Homework 5

Exercise 1: Q7M.6

$$
\begin{aligned}
|\psi_0\rangle &= \begin{bmatrix}\sqrt{4/5}\\ \sqrt{1/5}\end{bmatrix} = \sqrt{4/5}\ |+z\rangle + \sqrt{1/5}\ |-z\rangle \\
|\psi_t\rangle &= \sqrt{4/5}e^{-iEt/\hbar}\ |+z\rangle + \sqrt{1/5}\ |-z\rangle \\
|\langle +x\,|\,\psi_t\rangle|^2 &= \frac{2}{5} + \frac{1}{5}e^{iEt/\hbar} + \frac{1}{5}e^{-iEt/\hbar} + \frac{1}{10} = \frac{1}{2} + \frac{2}{5}cos(Et/\hbar)
\end{aligned}
$$

The probability that we will determine this electron's spin to be in the +x direction at time t is $\frac{1}{2} + \frac{2}{5}cos(Et/\hbar)$

Exercise 2: Q7D.4

$$
\begin{aligned}
|\psi_0\rangle &= \begin{bmatrix}\sqrt{1/2}\\ \sqrt{1/2}\end{bmatrix} = \sqrt{1/2}\ |+z\rangle + \sqrt{1/2}\ |-z\rangle \\
|\psi_t\rangle &= \sqrt{1/2}e^{-iEt/2\hbar}\ |+z\rangle + \sqrt{1/2}e^{+iEt/2\hbar}\ |-z\rangle \\
|\langle +x\,|\,\psi_t\rangle|^2 &= \frac{1}{4}(e^{iEt/\hbar} + 1 + 1 + e^{-iEt/\hbar}) = \frac{1}{2} + \frac{1}{2}cos(Et/\hbar)
\end{aligned}
$$

The probability is exactly same as we found in text example.

Exercise 3: QAM.3

Because $|\psi(0)\rangle = |+x\rangle$, therefore $|\psi(0)\rangle = \sqrt{1/2}\,|+z\rangle + \sqrt{1/2}\,|-z\rangle$
In this situation, $|\psi(t)\rangle = \sqrt{1/2}e^{-iE_+t/\hbar}\,|+z\rangle + \sqrt{1/2}e^{-iE_-t/\hbar}\,|-z\rangle$
For $\langle x\rangle$:

$$
\begin{aligned}
\langle +x \mid \psi(t)\rangle &= \sqrt{1/2}e^{-iE_+t/\hbar}\langle +x \mid +z\rangle + \sqrt{1/2}e^{-iE_-t/\hbar}\langle +x \mid -z\rangle \\
&= \sqrt{1/2}e^{-iE_+t/\hbar}\begin{bmatrix}\sqrt{1/2} & \sqrt{1/2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \sqrt{1/2}e^{-iE_-t/\hbar}\begin{bmatrix}\sqrt{1/2} & \sqrt{1/2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \\
&= \frac{1}{2}e^{-iE_+t/\hbar} + \frac{1}{2}e^{-iE_-t/\hbar}
\end{aligned}
$$

Similarly, $\langle -x \mid \psi(t)\rangle = \frac{1}{2}e^{-iE_+t/\hbar} - \frac{1}{2}e^{-iE_-t/\hbar}$. Therefore, the probability of these out come is:

$$
\begin{aligned}
|\langle -x \mid \psi(t)\rangle|^2 &= (\frac{1}{2}e^{-iE_+t/\hbar} - \frac{1}{2}e^{-iE_-t/\hbar})^*(\frac{1}{2}e^{-iE_+t/\hbar} - \frac{1}{2}e^{-iE_-t/\hbar}) \\
&= (\frac{1}{2}e^{iE_+t/\hbar} - \frac{1}{2}e^{iE_-t/\hbar})(\frac{1}{2}e^{-iE_+t/\hbar} - \frac{1}{2}e^{-iE_-t/\hbar}) \\
&= \frac{1}{2} - \frac{1}{4}(e^{i(E_+-E_-)t/\hbar} + e^{i(E_--E_+)t/\hbar}) \\
&= \frac{1}{2} - \frac{1}{2}cos(\omega t)
\end{aligned}
$$

$$
\begin{aligned}
|\langle +x \mid \psi(t)\rangle|^2 &= (\frac{1}{2}e^{-iE_+t/\hbar} + \frac{1}{2}e^{-iE_-t/\hbar})^*(\frac{1}{2}e^{-iE_+t/\hbar} + \frac{1}{2}e^{-iE_-t/\hbar}) \\
&= \frac{1}{2} + \frac{1}{2}cos(\omega t)
\end{aligned}
$$

The expectation value for S_x is:

$$\begin{aligned}\langle S_x\rangle &= \frac{\hbar}{2}(\frac{1}{2}+\frac{1}{2}cos(\omega t)) - \frac{\hbar}{2}(\frac{1}{2}-\frac{1}{2}cos(\omega t))\\ &= \frac{\hbar}{2}cos(\omega t)\end{aligned}$$

For $\langle y\rangle$:

$$\begin{aligned}\langle +y \mid \psi(t)\rangle &= \sqrt{1/2}e^{-iE_+t/\hbar}\langle +y \mid +z\rangle + \sqrt{1/2}e^{-iE_-t/\hbar}\langle +y \mid -z\rangle\\ &= \sqrt{1/2}e^{-iE_+t/\hbar}\begin{bmatrix}\sqrt{1/2} & -i\sqrt{1/2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \sqrt{1/2}e^{-iE_-t/\hbar}\begin{bmatrix}\sqrt{1/2} & -i\sqrt{1/2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ &= \frac{1}{2}e^{-iE_+t/\hbar} - i\frac{1}{2}e^{-iE_-t/\hbar}\end{aligned}$$

Similarly, $\langle -y \mid \psi(t)\rangle = \frac{1}{2}e^{-iE_+t/\hbar} + \frac{1}{2}e^{-iE_-t/\hbar}$. Therefore, the probability of these out come is:

$$\begin{aligned}|\langle -y \mid \psi(t)\rangle|^2 &= (\frac{1}{2}e^{-iE_+t/\hbar} + i\frac{1}{2}e^{-iE_-t/\hbar})^*(\frac{1}{2}e^{-iE_+t/\hbar} + i\frac{1}{2}e^{-iE_-t/\hbar})\\ &= (\frac{1}{2}e^{iE_+t/\hbar} - i\frac{1}{2}e^{iE_-t/\hbar})(\frac{1}{2}e^{-iE_+t/\hbar} + i\frac{1}{2}e^{-iE_-t/\hbar})\\ &= \frac{1}{2} - \frac{i}{4}(e^{i(E_+-E_-)t/\hbar} - e^{i(E_--E_+)t/\hbar})\\ &= \frac{1}{2} - \frac{1}{2}sin(\omega t)\end{aligned}$$

$$\begin{aligned}|\langle +y \mid \psi(t)\rangle|^2 &= (\frac{1}{2}e^{-iE_+t/\hbar} + \frac{1}{2}e^{-iE_-t/\hbar})^*(\frac{1}{2}e^{-iE_+t/\hbar} + \frac{1}{2}e^{-iE_-t/\hbar})\\ &= \frac{1}{2} + \frac{1}{2}sin(\omega t)\end{aligned}$$

The expectation value for S_y is:

$$\begin{aligned}\langle S_y\rangle &= \frac{\hbar}{2}(\frac{1}{2}+\frac{1}{2}sin(\omega t)) - \frac{\hbar}{2}(\frac{1}{2}-\frac{1}{2}sin(\omega t))\\ &= \frac{\hbar}{2}sin(\omega t)\end{aligned}$$

for $\langle S_z\rangle$:

$$\begin{aligned}\langle +z \mid \psi(t)\rangle &= \sqrt{1/2}e^{-iE_+t/\hbar}\langle +z \mid +z\rangle + \sqrt{1/2}e^{-iE_-t/\hbar}\langle +z \mid -z\rangle\\ &= \sqrt{1/2}e^{-iE_+t/\hbar}\\ \langle -z \mid \psi(t)\rangle &= \sqrt{1/2}e^{-iE_+t/\hbar}\langle -z \mid +z\rangle + \sqrt{1/2}e^{-iE_-t/\hbar}\langle -z \mid -z\rangle\\ &= \sqrt{1/2}e^{-iE_-t/\hbar}\end{aligned}$$

Therefore, we can get the expectation value of S_z:

$$\begin{aligned}\langle S_z\rangle &= \frac{\hbar}{2}|\langle +z \mid \psi\rangle|^2 - \frac{\hbar}{2}|\langle -z \mid \psi\rangle|^2\\ &= \frac{\hbar}{2}\end{aligned}$$

According to the calculation, we know that the $\langle S_z\rangle$ is a constant and $\langle S_x\rangle$ and $\langle S_y\rangle$ are functions depend on time. We can easily find that S is rotating about z axis on the xy plane. This motion is what we expect in classic model. So this result make sense.

Exercise 4: QAM.6

Because $|\psi(0)\rangle = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$, therefore $|\psi(0)\rangle = \frac{4}{5}|+z\rangle + -\frac{3}{5}|-z\rangle$

In this situation, $|\psi(t)\rangle = \frac{4}{5}e^{-iE_+t/\hbar}|+z\rangle - \frac{3}{5}e^{-iE_-t/\hbar}|-z\rangle$

For $\langle x\rangle$:

$$\begin{aligned}
\langle +x \mid \psi(t)\rangle &= 4/5e^{-iE_+t/\hbar}\langle +x \mid +z\rangle - 3/5e^{-iE_-t/\hbar}\langle +x \mid -z\rangle \\
&= 4/5e^{-iE_+t/\hbar}\begin{bmatrix}\sqrt{1/2} & \sqrt{1/2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} - 3/5e^{-iE_-t/\hbar}\begin{bmatrix}\sqrt{1/2} & \sqrt{1/2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \\
&= \frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}
\end{aligned}$$

Similarly, $\langle -x \mid \psi(t)\rangle = \frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}$. Therefore, the probability of these out come is:

$$\begin{aligned}
|\langle +x \mid \psi(t)\rangle|^2 &= (\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar})^*(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}) \\
&= (\frac{4}{5\sqrt{2}}e^{iE_+t/\hbar} + \frac{3}{5\sqrt{2}}e^{iE_-t/\hbar})(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}) \\
&= \frac{1}{50}(16 + 12e^{i(E_+-E_-)t/\hbar} + 12e^{i(E_--E_+)t/\hbar} + 9) \\
&= \frac{1}{2} - \frac{12}{25}cos(\omega t)
\end{aligned}$$

$$\begin{aligned}
|\langle -x \mid \psi(t)\rangle|^2 &= (\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar})^*(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}e^{-iE_-t/\hbar}) \\
&= \frac{1}{2} + \frac{12}{25}cos(\omega t)
\end{aligned}$$

The expectation value for S_x is:

$$\begin{aligned}
\langle S_x\rangle &= \frac{\hbar}{2}(\frac{1}{2} - \frac{12}{25}cos(\omega t)) - \frac{\hbar}{2}(\frac{1}{2} + \frac{12}{25}cos(\omega t)) \\
&= -\frac{12}{25}\hbar cos(\omega t)
\end{aligned}$$

For $\langle y\rangle$:

$$\begin{aligned}
\langle +y \mid \psi(t)\rangle &= 4/5e^{-iE_+t/\hbar}\langle +y \mid +z\rangle - 3/5e^{-iE_-t/\hbar}\langle +y \mid -z\rangle \\
&= 4/5e^{-iE_+t/\hbar}\begin{bmatrix}\sqrt{1/2} & -i\sqrt{1/2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} - 3/5e^{-iE_-t/\hbar}\begin{bmatrix}\sqrt{1/2} & -i\sqrt{1/2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \\
&= \frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}ie^{-iE_-t/\hbar}
\end{aligned}$$

Similarly, $\langle -y \mid \psi(t)\rangle = \frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}ie^{-iE_-t/\hbar}$. Therefore, the probability of these out come is:

$$\begin{aligned}
|\langle -y \mid \psi(t)\rangle|^2 &= (\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}ie^{-iE_-t/\hbar})^*(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}ie^{-iE_-t/\hbar}) \\
&= (\frac{4}{5\sqrt{2}}e^{iE_+t/\hbar} + \frac{3}{5\sqrt{2}}ie^{-iE_-t/\hbar})(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} - \frac{3}{5\sqrt{2}}ie^{-iE_-t/\hbar}) \\
&= \frac{1}{2} + \frac{12}{25}i(e^{i(E_+-E_-)t/\hbar} - e^{i(E_--E_+)t/\hbar}) \\
&= \frac{1}{2} + \frac{12}{25}sin(\omega t)
\end{aligned}$$

$$\begin{aligned}
|\langle +y \mid \psi(t)\rangle|^2 &= (\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}ie^{-iE_-t/\hbar})^*(\frac{4}{5\sqrt{2}}e^{-iE_+t/\hbar} + \frac{3}{5\sqrt{2}}ie^{-iE_-t/\hbar}) \\
&= \frac{1}{2} - \frac{12}{25}sin(\omega t)
\end{aligned}$$

The expectation value for S_y is:

$$\begin{aligned}
\langle S_y\rangle &= \frac{\hbar}{2}(\frac{1}{2} - \frac{12}{25}sin(\omega t)) - \frac{\hbar}{2}(\frac{1}{2} - \frac{12}{25}sin(\omega t)) \\
&= -\frac{12}{25}\hbar sin(\omega t)
\end{aligned}$$

for $\langle S_z\rangle$:

$$\begin{aligned}
\langle +z \mid \psi(t)\rangle &= 4/5e^{-iE_+t/\hbar}\langle +z \mid +z\rangle - 3/5e^{-iE_-t/\hbar}\langle +z \mid -z\rangle \\
&= 4/5e^{-iE_+t/\hbar} \\
\langle -z \mid \psi(t)\rangle &= 4/5e^{-iE_+t/\hbar}\langle -z \mid +z\rangle - 3/5e^{-iE_-t/\hbar}\langle -z \mid -z\rangle \\
&= -3/5e^{-iE_-t/\hbar}
\end{aligned}$$

Therefore, we can get the expectation value of S_z:

$$\begin{aligned}
\langle S_z\rangle &= \frac{\hbar}{2}|\langle +z \mid \psi\rangle|^2 - \frac{\hbar}{2}|\langle -z \mid \psi\rangle|^2 \\
&= \frac{7}{50}\hbar
\end{aligned}$$

According to the calculation, we know that the $\langle S_z\rangle$ is a constant and $\langle S_x\rangle$ and $\langle S_y\rangle$ are functions depend on time. We can easily find that S is rotating about z axis on the xy plane. This motion is what we expect in classic model. So this result make sense.

Exercise 5: QAD.5

According to the time evolution rule, we can assume that the energy eigenvector $|\psi_n\rangle = c_n e^{-iE_nt/\hbar}|E_n\rangle$, and we have a observable $|k\rangle$.

so the probability is :

$$\begin{aligned}
&|\langle k \mid \psi_n\rangle|^2 \\
=&|\langle k \mid E_n\rangle e^{-iE_nt/\hbar}|^2 \\
=&(\langle k \mid E_n\rangle^* c_n e^{iE_nt/\hbar})(\langle k \mid eE_n\rangle c_n e^{-iE_nt/\hbar}) \\
=&c_n^2(\langle k \mid E_n\rangle^*)(\langle k \mid E_n\rangle)
\end{aligned}$$

Because the probability does not depend on time, a energy eigenvector is a stationary state.