywhere if $\mathcal{B} \neq \emptyset$.*

Proof. It is known that the norm is piecewise differentiable around zero. We want to show that the norm is not differentiable at zero, which is the point $x^{(i)} = x^{(j)}$. Define $f(x) = f(x_1,x_2,x_3) := \sqrt{x_1^2 + x_2^2 + x_3^2}$ for some $x \in \mathbb{R}^3$, which is the 2-norm. Take for instance the direction $x = (h,0,0)^T$. Then

$$\lim_{h\to 0} \frac{f(0+h,0,0) - f(0,0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = \begin{cases} 1, & \text{if } h \to 0^+ \\ -1, & \text{if } h \to 0^- \end{cases}$$

Thus, as the limit does not exist in the origin, $E(X)$ is not differentiable everywhere.

□

The lack of differentiability does, however, not pose a problem practically. As the non-smooth point is at $\|x^{(i)} - x^{(j)}\| = 0$, this implies that the length of the bar is zero. Compression of the bar to a length of zero does not make physical sense, and thus we can ignore it for our computations.

The necessary optimality condition for (5) is given as

$$\langle \nabla E(X^*), X - X^* \rangle \geq 0 \quad \text{for all X within the constraint,} \tag{6}$$

where X^* is the local solution of the problem.

Theorem 5. *The minimisation problem* (5) *is in general non-convex if* $\mathcal{B} \neq \emptyset$.

Proof. This will be a proof by counterexample.
The two structures in figure 3 have been initialized with three fixed nodes, one free node, and bars between them. The free node has been initialized in two different ways, and the structures converged to two different local minimisers. The left figure has a total energy of around 0.08 and the right one −0.08. This means that there are at least two local minima, with different values. This means that the system cannot be convex, and thus (5) is not convex in general. □

Corollary 5.1. *The optimality condition in* (6) *is not sufficient if* $\mathcal{B} \neq \emptyset$.

Proof. This follows directly from Theorem 5 as the problem is not convex in general. □

Numerical results

As the minimisation problem (5) generally is non-convex, we know that there exist non-unique local minimisers. Therefore, we have to test the numerical scheme with good initialisations. A bad initialisation with a system with mass will most likely lead to a quite boring result where the structure simply collapses.

A tensegrity-dome is shown in figure 2, where we have used the same parameters for the BFGS algorithm as previously. This system is without mass, and thus it is expected to remain above ground. The solution converges in around 100 iterations. Additionally, note that the axis is scaled quite differently in the figure.

Another dome is shown, together with its given initialisation, in figure 4. This solution is quite close to its initial configuration and thus converged in only 18 steps. Moreover, the structure is created with weights on all free nodes $m_i = 0.01,\ i = M+1, \ldots, N$. All the bars have additional external force with $g\rho = 10^{-4}$. This structure is particularly interesting as it remains above ground after initialisation.

Free-standing structures

For the free-standing tensegrity structure, all the nodes are free, and the only constraint to the minimisation of the total energy function (1) is the inequality constraint given in (3). Thus, the optimisation problem becomes

$$\begin{aligned} \min_X E(X) = \sum_{e_{ij} \in \mathcal{B}} \left(E^{\text{bar}}_{\text{elast}}(e_{ij}) + E^{\text{bar}}_{\text{grav}}(e_{ij})\right) + \sum_{e_{ij} \in \mathcal{C}} \left(E^{\text{cable}}_{\text{elast}}(e_{ij})\right) + E_{\text{ext}}(X) \\ x_3^{(i)} \geq f(x_1^{(i)}, x_2^{(i)}), \qquad i = 1, \ldots, N. \end{aligned} \tag{7}$$

To formulate the first-order optimality conditions (KKT conditions) for (7), we define the Lagrangian for this problem. The optimisation problem is subject to the constraint (3), which can be reformulated as

$$C_i(X) := x_3^{(i)} - f(x_1^{(i)}, x_2^{(i)}) \geq 0, \quad i = 1, \ldots, N. \tag{8}$$

Let $\lambda_i \geq 0$ be the Lagrange multiplier associated with the constraint for node i, enforcing that the node must be above the ground. Hence, the Lagrangian $\mathcal{L}(X, \lambda)$ for the optimisation problem by adding the weighted constraints to the objective function is

$$\mathcal{L}(X, \lambda) = E(X) - \sum_{i=1}^{N} \lambda_i \left(x_3^{(i)} - f(x_1^{(i)}, x_2^{(i)})\right).$$

Thus, the first-order optimality conditions for (7), the KKT conditions, can be formulated as

$$\nabla_X E(X) - \sum_{i=1}^{N} \lambda_i \nabla_X (x_3^{(i)} - f(x_1^{(i)}, x_2^{(i)})) = 0 \tag{9.1}$$

$$x_3^{(i)} - f(x_1^{(i)}, x_2^{(i)}) \geq 0, \quad i = 1, \ldots, N \tag{9.2}$$

$$\lambda_i \geq 0, \quad i = 1, \ldots, N \tag{9.3}$$

$$\lambda_i (x_3^{(i)} - f(x_1^{(i)}, x_2^{(i)}) = 0, \quad i = 1, \ldots, N. \tag{9.4}$$

Proposition 3. *The KKT conditions formulated above are necessary for a local solution.*

Proof. The KKT conditions are necessary for (7) if $E(X)$ and $C_i(X)$ are continuously differentiable, and the LICQ condition holds. f is defined to be continuously differentiable, meaning that (8) is obviously continuously differentiable. Further, as discussed previously, $E(X)$ is continuously differentiable in all practical situations. We also need to show that LICQ holds, that is, we need to check if the gradient of the inequality constraint is linearly independent. The gradient of $C_i(X)$ is

$$\nabla_X C_i(X) = \left(-\frac{\partial f}{\partial x_1^{(i)}}, -\frac{\partial f}{\partial x_2^{(i)}}, 1 \right)^T,$$

which is zero everywhere, except in the i'th node. Also, since the third component of the gradient at node i is always equal to 1, this results in the set

$$\{C_i(X), \quad i = 1, 2, \dots, N\}$$

being linearly independent. Thus, LICQ holds, and the KKT conditions are necessary for a local solution. However, the KKT conditions are not sufficient for a local solution, because $E(X)$ is not convex.

□

Numerical results

The optimisation problem with the inequality constraint is different from the case of fixed nodes, and thus we use another numerical method for solving (7). We have chosen the quadratic penalty method as it is easy to implement. Moreover, we use the BFGS method within the algorithm, as we have already implemented it. The BFGS method uses the same parameters as before, and for the quadratic penalty, we use $\mu_0 = 10$ and $\mu_k = 10^k \mu_0$ for $k = 1, \dots, 5$.

We consider the constraint $f(x_1, x_2) = (x_1^2 + x_2^2)/20$ for the free-standing structures. Figure 5 shows two different minima after slightly different initialisations. The constraint is also plotted below to show that the method converges within the domain. The structure has a mass of the bars with $g\rho = 10^{-4}$. The left plot shows that the structure manages to stand upright on top of the constraint, whereas the right one shows a structural collapse after a slightly altered initialisation.

Conclusion

This project has shown various properties of minimising a total energy model for different tensegrity structures with two types of constraints. Most notably, we have proven that there exist solutions under some reasonable assumptions. Moreover, numerical results, using the BFGS algorithm with strong Wolfe line search, have shown convergence to minimisers. The non-convexity of the tensegrity domes and free-standing structures have been reflected in the non-uniqueness of the minimisers. Good initialisations have given interesting (stable) structures. It is clear that optimisation techniques are appropriate for modelling tensegrity structures, and the effectiveness of the BFGS algorithm with strong Wolfe line search has become evident. A similar approach is likely useful for other problems where optimisation methods can be applied.

References

[1] NASA. *Super Ball Bot.* Last accessed 9 March 2024. 2014. URL: `https://www.nasa.gov/image-article/super-ball-bot/`.

[2] Wikipedia. *Kurilpa Bridge.* Last accessed 9 March 2024. 2023. URL: `https://en.wikipedia.org/wiki/Kurilpa_Bridge`.

Appendix

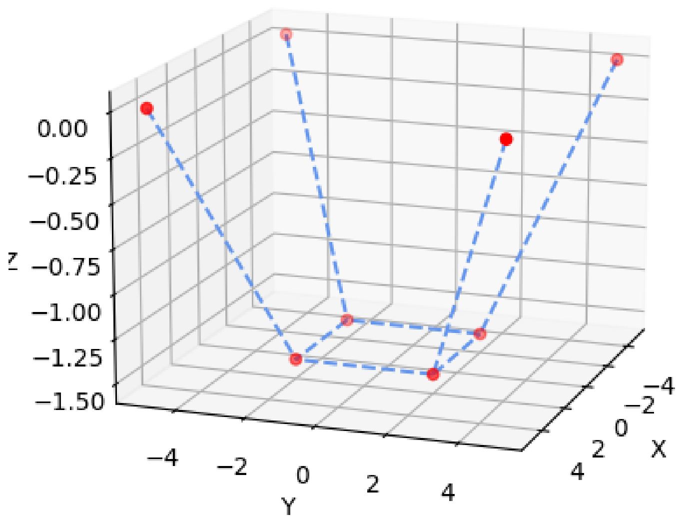

Figure 1: Solution for a cable net with four fixed nodes (with $x_3 = 0$). The free nodes have mass. The solution corresponds to the known analytic solution for the given initialisation.

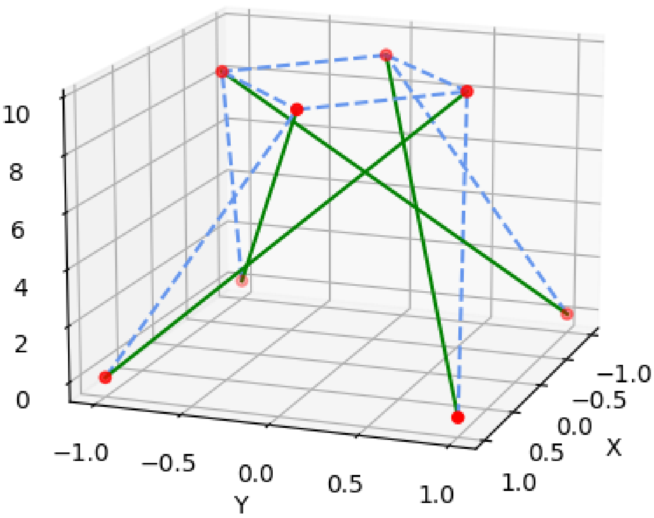

Figure 2: Solution of a tensegrity-dome. Cables are dashed blue lines, bars are green, and four fixed nodes (with $x_3 = 0$).

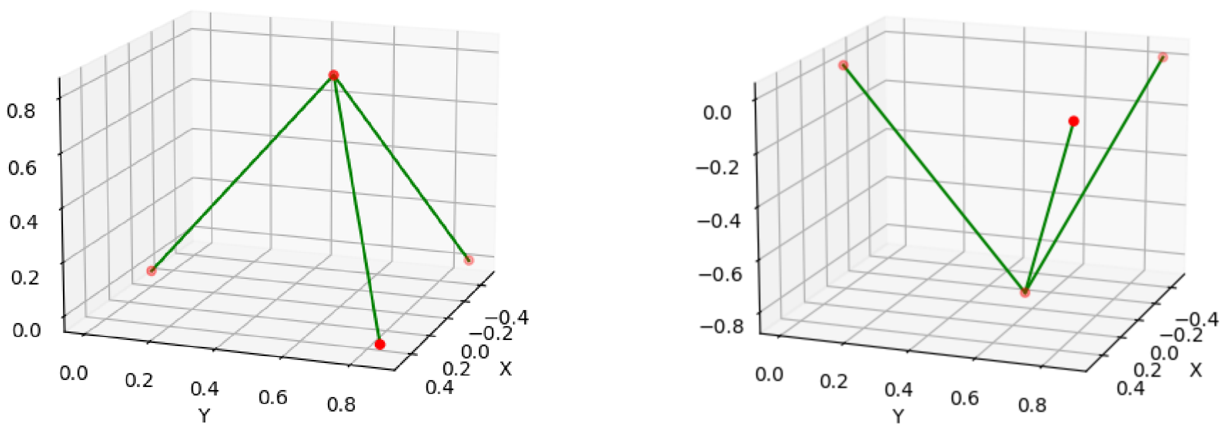

Figure 3: Two different local minimisers for the same (three) fixed nodes and bars.

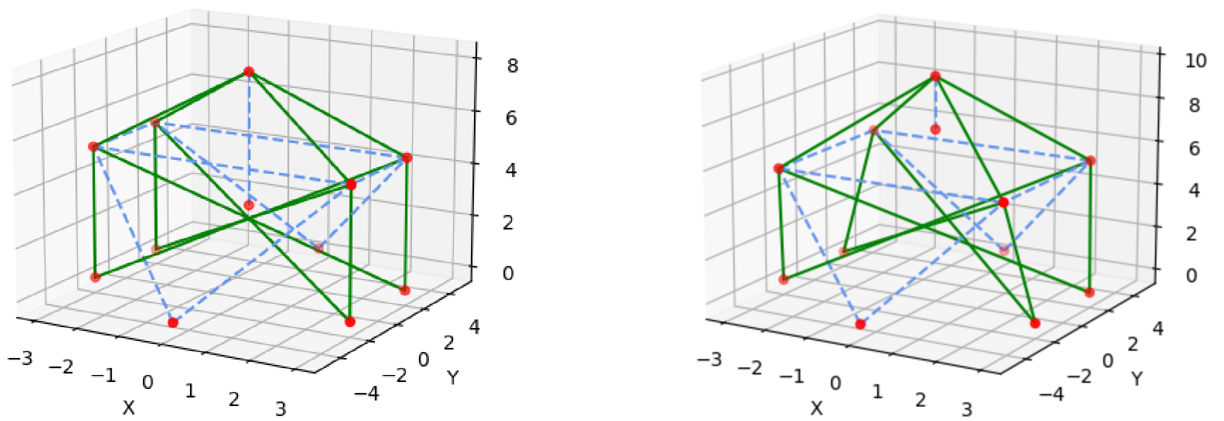

Figure 4: Initial configuration of the tensegrity-dome (left) and corresponding numerical solution. Blue dashed lines are cables, green lines are bars, and six fixed nodes (with $x_3 = 0$).

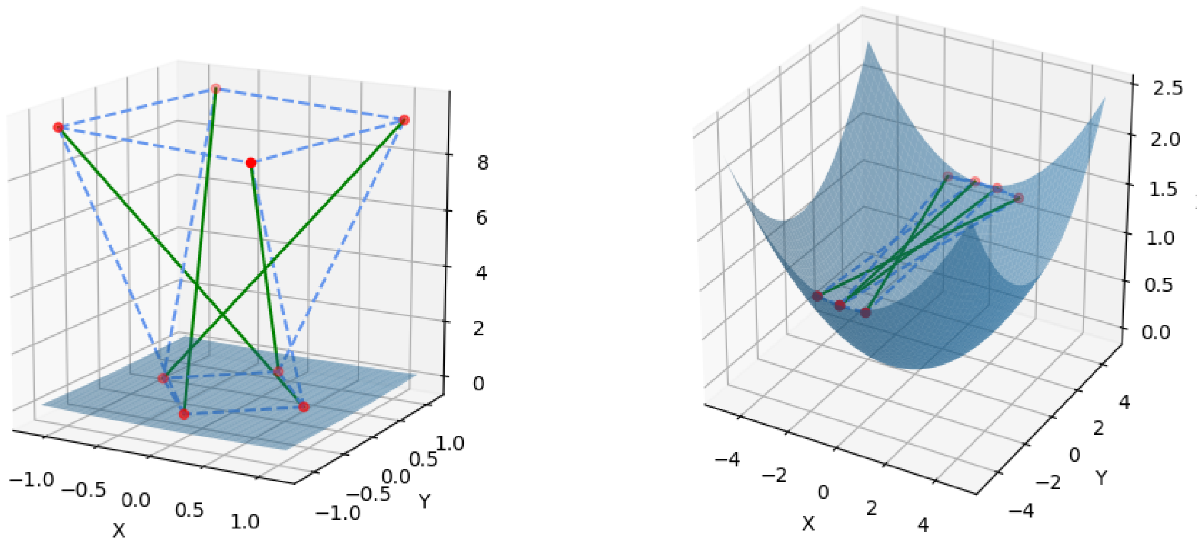

Figure 5: Two slightly different initialisations with the constraint plotted. Cables are dashed blue, bars in green, constraint $f(x_1, x_2) = (x_1^2 + x_2^2)/20$.