# CS130 - Transformations

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Identify what each of the following does to a point in homogeneous coordinates. You may choose from:

- uniform scale by $a$ (identify $a$)
- non-uniform scale by $a, b, c$ (identify $a, b, c$)
- translation by $a, b, c$ (identify $a, b, c$)
- rotation by angle $\theta$ about axis $a, b, c$ (identify $\theta, a, b, c$)
- reflections (identify the direction about the reflection is occurring)
- a sequence of the above (specify the operations in the order they are applied)

If the transformation cannot be obtained by applying a sequence of the above, explain why.

**1.** $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

This is a scaling of 4 on the x axis, 3 on the y axis, and 2 on the z axis.

**2.** $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

This reduces everything to zero vector. A scaling of sorts.

**3.** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

This is the identity matrix, nothing will change. A scaling of 1.

**4.** $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Reflect on the yz-plane.

**5.** $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Switch the x and y components of a coordinate. This can also be a rotation matrix around the z axis.

**6.** $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Another rotation matrix around the z axis.

**7.** $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Translate by 1 on the x axis

**8.** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

This will increase the "W" component of a vector, by x. But this doesn't do much.

**9.** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

This doubles the "W" component of a vector, by 2. A scaling operation, but doesn't do much to the vector/point itself.

**10.** $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

This translate the vector component x by 1, and the "W" component of the vector doubles.

**11.** $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

This places the x component in the z component, the y into the x, and the z into the y. This is not possible with the standard operations.

**12.** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

This makes the "W" component zero. This will cause this vector to no longer translate.

**13.** $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Let $\mathbf{v} = \mathbf{R}\mathbf{u}$ be the rotated version of $\mathbf{u}$. Use the dot product $\mathbf{v} \cdot \mathbf{u}$ to show that the angle between $\mathbf{v}$ and $\mathbf{u}$ is $\theta$.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

We know that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, so we need to show that $\cos^{-1}(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}) = \theta$. Let's first solve the fraction:

$$\frac{x^2\cos\theta - xy\sin\theta + xy\sin\theta + y^2\cos\theta}{\sqrt{x^2+y^2} \cdot \sqrt{x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin\theta + x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta}}$$

$$= \frac{(x^2+y^2)\cos\theta}{\sqrt{x^2+y^2} \cdot \sqrt{x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta)}}$$

$$= \frac{(x^2+y^2)\cos\theta}{x^2+y^2}$$

Inputting this into our original equation we get:

$$\cos^{-1}(\cos\theta) = \theta$$

Which is true

**14.** Show that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ for the $2 \times 2$ matrix in the previous problem.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**15.** Let $\mathbf{R}$ be a $3 \times 3$ matrix with columns $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Show that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ is equivalent to $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ being unit vectors and mutually orthogonal.

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \cdot \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 1$$

There we can see that each element of the resulting identity matrix represents the dot product between the two vectors. So $e_{11} = 1$ means that u dotted with u equals 1, our previous equation. Following this pattern further, we see that the only times that the elements are equal to 1 they are dotted with themselves. Otherwise they are zero showing that they are mutually orthogonal.

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

It can also be concluded that $\mathbf{u} \cdot \mathbf{u} = |u|^2 = 1$ therefore these are all unit vectors.

**16.** Let $\mathbf{R}$ be a matrix with $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and let $\mathbf{y} = \mathbf{R}\mathbf{x}$. Show that $\mathbf{x}$ and $\mathbf{y}$ must have the same length.

**17.** Let $\mathbf{R}$ be a matrix with $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and let $\mathbf{y} = \mathbf{R}\mathbf{x}$ and $\mathbf{v} = \mathbf{R}\mathbf{u}$. Show that $\mathbf{u} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{v}$ so that the dot product between vectors is preserved under rotation.

$$\mathbf{x} \cdot \mathbf{u} = R\mathbf{x} \cdot R\mathbf{u} = R(\mathbf{x} \cdot \mathbf{u})$$

Since the transpose of R is also it's inverse, it has a property such that it does not affect the scale of the vectors multiplied with it. Therefore $\mathbf{u} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{v}$ is valid.

**18.** Given the results of the previous two problems, explain why the angle between two vectors must also be preserved under rotation.

If the dot product does not affect scale, and therefore are equivalent, then it is also safe to assume that $\mathbf{u} \cdot \mathbf{x} = |\mathbf{x}||\mathbf{u}| \cos\theta_1 = |\mathbf{y}||\mathbf{v}| \cos\theta_2 = \mathbf{y} \cdot \mathbf{v}$. Since by 16 we determined that x and y have the same magnitude, we can conclude that $|\mathbf{x}||\mathbf{u}| \cos\theta_1 = |\mathbf{y}||\mathbf{v}| \cos\theta_2$ simplifies to $\cos\theta_1 = \cos\theta_2$. That must mean that the angle is conserved with the matrices $R$ and $R^T$