Small report on exam problem #1

Practical programming and numerical methods
Exam 2020

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1 Determining b_i, c_i and d_i

The cubic subspline is on the form

$$S_i(x) = y_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3. \quad (1)$$

The b_i, c_i and d_i coefficients are completely determined by the following three equations,

$$S_i(x_{i+1}) = y_{i+1} \quad (2)$$
$$S_i'(x_i) = y_i' \quad (3)$$
$$S_i'(x_{i+1}) = y_{i+1}'. \quad (4)$$

Solving these equations result in the following equations for the coefficients,

$$b_i = y_i' \quad (5)$$
$$c_i = \frac{3(p_i - b_i)}{\Delta x_i} - q_i \quad (6)$$
$$d_i = \frac{q_i - 2c_i}{3\Delta x_i}, \quad (7)$$

where $\Delta x_i \doteq x_{i+1} - x_i$, $p_i \doteq (y_{i+1} - y_i)/\Delta x_i$ and $q_i \doteq (y_{i+1}' - y_i')/\Delta x_i$.

2 Expanding to fourth order

To obtain a continuous second derivative, we expand the cubic subspline,

$$S_i(x) = y_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad (8)$$
$$+ e_i(x - x_i)^2(x - x_{i+1})^2. \quad (9)$$

Written on this form, nothing changes when we solve equation 2, 3 and 4, so all b_i, c_i and d_i coefficients stay the same.

The condition we want to fulfill to have a continuous second derivative is

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}). \quad (10)$$

This results in the recursive relation for e_i

$$e_i = \frac{c_{i+1} - c_i - 3d_i\Delta x_i + e_{i+1}(\Delta x_{i+1})^2}{(\Delta x_i)^2} \quad (11)$$

or

$$e_{i+1} = \frac{c_i - c_{i+1} + 3d_i\Delta x_i + e_i(\Delta x_i)^2}{(\Delta x_{i+1})^2}. \quad (12)$$

We have to choose a starting e-coefficient before we can use the recursive relations. Let's pick $e_0 = 0$. To determine all e_i's, first equation 12 is run recursively, and then afterwards equation 11 is run recursively backwards starting from $e_{n-1}/2$, where n is the provided amount of tabulated points (this becomes $e_{n-2}/2$ in C# due to the zero-indexing).

3 Remarks to BesselPlot.svg

The integral of the zeroth spherical Bessel function from 0 to x is known as the sine integral, and turns out to be quite complicated to solve. According to Wikipedia[1] it is known as the $Si(x)$ function,

$$Si(x) = \int_0^x j_0(x)dx = \int_0^x \frac{\sin(x)}{x}\mathrm{d}x. \quad (13)$$

It can be written out in a long series, but it is not easy to plot, since one apparently needs to include a long part of the series to get something that is somewhat accurate. Instead I have looked up some table values for it via wolframalpha.com, so there is a bit of data to compare the integrated subspline expression to.

The second derivative of the subspline is a bit wiggly. It is not precise around $x = 0$, but otherwise it follows the analytic second derivative of $j_0(x)$ fairly well. Increasing the amount of tabulated data points for $j_0(x)$ and $j_0'(x)$ will make the second derivative of the subspline follow the analytic expression very precisely (with the exception of just around $x = 0$). More tabulated data points can be generated by making the ODE solver take more steps when solving the spherical Bessel differential equation - for example by lowering the tolerated error.

References

[1] Wikipedia, `https://en.wikipedia.org/wiki/Trigonometric_integral`