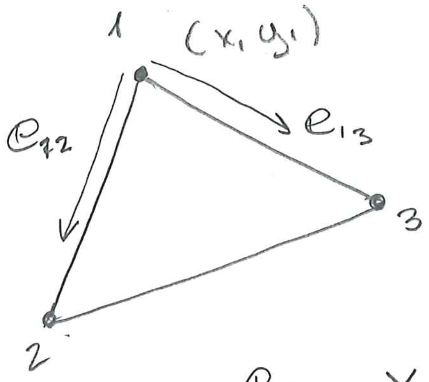
Constant Strain Triangle using Area Coordinates



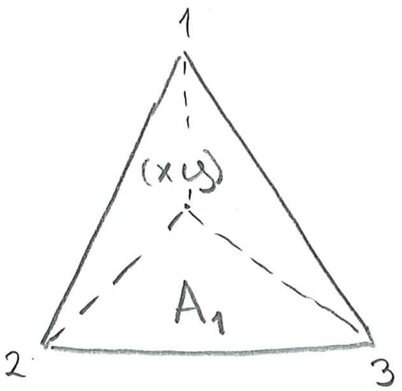
The area of the triangle is given by the cross-product of the side edge vectors:

$$\underline{2A} = \underset{e_{12}}{\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}} \times \underset{e_{13}}{\begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \end{bmatrix}} = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)$$

$$= x_2 y_3 - x_2 y_1 - x_1 y_3 + x_1 y_1 - x_3 y_2 + x_1 y_2 + x_3 y_1 - x_1 y_1$$

$$\left.\begin{aligned} &= 1\,(x_2 y_3 - x_3 y_2) \\ &\quad - 1\,(x_1 y_3 - x_3 y_1) \\ &\quad + 1\,(x_1 y_2 - x_2 y_1) \end{aligned}\right\} = \underline{\det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}$$

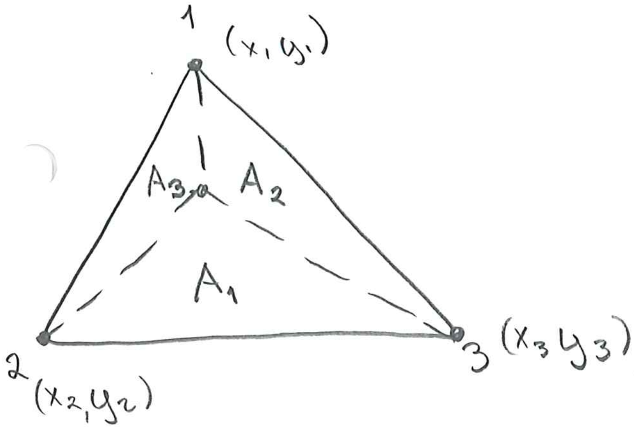
If we let the top point be an internal point (x, y) we have



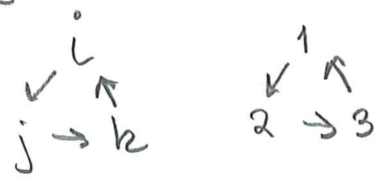
$$2A_1 = \det \begin{bmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$= (x_2 y_3 - x_3 y_2) - (x y_3 - x_3 y) + (x y_2 - x_2 y)$$

$$= \underbrace{(x_2 y_3 - x_3 y_2)}_{a_1} + x\underbrace{(y_2 - y_3)}_{b_1} + y\underbrace{(x_3 - x_2)}_{c_1}$$



Cyclic permutation



Through cyclic permutation of the nodes we have:

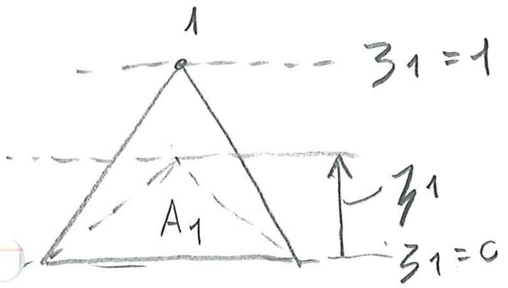
$$a_i = x_j y_k - x_k y_j$$

$$b_i = y_j - y_k$$

$$c_i = x_k - x_j$$

$$2A_i = a_i + b_i x + c_i y$$

Area coordinates: $\zeta_i = \dfrac{A_i}{A}$



The area coordinates ζ_i have unit value at node i and are zero at the line between j-k.

Partial derivatives at the area coordinates:

$$\frac{\partial \zeta_i}{\partial x} = \frac{b_i}{2A} = \frac{y_j - y_k}{2A}$$

$$\frac{\partial \zeta_i}{\partial y} = \frac{c_i}{2A} = \frac{x_k - x_j}{2A}$$

We can use the ζ_i as interpolation across the element interior:

Geometry: $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} = \mathbb{N}\,\mathbb{x}$$

Displacements $$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \mathbb{N}\,\mathbb{u}$$

Strains:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y}+v_{,x} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$\mathbb{E} = \mathbb{B}\,\mathbb{u}$$

The strains are constant over the element and integration to get the stiffness is quite simple:

$$\boxed{\mathbb{k}_e = \int_V \mathbb{B}^T \mathbb{C}\, \mathbb{B}\, dV = A h\, \mathbb{B}^T \mathbb{C}\, \mathbb{B}}$$

h is thickness of the element and the constitutive matrix for plane stress:

$$\mathbb{C} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$