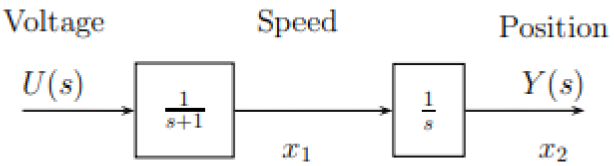
Homework 4

Henrik Lucander, 724140

Problem 1)

a)



$y(t) = x_2(t)$

$\dot{y}(t) = \dot{x}_2(t) = x_1(t)$

$\ddot{y}(t) = \dot{x}_1(t)$

transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+1)} = \frac{1}{s^2 + s}$$

we get:

$s^2 Y(s) + sY(s) = U(s)$

as $\ddot{y}(t) = \dot{x}_1(t)$ and $\dot{y}(t) = \dot{x}_2(t) = x_1(t)$ we get:

$sX_1(s) + X_1(s) = U(s)$

which equals:

$\dot{x}_1(t) = -x_1(t) + u(t)$

Therefore:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

b)

$$\phi(h) = e^{Ah} = e^{Ah}|_{t=h} = L^{-1}\{(sI - A)^{-1}\}|_{t=h} = L^{-1}\left\{\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\right)^{-1}\right\}|_{t=h} =$$

$$L^{-1}\left\{\begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s^2+s} & \frac{1}{s} \end{bmatrix}\right\}|_{t=h} = \begin{bmatrix} e^{-t} & 0 \\ 1 - e^{-t} & 1 \end{bmatrix}|_{t=h} = \begin{bmatrix} e^{-h} & 0 \\ 1 - e^{-h} & 1 \end{bmatrix}$$

$$\Gamma(h) = \int_0^h e^{As} ds B = \int_0^h \begin{bmatrix} e^{-s} & 0 \\ 1 - e^{-s} & 1 \end{bmatrix} ds \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \int_0^h \begin{bmatrix} e^{-s} \\ 1 - e^{-s} \end{bmatrix} ds = \begin{bmatrix} 1 - e^{-h} \\ h + e^{-h} - 1 \end{bmatrix}$$

$$x(kh + h) = \begin{bmatrix} e^{-h} & 0 \\ 1 - e^{-h} & 1 \end{bmatrix} x(kh) + \begin{bmatrix} 1 - e^{-h} \\ h + e^{-h} - 1 \end{bmatrix} u(kh)$$

$y(kh) = [0 \quad 1]x(kh)$

c)

$h = 1$

$u[k] = my_{ref} - L\hat{x}[k]$

$$\phi = \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} = \begin{bmatrix} e^{-1} & 0 \\ 1-e^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 0.368 & 0 \\ 0.632 & 1 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 1-e^{-h} \\ h+e^{-h}-1 \end{bmatrix} = \begin{bmatrix} 1-e^{-1} \\ e^{-1} \end{bmatrix} = \begin{bmatrix} 0.632 \\ 0.368 \end{bmatrix}$$

The system is reachable if the controllability matrix W_c is full rank

$$W_c = [\Gamma \quad \phi\Gamma] = \left[\begin{bmatrix} 1-e^{-1} \\ e^{-1} \end{bmatrix} \quad \begin{bmatrix} e^{-1} & 0 \\ 1-e^{-1} & 1 \end{bmatrix}\begin{bmatrix} 1-e^{-1} \\ e^{-1} \end{bmatrix}\right] = \left[\begin{bmatrix} 0.632 \\ 0.368 \end{bmatrix} \quad \begin{bmatrix} 0.368 & 0 \\ 0.632 & 1 \end{bmatrix}\begin{bmatrix} 0.632 \\ 0.368 \end{bmatrix}\right]$$
$$= \begin{bmatrix} 0.632 & 0.233 \\ 0.368 & 0.767 \end{bmatrix}$$

If the determinant of the controllability matrix W_c is non-zero, it is full rank

$$\det(W_c) = \begin{vmatrix} 0.632 & 0.233 \\ 0.368 & 0.767 \end{vmatrix} = 0.632 * 0.767 - 0.368 * 0.233 = 0.399 \neq 0$$

The rank is full; therefore, the system is reachable.

Thus, we can choose the feedback gain L such that the poles are placed as desired $0.5 \ \pm j0.5$ with Ackermann

$L = [0 \quad 1]W^{-1}P(\phi)$

we find W^{-1} and $P(\phi)$:

$$W^{-1} = \begin{bmatrix} 0.632 & 0.233 \\ 0.368 & 0.767 \end{bmatrix}^{-1} = \begin{bmatrix} 1.9223 & -0.5840 \\ -0.9223 & 1.5840 \end{bmatrix}$$

$$P(z) = \big(z-(0.5-0.5j)\big)\big(z-(0.5+0.5j)\big)$$
$$= z^2 - 0.5z - 0.5zj - 0.5z + 0.25 + 0.25j + 0.5zj - 0.25j - 0.25j^2 = z^2 - z + 0.5$$

$$P(\phi) = \phi^2 - \phi + 0.5I = \begin{bmatrix} 0.368 & 0 \\ 0.632 & 1 \end{bmatrix}^2 - \begin{bmatrix} 0.368 & 0 \\ 0.632 & 1 \end{bmatrix} + 0.5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.135 & 0 \\ 0.865 & 1 \end{bmatrix} - \begin{bmatrix} 0.368 & 0 \\ 0.632 & 1 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.267 & 0 \\ 0.233 & 0.5 \end{bmatrix}$$

$$L = [0 \quad 1]W^{-1}P(\phi) = [0 \quad 1]\begin{bmatrix} 1.9223 & -0.5840 \\ -0.9223 & 1.5840 \end{bmatrix}\begin{bmatrix} 0.267 & 0 \\ 0.233 & 0.5 \end{bmatrix} = [0.122 \quad 0.792]$$

The gain m needs to be chosen such that the output follows the reference input y_{ref}

We can take use of the final value theorem to obtain the gain m:

$$\lim_{k\to\infty} y[k] = \lim_{z\to 1}(z-1)Y(z)$$

We find $Y(z)$:

$$x[k+1] = \phi x[k] + \Gamma\big(u[k] + my_{ref}\big) = (\phi - \Gamma L)x[k] + \Gamma m y_{ref}$$

z-transform:

$$zX(z) = (\phi - \Gamma L)X(z) + \Gamma m * \frac{Y_{ref}(z)}{z-1}$$

$$(zI - \phi + \Gamma L)X(z) = \Gamma m * \frac{Y_{ref}(z)}{z-1}$$

$$y[k] = Cx[k]$$

z-transform:

$$Y(z) = CX(z)$$

insert X(z):

$$Y(z) = C * \frac{\Gamma m * \frac{Y_{ref}(z)}{z-1}}{(zI - \phi + \Gamma L)}$$

We get:

$$\begin{aligned}
\lim_{k\to\infty} y[k] &= \lim_{z\to 1}(z-1) * C * \frac{\Gamma m * \frac{Y_{ref}(z)}{z-1}}{(zI - \phi + \Gamma L)} = (z-1) * C * \frac{\Gamma m * Y_{ref}(z)}{(z-1)(zI - \phi + \Gamma L)} \\
&= \lim_{z\to 1} C(zI - \phi + \Gamma L)^{-1}\Gamma m Y_{ref}(z) \\
&= 1\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0.368 & 0\\0.632 & 1\end{bmatrix} + \begin{bmatrix}0.632\\0.368\end{bmatrix}\begin{bmatrix}0.122 & 0.792\end{bmatrix}\right)^{-1}\begin{bmatrix}0.632\\0.368\end{bmatrix} m Y_{ref}(z) \\
&= \begin{bmatrix}0\\1.263\end{bmatrix} m Y_{ref}(z) = 1.263 m Y_{ref}(z)
\end{aligned}$$

Therefore:

$$m = \frac{1}{1.263}$$

d)

Only u and y are measured. State observer can be designed separately.

For the observer, we need the gain K:

$$\begin{aligned}
\det(zI - \phi + KC) &= \det\left(\begin{bmatrix}z & 0\\0 & z\end{bmatrix} - \begin{bmatrix}0.368 & 0\\0.632 & 1\end{bmatrix} + \begin{bmatrix}k_1\\k_2\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}z - 0.368 & k_1\\-0.632 & z - 1 + k_2\end{bmatrix}\right) \\
&= (z - 0.368)(z - 1 + k_2) - (-0.632)k_1 \\
&= z^2 - z + k_2 z - 0.368z + 0.368 - 0.368k_2 + 0.632k_1 \\
&= z^2 + (k_2 - 1.368)z + (0.632k_1 - 0.368k_2 + 0.368)
\end{aligned}$$

$$k_2 = 1.368$$

$$k_1 = \frac{0.368k_2 - 0.368}{0.632} = 0.2143$$

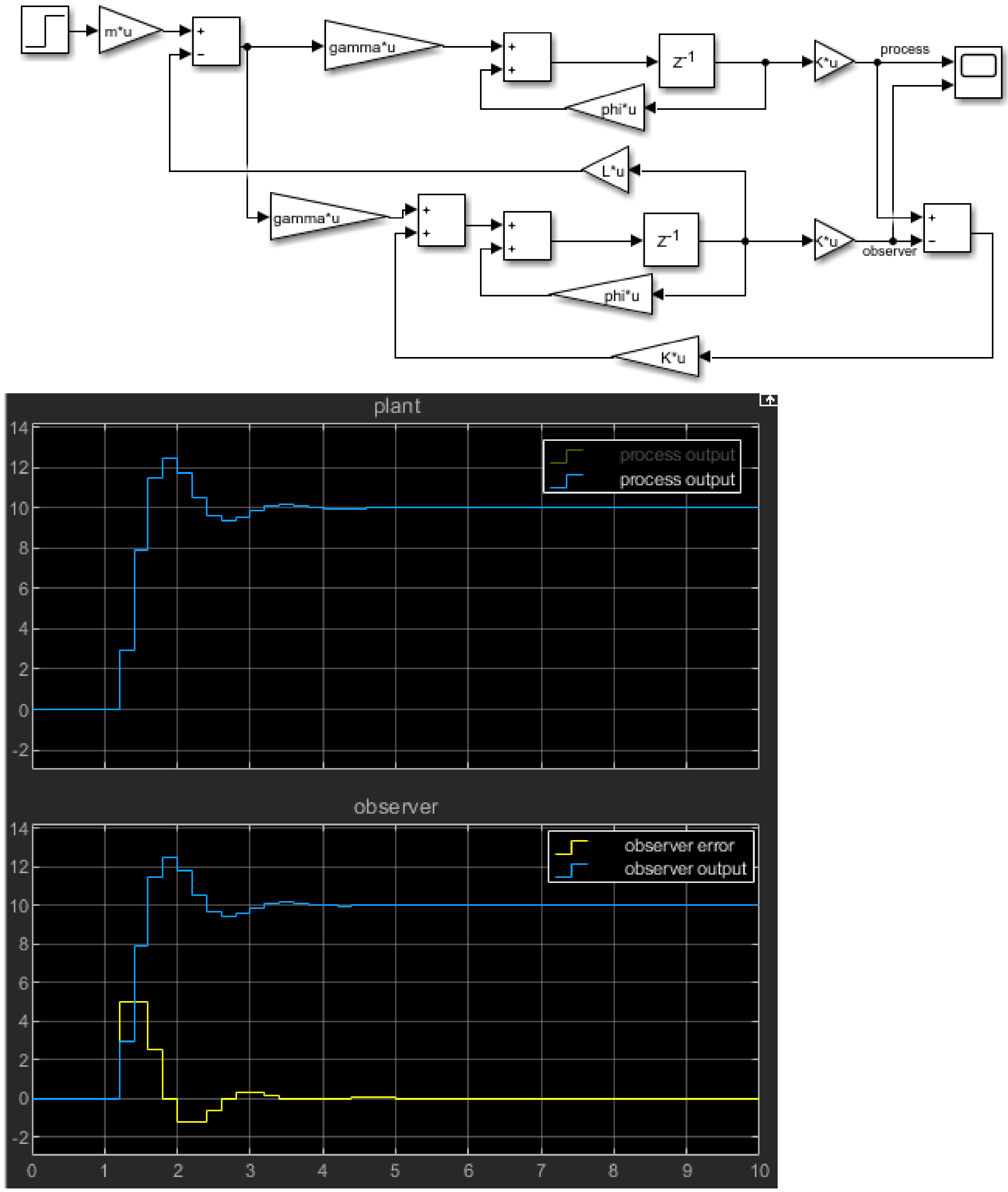
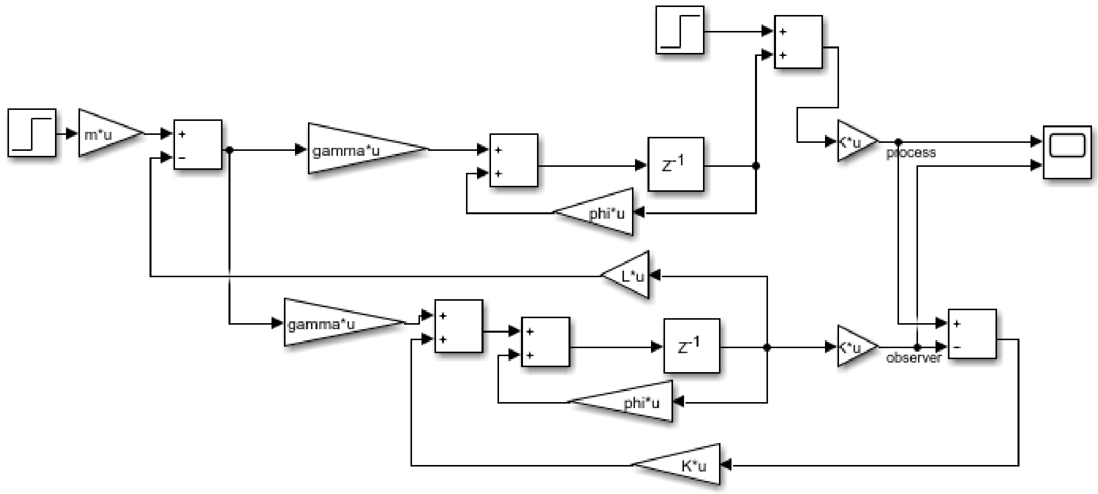


Figure 1. The final value is set to 10.

The process converges and the error goes to 0. Everything seems to work as intended.

e)



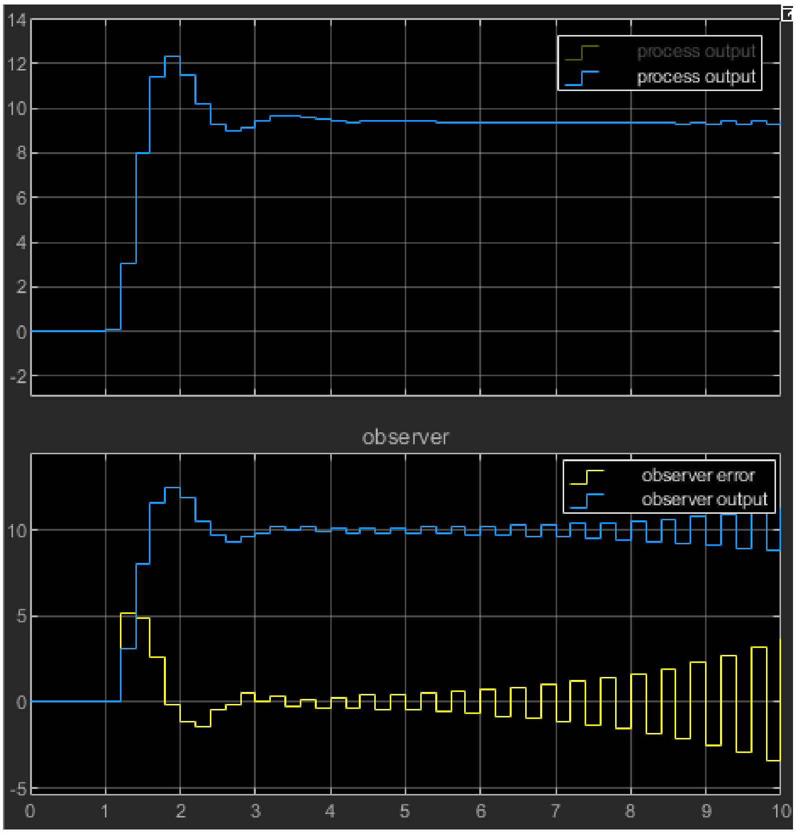


Figure 2.

My hypothesis was that the observation error would go higher and converge to 0 later. The system would remain stable.

However, it seems like the system can't handle the disturbance. After double checking the math and going through the lecture slides, I couldn't come up with any logical explanation why this happens….

```
phi = [0.368 0; 0.632 1];
gamma = [0.632; 0.368];
W = [0.632 0.233; 0.368 0.767];
P = phi^2-phi+0.5*eye(2);
L = [0 1]*inv(W)*P;
K = [1.368 0.2143];
m = 1/1.263;
C = [0 1];
C2 = 1;
```

m-code used. Math was done by hand/calculator. Included in the document.