CS130 - Reflections and transparency

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1. Given the vector $\mathbf{u} = \langle 1, 4 \rangle$ and the unit vector $\mathbf{n} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$, decompose $\mathbf{u}$ into component $\mathbf{u}_\perp$ perpendicular to $\mathbf{n}$ and component $\mathbf{u}_\parallel$ parallel to $\mathbf{n}$ such that $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\parallel$.

First we calculate the unit vector for u

$$\hat{u} = \langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle$$

We know that

$$u_\parallel = (n \cdot \hat{u})n$$

Then we can also determine

$$u_\perp = u - u_\parallel$$

Therefore using the numbers given to us we get:

$$u_\parallel = \langle \frac{-39}{25\sqrt{17}}, \frac{52}{25\sqrt{17}} \rangle = \langle -0.378, 0.504 \rangle$$

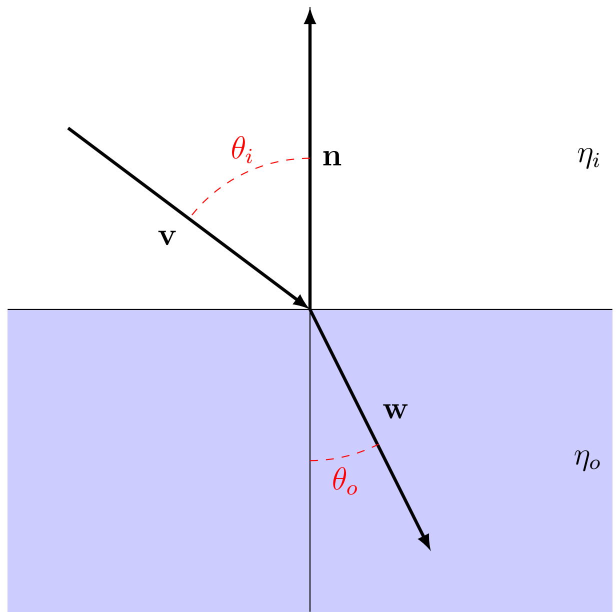
Using this we can calculate the perpendicular vector to be:

$$u_\perp = \langle 1, 4 \rangle - \langle \frac{-39}{25\sqrt{17}}, \frac{52}{25\sqrt{17}} \rangle$$

$$u_\perp = \langle 1.378, 3.496 \rangle$$

Therefore our entire equation for the vector u (considering the normal n) is:

$$u = u_\parallel + u_\perp = \langle -0.378, 0.504 \rangle + \langle 1.378, 3.496 \rangle$$



In the figure above, a ray originally in the air (index of refraction η_i) enters a transparent material (index of refraction η_o). The ray enters along direction $\mathbf{v}$ and leaves along direction $\mathbf{w}$. You are given that $\|\mathbf{v}\| = 1$ and $\|\mathbf{n}\| = 1$. You will construct $\mathbf{w}$ such that $\|\mathbf{w}\| = 1$. $\mathbf{w}$ lies in the same plane as $\mathbf{n}$ and $\mathbf{v}$.

2. Snell's law states that $\eta_i \sin\theta_i = \eta_o \sin\theta_o$. Express this equation in terms of the vectors $\mathbf{v}$, $\mathbf{n}$, and $\mathbf{w}$ using cross products (no dot products).

$$\eta_i||\mathbf{v} \times \mathbf{n}|| = \eta_o||\mathbf{w} \times \mathbf{n}||$$

3. Taking advantage of the fact that $\mathbf{w}$, $\mathbf{n}$, and $\mathbf{v}$ lie in the same plane, we can write $\mathbf{w} = a\mathbf{v} + b\mathbf{n}$. Using your result from the previous problem, solve for a. Note that you will only be able to solve for a up to a sign.

$$\eta_i||\mathbf{v} \times \mathbf{n}|| = \eta_o||(a\mathbf{v} + b\mathbf{n}) \times \mathbf{n}||$$

$$\eta_i||\mathbf{v} \times \mathbf{n}|| = \eta_o||(a\mathbf{v} \times \mathbf{n}) + (b\mathbf{n} \times \mathbf{n})||$$

This simplifies to:

$$\eta_i||\mathbf{v} \times \mathbf{n}|| = \eta_o||a\mathbf{v} \times \mathbf{n}||$$

Doing some algebra we get:

$$\frac{\eta_i}{\eta_o}||\mathbf{v} \times \mathbf{n}|| = \sqrt{a}||\mathbf{v} \times \mathbf{n}||$$

Therefore:

$$a = (\frac{\eta_i}{\eta_o})^2$$

4. Let $\mathbf{t}$ be a vector orthogonal to $\mathbf{n}$ as shown in the figure. Taking the dot product of $\mathbf{w} = a\mathbf{v} + b\mathbf{n}$ by $\mathbf{t}$, deduce the sign of a.

$$\mathbf{w} = a\mathbf{v} + b\mathbf{n}$$
$$\mathbf{w} \cdot \mathbf{t} = (a\mathbf{v} + b\mathbf{n}) \cdot \mathbf{t}$$
$$\mathbf{w} \cdot \mathbf{t} = (a\mathbf{v} \cdot \mathbf{t}) + (b\mathbf{n} \cdot \mathbf{t})$$

We know $\mathbf{t}$ is orthogonal to $\mathbf{n}$, therefore we can simplify this to:

$$\mathbf{w} \cdot \mathbf{t} = a\mathbf{v} \cdot \mathbf{t}$$

$$\mathbf{w} \cdot \mathbf{t} = a(\mathbf{v} \cdot \mathbf{t})$$

Therefore, a must be positive since $\mathbf{w}$ must be in the same relative direction as $\mathbf{v}$

5. Using $\|\mathbf{w}\|^2 = 1$ to derive a quadratic equation in b. Solve this for b, which should give you two solutions. We will select the solution we want later.

$$\|\mathbf{w}\|^2 = 1$$
$$\mathbf{w} \cdot \mathbf{w} = 1$$
$$(a\mathbf{v} + b\mathbf{n}) \cdot (a\mathbf{v} + b\mathbf{n}) = 1$$
$$a^2(\mathbf{v} \cdot \mathbf{v}) + b2a(\mathbf{n} \cdot \mathbf{v}) + b^2(\mathbf{n} \cdot \mathbf{n}) = 1$$

We know that $\|\mathbf{v}\|^2 = \|\mathbf{n}\|^2 = 1$, and that the dot product with itself is also the magnitude squared, therefore we can simplify to:

$$(a^2 - 1) + b(2a(\mathbf{n} \cdot \mathbf{v})) + b^2 = 0$$

Solving for b using the quadratic equation we get:

$$b = \frac{-2a(\mathbf{n} \cdot \mathbf{v}) \pm \sqrt{4a^2(\mathbf{n} \cdot \mathbf{v})^2 - 4(a^2 - 1)}}{2}$$
$$b = -a(\mathbf{n} \cdot \mathbf{v}) \pm \sqrt{a^2(\mathbf{n} \cdot \mathbf{v})^2 - a^2 + 1)}$$
$$b = -a(\mathbf{n} \cdot \mathbf{v}) \pm \sqrt{a^2(\cos^2 \theta_i - 1) + 1)}$$
$$b = -a(\mathbf{n} \cdot \mathbf{v}) \pm \sqrt{1 - a^2 \sin^2 \theta_i}$$

6. If $\mathbf{v} = -\mathbf{n}$, then we should get $\mathbf{w} = \mathbf{v}$ as our solution. Use this special case to deduce the correct sign for b. Using the a and b you derived, write out $\mathbf{w}$.

Our complete equation for w is

$$\mathbf{w} = (\frac{\eta_i}{\eta_o})^2\mathbf{v} + (-(\frac{\eta_i}{\eta_o})^2(\mathbf{n}\cdot\mathbf{v}) \pm \sqrt{1 - (\frac{\eta_i}{\eta_o})^4\|\mathbf{v}\times\mathbf{n}\|^2})\mathbf{n}$$

If $\mathbf{v} = -\mathbf{n}$ then:

$$\mathbf{w} = (\frac{\eta_i}{\eta_o})^2\mathbf{v} + ((\frac{\eta_i}{\eta_o})^2 \pm 1)\mathbf{n}$$

$$\mathbf{w} = a\mathbf{v} + (a \pm 1)\mathbf{n}$$

$$\mathbf{w} = a\mathbf{v} - (a \pm 1)\mathbf{v}$$

$$\mathbf{w} = a\mathbf{v} - (a\mathbf{v} \pm \mathbf{v})$$

Understanding that $\mathbf{w} = \mathbf{v}$ in this case, if a is positive, then b must be negative as well in order to cancel out the outside negative. Therefore we take the negative of the plus-minus symbol.

$$\mathbf{w} = a\mathbf{v} - (a\mathbf{v} - \mathbf{v})$$

$$\mathbf{w} = a\mathbf{v} - a\mathbf{v} + \mathbf{v}$$

So our final equation is:

$$\mathbf{w} = (\frac{\eta_i}{\eta_o})^2\mathbf{v} + (-(\frac{\eta_i}{\eta_o})^2(\mathbf{n}\cdot\mathbf{v}) - \sqrt{1 - (\frac{\eta_i}{\eta_o})^4\|\mathbf{v}\times\mathbf{n}\|^2})\mathbf{n}$$

7. Based on your formula for **w**, deduce the conditions under which complete internal reflection occurs.

For complete internal reflection to occur, the equation $(1 - (\frac{\eta_i}{\eta_o})^4\|\mathbf{v}\times\mathbf{n}\|^2) < 0$ must be satisfied. Which makes the square root complex.

8. What happens as the index of refraction of the sphere in `08.txt` is made closer to the index of refraction of the air? Support your conclusion by showing a sequence of renders. What happens when they are equal?

As 08.txt is given a index of refraction closer to that of air, the rays slowly become less and less refracted until finally (when they are equal) the initial view ray passes through as if there was no transparent object.