Homework #2

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Problem 1)

$b = average\ friction\ coefficient$

$u = controller's\ action$

a)

Forces applied to the car:

$$\sum F = ma$$

$$u - b\dot{x} = m\ddot{x}$$

$$\ddot{x} + \frac{b}{m}\dot{x} = \frac{u}{m}$$

and if $v(t) = \dot{x}(t)$ we can rewrite it as:

$$\boldsymbol{\dot{v}(t) + \frac{b}{m}v(t) = \frac{u(t)}{m}}$$

b)

$m = 1000kg$

$b = 100$

Design specification: 0 km/h to 100 km/h in 8 seconds with an overshoot less than 20%

i)

As stated in Problem 1 part a), the system can be represented by:

$$m\dot{v} + bv = u$$

We take the Laplace transform:

$$L\{m\dot{v} + bv\} = L\{u\}$$

$$msL\{v\} + bL\{v\} = L\{u\}$$

$$msV(s) + bV(s) = U(s)$$

$$(ms + b)V(s) = U(s)$$

We get:

$$G(s) = \frac{V(s)}{U(s)} = \frac{1}{ms + b}$$

The PI controller:

$$C_{PI}(s) = \frac{K_p s + K_i}{s}$$

Closed loop:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{C_{PI}(s)G(s)}{1 + C_{PI}(s)G(s)} = \frac{\frac{K_p s + K_i}{s} * \frac{1}{1000s + 100}}{1 + \frac{K_p s + K_i}{s} * \frac{1}{1000s + 100}} = \frac{\frac{K_p s + K_i}{1000s^2 + 100s}}{1 + \frac{K_p s + K_i}{1000s^2 + 100s}}$$
$$= \frac{K_p s + K_i}{1000s^2 + (100 + K_p)s + K_i} = \frac{0.001K_p s + 0.001K_i}{s^2 + (0.1 + 0.001K_p)s + 0.001K_i}$$

We need an overshoot less than 20%:

We can find ω_n and ζ from:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

I assume that the car "reaches 100km/h" when at the settling time t_s ,and not at the time it raises above 100km/h for the first time:

$$\zeta \geq 0.6\left(1 - \frac{M_p in\%}{100}\right) = 0.6\left(1 - \frac{20}{100}\right) = 0.48$$

$$t_s = \frac{4.6}{\zeta\omega_n}$$

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.48 * 8} \approx 1.198$$

Therefore, we want the following closed loop transfer function H(s):

$$H(s) = \frac{1.435}{s^2 + 1.15s + 1.435}$$

We get the following gains:

$$K_p = \frac{1.05}{0.001} = 1050$$

$$K_i = \frac{1.435}{0.001} = 1435$$

When simulating with these gain values, the overshoot is around 28% which is not acceptable. We increase the damping factor ζ by half:

$$\zeta = 0.72$$

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.72 * 8} \approx 0.799$$

$$H(s) = \frac{0.638}{s^2 + 1.15s + 0.638}$$

$$K_p = \frac{1.05}{0.001} = 1050$$

$$K_i = \frac{0.638}{0.001} = 638$$

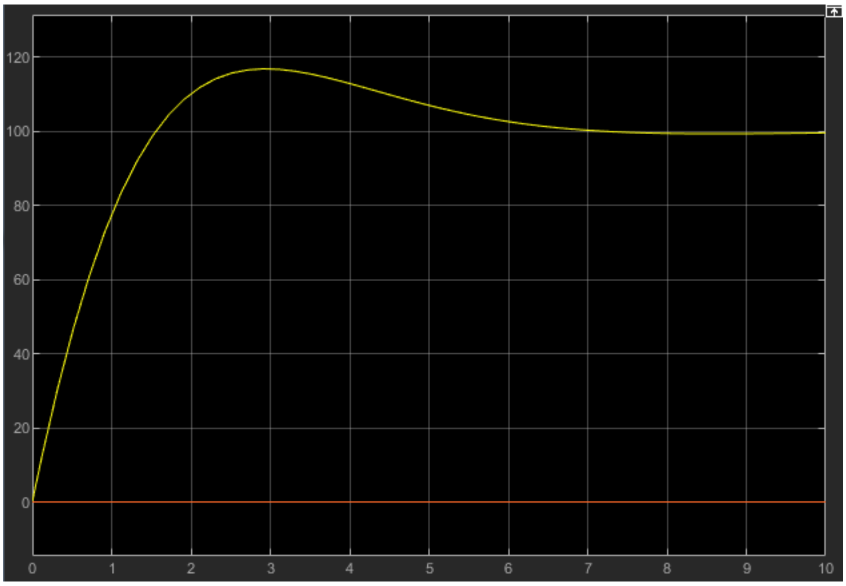
When simulating with these new gain values, the overshoot is 16,8% and a settling time of 7 seconds, which is acceptable.

Therefore, our PI controller is:

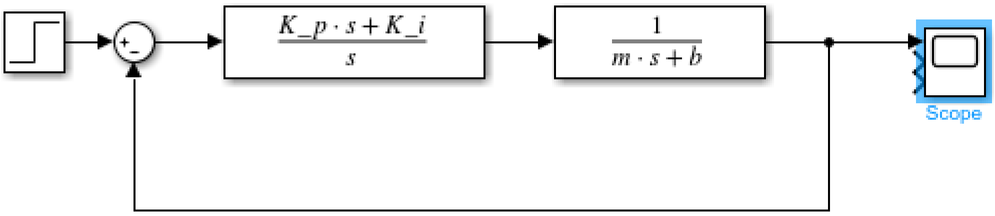
$$C_{PI}(s) = \frac{K_p s + K_i}{s} = \frac{1050s + 638}{s}$$

ii)

Simulink:



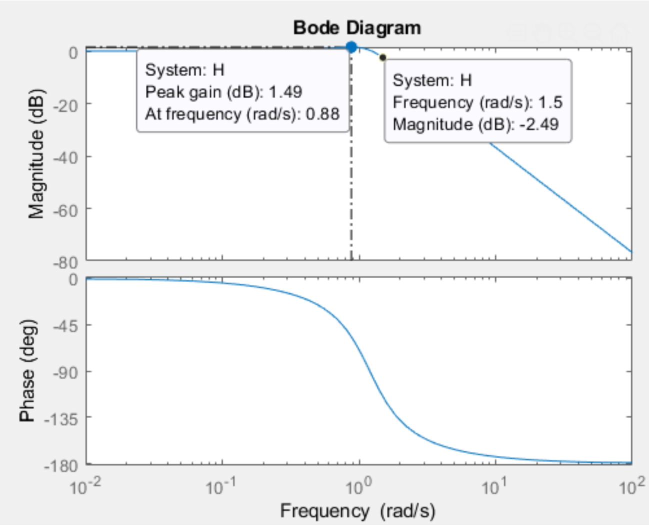
We get an overshoot of 16.8% and a settling time of 7 seconds.




```
%% Homework 2 - Digital and Optimal Control
%Henrik Lucande
%724140
close all
clear all
m = 1000;
b = 100;
K_p = 1050;
K_i = 638;
```

c)

We get the bandwidth from the bode diagram. Bandwidth is the frequency where magnitude drops -3dB from peak value. Therefore, bandwidth = 1.5rad/s = 0.2387 Hz



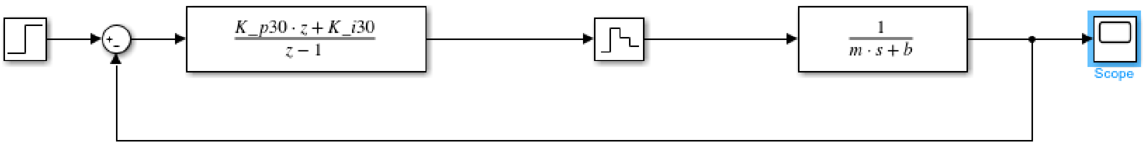
Discretization with Tustin method where $T_s = \frac{1}{30*0.2387} \approx 0.14$

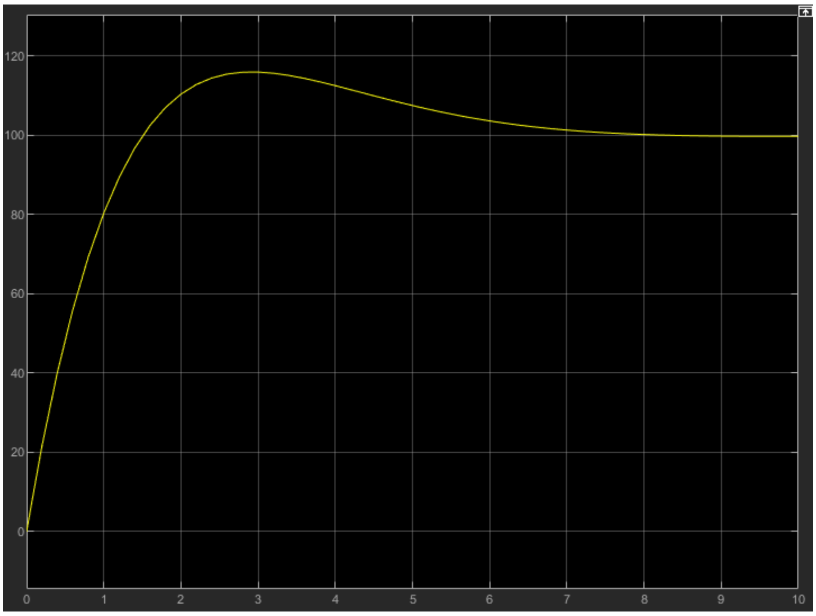
$$C(z) = C(s)\Big|_{s=\frac{2}{T_s}*\frac{z-1}{z+1}} = \frac{1050s + 638}{s}\Big|_{s=\frac{2}{T_s}*\frac{z-1}{z+1}} = \frac{1050\frac{2}{T_s}*\left(\frac{z-1}{z+1}\right) + 638}{\frac{2}{T_s}*\left(\frac{z-1}{z+1}\right)}$$

$$= \frac{(T_s*(z+1))*(1050\frac{2}{T_s}*\frac{z-1}{z+1} + 638)}{2(z-1)} = \frac{2100(z-1) + 638T_s z + 837T_s}{2(z-1)}$$

$$= \frac{2100z - 2100 + 89.32z + 117.18}{2(z-1)} = \frac{\mathbf{1094.66z - 991.41}}{\mathbf{z-1}}$$

With the discretized controller with a sampling rate 30 times the bandwidth, we got the following results:





As we can see from the plot above, performance is almost the same as with the continuous controller. With an 16% which is slightly less than the continuous one, but with a slightly slower settling time.

d)

i)

We discretize the controller with a sampling rate 6 times the bandwidth using the Tustin transformation.

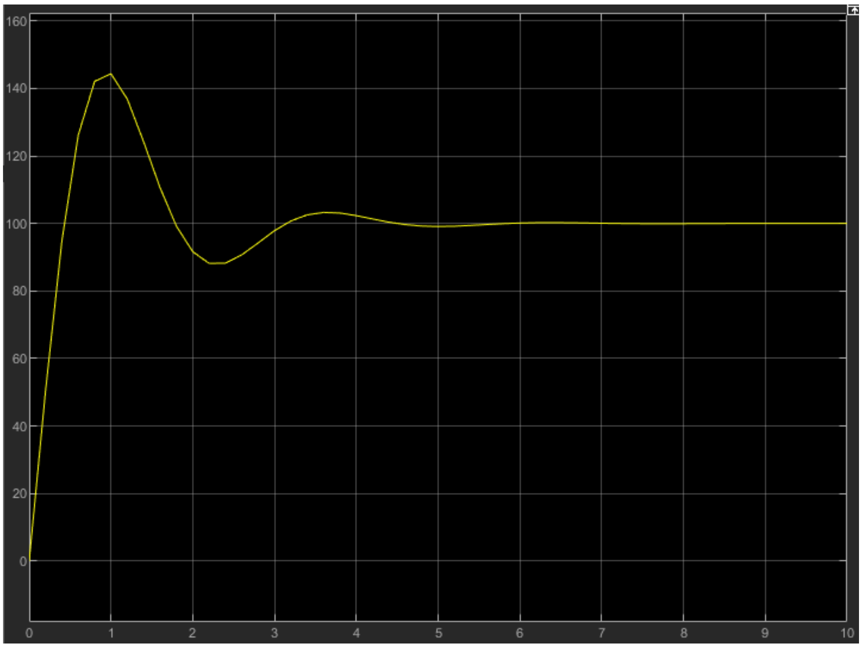
$$T_s = \frac{1}{6 * 0.2387} \approx 0.7$$

$$C(z) = C(s)\Big|_{s=\frac{2}{T_s}*\frac{z-1}{z+1}} = \frac{1050s + 638}{s}\Big|_{s=\frac{2}{T_s}*\frac{z-1}{z+1}} = \frac{1050\frac{2}{T_s} * \left(\frac{z-1}{z+1}\right) + 638}{\frac{2}{T_s} * \left(\frac{z-1}{z+1}\right)}$$

$$= \frac{(T_s * (z+1)) * (1050\frac{2}{T_s} * \frac{z-1}{z+1} + 638)}{2(z-1)} = \frac{2100(z-1) + 638T_s z + 837T_s}{2(z-1)}$$

$$= \frac{2100z - 2100 + 446.6z + 585.9}{2(z-1)} = \frac{\mathbf{2546.6z - 1514.1}}{\mathbf{z - 1}}$$

With this controller we get the following response:



As we can see from the plot above, the controller doesn't perform nearly as good as the continuous one. We get an overshoot of 44.3%, which is far from the system specifications.

ii)

We discretize the plant with $T_s = 0.7$:

$$G(z) = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z}Z\left\{\frac{0.001}{s(s+0.1)}\right\} = \frac{z-1}{z}Z\left\{\frac{0.01}{s} - \frac{0.01}{s+0.1}\right\}$$

$$= 0.01 * \frac{z-1}{z} * \left(\frac{z}{z-1} - \frac{z}{z - e^{-0.1*T_s}}\right) = \frac{\mathbf{0.00058}}{\mathbf{z - 0.942}}$$

Our digital controller:

$$C(z) = k_p + k_i * \frac{z}{z-1} = \frac{(k_p + k_i)z - k_p}{z-1}$$

Our closed loop transfer function therefore is:

$$H(z) = \frac{G(z)C(z)}{1 + G(z)C(z)} = \frac{\frac{0.00058}{z-0.942} * \frac{(k_p + k_i)z - k_p}{z-1}}{1 + \frac{0.00058}{z-0.942} * \frac{(k_p + k_i)z - k_p}{z-1}}$$
$$= \frac{0.00058(k_p + k_i)z - 0.00058k_p}{z^2 - z - 0.942z + 0.942 + 0.00058(k_p + k_i)z - 0.00058k_p}$$
$$= \frac{0.00058(k_p + k_i)z - 0.00058k_p}{z^2 + \left(0.00058\left(k_p + k_i\right) - 1.942\right)z + (0.942 - 0.00058k_p)}$$

We need to find the desired H(z) by using the zeta and omega from the continuous case:

$\zeta = 0.72$

$\omega_n = 0.799 = \frac{0.15\pi}{T}$

From the “s-plane to z-plane”-map we get:

$z_p = 0.7 + j0.2$

From which we get the desired H(z):

$$H(z) = \frac{(1 - z_p)(1 - \bar{z}_p)}{(z - z_p)(z - \bar{z}_p)} = \frac{0.13}{z^2 - 1.4z + 0.53}$$

Now we can get the following gains:

$0.53 = 0.942 - 0.00058k_p$

$k_p = \frac{0.53 - 0.942}{-0.00058} = 710.3$

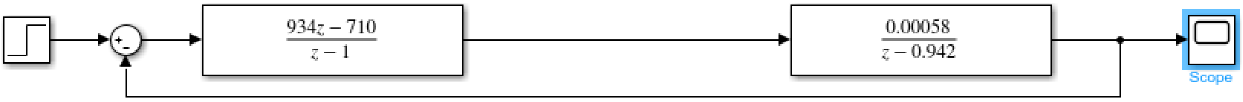
$-1.4 = 0.00058\left(k_p + k_i\right) - 1.942$

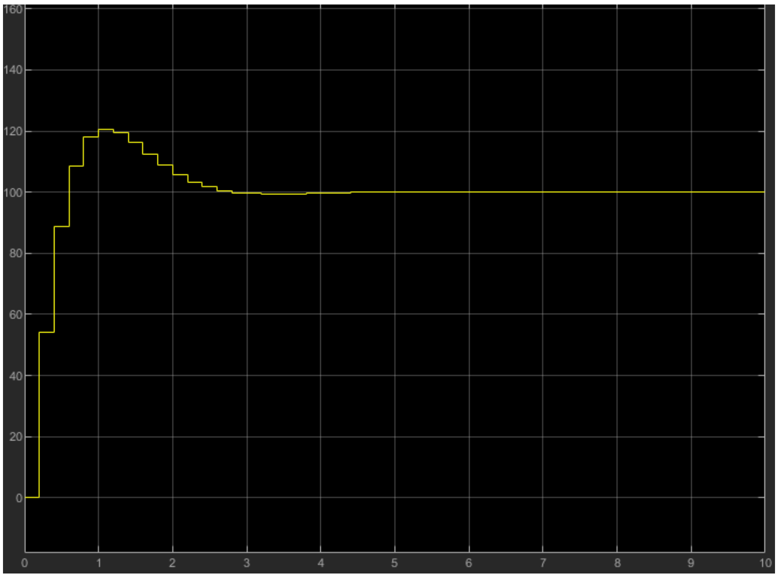
$k_i = \frac{-1.4 + 1.942 - 0.00058 * 710}{0.00058} = 224.5$

We get our final controller:

$$C(z) = k_p + k_i * \frac{z}{z-1} = \frac{(k_p + k_i)z - k_p}{z-1} = \boldsymbol{\frac{934z - 710}{z-1}}$$

Now we simulate:





As we can see from the plot above, the discretized plant with a sampling rate of 6 times the bandwidth with a discrete PI controller, does not meet the design specifications.

We get an overshoot of 20.7%

The m-code used in this homework is short as most was calculated for hand in this document (in the last simulation I just added the gains straight to Simulink):

```
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%Henrik Lucande
%724140
close all
clear all
m = 1000;
b = 100;
K_p = 1050;
K_i = 638;

%c)
%closed loop tf:
nom = 1.435;
den = [1 1.15 1.435];
H = tf(nom,den);
bode(H);
K_p30 = 1094.66;
K_i30 = -991.41;
K_p6 = 2546.6;
K_i6 = -1514.1;
```