Basic Amplitude Modulation theory [DRAFT]

heso

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1 Single-channel AM

A basic expression for a general sinusoidally AM-modulated RF signal $S_{RF}(t)$ can be written as:

$$S_{RF}(t) = A\sin(\omega_c t)\left(1 + m\sin(\omega_m t)\right) \tag{1}$$

where A is the amplitude, ω_c is the carrier (angular) frequency, m is the modulation index and ω_m is the modulation frequency.

Using trigonometric identities, this expression can be rewritten as

$$\begin{aligned} S_{RF}(t) &= A\sin(\omega_c t) + mA\sin(\omega_c t)\sin(\omega_m t) \\ &= A\sin(\omega_c t) + m\frac{A}{2}(\cos((\omega_c - \omega_m)t) - \cos((\omega_c + \omega_m)t)) \end{aligned} \tag{2}$$

The first term is the carrier, the second the lower sideband (LSB), and the third term the upper sideband (USB).

So modulation index can be calculated from

$$\begin{aligned} m &= 2\frac{\max\{V_{LSB}\}}{\max\{V_C\}} \\ &= 2\frac{\max\{V_{USB}\}}{\max\{V_C\}} \end{aligned} \tag{3}$$

1.1 Average power

Since the RMS of a sine wave $A\sin(\omega t)$ is $\mathrm{RMS}\{A\sin(\omega t)\} = A/\sqrt{2}$, we have the following expression for average carrier power P_C (in a 1 Ω system):

$$P_{C,avg} = \frac{A^2}{2} \tag{4}$$

Similarly, for upper and lower sideband average power P_{USB} and P_{LSB}:

$$\begin{aligned} P_{USB,avg} &= P_{LSB,avg} \\ &= \frac{m^2}{2}\frac{A^2}{4} \end{aligned} \tag{5}$$

Using the above equations, the total average power P_{avg} of $S_{RF}(t)$ can be rewritten as

$$\begin{aligned} P_{avg} &= P_{C,avg} + P_{USB,avg} + P_{LSB,avg} \\ &= P_{C,avg}(1 + \frac{m^2}{2}) \end{aligned} \tag{6}$$

1.2 Peak power

For the peak (instantaneous) power P_{pk} of $S_{RF}(t)$, it is obtained at the instant $S_{RF}(t)$ reaches its peak value, i.e. where sinusoids reach ± 1:

$$\begin{aligned} P_{pk} &= \max\{S_{RF}(t)\}^2 \\ &= \{A(1+m)\}^2 \\ &= A^2(1+m)^2 \\ &= P_{C,avg} \times 2(1+m)^2 \\ &= P_{C,pk} \times (1+m)^2 \end{aligned} \tag{7}$$

Note the difference between peak power and peak envelope power. Peak Envelope Power (PEP) is "the average power supplied to the antenna by a transmitter during one radio-frequency cycle at the crest of the modulation envelope taken under normal operating conditions".

PEP is by definition half of the peak (instantaneous) power.

1.3 PAR

$$\begin{aligned} \text{PAR} &= \frac{P_{pk}}{P_{avg}} \\ &= \frac{A^2(1+m)^2}{\frac{A^2}{2}(1+m^2/2)} \\ &= 2\frac{(1+m)^2}{1+m^2/2} \end{aligned} \tag{8}$$

1.4 Voice

It has been simulated that voice AM with modulation index 0.85 can be emulated as sinusoidal AM with modulation index m around 0.28, as far as average power is concerned. Denoting ordinary modulation index as $m_{peak} = 0.85$, and the average modulation index $m_{avg} = 0.28$, the expression for PAR becomes

$$\begin{aligned} \text{PAR} &= 2\frac{(1+m_{pk})^2}{1+m_{avg}^2/2} \\ &= 2\frac{(1+0.85)^2}{1+0.28^2/2} \\ &\approx 6.59\ (\sim 8.2\,\text{dB}) \end{aligned} \tag{9}$$

This has been verified by simulations.

2 Multi-channel AM

This section deals with multi-channel AM, i.e. different AM channels.

2.1 Average power

Average power of a N-channel AM signal is simply the added power of each channel:

$$\begin{aligned} P_{avg}(N) &= \sum_{i=1}^{N} P_{C,avg,i} \\ &= \sum_{i=1}^{N} A_i(1+\frac{m_i^2}{2}) \end{aligned} \tag{10}$$

So for same amplitude A and modulation index m, we have

$$\begin{aligned} P_{avg}(N) &= \sum_{i=1}^{N} A(1+\frac{m^2}{2}) \\ &= N \cdot A(1+\frac{m^2}{2}) \\ &= N \cdot P_{avg}(1) \end{aligned} \tag{11}$$

2.2 Peak power

Peak (instantaneous) power occurs when sum of amplitudes of each channel maximizes, with the theoretical value $A_1(1+m_1)+...+A_N(1+m_N)$. This translates into a peak power value of

$$P_{pk}(N) = (\sum_{i=1}^{N} A_i(1+m_i))^2 \tag{12}$$

And so for same amplitude A and modulation index m

$$\begin{aligned} P_{pk}(N) &= (NA(1+m))^2 \\ &= N^2A^2(1+m)^2 \\ &= N^2P_{pk}(1) \end{aligned} \tag{13}$$

This should be understood as a "peak-of-peaks", i.e. an upper bound for peak power.

2.3 PAR

$$\begin{aligned} \mathrm{PAR}_{\mathrm{MAX}}(N) &= \frac{P_{pk}(N)}{P_{avg}(N)} \\ &= \frac{N^2A^2(1+m)^2}{N\frac{A^2}{2}(1+m^2/2)} \\ &= 2N\frac{(1+m)^2}{1+m^2/2} \\ &= N \cdot \mathrm{PAR}(1) \end{aligned} \tag{14}$$

3 Compression

3.1 Compression model

Compression is modelled through an amplitude compressing function of the form

$$f(x) = x - \alpha x^3 \tag{15}$$

with α being positive. To find from α from IIP_3 [explain]:

$$\alpha = \frac{4}{3\left(IIP_3\right)^2} \tag{16}$$

3.2 Compression of AM

Inserting into the compression function yields:

$$\begin{aligned} f(A\sin(\varphi_c)\left(1+m\sin(\varphi_m)\right)) = & \; A\sin(\varphi_c)\left(1+m\sin(\varphi_m)\right) \\ & - \alpha(A\sin(\varphi_c)\left(1+m\sin(\varphi_m)\right))^3 \\ =& A\sin(\varphi_c)\left(1+m\sin(\varphi_m)\right)) \\ & - \alpha A^3(\sin(\varphi_c)(1+m\sin(\varphi_m)))^3 \end{aligned} \tag{17}$$

To proceed, it is noted that

$$\begin{aligned} \Big(\sin(x)(1+k\sin(y))\Big)^3 = & \frac{1}{32}(-3k^3\cos(x-3y)+k^3\cos(3x-3y) \\ & + 9k^3\cos(x-y) - 3k^3\cos(3x-y) - 9k^3\cos(x+y) \\ & + 3k^3\cos(3x+y) + 3k^3\cos(x+3y) - k^3\cos(3x+3y) \\ & - 18k^2\sin(x-2y) + 6k^2\sin(3x-2y) - 18k^2\sin(x+2y) \\ & + 6k^2\sin(3x+2y) + 36k^2\sin(x) - 12k^2\sin(3x) \\ & + 36k\cos(x-y) - 12k\cos(3x-y) - 36k\cos(x+y) \\ & + 12k\cos(3x+y) + 24\sin(x) - 8\sin(3x)) \end{aligned} \tag{18}$$

Setting $x = \varphi_c = \omega_c t$, $y = \varphi_m = \omega_m t$ and $k = m$, and neglecting terms that do not fall close to $1 \times \omega_c$ yields

$$\begin{aligned} \Big(\sin(\varphi_c)(1+m\sin(\varphi_m))\Big)^3 = & \frac{1}{32}\Big(- 3m^3\cos(\varphi_c-3\varphi_m) + 9m^3\cos(\varphi_c-\varphi_m) \\ & - 9m^3\cos(\varphi_c+\varphi_m) + 3m^3\cos(\varphi_c+3\varphi_m) \\ & - 18m^2\sin(\varphi_c-2\varphi_m) - 18m^2\sin(\varphi_c+2\varphi_m) \\ & + 36m^2\sin(\varphi_c) + 36m\cos(\varphi_c-\varphi_m) \\ & - 36m\cos(\varphi_c+\varphi_m) + 24\sin(\varphi_c)\Big) \\ = & \frac{1}{32}\Big(\\ & - 3m^3\cos(\varphi_c-3\varphi_m) \\ & - 18m^2\sin(\varphi_c-2\varphi_m) \\ & + (9m^3+36m)\cos(\varphi_c-\varphi_m) \\ & + (36m^2+24)\sin(\varphi_c) \\ & - (9m^3+36m)\cos(\varphi_c+\varphi_m) \\ & - 18m^2\sin(\varphi_c+2\varphi_m) \\ & + 3m^3\cos(\varphi_c+3\varphi_m)\Big) \end{aligned} \tag{19}$$

So returning to Equation (17) again,

$$
\begin{aligned}
f(A\sin(\varphi_c)\,(1+m\sin(\varphi_m))) =& A\sin(\varphi_c)\,(1+m\sin(\varphi_m))) \\
&-\alpha A^3(\sin(\varphi_c)(1+\sin(\varphi_m)))^3 \\
=& A\sin(\varphi_c)+m\frac{A}{2}(\cos(\varphi_c-\varphi_m)-\cos(\varphi_c+\varphi_m) \\
&-\frac{\alpha A^3}{32}\Big(\\
&-3m^3\cos(\varphi_c-3\varphi_m) \\
&-18m^2\sin(\varphi_c-2\varphi_m) \\
&+(9m^3+36m)\cos(\varphi_c-\varphi_m) \\
&+(36m^2+24)\sin(\varphi_c) \\
&-(9m^3+36m)\cos(\varphi_c+\varphi_m) \\
&-18m^2\sin(\varphi_c+2\varphi_m) \\
&+3m^3\cos(\varphi_c+3\varphi_m)\Big) \\
=& -3m^3\cos(\varphi_c-3\varphi_m) \\
&-18m^2\sin(\varphi_c-2\varphi_m) \\
&+\Big(m\frac{A}{2}-\alpha A^3\frac{9m^3+36m}{32}\Big)\cos(\varphi_c-\varphi_m) \\
&+\Big(A-\alpha A^3\frac{36m^2+24}{32}\Big)\sin(\varphi_c) \\
&-\Big(m\frac{A}{2}-\alpha A^3\frac{9m^3+36m}{32}\Big)\cos(\varphi_c+\varphi_m) \\
&-18m^2\sin(\varphi_c+2\varphi_m) \\
&+3m^3\cos(\varphi_c+3\varphi_m)\Big) \\
=& -3m^3\cos(\varphi_c-3\varphi_m) \\
&-18m^2\sin(\varphi_c-2\varphi_m) \\
&+m\frac{A}{2}\Big(1-\alpha A^2\frac{9m^2+36}{16}\Big)\cos(\varphi_c-\varphi_m) \\
&+A\Big(1-\alpha A^2\frac{36m^2+24}{32}\Big)\sin(\varphi_c) \\
&-m\frac{A}{2}\Big(1-\alpha A^2\frac{9m^2+36}{16}\Big)\cos(\varphi_c+\varphi_m) \\
&-18m^2\sin(\varphi_c+2\varphi_m) \\
&+3m^3\cos(\varphi_c+3\varphi_m)\Big)
\end{aligned} \tag{20}
$$

It is seen that the basic LSB and USB contributions vanish for

$$
\begin{gathered}
1-\alpha A^2\frac{9m^2+36}{16}=0 \\
\Updownarrow \\
\alpha_0=\frac{16}{A^2(9m^2+36)}
\end{gathered} \tag{21}
$$

and so for $A=1$:

$$
\alpha_0=\frac{16}{9m^2+36} \tag{22}
$$

which corresponds to an $IIP_{3,0}$ of

$$
\begin{aligned}
IIP_{3,0} &= \sqrt{\frac{4}{3\alpha_0}} \\
&= \sqrt{\frac{4}{3}\frac{9m^2+36}{16}} \\
&= \frac{1}{2}\sqrt{3m^2+12}
\end{aligned}
\tag{23}
$$

Consequently, for the case $A = 1$ and $m = 0.85$, we get $\alpha_0 \simeq 0.376$, and $IIP_{3,0} \simeq 1.882V$, corresponding to a power of $(1.882V)^2/1\Omega = 3.54$ W in a 1Ω system, or 71 mW in a 50Ω system.

From above, an effective/apparent modulation index can be calculated as

$$
\begin{aligned}
m_{eff} &= 2\frac{\max\{V_{LSB}\}}{\max\{V_C\}} \\
&= 2\frac{m\frac{A}{2} - \alpha A^3 \frac{9m^3+36m}{32}}{A - \alpha A^3 \frac{36m^2+24}{32}} \\
&= \frac{m - \alpha A^2 \frac{9m^3+36m}{16}}{1 - \alpha A^2 \frac{36m^2+24}{32}} \\
&= m\frac{1 - \alpha A^2 \frac{9m^2+36}{16}}{1 - \alpha A^2 \frac{36m^2+24}{32}}
\end{aligned}
\tag{24}
$$

So for $A = 1$:

$$
m_{eff} = m\frac{1 - \alpha \frac{9m^2+36}{16}}{1 - \alpha \frac{36m^2+24}{32}}
\tag{25}
$$

Plotting m_{eff} versus α yields Figure 1.

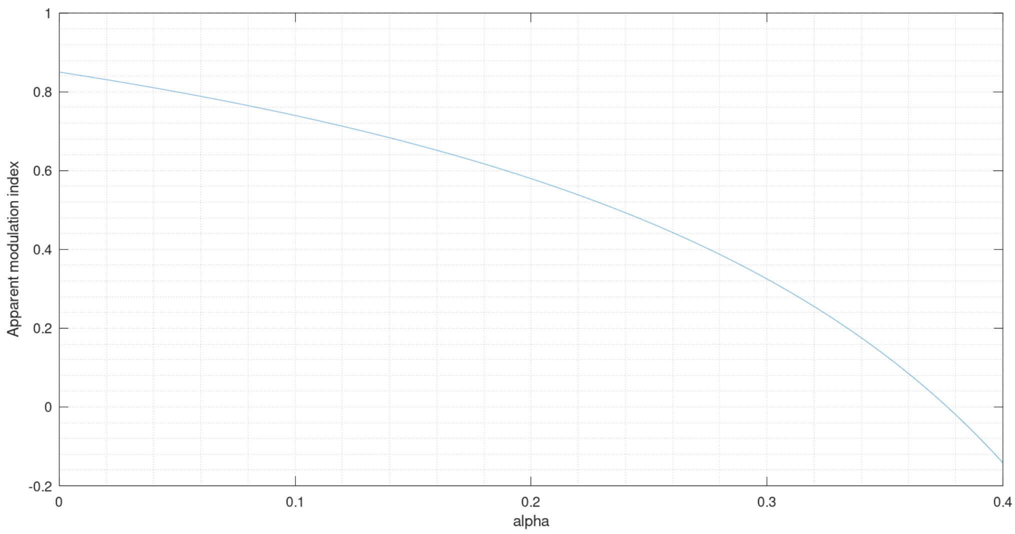

Figure 1: Apparent modulation index vs alpha-parameter. [TODO: Add units]

Note how the apparent modulation index is zero at $\alpha = \alpha_0$, as predicted.

Inserting the expression for α in terms of IIP_3 into Equation (28) yields

$$\begin{aligned} m_{eff} &= m\frac{1-\frac{4}{3IIP_3^2}\frac{9m^2+36}{16}}{1-\frac{4}{3IIP_3^2}\frac{36m^2+24}{32}} \\ &= m\frac{IIP_3^2-\frac{3}{4}m^2-3}{IIP_3^2-\frac{3}{2}m^2-1} \end{aligned} \tag{26}$$

There is obviously a singularity at $IIP_3 = \sqrt{\frac{3}{2}m^2+1}$ ($\simeq 1.4435$ for $m = 0.85$), and a zero at $IIP_3 = \sqrt{\frac{3}{4}m^2+3}$ ($\simeq 1.8820$ for $m = 0.85$).

Furthermore, plotting m_{eff} versus IIP_3 yields Figure 2. From the figure, it

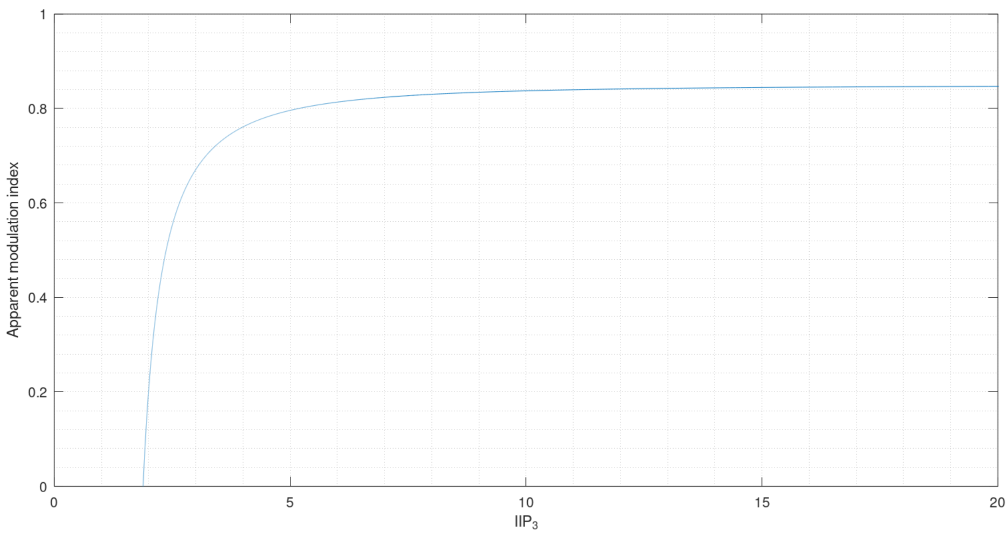

Figure 2: Apparent modulation index vs IIP3. Curve at and below singularity not shown. [TODO: Add units]

appears that to avoid noticeable modulation index degradation, ist best to keep $IIP_3 > 10$V, corresponding to $(10V)^2/1\Omega = 100$ W in a 1Ω system, or 2 W in a 50Ω system.

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