Exercise 5

1. Bias and variance of ridge regression

From the lecture we know:

$$\hat{\beta}_\tau = S_\tau^{-1} X^T y$$

using $E[y] = X\beta^*$ we get for the expectation:

$$E[\hat{\beta}_\tau] = E[S_\tau^{-1} X^T y] = S_\tau^{-1} X^T E[y] = S_\tau^{-1} X^T X \beta^* = S_\tau^{-1} S \beta^*$$

for the covariance we use $\mathrm{Cov}[AY] = A\mathrm{Cov}[Y]A^T$:

$$\mathrm{Cov}[\hat{\beta}_\tau] = \mathrm{Cov}[S_\tau^{-1} X^T y] = S_\tau^{-1} X^T \mathrm{Cov}[y] X S_\tau^{-1} = \sigma^2 S_\tau^{-1} X^T X S_\tau^{-1} = S_\tau^{-1} S S_\tau^{-1} \sigma^2$$

with $\mathrm{Cov}[y] = \mathrm{Cov}[\epsilon] = \sigma^2$