① This color is a comment

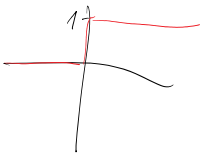
1. logical OR

$z_{in} = (0\ 1\ 1\ 0\ 1)$ for example

$\beta = (1\ 1\ 1\ 1\ 1) = \{1\}^D$

$b = -0.5$ (to avoid exactly 0)

activation function f = step function Θ



e.g. $z' = z_{in} \cdot \beta + b = (0+1+1+0+1) - 0,5 = 2.5$

$f(z') = 1$ ✓

2. masked logical OR

$z_{in} = (1\ 0\ 1)$ $\beta = c$

$c = (1\ 0\ 1)$ $b = -0,5$

activation function $f = \Theta$

$z' = z_{in}\beta + b = (1+0+1) - 0,5 = 1,5$ $\Theta(1.5) = 1$ ✓

3. perfect match

$z_{in} = (1\ 1\ 0)$ for example $\beta_i = \begin{cases} -1 & \text{if } c_i = 0 \\ +1 & \text{if } c_i = 1 \end{cases} = (1\ 1\ -1)$

$c = (1\ 1\ 0)$ for example $b = -\sum_i c_i + 1$

activation function f = ReLu

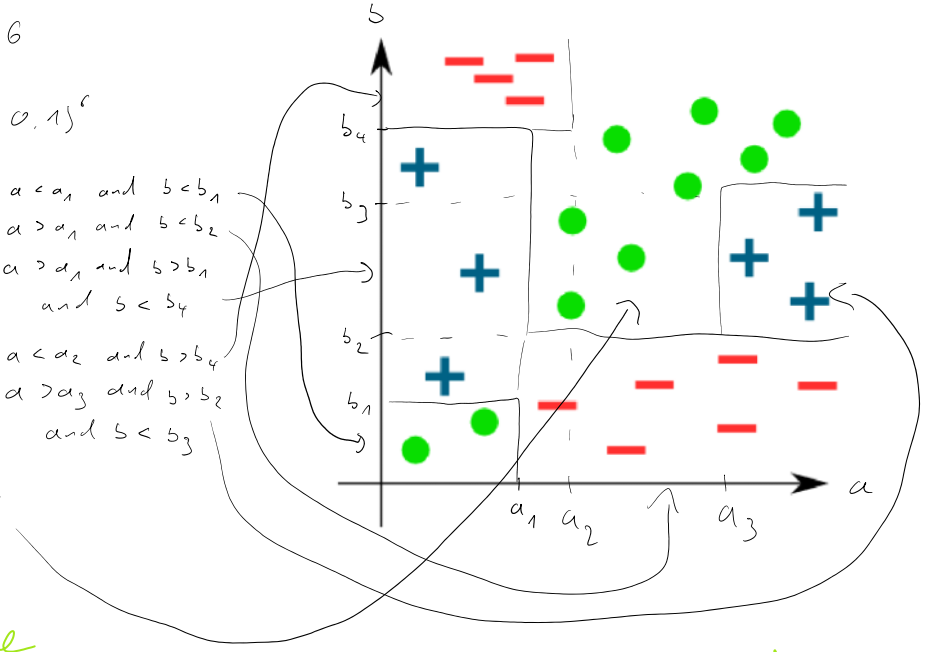
$z' = z_{in}\beta + b = (1+1+0) - 1 = 1$ $f(1) = 1$

I think this solution should work as well.

$X = (a, b)$ $M = 6$

first layer $f(x) \rightarrow \{0,1\}^6$

$f(a,b) = \begin{cases} (100\ 000) & \text{if } a < a_1 \text{ and } b < b_1 \\ (010\ 000) & \text{if } a > a_1 \text{ and } b < b_2 \\ (001\ 000) & \text{if } a > a_1 \text{ and } b > b_1 \text{ and } b < b_4 \\ (000\ 100) & \text{if } a < a_2 \text{ and } b > b_4 \\ (000\ 010) & \text{if } a > a_3 \text{ and } b > b_2 \text{ and } b < b_3 \\ (000\ 001) & \text{else} \end{cases}$



M=3 would have been enough, but it was not asked to have a minimal M_i, so this should be correct.

Given the input vectors above this should work

Second layer not necessary:

third layer has 3 Neurons

1. z_{out_1} = masked OR (z_{in}, 010100) red minus
2. z_{out_2} = masked OR (z_{in}, 001010) blue plus
3. z_{out_3} = masked OR (z_{in}, 100001) green circle

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other option $M = 7$ 7 neurons: one for $a > a_1$, one for $a > a_2$ ...

$f_1(a,s) = \Theta(a - a_1)$ $\Rightarrow$ $\beta = (1,0)$: $z_{in} = x = (a,s)$ $z' = z_{in}\beta + b = a - a_1$

$b = -a_1$ $f = \Theta$

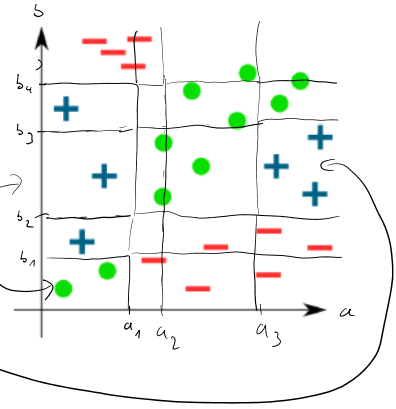
$f_2(a,s) = \Theta(a - a_2)$
$f_3(a,s) = \Theta(a - a_3)$
$f_4(a,s) = \Theta(s - b_1)$
$f_5(a,s) = \Theta(b - b_2)$
$f_6(a,s) = \Theta(s - b_3)$
$f_7(a,s) = \Theta(s < b_4)$

examples:

this would be (0000000)

this would be (1111100)

this would be (0001100)



(perfect match)

Second layer: as many Neurons as regions (4·5 = 20 here)

each neuron represents one Region

1. z_{out} = perfect match (z_{in}, (0000000))
2. ...
3. ...

third layer: 3 masked OR Neurons:

z_{out_1} = masked OR (z_{in}, $\{0,1\}^{20}$) with 1 where region contains red minus

z_{out_2} = masked OR (z_{in}, $\{0,1\}^{20}$) with 1 where region contains blue plus

z_{out_3} = masked OR (z_{in}, $\{0,1\}^{20}$) with 1 where region contains green circle

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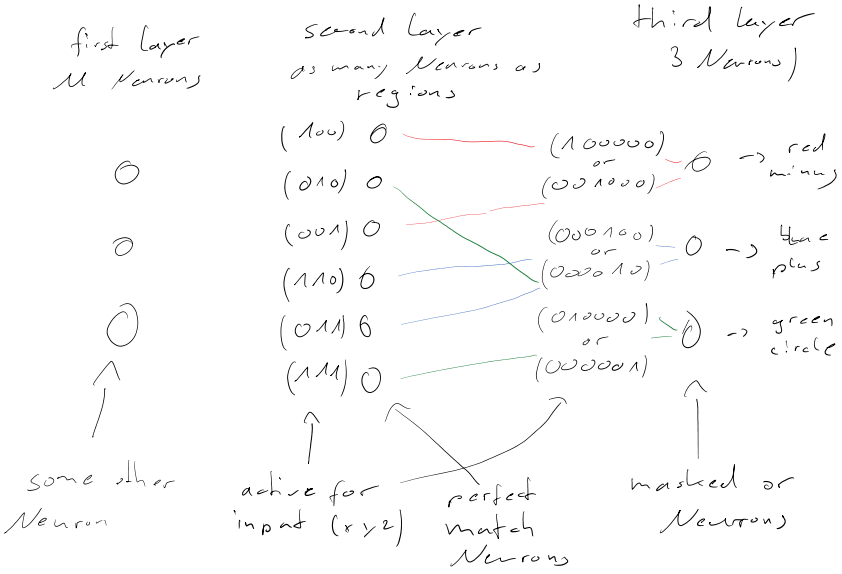
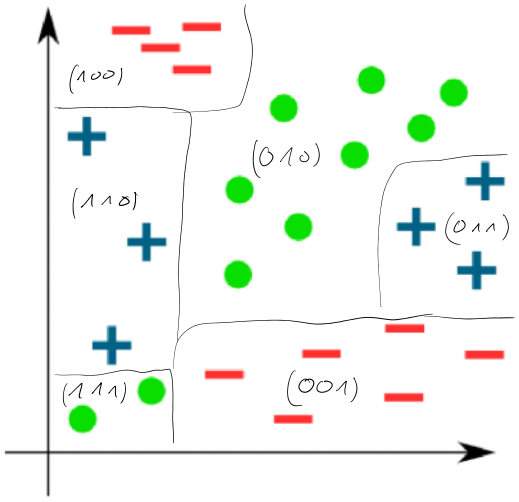
third option $M = 3 = \text{ceil}(\log_2(\#\text{Regions}))$





For more dimensions you need more decision Regions and M gets larger. You always need M Neurons in the first layer mapping Features to distinct M-dimensional vectors. The second layer has as many Neurons as decision Regions and each M-dim vector gets mapped to a specific Neuron representing that region. The last layer has as many Neurons as there are classes. Using the masked OR all Neurons corresponding to a class are mapped to one Neuron representing that class.

This gets difficult because there will be a lot of decision Regions and a lot of necessary Neurons. Also overfitting will be a problem.

2

a network with L layers

$$
\begin{aligned}
Z_0 &= X \\
\tilde{Z}_l &= Z_{l-1} \cdot B_l + b_l \\
Z_l &= \phi_l(\tilde{Z}_l)
\end{aligned}
$$

$$
\begin{aligned}
Z_L &= Z_{L-1} \cdot B_L + b_L \\
&= (Z_{L-2} \cdot B_{L-1} + b_{L-1}) \cdot B_L + b_L \\
&\dots \\
&= Z_{L-2} \cdot \underbrace{B_{L-1} \cdot B_L}_{\text{new } B} + \underbrace{b_{L-1} \cdot B_L}_{\text{new } b} + b_L \\
&= Z_{L-2} \, B + b \\
&\dots \quad \text{repeat for all } L \text{ layers} \\
&= Z_0 \, B' + b'
\end{aligned}
$$

same as one layer