1. (Math) Gaussian function is

$G(x, y; \sigma) = \frac{1}{2\pi\sigma^2} exp(-\frac{x^2+y^2}{2\sigma^2})$

The scale-normalized Laplacian of Gaussian (LOG) is

$LoG = \sigma^2 \Delta^2 G$

Please verify that Difference of Gaussian (DOG)

$DoG = G(x, y; k\sigma) - G(x, y; \sigma)$

can be a good approximation of Log.

Prove:

Laplacian Operator is

$\Delta^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Apply Laplacian Operator to the two dimensional gaussian:

$\Delta^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \frac{-2\sigma^2+x^2+y^2}{2\pi\sigma^6} e^{-(x^2+y^2)/2\sigma^2}$

Take the first partial derivative about σ of a two-dimensional gaussian.

$\frac{\partial G}{\partial \sigma} = \frac{-2\sigma^2+x^2+y^2}{2\pi\sigma^5} e^{-(x^2+y^2)/2\sigma^2}$

It is easy to find that

$\frac{\partial G}{\partial \sigma} = \sigma\Delta^2 G$

Since $DoG = G(x, y; k\sigma) - G(x, y; \sigma)$

We can get that $\frac{\partial G}{\partial \sigma} = \lim_{\Delta\sigma \to 0} \frac{G(x,y,\sigma+\Delta\sigma)-G(x,y,\sigma)}{(\sigma+\Delta\sigma)-\sigma} \approx \frac{G(x,y,k\sigma)-G(x,y,\sigma)}{k\sigma-\sigma}$

Therefore,

$\sigma\Delta^2 G = \frac{\partial G}{\partial \sigma} \approx \frac{G(x,y,k\sigma)-G(x,y,\sigma)}{k\sigma-\sigma}$

That is $G(x, y, k\sigma) - G(x, y, \sigma) \approx (k-1)\sigma^2\Delta^2 G$

Thus we can use DoG as an approximation of Log.

2.(Math) In the lecture, we talked about the least square method to solve an over-determined linear system $Ax = b, A \in R^{m*n}, m > n, rank(A) = n$. The closed form solution is $x = (A^T A)^{-1} A^T b$. Try to prove that $A^T A$is non-singular (or in other words, it is invertible).

(1) First, we need to prove that $rank(A^T A) = rank(A)$:

We assume that α is the solution of $Ax = 0$, then $A^T A\alpha = A^T(A\alpha) = 0$

That is to say, the solution of $Ax = 0$ is also the solution of $A^T Ax = 0$;

On the contrary, if α is the solution of $A^T Ax = 0$, that is $A^T A\alpha = 0$

So, $\alpha^T A^T A\alpha = (A\alpha)^T (A\alpha) = 0$, Therefore, $A\alpha = 0$

That is to say, the solution of $A^T Ax = 0$ is also the solution of $Ax = 0$.

In conclusion, the solutions of $A^T Ax = 0$ and $Ax = 0$ are the same.

Therefore, $rank(A) = rank(A^T A) = n$.

(2) From the given information we know that $rank(A) = n$.

Based on what proved avove, $rank(A^T A) = rank(A) = n$

Plus, $A^T A$ is a matrix of $n * n$

Therefore, $A^T A$ is a non-singular matrix. In other words, it is invertible.