Unitārās telpas

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$$\begin{aligned} &R && C \\ &x_e^{\intercal} \cdot F_e \cdot y_e && f(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \cdot \overline{x}_i \cdot y_j = x_e^* \cdot F_e \cdot y_e \\ & && A^* = \overline{A^{\intercal}} \end{aligned}$$

$$\begin{aligned} \text{Simetriska matrica}: a_{ij} &= a_{ji} & \text{Ermita(Hermit) matrica}: a_{ij} &= \overline{a}_{ji} \\ A^{\intercal} &= A & A^* &= A \\ & & F(x) &= f(x,x) \in R \\ & & (x,y) = \overline{x}_1 \cdot y_1 + \overline{x}_2 \cdot y_2 + \ldots + \overline{x}_n \cdot y_n &= x^* \cdot y \end{aligned}$$

$$\begin{aligned} &\text{Ortogonāls operators} && \text{Unitārs operators}: \\ &A^{\intercal} = A^{-1} && (x,y) = (A(x), A(y)) \end{aligned}$$

$$\begin{aligned} &\text{Simetrisks operators} && \text{Ermita operators} \\ &(A(x), y) = (x, A(y)) && (A(x), y) = (x, A(y)) \end{aligned}$$

Ermita operatoram visas īpašvērtības ir reālas, eksistē ortonormēta īpašvektoru bāze.

C
$A : A^* = A$
$\forall_{x,y}((Ax, y) = (x, Ay))$
λ - A īpašvertība, x - īpasvektors

$A \cdot x = \lambda \cdot x$
$\overline{A \cdot x} = \overline{\lambda \cdot x}$
$\overline{A} \cdot \overline{x} = \overline{\lambda} \cdot \overline{x}$
$(\overline{A} \cdot \overline{x})^\intercal = (\overline{\lambda} \cdot \overline{x})^\intercal$
$x^* \cdot A^* = \overline{\lambda} \cdot x^*$

$x^* \cdot A \cdot x = x^* \cdot (A \cdot x) = x^* \cdot (\lambda \cdot x) = \lambda \cdot x^* \cdot x = \lambda \cdot |x|^2$

$\|$
$x^* \cdot A \cdot x = (x^* \cdot A^*) \cdot x = (\overline{\lambda} \cdot x^*) \cdot x = \overline{\lambda} \cdot x^* \cdot x = \overline{\lambda} \cdot |x|^2$
$\lambda = \overline{\lambda}$, tātad $\lambda \in R$

$$A \rightarrow \begin{pmatrix} 1 & & & \text{\Large 0} \\ & 1 & & \\ & & \ddots & \\ & \text{\Large 0} & & 1 \end{pmatrix} \tag{1}$$
$$D = P_{s'f}^{-1} \cdot A_{s's} \cdot P_{se} \tag{2}$$
$$A_{s's} = P_{s'f} \cdot D \cdot P_{se}^{-1} \tag{3}$$

Singulārvērtību izjaukums (dekompozīcija)
SVD

Singular value decomposition

Dots: lin. operators $A : L_1 \to L_2, \;\; L_1, L_2$ - Eiklīda vai unitāras

Teorēma (SVD): katrai reālai (kompleksai) mxn matricai A eksistē tādas ortogonālas (unitāras) matricas U un V un tāda diagonālmatrica $\sum$, ka

$A = U \cdot \Sigma \cdot V^*$
m x n = m x m * m x n * n x n

$$\Sigma = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ & 0 & & \\ & & & \sigma_m \end{pmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$

Pozitīvās vērtības σ_i sauc par operatora A singulārvērtībām, matricas U i-to stabiņu sauc par kreiso singulārvektoru, kas atbilst σ_i, matricas V i-to stabiņu sauc par labējo singulārvektoru, kas atbilst σ_i

Pier.

$A^* \cdot A$ – simetriska (Ermita)

$(A^* \cdot A)^* = A^* \cdot A^{**} = A^* \cdot A$

$A^* \cdot A$ – nenegatīvi noteikta

$x^* \cdot A^* \cdot A \cdot x = (A \cdot x)^* \cdot (A \cdot x) = (Ax, Ax) = |A \cdot x|^2 \geq 0$

$A^*A = \underset{n\times n}{V} \cdot D \cdot \underset{n\times n}{V^*} = V \cdot \Sigma \cdot \Sigma^* \cdot V^*$

$$\underset{n\times n}{D} = \begin{pmatrix} d_1 & & \text{\Huge 0} \\ & \ddots & \\ \text{\Huge 0} & & d_n \end{pmatrix}$$

$d_i \in R$

$d_1 \geq 0$

$d_1 \geq \ldots \geq d_n$

$\sigma_i = \sqrt{d_i}$

$$D = S^2 = S^\intercal \cdot S = S^* \cdot S = \Sigma^* \cdot \Sigma = \Sigma \cdot \Sigma^*,$$

$$S = \begin{pmatrix} \sigma_1 & & \text{\Huge 0} \\ & \ddots & \\ \text{\Huge 0} & & \sigma_n \end{pmatrix}$$

1. Izrēķina $A^* \cdot A$
2. Atrod $A^* \cdot A$ īpašvērtības, dabū D, Σ
3. Atrod atbilstošas īpašvektorus (ortonormētu sistēmu), dabū V (stabiņi- īpašvektori)

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$A^\intercal = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{pmatrix}$$

$$\begin{cases} (u_1, x) = 0 \\ \dots\dots\dots \\ (u_r, x) = 0 \end{cases}$$

$$A^\intercal \cdot A = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix} \tag{4}$$

$$\chi_{A^\intercal \cdot A}(\lambda) = \lambda^2 - 18\lambda \tag{5}$$

$$\lambda_1 = 18, \lambda_2 = 0 \tag{6}$$

$$D = \begin{pmatrix} 18 & 0 \\ 0 & 0 \end{pmatrix} \tag{7}$$

FAS + Grama - Šmita process

$$\Sigma = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 1\sqrt{2} & 1\sqrt{2} \\ -1\sqrt{2} & 1\sqrt{2} \end{pmatrix}$$

$$\lambda_1 = 18 \ldots v_1 = \begin{pmatrix} 1\sqrt{2} \\ -1\sqrt{2} \end{pmatrix}$$

$$\lambda_2 = 0 \ldots v_2 = \begin{pmatrix} 1\sqrt{2} \\ 1\sqrt{2} \end{pmatrix}$$

$$u_1 = \frac{1}{3\sqrt{2}} \cdot \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1\sqrt{2} \\ -1\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

$$u_i = \frac{1}{\sigma_i} \cdot A \cdot v_i \tag{8}$$

$$\text{līdz pēdējam nenulles} \quad \sigma_i \tag{9}$$

$$\text{dabūjam} \quad u_1, u_2, \ldots, u_r \tag{10}$$

$$u_{r+1}, \ldots, u_m - \text{papildinām līdz ortonormētai bāzei} \tag{11}$$

$$U = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} 1/3 & -2/3 & 2/3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 \end{pmatrix} \tag{13}$$

$$\tag{14}$$

$$FAS : (2; 1; 0) \tag{15}$$

$$(-2; 0; 1) \tag{16}$$

$$(-2; 0; 1) - \frac{(-2) \cdot 2 + 0 \cdot 1 + 1 \cdot 0}{2^2 + 1^2 + 0} \cdot (2; 1; 0) = \tag{17}$$

$$= (-2; 0; 1) + \frac{4}{5} \cdot (2; 1; 0) = \tag{18}$$

$$= (-2/5; 4/5; 1) \sim (-2; 4; 5) \tag{19}$$