CS130 - Barycentric coordinates

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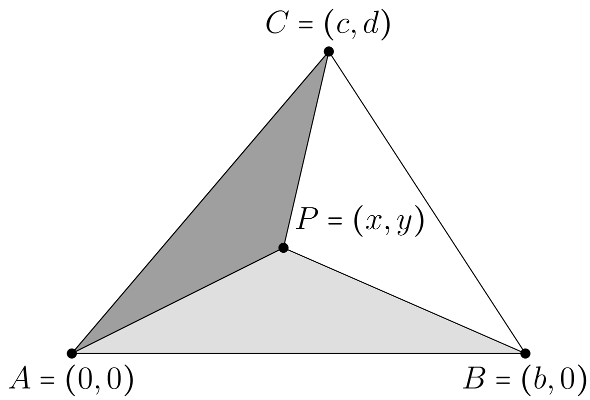
1. In class, we formulated the barycentric coordinates through ratios of triangle areas. We then implicitly assumed that $P = \alpha A + \beta B + \gamma C$. That is, the barycentric coordinates have the property that they interpolate the vertices of the triangle to the point P. In this problem, you will prove this property.

For the triangle ABC illustrated, show that when the barycentric coordinates are determined through ratios of triangle areas, they interpolate the vertices to the point P. That is, show that $P = \alpha A + \beta B + \gamma C$.



α can be considered the distance between the edge of the triangle opposite point A and point B. However that distance is greater than 1. So we must divide it by the total distance between that edge and point A. This would look like:

$$\alpha = \frac{|P - P_{ea}|}{|A - A_{ea}|}$$

However calculating P_{ea} and A_{ea} (the points on the edge opposite A) would be rather difficult, so we must look at the area of a triangle formula. If we treat the two triangles ABC and APC, and compare their formulas:

$$area(\triangle ABC) = \frac{1}{2} b_{BC} h_a$$

$$area(\triangle PBC) = \frac{1}{2}b_{BC}h_p$$

However, we know that $h_a = |A - A_{ea}|$ and $h_p = |P - P_{ea}|$, therefore we could rewrite these as:

$$area(\triangle ABC) = \frac{1}{2}b_{BC} \cdot |A - A_{ea}|$$

$$area(\triangle PBC) = \frac{1}{2}b_{BC} \cdot |P - P_{ea}|$$

And we could even compare them in the same manner as the original method:

$$\alpha = \frac{\frac{1}{2}b_{BC} \cdot |P - P_{ea}|}{\frac{1}{2}b_{BC} \cdot |A - A_{ea}|}$$

Or more simplified:

$$\alpha = \frac{area(\triangle PBC)}{area(\triangle ABC)}$$

And this logic can be applied to β and γ as well.

2. The transformation $\mathbf{x} \to \mathbf{M}\mathbf{x} + \mathbf{b}$ is called an *affine* transformation, where $\mathbf{M}$ is a matrix and $\mathbf{b}$ is a vector. Let P be a point inside triangle ABC. The transformed point $P' = \mathbf{M}P + \mathbf{b}$ is inside the triangle with vertices $A' = \mathbf{M}A + \mathbf{b}$, $B' = \mathbf{M}B + \mathbf{b}$, and $C' = \mathbf{M}C + \mathbf{b}$. Show that P' has the same barycentric coordinates (in $A'B'C'$) as P (in ABC). That is, barycentric coordinates are preserved under affine transformations.

Since the formula for a single coordinate is the same for all coordinates, let us take α as an example:

$$\alpha = \frac{area(\triangle P'B'C')}{area(\triangle A'B'C')}$$

Using our area formula:

$$\alpha = \frac{\frac{1}{2}b_{B'C'}h_{P'}}{\frac{1}{2}b_{B'C'}h_{A'}}$$

$$\alpha = \frac{h_{P'}}{h_{A'}}$$

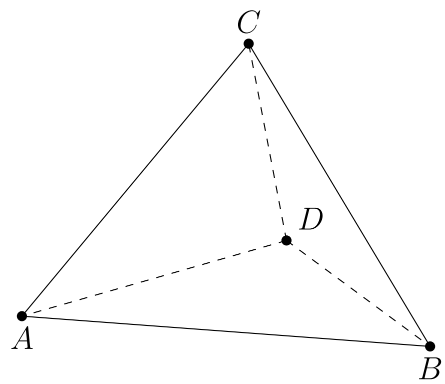
This is the same thing as before. We know that the triangle and the point are simply shifted around by $\mathbf{b}$ and scaled by $\mathbf{M}$, therefore the ratio between the heights will remain the same. That is clear to see from:

$$h_{P'} = |P' - P'_{ea}| = |\mathbf{M}P + \mathbf{b} - (\mathbf{M}P_{ea} + \mathbf{b})| = |\mathbf{M}(P - P_{ea}) + 2\mathbf{b}|$$

$$h_{A'} = |A' - A'_{ea}| = |\mathbf{M}A + \mathbf{b} - (\mathbf{M}A_{ea} + \mathbf{b})| = |\mathbf{M}(A - A_{ea}) + 2\mathbf{b}|$$

The ratio between these two will be the same as the original ratio.

3.



How might one formulate barycentric coordinates for a tetrahedron? Suggest formulas for computing them.

If the volume of a tetrahedron formula involves use of a height component, one can possibly use the volume in the same manner as the triangular barycentric coordinate formulas.

$$\alpha = \frac{volume(PBCD)}{volume(ABCD)}, \text{ and so on for the rest}$$